SPECTRUM OF DISCRETE SCHRÖDINGER OPERATORS

FUXIN WEI

Spectrum of Discrete Schrödinger Operators

by

©Fuxin Wei

A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics and Statistics

Memorial University of Newfoundland

October 2013

St. John's Newfoundland

Abstract

The purpose of this thesis is to explore the spectra of discrete Schrödinger operators of a special form. We consider the specific way to identify the Schrödinger operator H_{x_0} in our model with an element in the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$, which is generated by the commutative unital C*-algebra $C(X)$ and countable discrete group $\mathbb{Z}$ via the action α with respect to the universal norm. We show that the crossed product $C(Y) \rtimes_\alpha \mathbb{Z}$, where $Y = \overline{Orb_\varphi(x_0)}$, is isomorphic to the concrete algebra $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z})$, where σ_{x_0} is an integrated representation of $C(X) \rtimes_\alpha \mathbb{Z}$ induced by point evaluation μ_{x_0}. As a corollary, we conclude that the spectra of the Schrödinger operators H_{x_0} and H_{x_1} are the same when the closures of the orbits of the two points $x_0, x_1 \in X$ are the same, and apply this result to some special kinds of systems. After considering the classification of the spectra of discrete Schrödinger operators, we give some examples to show the calculation of the spectrum by using K-theory.

Acknowledgements

I am grateful to my supervisor, Dr. Zhuang Niu, and my co-supervisor, Dr. Mikhail V. Kotchetov, for their professional guidance, patience and encouragement.

Table of Contents

Introduction

The Schrödinger equation was proposed by physicist Erwin Schrödinger in 1926. There are two types of Schrödinger equations, time-dependent and time-independent. The time-independent Schrödinger equation is used when dealing with stationary states, i.e., the states that do not change over time, so the wavefunction is a function of position. In the time-dependent Schrödinger equation, the wavefunction is a function of position and time.

We have that Kinetic Energy (T) + Potential Energy (V)=Total Energy (E) from classical mechanics. The Schrödinger equation uses this fundamental principle in terms of its wavefunction:

$$\hat{H}\psi_n = E_n\psi_n,$$

where ψ_n is the wavefunction, $\hat{H}$ is the Hamiltonian operator, and E_n is the nth energy eigenvalue corresponding to ψ_n (solutions exist for the time-independent Schrödinger equation only for certain values of energy). In the time-independent Schrödinger equation, the Hamiltonian operator is equivalent to the total energy operator.

In this thesis, we consider the one-dimensional discrete Schrödinger operator $H = H_{x_0}$ of the form

$$(H_{x_0}\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{0.0.1}$$

on $\ell^2(\mathbb{Z})$, where the potential $V : \mathbb{Z} \to \mathbb{R}$ is given by

$$V(n) = f(\varphi^n(x_0)),$$

with the point x_0 in a compact metric space X, φ a homeomorphism of X onto itself and f a continuous function from X to $\mathbb{R}$.

The operator H is a self-adjoint bounded operator on $\ell^2(\mathbb{Z})$, and hence its spectrum is a non-empty compact set in the real line.

In the first chapter, the basic definitions of $*$-algebras, normed algebras, Banach algebras and C*-algebras are reviewed; the equivalence of a topological system (X, φ) and a dynamical system $(C(X), \alpha)$ is shown (see e.g. [29]), where $C(X)$ is the C*-algebra of all continuous complex-valued functions on X and α is an automorphism of $C(X)$ of the form (1.2.1). Moreover, given a single automorphism α in $\mathrm{Aut}(C(X))$, we know it gives rise to an action of the group $\mathbb{Z}$ on $C(X)$ (see Definition 1.3.7) by $\alpha_n := \alpha^n$. In the following sections, we also denote by α this action of $\mathbb{Z}$ induced by a single automorphism α. In this way, hence, we form a C*-dynamical system $(C(X), \mathbb{Z}, \alpha)$ (see Definition 1.3.10) and then obtain the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ (see e.g. Theorem 1.4.1, [24], [30]). Considering the map $\sigma_{x_0} = \mu_{x_0} \rtimes \Lambda$, which is the representation (see Definition 1.3.13 and Equation (1.4.1)) of $C(X) \rtimes_\varphi \mathbb{Z}$ on the Hilbert space $\ell^2(\mathbb{Z})$ corresponding to the covariant representation $(\tilde{\mu}_{x_0}, \Lambda)$ (see Definition 1.3.16 and Equations (1.4.3)) induced by a representation μ_{x_0} of $C(X)$ on $\mathbb{C}$, we will identify the Schrödinger operator H_{x_0} of the form (0.0.1) with the image of an element in the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ under the representation σ_{x_0}.

In the second chapter, the properties of short exact sequences of C*-algebras are presented and then used to show (see Theorem 2.2.1) that $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z})$ is isomorphic to $C(X) \rtimes_\alpha \mathbb{Z}$ where $Y = \overline{Orb_\varphi(x_0)}$, and $Orb_\varphi(x_0)$ is the orbit of the point x_0 in X

under φ. As a consequence, the spectrum of the Schrödinger operator H_{x_0} of the form (0.0.1) is determined by Y (see Corollary 2.2.6). Moreover, this result is applied to three kinds of dynamical systems: minimal systems, almost minimal systems and essentially minimal systems.

It is important to label gaps in the spectrum of the Schrödinger operator H_{x_0} of the form (0.0.1), where a gap means a connected component in the set $\mathbb{R} \setminus \mathrm{sp}(H_{x_0})$. In the last chapter, we review K_0-groups, K_1-groups and the Pimsner-Voiculescu sequence of a C*-algebra, which is the main tool to label gaps in the spectrum (see e.g. [3], [4], [6],[9], [10], [11], [13], [17], [20], [21]). In the end, some special Schrödinger operators whose spectra are Cantor sets will be given as examples for the calculation of the spectrum (see e.g. [5], [6], [7], [25]).

Chapter 1

Realization of the Schrödinger operators in crossed products

1.1 Preliminaries to C*-algebras

In this part, we will review the definitions of different kinds of algebras and the spectrum of an element in a C*-algebra.

Definition 1.1.1. *An* algebra *(over $\mathbb{C}$) is a vector space A endowed with a product $A \times A \to A$, $(a, b) \mapsto ab$ such that*

(i) $a(bc) = (ab)c$ *for all* $a, b, c \in A$ (associativity),

(ii) $a(b + c) = ab + ac$ *and* $(b + c)a = ba + ca$ *for all* $a, b, c \in A$ (distributivity),

(iii) $(\alpha a)(\beta b) = (\alpha\beta)(ab)$ *for all* $\alpha, \beta \in \mathbb{C}$ *and* $a, b \in A$

(compatibility with scalar multiplication).

Definition 1.1.2. (a) *A* $*$-algebra *is an algebra A provided with a map $* : A \to A$, $a \mapsto a^*$ such that, for all $a, b \in A$ and $\alpha \in \mathbb{C}$,*

(i) $(a+b)^* = a^* + b^*$,

(ii) $(\alpha a)^* = \overline{\alpha} a^*$,

(iii) $(ab)^* = b^* a^*$,

(iv) $(a^*)^* = a$.

The mapping $a \mapsto a^*$ *is called the* involution.

(b) *A* normed algebra *is an algebra* A *with a norm* $\|\cdot\| : A \to \mathbb{R}^+$ *(with the convention that* 0 *belongs to* $\mathbb{R}^+$*),* $a \mapsto \|a\|$ *such that* $\|ab\| \leq \|a\|\|b\|$ *for all* $a, b \in A$.

(c) *A* Banach algebra *is a normed algebra which is complete in its norm.*

(d) *A* C*-algebra A *is a Banach algebra which is at the same time a* $*$*-algebra such that the norm satisfies*

$$\|a^* a\| = \|a\|^2. \tag{1.1.1}$$

for all $a \in A$.

An algebra A is *unital* if it has a multiplicative identity, which will be denoted by e or e_A. It follows from the condition (1.1.1) that $\|e\| = 1$ for any nontrivial unital C*-algebra. A C*-algebra is said to be *separable* if it contains a countable dense subset. A *sub-C*-algebra* of a C*-algebra A is a non-empty closed subset of A which is a $*$-algebra with respect to the operations given on A.
Let A be a C*-algebra, and let F be a subset of A. The sub-C*-algebra of A *generated by* F, denoted by $C^*(F)$, is the intersection of all sub-C*-algebras of A that contain F. We write $C^*(a_1, a_2 \dots, a_n)$ instead of $C^*(\{a_1, a_2 \dots, a_n\})$, when $a_1, a_2, \dots, a_n \in A$.

Theorem 1.1.3. *Let* A *be a* C*-*algebra. Then the involution is isometric,* i.e., $\|a\| = \|a^*\|$.

Proof. By the second and third axioms for C*-algebra A, we know that $\|a^*a\| = \|a\|^2 \leq \|a^*\|\|a\|$ for any element $a \in A$. If $\|a\| = 0$, then $a = a^* = 0$, so $\|a\| = \|a^*\|$. If $\|a\| > 0$, then we obtain $\|a\| \leq \|a^*\|$. On the other hand, since $(a^*)^* = a$, in the similar way, we obtain that $\|a^*\| \leq \|a\|$. Thus, $\|a\| = \|a^*\|$ for all $a \in A$. $\square$

An element a in a C*-algebra A is called

(i) *self-adjoint* if $a^* = a$;

(ii) a *projection* if $a = a^* = a^2$;

(iii) *normal* if $a^*a = aa^*$;

(iv) (if A is unital) *unitary* if $a^*a = e_A = aa^*$,

(v) (if A is unital) *invertible* if there is an element b in A such that $ab = ba = e_A$.

If A is a C*-algebra, then the *unitalization* of A is the unique (up to canonical isometric $*$-isomorphism) C*-algebra $\widetilde{A}$ with multiplicative unit which contains A as a closed, two-sided ideal of linear codimension one. The algebra $\widetilde{A}$ can be constructed as follows (see e.g. [22, Page 5]): If A is contained in a unital C*-algebra B whose unit e_B does not belong to A, then $\widetilde{A}$ is equal (or isomorphic) to the sub-C*-algebra $A + \mathbb{C} \cdot e_B$ of B. If A has a unit e_A, and if $e_{\widetilde{A}}$ is the unit in $\widetilde{A}$, then $f = e_{\widetilde{A}} - e_A$ is a projection in $\widetilde{A}$, and

$$\widetilde{A} = \{a + \alpha f : a \in A, \ \alpha \in \mathbb{C}\}.$$

Definition 1.1.4. *If A is any unital C*-algebra, the* spectrum *of an element $a \in A$ is the set*

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid \lambda e_A - a \text{ is not invertible in } A\}.$$

If A is not unital, then $\mathrm{sp}(a)$ is defined to be the spectrum of a in the unitalization $\widetilde{A}$. (It follows that if A is non-unital, then $0 \in \mathrm{sp}(a)$ for every $a \in A$).

The spectral radius *of a is*

$$r(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}.$$

The spectrum $\mathrm{sp}(a)$ is a nonempty compact subset of $\mathbb{C}$, and the spectral radius satisfies $r(a) \leq \|a\|$ (see e.g. [16, Lemma 1.2.4]).

The following theorem can be found in [23, Theorem 18.9].

Theorem 1.1.5 (Spectral Radius Formula). *For every $x \in A$,*

$$\lim_{n\to\infty} \|x^n\|^{1/n} = r(x). \tag{1.1.2}$$

For a normal element $a \in A$, Equation (1.1.2) reduces to $r(a) = \|a\|$. By (1.1.1), we have that $\|a\| = \sqrt{\|a^*a\|} = \sqrt{r(a^*a)}$, or, equivalently,

$$\|a\|^2 = \sup\{\lambda \in \mathbb{C} \mid \lambda e_A - a^*a \text{ is not invertible in } A\}.$$

This implies that the norm in a C*-algebra is uniquely determined by product and involution.

Definition 1.1.6. *An element a in a* C*-*algebra A is* positive *if it is normal and* $\mathrm{sp}(a) \subseteq \mathbb{R}^+$. *We will write $a \geq 0$ to indicate that a is positive in this sense.*

The set of positive elements in A is denoted by A^+. An element a in A is positive if and only if $a = x^*x$ for some $x \in A$ (see e.g. [22, page 6]).

1.2 The equivalence of a dynamical system and a topological system

A classical dynamical system consists of a compact Hausdorff space X and a homeomorphism φ of X onto itself.

Note that the Schrödinger operator H_{x_0} of the form (0.0.1) is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$ and is determined by the topological system (X, φ, x_0).

Given a topological system (X, φ), consider the C*-algebra $C(X)$ of all the continuous complex-valued functions on a compact Hausdorff space X. Define

$$\alpha(f) = f \circ \varphi^{-1}. \tag{1.2.1}$$

Obviously, $\alpha \in \mathrm{Aut}(C(X))$. Let us show, conversely, that a dynamical system $(C(X), \alpha)$, $\alpha \in \mathrm{Aut}(C(X))$, induces a corresponding topological system (X, φ) such that (1.2.1) holds.

Definition 1.2.1. *Let A be an algebra over $\mathbb{C}$. A* multiplicative linear functional *is a nonzero linear functional $\varphi : A \to \mathbb{C}$ such that*

$$\varphi(xy) = \varphi(x)\varphi(y), \quad \forall x,\, y \in A.$$

Multiplicative linear functionals are also called characters *of A.*

Definition 1.2.2. *Let A be a Banach algebra. A* left ideal (right ideal) *of A is a closed linear subalgebra $I \subseteq A$ for which $a \in I$ implies that $ba \in I$ ($ab \in I$) for all $b \in A$.*

An ideal *in A is a subspace that is both a left and a right ideal (*i.e.*, a two-sided ideal). If $I \neq A$, I is a proper ideal.* Maximal ideals *are proper ideals which are not contained* *in any larger proper ideals.*

Definition 1.2.3. *Let $\varphi : A \to B$ be a map between* C**-algebras A and B. The map $\varphi : A \to B$ is called a* $*$-homomorphism *if it is a linear and multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$; the map $\varphi : A \to B$ is called a* $*$-isomorphism *if it is a $*$-homomorphism which is bijective; the map $\varphi : A \to B$ is called an* isometric $*$-isomorphism *if it is a $*$-isomorphism which is isometric.*

Theorem 1.2.4 ([15, Theorem 1.5.15]). *Every $*$-homomorphism $\varphi : A \to B$ of* C**-algebras is norm-decreasing, and $\varphi(A)$ is always a sub-*C**-algebra of B. If φ is injective, then it is an isometry.*

By Theorem 1.2.4, we know that the kernel of a $*$-homomorphism $\varphi : A \to B$ of C*-algebras is closed in A and the image is closed in B.

The kernel of a $*$-homomorphism of C*-algebras $\pi : A \to B$ is an ideal in A. Conversely, for any ideal $I \subseteq A$, A/I is a C*-algebra and I is the kernel of the quotient map from A to A/I.

Let $\mathcal{H}$ be a Hilbert space. Denote by $B(\mathcal{H})$ the C*-algebra of all bounded linear operators on $\mathcal{H}$ and by $U(\mathcal{H})$ the group of all unitary operators on $\mathcal{H}$.

The following theorem is known as Gelfand-Naimark-Segal Theorem that can be found in e.g. [22, Theorem 1.1.3], in which the GNS-construction is used to construct a Hilbert space $\mathcal{H}$ for a given C*-algebra A such that A can be isometrically embedded into $B(\mathcal{H})$ as a sub-C*-algebra.

Theorem 1.2.5 (Gelfand-Naimark-Segal Theorem). *For each* C**-algebra A there exist a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism φ from A into $B(\mathcal{H})$. If A is separable, then H can be chosen to be a separable Hilbert space.*

The following Gelfand-Naimark Theorem (see e.g. [22, Theorem 1.2.3]) gives a universal model for any commutative C*-algebra.

Theorem 1.2.6 (Gelfand-Naimark Theorem). *Every commutative* C*-*algebra is isometrically* $*$-*isomorphic to the* C*-*algebra* $C_0(X)$ *for some locally compact Hausdorff space* X.

Recall that $C_0(X)$ is the C*-algebra of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity: for each $\varepsilon > 0$ there is a compact subset K of X such that $|f(x)| \leq \varepsilon$ for all $x \in X \setminus K$. The norm on $C_0(X)$ is the supremum norm. If X is compact, then $C_0(X) = C(X)$.

In addition to Gelfand-Naimark Theorem, we have the following properties (see [22, Page 7]):

(i) The C*-algebra $C_0(X)$ is unital if and only if X is compact.

(ii) X and Y are homeomorphic if and only if $C_0(X)$ and $C_0(Y)$ are isometrically $*$-isomorphic.

(iii) There is a bijective correspondence between open sets of X and ideals in $C_0(X)$, which is set up as follows: the ideal corresponding to the open subset U of X is the set $\{f \in C_0(X) \mid f \text{ vanishes on } U^c\} \cong C_0(U)$.

In the following, denote by $\Delta(A)$ the set of all nonzero characters of a commutative Banach algebra A. It is a compact Hausdorff space with respect to the following topology: A neighborhood basis at $\varphi_0 \in \Delta(A)$ is given by the collection of sets

$$U(\varphi_0, x_1, \cdots, x_n, \epsilon) = \{\varphi \in \Delta(A) \mid |\varphi(x_i) - \varphi_0(x_i)| < \epsilon,\ 1 \leq i \leq n\},$$

where $\epsilon > 0$, $n \in \mathbb{N}$, and $x_1, \cdots, x_n$ are arbitrary elements of A. This topology on $\Delta(A)$ is called the *Gelfand topology.*

If A has an identity, then $\Delta(A)$ is also called the *maximal ideal space* of A.

Definition 1.2.7 (Gelfand transform). *Suppose that A is a commutative Banach algebra with $\Delta(A)$ nonempty. Then the Gelfand transform of $a \in A$ is the function*

$$\hat{a} : \Delta(A) \to \mathbb{C}$$

given by $\hat{a}(h) = h(a)$.

The space X corresponding to the commutative C*-algebra A in Gelfand-Naimark Theorem can be chosen as the space $\Delta(A)$ with Gelfand topology, and then the Gelfand transform provides an isometric $*$-isomorphism $A \to C_0(X)$.

Let X be a compact Hausdorff space. By Gelfand-Naimark Theorem, we have that $C(X)$ is isomorphic to $C(\Delta(C(X)))$, hence X is homeomorphic to $\Delta(C(X))$. Explicitly, for any $x \in X$,

$$\mu_x : C(X) \to \mathbb{C}, \ \mu_x(f) = f(x)$$

is a character of $C(X)$, and the mapping $x \mapsto \mu_x$ is a homeomorphism $X \to \Delta(C(X))$.

Definition 1.2.8. *The function μ_x is called a* point evaluation *of $x \in X$.*

Any $\alpha \in \mathrm{Aut}(C(X))$ defines a permutation $\widehat{\alpha} : \Delta(C(X)) \to \Delta(C(X))$ as follows:

$$\xi \overset{\widehat{\alpha}}{\mapsto} \xi \circ \alpha \text{ for any } \xi \in \Delta(C(X)).$$

Since the set $\Delta(C(X))$ coincides with the set of all point evaluations of $C(X)$, i.e., $\Delta(C(X)) = \{\mu_x : x \in X\}$, the permutation $\widehat{\alpha}$ induces a permutation φ of X by $\mu_x \overset{\hat{\alpha}}{\mapsto} \mu_x \circ \alpha =: \mu_{\varphi^{-1}(x)}$. Thus, we have that $\alpha(f) = f \circ \varphi^{-1}$ for all $f \in C(X)$, and then one can show that φ is a homeomorphism and uniquely determined by α.

As discussed above, a topological system (X, φ) corresponds to a dynamical system $(C(X), \alpha)$ uniquely, which implies that studying a topological system (X, φ) is equivalent to investigating the corresponding dynamical system $(C(X), \alpha)$.

In the following sections, we always denote by α the automorphism of $C(X)$ corresponding to the homeomorphism $\varphi : X \to X$.

1.3 C*-dynamical systems

Now we review some basic definitions and properties related to the concept of a C*-dynamical system.

Definition 1.3.1. *A* topological group *is a group $(G, \cdot)$ together with a topology τ such that*

(i) *points are closed in (G, τ), and*

(ii) *the map $G \times G \to G$, $(s, r) \mapsto sr^{-1}$ is continuous.*

Example 1.3.2. *Any group G equipped with the discrete topology is a topological group, for example, $\mathbb{Z}^n$. The groups $\mathbb{R}^n$ and $\mathbb{T}^n = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n \mid |z_i| = 1 \text{ for all } i\}$. are topological groups in their usual topologies.*

Definition 1.3.3. *A (locally) compact group is a topological group for which the underlying topology is (locally) compact.*

Example 1.3.4. *Any discrete group is locally compact; $\mathbb{R}^n$ is locally compact; $\mathbb{T}^n$ is compact.*

Definition 1.3.5 (Group actions on sets)**.** *A group G acts on the left on a set X if there is a map $G \times X \to X$,*

$$(s, x) \mapsto s \cdot x \tag{1.3.1}$$

such that

$$e \cdot x = x \quad \text{and} \quad s \cdot (r \cdot x) = sr \cdot x$$

for all $s, r \in G$ and all $x \in X$, where e is the identity element of G.

If G is a topological group and X is a topological space, then we say the action is continuous if the mapping (1.3.1) is continuous. In this case, X is called a left *G-space* and the pair (G, X) is called a *transformation group.* If both G and X are locally compact, then (G, X) is called a *locally compact transformation group.*

Example 1.3.6. *Let φ be a homeomorphism of a space X onto itself. Then $\mathbb{Z}$ acts on X by $n \cdot x := \varphi^n(x)$, and $(\mathbb{Z}, X)$ is a transformation group.*

Definition 1.3.7 (Group actions on C*-algebras)**.** *Let G be a topological group and let A be a C*-algebra. A map $\alpha : G \to \mathrm{Aut}(A)$, $t \mapsto \alpha_t$ is called an* action of G on A *if*

(1) *for any $t, s \in G$, we have $\alpha_t \circ \alpha_s = \alpha_{ts}$,*

(2) *for any $a \in A$, the map $G \to A$, $t \mapsto \alpha_t(a)$ is continuous.*

Let (G, X) be a locally compact transformation group. Then for each $s \in G$, the map $x \mapsto s \cdot x$ is a homeomorphism of X. Therefore we obtain a homomorphism

$$\alpha : G \to \mathrm{Aut}(C_0(X)) \tag{1.3.2}$$

defined by

$$\alpha_s(f)(x) := f(s^{-1} \cdot x).$$

Indeed,

$$\alpha_{ts} = \alpha_t \circ \alpha_s \tag{1.3.3}$$

since

$$\alpha_{ts}(f)(x) = f(s^{-1}t^{-1} \cdot x) = \alpha_s(f)(t^{-1} \cdot x) = \alpha_t(\alpha_s(f))(x).$$

The set $\mathrm{Aut}(A)$ of automorphisms of a C*-algebra A is a group under composition.

Definition 1.3.8. *The* point-norm topology *on* $\mathrm{Aut}(A)$ *is the topology of point-wise convergence of functions on* A*; thus* $\alpha_i \to \alpha$ *in the point-norm topology if and only if* $\|\alpha_i(a) - \alpha(a)\| \to 0$ *for all* $a \in A$.

Lemma 1.3.9 ([30, Lemma 2.5])**.** *Suppose that* (G, X) *is a locally compact transformation group and that* $\mathrm{Aut}(C_0(X))$ *is given the point-norm topology. Then the associated homomorphism (1.3.2) of* G *into* $\mathrm{Aut}(C_0(X))$ *is continuous.*

Definition 1.3.10. *A* C*-dynamical system *is a triple* (A, G, α) *consisting of a* C*-*algebra* A, *a locally compact group* G *and an action* α *of* G *on* A.

Example 1.3.11. *For any* $\alpha \in \mathrm{Aut}(A)$, *we can define an action of* $\mathbb{Z}$ *on* A *by* $n \cdot a = \alpha^n(a)$. *Such a dynamical system* $(A, \mathbb{Z}, \alpha)$, *with* $A = C(X)$, *already appeared in Section 1.2.*

Equation (1.3.3) and Lemma 1.3.9 tell us that the homomorphism (1.3.2) is an action of the topological group G on the C*-algebra $C_0(X)$, which means that there is a C*-dynamical system $(C_0(X), G, \alpha)$ induced by a transformation group (G, X). The following proposition tells us that, conversely, a transformation group (G, X) can be induced by a C*-dynamical system $(C_0(X), G, \alpha)$.

Proposition 1.3.12 ([30, Lemma 2.7])**.** *Suppose that* $(C_0(X), G, \alpha)$ *is a* C*-*dynamical system (with* X *locally compact). Then there is a transformation group* (G, X) *such that*

$$\alpha_s(f)(x) = f(s^{-1} \cdot x).$$

Definition 1.3.13 (A representation of a C*-algebra)**.** *A* representation π *of a* C*-*algebra* A *on a Hilbert space* $\mathcal{H}$ *is a* $*$*-homomorphism of* A *into* $B(\mathcal{H})$. *A representation* π *is* nondegenerate *if the set*

$$\{\pi(a)\xi \mid a \in A,\ \xi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$. We say that a representation π is faithful *if it is an injective map.*

Definition 1.3.14 (A unitary representation of a group). *A* unitary representation *of a group G on a Hilbert space $\mathcal{H}$ is a homomorphism $U : G \to U(\mathcal{H})$ such that the map $t \mapsto U_t(x)$ is continuous for every vector $x \in \mathcal{H}$.*

For a locally compact group G with a left Haar measure μ (see [30, Section 1.3]), denote by $L^2(G)$ the Hilbert space of equivalence classes of Borel measurable functions f on G with complex values such that $\int_G |f(s)|^2 d\mu(s) < \infty$; denote by $L^\infty(G)$ the Banach space of all essentially bounded functions $G \to \mathbb{C}$ with respect to the Haar measure. More generally, let $\mathcal{H}$ be a complex Hilbert space. We define $L^2(G, \mathcal{H})$ to be the space of equivalence classes of Borel measurable functions f on G with values in $\mathcal{H}$. If G is a countable discrete group, then the Haar measure is just counting measure, so we can think of elements of $L^2(G)$ and $L^2(G, \mathcal{H})$ as sequences indexed by G. The traditional notation for $L^2(G)$ and $L^2(G, \mathcal{H})$ in this case is $\ell^2(G)$ and $\ell^2(G, \mathcal{H})$, respectively.

Example 1.3.15. *Let G be a locally compact group and consider $L^2(G)$. Then, for $r \in G$, the translation $\lambda(r)$ is defined by*

$$\lambda(r)f(s) := f(r^{-1}s), \quad f \in L^2(G).$$

Since translation $\lambda(r)$ is a unitary operator on $L^2(G)$, it follows that $\lambda : G \to U(L^2(G))$ is a representation of G. It is called the left regular representation. *More generally, we can define a left regular representation of G on $L^2(G, \mathcal{H})$ for any Hilbert space $\mathcal{H}$.*

Definition 1.3.16 (Covariant representation). *Suppose that (A, G, α) is a C^*-dynamical system and that $\mathcal{H}$ is a Hilbert space. Then a* covariant representation *of (A, G, α) into $B(\mathcal{H})$ is a pair (π, U) consisting of a representation $\pi : A \to B(\mathcal{H})$ and a unitary* *representation $U : G \to U(\mathcal{H})$ on the same Hilbert space such that*

$$U_s \pi(a) U_s^* = \pi(\alpha_s(a)) \text{ for all } a \in A, \ s \in G.$$

Example 1.3.17. *Covariant representations of a dynamical system* $(\mathbb{C}, G, \mathrm{triv})$ *correspond to unitary representations of* G.

1.4 Construction of crossed products

In the following, we will concentrate on a C*-dynamical system (A, G, α), where A is a unital separable C*-algebra and G is a countable discrete group.

Consider the space $C_c(G, A)$ of continuous functions from G to A with compact support, i.e.,

$$C_c(G, A) = \{f = \sum_{t \in G} a_t u_t \mid a_t \in A \text{ for all } t \in G, \text{ and } a_t \neq 0 \text{ for only finitely many } t \in G\},$$

where $\{u_t : t \in G\}$ is the basis of the space $C_c(G, A)$ given by

$$u_t(s) = \begin{cases} e_A, & \text{if } t = s, \\ 0, & \text{if } t \neq s. \end{cases}$$

Instead of the usual point-wise multiplication, we define the multiplication of $C_c(G, A)$ by the formal rules

$$u_t a = \alpha_t(a) u_t \text{ and } u_t u_s = u_{ts}.$$

Thus we obtain the *twisted convolution product*:

$$fg = \sum_{s \in G} \Big(\sum_{t \in G} a_t \alpha_t(b_{t^{-1}s}) \Big) u_s \in C_c(G, A).$$

where $f = \sum_{t\in G} a_t u_t$ and $g = \sum_{s\in G} b_s u_s \in C_c(G, A)$.

The involution is determined by the rule $u_t^* = u_{t^{-1}}$, so

$$f^* = \Big(\sum_{t\in G} a_t u_t\Big)^* = \sum_{t\in G} \alpha_t(a_{t^{-1}}^*) u_t \in C_c(G, A).$$

One can check that the space $C_c(G, A)$ endowed with the multiplication and involution as above is a $*$-algebra.

Note that a covariant representation (π, U) of the C*-dynamical system (A, G, α) induces a $*$-representation of $C_c(G, A)$ by

$$\sigma(f) = \sigma\Big(\sum_{t\in G} a_t u_t\Big) = \sum_{t\in G} \pi(a_t) U_t. \tag{1.4.1}$$

Indeed,

$$\sigma(f)^* = \sum_{t\in G} U_t^* \pi(a_t)^* = \sum_{t\in G} U_{t^{-1}} \pi(a_t^*) U_t U_{t^{-1}} = \sum_{s\in G} \pi(\alpha_s(a_{s^{-1}}^*)) U_s = \sigma(f^*),$$

and

$$\sigma(f)\sigma(g) = \sum_{t\in G}\sum_{u\in G} \pi(a_t) U_t \pi(b_u) U_u = \sum_{s\in G}\Big(\sum_{t\in G} \pi(a_t \alpha_t(b_{t^{-1}s}))\Big) U_s = \sigma(fg).$$

This representation σ is denoted by $\pi \rtimes U$ and called the *integrated representation* induced by (π, U).

Conversely, when A is unital, a $*$-representation σ of $C_c(G, A)$ yields a covariant representation (π, U) of (A, G, α) as follows:

$$\pi(a) = \sigma(a u_e) \text{ and } U_s = \sigma(u_s).$$

For indeed,

$$U_s\pi(a)U_s^* = \sigma(s)\sigma(ae)\sigma(s^*) = \sigma(sas^{-1}) = \sigma(\alpha(a)e) = \pi(\alpha(a)).$$

Theorem 1.4.1 ([30, Lemma 2.27]). *Suppose that (A, G, α) is a C*-dynamical system. For each $f \in C_c(G, A)$, define*

$$\|f\| := \sup\{\|(\pi \rtimes U)(f)\| \mid (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}. \quad (1.4.2)$$

Then $\|\cdot\|$ is a norm on $C_c(G, A)$ called the universal norm. *The universal norm is dominated by the $\|\cdot\|_1$-norm.*

The $\|\cdot\|_1$-norm on $C_c(G, A)$ is defined as $\|f\|_1 = \sum_{t\in G} \|a_t\|_A$, for all $f = \sum_{t\in G} a_t u_t \in C_c(G, A)$.

Definition 1.4.2. *The completion of $C_c(G, A)$ with respect to $\|\cdot\|$ is a C*-algebra called the* crossed product *of A by G and denoted by $A \rtimes_\alpha G$.*

In the following, denote by $A \overset{\|\cdot\|}{=\!=} \overline{B}$ the C*-algebra A generated by a $*$-algebra B with respect to the norm $\|\cdot\|$. We will denote by $\|\cdot\|$ the universal norm on the $*$-algebra $C_c(G, A)$ and by $\|\cdot\|_I$ the universal norm on the $*$-algebra $C_c(G, I)$, where I is an α-invariant ideal in the C*-algebra A. Then, by the definition of crossed products, we know that $A \rtimes_\alpha G \overset{\|\cdot\|}{=\!=} \overline{C_c(G, A)}$ and $I \rtimes_\alpha G \overset{\|\cdot\|_I}{=\!=} \overline{C_c(G, I)}$

The integrated representation $\sigma = \pi \rtimes U$ of $C_c(G, A)$ extends uniquely to a representation of $A \rtimes G$, also denoted by $\pi \rtimes U$.

In general, it is not obvious that there are any covariant representations of a given dynamical system, although Theorem 1.4.1 implies, in particular, that they must exist. On the other hand, the GNS theory (see e.g. Theorem 1.2.5, [1, Section 1.6], [8, Page 29], [12, Page 357]), constructs lots of representations of a given C*-algebra. But it could be difficult to find the universal norm, so it will be useful to display a concrete realization of the crossed product $A \rtimes_\alpha G$. This is done via regular representations which give rise to the reduced norm on $C_c(G, A)$. At first, let us look at the way in which the regular representations are obtained.

Let π be any $*$-representation of A on a Hilbert space $\mathcal{H}$. We form the Hilbert space $\ell^2(G, \mathcal{H}) = L^2(G, \mathcal{H})$ of all square summable functions x from G into $\mathcal{H}$ with the norm

$$\|x\|_2^2 = \sum_{t \in G} \|x(t)\|^2.$$

Define a covariant representation $(\tilde{\pi}, \Lambda)$ of the C*-dynamical system (A, G, α) on $\ell^2(G, \mathcal{H})$ by

$$\begin{aligned}(\tilde{\pi}(a)x)(s) &= \pi(\alpha_s^{-1}(a))(x(s)),\\ (\Lambda_t x)(s) &= x(t^{-1}s)\end{aligned} \tag{1.4.3}$$

for all $a \in A$, $x \in \ell^2(G, \mathcal{H})$ and $s,\ t \in G$.

Indeed, $\tilde{\pi}$ is a $*$-representation of A, Λ is a left regular representation of G, and the covariance condition is also satisfied since we have that

$$\begin{aligned}(\Lambda_t \tilde{\pi}(A) \Lambda_t^* x)(s) &= (\tilde{\pi}(A)\Lambda_t^* x)(t^{-1}s) = \pi(\alpha_{t^{-1}s}^{-1}(A))(\Lambda_t^* x(t^{-1}s))\\ &= \pi(\alpha_s^{-1}\alpha_t(A))(x(s)) = (\tilde{\pi}(\alpha_t(A))x)(s)\end{aligned}$$

for all $x \in \ell^2(G, \mathcal{H})$ and $s,\ t \in G$.

By Equations (1.4.1) and (1.4.3), the integrated representation σ of $C_c(G, A)$ induced by covariant representation $(\tilde{\pi}, \Lambda)$ has the form:

$$\begin{aligned}(\sigma(f)x)(s) &= \sum_{t\in G}(\tilde{\pi}(a_t)\Lambda_t x)(s) = \sum_{t\in G}\pi(\alpha_s^{-1}(a_t))(\Lambda_t x(s)) \\ &= \sum_{t\in G}\pi(\alpha_s^{-1}(a_t))x(t^{-1}s) \end{aligned} \tag{1.4.4}$$

for $f \in C_c(G, A)$ with the finite form $f = \sum_{t\in G} a_t u_t$, $x \in \ell^2(G, \mathcal{H})$ and $s,\ t \in G$. The kernel of the integrated representation $\sigma = \tilde{\pi} \rtimes \Lambda$ of $A \rtimes_\alpha G$ induced by the representation π of A is determined by the kernel of π (see [30, Chapter 5]). In particular, if π and π' are both faithful representations of A, then the integrated representations $\tilde{\pi} \rtimes \Lambda$ and $\tilde{\pi}' \rtimes \Lambda'$ have the same kernel, and

$$\|(\tilde{\pi} \rtimes \Lambda)(f)\| = \|(\tilde{\pi}' \rtimes \Lambda')(f)\| \text{ for all } f \in A \rtimes_\alpha G.$$

This makes the following definition reasonable because it is independent of the choice of a faithful representation π.

Definition 1.4.3. *If (A, G, α) is a C*-dynamical system, then the* reduced norm *on $C_c(G, A)$ is given by*

$$\|f\|_r := \|(\tilde{\pi} \rtimes \Lambda)(f)\|,$$

where $\tilde{\pi} \rtimes \Lambda$ is the representation of $C_c(G, A)$ on $\ell^2(G, \mathcal{H})$ induced by any faithful representation π of A on the Hilbert space $\mathcal{H}$. The completion $A \rtimes_{\alpha,r} G$ of $C_c(G, A)$ with respect to $\|\cdot\|_r$ is called the reduced crossed product.

Below we discuss an important case where $A \rtimes_{\alpha,r} G$ coincides with $A \rtimes_\alpha G$.

Lemma 1.4.4 ([30, Lemma 7.8]). *Suppose that (A, G, α) is a C*-dynamical system. Then the reduced crossed product $A \rtimes_{\alpha,r} G$ is (isomorphic to) the quotient of $A \rtimes_\alpha G$ by the kernel of $\tilde{\pi} \rtimes \Lambda$ for any faithful representation π of A.*

Definition 1.4.5. *A* positive linear functional *on a* C*-*algebra is a linear functional such that* $f(a) \geq 0$ *whenever* $a \geq 0$. *A* state *is a positive linear functional of norm* 1.

If G is a locally compact topological group, a *mean* on G is a state on the C*-algebra $L^\infty(G)$.

Definition 1.4.6. *A group G is called* amenable *is there is a left translation invariant mean for G. Here,* left invariance *indicates that* $\mu(g_s) = \mu(g)$ *for all* $g \in L^\infty(G)$ *and* $s \in G$, *where* $g_s \in L^\infty(G)$ *given by* $g_s(t) = g(s^{-1}t)$ *for all* $t \in G$.

Example 1.4.7. *All finite groups and all abelian groups are amenable.*

Theorem 1.4.8 ([30, Theorem 7.13]). *If G is amenable, then the reduced norm* $\|\cdot\|_r$ *coincides with the universal norm on* $C_c(G, A)$, *and hence* $A \rtimes_{\alpha,r} G = A \rtimes_\alpha G$. *In particular, if π is a faithful representation of A, then* $\tilde{\pi} \rtimes \Lambda$ *is a faithful representation of* $A \rtimes_\alpha G$.

Remark 1.4.9. *Let A be a* C*-*algebra. The group $\mathbb{Z}$ is an amenable group, so the full crossed product* $A \rtimes_\alpha \mathbb{Z}$ *and the reduced crossed product* $A \rtimes_{\alpha,r} \mathbb{Z}$ *are identical.*

A universal C*-*algebra* is a C*-algebra characterized by a universal property (see [31]). A universal C*-algebra can be expressed as a presentation, in terms of generators and relations. For example, the universal C*-algebra generated by a unitary element u has presentation $\langle u \mid u^*u = uu^* = 1 \rangle$.

By [26], [27] and [28], the crossed product $A \rtimes_\alpha \mathbb{Z}$ also has the following interpretation:

Remark 1.4.10. *Let A be a unital* C*-*algebra, and let* $\alpha \in \mathrm{Aut}(A)$. *Then the crossed product* $A \rtimes_\alpha \mathbb{Z}$ *is the universal* C*-*algebra generated by A and a unitary u subject to the relations* $uau^* = \alpha(a)$ *for all* $a \in A$.

If $A = C(X)$, where X is a compact Hausdorff space, one has that

$$C(X) \rtimes_\alpha \mathbb{Z} = \langle\, C(X), u \mid u^*u = uu^* = 1,\ ufu^* = \alpha(f) \text{ for all } f \in C(X) \,\rangle$$

$$\overset{\|\cdot\|}{=\!=} \overline{\{\sum_{n\in\mathbb{Z}} f_n u^n \mid f_n \in C(X) \text{ for all } n \in \mathbb{Z}, \text{ and } f_n \neq 0 \text{ for only finitely many } n \in \mathbb{Z}\}}.$$

1.5 Identification of the Schrödinger operator in $C(X) \rtimes_\alpha \mathbb{Z}$

We will identify the Schrödinger operator of the form (0.0.1) for a given self-adjoint element f in $C(X)$, i.e., a real-valued continuous function on X. The space $\ell^2(\mathbb{Z})$ has the standard basis

$$\{\ldots, e_{-1}, e_0, e_1, e_2, \ldots\}.$$

Then every element $H \in B(\ell^2(\mathbb{Z}))$ corresponds to a unique matrix M_H. Thus, H_{x_0} of the form (0.0.1) corresponds to the matrix $M_{H_{x_0}}$:

$$\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & f(\varphi^{-1}(x_0)) & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 1 & f(x_0) & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & f(\varphi(x_0)) & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & f(\varphi^2(x_0)) & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & f(\varphi^3(x_0)) & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Choosing the Hilbert space $\mathcal{H} = \mathbb{C}$, we have that $B(\mathcal{H}) = B(\mathbb{C}) = \mathbb{C}$. For a representation π of $C(X)$ on $\mathbb{C}$, by Formulas (1.4.3), we obtain a regular representation $(\widetilde{\pi}, \Lambda)$

of $(C(X), \mathbb{Z}, \alpha)$ on $\ell^2(\mathbb{Z})$:

$$(\widetilde{\pi}(f)\psi)(n) = \pi(\alpha_n^{-1}(f))(\psi(n)),$$

$$(\Lambda_m \psi)(n) = \psi(n-m).$$

By Gelfand-Naimark Theorem, we know that there is an element $x_0 \in X$ such that $\pi = \mu_{x_0}$, hence

$$\pi(\alpha_n^{-1}(f)) = \mu_{x_0}(\alpha_n^{-1}(f)) = \alpha_n^{-1}(f)(x_0) = f(\varphi^n(x_0)).$$

Considering the integrated representation $\sigma_{x_0} = \widetilde{\pi} \rtimes \Lambda$ of $C_c(\mathbb{Z}, C(X))$ on $\ell^2(\mathbb{Z})$, we obtain from (1.4.4) that $\sigma_{x_0}(u) = \Lambda_1$ and $\sigma_{x_0}(u^*) = \Lambda_{-1}$.

Therefore, we know that the element $\sigma_{x_0}(u) = \Lambda_1 \in B(\ell^2(\mathbb{Z}))$ corresponds to the matrix

$$M_u = \Lambda_1 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

the element $\sigma_{x_0}(u^*) = \Lambda_{-1} \in B(\ell^2(\mathbb{Z}))$ corresponds to the matrix

$$M_{u^*} = \Lambda_{-1} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

and the element $\sigma_{x_0}(f) = \widetilde{\pi}(f) \in B(\ell^2(\mathbb{Z}))$ corresponds to the matrix $M_f = \widetilde{\pi}(f)$:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f(\varphi^{-1}(x_0)) & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & f(x_0) & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & f(\varphi(x_0)) & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & f(\varphi^2(x_0)) & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & f(\varphi^3(x_0)) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence, the operator $\sigma_{x_0}(u + u^* + f) \in B(\ell^2(\mathbb{Z}))$ corresponds to the matrix M_{u+u^*+f}:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f(\varphi^{-1}(x_0)) & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & f(x_0) & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & f(\varphi(x_0)) & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & f(\varphi^2(x_0)) & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & f(\varphi^3(x_0)) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we obtain that the Schrödinger operator H_{x_0} of the form (0.0.1) and the operator $\sigma_{x_0}(u + u^* + f)$ correspond to the same matrix with respect to the same canonical basis $\{\ldots, e_{-1}, e_0, e_1, e_2, \ldots\}$ of the space $\ell^2(\mathbb{Z})$, so $H_{x_0} = \sigma_{x_0}(u + u^* + f)$.

Under some conditions (see e.g. minimal systems or almost minimal systems in Section 2.3), the integrated representation $\sigma_{x_0} = \widetilde{\pi} \rtimes \Lambda$ is faithful. Then we can identify $u + u^* + f \in C(X) \rtimes_\alpha \mathbb{Z}$ and H_{x_0}. The relationship between the spectra of $u + u^* + f$ and H_{x_0} will be discussed in the following sections even if the integrated representation σ_{x_0} is not faithful.

Remark 1.5.1. *If $f \in C(X)$ is self-adjoint,* i.e., *real-valued, then $u + u^* + f$ is self-adjoint, and $H_{x_0} = \sigma_{x_0}(u + u^* + f)$ is a self-adjoint operator in $B(\ell^2(\mathbb{Z}))$.*

Chapter 2

Spectrum of discrete Schrödinger operators

2.1 Short exact sequences for crossed products

In this section, the basic concepts and results related to short exact sequences for crossed products are shown.

Definition 2.1.1. *A sequence of* C**-algebras and* $*$*-homomorphisms*

$$\cdots \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+1} \longrightarrow \cdots$$

is said to be exact *if* $\mathrm{Im}(\varphi_n) = \mathrm{Ker}(\varphi_{n+1})$ *for all* n*. And exact sequence of the form*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{2.1.1}$$

is called short exact.

Remark 2.1.2 ([22, Page 4]). *If I is an ideal in A, then*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{q} A/I \longrightarrow 0$$

is a short exact sequence, where ι is the inclusion mapping and q is the quotient mapping. Conversely, given (2.1.1), $\varphi(I)$ is an ideal in A, the C-algebra B is isomorphic to $A/\varphi(I)$, and we have a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\cong"', "\varphi"] & A \arrow[r, "\psi"] \arrow[d, equal] & B \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \varphi(I) \arrow[r, hook, "\iota"] & A \arrow[r, "q"] & A/\varphi(I) \arrow[r] & 0 \end{tikzcd} \tag{2.1.2}$$

Remark 2.1.3. *Let (A, G, α) be a dynamical system. Let $I_G(A)$ denote the set of α-invariant (closed two-sided) ideals in A.*

If $I \in I_G(A)$, then each α_s restricts to an automorphism of I and we obtain a dynamical system (I, G, α) as well as a quotient system $(A/I, G, \alpha)$ defined in the following way:

$$\alpha_s^I(a + I) := \alpha_s(a) + I.$$

Remark 2.1.4. *Recall that an* equivariant map *is a function between two sets that commutes with the action of a group. Specifically, let G be a group, and let X and Y be two G-sets. A function $f : X \to Y$ is said to be* equivariant *if*

$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and all $x \in X$.

Theorem 2.1.5 ([30, Corollary 2.48]). *Suppose that (A, G, α) and (B, G, β) are dynamical systems and that $\varphi : A \to B$ is an equivariant homomorphism. Then there is* *a homomorphism* $\varphi \rtimes \mathrm{id} : A \rtimes_\alpha G \to B \rtimes_\beta G$ *mapping* $C_c(G, A)$ *into* $C_c(G, B)$ *such that*

$$(\varphi \rtimes \mathrm{id})(f)(s) = \varphi(f(s)).$$

Since the inclusion map $\iota : I \to A$ and the quotient map $q : A \to A/I$ are equivariant homomorphisms, by Theorem 2.1.5, we can define homomorphisms

$$\iota \rtimes \mathrm{id} : I \rtimes_\alpha G \to A \rtimes_\alpha G, \tag{2.1.3}$$

and

$$q \rtimes \mathrm{id} : A \rtimes_\alpha G \to A/I \rtimes_\alpha G. \tag{2.1.4}$$

Note that $C_c(G, I)$ sits in $C_c(G, A)$ as a $*$-closed two-sides ideal. Therefore the closure with respect to the same norm is an ideal of $A \rtimes_\alpha G$ which is denoted by $\mathrm{Ex}I$. The next lemma will allow us to identify $\mathrm{Ex}I$ and $I \rtimes_\alpha G$.

Lemma 2.1.6 ([30, Lemma 3.17]). *If* (A, G, α) *is a dynamical system and if* I *is an* α*-invariant ideal in* A*, then* $\iota \rtimes \mathrm{id}$ *is an isometric* $*$*-isomorphism of* $I \rtimes_\alpha G$ *onto* $\mathrm{Ex}I$*.*

Proposition 2.1.7 ([30, Proposition 3.19]). *Suppose that* (A, G, α) *is a dynamical system and* I *is an* α*-invariant ideal in* A*. Then* $\iota \rtimes \mathrm{id}$ *is an isomorphism identifying* $I \rtimes_\alpha G$ *with* $\mathrm{Ex}I = \mathrm{Ker}\,(q \rtimes \mathrm{id})$ *and we have a short exact sequence*

$$0 \longrightarrow I \rtimes_\alpha G \xrightarrow{\iota \rtimes \mathrm{id}} A \rtimes_\alpha G \xrightarrow{q \rtimes \mathrm{id}} A/I \rtimes_\alpha G \longrightarrow 0.$$

of C*-*algebras.*

According to [24, Page 240], we know that $I \rtimes_\alpha G$ is the smallest ideal in $A \rtimes_\alpha G$ containing the α-invariant ideal in A.

Corollary 2.1.8 ([8, Proposition III.3.3]). *Suppose that X is a compact Hausdorff space and Y is a nonempty closed invariant subset of topological system (X, φ), then the ideal $C_0(X \setminus Y) = \{f \in C(X) \mid f|_Y = 0\}$ generates a proper ideal of $C(X) \rtimes_\alpha \mathbb{Z}$.*

Recall that there is a dynamical system $(C(X), \alpha)$ corresponding to a topological dynamical system (X, φ).

Corollary 2.1.9. *Suppose that X is a compact Hausdorff space and Y is a closed invariant subset of topological system (X, φ), then we have a short exact sequence*

$$0 \longrightarrow C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z} \xrightarrow{\iota \rtimes \mathrm{id}} C(X) \rtimes_\alpha \mathbb{Z} \xrightarrow{q \rtimes \mathrm{id}} C(Y) \rtimes_\alpha \mathbb{Z} \longrightarrow 0. \tag{2.1.5}$$

of C**-algebras.*

By Lemma 2.1.6, we know that $C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z}$ and $\mathrm{Ex}\, C_0(X \setminus Y)$ are isometrically $*$-isomorphic.

2.2 Spectrum of the Schrödinger operator

For a representation μ_{x_0} of $C(X)$ on the Hilbert space $\mathcal{H} = \mathbb{C}$, there is a regular covariant representation $(\tilde{\mu}_{x_0}, \Lambda)$ on the Hilbert space $\ell^2(\mathbb{Z})$, and then there is an integrated representation $\sigma_{x_0} = \tilde{\mu}_{x_0} \rtimes \Lambda$ of $C(X) \rtimes_\alpha \mathbb{Z}$ on $\ell^2(\mathbb{Z})$. We will use these notations in the following results.

A sub-C*-algebra of $B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, is called a *concrete* C**-algebra.*

Theorem 2.2.1. *For any point $x_0 \in X$, denote $Y = \overline{Orb_\varphi(x_0)}$. Let π_{x_0} be the representation of $C(X)$ corresponding to the point x_0. Then, the crossed product $C(Y) \rtimes_\alpha \mathbb{Z}$ is isomorphic to the concrete algebra $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z})$.*

Proof. Since there is a short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\sigma_{x_0}) \xrightarrow{\iota} C(X) \rtimes_\alpha \mathbb{Z} \xrightarrow{\sigma_{x_0}} \sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \longrightarrow 0\,,$$

in order to show $C(Y) \rtimes_\alpha \mathbb{Z}$ is isomorphic to the concrete algebra $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z})$, by the commutative diagram (2.1.2), we only need to show

$$\mathrm{Ker}(\sigma_{x_0}) = C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z},$$

where we have identified $C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z}$ with an ideal of $C(X) \rtimes_\alpha \mathbb{Z}$ using $\iota \rtimes \mathrm{id}$ as in short exact sequence (2.1.5). Choosing $\mathcal{H} = \ell^2(\mathbb{Z})$, we define a map $\pi : C(Y) \to B(\mathcal{H})$,

$$f \mapsto M_f = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f(\varphi^{-1}(x_0)) & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & f(x_0) & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & f(\varphi(x_0)) & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & f(\varphi^2(x_0)) & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & f(\varphi^3(x_0)) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously, the map $\pi : C(Y) \to B(\mathcal{H})$ is a $*$-homomorphism. We claim that it is faithful. Indeed, if $M_f = 0$, then $f|_{\mathrm{Orb}_\varphi(\mathrm{x}_0)} = 0$ and hence $f = 0$ since f is continuous on Y. The faithful representation π of $C(Y)$ on $\mathcal{H}$ induces the integrated representation $\tilde{\pi} \rtimes \Lambda$ of $C_c(\mathbb{Z}, C(Y))$ on the Hilbert space $\ell^2(\mathbb{Z}, \mathcal{H})$ and then $\tilde{\pi} \rtimes \Lambda$ extends to a representation also denoted by $\tilde{\pi} \rtimes \Lambda : C(Y) \rtimes_\alpha \mathbb{Z} \to B(\ell^2(\mathbb{Z}, \mathcal{H}))$. Since $\mathbb{Z}$ is amenable, $\tilde{\pi} \rtimes \Lambda$ is faithful by Theorem 1.4.8. Let $q : C(X) \to C(Y)$ be the restriction $f \mapsto f|_Y$ as in (2.1.5). The composition map $\pi q : C(X) \to B(\mathcal{H})$, $f \mapsto M_{f|_Y}$ induces the integrated representation $\widetilde{\pi q} \rtimes \Lambda = (\tilde{\pi} \rtimes \Lambda)(q \rtimes \mathrm{id})$ on $\ell^2(\mathbb{Z}, \mathcal{H})$.

Since $\widetilde{\pi} \rtimes \Lambda$ is faithful, by (2.1.5), we have that

$$\operatorname{Ker}(\widetilde{\pi q} \rtimes \Lambda) = \operatorname{Ker}(q \rtimes \mathrm{id}) = C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z} \tag{2.2.1}$$

Let $f = \sum_{n\in\mathbb{Z}} f_n u^n \in C_c(\mathbb{Z}, C(X))$. We identify $\ell^2(\mathbb{Z}, \mathcal{H})$ with $\ell^2(\mathbb{Z}^2)$ by setting $\Psi(k,j) = \Psi(k)(j)$. We compute:

$$(\widetilde{\pi q}(f_n)\Psi)(k) = (\pi q)(\alpha_k^{-1}(f_n))\Psi(k) = (\pi q)(f_n\varphi^k))\Psi(k).$$

We know that

$$(\pi q)(f_n\varphi^k) = M_{f_n\varphi^k} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f_n(\varphi^{k-1}(x_0)) & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & f_n(\varphi^{k}(x_0)) & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & f_n(\varphi^{k+1}(x_0)) & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & f_n(\varphi^{k+2}(x_0)) & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & f_n(\varphi^{k+3}(x_0)) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

so

$$\widetilde{\pi q}(f_n)(\Psi)(k) = M_{f_n\varphi^k}\Psi(k) = \begin{pmatrix} \vdots \\ f_n(\varphi^{k-1}(x_0)) \cdot \Psi(k,-1) \\ f_n(\varphi^{k}(x_0)) \cdot \Psi(k,0) \\ f_n(\varphi^{k+1}(x_0)) \cdot \Psi(k,1) \\ \vdots \end{pmatrix}.$$

Hence, we obtain

$$((((\widetilde{\pi q}) \rtimes \Lambda)f)\Psi)(k) = \sum_{n\in\mathbb{Z}} \widetilde{\pi q}(f_n)(\Lambda_n\Psi)(k) = \sum_{n\in\mathbb{Z}} \begin{pmatrix} \vdots \\ f_n(\varphi^{k-1}(x_0)) \cdot \Psi(k-n,-1) \\ f_n(\varphi^{k}(x_0)) \cdot \Psi(k-n,0) \\ f_n(\varphi^{k+1}(x_0)) \cdot \Psi(k-n,1) \\ \vdots \end{pmatrix}, \tag{2.2.2}$$

that is, $(((\widetilde{\pi q} \rtimes \Lambda)f)\Psi)(k)(j) = \sum_{n\in\mathbb{Z}} f_n(\varphi^{k+j}(x_0)) \cdot \Psi(k-n,j)$, for k, $j \in \mathbb{Z}$. Writing $\Psi \in \ell^2(\mathbb{Z}^2)$ as $\Psi = (\cdots, \psi_{-1}, \psi_0, \psi_1, \cdots)$ where $\psi_j = \Psi(\cdot, j)$. The result can be restated as

$$(((\widetilde{\pi q} \rtimes \Lambda)f)\Psi)(k)(j) = \sum_{n\in\mathbb{Z}} f_n(\varphi^{k+j}(x_0)) \cdot \psi_j(k-n)$$

On the other hand, we have

$$((\sigma_{\varphi^j(x_0)}(f))\psi_j)(k) = \sum_{n\in\mathbb{Z}} \mu_{\varphi^j(x_0)}(f_n\varphi^k)(\Lambda_n\psi_j)(k) = \sum_{n\in\mathbb{Z}} f_n(\varphi^{k+j}(x_0))\psi_j(k-n). \tag{2.2.3}$$

By Equations (2.2.2) and (2.2.3), we obtain that

$$(((\widetilde{\pi q} \rtimes \Lambda)f)\Psi)(k)(j) = ((\sigma_{\varphi^j(x_0)}(f))\psi_j)(k),$$

or, equivalently,

$$((\widetilde{\pi q} \rtimes \Lambda)f)\Psi = (\cdots, (\sigma_{\varphi^{j-1}(x_0)}(f))\psi_{j-1}, (\sigma_{\varphi^j(x_0)}(f))\psi_j, (\sigma_{\varphi^{j+1}(x_0)}(f))\psi_{j+1}, \cdots).$$

In other words, and $(\widetilde{\pi q} \rtimes \Lambda)(f) \in B(\ell^2(\mathbb{Z}, \ell^2(\mathbb{Z})))$ corresponds to the diagonal matrix

$$\mathrm{diag}(\cdots, \sigma_{\varphi^{-1}(x_0)}(f), \sigma_{x_0}(f), \sigma_{\varphi(x_0)}(f), \cdots),$$

for all $f \in C_c(\mathbb{Z}, C(X))$. The integrated representation $\widetilde{\pi q} \rtimes \Lambda$ is continuous on $C(X) \rtimes_\alpha \mathbb{Z}$ and $\|\mathrm{diag}(\cdots, A_{-1}, A_0, A_1, \cdots)\| = \sup_{j \in \mathbb{Z}}\{\|A_j\|\}$ where A_k is a bounded operator in $B(\ell^{\in}(\mathbb{Z}))$, $k \in \mathbb{Z}$.

The $*$-algebra $C_c(\mathbb{Z}, C(X))$ is dense in $C(X) \rtimes_\alpha \mathbb{Z}$, therefore,

$$(\widetilde{\pi q} \rtimes \Lambda)(f) = (\cdots, \sigma_{\varphi^{-1}(x_0)}, \sigma_{x_0}, \sigma_{\varphi(x_0)}, \cdots)(f),$$

for all $f \in C(X) \rtimes_\alpha \mathbb{Z}$. Because $\mathrm{Ker}(\sigma_{x_0}) = \mathrm{Ker}(\sigma_{\varphi^j(x_0)})$, for any $j \in \mathbb{Z}$, we have $\mathrm{Ker}(\sigma_{x_0}) = \mathrm{Ker}(\widetilde{\pi q} \rtimes \Lambda) = C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z}$. □

Corollary 2.2.2. *If two points, say x_0 and x_1, have the same closure of orbits, then the concrete algebras in the corresponding representations are isomorphic. Moreover, there exists a $*$-isomorphism that maps H_{x_0} to H_{x_1}.*

Proof. For any two points x_0 and x_1 in X, if the closures of their orbits under φ are the same, denote them by Y. By Theorem 2.2.1, we have that $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \cong C(Y) \rtimes_\alpha \mathbb{Z} \cong \sigma_{x_1}(C(X) \rtimes_\alpha \mathbb{Z})$.

For any $x \in X$, σ_x is the integrated representation induced by the covariant representation $(\tilde{\mu}_x, \Lambda)$, and it is continuous. Define a bounded linear map $\Psi : \sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \to \sigma_{x_1}(C(X) \rtimes_\alpha \mathbb{Z})$ such that $\Psi(\sigma_{x_0}(u)) = \sigma_{x_1}(u)$, denote by Λ_1 the elements $\sigma_{x_0}(u)$ and $\sigma_{x_1}(u)$, where u is unitary and satisfies $ufu^* = \alpha(f)$ for all $f \in C(X)$, and for $f = \sum_{n \in \mathbb{Z}} f_n u^n \in C_c(\mathbb{Z}, C(X))$, $\sigma_{x_0}(f) = \sum_{n \in \mathbb{Z}} \tilde{\mu}_{x_0}(f_n)\Lambda_n$, we have $\Psi(\sigma_{x_0}(f)) = \sum_{n \in \mathbb{Z}} \tilde{\mu}_{x_1}(f_n)\Lambda_n = \sigma_{x_1}(f)$. This implies that $\Psi : \sigma_{x_0}(C_c(\mathbb{Z}, C(X))) \to \sigma_{x_1}(C_c(\mathbb{Z}, C(X)))$

is bijective. We know that $\widetilde{\mu}_{x_0}(f) \in B(\ell^2(\mathbb{Z}))$ corresponds to the matrix:

$$M_{x_0} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & f(\varphi^{-1}(x_0)) & 0 & 0 & 0 & \cdots \\ \cdots & 0 & f(x_0) & 0 & 0 & \cdots \\ \cdots & 0 & 0 & f(\varphi(x_0)) & 0 & \cdots \\ \cdots & 0 & 0 & 0 & f(\varphi^2(x_0)) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that $\Psi : \sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \to \sigma_{x_1}(C(X) \rtimes_\alpha \mathbb{Z})$ is a $*$-isomorphism. Writing another $g = \sum_{m \in \mathbb{Z}} g_m u^m \in C_c(\mathbb{Z}, C(X))$, we have

$$fg = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} f_n \alpha_n(g_m) u^{m+n} \in C_c(\mathbb{Z}, C(X)),$$

and

$$\Psi(\sigma_{x_0}(f)\sigma_{x_0}(g)) = \Psi(\sigma_{x_0}(fg)) = \sigma_{x_1}(fg) = \sigma_{x_1}(f)\sigma_{x_1}(g) = \Psi(\sigma_{x_0}(f))\Psi(\sigma_{x_0}(g)). \tag{2.2.4}$$

Moverover, we have

$$\Psi(\sigma_{x_0}(f)^*) = \Psi(\sum_{n \in \mathbb{Z}} \Lambda_n^* \widetilde{\mu}_{x_0}(f_n)^*) = \sum_{n \in \mathbb{Z}} \Lambda_n^* \Psi(\widetilde{\mu}_{x_0}(f_n))^* = \Psi(\sigma_{x_0}(f))^*. \tag{2.2.5}$$

Because of the continuity of and Ψ and σ_x for any $x \in X$, the (2.2.4) and (2.2.5) hold for all $f \in C(X) \rtimes_\alpha \mathbb{Z}$. Hence, $\Psi : \sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \to \sigma_{x_1}(C(X) \rtimes_\alpha \mathbb{Z})$ is a $*$-isomorphism and $\Psi(H_{x_0}) = H_{x_1}$. □

The following theorem can be found in e.g. [2, page 31].

Theorem 2.2.3 (Spectral Permanence Theorem). *Suppose A is a unital* C**-algebra* *and $B \subseteq A$ is a sub-*C*-algebra containing the identity of A. Then for all $x \in B$,*

$$\mathrm{sp}_B(x) = \mathrm{sp}_A(x).$$

Remark 2.2.4. *The Spectrum Permanence Theorem 2.2.3 is equivalent to the statement that, for any $x \in A$, x is invertible in B if and only if it is invertible in A.*

Proposition 2.2.5. *Let A and B be unital* C*-algebras, and let $\varphi : A \to B$ be a unital $*$-homomorphism. Then $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$ for all $a \in A$, and $\mathrm{sp}(\varphi(a)) = \mathrm{sp}(a)$ for all $a \in A$ if φ is injective.*

Proof. Since $\varphi : A \to B$ is a unital $*$-homomorphism, $\varphi(A) \subseteq B$ is a sub-C*-algebra containing the identity element in B. By Spectral Permanence Theorem 2.2.3, we know that $\mathrm{sp}_B(x) = \mathrm{sp}_{\varphi(A)}(x)$ for all $x \in \varphi(A)$.

Let $a \in A$ and let $\lambda \in \mathrm{sp}_B(\varphi(a))$, i.e., $\lambda e_B - \varphi(a)$ is not invertible in B. We claim that $\lambda e_A - a$ is not invertible. Suppose $\lambda e_A - a$ is invertible, i.e., there exists an element $b \in A$ such that $(\lambda e_A - a)b = b(\lambda e_A - a) = e_A$, since $\varphi(e_A) = e_B$, we have that $(\lambda e_B - \varphi(a))\varphi(b) = \varphi(b)(\lambda e_B - \varphi(a)) = e_B$, which implies that $\lambda e_B - \varphi(a)$ is invertible in B, and we obtain a contradiction. Thus

$$\mathrm{sp}_B(\varphi(a)) \subseteq \mathrm{sp}_A(a)$$

for all $a \in A$.

If φ is injective, there is a $*$-isomorphism $\psi : \varphi(A) \to A$ satisfying $\psi(\varphi(a)) = a$ for all $a \in A$, according to the above result, we know that $\mathrm{sp}_A(\psi(\varphi(a))) \subseteq \mathrm{sp}_{\varphi(A)}(\varphi(a))$ for any $\varphi(a) \in \varphi(A)$, which implies that $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_B(\varphi(a))$, hence $\mathrm{sp}_B(\varphi(a)) = \mathrm{sp}_A(a)$. $\square$

Corollary 2.2.6. *If two points have the same closure of orbits, then the spectra of the two Schödinger operators are the same in the corresponding representations.*

Proof. A Schrödinger operator H_x can be identified with the element $u + u^* + f \in C(X) \rtimes_\alpha \mathbb{Z}$ via the representation σ_x corresponding to the point $x \in X$. For any two points x_0 and x_1 in X, we know that

$$H_{x_0} = \sigma_{x_0}(u + u^* + f) \text{ and } H_{x_1} = \sigma_{x_1}(u + u^* + f).$$

By Corollary 2.2.2, there is an isomorphism

$$\Psi : \sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z}) \to \sigma_{x_1}(C(X) \rtimes_\alpha \mathbb{Z})$$

such that $\Psi(H_{x_0}) = H_{x_1}$, by Proposition 2.2.5, we know that $\mathrm{sp}(H_{x_0}) = \mathrm{sp}(H_{x_1})$. $\square$

In Definition 2.2.7, Theorem 2.2.8 and Theorem 2.2.9, we denote the topological system (X, φ) by Σ, where X is a compact Hausdorff space and φ is a homeomorphism of X onto itself, and we denote the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ by $A(\Sigma)$, where α is the automorphism induced by φ.

Definition 2.2.7. *For $\Sigma = (X, \varphi)$, a point $x \in X$ is called* aperiodic *if, for every nonzero $n \in \mathbb{Z}$, we have $\varphi^n(x) \neq x$. The system Σ is called* topologically free *if the set of its aperiodic points is dense in X.*

Theorem 2.2.8 ([29, Theorem 5.4]). *For $\Sigma = (X, \varphi)$ the following three properties are equivalent:*

(1) *Σ is topologically free;*

(2) *For any ideal I of $A(\Sigma)$, $I \cap C(X) \neq \{0\}$ if and only if $I \neq \{0\}$;*

(3) *$C(X)$ is a maximal abelian sub-C*-algebra of $A(\Sigma)$.*

For a C^*-dynamical system (A, G, α) with G is discrete, we say A *separates the ideals in the reduced crossed product* $A \rtimes_{\alpha,r} G$ if the map $I \mapsto I \cap A$, from the ideals in $A \rtimes_{\alpha,r} G$ into the invariant ideals in A, is injective.

The condition (2) in Theorem 2.2.8 implies Σ is topologically free. However, topological freeness is not sufficient to ensure the separation of ideals (see e.g. [24, Page 238]).

Theorem 2.2.9. *For $\Sigma = (X, \varphi)$, suppose that $C(X)$ separates the ideals in $C(X) \rtimes_\alpha \mathbb{Z}$. Then proper nonempty closed invariant subsets correspond to nonzero proper ideals of $C(X) \rtimes_\alpha \mathbb{Z}$.*

Proof. Let Y be a proper closed invariant subset of X. By Corollary 2.1.8, we know that $C_0(X \setminus Y)$ generates a proper ideal $C_0(X \setminus Y) \rtimes_\alpha \mathbb{Z}$ of $C(X) \rtimes_\alpha \mathbb{Z}$.

Let I be a proper closed ideal of $C(X) \rtimes_\alpha \mathbb{Z}$. Since $C(X)$ separates the ideals in $C(X) \rtimes_\alpha \mathbb{Z}$, it follows that $I \cap C(X) \neq \{0\}$. It is not difficult to see that $I \cap C(X)$ is a closed ideal of $C(X)$ that is invariant under α and its inverse. It is proper since $I \cap C(X) = C(X)$ would imply that $I = C(X) \rtimes_\alpha \mathbb{Z}$. By (iii) on page 4, there exists some proper nonempty closed subset Y_I of X such that $I \cap C(X) = \{f \in C(X) \mid f(x) = 0, \forall x \in Y_I\} = C_0(X \setminus Y_I)$. It also follows that Y_I is invariant under φ and its inverse, since $I \cap C(X)$ is invariant under α and its inverse. Since $(C_0(X \setminus Y_I) \rtimes_\alpha \mathbb{Z}) \cap C(X) = C_0(X \setminus Y_I)$ and $C(X)$ separates the ideas, we conclude $I = C_0(X \setminus Y_I) \rtimes_\alpha \mathbb{Z}$. □

2.3 Some special kinds of systems

In this section, the results above are applied to three kinds of topological systems: minimal systems, almost minimal systems and essentially minimal systems.

2.3.1 Minimal system

Definition 2.3.1. *A* C*-*algebra A is called* simple *if the only ideals in A are the two trivial ideals 0 and A. A topological dynamical system (X, φ) is called* minimal *if the orbit of every point is dense in X.*

The following result, see [8, Theorem VIII.3.9], shows the relationship between the minimality of the topological system and the simplicity of the crossed product, which is also proved in [29, Theorem 5.3].

Theorem 2.3.2. *Let (X, φ) be a dynamical system on an infinite compact Hausdorff space X. Then, the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ is simple if and only if φ is minimal.*

If the topological system (X, φ) is minimal, then the spectrum of Schrödinger operator $H_x = \sigma_x(u + u^* + f)$ is the same for all $x \in X$ by Corollary 2.2.6. For any $x \in X$, the closure of the orbit is the whole X, so Theorem 2.2.1 implies that the integrated representation σ_x of $C(X) \rtimes_\alpha \mathbb{Z}$ on the Hilbert space $\ell^2(\mathbb{Z})$ is faithful. According to Proposition 2.2.5, we obtain that $\mathrm{sp}(H_x)) = \mathrm{sp}(u + u^* + f)$ for any $x \in X$.

2.3.2 Almost minimal system

Definition 2.3.3. *We say that (X, φ) is* almost minimal *if it satisfies the following conditions:*

(i) *there is a fixed point and*

(ii) *the orbit of any other point is dense.*

For an almost minimal system (X, φ), say x_0 is the fixed point in X. There are only two nonempty closed φ-invariant subsets, $\{x_0\}$ and X. We know that $\sigma_{x_0}(C(X) \rtimes_\alpha \mathbb{Z})$ is isomorphic to $C(\{x_0\}) \rtimes_\alpha \mathbb{Z}$, which is the group C*-algebra of $\mathbb{Z}$. In this case, we have

$$\sigma_{x_0}(u + u^* + f) = \Lambda_1 + \Lambda_{-1} + f(x_0) \cdot \mathrm{id}_{B(\ell^2(\mathbb{Z}))},$$

so the spectrum of H_{x_0} is a shift of the spectrum of $\Lambda_1 + \Lambda_{-1}$. Because the closures of their orbits are the whole X, so the integrated representation σ_x for $x \in X \setminus \{x_0\}$ is faithful, by Proposition 2.2.5, the spectra of the Schrödinger operators $H_x = \sigma_x(u + u^* + f)$ for any point $x \in X \setminus \{x_0\}$ are the same as the spectrum of $(u + u^* + f)$.

2.3.3 Essentially minimal system

Definition 2.3.4. *A set Z in X is* minimal *if it is minimal among closed, φ-invariant, nonempty sets.*

Proposition 2.3.5. *The closure of the orbit of a point x is a nonempty φ-invariant closed subset of X.*

Proof. Consider

$$\begin{aligned} Y &= \overline{Orb_\varphi(x)} \\ &= \overline{\{\cdots, \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \cdots\}} \\ &= \overline{\{\cdots, \varphi^{2}(x), \varphi(x), x, \varphi^{-1}(x), \varphi^{-2}(x), \cdots\}} \\ &= \overline{Orb_{\varphi^{-1}}(x)} \end{aligned}$$

Clearly, Y is nonempty and closed. To show $\varphi(Y) = Y$, we need to show $\varphi(Y) \subseteq Y$ and $Y \subseteq \varphi(Y)$.

For any point $y \in Y = \overline{Orb_\varphi(x)}$, there are two cases: if there is $n \in \mathbb{Z}$ such that $y = \varphi^n(x)$, then $\varphi(y) = \varphi^{n+1}(x) \in Y$; if $y \neq \varphi^n(x)$ for any $n \in \mathbb{Z}$, there is a sequence $\{t_n\}_{n\geq 1} \subseteq \mathbb{Z}$ such that $\varphi^{t_n}(x) \to y$ as $n \to \infty$. Since φ is a homeomorphism of X, then $\varphi^{t_n+1}(x) \to \varphi(y)$. Since Y is closed, $\varphi(y) \in Y$. We have proved $\varphi(Y) \subseteq Y$.

Similarly, we obtain $\varphi^{-1}(Y) \subseteq Y$, equivalently, $Y \subseteq \varphi(Y)$. Thus, $\varphi(Y) = Y$. $\square$

Denote $W^+(x)$ and $W^-(x)$ the sets of accumulation points of the sequences $\{\varphi^n(x) \mid n \geq 0\}$ and $\{\varphi^n(x) \mid n \leq 0\}$, respectively.

Theorem 2.3.6 ([13, Theorem 1.1]). *Let (X, φ) be a topological system and let y be any point of X. Then the following are equivalent.*

(i) *For every point x in X, y is in $W^+(x)$.*

(ii) *For every point x in X, y is in $W^-(x)$.*

(iii) *For every neighborhood U of y,*

$$\cup_{n\in\mathbb{Z}} \varphi^n(U) = X.$$

(iv) *X contains a unique minimal set Y and $y \in Y$.*

Definition 2.3.7 ([13]). *We say that (X, φ, y) is* essentially minimal *if it satisfies the conditions above. We also say that (X, φ) is* essentially minimal *if it has a unique minimal set.*

Obviously, an almost minimal system is an essentially minimal system, with $Y = \{x_0\}$.

Remark 2.3.8. *By Proposition 2.3.5, we know that $\overline{Orb_\varphi(y)}$ is a closed, φ-invariant set. Since $\overline{Orb_\varphi(y)} \subseteq Y$, we obtain, by the minimality of Y, that $Y = \overline{Orb_\varphi(y)}$. By the equivalent conditions in Theorem 2.3.6, one has that*

(a) *For any point x in Y, $\overline{Orb_\varphi(x)} = Y$, since* (iv)*;*

(b) *for a point x in $X \setminus Y$, $Y \subsetneq \overline{Orb_\varphi(x)}$, since* (i).

Suppose that Y is the unique minimal set in X. For any $x \in Y$, we obtain that $\mathrm{sp}(\sigma_x(u + u^* + f)) = \mathrm{sp}(u + u^* + f|_Y)$. There may be many different closures of the orbits of points in $X \setminus Y$, so it is more complicated to classify the spectra of Schrödinger operators on an essentially minimal system.

Chapter 3

The method to label gaps in the spectrum of Schrödinger operators

3.1 The K_0-group of a C*-algebra

First, we will review the basic definitions and properties of K_0-group of a C*-algebra.

Definition 3.1.1 (Homotopy). *Let X be a topological space. Two points a, b in X are* homotopic *in X, denoted by $a \sim_h b$ in X, if there is a continuous function*

$$v : [0, 1] \to X$$

such that

$$v(0) = a \text{ and } v(1) = b.$$

The relation $\sim_h$ is an equivalence relation on X. The continuous function v is called a continuous path *from a to b.*

Denote by $\mathcal{P}(A)$ the set of all projections in a C*-algebra A and, if A is unital, denote by $\mathcal{U}(A)$ the group of unitary elements in A. We have the homotopy equivalence relation $\sim_h$ on $\mathcal{P}(A)$ and $\mathcal{U}(A)$.

Consider the following equivalence relations on $\mathcal{P}(A)$ (see e.g. [22, Page 21]):

· $p \sim q$ is there exists v in A with $p = v^*v$ and $q = vv^*$ (*Murray-von Neumann equivalence*),

· $p \sim_u q$ if there exists a unitary element u in $\mathcal{U}(\tilde{A})$ with $q = upu^*$ (*unitary equivalence*)

The relationship between these equivalence relations are shown in the following Propositions.

Proposition 3.1.2 ([22, Proposition 2.2.2])). *Let p, q be projections in a unital* C*-*algebra A. Then following consitions are equivalent:*

(i) $p \sim_u q$.

(ii) $q = upu^*$ *for some unitary u in A.*

(iii) $p \sim q$ *and* $e_A - p \sim e_A - q$.

Proposition 3.1.3 ([22, Proposition 2.2.7])). *Let p, q be projections in a* C*-*algebra A.*

(i) *If $p \sim_h q$, then $p \sim_u q$.*

(ii) *If $p \sim_u q$, them $p \sim q$.*

For the partial ordering of projections of a C*-algebra, recall that for projections P and Q in an abstract algebra A, $P \leq Q$ if $PQ = QP = P$. If A is a sub-C*-algebra of $B(\mathcal{H})$ then $P \leq q$ if and only if $P(\mathcal{H}) \subset Q(\mathcal{H})$.

3.1.1 Matrix algebras

Denote by $M_{m,n}(A)$ the set of all rectangular $m \times n$ matrices:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

with entries $a_{ij} \in A$, $i = 1, 2, \cdots, m$ and $j = 1, 2. \cdots, n$. In particular, denote by $M_n(A)$ the set $M_{n,n}(A)$. Equip $M_n(A)$ with the usual entry-wise vector space operations and matrix multiplication. Also, set

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & a_{21}^* & \cdots & a_{m1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{m2}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{mn}^* \end{pmatrix}.$$

In order to define a C*-norm on $M_n(A)$, by Theorem 1.2.5, we can choose a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism $\varphi : A \to B(\mathcal{H})$. Let $\varphi_n : M_n(A) \to B(\mathcal{H}^n)$ be given by

$$\varphi_n \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \varphi(a_{11})\xi_1 + \cdots + \varphi(a_{1n})\xi_n \\ \varphi(a_{21})\xi_1 + \cdots + \varphi(a_{2n})\xi_n \\ \vdots \\ \varphi(a_{n1})\xi_1 + \cdots + \varphi(a_{nn})\xi_n \end{pmatrix}, \ \xi_j \in \mathcal{H}.$$

Define a norm on $M_n(A)$ by $\|a\| = \|\varphi_n(a)\|$ for a in $M_n(A)$. With these operations, $M_n(A)$ becomes a C*-algebra, the norm is independent of the choice of isometric

$*$-homomorphism φ. We shall use the abbreviation

$$\operatorname{diag}(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

for a *diagonal matrix*, where $a_1, a_2, \cdots, a_n$ are in A.

3.1.2 Semigroups of projections

Definition 3.1.4 (The semigroup $\mathcal{P}_\infty(A)$). *Put*

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)),\ \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A),$$

where A is a C**-algebra and n is a positive integer. We view the sets $\mathcal{P}_n(A)$ for all $n \in \mathbb{Z}$ as being pairwise disjoint.*

Define the relation $\sim_0$ on $\mathcal{P}_\infty(A)$ as follows: suppose that p is a projection in $\mathcal{P}_n(A)$ and q is a projection in $\mathcal{P}_m(A)$. Then $p \sim_0 q$ if there is an element v in $M_{m,n}(A)$ with

$$p = v^*v \text{ and } q = vv^*.$$

Note that the equivalence relation is the Murray-von Neumann equivalence in $\mathcal{P}_\infty(A)$.

Remark 3.1.5. *Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by*

$$p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

so that $p \oplus q$ belongs to $\mathcal{P}_{n+m}(A)$ when p is in $\mathcal{P}_n(A)$ and q is in $\mathcal{P}_m(A)$.

Proposition 3.1.6 ([22, Proposition 2.3.2]). *Let p, q, r, p' q' be projections in $\mathcal{P}_\infty(A)$ for some* C**-algebra A.*

(i) $p \sim_0 p \oplus 0_n$ *for every natural number n, where 0_n is the zero element of $M_n(A)$,*

(ii) *if $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$,*

(iii) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$,

(iv) $p \oplus q \sim_0 q \oplus p$.

Definition 3.1.7 (The semigroup $\mathcal{D}(A)$). *With $(\mathcal{P}_\infty(A), \sim_0, \oplus)$ as in the definition 3.1.4, set*

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/ \sim_0 .$$

For each p in $\mathcal{P}_\infty(A)$, let $[p]_\mathcal{D}$ in A denote the equivalence class containing p. Define addition on $\mathcal{D}(A)$ by

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}, \quad p, q \in \mathcal{P}_\infty(A).$$

It follows from Proposition 3.1.6 that this operation is well-defined and the $(\mathcal{D}(A), +)$ is an abelian semigroup.

3.1.3 The Grothendieck group of a commutative semigroup

Definition 3.1.8. *Let $(S, +)$ be an abelian semigroup. Define an equivalence relation $\sim$ on $S \times S$ by $(x_1, y_1) \sim (x_2, y_2)$ if there exists z in S such that*

$$x_1 + y_2 + z = x_2 + y_1 + z.$$

Denote $G(S)$ the quotient $S \times S/\sim$, and let $\langle x, y\rangle$ denote the equivalence class in $G(S)$ containing (x, y) in $S \times S$. The operation

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$$

is well-defined and turns $(G(S), +)$ into an abelian group. The group $G(S)$ is called the Grothendieck group *of S.*

Given $y \in S$, define $\gamma_S : S \to G(S)$ by

$$\gamma_S(x) = \langle x + y, y\rangle.$$

This map does not depend on y and it is called the Grothendieck map *from S to $G(S)$.*

Remark 3.1.9. *Note that $-\langle x, y\rangle = \langle y, x\rangle$ and that $\langle x, x\rangle = 0$ for all x, y in S.*

3.1.4 The functor K_0 for unital C*-algebras

Definition 3.1.10 (The K_0-group for a unital C*-algebra)**.** *Let A be a unital* C**-algebra, and let $(\mathcal{D}(A), +)$ be the abelian semigroup from Definition 3.1.7. Define $K_0(A)$ to be the Grothendieck group of $\mathcal{D}(A)$,* i.e,

$$K_0(A) = G(\mathcal{D}(A)).$$

Define $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by

$$[p]_0 = \gamma([p]_{\mathcal{D}}) \in K_0(A), \quad p \in \mathcal{P}_\infty(A),$$

where $\gamma : \mathcal{D}(A) \to K_0(A)$ is the Grothendieck map.

Proposition 3.1.11 ([22, Proposition 3.1.7]). *Let A be a unital* C*-*algebra. Then*

$$
\begin{aligned}
K_0(A) &= \{[p]_0 - [q]_0 : p,\, q \in \mathcal{P}_\infty(A)\} \\
&= \{[p]_0 - [q]_0 : p,\, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}. \qquad (3.1.1)
\end{aligned}
$$

Proposition 3.1.12 ([22, Proposition 3.1.8]). *Let A be a unital* C*-*algebra, let G be an abelian group, and suppose that $\nu : \mathcal{P}_\infty(A) \to G$ is a map that satisfies*

(i) $\nu(p \oplus q) = \nu(p) + \nu(q)$*for all projections* $p,\ q \in \mathcal{P}_\infty(A)$,

(ii) $\nu(0_A) = 0$,

(iii) *if p, q belong to $\mathcal{P}_n(A)$ for some n and $p \sim_h q$ in $\mathcal{P}_n(A)$, then $\nu(p) = \nu(q)$.*

Then there is a unique group homomorphism $\alpha : K_0(A) \to G$ which makes the diagram

$$
\begin{tikzcd}
\mathcal{P}_\infty(A) \arrow[d, "{[\cdot]_0}"'] \arrow[dr, "\nu"] & \\
K_0(A) \arrow[r, "\alpha"'] & G.
\end{tikzcd}
$$

commutative.

Let A and B be unital C*-algebras, and let $\varphi : A \to B$ be a $*$-homomorphism. Associate to φ a group homomorphism $K_0(\varphi) : K_0(A) \to K_0(B)$ as follows. The $*$-homomorphism φ extends to a $*$-homomorphism $\varphi : M_n(A) \to M_n(B)$ for each n. A $*$-homomorphism maps projections to projections, and so φ maps $\mathcal{P}_\infty(A)$ to $\mathcal{P}_\infty(B)$. Define $\nu : \mathcal{P}_\infty(A) \to K_0(B)$ by $\nu(p) = [\varphi(p)]_0$ for p in $\mathcal{P}_\infty(A)$. Then ν satisfies conditions (i), (ii), and (iii) in Proposition 3.1.12, and ν therefore factors uniquely through a group homomorphism $\varphi_* : K_0(A) \to K_0(B)$ given by

$$\varphi_*([p]_0) = [\varphi(p)]_0, \quad p \in \mathcal{P}_\infty(A),$$

and commonly denoted by $K_0(\varphi)$. In other words, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_0}"'] & \mathcal{P}_\infty(B) \arrow[d, "{[\cdot]_0}"] \\ K_0(A) \arrow[r, "K_0(\varphi)"'] & K_0(B) \end{tikzcd}.$$

Definition 3.1.13 (Traces and K_0). *Let A be a C^*-algebra. A* bounded trace *on A is a bounded positive linear map $\tau : A \to \mathbb{C}$ with the* trace property*:*

$$\tau(ab) = \tau(ba), \quad a, b \in A. \tag{3.1.2}$$

For every trace τ on a C*-algebra A there is precisely one trace τ_n on $M_n(A)$ that satisfies $\tau_n(\mathrm{diag}(a, 0, \cdots, 0)) = \tau(a)$ for all a in A. Explicitly, τ_n is given by

$$\tau_n \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{i=1}^{n} \tau(a_{ii}).$$

A trace τ on a C*-algebra A induces a function $\tau : \mathcal{P}_\infty(A) \to \mathbb{C}$, and this function satisfies conditions (i), (ii), and (iii) in Proposition 3.1.12, and so there is a unique group homomorphism

$$K_0(\tau) : K_0(A) \to \mathbb{C}$$

satisfying

$$K_0(\tau)([p]_0) = \tau(p), \quad p \in \mathcal{P}_\infty(A). \tag{3.1.3}$$

3.1.5 The ordered abelian group $K_0(A)$

Definition 3.1.14. *A pair (G, G^+) is called an* ordered abelian group *if G is an abelian group, G^+ is a subset of G, and*

$$\text{(i) } G^+ + G^+ \subseteq G^+, \quad \text{(ii) } G^+ \cap (-G^+) = \{0\}, \quad \text{(iii) } G^+ - G^+ = G. \tag{3.1.4}$$

Define a relation $\leq$ on G by $x \leq y$ if $y - x$ belongs to G^+.

Conditions (i) and (ii) in (3.1.4) imply that $(G, \leq)$ is a (partially) ordered set. The set G^+ is called the *positive cone* of G. The *negative cone* is $G^- := -G^+$.

Definition 3.1.15. *For a* C*-*algebra A, the* positive cone *of $K_0(A)$ is*

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\} \subseteq K_0(A).$$

Remark 3.1.16. *For two projections P and Q in* C*-*algebra A, if $P \leq Q$, then* $[P]_0 \leq [Q]_0$.

Definition 3.1.17. *A projection p in a* C*-*algebra A is said to be* infinite *if it is equivalent to a proper subprojection of itself,* i.e., *if there is a projection q in A such that $p \sim q < p$. If p is not infinite, then p is said to be* finite.

A unital C*-*algebra is said to be* finite *if its unit e_A is a finite projection. Otherwise A is called* infinite. *If $M_n(A)$ is finite for all positive integers n, then A is* stabely finite.

If A is a C*-*algebra without a unit, then A is called* finite/stably finite/infinite *if its unitization $\widetilde{A}$ is* finite/stably finite/infinite.

Proposition 3.1.18 ([22, Proposition 5.1.5]). *Let A be a unital* C*-*algebra.*

(i) $K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+$.

(ii) $K_0(A)^+ - K_0(A)^+ = K_0(A)$.

(iii) *if A is stably finite, then* $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$.

Thus,if A is unital and stably finite, then $(K_0(A), K_0(A)^+)$ *is an ordered abelian group.*

3.2 Continuous function calculus

The following fundamental results can be found in e.g. [23, Theorem 18.6].

Theorem 3.2.1. *Let A be a unital complex Banach algebra. For every $a \in A$, its spectrum,* $\mathrm{sp}(a)$, *is compact and not empty.*

Theorem 3.2.2 (Continuous Function Calculus). *Let A be a unital* C*-*algebra. To each normal element $a \in A$, there is one and only one isometic $*$-isomorphism*

$$\begin{aligned} C(\mathrm{sp}(a)) &\rightarrow C^*(a,1) \subseteq A, \\ f &\mapsto f(a), \end{aligned} \tag{3.2.1}$$

which maps ι to a, where ι in $C(\mathrm{sp}(a))$ is given by $\iota(z) = z$ for all $z \in \mathrm{sp}(a)$.

The unique $*$-isomorphism in Continuous Function Calculus Theorem 3.2.2 is called *Galfand-Naimark map.*

It follows that the spectrum of a self-adjoint element (for example, the Schrödinger operator) is contained in $\mathbb{R}$.

A *gap* of the spectrum of a self-adjoint element H is a connected component of $\mathbb{R} \setminus \mathrm{sp}(H)$. For any point $E \in \mathbb{R} \setminus \mathrm{sp}(H)$, i.e., if E lies in a gap g of the spectrum, there is one corresponding projection $\chi_{(-\infty,E]} \in C(\mathrm{sp}(H))$ with the form

$$\chi_{(-\infty,E]}(x) = \begin{cases} 1, & \text{if } x \in (-\infty, E] \cap \mathrm{sp}(H); \\ 0, & \text{if } x \in (E, +\infty) \cap \mathrm{sp}(H). \end{cases} \tag{3.2.2}$$

Observe that the projection $\chi_{(-\infty,E]}(H)$ is independent of the value of $E \in g$.

By Theorem 3.2.2, there is one and only one $*$-isomorphism Φ:

$$\begin{aligned} C(\mathrm{sp}(H)) &\rightarrow C^*(H,1) \subseteq A, \\ f &\mapsto f(H). \end{aligned}$$

The $*$-isomorphism Φ gives a bijection between the projections of these C*-algebras:

$$\mathcal{P}(C(\mathrm{sp}(H))) \rightarrow \mathcal{P}(C^*(H,1)) \subseteq \mathcal{P}(A).$$

Then, for any $\chi_{(-\infty,E]} \in \mathcal{P}(C(\mathrm{sp}(H)))$, we have that

$$\chi_{(-\infty,E]} \mapsto \chi_{(-\infty,E]}(H) \in \mathcal{P}(A) \subseteq \mathcal{P}_\infty(A) \xrightarrow{[\cdot]_0} K_0(A).$$

where $[\cdot]_0$ is the map as in Definition 3.1.10.

Therefore,

$$[\chi_{(-\infty,E]}(H)]_0 \in K_0(A).$$

We say that $f \in C(X)$ is *positive*, which is denoted by $f \geq 0$, if $f(x) \geq 0$ for all $x \in X$, and $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$ (This agrees with the general definition of positive elements in a C*-algebra — see Definition 1.1.6). For any two points in $\mathbb{R}$, say E_1 and E_2, suppose that $E_1 > E_2$, then there are two corresponding projections $\chi_{(-\infty,E_1]}$ and $\chi_{(-\infty,E_2]}$ defined as (3.2.2) in $\mathcal{P}(C(\mathrm{sp}(H)))$. Clearly, $\chi_{(-\infty,E_1]} \geq \chi_{(-\infty,E_2]}$

and $\chi_{(-\infty,E_1]} - \chi_{(-\infty,E_2]}$ is also a projection, so the set $\{\chi_{(-\infty,E]} \mid E$ is in a gap$\}$ is totally ordered and then $\{\chi_{(-\infty,E]}(H) \mid E$ is in a gap$\}$ is totally ordered in $\mathcal{P}(A)$. Hence, $\{[\chi_{(-\infty,E]}(H)]_0 \mid E$ is in a gap$\}$ is a totally ordered set in the positive cone $K_0(A)^+$, which implies that every element in the totally ordered set $\{\chi_{[\chi_{(-\infty,E]}(H)]_0} \mid E$ is in a gap$\}$ corresponds to a gap in the spectrum of H, so we can use $[\chi_{(-\infty,E]}(H)]_0$ to label the gap containing E.

3.3 The trace induced by special kind of measures

Even though the set $\{\chi_{[\chi_{(-\infty,E]}(H)]_0} \mid E$ is in a gap$\} \subset K_0(A)^+$ can be used to label the gaps in the spectrum of H, in general, the group $K_0(A)$ itself is not easy to calculate. In the following, we will exhibit a concrete way to label gaps in the spectrum.

Recall that (X, φ) determines a C*-dynamical system $(C(X), \mathbb{Z}, \alpha)$, where $\alpha_n(f) := f \circ \varphi^n$. In the following, we consider the minimal system (X, φ). Here we will exhibit some properties of the associated crossed product $C(X) \rtimes_\alpha \mathbb{Z}$, which is stably finite. A finite Borel measure μ on X is *translation invariant* for φ if $\mu(\varphi^{-1}(E)) = \mu(E)$ for every Borel subset of X.

Theorem 3.3.1 ([8, Proposition VIII.3.1]). *Let (X, φ) be a classical dynamical system. Then there is a Borel probability measure on X which is translation invariant for φ.*

A translation invariant probability measure μ is said to be *ergodic* if whenever E is a translation invariant measurable set, then $\mu(E) = 0$ or $\mu(E) = 1$.

Proposition 3.3.2 ([8, Proposition VIII.3.2]). *Every dynamical system (X, φ) has an ergodic measure.*

From [11, Page 321], we have that any ergodic probability measure μ on X induces a trace τ_μ on the algebra $C_c(\mathbb{Z}, C(X))$:

$$\tau_\mu(f) := \int_X d\mu(\omega) f(0)(\omega), \quad f \in C_c(\mathbb{Z}, C(X)). \tag{3.3.1}$$

Then τ_μ extends as a trace on the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$, and then τ_μ induces a homomorphism $K_0(\tau_\mu)$, by (3.1.3), from $K_0(C(X) \rtimes_\alpha \mathbb{Z})$ to the real line:

$$K_0(\tau_\mu)([P_1]_0 - [P_2]_0) = \tau_\mu(P_1) - \tau_\mu(P_2),$$

where P_1 and P_2 are projections in $M_n(C(X) \rtimes_\alpha \mathbb{Z})$. In other words, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{P}_\infty(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[d, "{[\cdot]_0}"'] \arrow[dr, "\tau_\mu"] & \\ K_0(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[r, "K_0(\tau_\mu)"'] & \mathbb{R}. \end{tikzcd}$$

Since τ_μ is bounded, positive and linear, if $P \in \mathcal{P}_\infty(C(X) \rtimes_\alpha \mathbb{Z})$ is positive, $\tau_\mu(P) = K_0(\tau_\mu)([P]_0) \in [0, 1]$. Then the set

$$\{K_0(\tau_\mu)([\chi_{(-\infty,E]}(H)]_0) = \tau_\mu(\chi_{(-\infty,E]}) \in [0, 1] \mid E \text{ is a gap}\},$$

which is totally ordered, can be used to label gaps in the spectrum of H.

3.4 The K_1-group and calculation of the Gap-Labels

There is a useful tool for computing the K_0-group of the crossed products. We will need to define the K_1-group of a C*-algebra A.

Definition 3.4.1. *Let A be a unital* C*-*algebra. Set*

$$\mathcal{U}_n(A) = \mathcal{U}(M_n(A)), \ \ \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).$$

Define a binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ by

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(A), \ \ u \in \mathcal{U}_n(A), \ u \in \mathcal{U}_m(A).$$

Define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows: for $u \in \mathcal{U}_n(A)$, $u \in \mathcal{U}_m(A)$, write $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that

$$u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$$

in $u \in \mathcal{U}_k(A)$, where 1_r is the unit in $M_r(A)$.

Lemma 3.4.2 ([22, Lemma 8.1.2])**.** *Let A be a unital* C*-*algebra.*

(i) *$\sim_1$ is an equivalence relation on $\mathcal{U}_\infty(A)$,*

(ii) *$u \sim_1 u \oplus 1_n$ for all $u \in \mathcal{U}_\infty(A)$ and $n \in \mathbb{N}$,*

(iii) *$u \oplus v \sim_1 v \oplus u$ for all $u, v \in \mathcal{U}_\infty(A)$,*

(iv) *if $u, v, u', v' \in \mathcal{U}_\infty(A)$, $u \sim_1 u'$, and $v \sim_1 v'$, then $u \oplus v \sim_1 u' \oplus v'$,*

(v) *if $u, v \in \mathcal{U}_\infty(A)$ for some n, then $uv \sim_1 vu \sim_1 u \oplus v$,*

(vi) *$(u \oplus v) \oplus w = u \oplus (v \oplus w)$ for all $u, v, w \in \mathcal{U}_\infty(A)$.*

Definition 3.4.3 (The K_1-group for a unital C*-algebra)**.** *For a unital* C*-*algebra A define*

$$K_1(A) = \mathcal{U}_\infty(A)/\sim_1 .$$

Let $[u]_1$ *in* $K_1(A)$ *denote the equivalence class containing* u *in* $\mathcal{U}_\infty(A)$. *Denote a binary operation* $+$ *on* $K_1(A)$ *by*

$$[u]_1 + [v]_1 = [u \oplus v]_1,$$

where u, v *belong to* $\mathcal{U}_\infty(A)$. *Lemma 3.4.2 shows that* $+$ *is well-defined, commutative, associative, has zero element* $[1]_1$, *and that*

$$0 = [1_n] = [uu^*]_1 = [u]_1 + [u^*]_1$$

for each u *in* $\mathcal{U}_n(A)$. *This shows that* $(K_1(A), +)$ *is an abelian group, and* $-[u]_1 = [u^*]_1$ *for all* u *in* $\mathcal{U}_\infty(A)$.

As in the case of K_0, a homomorphism φ from A to B induces a homomorphism $\varphi_* = K_1(\varphi)$ of $K_1(A)$ into $K_1(B)$. This makes K_1 a covariant functor from the category of C*-algebras into the category of abelian groups. We have the *Pimsner-Voiculescu sequence* for a C*-algebra A and its crossed product with $\mathbb{Z}$.

Theorem 3.4.4 ([8, Theorem VIII.5.1]). *Suppose that* α *is an automorphism of a* C*-*algebra* A. *Then there is a cyclic six term exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "\iota_*"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[r] & K_1(A) \arrow[d, "\mathrm{id}_* - \alpha_*"] \\
K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\mathrm{id}_* - \alpha_*"] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[l] & K_1(A) \arrow[l, "\iota_*"]
\end{tikzcd}$$

Lemma 3.4.5 ([11, Lemma 7]). *Let* X *be a compact, metrizable, totally disconnected topological space. Then*

$$K_0(C(X)) \cong C(X, \mathbb{Z}), \quad K_1(C(X)) = 0.$$

Remark 3.4.6. *The positive cone of abelian group $K_0(C(X))$ is $C(X,\mathbb{Z})^+ = \{f \in C(X,\mathbb{Z}) \mid f \geq 0\}$.*

Application of the Pimsner-Voiculescu sequence yields:

$$\begin{tikzcd} C(X,\mathbb{Z}) \arrow[r, "i^*"] & K_0(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[r] & 0 \arrow[d] \\ C(X,\mathbb{Z}) \arrow[u, "\mathrm{id}-\alpha^*"] & K_1(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[l, "\mathrm{ind}"] & 0 \arrow[l] \end{tikzcd}$$

Proposition 3.4.7 ([11, Proposition 1]). *Let X be as in Lemma 3.4.5 and $\alpha(f) = f \circ \varphi^{-1}$. Then,*

$$K_0(C(X) \rtimes_\alpha \mathbb{Z}) \cong C(X,\mathbb{Z})/\{f - f \circ \varphi^{-1}\}, \tag{3.4.1}$$

and

$$K_1(C(X) \rtimes_\alpha \mathbb{Z}) \cong \{f \in C(X,\mathbb{Z}) : f = f \circ \varphi^{-1}\}. \tag{3.4.2}$$

Therefore, the totally ordered set $\{[\chi_{(-\infty,E]}(H)]_0 \mid E \text{ is in a gap}\}$ is in the abelian semigroup $C(X,\mathbb{Z})^+/\{f - f\circ\varphi^{-1}\}$ and can be applied to label the possible gaps in the spectrum of the discrete Schrödinger operator with the form (0.0.1). Theoretically, by the existence of ergodic measure μ of a classical dynamical system, we can produce the trace τ_μ on $C(X) \rtimes_\alpha \mathbb{Z}$ and the induced homomorphism $K_0(\tau_\mu)$ of the group $K_0(C(X) \rtimes_\alpha \mathbb{Z})$, then the totally ordered set $K_0(\tau_\mu)(\{[\chi_{(-\infty,E]}(H)]_0 \mid E \text{ is in a gap}\}) \cap [0,1]$ in the set $K_0(\tau_\mu)(K_0(C(X) \rtimes_\alpha \mathbb{Z}))$ gives a labeling of the spectrum of H.

3.5 Labeling on the gaps of a Cantor set

In general, the set $\{K_0(\tau_\mu)([\chi_{(-\infty,E]}(H)]_0) = \tau_\mu(\chi_{(-\infty,E]}) \in [0,1] \mid E \text{ is a gap}\}$ we use to label gaps in the spectrum of H is difficult to compute. In the following, let us look at some examples of Schrödinger operators of the form (0.0.1) in one dimension that have been designed leading to a Cantor spectrum.

An example of a one-dimensional discrete Schrödinger operator is the Almost Mathieu operator on $\ell^2(\mathbb{Z})$, in which the potential given by

$$V(n) = 2\lambda\cos(2\pi(\alpha n + \omega))\psi(n),$$

where $\alpha, \omega \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $\lambda > 0$. By [19], for irrational α, it is known that the spectrum of $H_{\alpha,\omega,\lambda}$ is a Cantor set of the real line.

It is shown in [7] that one dimensional Schödinger operator on $\ell^2(\mathbb{Z})$ with potential given by

$$V(n) = \lambda\chi_{[1-\alpha,1[}(x + n\alpha), \quad \alpha \notin \mathbb{Q}$$

has a Cantor spectrum of zero Lebesgue measure for any irrational α and any $\lambda > 0$. Moreover, as shown in [25], the spectrum of the discrete Schödinger operator with the potential given by

$$V(n) = \mu\chi_{[-\omega^3,\omega^2[}((n-1)\omega),$$

where $\omega = (\sqrt{5}-1)/2$ and χ_I is the characteristic function of the interval I, is also a Cantor set for $|\mu| \geq 4$.

Bibliography

[1] Arveson, W., *An Invitation to* C*-*Algebras*, Volume **39** of Applied Mathematical Sciences, Volume **39** of Graduate Texts in Mathematics, Springer, 1976.

[2] Arveson, W., *A Short Course on Spectral Theory*, Volume **209** of Graduate texts in mathematics, Springer-Verlag New York Berlin Heideberg (2002).

[3] Avila, A., Bochi, J. and Damanik, D., *Opening Gaps in the Spectrum of Strictly Ergodic Schrödinger Operators*, (12 Mar 2009). Cite as: arXiv:0903.2281v1 [math.DS].

[4] Bratteli, O., *Inductive Limits of Finite-Dimensional* C*-*Algebras*, AMS **171** (1972), 195-234.

[5] Bellissard, J. *K-theory of* C*-*Algebras in Solid State Physics*, Statistical Mechanics and Field Theory: Mathematical Aspects (T. C. Dorlas, N. M. Hugenholtz, and M. Winnink, eds.), Lecture Notes in Physics 257, Springer Verlag, (1986), 99-156.

[6] Bellissard, J., Bovier, A. and Ghex, J. M., *Gap Labelling Theorems for One Dimensional Discrete Schrödinger Operators*, Reviews in Mathematical Physics **Vol. 4, No. 1** (1992), 1-37.

[7] Bellissard. J., Iochum. B., Schoppola. E. and Testard. D., *Spectral Properties of One dimensional Quasi-Crystals*, Commun.Math. Phys. **125** (1989), 527-543.

[8] Davidson, K. R., C*-*Algebras by Example*, Fields Institute Monographs **No. 6** (1996). AMS.

[9] Durand, F., Host, B. and Skau, C., *Substitution Dynamical Systems, Bratteli Diagrams and Dimension Groups*, Ergodic Theory and Dynamical Systems **No. 19** (1994), 953-993.

[10] Damanik, D. and Lenz, D., *Uniform Spectral Properties of One-Dimensional Quasicrystals, I. Absence of Eigenvalues, Commun*, Math. Phys. **207** (1999), 687-696.

[11] Elst, A. V., *Gap-Labelling Theorems for Schrödinger Operators on the Square and Cubic Lattice*, Reviews in Mathematical Physics, **Vol. 6, No. 2** (1994), World Scientific Publishing Company, 319-342.

[12] Helemskii, A. Ya., *Lectures and Exercises on Functional Analysis*, Volume **233** of Translations of Mathematical Monographs, American Mathematical Soc., 2006.

[13] Herman, R. H., Putnam, I. F. and Skau, C. F., *Ordered Bratteli Diagrams, Dimension Groups and Topological Dynamics*, International Journal of mathematics **Vol. 3, No. 6** (1992), 827-864.

[14] Kadison, V. R. and Ringrose, R. J., *Fundamentals of the Theory of Operator Algebras. V2: Advanced Theory*, Pure and Applied Mathematics, Academic Press, 1986.

[15] Lin, H., *An Introduction to the Classification of Amenable* $\mathbb{C}^*$-*Algebras*, World Scientific, 2001.

[16] Murphy, J. G., C*-*algebras and operator theory*, Academic Press, London, 1990.

[17] O′ Donovan, D. P., *Weighted Shifts and Covariance Algebras*, Trans. Amer. Math. Soc. **208** (1975), 1-25.

[18] Power, S. C., *Simplicity of* C*-*Algebras of Minimal Dynamical Systems*, J. London Math. Soc. **18** (1978), 534-538.

[19] Puig, J. *Cantor Spectrum for the Almost Mathieu Operator*, Commun. Math. Phys. **244** (2004), 297-309

[20] Pimsner, M. and Voiculescu, D., *Exact Sequences for K-groups and Ext-Groups of Certain Cross-Product* C*-*Algebras*, J. Operator Theory **4** (1980), 93-118.

[21] Poon, Y. T., *AF Subalgebras of Certain Crossed Products*, Rocky Mountain Journal of Mathematics **Vol. 20, No. 2** (1990), Spring.

[22] Rordam, M., Larsen, F. and Laustsen, N.J., *An Introduction to K-Theory for* C*-*Algebras*, London mathmatical Society Student Texts **49**, Cambridge University Press, 2000.

[23] Rudin, W., *Real and Complex Analysis, Third Edition*, McGraw-Hill, Inc. New York, (1987).

[24] Sierakowski, A., *The Ideal Structure of Reduced Crossed Products*, Münster J. of Math **3** (2010), 237-262.

[25] Sütő, A., *The Spectrum of a Quasiperiodic Schrödinger Operator*, Comun. Math. Phys. **111** (1987), 409-415.

[26] Svensson, C., Silvestrov, S. and Jeu, M., *Dynamical Systems and Commutants in Crossed Products*, Internet. J. Math. **18** (2007), 455-471.

[27] Svensson, C., Silvestrov, S. and Jeu, M., *Dynamical Systems Associated with Crossed products*, Acta Appl. Math. **108** (2009), 547-559.

[28] Svensson, C., Silvestrov, S. and Jeu, M., *Connections between Dynamical Systems and Crossed Products of Banach Algebras by* $\mathbb{Z}$, pages 391–401 in: Methods of Spectral Analysis in Mathematical Physics (Conference on Operator theory, Analysis and Mathematical Physics (OTAMP) 2006, Lund, Sweden), J. Janas, P. Kurasov, A. Laptev, S. Naboko, G. Stolz (eds.), Operator Theory: Advances and Applications **186**, Birkhäuser, Basel, 2009.

[29] Tomiyama, J., *The Interplay between Topological Dynamics and Theory of* C*-*Algebras*, Lecture note **No. 2** (1992), Res. Inst. Math. Seoul.

[30] Williams, D. P., *Crossed Products of* C*-*Algebras*, Mathematical Surveys and Monographs, **Vol. 134** (2007). AMS.

[31] *Universal Property*, (n.d.). Available at: `http://en.wikipedia.org/wiki/Universal_property`