DECOMPOSITIONS OF MATRICES AND LINEAR TRANSFORMATIONS

LU WANG

Decompositions of Matrices and Linear Transformations

by

© Lu Wang

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Abstract

The aim of this thesis is to discuss how to express a matrix (or a linear transformation) as the sum of two invertible matrices (or invertible linear transformations) with some constraints. The work for this thesis is two-fold. Firstly, it is proved that if R is a semilocal ring or an exchange ring with primitive factors Artinian then R satisfies the Goodearl-Menal condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. These results correct two existing results in the literature. Secondly, for the ring R of linear transformations of a right vector space over a division ring D, two results are proved in this thesis: (1) If $|D| > 3$, then for any $a \in R$ there exists a unit u of R such that both $a + u$ and $a - u^{-1}$ are units of R; (2) If $|D| > 2$, then for any $a \in R$ there exists a unit u of R such that both $a - u$ and $a - u^{-1}$ are units of R. Result (1) extends a result of H. Chen [7] that the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 3$ satisfies the condition that for any $a \in R$, there exists $u \in U(R)$ such that $a + u$ and $a - u^{-1} \in U(R)$. And result (2) answers a question raised by H. Chen [7] whether the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 2$ satisfies the condition that for any $a \in R$, there exists $u \in U(R)$ such that $a - u$ and $a - u^{-1} \in U(R)$. Connections of these conditions with some well-known conditions in ring theory are also discussed.

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Chapter 1

Introduction

Throughout the thesis, R denotes an associative ring with identity, $U(R)$ denotes the group of units of R, $J(R)$ denotes the Jacobson radical of R. $\mathbb{Z}_n$ denotes the ring of integers modulo n and $\mathbb{M}_n(R)$ denotes the $n \times n$ matrix ring over R. We write $I \lhd R$ to mean that I is an ideal of R.

In 1954, Zelinsky [26] proved that every element in the ring of linear transformations of a right vector space V_D over a division ring D is the sum of two units unless $D \cong \mathbb{Z}_2$ and the dimension of V_D is one. Also, in 1998, Nicholson and Varadarajan [19] proved that every linear transformation over a countable-dimensional vector space is the sum of an idempotent and an automorphism. In addition, in 2010, H. Chen [7] proved that the ring R of linear transformations of a countably generated right vector space over a division ring D with $|D| > 3$ satisfies the condition that for any $a \in R$, there exists $u \in U(R)$ such that $a + u$ and $a - u^{-1} \in U(R)$. H. Chen [7] also raised the question whether the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 2$ satisfies the condition that for any $a \in R$, there exists $u \in U(R)$ such that $a - u$ and $a - u^{-1} \in U(R)$. These are the motivation for us to discuss the decompositions of matrices and linear transformations in this thesis.

A ring R is said to satisfy unit 1-stable range if whenever $aR+bR = R$, there exists $u \in U(R)$ such that $a+bu \in U(R)$. In 1984, Menal and Moncasi [16] proved that if R satisfies unit 1-stable range, then $K_1(R) = U(R)/V(R)$, where $V(R)$ is the subgroup of $U(R)$ generated by $\{(ab+1)(ba+1)^{-1} : ab+1 \in U(R)\}$. Here $K_1(R)$ denotes the K_1-group of R, which is an important topic in homological algebra, topology, algebraic geometry and etc. Notice that for a ring R, $K_1(R) = GL(R)/[GL(R), GL(R)]$, where $GL(R)$ is the direct limit of $GL_n(R)$, the group of invertible matrices in $\mathbb{M}_n(R)$, and $[GL(R), GL(R)]$ is the commutator subgroup of $GL(R)$.

Later in 1988, Goodearl and Menal [9] showed that the unit 1-stable range is always satisfied by a ring R with the condition that for any $x, y \in R$, there exists a unit u of R such that both $x - u$ and $y - u^{-1}$ are units of R. The latter condition was called the Goodearl-Menal condition by H.Chen [5] in 2001. Goodearl and Menal [9] also provided many classes of rings which satisfiy the Goodearl-Menal condition. Here we recall some concepts in ring theory used in this thesis. A ring is called simple if it is a non-zero ring that has no (two-sided) ideal besides the zero ideal and itself. A ring R is semilocal if $R/J(R)$ is semisimple Artinian. The notion of an exchange ring was introduced by Warfield [24] via the exchange property of modules. Here we use an equivalent condition of an exchange ring obtained independently by Goodearl [10] and Nicholson [18]: a ring R is an exchange ring if and only if for each $a \in R$ there exists $e^2 = e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. And an exchange ring with primitive factors Artinian is an exchange ring whose primitive factors are

Artinian. A ring R is called right self-injective if every R-homomorphism from a right ideal of R into R can be extended to an R-homomorphism from R to R. A ring R is called strongly π-regular if the descending chain $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ is stable for all $a \in R$.

As mentioned before, a ring R is said to satisfy the Goodearl-Menal condition if for any $x, y \in R$, there exists a unit u of R such that both $x-u$ and $y-u^{-1}$ are units of R. For brevity, we will use the term GM-condition for the Goodearl-Menal condition. The class of ring satisfying the GM-condition is closed under direct products and homomorphic images. Besides the GM-condition we are concerned with the following two conditions on R:

Condition (P). A ring R is said to satisfy (P) if for any $a \in R$, there exists $u \in U(R)$ such that $a+u, a-u^{-1} \in U(R)$.

Condition (Q). A ring R is said to satisfy (Q) if for any $a \in R$, there exists $u \in U(R)$ such that $a-u, a-u^{-1} \in U(R)$.

In this thesis, we mainly concern about rings with the GM-condition and the ring of linear transformations with Conditions (P) and (Q). It is proved that if R is a semilocal ring or an exchange ring with primitive factors Artinian, then R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. As a consequence, it is proved that, if R is a ring such that $R/J(R)$ is right self-injective strongly π-regular, then R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. These results correct two existing results (see [6, Theorem 3.4] and [6, Theorem 4.1]), and disprove a claim in [6] (see [6, p.753]) and a claim in [7] (see [7, p.432]).

The incorrect statement that a semilocal ring R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ has been implicitly used in making/proving several claims about rings with related conditions (see [6, p.753], [7, Proposition 9], [7, p.6]), so the argument's validity needs to be established. And this is done here.

Let R be the ring of linear transformations of a right vector space over a division ring D. We proved that if $|D| > 3$, then R satisfies (P); If $|D| > 2$, then R satisfies (Q). Connections of these conditions with some well-known conditions in ring theory are briefly discussed.

Chapter 2

Rings with the Goodearl-Menal condition

First let us recall a ring R satisfies the GM-condition if for any $x, y \in R$, there exists a unit u of R such that both $x - u$ and $y - u^{-1}$ are units of R.

The unit 1-stable range condition has been discussed by several authors. For example, Menal and Moncasi [8] proved that if R satisfies unit 1-stable range, then $K_1(R) = U(R)/V(R)$, where $V(R)$ is the subgroup of $U(R)$ generated by $\{(ab + 1)(ba + 1)^{-1} : ab + 1 \in U(R)\}$. Later Goodearl and Menal [9] proved that if a ring R satisfies the GM-condition then it satisfies unit 1-stable range. They also provided many classes of rings satisfying the GM-condition. In this chapter, we prove that if R is a semilocal ring, then R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. As consequences, we also prove that, for an exchange ring R with primitive factors Artinian or a ring R such that $R/J(R)$ is right self-injective strongly π-regular, R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. Applications of these results are discussed.

2.1 Semilocal rings

We begin with the following example.

Example 2.1.1 *The ring $\mathbb{M}_2(\mathbb{Z}_2)$ does not satisfy the GM-condition.*

Proof. We have $U(\mathbb{M}_2(\mathbb{Z}_2)) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$.

It is easy to check that, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, there does not exist a unit U of $\mathbb{M}_2(\mathbb{Z}_2)$ such that $A - U, B - U^{-1}$ are units of $\mathbb{M}_2(\mathbb{Z}_2)$. □

It is clear that $\mathbb{M}_2(\mathbb{Z}_2)$ is semilocal. Also since it is simple, no homomorphic image of $\mathbb{M}_2(\mathbb{Z}_2)$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Thus $\mathbb{M}_2(\mathbb{Z}_2)$ is a counter-example to H.Chen's result [6] that a semilocal ring R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and this raises the question of which semilocal rings satisfy the GM-condition. The main result in this chapter is the following

Theorem 2.1.2 *Let R be a semilocal ring. The following are equivalent:*

1. R satisfies the GM-condition.

2. No homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$.

The proof of the theorem relies on the following theorem and three lemmas.

Theorem 2.1.3 *If $\mathbb{M}_n(R)$ and $\mathbb{M}_m(R)$ both satisfy the GM-condition, then $\mathbb{M}_{n+m}(R)$ satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_{n+m}(R)$. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{11}, B_{11} \in \mathbb{M}_n(R)$, $A_{22}, B_{22} \in \mathbb{M}_m(R)$, A_{12} and B_{12} are $n \times m$ matrices, A_{21} and B_{21} are $m \times n$ matrices. By our assumption, there exists a unit U_1 in $\mathbb{M}_n(R)$ such that $X := A_{11} - U_1$, $Y := B_{11} - U_1^{-1}$ are units of $\mathbb{M}_n(R)$. Now $A_{22} - A_{21}X^{-1}A_{12}$, $B_{22} - B_{21}Y^{-1}B_{12}$ are matrices in $\mathbb{M}_m(R)$. By assumption, there exists a unit U_2 of $\mathbb{M}_m(R)$ such that

$$X' := (A_{22} - A_{21}X^{-1}A_{12}) - U_2$$
$$Y' := (B_{22} - B_{21}Y^{-1}B_{12}) - U_2^{-1}$$

are units of $\mathbb{M}_m(R)$. Then, $U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ is a unit of $\mathbb{M}_{n+m}(R)$ such that

$$A - U = \begin{pmatrix} X & A_{12} \\ A_{21} & A_{21}X^{-1}A_{12} + X' \end{pmatrix} = \begin{pmatrix} I & 0 \\ A_{21}X^{-1} & I \end{pmatrix}\begin{pmatrix} X & A_{12} \\ 0 & X' \end{pmatrix},$$

$$B - U^{-1} = \begin{pmatrix} Y & B_{12} \\ B_{21} & B_{21}Y^{-1}B_{12} + Y' \end{pmatrix} = \begin{pmatrix} I & 0 \\ B_{21}Y^{-1} & I \end{pmatrix}\begin{pmatrix} Y & B_{12} \\ 0 & Y' \end{pmatrix}$$

are units of $\mathbb{M}_{n+m}(R)$. This completes the proof. □

Lemma 2.1.4 *If D is a division ring with $|D| \geq 4$, then $\mathbb{M}_n(D)$ satisfies the GM-condition for all $n \geq 1$.*

Proof. By Theorem 2.1.3, it suffices to show that D satisfies the GM-condition. Let $x, y \in D$. If $x = 0$, choose $0 \neq u \in D$ such that $u^{-1} \neq y$ and then we have that $x - u \neq 0$ and $y - u^{-1} \neq 0$. If $y = 0$ choose $0 \neq u \in D$ such that $u \neq x$ and then we have $x - u \neq 0$ and $y - u^{-1} \neq 0$. If $x \neq 0$ and $y \neq 0$, choose $0 \neq u \in D$ such that $u \neq x$ and $u \neq y^{-1}$, and we have $x - u \neq 0$ and $y - u^{-1} \neq 0$. So D satisfies the GM-condition. □

Next we show that $\mathbb{M}_n(\mathbb{Z}_3)$ $(n \geq 2)$ and $\mathbb{M}_n(\mathbb{Z}_2)$ $(n \geq 3)$ satisfy the GM-condition. The idea in proving Lemma 2.1.4 does not apply to these cases, because none of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$ satisfies the GM-condition. The long verifications of the next two examples are given in the Appendix.

Example 2.1.5 $\mathbb{M}_2(\mathbb{Z}_3)$ *and* $\mathbb{M}_3(\mathbb{Z}_3)$ *satisfy the GM-condition.*

Example 2.1.6 $\mathbb{M}_3(\mathbb{Z}_2)$, $\mathbb{M}_4(\mathbb{Z}_2)$ *and* $\mathbb{M}_5(\mathbb{Z}_2)$ *satisfy the GM-condition.*

Theorem 2.1.3 can be used to show that $\mathbb{M}_n(\mathbb{Z}_3)$ $(n \geq 2)$ and $\mathbb{M}_n(\mathbb{Z}_2)$ $(n \geq 3)$ satisfy the GM-condition based on Examples 2.1.5 and 2.1.6.

Lemma 2.1.7 $\mathbb{M}_n(\mathbb{Z}_3)$ *satisfies the GM-condition for all* $n \geq 2$.

Proof. For any $n \geq 2$, $n = 2s$ or $n = 3s$ or $n = 2s + 3$, where s is a positive integer. It follows from Example 2.1.5 and Theorem 2.1.3 that $\mathbb{M}_n(\mathbb{Z}_3)$ satisfies the GM-condition. □

Lemma 2.1.8 $\mathbb{M}_n(\mathbb{Z}_2)$ *satisfies the GM-condition for all* $n \geq 3$.

Proof. For any $n \geq 3$, $n = 3s$ or $n = 4s$ or $n = 5s$ or $n = 3s + 4$ or $3s + 5$, where s is a positive integer. It follows from Example 2.1.6 and Theorem 2.1.3 that $\mathbb{M}_n(\mathbb{Z}_2)$

satisfies the GM-condition. □

We are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2.
Suppose that R satisfies the GM-condition. If S is a nonzero homomorphic image of R, then S satisfies the GM-condition. Because none of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$ satisfies the GM-condition, S is not isomorphic to either of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$.

Conversely, suppose that no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. Notice that R satisfies the GM-condition if and only if $R/J(R)$ satisfies the GM-condition. So we may assume that $J(R) = 0$. Since R is semilocal, $R = \mathbb{M}_{n_1}(D_1)\oplus\cdots\oplus\mathbb{M}_{n_s}(D_s)$, where $s \geq 1$, $n_i \geq 1$ and D_i is a division ring for $i = 1, \ldots, s$. By our assumption, no homomorphic image of $\mathbb{M}_{n_1}(D_1)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. Thus, either $n_1 = 1$ with $|D_1| > 3$, or $n_1 = 2$ with $|D_1| > 2$, or $n_1 \geq 3$. Hence, by Lemmas 2.1.4, 2.1.7 and 2.1.8, $\mathbb{M}_{n_1}(D_1)$ satisfies the GM-condition. It is similar to show that $\mathbb{M}_{n_i}(D_i)$ satisfies the GM-condition for $i = 2, \ldots, s$. Hence R satisfies the GM-condition. □

2.2 Exchange rings with primitive factors Artinian

Theorem 2.2.1 *Let R be an exchange ring with primitive factors Artinian. The following are equivalent:*

1. *R satisfies the GM-condition.*

2. *No homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$.*

Proof. As seen in the proof of Theorem 2.1.2, (1) implies (2) for any arbitrary ring. Now suppose that (2) holds. Assume on the contrary that R does not satisfy the GM-condition. Then there exist $x, y \in R$ such that, for each $u \in U(R)$, either $x - u \notin U(R)$ or $y - u^{-1} \notin U(R)$. For an ideal I of R and $r \in R$, we write $\overline{R} = R/I$ and $\bar{r} = r + I \in \overline{R}$. Thus,

$$\mathcal{F} = \left\{ I \triangleleft R : \bar{x} - \bar{u} \notin U(\overline{R}) \text{ or } \bar{y} - \bar{u}^{-1} \notin U(\overline{R}) \ \forall \bar{u} \in U(\overline{R}) \right\}$$

is not empty. It is easily seen that $\mathcal{F}$ is an inductive set, so by Zorn's Lemma $\mathcal{F}$ has a maximal element, say I. Because every unit of $(R/I)/J(R/I)$ is lifted to a unit of R/I, the maximality of I implies that $J(R/I) = 0$. We next show that R/I is indecomposable. In fact, if R/I is decomposable, then there exist ideals I_1, I_2 of R such that $I \subsetneqq I_i \subsetneqq R$ $(i = 1, 2)$ and

$$R/I \cong R/I_1 \bigoplus R/I_2 \text{ via } r + I \mapsto (r + I_1, r + I_2).$$

By the maximality of I, there exists a unit $v + I_1$ in $U(R/I_1)$ with inverse $v' + I_1$ and a unit $w + I_2$ in $U(R/I_2)$ with inverse $w' + I_2$ such that $(x + I_1) - (v + I_1), (y + I_1) - (v' + I_1) \in U(R/I_1)$ and $(x + I_2) - (w + I_2), (y + I_2) - (w' + I_2) \in U(R/I_2)$. Thus, $(v + I_1, w + I_2)$ is a unit of $R/I_1 \oplus R/I_2$ with inverse $(v' + I_1, w' + I_2)$ and, moreover, $(x + I_1, x + I_2) - (v + I_1, w + I_2)$ and $(y + I_1, y + I_2) - (v' + I_1, w' + I_2)$ are units of $R/I_1 \oplus R/I_2$. This shows that there exists a unit $u + I$ in R/I with inverse $u' + I$ such that $(x + I) - (u + I)$ and $(y + I) - (u' + I)$ are units of R/I. This contradiction shows that R/I is indecomposable. Thus R/I is a semiprimitive, indecomposable exchange ring with primitive factors Artinian. Now by Menal [4, Lemma 1], R/I is a simple Artinian ring. Because R/I does not satisfy the GM-condition, by Theorem 2.1.2 $R/I = \mathbb{Z}_2$ or $R/I = \mathbb{Z}_3$ or $R/I = \mathbb{M}_2(\mathbb{Z}_2)$. This contradicts (2). □

Every one-sided perfect ring (in particular, one-sided Artinian ring) is strongly π-regular. A von Neumann regular ring in which every idempotent is central is called a strongly regular ring.

Corollary 2.2.2 *Let R be a ring such that $R/J(R)$ is right self-injective strongly π-regular. The following are equivalent:*

1. *R satisfies the GM-condition.*

2. *No homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. To show the implication (2) $\Rightarrow$ (1), we can assume that $J(R) = 0$. Then by [12], R is a finite direct product of matrix rings over strongly regular rings. Thus, one can easily show that every primitive image of R is Artinian. Hence (1) holds by Theorem 2.2.1. □

2.3 Some consequences

Recall that a ring R satisfies (P) if for each $a \in R$ there exists $u \in U(R)$ such that $a + u, a - u^{-1} \in U(R)$, and a ring R satisfies (Q) if for each $a \in R$ there exists $u \in U(R)$ such that $a - u, a - u^{-1} \in U(R)$. These conditions have been discussed in [7]. It is clear that $\mathbb{Z}_2$ does not satisfy (Q) and that neither $\mathbb{Z}_2$ nor $\mathbb{Z}_3$ satisfy (P). Moreover, the GM-condition implies both (P) and (Q), and the classes of rings which satisfy (P) and (Q) are closed under direct products and homomorphic images. In [7, Proposition 9], the author gave a proof of the claim that a semilocal ring satisfies (P) iff no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and at the end of the article, made the claim that a semilocal ring R satisfies (Q) iff R satisfies unit 1-stable range.

The claim and its proof are implicitly involved with the use of the incorrect statement that a semilocal ring R satisfies the GM-condition iff no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ (see the last paragraph of [7]), so their validity need be clarified.

Proposition 2.3.1 *Let R be a semilocal ring. The following are equivalent:*

1. *R satisfies (P).*

2. *$1+u, 1-u \in U(R)$ for some $u \in U(R)$.*

3. *No homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. (1) $\Rightarrow$ (2). By (1), there exists $u \in U(R)$ such that $1+u, 1-u^{-1} \in U(R)$. It follows that $1+u, 1-u \in U(R)$.

(2) $\Rightarrow$ (3). If (2) holds for R then (2) holds for any nonzero homomorphic image of R. But neither $\mathbb{Z}_2$ nor $\mathbb{Z}_3$ satisfy (2). So no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

(3) $\Rightarrow$ (1). Noting that R satisfies (P) iff $R/J(R)$ satisfies (P), we may assume that $J(R) = 0$. Thus, $R = \mathbb{M}_{n_1}(D_1) \oplus \cdots \oplus \mathbb{M}_{n_s}(D_s)$, where $s \geq 1$, $n_i \geq 1$ and D_i is a division ring for $i = 1, \ldots, s$. By (3), no homomorphic image of $\mathbb{M}_{n_1}(D_1)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Thus, either $n_1 = 1$ with $|D_1| > 3$, or $n_1 \geq 2$.

If $|D_1| > 3$ or $n_1 \geq 3$, then $\mathbb{M}_{n_1}(D_1)$ satisfies the GM-condition by Theorem 2.1.2, so it satisfies (P). Moreover, $\mathbb{M}_2(\mathbb{Z}_3)$ satisfies the GM-condition by Theorem 2.1.2, so it satisfies (P). Lastly, $\mathbb{M}_2(\mathbb{Z}_2)$ satisfies (P) by [7, Example 8]. This shows that $\mathbb{M}_{n_1}(D_1)$ satisfies (P). It is similar to show that $\mathbb{M}_{n_i}(D_i)$ satisfies (P) for $i = 2, \ldots, s$. Hence R satisfies (P). $\square$

Proposition 2.3.2 *Let R be a semilocal ring. The following are equivalent:*

1. *R satisfies (Q).*

2. *No homomorphic image of R is isomorphic to $\mathbb{Z}_2$.*

3. *R satisfies unit 1-stable range.*

4. *Every element of R is the sum of two units.*

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (2). These are clear.

(2) $\Rightarrow$ (1). Because R satisfies (Q) iff $R/J(R)$ satisfies (Q), we may assume that $J(R) = 0$. Thus, $R = \mathbb{M}_{n_1}(D_1) \oplus \cdots \oplus \mathbb{M}_{n_s}(D_s)$, where $s \geq 1$, $n_i \geq 1$ and D_i is a division ring for $i = 1, \ldots, s$. By (2), no homomorphic image of $\mathbb{M}_{n_1}(D_1)$ is isomorphic to $\mathbb{Z}_2$. Thus, either $n_1 = 1$ with $|D_1| > 2$, or $n_1 \geq 2$.

It is clear that $\mathbb{Z}_3$ satisfies (Q). By Theorem 2.1.2, every division ring D with $|D| > 3$ satisfies the GM-condition, and hence satisfies (Q). Thus, $\mathbb{M}_{n_1}(D_1)$ satisfies (Q) if $n_1 = 1$ and $|D_1| > 2$. If $|D_1| \geq 3$ and $n_1 \geq 2$, then $\mathbb{M}_{n_1}(D_1)$ satisfies the GM-condition by Theorem 2.1.2, and hence satisfies (Q). Finally for any $n \geq 2$, Proposition 2.3.1 shows that $\mathbb{M}_n(\mathbb{Z}_2)$ satisfies (P) and hence satisfies (Q) because $2 = 0$ in $\mathbb{M}_n(\mathbb{Z}_2)$. Therefore, $\mathbb{M}_{n_1}(D_1)$ satisfies (Q). It is similar to show that $\mathbb{M}_{n_i}(D_i)$ satisfies (Q) for $i = 2, \ldots, s$. Hence R satisfies (Q).

(2) $\Leftrightarrow$ (3). This was proved by Wu [25, Corollary 4]. Since Wu's article is published in Chinese, we include a proof for the readers convenience. Notice that R

satisfies unit 1-stable range iff $R/J(R)$ satisfies unit 1-stable range and the class of rings satisfying unit 1-stable range is closed under direct products and direct summands.

Suppose (2) holds. To show (3), we can assume that $J(R) = 0$. Thus, $R = \mathbb{M}_{n_1}(D_1)\oplus\cdots\oplus\mathbb{M}_{n_s}(D_s)$, where $s \geq 1$, $n_i \geq 1$ and D_i is a division ring for $i = 1, \ldots, s$. By (2), no homomorphic image of $\mathbb{M}_{n_1}(D_1)$ is isomorphic to $\mathbb{Z}_2$. Thus, either $n_1 = 1$ with $|D_1| > 2$, or $n_1 \geq 2$. Since the GM-condition implies unit 1-stable range, by Theorem 2.1.1 to see that $\mathbb{M}_{n_1}(D_1)$ satisfies unit 1-stable range we only need to show that $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$ satisfy unit 1-stable range. But it can easily be verified that $\mathbb{Z}_3$ and $\mathbb{M}_2(\mathbb{Z}_2)$ satisfy unit 1-stable range. So $\mathbb{M}_{n_1}(D_1)$ satisfies unit 1-stable range. Similarly, $\mathbb{M}_{n_i}(D_i)$ satisfies unit 1-stable range for $i = 2, \ldots, s$. Hence R satisfies unit 1-stable range.

Suppose that (2) does not hold. Then $R/I \cong \mathbb{Z}_2$ for an ideal I of R. Hence $I \supseteq J(R)$ and so $\mathbb{Z}_2$ is a homomorphic image $R/J(R)$. Since R is semilocal, it follows that $\mathbb{Z}_2$ is isomorphic to a direct summand of the ring $R/J(R)$. Since $\mathbb{Z}_2$ does not satisfy unit 1-stable range, we deduce that $R/J(R)$ does not satisfy unit 1-stable range, and hence R does not satisfy unit 1-stable range. $\square$

Arguing as in proving Theorem 2.2.1, one can show the following

Theorem 2.3.3 *Let R be an exchange ring with primitive factors Artinian.*

1. *The following are equivalent:*

   (a) R satisfies (P).

   (b) $1+u, 1-u \in U(R)$ for some $u \in U(R)$.

(c) No homomorphic image of R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

2. *The following are equivalent:*

   (a) R satisfies (Q).

   (b) No homomorphic image of R is isomorphic to $\mathbb{Z}_2$.

   (c) R satisfies unit 1-stable range.

   (d) Every element of R is the sum of two units.

Chapter 3

The ring of linear transformations

Recall that a ring R satisfies Condition (P) if for any $a \in R$, there exists $u \in U(R)$ such that $a + u, a - u^{-1} \in U(R)$, and a ring R satisfies Condition (Q) if for any $a \in R$, there exists $u \in U(R)$ such that $a - u, a - u^{-1} \in U(R)$.

Many authors have discussed the decomposition of linear transformations. For example, Zelinsky [26] proved that every linear transformation of a right vector space over a division ring D is a sum of two automorphisms unless $D = \mathbb{Z}_2$ and $\dim(V) = 1$. Nicholson and Varadarajan [19] proved that every linear transformation over a countably generated vector space is the sum of an idempotent and an automorphism. Also Chen [7] proved that the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 3$ satisfies (P). Chen [7] also raised the question whether the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 2$ satisfies (Q). In this chapter, extending Chen's work, we prove that the ring of linear transformations of a right vector space over a division ring D with $|D| > 3$ satisfies (P) and we anwser Chen's question by showing that the ring of linear transformations of a right vector space over a division ring D with $|D| > 2$ satisfies (Q).

3.1 Condition (P)

The following theorem is an improvement of the main result of [7, Theorem 5] that the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| > 3$ satisfies (P).

Theorem 3.1.1 *Let* $\mathrm{End}(V_D)$ *be the ring of linear transformations of a right vector space* V *over a division ring* D*. If* $|D| > 3$*, then* $\mathrm{End}(V_D)$ *satisfies (P).*

To prove this theorem, the following lemma is needed.

For a countably infinite dimensional right vector space V_D, a linear transformation $f \in \mathrm{End}(V_D)$ is called a *shift operator* if there exists a basis $\{v_1, v_2, \ldots, v_n, \ldots\}$ of V such that $f(v_i) = v_{i+1}$ for all i.

Lemma 3.1.2 *[7] Let* V *be a countably infinite dimensional right vector space over a division ring* D *and* $f \in \mathrm{End}(V_D)$ *be a shift operator. Then there exists* $g \in U(\mathrm{End}(V_D))$ *such that* $f + g, f - g^{-1} \in U(\mathrm{End}(V_D))$*.*

Proof. By fixing a basis of V_D, we can identify f with a matrix

$$A = \begin{pmatrix} X & 0 & 0 & \cdots \\ Y & X & 0 & \cdots \\ 0 & Y & X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ where } X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $B = \begin{pmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ 0 & 0 & X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ Y & 0 & 0 & \cdots \\ 0 & Y & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. Then $B^2 = C^2 = 0$ and $A = B+C$. Thus, $1+B$ is invertible with inverse $1-B$. We see that $A+(1-B) = 1+C$ and $A-(1-B)^{-1} = A-(1+B) = C-1$ are invertible. □

Proof of Theorem 3.1.1. Let $f \in \mathrm{End}(V_D)$. Let $\mathcal{S}$ be the set of all ordered pairs (W, g), where W is an f-invariant subspace of V and $g, f|_W + g, f|_W - g^{-1}$ are units of $\mathrm{End}(W_D)$ (where $f|_W$ is restriction of f to W). Clearly, $((0), 1) \in \mathcal{S}$.

Define a partial ordering on $\mathcal{S}$ by setting $(W', g') \leq (W, g)$ whenever both are in $\mathcal{S}$, $W' \subseteq W$ and $g' = g|_{W'}$.

Suppose that $\{(W_\alpha, g_\alpha) : \alpha \in \Lambda\}$ is a totally ordered subset of $\mathcal{S}$. We define $g \in \mathrm{End}((\cup W_\alpha)_D)$ by setting $g(x) = g_\alpha(x)$ ($\alpha \in \Lambda, x \in W_\alpha$), and it is easy to see that $(\cup W_\alpha, g) \in \mathcal{S}$ and $(W_\alpha, g_\alpha) \leq (\cup W_\alpha, g)$ for all $\alpha \in \Lambda$. It follows from Zorn's Lemma that there exists a maximal element (U, h) in $\mathcal{S}$; we prove this theorem by showing that $U = V$. We assume that $U \neq V$, and show that this leads to a contradiction.

Let us fix $x \in V \backslash U$. Let $V_0 := U + K$ where K is the subspace of V spanned by $\{x, f(x), f^2(x), \ldots\}$, and write $V_0 = U \oplus N$ where N is a nonzero subspace of V_0. Since U is f-invariant, there is a linear transformation $\bar{f} : V_0/U \to V_0/U$ given by $\bar{f}(\bar{v}) = \overline{f(v)}$ (for $v \in V_0$). Let $\pi : V_0 \to N$ be the projection on N along U. There is a natural isomorphism $\varphi : V_0/U \to N$ such that $\varphi(\bar{v}) = \pi(v)$ (for $v \in V_0$). Thus $\theta := \varphi \bar{f} \varphi^{-1} \in \mathrm{End}(N_D)$, and so $\theta\varphi = \varphi\bar{f}$. Since V_0/U is spanned by $\{\bar{x}, \bar{f}(\bar{x}), \bar{f}^2(\bar{x}), \ldots\}$, N is spanned by $\{\varphi(\bar{x}), \varphi(\bar{f}(\bar{x})), \varphi(\bar{f}^2(\bar{x})), \ldots\} = \{\varphi(\bar{x}), \theta\varphi(\bar{x}), \theta^2\varphi(\bar{x}), \ldots\}$. Thus, either $\theta \in \mathrm{End}(N_D)$ is a shift operator or N_D is finite dimensional. So, by Lemma 3.1.2 and Proposition 2.3.1, there exists $\alpha \in U(\mathrm{End}(N_D))$ such that $\theta+\alpha$ and $\theta-\alpha^{-1}$ are all units of $\mathrm{End}(N_D)$. Let $g : V_0 \to V_0$ be given by $g(u+v) = h(u)+\alpha(v)$ $(u \in U, v \in N)$. Then g is a unit of $\mathrm{End}((V_0)_D)$. We next show that $f+g$ and $f-g^{-1}$ are units of $\mathrm{End}((V_0)_D)$.

For $u \in U$ and $v \in N$, we have

$$(*) \qquad (f+g)(u+v) = (f+h)(u) + [f(v)+\alpha(v)].$$

Applying π to both sides of $(*)$ gives

$$\begin{aligned}\pi(f+g)(u+v) &= \pi f(v)+\alpha(v) = \varphi\overline{f(v)}+\alpha(v) = \varphi\bar{f}(\bar{v})+\alpha(v)\\ &= \theta\varphi(\bar{v})+\alpha(v) = \theta\pi(v)+\alpha(v) = \theta(v)+\alpha(v)\\ &= (\theta+\alpha)(v).\end{aligned}$$

If $(f+g)(u+v)=0$, then $(\theta+\alpha)(v)=0$ and so $v=0$. It follows from $(*)$ that $(f+h)(u)=0$, and hence $u=0$. Thus, $f+g : V_0 \to V_0$ is one-to-one.

Clearly, $U \subseteq \mathrm{Im}(f+g)$. For any $w \in N$, there exists $v \in N$ such that $(\theta+\alpha)(v) = w$. Thus, $w = (\theta+\alpha)(v) = \pi(f+g)(u+v) \in \mathrm{Im}(f+g)$ $\big($as $U \subseteq \mathrm{Im}(f+g)\big)$. So $f-g : V_0 \to V_0$ is onto. Hence $f+g$ is a unit of $\mathrm{End}((V_0)_D)$.

It is similar to show that $f-g^{-1}$ is a unit of $\mathrm{End}((V_0)_D)$.

Thus, $(V_0, g) \in \mathcal{S}$ and $(U,h) \le (V_0, g)$, contradicting the maximality of (U,h). So $U = V$ and the proof is complete. □

There remains a question to be considered:

Question 3.1.3 *Let $D=\mathbb{Z}_2$ or $D=\mathbb{Z}_3$ and let V_D be a right vector space of infinite dimension. Dose End(V_D) satisfy (P)?*

3.2 Condition (Q)

In Chapter 2, we proved that, for a semilocal ring R, R satisfies the GM-condition if and only if no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{M}_2(\mathbb{Z}_2)$. Also, R satisfies (P) if and only if no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In addition, R satisfies (Q) if and only if R satisfies unit 1-stable range if and only if no homomorphic image of R is isomorphic to $\mathbb{Z}_2$. Clearly, under the condition of a semilocal ring, the GM-condition implies Condition (P), and Condition (P) implies Condition (Q) and unit 1-stable range. It is easy to verify that the ring $\mathbb{Z}_3$ satisfies (Q), but not (P). But it is unknown whether (P) implies (Q). We first give a sufficient condition for (P) to imply (Q). A ring R is called *right continuous* if every right ideal is essential in a direct summand of R_R and every right ideal isomorphic to a direct summand of R_R is itself a direct summand.

Theorem 3.2.1 *Let $R/J(R)$ be a right continuous ring. If R satisfies (P), then it satisfies (Q).*

Proof. Because every unit of $R/J(R)$ can be lifted to a unit of R, R satisfies (P) (resp. (Q)) if and only if $R/J(R)$ satisfies (P) (resp. (Q)). Thus, we can assume that R is semiprimitive, right continuous. By Utumi [21], R is von Neumann regular; so 2 is a regular element of R. By [27, Lemma 7], $R = S \times T$ where 2 is a unit of S and 2 is a nilpotent element of T. Thus $2 \in J(T) \subseteq J(R)$. Since $J(R) = 0$, $2 = 0$ in T. Since R satisfies (P), T satisfies (P). This, together with the fact that $2 = 0$ in T, implies that T satisfies (Q). It remains to show that S satisfies (Q). Because R is right continuous, S is right continuous. So every element of S is the sum of an idempotent and a unit by [2, Theorem 3.9], and $2 \in U(S)$. Thus, by [3, Theorem 11], for any $a \in S$, $a = u+v$ where $u \in U(S)$ and $v^2 = 1$. This shows $a-v = a-v^{-1} = u \in U(S)$. So S satisfies (Q). Hence $R = S \times T$ satisfies (Q). □

As a consequence of Theorem 3.2.1, the following theorem is an affirmative answer to Chen's question [7, p.6] whether the ring of linear transformations of a countably generated right vector space over a division ring of more than two elements satisfies (Q).

Theorem 3.2.2 *Let* $\mathrm{End}(V_D)$ *be the ring of linear transformations of a right vector space* V *over a division ring* D*. If* $|D| > 2$*, then* $\mathrm{End}(V_D)$ *satisfies* (Q)*.*

Proof. Let $R = \mathrm{End}(V_D)$. It is well-known that R is a right self-injective, von Neumann regular ring. So $R/J(R) = R$ is right continuous. If $|D| > 3$, then R satisfies (P) by Theorem 3.1.1, so R satisfies (Q) by Theorem 3.2.1. Thus, we can assume that $|D| = 3$, i.e., $D \cong \mathbb{Z}_3$. Since R is right self-injective, every element of R is the sum of an idempotent and a unit by [2, Theorem 3.9]. Since $D \cong \mathbb{Z}_3$, 2 is a unit of R. Hence, by [3, Theorem 11], for any $a \in R$, $a = u + v$, where $u \in U(R)$ and $v^2 = 1$. This shows $a - v = a - v^{-1} = u \in U(R)$. Hence R satisfies (Q). □

We have been unable to answer the following

Question 3.2.3 *Let* $D = \mathbb{Z}_2$ *and let* V_D *be a right vector space of infinite dimension. Dose End(*V_D*) satisfy (*Q*)?*

Appendix A

Here we verify that the rings $\mathbb{M}_2(\mathbb{Z}_3)$, $\mathbb{M}_3(\mathbb{Z}_3)$, $\mathbb{M}_3(\mathbb{Z}_2)$, $\mathbb{M}_4(\mathbb{Z}_2)$ and $\mathbb{M}_5(\mathbb{Z}_2)$ all satisfy the GM-condition. The complete set of $n \times n$ matrix units is denoted by $\{E_{ij} : 1 \leq i,j \leq n\}$. For any i,j with $1 \leq i,j \leq n$ and any $a \in R$, we let $P_{ij} = I - E_{ii} - E_{jj} + E_{ij} + E_{ji}$ and $T_{ij}(a) = I + aE_{ij}$. Note that $P_{ij}^2 = I = T_{ij}(a)T_{ij}(-a)$ in $\mathbb{M}_n(R)$. The transpose of a square matrix ring A is denoted by A^T. An $n \times n$ companion matrix over R is a matrix in $\mathbb{M}_n(R)$ of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}.$$

Let $a, b \in R$. If there exists $u \in U(R)$ such that $a - u, b - u^{-1} \in U(R)$, then we write $a \leftrightarrow b$ (or $a \overset{u}{\leftrightarrow} b$ to emphasize the unit u). The next lemma is obvious.

Lemma A.1 *Let $a, b \in R$, $u, x, y \in U(R)$, and let σ be an automorphism or anti-automorphism of R. Then:*

1. *$a \overset{u}{\leftrightarrow} b$ iff $\sigma(a) \overset{\sigma(u)}{\leftrightarrow} \sigma(b)$.*

2. *$a \overset{u}{\leftrightarrow} b$ iff $xay \overset{xuy}{\leftrightarrow} y^{-1}bx^{-1}$.*

The following well-known result is needed.

Lemma A.2 *[13, p.192] Let F be a field and $n \geq 2$. Then every $A \in \mathbb{M}_n(F)$ is similar to its rational canonical form $B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_s \end{pmatrix}$, where $s \geq 1$, B_i is a companion matrix of size n_i, and, when necessary we can assume that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$.*

For a field F, the rank of any $A \in \mathbb{M}_n(F)$ is denoted by $\operatorname{rank}(A)$.

Remark A.3 *Let F be a field and let $A \in \mathbb{M}_n(F)$ with $\operatorname{rank}(A) = n$. To show $A \leftrightarrow B$ for all $B \in \mathbb{M}_n(F)$ it suffices to assume that $A = I$ and B is an arbitrary rational canonical form.*

Proof. Since $rank(A) = n$, there exist units X, Y in $\mathbb{M}_n(F)$ such that $XAY = I$. By Lemma A.2, there exists a unit Z in $\mathbb{M}_n(F)$ such that $Z(Y^{-1}BX^{-1})Z^{-1}=B'$, where B' is the rational canonical form of $Y^{-1}BX^{-1}$. Then $(ZX)A(YZ^{-1}) = ZIZ^{-1} = I$ and $(YZ^{-1})^{-1}B(ZX)^{-1} = Z(Y^{-1}BX^{-1})Z^{-1} = B'$. By Lemma A.1(2), we know that $A \leftrightarrow B$ if and only if $(ZX)A(YZ^{-1}) \leftrightarrow (YZ^{-1})^{-1}B(ZX)^{-1}$, that is $I \leftrightarrow B'$. Hence, $I \leftrightarrow C$ for every rational canonical form C in $\mathbb{M}_n(F)$ implies that $A \leftrightarrow B$ for all $B \in \mathbb{M}_n(F)$. □

We show that $\mathbb{M}_n(\mathbb{Z}_3)$ with $n = 2, 3$ satisfies the GM-condition.

Example A.4 *$\mathbb{M}_2(\mathbb{Z}_3)$ satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_2(\mathbb{Z}_3)$. We need to show that $A \leftrightarrow B$. This is certainly true if $A = 0$. So we assume that $A \neq 0$. Because A is equivalent to either $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or I, by Lemma A.1(1) we can assume that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $A = I$.

Case 1: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc + a + d + 1 \neq 0$, let $U = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and then $A \overset{U}{\leftrightarrow} B$. So we may assume that $ad - bc + a + d + 1 = 0$. If $b \neq 0$, let $U = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ and then $A \overset{U}{\leftrightarrow} B$. So we can also assume $b = 0$. If $c \neq 0$, let $U = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and then $A \overset{U}{\leftrightarrow} B$. So we can also assume that $c = 0$. If $a \neq 1$, let $U = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}$ and then $A \overset{U}{\leftrightarrow} B$. So we can further assume that $a = 1$. It follows that $d = 2$ and so $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. But we see that $A \overset{U}{\leftrightarrow} B$ for $U = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Case 2: $A = I$. By Remark A.3 we can assume that B coincides with its rational canonical form. Thus, either $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $B = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{Z}_3$. For $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, choose $U = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}$ when $a \neq 0$ and choose $U = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ when $a = 0$, and we see that $A \overset{U}{\leftrightarrow} B$. For $B = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$, choose $U = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ when $b - a + 1 \neq 0$ and choose $U = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ when $b - a + 1 = 0$, and we see that $A \overset{U}{\leftrightarrow} B$. □

A ring R is said to satisfy the 2-sum property if every element of R is the sum of two units.

Lemma A.5 *Let $A_1, B_1 \in \mathbb{M}_n(R)$, and $A = \begin{pmatrix} A_1 & * \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & * \\ * & * \end{pmatrix} \in \mathbb{M}_{n+m}(R)$. If $A_1 \leftrightarrow B_1$ in $\mathbb{M}_n(R)$ and $\mathbb{M}_m(R)$ satisfies the 2-sum property, then $A \leftrightarrow B$ in*

$\mathbb{M}_{n+m}(R)$.

Proof. Write $B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. By hypothesis, there is a unit U_1 of $\mathbb{M}_n(R)$ such that

$$X := A_1 - U_1 \text{ and } Y := B_1 - U_1^{-1}$$

are units of $\mathbb{M}_n(R)$. Thus $B_{22} - B_{21}Y^{-1}B_{12} \in \mathbb{M}_m(R)$. Since $\mathbb{M}_m(R)$ satisfies the 2-sum property, there is a unit U_2 of $\mathbb{M}_m(R)$ such that $Z := (B_{22} - B_{21}Y^{-1}B_{12}) - U_2^{-1}$ is a unit of $\mathbb{M}_m(R)$. Then $U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ is a unit of $\mathbb{M}_{n+m}(R)$ such that

$$A - U = \begin{pmatrix} X & * \\ 0 & -U_2 \end{pmatrix} \text{ and}$$

$$B - U^{-1} = \begin{pmatrix} Y & B_{12} \\ B_{21} & B_{21}Y^{-1}B_{12} + Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ B_{21}Y^{-1} & I \end{pmatrix} \begin{pmatrix} Y & B_{12} \\ 0 & Z \end{pmatrix}$$

are units of $\mathbb{M}_{n+m}(R)$. So $A \overset{U}{\leftrightarrow} B$. □

Example A.6 *$\mathbb{M}_3(\mathbb{Z}_3)$ satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_3(\mathbb{Z}_3)$. We need to show that $A \leftrightarrow B$. As done in Example A.4, we can assume that $A = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$ with $s < 3$ or $A = I$. If $A = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$ where $s < 3$, then $A \leftrightarrow B$ by Lemma A.5 because $\mathbb{M}_2(\mathbb{Z}_3)$ satisfies the GM-condition (by Example A.4) and $\mathbb{Z}_3$ satisfies the 2-sum property. Hence we can assume $A = I$. By Lemma A.2 and Remark A.3, we can assume that B is one of the following matrices:

$$B_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad B_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b & c \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}, \text{ where } a, b, c \in \mathbb{Z}_3.$$

Case 1: $B = B_1$. If $abc = 0$, we can assume that $c = 0$ by Lemma A.1(1). Then Lemma A.5 shows that $A \leftrightarrow B$. So we can assume that $abc \neq 0$. If one of a, b, c is 1, we can assume that $c = 1$. By Example A.4, there exists a unit U_1 of $\mathbb{M}_2(\mathbb{Z}_3)$ such that $I_2 \overset{U_1}{\leftrightarrow} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $\mathbb{M}_2(\mathbb{Z}_3)$, and hence $A \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} U_1 & 0 \\ 0 & 2 \end{pmatrix}$. So we can further assume that $a = b = c = 2$. But we see $A \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ with $U^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Case 2: $B = B_2$. Choose $0 \neq x \in \mathbb{Z}_3$ such that $x \neq (a-1)(b-c)$, and we see that $A \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} 0 & 0 & 2x \\ 0 & 2 & 0 \\ 1 & 0 & x \end{pmatrix}$ with $U^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2x & 0 & 0 \end{pmatrix}$.

Case 3: $B = B_3$. Choose $x \in \mathbb{Z}_3$ such that $x \neq 2 - a + b - c$. Then we have $A \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$ with $U^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2x & 0 & 2 \end{pmatrix}$. □

Lemma A.7 *Let $A, B \in \mathbb{M}_3(\mathbb{Z}_2)$ with $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & 0 & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}$. Then $A \overset{U}{\leftrightarrow} B$ for some unit U of $\mathbb{M}_3(\mathbb{Z}_2)$.*

Proof. Take $U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & b_{23} \\ 1 & b_{32} & b_{33} \end{pmatrix}$. Then $A \overset{U}{\leftrightarrow} B$ holds. □

Lemma A.8 *Let $A, B \in \mathbb{M}_3(\mathbb{Z}_2)$ with $A = E_{11}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$. Then $A \overset{U}{\leftrightarrow} B$ for some unit U of $\mathbb{M}_3(\mathbb{Z}_2)$.*

Proof. By Lemma A.7, we can assume that either $b_{13} = 1$ or $b_{31} = 1$. If $b_{13} = b_{31} = 1$, then $T_{32}(b_{21})P_{23}BP_{23}T_{23}(b_{12}) = T_{32}(b_{21})P_{23} \begin{pmatrix} b_{11} & 1 & b_{12} \\ b_{21} & b_{23} & 0 \\ 1 & 0 & b_{32} \end{pmatrix} T_{23}(b_{12}) =$

$T_{32}(b_{21})\begin{pmatrix} b_{11} & 1 & b_{12} \\ 1 & 0 & b_{32} \\ b_{21} & b_{23} & 0 \end{pmatrix}T_{23}(b_{12}) = \begin{pmatrix} b_{11} & 1 & b_{12} \\ 1 & 0 & b_{32} \\ 0 & b_{23} & * \end{pmatrix}T_{23}(b_{12}) = \begin{pmatrix} b_{11} & 1 & b_{12} \\ 1 & 0 & b_{32} \\ 0 & b_{23} & * \end{pmatrix}$. So $A \leftrightarrow T_{32}(b_{21})P_{23}BP_{23}T_{23}(b_{12})$ by Lemma A.7. But since $T_{23}(b_{12})P_{23}AP_{23}T_{32}(b_{21}) = T_{23}(b_{12})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}T_{32}(b_{21}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A$, we have $A \leftrightarrow B$ by Lemma A.1(2). Hence we can assume that one of b_{13} and b_{31} is 1 and the other is 0. Because $A^T = A$, $A \leftrightarrow B$ iff $A \leftrightarrow B^T$ by Lemma A.1(1). Hence without loss of generality, we can assume that $b_{13} = 1$ and $b_{31} = 0$. If $b_{12} = 1$, then $BT_{23}(1) = \begin{pmatrix} b_{11} & 1 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & b_{32} & b_{32} \end{pmatrix}$; so $A \leftrightarrow BT_{23}(1)$ by Lemma A.7. But, since $T_{23}(1)A = A$, we have $A \leftrightarrow B$ by Lemma A.1(2). Hence we can finally assume that $b_{13} = 1$, $b_{31} = 0$, and $b_{12} = 0$. Then $T_{21}(b_{23})BP_{23} = T_{21}(b_{23})\begin{pmatrix} b_{11} & 1 & 0 \\ b_{21} & b_{23} & 0 \\ 0 & 0 & b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & 1 & 0 \\ * & 0 & 0 \\ 0 & 0 & b_{32} \end{pmatrix}$, and hence $A \leftrightarrow T_{21}(b_{23})BP_{23}$ by Lemma A.7. But, since $P_{23}AT_{21}(b_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}T_{21}(b_{23}) = A$, we have $A \leftrightarrow B$ by Lemma A.1(2). The proof is complete. □

Now we prove that $\mathbb{M}_n(\mathbb{Z}_2)$ with $3 \le n \le 5$ satisfies the GM-condition.

Example A.9 $\mathbb{M}_3(\mathbb{Z}_2)$ *satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_3(\mathbb{Z}_2)$. We need to show that $A \leftrightarrow B$. This is clearly true if $A = 0$ or $B = 0$. So we suppose $A \neq 0$ and $B \neq 0$.

Case 1: $\text{rank}(A) = 1$ or $\text{rank}(B) = 1$. Without loss of generality we assume $\text{rank}(A) = 1$. By Lemma A.1(2), we can further assume $A = E_{11}$. If B_{22} be the lower right 2×2 supmatrix of B, then by Lemma A.2 there is a unit P of $\mathbb{M}_2(\mathbb{Z}_2)$ such that $PB_{22}P^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $PB_{22}P^{-1} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ where $a, b \in \mathbb{Z}_2$. Let $U = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$.

Then $A = UAU^{-1}$ and $UBU^{-1} = \begin{pmatrix} * & * \\ * & PB_{22}P^{-1} \end{pmatrix}$. Hence, by Lemma A.1(2), we can assume that either $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & a & 0 \\ b_{31} & 0 & b \end{pmatrix}$ or $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & 1 \\ b_{31} & a & b \end{pmatrix}$.

If $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & a & 0 \\ b_{31} & 0 & b \end{pmatrix}$, then $P_{23}B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{31} & 0 & b \\ b_{21} & a & 0 \end{pmatrix}$, so $A \leftrightarrow P_{23}B$ by Lemma A.8. But since $AP_{23} = A$, we have $A \leftrightarrow B$ by Lemma A.1(2).

If $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & 1 \\ b_{31} & a & b \end{pmatrix}$, then $T_{32}(b)B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & 1 \\ * & a & 0 \end{pmatrix}$, so $A \leftrightarrow T_{32}(b)B$ by Lemma A.8. But since $AT_{32}(b) = A$, we have $A \leftrightarrow B$ by Lemma A.1(2).

Case 2: $\operatorname{rank}(A) = 2$ or $\operatorname{rank}(B) = 2$. Without loss of generality we assume $\operatorname{rank}(A) = 2$. By Lemma A.1(2), we can further assume $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If B_{11} is the upper left 2×2 supmatrix of B, then by Lemma A.2 there is a unit P of $\mathbb{M}_2(\mathbb{Z}_2)$ such that $PB_{11}P^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $PB_{11}P^{-1} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ where $a, b \in \mathbb{Z}_2$. Let $U = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$. Then $A = UAU^{-1}$ and $UBU^{-1} = \begin{pmatrix} PB_{11}P^{-1} & * \\ * & * \end{pmatrix}$. Hence, by Lemma A.1(2), we can assume that either $B = \begin{pmatrix} a & 0 & b_{13} \\ 0 & b & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ or $B = \begin{pmatrix} 0 & 1 & b_{13} \\ a & b & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$.

Subcase 1: $B = \begin{pmatrix} 0 & 1 & b_{13} \\ a & b & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. We have

$$A' := T_{13}(b_{13})P_{12}AT_{21}(b)T_{31}(b_{32}) = T_{13}(b_{13})P_{12}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}T_{21}(b)T_{31}(b_{32}) = T_{13}(b_{13})$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_{21}(b)T_{31}(b_{32}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_{21}(b)T_{31}(b_{32}) = \begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_{31}(b_{32}) = \begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $B' := T_{31}(b_{32})T_{21}(b)BP_{12}T_{13}(b_{13}) = T_{31}(b_{32})T_{21}(b)\begin{pmatrix} 1 & 0 & b_{13} \\ b & a & b_{23} \\ b_{32} & b_{31} & b_{33} \end{pmatrix} T_{13}(b_{13}) =$

$T_{31}(b_{32})T_{21}(b)\begin{pmatrix} 1 & 0 & 0 \\ b & a & * \\ b_{32} & b_{31} & * \end{pmatrix} = T_{31}(b_{32})\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & * \\ b_{32} & b_{31} & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & * \\ 0 & b_{31} & * \end{pmatrix}$. If $a = 0$, then $A' \leftrightarrow B'$ by Lemma A.7; so $A \leftrightarrow B$ by Lemma A.1(2). If $a = 1$, then $A'T_{32}(b_{31}) = A'$ and $T_{32}(b_{31})B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b'_{23} \\ 0 & 0 & b'_{33} \end{pmatrix}$. Let $U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & b'_{23}+1 \\ 1 & 1 & b'_{33} \end{pmatrix}$. Then

$U = \begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $A'T_{32}(b_{31}) - U$ and $T_{32}(b_{31})B' - U^{-1}$ are units of $\mathbb{M}_3(\mathbb{Z}_2)$. That is, $A'T_{32}(b_{31}) \overset{U}{\leftrightarrow} T_{32}(b_{31})B'$. This implies $A' \leftrightarrow B'$, which in turn implies $A \leftrightarrow B$ by Lemma A.1(2).

Subcase 2: $B = \begin{pmatrix} a & 0 & b_{13} \\ 0 & b & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. We have $A_1 := AP_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$B_1 := P_{12}B = \begin{pmatrix} 0 & b & b_{23} \\ a & 0 & b_{13} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. By Lemma A.1(2), to show $A \leftrightarrow B$ it suffices to show $A_1 \leftrightarrow B_1$.

(1) If $b_{23} = b_{31} = 0$, then $A_1 \leftrightarrow B_1$ by Lemma A.7.

(2) Suppose $b_{23} = 0$ and $b_{31} = 1$. Then $B_1 = \begin{pmatrix} 0 & b & 0 \\ a & 0 & b_{13} \\ 1 & b_{32} & b_{33} \end{pmatrix}$. Let $U^{-1} =$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & b_{13} \\ 1 & b_{32}+1 & b_{33} \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & b_{13} \\ 0 & 0 & 1 \end{pmatrix}$. Then $U = \begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 1 & b_{13} \\ 0 & 1 & b_{13} \\ 0 & 0 & 1 \end{pmatrix}$.
It can be checked that $A_1 \overset{U}{\leftrightarrow} B_1$ if $a = 1$ and that $T_{13}(b_{33})A_1 \overset{V}{\leftrightarrow} B_1T_{13}(b_{33})$ if $a = 0$. Hence $A_1 \leftrightarrow B_1$ by Lemma A.1(2).

(3) Suppose $b_{23} = 1$ and $b_{31} = 0$. As seen in (2), $A_1^T \leftrightarrow B_1^T$, so $A_1 \leftrightarrow B_1$ by Lemma A.1(1).

(4) Suppose $b_{23} = 1$ and $b_{31} = 1$. Then $B_1 = \begin{pmatrix} 0 & b & 1 \\ a & 0 & b_{13} \\ 1 & b_{32} & b_{33} \end{pmatrix}$. If $a = 1$, then $T_{32}(1)B_1 = \begin{pmatrix} 0 & b & 1 \\ 1 & 0 & b_{13} \\ 0 & b_{32} & * \end{pmatrix}$ and $A_1T_{32}(1) = A_1$; so as done in (3), $A_1T_{32}(1) \leftrightarrow T_{32}(1)B_1$, showing $A_1 \leftrightarrow B_1$. If $b = 1$, then $B_1T_{23}(1) = \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & b_{13} \\ 1 & b_{32} & * \end{pmatrix}$ and $T_{23}(1)A_1 = A_1$; so as done in (2), $T_{23}(1)A_1 \leftrightarrow B_1T_{23}(1)$, showing $A_1 \leftrightarrow B_1$. If $a = b = 0$, then $B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & b_{13} \\ 1 & b_{32} & b_{33} \end{pmatrix}$. Let $x = b_{13} + b_{33} + b_{13}b_{32} + 1$. Then $U := \begin{pmatrix} 1 & 1 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U^{-1}$ and, moreover, $A_1 \overset{U}{\leftrightarrow} B_1$. Hence in subcase 2, we have proved $A_1 \leftrightarrow B_1$.

Case 3: $\operatorname{rank}(A) = \operatorname{rank}(B) = 3$. In view of Remark A.3, we can assume that $A = I_3$ and B is one of the following matrices: $B_1 = I_3$, $B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & c \end{pmatrix}$, $B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & b & c \end{pmatrix}$, where $b, c \in \mathbb{Z}_2$. Let $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with $U^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then $A \leftrightarrow B$ follows from the facts that $I_3T_{31}(1) \overset{U}{\leftrightarrow} T_{31}(1)B_1$, $T_{13}(1)I_3T_{31}(1) \overset{U}{\leftrightarrow} T_{31}(1)B_2T_{13}(1)$, and $I_3T_{32}(c)T_{31}(1) \overset{U}{\leftrightarrow} T_{31}(1)T_{32}(c)B_3$. □

Lemma A.10 *Let $A, B \in \mathbb{M}_{n+1}(\mathbb{Z}_2)$ and suppose $\mathbb{M}_n(\mathbb{Z}_2)$ satisfies the GM-condition. If $\operatorname{rank}(A) \le n$ and $\operatorname{rank}(B) \le n$, then $A \leftrightarrow B$ in $\mathbb{M}_{n+1}(\mathbb{Z}_2)$.*

Proof. By Lemma A.1(2), we can assume that $A = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$ with $s \le n$ and $\operatorname{rank}(B) \le n$. There exist units P and Q of $\mathbb{M}_{n+1}(\mathbb{Z}_2)$ such that $PBQ = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ where $k \le n$. Hence $PB = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} = \begin{pmatrix} B_1 & * \\ 0 & 0 \end{pmatrix}$ and $AP^{-1} = \begin{pmatrix} A_1 & * \\ 0 & 0 \end{pmatrix}$, where $A_1, B_1 \in \mathbb{M}_n(\mathbb{Z}_2)$. Since $\mathbb{M}_n(\mathbb{Z}_2)$ satisfies the GM-condition, there is a unit U_1 of

$\mathbb{M}_n(\mathbb{Z}_2)$ such that $A_1 \overset{U_1}{\leftrightarrow} B_1$. Hence $A \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix}$. □

Lemma A.11 *For $a, b \in \mathbb{Z}_2$, $I_2 \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $I_2 \leftrightarrow \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}$ in $\mathbb{M}_2(\mathbb{Z}_2)$.*

Proof. Let $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. To show $I_2 \leftrightarrow B$ we can assume that $B = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$ where $s \leq 2$ by Lemma A.1(2), and so $I_2 \overset{U}{\leftrightarrow} B$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Moreover, $I_2 \overset{U}{\leftrightarrow} \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. □

Example A.12 $\mathbb{M}_4(\mathbb{Z}_2)$ *satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_4(\mathbb{Z}_2)$. We need to show that $A \leftrightarrow B$. In view of Example A.9 and Lemma A.10, we can assume that either $\text{rank}(A) = 4$ or $\text{rank}(B) = 4$. Without loss of generality we assume that $\text{rank}(A) = 4$. By Remark A.3, we may further assume $A = I_4$ and B is one of the following matrices:

$$B_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, B_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix},$$

$$B_4 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & c & d \end{pmatrix}, B_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}_2$. By Lemma A.11, there exist units P_1, P_2 of $\mathbb{M}_2(\mathbb{Z}_2)$ such that $I_2 \overset{P_1}{\leftrightarrow} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $I_2 \overset{P_2}{\leftrightarrow} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ in $\mathbb{M}_2(\mathbb{Z}_2)$. It follows that $I_4 \overset{P}{\leftrightarrow} B_1$ where $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$. It is similar to see that $A \leftrightarrow B_2$ when $d = 1$. Let $U_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, U_2 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, U_3 = U_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, U_5 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To verify $A \leftrightarrow B_2$, we can assume $d = 0$. We see that

$$T_{21}(1)I_4P_{12}T_{21}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_{12}T_{21}(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T_{21}(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{U_2}{\leftrightarrow} \begin{pmatrix} b & b & 0 & 0 \\ a-b & b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix} = \begin{pmatrix} 0 & b & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix} T_{21}(1) = T_{21}(1) \begin{pmatrix} 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix} T_{21}(1) =$$

$T_{21}(1)P_{12}B_2T_{21}(1)$, if $c = 0$ and that $T_{21}(1)I_4P_{12} \overset{V_3}{\leftrightarrow} P_{12}B_2T_{21}(1)$ if $c = 1$. So $A \leftrightarrow B_2$ follows.

$$\text{Because } I_4T_{21}(b)T_{12}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} T_{12}(1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \overset{U_3}{\leftrightarrow} \begin{pmatrix} a & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix}$$

$$= T_{12}(1) \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & d \end{pmatrix} = T_{12}(1)T_{21}(b)B_3, \text{ we see } I_4 \leftrightarrow B_3.$$

$$\text{Because } I_4T_{43}(d)T_{42}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & d & 1 \end{pmatrix} T_{42}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & d & 1 \end{pmatrix} \overset{U_4}{\leftrightarrow} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & c-1 & 0 \end{pmatrix}$$

$$= T_{42}(1) \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & c & 0 \end{pmatrix} = T_{42}(1)T_{43}(d)B_4, \text{ we have } I_4 \leftrightarrow B_4.$$

Finally, $I_4 \leftrightarrow B_5$, since $I_4 T_{41}(1) T_{43}(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} T_{43}(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & d & 1 \end{pmatrix}$

$$\overset{U_5}{\leftrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b-1 & c & 0 \end{pmatrix} = T_{43}(d) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b-1 & c & d \end{pmatrix} = T_{43}(d) T_{41}(1) B_5. \qquad \square$$

Example A.13 $\mathbb{M}_5(\mathbb{Z}_2)$ *satisfies the GM-condition.*

Proof. Let $A, B \in \mathbb{M}_5(\mathbb{Z}_2)$. We need to show that $A \leftrightarrow B$. In view of Example A.12 and Lemma A.10, we can assume that either $\operatorname{rank}(A) = 5$ or $\operatorname{rank}(B) = 5$. Without loss of generality we assume that $\operatorname{rank}(A) = 5$. By Remark A.3, we may further assume $A = I$ and B is one of the following matrices:

$$B_1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{pmatrix}, B_2 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & d & e \end{pmatrix}, B_3 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & c & d & e \end{pmatrix},$$

$$B_4 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & b & c & d & e \end{pmatrix}, B_5 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & d & e \end{pmatrix}, B_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & c & d & e \end{pmatrix},$$

where $a, b, c, d, e \in \mathbb{Z}_2$. By Lemma A.11 and Example A.9, there exists a unit U_1 of $\mathbb{M}_2(\mathbb{Z}_2)$ and a unit U_2 of $\mathbb{M}_3(\mathbb{Z}_2)$ such that $I_2 \overset{U_1}{\leftrightarrow} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $I_3 \overset{U_2}{\leftrightarrow} \begin{pmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}$. It follows that $I_5 \overset{U}{\leftrightarrow} B_1$ where $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$. It is similar to see that $I_5 \leftrightarrow B_i$ for $i = 2, 3,$

and $I_5 \leftrightarrow B_6$ when $b = 1$. Let $U_4 = U_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $U_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Because $I_5T_{54}(e)T_{52}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e & 1 \end{pmatrix} T_{52}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & e & 1 \end{pmatrix} \overset{U_4}{\leftrightarrow} \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & b & c-1 & d & 0 \end{pmatrix} =$

$T_{52}(1) \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & b & c & d & 0 \end{pmatrix} = T_{52}(1)T_{54}(e)B_4$, we have $I_5 \leftrightarrow B_4$.

To verify $I_5 \leftrightarrow B_5$, we first assume $c = 1$. Then there is a unit P of $\mathbb{M}_5(\mathbb{Z}_2)$ such that $PB_5P^{-1} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ where $X = \begin{pmatrix} 0 & 1 \\ b & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & d & e \end{pmatrix}$, and we see that $I_5 \leftrightarrow PB_5P^{-1}$ by arguing as in proving $I_5 \leftrightarrow B_1$. So by Lemma A.1(2), $I_5 = P^{-1}I_5P \leftrightarrow P^{-1}(PB_5P^{-1})P = B_5$. Hence, we can assume that $c = 0$. Then we have

$I_5T_{52}(1)T_{54}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} T_{54}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & e & 1 \end{pmatrix} \overset{U_5}{\leftrightarrow} \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & d & 0 \end{pmatrix} =$

$T_{54}(e) \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & d & 0 \end{pmatrix} = T_{54}(e)T_{52}(1)B_5$, so $I_5 \leftrightarrow B_5$ follows. To verify $I_5 \leftrightarrow B_6$, we can assume $b = 0$. Then $I_5 \leftrightarrow B_6$ follows from the fact that $I_5T_{51}(1)T_{54}(e) =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} T_{54}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & e & 1 \end{pmatrix} \overset{U_6}{\leftrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & c & d & 0 \end{pmatrix} = T_{54}(e) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & c & d & e \end{pmatrix} =$$

$T_{54}(e)T_{51}(1)B_6$. The proof is completed. □

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