DYNAMICS OF SOME NEURAL NETWORK MODELS

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Dynamics of Some Neural Network Models

by

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Abstract

This Ph.D. dissertation consists of four chapters and mainly deals with the dynamics of several neural network models described by delay differential equations, difference equations and stochastic differential equations.

In Chapter 1, some background of neural networks and the motivations of this work are briefly addressed.

In Chapter 2, Liapunov functional method and the theory of monotone dynamical systems are employed to obtain some delay independent and delay dependent stability results for the general continuous–time Cohen–Grossberg neural networks with distributed delays. Detailed local stability and bifurcation analysis are also given in this chapter for the bidirectional associative memory (BAM) neural networks with and without self–connections.

Chapter 3 is devoted to the study of discrete–time neural networks with delays. Specifically, we first derive some global stability results for the discrete–time neural networks with variable delays and then investigate the capacity of the discrete-time BAM neural networks by giving the number of all possible stable periodic solutions.

The stochastic neural networks are studied in Chapter 4, in which some criteria for the almost sure exponential stability, mean square exponential stability are established for stochastic Cohen–Grossberg neural networks with and without delays.

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Chapter 1

Introduction

What is a neural network? As stated by Hirsch [46], a neural network could be many things: a piece of hardware, a computer, an algorithm and so on. In this thesis, we only consider artificial neural networks, that is, networks of interconnected elements behaving like biological neurons. Such a network is usually described by a system of differential equations (continuous–time) or difference equations (discrete–time). For each single neuron, the simple structure results in a very simple mathematical formula so that the function of a single neuron can be easily fulfilled by a simple electrical element. However, when many such simple neurons are connected to form a network, which results in a system of coupled differential equations (or difference equations), the whole network could have very rich dynamics and thus could admit various applications.

We plan to discuss three types of artificial neural networks: continuous-time neural networks; discrete-time neural networks and stochastic neural networks. They all have the following common characteristics:

(i) to each neuron i, a variable is associated to represent its state or activation;

(ii) a real valued weight w_{ij} is assigned, with $|w_{ij}|$ giving the strength of the connection between two neurons i and j, and the sign of w_{ij} telling whether the connection is excitatory ($w_{ij} > 0$) or inhibitory ($w_{ij} < 0$);

(iii) each neuron is assigned a nonlinear function (activation function, or transfer function), which yields an output to other neurons in the network.

The first mathematical model of biological neurons was proposed by McCulloch and Pitts in 1943 [76], in which the evolution of the network is governed by the system of difference equations

$$x_i(t+1) = s\left(\sum_{j=1}^{n} w_{ij}x_j(t) - \theta_i\right), i = 1, 2, ..., n, \tag{1.0.1}$$

where x_i is the state variable associated with neuron i, w_{ij} represents the synaptic coupling strengths between neurons j and i. θ_i is a threshold and the transfer function $s(x)$ is the unit step function. Although McCulloch and Pitts showed that such a network can carry out any logical calculation and thus can be viewed as a kind of computer working in parallel manner, there was not a good algorithm for choosing the synaptic couplings for the desired output until 1961, when Caianiello [8] introduced a learning algorithm based on the well-known Hebb's learning rule. For many other neural network models, we refer to [27].

The theory and applications of neural networks have been greatly developed since 1980s after Cohen and Grossberg's paper [24] and Hopfield's paper [50] were published. In [24], Cohen and Grossberg discussed a network model now known as the Cohen–Grossberg neural network (CGNN) model described by a system of ordinary differential equations

$$\dot{x}_i(t) = a_i(x_i(t)) \left(b_i(x_i(t)) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t)) \right), \quad i = 1, \cdots, n, \tag{1.0.2}$$

In [50], Hopfield proposed an additive network known as the Hopfield neural network (HNN) described by the following system

$$C_i \dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij} s_j(x_j(t)) + J_i, \quad i = 1, 2, \ldots, n, \tag{1.0.3}$$

which was later implemented by electric circuits to fulfill various tasks such as linear programming, solving the sales-man problem [95], etc. Due to their promising potential for the tasks of classification, associative memory, parallel computations, and their ability to solve difficult optimization problems, (1.0.2) and (1.0.3) have attracted great attention from the scientific community. Various generalizations and modifications of (1.0.2) and (1.0.3) have then been proposed and studied, among which is the incorporation of time delay into the model. In fact, due to the finite speeds of the switching and transmission of signals in a network, time delays do exist in a working network and thus should be incorporated into the model equations of the network. More detailed justifications for introducing delays into model equations of neural networks can be found in [73], [80] and the recent book [106].

Marcus and Westervelt [73] first introduced a single delay into (1.0.3) and considered the following system of delay differential equations

$$C_i \dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij} s_j(x_j(t - \tau)) + J_i, \quad i = 1, 2, \ldots, n. \tag{1.0.4}$$

They observed both experimentally and numerically [73] that delay could destroy a stable network and cause sustained oscillations and thus, could be harmful. System

(1.0.4) has also been studied by Wu [107], Wu and Zou [112]. Recently Gopalsamy and He [36], and van den Driessche and Zou [97] studied a further generalized version with multiple delays

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} w_{ij} s_j(x_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \ldots, n. \tag{1.0.5}$$

For the Cohen-Grossberg model (1.0.2), Ye, Michel and Wang [119] introduced delays by considering the following system of delay differential equations

$$\dot{x}_i(t) = -a_i(x_i(t)) \left(b_i(x_i(t)) - \sum_{k=0}^{K} \sum_{j=1}^{n} w_{ij}^{(k)} s_j(x_j(t-\tau_k)) \right), \quad i = 1, 2, \ldots, n. \tag{1.0.6}$$

Established in the pioneering work of Cohen and Grossberg [24] and Hopfield [50] was the "globally asymptotic stability" of systems (1.0.2) and (1.0.3), respectively, in the sense that given any initial conditions, the solution of the system (1.0.2) (or (1.0.3)) will converge to some equilibrium of the corresponding system. Such a "global stability" in Hopfield [50] and Cohen and Grossberg [24] was obtained by considering some energy functions under some assumptions, among which the *symmetry of the connection matrix* T is crucial.

When it comes to the delayed systems (1.0.4), (1.0.5) and (1.0.6), it is natural to expect that such a global stability remains if the delays are sufficiently small. Indeed, such an expectation was confirmed in [118] and [119] under a certain type of *symmetry* conditions on the connection matrices. When a network is designed for the purpose of associative memories, it is required that the system have a set of stable equilibria, each of which corresponds to an addressable memory. The global stability confirmed in [50], [24], [118] and [119] is necessary and crucial for associative memory networks. However, an obvious drawback of the above work is the lack of description or even estimates for the basin of attraction of each stable equilibrium. In other words, given a set of initial conditions, one knows that the solution will converge to some equilibrium, but *does not* know exactly to which equilibrium it will converge. In terms of associative memories, one *does not* know precisely what initial conditions are needed in order to retrieve a particular pattern stored in the network. Furthermore, the work of [118] and [119] cannot tell what would happen when the delays increase. Usually large delay could destroy the stability of an equilibrium in a network. Even if sometimes the delay does not change the stability of an equilibrium, it could affect the basin of attraction of a stable equilibrium. For such a topic, see [2], Pakdaman et al [83] and [84], or Wu [106].

On the other hand, in applications of neural networks to parallel computation, signal processing and other problems involving the solutions of optimization problems, it is required that there be a well-defined computable solution for all possible initial states. In other words, it is required that the network should have a *unique* equilibrium that is globally attractive. In fact, earlier applications of neural networks to optimization problems have suffered from the existence of a complicated set of equilibria [95]. Thus, the global attractivity of a unique equilibrium for the system is of great importance for both practical and theoretical purposes, and has been the major concern of many authors. For the Hopfield type neural networks, see, for example, Bélair [3], Cao and Wu [12], Gopalsamy and He [36], Hirsch [47], van den Driessche and Zou [97], Lu[69], Matsuoka [74], Guan, Chen and Qin [38]. But, to the best of our knowledge, for the Cohen-Grossberg type neural networks, results appeared in the literature are very few (See, [98], [99], [102], [103] and [119]), especially when multiple delays (infinitely distributed delays, finitely distributed delays, variable delays) are involved in the model.

In designing and implementing a network, it is preferable and desirable that the neural network not only converge, but also converge as fast as possible. It is well known that exponential stability gives a fast convergence rate to the equilibrium. Therefore, we expect to obtain some exponential stability results as well.

As we mentioned before, the delays in a network have impact on its dynamics. One way to see how the delays will affect the dynamics of a neural network is to carry out a bifurcation analysis by viewing some delay as the bifurcation parameter. In this work, we will consider the bidirectional associative memory (BAM for short) neural networks with multiple delays and study the Hopf bifurcations caused by the self–connection delay and off-diagonal connection delays, respectively.

In associative memory, the capacity of the network is a big issue that needs to be concerned with. Thus, in this work, we will study the capacity of discrete–time BAM neural networks. Our results show that even though the discrete–time BAM neural networks allow relatively small equilibrium capacity, the delayed discrete–time BAM neural networks can admit large capacity for stable periodic solution under certain conditions.

We also notice that researchers have paid little attention to the study of stochastic neural networks. This encourages us to explore the dynamics of stochastic neural networks. Note that a stochastic neural network can be viewed as a deterministic neural network with stochastic perturbations, it is therefore of importance to identify the role of stochastic perturbation in the dynamics of neural networks. To this end, we shall mainly study the stability including almost sure exponential stability and mean square exponential stability for both stochastic Cohen–Grossberg neural networks with and without delays.

Throughout this thesis, the following general notations are adopted:

$\mathbb{R}$: the set of real numbers

$\mathbb{R}_+$: the set of all nonnegative real numbers

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}$

$\mathbb{R}^{n\times n}$: the set of $n \times n$ real matrices

$x^T = (x_1, \cdots, x_n)$: the transpose of x

W^T : the transpose of the matrix W

$||W||_2 = \left(\max\{|\lambda| : \lambda \text{ is an eigenvalue of } W^T W\}\right)^{1/2}$

$N(a) = \{a, a+1, \cdots, \}$, $N(a, b) = \{a, a+1, \cdots, b-1, b\}$, where$a < b$

$\sigma(W)$: the (point) sepctrum set of an matrix M

$\rho(W)$: the spectral radius of the matrix W

$C : C([-\tau, 0], R^n)$

$||\phi|| = \sup\{|\phi(\theta)|, \theta \in [-\tau, 0], \phi \in C\}$

$Lip(f) = \sup\left\{\left|\frac{f(u)-f(v)}{u-v}\right|, u \neq v, u, v \in R\right\}$.

Other notations will be specified in the context.

The rest of the thesis is organized as follows: In Chapter 2, the Liapunov functional method is first employed to discuss the stability of the general continuous–time Cohen–Grossberg neural networks with distributed delays. The Hopfield neural networks with infinite (finite) distributed delays, periodic inputs, are then studied. Moreover, the detailed local stability and bifurcation analysis are also given in this chapter for the bidirectional associative memory (BAM) neural networks with and without delayed self–connections. The exponential stability of discrete–time neural networks with variable delays and the capacity of stable periodic solutions in the discrete-time BAM neural networks are investigated in Chapter 3. The almost sure exponential stability, mean square exponential stability of the stochastic neural networks with and without delays are established in Chapter 4.

Chapter 2

Dynamics of Continuous–Time Neural Network Models with Multiple Delays

In this chapter, we study the (global) stability of some general continuous-time neural networks with multiple delays. We also consider the local stability and Hopf bifurcation of delayed BAM neural networks with and without delayed self-connections.

In Section 2.1, some global stability results for the Cohen–Grossberg neural networks with infinite distributed delays and finite distributed delays are established by using Liapunov functional method.

Section 2.2 is devoted to the stability of Hopfield neural networks with infinite and finite distributed delays. By analyzing the associated characteristic equation, we obtain the local stability for Hopfield neural networks with infinite distributed delays. Moreover, the stabilization roles of inhibitory self–connections for Hopfield neural networks with finite distributed delays are identified via the theory of monotone dynamical systems.

In Section 2.3, we first apply our results obtained in Section 2.1 to the BAM neural networks to obtain some global stability results and then investigate the local stability and Hopf bifurcation and establish the corresponding algorithms to determine the direction and stability of the Hopf bifurcation. Some examples and numerical simulations are also presented.

2.1 Continuous-time Cohen–Grossberg neural networks with distributed delays

Cohen and Grossberg [24] proposed a neural network model (CGNN) in 1983 described by the following system

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}g_j(x_j(t))\right), i \in N(1,n) \tag{2.1.1}$$

where $i \in N(1,n)$ and $n \geq 2$ is the number of neurons in the network; $x_i(t)$ describes the *activation* of the *ith* neuron at time t; a_i represents an amplification function and the function b_i can include a constant term indicating a fixed input to the network; the $n \times n$ connection matrix $A = (a_{ij})$ tells how the neurons are connected in the network; the activation functions $g_j, j \in N(1,n)$ show how the neurons react to the input. As pointed out in [24], the system (2.1.1) includes a number of models from neurobiology, population biology and evolution theory, among which is the Hopfield neural network (HNN) model

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij}g_j(x_j(t)) + I_i, \ i \in N(1,n) \tag{2.1.2}$$

where I_i $(i \in N(1,n))$ is a fixed input from outside of the network. Systems (2.1.1) and (2.1.2) have attracted great attention of the scientific community and have been extensively investigated, see, for example, [3], [9], [12], [13], [14], [15], [16], [17], [31], [32], [36], [38], [45], [56], [58], [64], [69], [73], [74], [79], [97], [101], [102], [103], [107], [113], [114], [115], [116], [117] and [119]. Instead of considering discrete time delays like in [36], [97], we incorporate time delays which are continuously distributed over an infinite interval reflecting the fact that the distant past has less influence compared to the most recent neurons' states on the current states of system (2.1.1) and obtain the following CGNN model with infinite distributed delays

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} g_j\Big(\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds\Big)\right), \tag{2.1.3}$$

where the delay kernel functions $k_{ij}(t)$ are assumed to be piecewise continuous and satisfy

$$k_{ij}(t) \geq 0,\ t \geq 0,\ \int_0^{\infty} k_{ij}(t)dt = 1,\ \int_0^{\infty} t k_{ij}(t)dt < \infty. \tag{2.1.4}$$

We now give some assumptions which will be used later:

(H_1) For each $i \in \{1, 2, \cdots, n\}$, a_i is continuous and satisfies $0 < \underline{\alpha}_i \leq a_i(u) \leq \overline{\alpha}_i$.

(H_2) For each $i \in \{1, 2, \cdots, n\}$, b_i is continuous and strictly increasing.

The activation functions $g_i(x)$, $i \in N(1, n)$, are typically assumed to be sigmoid, that is, they are required to satisfy the following

(A_1) $g_i \in C^2(\mathbb{R})$, $g_i'(x) > 0$, for all $x, \sup_{x\in\mathbb{R}} g_i'(u) = g_i'(0) = 1, i \in N(1, n)$;

(A_2) $g_i(0) = 0$ and $\lim_{x\to\pm\infty} g_i(x) = \pm 1$.

One commonly used such function is $g(x) = \tanh(x)$.

The above assumptions (A_1) and (A_2) imply the monotonicity and smoothness of the activation functions. However, as argued in [97], for some purposes, non-monotone and non-differentiable activation functions would be better candidates and have been frequently adopted in implementation. Such a practical motivation suggests that sometimes we may consider the following replacement for (A_1) and (A_2):

(S_1) For each $i \in N(1,n)$, $g_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with a Lipschitz constant L_i;

(S_2) For each $i \in N(1,n)$, $|g_i(x)| \leq M_i$, $x \in \mathbb{R}$ for some constant $M_i > 0$.

Clearly, $(S_1) - (S_2)$ implies $(A_1) - (A_2)$.

The initial conditions associated with (2.1.3) are of the form

$$x_i(s) = \phi_i(s), i \in N(1,n), \tag{2.1.5}$$

where $(\phi_1, \phi_2, \cdots, \phi_n) =: \phi$ belonging to the Banach space BC of bounded and continuous functions that map $(-\infty, 0]$ into $\mathbb{R}^n$, with the uniform norm $||\phi||_\infty = \sup_{s\leq 0} |\phi(s)|$, where $|\cdot|$ is a chosen norm on $\mathbb{R}^n$. For the general standard existence, uniqueness, continuation of results for the system (2.1.3), we refer to [37], [39], [42], [108].

We first establish an existence result for an equilibrium of system (2.1.3).

Theorem 2.1.1. *If $(H_1) - (H_2)$ and $(S_1) - (S_2)$ (or $(A_1) - (A_2)$) hold, then there exists at least one equilibrium for system (2.1.3).*

Proof. By (H_1), we know that x^* is an equilibrium of (2.1.3) if and only if $x = x^* \equiv (x_1^*, \cdots, x_n^*)^T$ is a solution of the equations

$$b_i(x_i) - \sum_{j=1}^{n} a_{ij} g_j(\int_{-\infty}^{t} k_{ij}(t-s) x_j ds) = 0,\ i \in N(1,n). \tag{2.1.6}$$

From (S_2), we have

$$\left| \sum_{j=1}^{n} a_{ij} g_j(\int_{-\infty}^{t} k_{ij}(t-s) x_j ds) \right| \leq \sum_{j=1}^{n} |a_{ij}| M_j =: P_i, i \in N(1,n).$$

Since (H_2) holds, then b_i^{-1} exists and is strictly increasing. Now consider

$$x_i = h_i(x_1, x_2, \cdots, x_n) := b_i^{-1} \left(\sum_{j=1}^{n} a_{ij} g_j(\int_{-\infty}^{t} k_{ij}(t-s) x_j ds) \right)$$

for $i \in N(1,n)$. We have

$$|h_i(x_1, x_2, \cdots, x_n)| \leq \max\{|b_i^{-1}(P_i)|, |b_i^{-1}(-P_i)|\} =: D_i,\ i \in N(1,n).$$

It follows that $(h_1, h_2, \cdots, h_n)^T$ maps a bounded set $D := [-D_1, D_1] \times [-D_2, D_2] \times \cdots \times [-D_n, D_n]$ to itself. Then the existence of the equilibrium follows from the Brouwer's fixed point theorem (Theorem 3.2, [26]) and the proof is thus completed.
□

Let x^* be an equilibrium of (2.1.3) and $u(t) = x(t) - x^*$. Substituting $x(t) = u(t) + x^*$ into (2.1.3) leads to

$$\dot{u}_i(t) = -\alpha_i(u_i(t)) \left(\beta_i(u_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(\int_{-\infty}^{t} k_{ij}(t-s) u_j(s) ds) \right) \tag{2.1.7}$$

for $i \in N(1,n)$ where $\alpha_i(u_i(t)) = a_i(u_i(t) + x_i^*)$, $\beta_i(u_i(t)) = b_i(u_i(t) + x_i^*) - b_i(x_i^*)$, $f_j = g_j(\int_{-\infty}^{t} k_{ij}(t-s) u_j(s) ds + x_j^*) - g_j(x_j^*)$.

Next, we will use Liapunov functional method to establish some sufficient conditions to guarantee the global stability of the equilibrium of (2.1.3). Such Liapunov functional method to achieve the global stability for functional differential equations with infinite delay has been used in [6], [18], [62], [63].

Theorem 2.1.2. *Assume that* $(H_1), (H_2)$ *and* $(S_1), (S_2)$ *hold. If there exist* $\gamma_i > 0$, *and* $q_i > 0$ *such that*

$$\frac{b_i(u) - b_i(v)}{u - v} \geq \gamma_i \textit{ for } u \in \mathbb{R}, \quad i \in N(1,n) \tag{2.1.8}$$

and

$$\mu := \min_{1 \leq i \leq n} \left\{ \underline{\alpha}_i \gamma_i q_i - L_i \sum_{j=1}^{n} \overline{\alpha}_j |a_{ji}| q_j \right\} > 0, \tag{2.1.9}$$

then every solution of (2.1.3) will finally approach to the equilibrium x^*, *namely,*

$$x_i(t) \to x_i^* \textit{ as } t \to \infty, i \in N(1,n). \tag{2.1.10}$$

Proof. Combining (2.1.7) with (2.1.8), we can estimate the upper righthand derivative of $|u_i(t)|$ as below

$$\begin{aligned} D^+|u_i(t)| &= sgn(u_i(t))\dot{u}_i(t) \\ &\leq -\underline{\alpha}_i \gamma_i |u_i(t)| + \overline{\alpha}_i \sum_{j=1}^{n} L_j |a_{ij}| \int_{-\infty}^{t} k_{ij}(t-s)|u_j(s)|ds \\ &= -\underline{\alpha}_i \gamma_i |u_i(t)| + \overline{\alpha}_i \sum_{j=1}^{n} L_j |a_{ij}| \int_{0}^{\infty} k_{ij}(s)|u_j(t-s)|ds, \end{aligned} \tag{2.1.11}$$

where

$$sgn(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1 & x < 0. \end{cases}$$

Let $V(t) = V(u)(t)$ be defined as

$$V(t) = \sum_{i=1}^{n} \left(q_i|u_i(t)| + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j|a_{ij}| \int_0^\infty k_{ij}(s) \int_{t-s}^{t} |u_j(\omega)| d\omega ds \right). \qquad (2.1.12)$$

Then

$$\min_{1\le i\le n}\{q_i\} \max_{1\le i\le n}\{|u_i(t)|\} \le V(u)(t)$$

and the upper righthand derivative of $V(t)$ along the solution of (2.1.7) satisfies the following

$$\begin{aligned} D^+V(t) &\le \sum_{i=1}^{n} q_i \left(-\underline{\alpha}_i\gamma_i|u_i(t)| + \overline{\alpha}_i \sum_{j=1}^{n} L_j|a_{ij}| \int_0^\infty k_{ij}(s)|u_j(t-s)|ds \right) \\ &\quad + \sum_{i=1}^{n} \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j|a_{ij}| \int_0^\infty k_{ij}(s) \left(|u_j(t)| - |u_j(t-s)|\right) ds \\ &\le \sum_{i=1}^{n} \left(-\underline{\alpha}_i\gamma_i q_i|u_i(t)| + \overline{\alpha}_i q_i \sum_{j=1}^{n} |a_{ij}|L_j|u_j(t)| \right) \\ &= -\sum_{i=1}^{n} \left(\underline{\alpha}_i\gamma_i q_i - L_i \sum_{j=1}^{n} \overline{\alpha}_j|a_{ji}|q_j \right) |u_i(t)| \\ &\le -\mu \sum_{i=1}^{n} |u_i(t)|. \end{aligned} \qquad (2.1.13)$$

This shows the zero solution of (2.1.7) is stable and $\sum_{i=1}^{n} |u_i(t)|$ is bounded for all $t \ge 0$, thus the solutions of (2.1.7) exist globally. Moreover, we have

$$V(t) + \mu \int_0^t \left(\sum_{i=1}^{n} |u_i(s)| \right) ds \le V(0).$$

On the other hand,

$$\begin{aligned} V(0) &= \sum_{i=1}^{n} \left(q_i|u_i(0)| + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j|a_{ij}| \int_0^\infty k_{ij}(s) \int_{-s}^{0} |u_j(\omega)| d\omega ds \right) \\ &\le \sum_{i=1}^{n} \left(q_i + L_i \sum_{j=1}^{n} \overline{\alpha}_j q_j|a_{ji}| \int_0^\infty s k_{ji}(s) ds \right) \sup_{\omega\in(-\infty,0]} |\phi_i(\omega) - x_i^*| < \infty, \end{aligned}$$

which implies that

$$f(t) := \sum_{i=1}^{n} |u_i(t)| \in L^1(0,\infty). \tag{2.1.14}$$

It is clear that $f(t)$ is nonnegative and is defined on $[0,\infty)$. Estimate implies the boundedness of $f(t)$, which, together with (2.1.7), leads to the boundedness of $\dot{u}_i(t),\ i = 1,2,\cdots,n$. This in turn implies that $f(t)$ is uniformly continuous on $[0,\infty)$. By Lemma 1. 2. 2 of [34], we obtain

$$\sum_{i=1}^{n} |u_i(t)| \to 0 \ \text{ as } t \to \infty, \tag{2.1.15}$$

which shows that (2.1.10) holds. The proof is complete. □

Remark 2.1.1. *As stated in [62], [63], the space BC may cause problems for the usual well-posedness questions and compactness of solution semiflow related to functional differential equations with infinite delays. It is suggested in [62], [63], instead of BC, a more friendly space UC_g (the definition can be found in [39], [42], [62], [63], [108]), can be adopted. However, in Theorem 2.1.2 (also in Theorem 2.1.3), we use estimates of solutions in $\mathbb{R}^n$ in the proof of the global convergence result. Thus, neither the choice of BC nor UC_g will cause any problem for our global convergence property.*

Corollary 2.1.1. *If (2.1.8) holds and (2.1.9) is replaced by*

$$\mu_1 := \min_{1\le i\le n} \left\{ \underline{\alpha}_i \gamma_i - L_i \sum_{j=1}^{n} \overline{\alpha}_j |a_{ji}| \right\} > 0, \tag{2.1.16}$$

then the equilibrium x^* of (2.1.3) is globally asymptotically stable.

Proof. Condition (2.1.16) implies (2.1.9) holds for $q_i = 1, i \in N(1,n)$, and thus this corollary follows directly from Theorem 2.1.2. $\square$

By using a different Liapunov functional, we have

Theorem 2.1.3. *Assume that* $(H_1), (H_2)$, $(S_1), (S_2)$ *and (2.1.8) hold. If there exist positive real numbers* $p_i > 0$, *and* $\eta_i > 0$ *such that for* $i \in N(1,n)$

$$2\underline{\alpha}_i \gamma_i p_i - \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| L_j \eta_j - \frac{L_i}{\eta_i} \sum_{j=1}^{n} |a_{ji}| \overline{\alpha}_j p_j > 0, \tag{2.1.17}$$

then the equilibrium x^* *of (2.1.3) is globally asymptotically stable.*

Proof. Define $V(t) = V(u)(t)$ by

$$V(t) = \sum_{i=1}^{n} \left(p_i u_i^2(t) + \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| \frac{L_j}{\eta_j} \int_0^{\infty} k_{ij}(s) \int_{t-s}^{t} u_j^2(\omega) d\omega ds \right). \tag{2.1.18}$$

Now we can estimate the upper righthand derivative of $V(t)$ along the solution of (2.1.7) as follows

$$\begin{aligned}
D^+V(t) \quad &\leq \quad \sum_{i=1}^{n} p_i \left(-2\underline{\alpha}_i \gamma_i u_i^2(t) + 2\overline{\alpha}_i |u_i| \sum_{j=1}^{n} L_j |a_{ij}| \int_0^{\infty} k_{ij}(s) |u_j(t-s)| ds \right) \\
&\quad + \sum_{i=1}^{n} \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| \frac{L_j}{\eta_j} \int_0^{\infty} k_{ij}(s) \left(u_j^2(t) - u_j^2(t-s) \right) ds \\
&\leq \quad \sum_{i=1}^{n} \Bigg(-2\underline{\alpha}_i \gamma_i p_i u_i^2(t) + \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| L_j \int_0^{\infty} k_{ij}(s) (\eta_j u_i^2(t) + \\
&\quad \frac{u_j^2(t-s)}{\eta_j}) ds \Bigg) + \sum_{i=1}^{n} \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| \frac{L_j}{\eta_j} \int_0^{\infty} k_{ij}(s) \left(u_j^2(t) - u_j^2(t-s) \right) ds \\
&= \quad -\sum_{i=1}^{n} \left(2\underline{\alpha}_i \gamma_i p_i - \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| L_j \eta_j - \frac{L_i}{\eta_i} \sum_{j=1}^{n} |a_{ji}| \overline{\alpha}_j p_j \right) u_i^2(t).
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1.2. □

Varying the parameters in Theorem 2.1.3, we immediately have

Corollary 2.1.2. *Assume that $(H_1)-(H_2)$, $(S_1)-(S_2)$ and (2.1.8) hold. If one of the following conditions holds for some positive real numbers $\eta_i, p_i, i \in N(1,n)$, then the equilibrium x^* of (2.1.3) is globally asymptotically stable.*

$$2\underline{\alpha}_i\gamma_i - \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|L_j\eta_j - \frac{L_i}{\eta_i}\sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j > 0; \tag{2.1.19}$$

$$2\underline{\alpha}_i\gamma_i p_i - \overline{\alpha}_i p_i\sum_{j=1}^{n}|a_{ij}|L_j - L_i\sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j p_j > 0; \tag{2.1.20}$$

$$2\underline{\alpha}_i\gamma_i - \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|L_j - L_i\sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j > 0; \tag{2.1.21}$$

$$2\underline{\alpha}_i\gamma_i p_i - \overline{\alpha}_i p_i\sum_{j=1}^{n}|a_{ij}| - L_i^2\sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j p_j > 0; \tag{2.1.22}$$

$$2\underline{\alpha}_i\gamma_i p_i - \overline{\alpha}_i p_i\sum_{j=1}^{n}|a_{ij}|L_j^2 - \sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j p_j > 0; \tag{2.1.23}$$

$$2\underline{\alpha}_i\gamma_i - \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}| - L_i^2\sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j > 0; \tag{2.1.24}$$

$$2\underline{\alpha}_i\gamma_i - \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|L_j^2 - \sum_{j=1}^{n}|a_{ji}|\overline{\alpha}_j > 0. \tag{2.1.25}$$

Proof. The above conditions can be obtained by letting $p_i = 1$; $\eta_i = 1$; $p_i = \eta_i = 1$; $\eta_i = \frac{1}{L_i}$; $\eta_i = L_i$, $p_i = 1, \eta_i = \frac{1}{L_i}$; $p_i = 1, \eta_i = L_i$, in (2.1.17), respectively. □

It is well known that exponential convergence to an equilibrium means fast convergence in the network, and thus, is desirable in the implementation of artificial neural networks. Regarding the global exponential stability of the equilibrium of (2.1.3), we have

Theorem 2.1.4. *In addition to the conditions in Theorem 2.1.2, if we further assume that*

$$k_{ij}(t) \leq e^{-\delta t},\ t > T_0, i, j \in N(1, n) \tag{2.1.26}$$

holds for some positive numbers δ and T_0, then the equilibrium of (2.1.3) is globally exponentially asymptotically stable. More precisely, we have

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq C_1 e^{-\sigma_1 t} \sum_{j=1}^{n} \left(\sup_{\omega \in (-\infty, 0]} |\phi_j(\omega) - x_j^*| \right),\ t > 0, \tag{2.1.27}$$

where $C_1 > 0$ and $\sigma_1 > 0$ will be specified later.

Proof. From (2.1.11), we have

$$D^+|u_i(t)| \leq -\underline{\alpha}_i \gamma_i |u_i(t)| + \overline{\alpha}_i \sum_{j=1}^{n} |a_{ij}| L_j \int_0^\infty k_{ij}(s) |u_j(t-s)| ds$$

Since (2.1.9) holds, we can choose a positive real number $\sigma_1 \in (0, \delta)$ such that

$$\nu := \min_{1 \leq i \leq n} \left(q_i \underline{\alpha}_i \gamma_i - q_i \sigma_1 - L_i \sum_{j=1}^{n} \overline{\alpha}_j q_j |a_{ji}| \int_0^\infty k_{ji}(s) e^{\sigma_1 s} ds \right) > 0. \tag{2.1.28}$$

Let $y_i(t) = e^{\sigma_1 t}|u_i(t)|$, a direct calculation shows that

$$\begin{aligned} D^+ y_i(t) &= e^{\sigma_1 t} \left(\sigma_1 |u_i(t)| + D^+|u_i(t)| \right) \\ &\leq e^{\sigma_1 t} \left(\sigma_1 |u_i(t)| - \underline{\alpha}_i \gamma_i |u_i(t)| + \overline{\alpha}_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^\infty k_{ij}(s) |u_j(t-s)| ds \right) \\ &= -(\underline{\alpha}_i \gamma_i - \sigma_1) y_i(t) + \overline{\alpha}_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^\infty k_{ij}(s) e^{\sigma_1 s} y_j(t-s) ds. \end{aligned}$$

Define

$$V(t) = V(y)(t) = \sum_{i=1}^{n} \left(q_i y_i(t) + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^{\infty} k_{ij}(s) e^{\sigma_1 s} \int_{t-s}^{t} y_j(\omega) d\omega ds \right). \tag{2.1.29}$$

It is easy to show that

$$\begin{aligned} D^+V(t) &= \sum_{i=1}^{n} \left(q_i D^+ y_i(t) + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^{\infty} k_{ij}(s) e^{\sigma_1 s} (y_j(t) - y_j(t-s)) ds \right) \\ &\leq \sum_{i=1}^{n} \left(-q_i(\underline{\alpha}_i \gamma_i - \sigma_1) y_i(t) + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^{\infty} k_{ij}(s) e^{\sigma_1 s} y_j(t) ds \right) \\ &= -\sum_{i=1}^{n} \left(q_i(\underline{\alpha}_i \gamma_i - \sigma_1) - L_i \sum_{j=1}^{n} \overline{\alpha}_j q_j |a_{ji}| \int_0^{\infty} k_{ji}(s) e^{\sigma_1 s} ds \right) y_i(t) \\ &\leq -\nu \sum_{i=1}^{n} y_i(t) \\ &\leq 0, \end{aligned}$$

which indicates that

$$V(t) \leq V(0).$$

Hence, we have

$$\begin{aligned} \sum_{i=1}^{n} q_i y_i(t) &\leq V(t) \leq V(0) \\ &\leq \sum_{i=1}^{n} \left(q_i y_i(0) + \overline{\alpha}_i q_i \sum_{j=1}^{n} L_j |a_{ij}| \int_0^{\infty} k_{ij}(s) s e^{\sigma_1 s} ds \sup_{\omega \in (-\infty, 0)} y_j(\omega) \right) \\ &\leq \sum_{i=1}^{n} \left(q_i + L_i \sum_{j=1}^{n} \overline{\alpha}_j q_j |a_{ji}| \int_0^{\infty} k_{ji}(s) s e^{\sigma_1 s} ds \right) \sup_{\omega \in (-\infty, 0]} y_i(\omega). \end{aligned}$$

By virtue of (2.1.26) and (2.1.4), it follows that

$$\begin{aligned} \hat{k}_{ji} : &= \int_0^{\infty} k_{ji}(s) s e^{\sigma_1 s} ds = (\int_0^{T_0} + \int_{T_0}^{\infty}) k_{ji}(s) s e^{\sigma_1 s} ds \\ &\leq T_0 e^{\sigma_1 T_0} + \int_0^{\infty} s e^{-(\delta - \sigma_1)s} ds = T_0 e^{\sigma_1 T_0} + \frac{1}{(\delta - \sigma_1)^2} < \infty. \end{aligned}$$

Putting

$$C_1 := \frac{\max_{1\le i\le n}\left\{q_i + L_i\sum_{j=1}^{n}\overline{\alpha}_j q_j|a_{ji}|\hat{k}_{ji}\right\}}{\min_{1\le i\le n}\{q_i\}}$$

and noting that

$$\sup_{\omega\in(-\infty,0]} y_i(\omega) = \sup_{\omega\in(-\infty,0]} e^{\sigma_1\omega}|\phi_i(\omega)-x_i^*| \le \sup_{\omega\in(-\infty,0]}|\phi_i(\omega)-x_i^*|,$$

we have

$$\sum_{i=1}^{n} y_i(t) \le C_1\sum_{j=1}^{n}\sup_{\omega\in(-\infty,0]}|\phi_j(\omega)-x_j^*|,$$

which implies that

$$\sum_{i=1}^{n}|x_i(t)-x_i^*| \le C_1 e^{-\sigma_1 t}\sum_{j=1}^{n}\sup_{\omega\in(-\infty,0]}|\phi_j(\omega)-x_j^*|. \tag{2.1.30}$$

Thus the proof is complete. □

Similarly, we have

Theorem 2.1.5. *Assume that all conditions in Theorem 2.1.3 and (2.1.26) are satisfied. Then the equilibrium of (2.1.3) is globally exponentially stable in the sense that the following inequality holds.*

$$\sum_{i=1}^{n}(x_i(t)-x_i^*)^2 \le C_2 e^{-\sigma_2 t}\sum_{j=1}^{n}\left(\sup_{\omega\in(-\infty,0]}(\phi_j(\omega)-x_j^*)^2\right),\ t>0, \tag{2.1.31}$$

where $C_2 > 0$ and $\sigma_2 > 0$ are given by

$$C_2 = \frac{\max_{1\le i\le n}\left\{p_i + \frac{L_i}{\eta_i}\sum_{j=1}^{n}\overline{\alpha}_j p_j|a_{ji}|\hat{k}_{ji}\right\}}{\min_{1\le i\le n}\{p_i\}},$$

and

$$\sigma_2 < \sup\left\{\sigma \in (0,\delta) : 2\underline{\alpha}_i p_i \gamma_i - p_i\sigma - \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| L_j \eta_j \right.$$
$$\left. -\frac{L_i}{\eta_i}\sum_{j=1}^{n} \overline{\alpha}_i p_j |a_{ji}| \int_0^\infty k_{ji}(s) e^{\sigma s} ds > 0, i \in N(1,n)\right\}.$$

Corollary 2.1.1 together with Theorem 2.1.4 immediately gives

Corollary 2.1.3. *If all conditions of Corollary 2.1.1 and (2.1.26) are satisfied, then the equilibrium of (2.1.3) is globally exponentially stable.*

Combining Corollary 2.1.2 and Theorem 2.1.5, we have

Corollary 2.1.4. *If all conditions of Corollary 2.1.2 and (2.1.26) are satisfied, then the equilibrium of (2.1.3) is globally exponentially stable.*

If the kernel functions $k_{ij}(t)$ are assumed to take some special forms in (2.1.3), such as

$$k_{ij}(t) = \begin{cases} l_{ij}(t), & t \in [0, \tau_{ij}] \\ 0, & otherwise, \end{cases}$$

then the duration intervals for time delays are finite, and thus the corresponding Cohen-Grossberg neural network model can be described by

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} g_j\left(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s)x_j(s)ds\right)\right), \qquad (2.1.32)$$

where $i \in N(1,n)$ and the delay kernel functions $l_{ij}(t)$ are subject to

$$l_{ij}(t) \geq 0, \quad \text{for } t \geq 0, \int_0^{\tau_{ij}} l_{ij}(t)dt = 1. \qquad (2.1.33)$$

Using similar arguments, we have

Theorem 2.1.6. *Suppose that* $(H_1), (H_2), (S_1), (S_2)$, *(2.1.8) and (2.1.9) hold. Then the equilibrium* x^* *of (2.1.32) is globally exponentially stable in the sense that*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq C_3 e^{-\sigma_3 t} \sum_{j=1}^{n} \left(\max_{\omega \in [-\tau, 0]} |\psi_j(\omega) - x_j^*| \right), t > 0 \tag{2.1.34}$$

with $\tau = \max(\tau_{ij}, i, j \in \{1, 2, \cdots, n\})$, $\sigma_3 > 0$ *such that*

$$\min_{1 \leq i \leq n} \left\{ \underline{\alpha}_i \gamma_i q_i - q_i \sigma_3 - L_i \sum_{j=1}^{n} \overline{\alpha}_j q_j |a_{ji}| \int_0^{\tau_{ji}} l_{ji}(s) e^{\sigma_3 s} ds \right\} > 0$$

and

$$C_3 := \frac{\max\limits_{1 \leq i \leq n} \left\{ q_i + L_i \sum\limits_{j=1}^{n} \overline{\alpha}_j q_j |a_{ji}| \tau_{ji} e^{\sigma_3 \tau_{ji}} \right\}}{\min\limits_{1 \leq i \leq n} \{q_i\}}.$$

Theorem 2.1.7. *Assume that* $(H_1), (H_2)$, $(S_1), (S_2)$ *and (2.1.8) hold. If there exist positive real numbers* $p_i > 0$, *and* $\eta_i > 0$ *such that (2.1.17) holds, then the equilibrium* x^* *of (2.1.32) is globally exponentially stable with*

$$\sum_{i=1}^{n} (x_i(t) - x_i^*)^2 \leq C_4 e^{-\sigma_4 t} \sum_{j=1}^{n} \left(\sup_{\omega \in (-\tau, 0]} (\phi_j(\omega) - x_j^*)^2 \right), \ t > 0, \tag{2.1.35}$$

where $\sigma_4 > 0$ *such that*

$$\min_{1 \leq i \leq n} \left\{ 2\underline{\alpha}_i p_i \gamma_i - p_i \sigma_4 - \overline{\alpha}_i p_i \sum_{j=1}^{n} |a_{ij}| L_j \eta_j \ - \frac{L_i}{\eta_i} \sum_{j=1}^{n} \overline{\alpha}_i p_j |a_{ji}| \int_0^{\tau} l_{ji}(s) e^{\sigma_4 s} ds \right\} > 0$$

and

$$C_4 = \frac{\max\limits_{1 \leq i \leq n} \left\{ p_i + \frac{L_i}{\eta_i} \sum\limits_{j=1}^{n} \overline{\alpha}_j p_j |a_{ji}| \tau_{ji} e^{\sigma_4 \tau_{ji}} \right\}}{\min\limits_{1 \leq i \leq n} \{p_i\}}.$$

2.2 Hopfield neural networks (HNNs) with distributed delays

Letting $a_i \equiv 1, b_i(x) = b_i x + I_i, i \in N(1,n)$ with $b_i > 0, i \in N(1,n)$ in (2.1.3) leads to the Hopfield neural network model with distributed delays

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds) + I_i, i \in N(1,n). \tag{2.2.1}$$

In this section, we will first consider the case where the inputs $I_i, i \in N(1,n)$, are constants and then study the case where the inputs $I_i, i \in N(1,n)$, are functions of time t and are periodic with period ω, that is, $I_i(t) = I_i(t+\omega)$ for all t.

Note that the models with discrete delays can be included in our models by choosing suitable kernel functions, for example, taking the delay kernel functions $k_{ij}(t)$ as

$$k_{ij}(t) = \delta(t - \tau_{ij}), \ i,j \in N(1,n). \tag{2.2.2}$$

System (2.2.1) has been briefly studied by Burton [7], Gopalsamy and He [36] and Mohamad and Gopalsamy [79]. However, as we will see, their conclusions can be included in our results as special cases. An application of system (2.2.1) can be found in Tank and Hopfield [95]. Clearly the Hopfield neural networks with distributed delays (2.2.1) has the same equilibria as the system (2.1.3) does and hence based on the same assumptions, we know (2.2.1) admits at least one equilibrium.

2.2.1 Global stability of HNNs

Since (2.2.1) is a special case of (2.1.3),results established in Section 2.1 all apply to (2.2.1). For convenience of applications, in this subsection, we state some criteria for the global stability of (2.2.1). First, applying Theorem 2.1.2 and Theorem 2.1.4 to system (2.2.1), we have

Theorem 2.2.1. *If there exist* $q_i > 0, i \in N(1,n)$ *such that*

$$\nu := \min_{1\leq i\leq n}\left\{b_i q_i - L_i \sum_{j=1}^{n} |a_{ji}| q_j\right\} > 0, \tag{2.2.3}$$

then the equilibrium x^* *is globally asymptotically stable. In addition, if (2.1.26) holds, the* x^* *is globally exponentially stable.*

Letting $q_i = 1, i \in N(1,n)$ in (2.2.3), we have

Corollary 2.2.1. *If*

$$\nu_1 := \min_{1\leq i\leq n}\left\{b_i - L_i \sum_{j=1}^{n} |a_{ji}|\right\} > 0 \tag{2.2.4}$$

holds, then we have the same results as Theorem 2.2.1.

Remark 2.2.1. *Corollary 2.2.1 coincides with Theorem 3.3 in [79].*

Applying Theorem 2.1.3 and Theorem 2.1.5 to system (2.2.1), we have

Theorem 2.2.2. *If there exist* $p_i > 0, \eta_i > 0, i \in N(1,n)$ *such that*

$$\min_{1\leq i\leq n}\left\{2b_i p_i - p_i \sum_{j=1}^{n} |a_{ij}| L_j \eta_j - \frac{L_i}{\eta_i} \sum_{j=1}^{n} |a_{ji}| p_j\right\} > 0, \tag{2.2.5}$$

then the equilibrium x^* *is globally asymptotically stable. In addition, if (2.1.26) holds, the* x^* *is globally exponentially stable.*

2.2.2 Local stability of HNNs

This subsection is dedicated to local stability analysis. Instead of (S_1) and (S_2), we assume that (A_1) and (A_2) hold and $b_i > 0$ for $i \in N(1,n)$. Linearizing (2.2.1) at an equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ gives

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j^{'}(x_j^*) \int_0^\infty k_{ij}(s) x_j(t-s) ds. \tag{2.2.6}$$

Then the characteristic equation (for the characteristic equation of functional differential equations with infinite delays, we refer to [94]) is

$$F(\lambda) = 0, \tag{2.2.7}$$

where

$$F(\lambda) = det \begin{pmatrix} -\lambda - b_1 + c_{11}k_{11}(\lambda) & \cdots & c_{1n}k_{1n}(\lambda) \\ c_{21}k_{21}(\lambda) & \cdots & c_{2n}k_{2n}(\lambda) \\ \vdots & \vdots & \vdots \\ c_{n1}k_{n1}(\lambda) & \cdots & -\lambda - b_n + c_{nn}k_{nn}(\lambda) \end{pmatrix}, \tag{2.2.8}$$

in which $c_{ij} = a_{ij} g_j^{'}(x_j^*)$ and $k_{ij}(\lambda) = \int_0^\infty k_{ij}(s) e^{-\lambda s} ds$.

Theorem 2.2.3. *If $F(0) \neq 0$ and there exist $q_i > 0, i \in N(1,n)$ such that*

$$(-b_i + |c_{ii}|)q_i + \sum_{j \neq i} |c_{ij}| q_j \leq 0, \ \textit{for } i \in N(1,n), \tag{2.2.9}$$

then the equilibrium of (2.2.1) is asymptotically stable [41].

Proof. Let λ be a root of (2.2.7). Then λ is an eigenvalue of the matrix $D = (d_{ij})$ with $d_{ii} = -b_i + c_{ii}k_{ii}(\lambda)$, $d_{ij} = c_{ij}k_{ij}(\lambda), i \neq j$ for $i, j \in N(1,n)$. Let $\hat{D} = (\hat{d}_{ij})$ with $\hat{d}_{ij} = q_i^{-1} d_{ij} q_j$. Then $\hat{D} = Q^{-1}DQ$, where $Q = diag(q_1, q_2, \cdots, q_n)$. So $\hat{D}$ and D are similar and thus have the same eigenvalues. Let $\hat{\lambda}$ be an eigenvalue of $\hat{D}$. Applying the Gershgorin's theorem [33] to $\hat{D}$, we know that for some $i \in \{1, 2, \cdots, n\}$

$$|\hat{\lambda} - \hat{d}_{ii}| \leq \sum_{j \neq i} |\hat{d}_{ij}|,$$

that is,

$$|\hat{\lambda} - d_{ii}| \leq \sum_{j \neq i} q_i^{-1} |d_{ij}| q_j.$$

Therefore, we have

$$\mathbf{R}e(\lambda) = \mathbf{R}e(\hat{\lambda}) \leq \mathbf{R}e(d_{ii}) + \sum_{j \neq i} q_i^{-1} |d_{ij}| q_j.$$

If $\mathbf{R}e(\lambda) \geq 0$, then

$$\begin{aligned} \mathbf{R}e(d_{ii}) &= \mathbf{R}e(-b_i + c_{ii} k_{ii}(\lambda)) \\ &\leq -b_i + |c_{ii}| \mathbf{R}e(k_{ii}(\lambda)) \\ &\leq -b_i + |c_{ii}|, \end{aligned}$$

and

$$|d_{ij}| = |c_{ij} k_{ij}(\lambda)| \leq |c_{ij}|.$$

Hence, we have

$$\mathbf{R}e(\lambda) \leq -b_i + |c_{ii}| + \sum_{j \neq i} q_i^{-1} |c_{ij}| q_j.$$

Multiplying both sides of the above inequalities by q_i, we get

$$q_i \mathbf{R}e(\lambda) \leq (-b_i + |c_{ii}|) q_i + \sum_{j \neq i} |c_{ij}| q_j,$$

which implies that $\mathbf{R}e(\lambda) \leq 0$. From the analysis, we know the equality occurs only when λ is real and $F(0) \neq 0$ implies λ can not be 0. So we must have $\mathbf{R}e(\lambda) < 0$.

This is a contradiction with our assumption $\mathbf{R}e(\lambda) \geq 0$. Thus we show that all the roots of $F(\lambda)$ have negative real parts, which implies that the equilibrium x^* is locally asymptotically stable [41]. $\square$

Remark 2.2.2. *If we denote $K = -B + |C|$ with $B = diag(b_1, b_2, \cdots, b_n)$ and $|C| = (|c_{ij}|)$ and use a similar argument as in [9], then Theorem 2.2.3 can be modified to: If $-K$ is weakly diagonally dominant in the sense of [49] and $F(0) \neq 0$, then the conclusion in Theorem 2.2.3 still holds.*

2.2.3 HNNs with finite distributed delays

In this subsection, we study the Hopfield type neural networks with finite distributed delays described by

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} g_j \left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s) x_j(s) ds \right) + J_i,\ i \in N(1,n), \quad (2.2.10)$$

where $\tau_{ij} \geq 0$ for $i,\ j \in N(1,n)$ with $\tau = \max\{\tau_{ij}, i, j \in N(1,n)\}$ and the delay kernels satisfy

$$k_{ij}(t) \geq 0,\ \int_{0}^{\tau_{ii}} k_{ii}(t) dt = 1, i, j \in N(1,n).$$

Note that (2.2.10) includes

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + J_i,\ i \in N(1,n), \quad (2.2.11)$$

and

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} g_j \left(x_j(t-\tau_{ij}) \right) + J_i,\ i \in N(1,n), \quad (2.2.12)$$

as special cases. We will establish some global attractivity results for (2.2.10), which can not be derived from the results in Section 2.1. Instead, we will employ the ideas and techniques in van den Driessche, Wu and Zou [96] and van den Driessche and Zou [97] to show how to stabilize the Hopfield neural networks (2.2.10) with general activation functions and distributed delays via the self–inhibitory connections.

Unlike in [96] and [97], we do not require any differentiability for the activation functions and the time delays are not necessary to be fixed constants, indeed they are distributed over a finite interval. We will see later, by applying our results to (2.2.12), that the results in [96] and [97] are reproduced with a better estimate for the smallness of effective delays.

The initial conditions associated with (2.2.11) are set to be

$$x_i(0) = \phi_i(0), i \in N(1,n) \tag{2.2.13}$$

and the associated ones for (2.2.10) are

$$x_i(s) = \phi_i(s),\ s \in [-\tau, 0],\ i \in N(1,n). \tag{2.2.14}$$

We assume that for each $i \in N(1,n)$

$$b_i \text{ is continuous with } \frac{b_i(u) - b_i(v)}{u - v} \geq m_i > 0. \tag{2.2.15}$$

By a similar proof to that of Theorem 2.1.1, we can establish the following existence result for an equilibrium of system (2.2.11).

Theorem 2.2.4. *Assume that $g_i, i \in N(1,n)$ satisfy (S_2) and (2.2.15) holds, then for every input J, there exists an equilibrium for system (2.2.11).*

2.2.3.1 Global stability of (2.2.11)

Let x^* be an equilibrium of (2.2.11). Substituting $x(t) = u(t)+x^*$ into (2.2.11) leads to

$$\dot{u}_i(t) = -\left[b_i(u_i(t)+x_i^*) - b_i(x_i^*) - \sum_{j=1}^{n} a_{ij}(g_j(u_j(t)+x_j^*) - g_j(x_j^*))\right], \quad (2.2.16)$$

which can be denoted by

$$\dot{u}_i(t) = -\beta_i(u_i(t)) + \sum_{j=1}^{n} a_{ij}s_j(u_j(t)), i \in N(1,n) \quad (2.2.17)$$

where $\beta_i(u_i(t)) = b_i(u_i(t)+x_i^*) - b_i(x_i^*)$, $s_j(u_j(t)) = g_j(u_j(t)+x_j^*) - g_j(x_j^*)$. If we let $u = (u_1, \cdots, u_n)^T \in R^n$, $T = [a_{ij}]_{n\times n}$, $s(u) = (s_1(u_1), \cdots, s_n(u_n))^T$, $B(u) = (\beta_1(u_1), \cdots, \beta_n(u_n))^T \in R^n$, then system (2.2.17) can be rewritten as

$$\dot{u}(t) = -B(u(t)) + Ts(u(t)). \quad (2.2.18)$$

It is obvious that x^* is globally asymptotically (exponentially) stable for (2.2.11) if and only if the trivial solution $u = 0$ of (2.2.17) or (2.2.18) is globally asymptotically (exponentially) stable. Moreover, the uniqueness of the equilibrium of (2.2.11) follows from its global asymptotic (exponential) stability.

From Theorem 2 of [31], we have

Theorem 2.2.5. *Suppose* (2.2.15), (S_1) *and* (S_2) *are satisfied. Assume also that for each* $i \in N(1,n)$,

$$us_i(u) > 0 \text{ when } u \neq 0. \quad (2.2.19)$$

Then, for every input J, system (2.2.11) has a unique equilibrium x^* which is globally asymptotically stable if the matrix W^* defined by

$$W^* = diag(\frac{p_1 m_1}{L_1}, \cdots, \frac{p_n m_n}{L_n}) - \frac{1}{2}\left(PT + T^T P\right) \quad (2.2.20)$$

is positive definite for some $P = diag(p_1, \cdots, p_n)$ *with* $p_i > 0,\ i \in N(1, n)$.

A direct corollary of this theorem is

Corollary 2.2.2. *Suppose* (2.2.15), (S_1) *and* (S_2) *and (2.2.19) hold. If for some* $p_i > 0, i \in N(1, n)$

$$a_{ii}p_i + \frac{1}{2}\sum_{j\neq i} |p_i a_{ij} + p_j a_{ji}| < p_i \frac{m_i}{L_i},\ i \in N(1, n), \tag{2.2.21}$$

holds, then the equilibrium x^* *of (2.2.11) is globally asymptotically stable.*

If (2.2.21) is strengthened a little bit, we can actually obtain the global exponential stability, as is stated in the following theorem

Theorem 2.2.6. *Suppose* (2.2.15), (S_1) *and* (S_2) *and (2.2.19) hold. If*

$$p_i a_{ii} + \sum_{j\neq i} |p_j a_{ji}| < p_i \frac{m_i}{L_i},\ i \in N(1, n). \tag{2.2.22}$$

holds for some positive numbers $p_1, p_2, \cdots, p_n$*, then, for every input* J*, system(2.2.11) has a unique equilibrium which is globally exponentially stable in the sense that*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq C_1 e^{-\sigma_1 t}, \tag{2.2.23}$$

where C_1, σ_1 *will be specified later.*

Proof. Let $\delta_i > 0$ be defined by

$$\delta_i := m_i p_i - \max(0, L_i(p_i a_{ii} + \sum_{j\neq i} p_j |a_{ji}|)$$

and let $\sigma_1 > 0$ be a number such that

$$\delta := \min_{1\leq i\leq n} \{\delta_i - p_i\sigma_1\} > 0, \text{ for } i \in N(1, n).$$

Define $V(t) = V(u(t))$ by

$$V(t) = \sum_{i=1}^{n} p_i e^{\sigma_1 t} |u_i|. \tag{2.2.24}$$

Then

$$\begin{aligned}
D^+_{(2.2.11)} V(t) &= \sum_{i=1}^{n} p_i e^{\sigma_1 t} \left[\sigma_1 |u_i(t)| - sign(u_i(t))[\beta_i(u_i(t)) - \sum_{j=1}^{n} a_{ij} s_j(u_j(t))] \right] \\
&\leq \sum_{i=1}^{n} e^{\sigma_1 t} \left(p_i \sigma_1 |u_i(t)| - p_i m_i |u_i(t)| + p_i a_{ii} |s_i(u_i(t))| + \sum_{j \neq i} p_i |a_{ij}| |s_j(u_j(t))| \right) \\
&= \sum_{i=1}^{n} e^{\sigma_1 t} \left(p_i \sigma_1 |u_i(t)| - p_i m_i |u_i(t)| + p_i a_{ii} |s_i(u_i(t))| + \sum_{j \neq i} p_j |a_{ji}| |s_i(u_i(t))| \right) \\
&= \sum_{i=1}^{n} e^{\sigma_1 t} \left(p_i \sigma_1 |u_i(t)| - p_i m_i |u_i(t)| + \left(p_i a_{ii} + \sum_{j \neq i} p_j |a_{ji}| \right) |s_i(u_i)| \right) \\
&\leq -\delta e^{\sigma_1 t} \sum_{i=1}^{n} |u_i(t)|
\end{aligned}$$

This shows that $V(t)$ is a Liapunov function and hence

$$V(t) \leq V(0) = \sum_{i=1}^{n} p_i |\phi_i(0) - x_i^*| =: C_0 < \infty \tag{2.2.25}$$

and thus we have

$$\sum_{i=1}^{n} |u_i(t)| = \sum_{i=1}^{n} |x_i(t) - x_i^*| \leq C_1 e^{-\sigma_1 t} \tag{2.2.26}$$

with $C_1 := \dfrac{C_0}{\min\limits_{1 \leq i \leq n} \{p_i\}}$. $\square$

Remark 2.2.3. *Corollary 2.2.2 and Theorem 2.2.6 show that the self-inhibitory connections do play an important role in stabilizing a network.*

2.2.3.2 Global attractivity of HNNs with finite distributed delays

Note that the delays do not change the equilibrium, but the stability may be lost if the delays are too large. It is natural to expect that the stable equilibrium of the system without delays remains stable for the delayed system when the delays are sufficiently small. In order to give an estimation for the smallness, we are going to use the powerful theory of monotone dynamic systems. To be more precise, we shall use the related theory about the nonstandard ordering –exponential ordering.

In the following, as in [91], the partial order $\leq$ on $\mathbb{R}^n$ will be the usual componentwise ordering. The partial order $\phi \leq \psi$ on $C := C([-\tau, 0], \mathbb{R}^n)$ will mean $\phi(\theta) \leq \psi(\theta)$ for each $\theta \in [-\tau, 0]$. The inequality $x < y$ ($x \ll y$) between two vectors in $\mathbb{R}^n$ will mean $x \leq y$ and $x_i < y_i$ for some (all) $i \in N(1, n)$. The inequality $\phi < \psi$ in C will mean that $\phi \leq \psi$ and $\phi \neq \psi$, and $\phi \ll \psi$ will mean that $\phi(\theta) \ll \psi(\theta)$ for all $\theta \in [-\tau, 0]$.

Let D be an $n \times n$ essentially nonnegative matrix, that is, $D + \lambda I$ is entry-wise nonnegative for all sufficiently large λ. Define

$$K_D = \left\{ \psi \in C : \psi \geq 0 \text{ and } \psi(t) \geq e^{D(t-s)}\psi(s), -\tau \leq s \leq t \leq 0 \right\}.$$

It can be seen that K_D is a normal cone and thus it induces a partial order on C denoted by $\leq_D$, in the usual way, that is, if $\phi \leq_D \psi$ if and only if $\psi - \phi \in K_D$ and $\phi <_D \psi$ means $\phi \leq_D \psi$ and $\phi \neq \psi$, $\phi \ll_D \psi$ is similarly defined.

Consider the functional differential equation

$$x'(t) = f(x_t), \tag{2.2.27}$$

where $f : C \to \mathbb{R}^n$ is globally Lipschitz continuous. By the fundamental theory of

FDES [41], the system (2.2.27) generates a semi-flow Φ on C by

$$\Phi(t,\phi) = \Phi_t(\phi) = x_t(\phi),\ t \geq 0,\ \phi \in C$$

for those t for which $x_t(\phi)$ is defined.

The following theorem is from Smith and Thieme [93]

Theorem 2.2.7. *Assume the following conditions are satisfied*

(I_D) *If $\phi, \psi \in C$ satisfy $\phi \leq_D \psi$ and K is a proper subset of $N(1,n)$ such that $\phi_k \ll \psi_k, k \in K$ and $\phi_k(0) = \psi_k(0)$ for $k \in N(1,n) - K$ then for some $p \in N(1,n) - K$*

$$f_p(\psi) > f_p(\phi).$$

(SM_D) *If $\phi, \psi \in C$ satisfy $\phi \leq_D \psi$ and $\phi \ll \psi$ then*

$$f(\psi) - f(\phi) \gg D(\psi(0) - \phi(0)).$$

Then Φ is strongly order preserving (SOP) on C under $\leq_D$.

Using the above theorem, we will show that the semiflow Φ generated by the solution of (2.2.10) is SOP under the exponential ordering, if the diagonal delays corresponding to negative self–connections are sufficiently small.

Theorem 2.2.8. *Assume that $b_i(u)$ is Lipschitz continuous with $Lip(b_i) = \gamma_i$ for $i \in N(1,n)$, $a_{ij} \geq 0$ for $i \neq j$, T is irreducible and g_i satisfies*

$$0 \leq \frac{g_i(u) - g_i(v)}{u - v} \leq L_i,\ i \in N(1,n). \tag{2.2.28}$$

If the diagonal delays τ_{ii} corresponding to negative a_{ii} are sufficiently small, satisfying

$$\tau_{ii} \leq \frac{1}{r_i^*}, \tag{2.2.29}$$

where r_i^* will be given below, then the semi-flow Φ generated by the solution of (2.2.10) is SOP under $\leq_D$.

Proof. Take $D = diag(d_1, \cdots, d_n)$ with

$$d_i = -\gamma_i - r_i, \ i \in N(1,n), \tag{2.2.30}$$

where $r_i > 0, \ i \in N(1,n)$, are constants to be specified later, then D is essentially nonnegative. For the system (2.2.10), we have

$$f_i(\phi) = -b_i(\phi_i(0)) + \sum_{j=1}^{n} a_{ij} g_j \left(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\phi_j(s)ds \right) + J_i, \ i \in N(1,n). \tag{2.2.31}$$

Let $\phi, \psi \in C$ satisfy $\phi \leq_D \psi$ and $\phi \ll \psi$. Then

$$\begin{aligned}
f_i(\psi) - f_i(\phi) &= -b_i(\psi_i(0)) + \sum_{j=1}^{n} a_{ij} g_j \left(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\psi_j(s)ds \right) \\
&\quad + b_i(\phi_i(0)) - \sum_{j=1}^{n} a_{ij} g_j \left(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\phi_j(s)ds \right) \\
&= -\left[b_i(\psi_i(0) - b_i(\phi_i(0)) \right] \\
&\quad + \sum_{j=1}^{n} a_{ij} \left[g_j\left(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\psi_j(s)ds\right) - g_j\left(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\phi_j(s)ds\right) \right] \\
&\geq -\gamma_i[\psi_i(0) - \phi_i(0)] + a_{ii}[g_i(\int_{-\tau_{ii}}^{0} k_{ii}(-s)\psi_i(s)ds) \\
&\quad - g_i(\int_{-\tau_{ii}}^{0} k_{ii}(-s)\phi_i(s)ds)] + \sum_{j \neq i} a_{ij}[g_j(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\psi_j(s)ds) \\
&\quad - g_j(\int_{-\tau_{ij}}^{0} k_{ij}(-s)\phi_j(s)ds)].
\end{aligned}$$

Clearly if the matrix T is a nonnegative matrix, then (SM_D) holds. In the following we therefore may assume that $a_{ii} < 0$ for some $i \in N(1,n)$. Then for such an i,

$$\begin{aligned}
& f_i(\psi) - f_i(\phi) - d_i[\phi_i(0) - \psi_i(0)] \\
& \geq -\gamma_i[\psi_i(0) - \phi_i(0)] + a_{ii}[g_i(\textstyle\int_{-\tau_{ii}}^0 k_{ii}(-s)\psi_i(s)ds) - g_i(\int_{-\tau_{ii}}^0 k_{ii}(-s)\phi_i(s)ds)] \\
& \quad +(\gamma_i + r_i)[\psi_i(0) - \phi_i(0)] \\
& = r_i[\psi_i(0) - \phi_i(0)] + a_{ii}[g_i(\textstyle\int_{-\tau_{ii}}^0 k_{ii}(-s)\psi_i(s)ds) - g_i(\int_{-\tau_{ii}}^0 k_{ii}(-s)\phi_i(s)ds)] \\
& \geq r_i[\psi_i(0) - \phi_i(0)] + a_{ii}L_i \textstyle\int_{-\tau_{ii}}^0 k_{ii}(-s)[\psi_i(s) - \phi_i(s)]ds
\end{aligned}$$

On the other hand $\phi \leq_D \psi$ implies that $\psi - \phi \in K_D$ and hence

$$\psi(0) - \phi(0) \geq e^{-Ds}[\psi(s) - \phi(s)], \quad \text{for } s \in [-\tau, 0],$$

that is

$$\psi_i(s) - \phi_i(s) \leq e^{-(r_i+\gamma_i)s}[\psi_i(0) - \phi_i(0)], \quad \text{for } i \in N(1,n).$$

This indicates that

$$\begin{aligned}
\int_{-\tau_{ii}}^0 k_{ii}(-s)[\psi_i(s) - \phi_i(s)]ds &\leq \int_{-\tau_{ii}}^0 k_{ii}(-s)e^{-(r_i+\gamma_i)s}[\psi_i(0) - \phi_i(0)]ds \\
&\leq [\psi_i(0) - \phi_i(0)]e^{(r_i+\gamma_i)\tau_{ii}}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& f_i(\psi) - f_i(\phi) - d_i[\psi_i(0) - \phi_i(0)] \\
& \geq \left(r_i + a_{ii}L_i e^{\tau_{ii}(r_i+\gamma_i)}\right)[\psi_i(0) - \phi_i(0)] \\
& > 0.
\end{aligned}$$

provided that

$$r_i > |a_{ii}|L_i e^{\tau_{ii}(r_i+\gamma_i)}, \ i \in N(1,n). \tag{2.2.32}$$

Inequality (2.2.32) is satisfied if and only if

$$\tau_{ii} < \frac{ln\frac{r_i}{|a_{ii}|L_i}}{\gamma_i + r_i}, \ i \in N(1,n). \tag{2.2.33}$$

Now let

$$\tau_i(s) := \frac{ln\frac{s}{|a_{ii}|L_i}}{\gamma_i + s}, \; i \in N(1,n). \tag{2.2.34}$$

A simple calculation shows that $\ddot{\tau}_i(s) < 0$ for $s > |a_{ii}|L_i$, $\dot{\tau}_i(|a_{ii}|L_i) > 0$ and $\tau_i(|a_{ii}|L_i) = 0$. Therefore $\tau_i(s)$ attains its maximal value $\frac{1}{r_i^*}$ for $s > |a_{ii}|L_i$ at r_i^*, where r_i^* is the unique positive root of equation

$$h(s) := 1 + \frac{\gamma_i}{s} - ln\frac{s}{|a_{ii}|L_i} = 0, \text{ for } s|a_{ii}|L_i.$$

Taking $r_i = r_i^*$ in (2.2.30) for $i \in N(1,n)$, then (SM_D) holds if the diagonal delays τ_{ii} corresponding to the negative a_{ii} satisfy

$$\tau_{ii} < \frac{1}{r_i^*}, i \in N(1,n). \tag{2.2.35}$$

The property (I_D) can be easily verified under the assumption that the connection matrix T is irreducible. Thus the proof is complete. □

Since Corollary 2.2.2 and Theorem 2.2.6 imply the uniqueness of the equilibrium x^* of system (2.2.11), which shows that under the same assumptions, the equilibrium x^* of system (2.2.10) is unique. Note that the phase space here is $X = C = C([-\tau, 0], \mathbb{R}^n)$ and it is easily seen that every non-equilibrium point in X can be approximated from below and from above under the exponential ordering. Also from the boundedness of the activation functions, it is easy to show that every bounded set $B \subset X$ has a bounded orbit and thus the relatively weak compactness requirement (**C**) in [91] is met. Thus Theorem 2.2.8, together with Theorem 2. 3. 1 of Smith [91], immediately gives

Theorem 2.2.9. *Assume that (2.2.28) holds and b_i satisfies*

$$m_i \leq \frac{b_i(u) - b_i(v)}{u - v} \leq \gamma_i, \ i \in N(1, n),$$

T is irreducible and $a_{ij} \geq 0$ for $i \neq j$. If either (2.2.21) or (2.2.22) holds, then the system (2.2.10) has a unique equilibrium which is globally attractive provided the diagonal delays τ_{ii} corresponding to negative a_{ii} are sufficiently small satisfying (2.2.29).

Letting $s = e|a_{ii}|L_i$ in (2.2.34), we have

Corollary 2.2.3. *Under the same assumptions as the above theorem except that (2.2.29) is replaced by*

$$\tau_{ii} \leq \frac{1}{\gamma_i + e|a_{ii}|L_i}, \ i \in N(1, n). \tag{2.2.36}$$

Then we have the same conclusion.

Note that in Theorem 2.2.9 and Corollary 2.2.3, the connection matrix T is supposed to be irreducible and the off-diagonal terms $a_{ij} \geq 0, \ j \neq i$. Motivated by [96], we will remove those restrictions. Indeed, we have

Theorem 2.2.10. *If either (2.2.22) or*

$$p_i a_{ii} + \sum_{j \neq i}^{n} \frac{p_i |a_{ij}| + p_j |a_{ji}|}{2} < p_i \frac{m_i}{L_i}, \ i \in N(1, n), \tag{2.2.37}$$

for some $p_i > 0, \ i \in N(1, n)$, then system (2.2.10) has a unique equilibrium which is globally attractive provided the diagonal delays τ_{ii} corresponding to negative a_{ii} are sufficiently small such that (2.2.29) holds.

Proof. Let $a^+ = max(a,0)$ and $a^- = max(-a,0)$. Then a^+ and a^- are nonnegative and $a = a^+ - a^-$, $|a| = a^+ + a^-$. Define $n \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ by

$$A_{ij} = \begin{cases} a_{ii}, & \text{for } j = i \\ a_{ij}^+ + s, & \text{for } j \neq i \end{cases}, \quad B_{ij} = \begin{cases} 0, & \text{for } j = i \\ a_{ij}^- + s, & \text{for } j \neq i \end{cases} \tag{2.2.38}$$

where $s > 0$ will be specified later. Now system (2.2.10) can be rewritten as

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} A_{ij} g_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)x_j(s)ds\right) - \sum_{j=1}^{n} B_{ij} g_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)x_j(s)ds\right) + J_i. \tag{2.2.39}$$

Let $y_i = -x_i$, $i \in N(1,n)$. Then (2.2.39) can be embedded into the following system with dimension $2n$:

$$\begin{cases} \dot{x}_i(t) &= -b_i(x_i(t)) + \sum_{j=1}^{n} A_{ij} g_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)x_j(s)ds\right) \\ & \quad + \sum_{j=1}^{n} B_{ij} f_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)y_j(s)ds\right) + J_i \\ \dot{y}_i(t) &= -q_i(y_i(t)) + \sum_{j=1}^{n} B_{ij} g_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)x_j(s)ds\right) \\ & \quad + \sum_{j=1}^{n} A_{ij} f_j\left(\int_{t-\tau_{ij}}^{t} k_{ij}(t-s)y_j(s)ds\right) - J_i, \end{cases} \quad i \in N(1,n) \tag{2.2.40}$$

where $q_i(x)$ and $f_i(x)$ are defined by $q_i(x) = -b_i(-x)$ and $f_i(x) = -g_i(-x)$, respectively, for $i \in N(1,n)$ and $x \in R$. Clearly q_i has the same property as b_i does and f_i has the same property as g_i does. Define $u_i(t), i \in N(1,2n)$ by

$$u_i(t) = x_i(t),\ u_{n+i}(t) = y_i(t),\ i \in N(1,n),$$

$b_i^*(u), i \in N(1,2n)$ by

$$b_i^*(u) = b_i(u),\ b_{n+i}^*(u) = q_i(u),\ i \in N(1,n),$$

and $h_i(u)$, $i \in N(1,2n)$ by

$$h_i(u) = g_i(u),\ h_{n+i}(u) = f_i(u),\ i \in N(1,n).$$

Then (2.2.40) can be rewritten as

$$\dot{u}_i(t) = -b_i^*(u_i(t)) + \sum_{j=1}^{2n} w_{ij} h_j\Big(\int_{t-\delta_{ij}}^{t} K_{ij}(t-s)u_j(s)ds\Big) + I_i,\ i \in N(1,2n) \quad (2.2.41)$$

where $I_i = J_i$ and $I_{n+i} = -J_i$ for $i \in N(1,n)$, the $2n \times 2n$ matrix $W = (w_{ij})$ is given by

$$W = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

and δ_{ij}, K_{ij} for $i,j \in N(1,2n)$ are given by

$$\delta_{ij} = \delta_{n+i,j} = \delta_{i,n+j} = \delta_{n+i,n+j} = \tau_{ij},\ i,j \in N(1,n),$$

and

$$K_{ij} = K_{n+i,j} = K_{i,n+j} = K_{n+i,n+j} = k_{ij},\ i,j \in N(1,n).$$

It is obvious now that $A_{ij} > 0$ and $B_{ij} > 0$ for $i,j \in N(1,n)$ and $j \neq i$ and thus $w_{ij} > 0$ for $i,j \in N(1,2n)$ and W is irreducible. So if condition (2.2.37) holds for some $p_i > 0,\ i \in N(1,n)$, then we are able to choose a sufficient small $s > 0$ such that

$$a_{ii}p_i + \sum_{j\neq i}^{n} \frac{p_i|a_{ij}| + p_j|a_{ji}|}{2} + \frac{s}{2}\sum_{j\neq i}^{n}(p_i + p_j) < p_i\frac{m_i}{L_i},\ i \in N(1,n) \quad (2.2.42)$$

which implies that for $p_i^* > 0, i \in N(1,2n)$ with $p_i^* = p_i$ and $p_{n+i}^* = p_i$ for $i \in N(1,n)$, we have

$$p_i^* w_{ii} + \sum_{j\neq i}^{2n} \frac{p_i^*|w_{ij}| + p_j^*|w_{ji}|}{2} < p_i^*\frac{m_i}{L_i},\ i \in N(1,2n). \quad (2.2.43)$$

If (2.2.22) is true for some $p_i > 0, i \in N(1,n)$, then similarly we can find sufficient small $s > 0$ such that

$$a_{ii}p_i + \sum_{j\neq i}^{n} p_j|a_{ji}| + s\sum_{j\neq i}^{n} p_j < p_i\frac{m_i}{L_i}, i \in N(1,n), \quad (2.2.44)$$

which implies that for $p_i^* > 0, i \in N(1,2n)$ with $p_i^* = p_i$ and $p_{n+i}^* = p_i$ for $i \in N(1,n)$, we have

$$p_i^* w_{ii} + \sum_{j\neq i}^{2n} p_i^* |w_{ji}| < p_i^* \frac{m_i}{L_i}, \ i \in N(1,2n). \tag{2.2.45}$$

Applying Theorem 2.2.9 to the system (2.2.41) immediately completes the proof. □

2.2.3.3 Examples

Example 2.2.1. *Consider*

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = -\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \tanh(\frac{2}{\sqrt{3}}x_1(t)) \\ \tanh(\frac{2}{\sqrt{3}}x_2(t)) \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \tag{2.2.46}$$

Here, $T = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$, $b_i(x_i(t)) = x_i(t)$, $s_i(x_i) = \tanh(2x_i/\sqrt{3})$, $m_i = 1$, $L_i = \frac{2}{\sqrt{3}}$ for $i = 1,\ 2$. Clearly, by choosing $p_1 = p_2 = 1$, all conditions of Corollary 2.2.2 are satisfied. Therefore, the system (2.2.46) has an equilibrium, which is globally asymptotically stable. We point out that Corollary 2 of [69] cannot be applied to system(2.2.46).

Note that the stabilizing role of inhibitory self–connection given in (2.2.21) and (2.2.22) is conditional for the delayed network (2.2.10), in the sense that the corresponding delays τ_{ii} must be small. This is demonstrated in the following example.

Example 2.2.2. *Consider the CNN networks with two neurons*

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) - 0.5f(x_1(t-\tau_{11})) + 1.7f(x_2(t-\tau_{12})) \\ \dot{x}_2(t) &= -x_2(t) + 1.25f(x_1(t-\tau_{21})) - 0.6f(x_2(t-\tau_{22})) \end{cases} \tag{2.2.47}$$

where $f(s) = \frac{1}{2}\left(|s+1| - |s-1|\right)$.

Applying Theorem 2.2.9 to this example, we know that the system (2.2.47) has a unique equilibrium $(x_1, x_2) = (0, 0)$ which is globally asymptotically stable whenever the diagonal delays are small, satisfying $\tau_{11} \leq 0.463$, $\tau_{22} \leq 0.408$. This conclusion is shown by Figure 2.1. However, if (2.2.29) is not satisfied, then it is possible for system (2.2.47) to have a solution which is not asymptotically stable. Indeed, if we let $\tau_{11} = 1.2$, $\tau_{12} = 0.5$, $\tau_{21} = 2.0$, $\tau_{22} = 1.0$, the numerical simulation Figure. 2.2 shows that system (2.2.47) has a periodic solution. While Corollary 2.2.3 gives a smaller estimation of the diagonal delays as $\tau_{11} \leq 0.424$, $\tau_{22} \leq 0.38$.

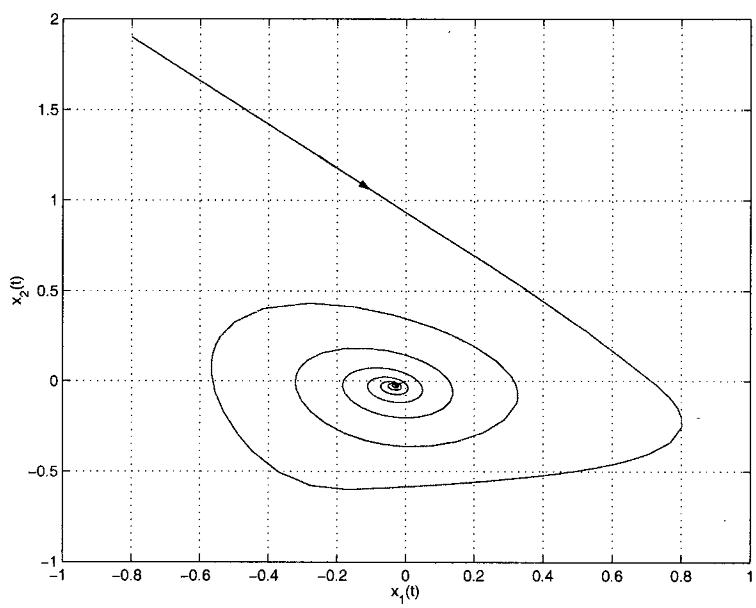

Figure 2.1: Numerical simulation for (2.2.47). Here we choose: $J_1 = J_2 = 0$ and the related data as: $t_{11} = -0.5$, $t_{12} = 1.7$, $t_{21} = 1.25$, $t_{22} = -0.6$ and $\tau_{11} = 0.45$, $\tau_{12} = 0.5$, $\tau_{21} = 2.0$, $\tau_{22} = 0.4$; the initial data are: $x_1(s) = -0.8, x_2(s) = 1.9$ for $s \in [-2, 0]$.

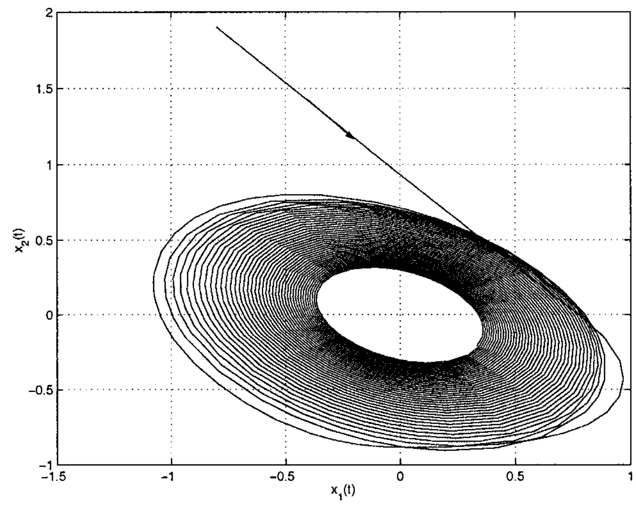

Figure 2.2: Numerical simulation for (2.2.47) with $\tau_{11} = 1.2$ and $\tau_{22} = 1.0$.

2.2.4 HNNs with periodic inputs

In this subsection, we consider the HNNs with periodic inputs,

$$\dot{x}_i(t) = -b_i x_i(t) - \sum_{j=1}^{n} a_{ij} g_j\Big(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s)x_j(s)ds\Big) + I_i(t), t \geq 0, i \in N(1,n), \quad (2.2.48)$$

where $I_i(t) = I_i(t+\omega)$ for $t \geq 0$ and the activation functions $g_j, j \in N(1,n)$ satisfy (S_1) and (S_2). The initial conditions associated with (2.2.48) are given by

$$x_i(s) = \phi_i(s), s \in [-\tau, 0]$$

with $||\phi|| = \max_{1 \leq i \leq n} ||\phi_i||$, $||\phi_i|| = \max_{s \in [-\tau, 0]} |\phi_i(s)|$ for $i \in N(1,n)$. Letting

$$Q_i = P_i + \max_{s \in [0,\omega]} |I_i(s)|,$$

where $P_i := \sum_{j=1}^{n} |a_{ij}| M_j$, then from (2.2.48), we have

$$\dot{x}_i(t) \leq -b_i x_i(t) + Q_i,$$

which gives

$$x_i(t) \le (x_i(0) + \frac{Q_i}{b_i})e^{-b_i t} - \frac{Q_i}{b_i}. \tag{2.2.49}$$

Therefore, we have

$$|x_i(t)| \le 2\frac{Q_i}{b_i} + |x_i(0)|e^{-b_i t} \le 2\frac{Q_i}{b_i} + |x_i(0)|, \tag{2.2.50}$$

which shows that the solution of (2.2.48) is defined on $[-\tau, \infty)$. Denote $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$. Let $(T(t)\phi)(\theta) = x(t+\theta, \phi)$ for $\theta \in [-\tau, 0]$. Then $T(t)\phi$ is defined for all $t \ge 0$ and is an $\omega-$periodic process [41]. From (2.2.50), we can see $T(t)$ is also a bounded map (mapping bounded sets to bounded sets) and point dissipative. Hence from Theorem 4.1.11 in [40], we have

Theorem 2.2.11. *Assume that the activation functions $g_i, i \in N(1,n)$ in (2.2.48) are continuous and satisfy (S_2), then (2.2.48) has an ω-periodic solution.*

Denote, by $\bar{x}(t) = (\bar{x}_1(t), \cdots, \bar{x}_n(t))^T$, the $\omega-$periodic solution claimed in Theorem 2.2.11. We will prove that the ω-periodic solution $\bar{x}(t)$ is globally exponentially stable.

Theorem 2.2.12. *Assume that (S_1) and (S_2) hold. If (2.2.3) or (2.2.5) holds, then the ω-periodic solution $\bar{x}(t)$ (2.2.48) is globally exponentially stable.*

Proof. Let $x(t)$ be a solution of (2.2.48) other than $\bar{x}(t)$. Then we have

$$\begin{aligned} D^+[x_i(t) - \bar{x}_i(t)] &= -b_i\,[x_i(t) - \bar{x}_i(t)] + \textstyle\sum_{j=1}^n a_{ij} g_j(\int_{t-\tau_{ij}}^t l_{ij}(t-s)x_j(s)ds) \\ &\quad - \textstyle\sum_{j=1}^n a_{ij} g_j(\int_{t-\tau_{ij}}^t l_{ij}(t-s)\bar{x}_j(s)ds) \end{aligned} \tag{2.2.51}$$

Letting $u_i(t) = x_i(t) - \bar{x}_i(t)$, we then have

$$D^+|u_i(t)| \le -b_i|u_i(t)| + \sum_{j=1}^n |a_{ij}|L_j \int_{t-\tau_{ij}}^t l_{ij}(t-s)|u_j(s)|ds.$$

Using similar arguments to that in previous subsection, we complete the proof. □

For the HNNs with infinite distributed delays and periodic inputs

$$\dot{x}_i(t) = -b_i x_i(t) - \sum_{j=1}^{n} a_{ij} g_j(\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds) + I_i(t), t \geq 0, \tag{2.2.52}$$

with $I_i(t) = I_i(t+\omega), i \in N(1,n)$, using the result in [75], we have

Theorem 2.2.13. *Assume that* (S_1) *and* (S_2) *hold. Then system (2.2.52) admits an* ω*-periodic solution. Moreover, if (2.2.3) or (2.2.5) holds, the* ω*-periodic solution is globally stable. In addition, if (2.1.26) holds, then the* ω*-periodic solution is globally exponentially stable.*

2.3 Stability and Hopf Bifurcations of Bidirectional Associative Memory Neural Networks (BAMNNs)

In this section, we consider the delayed bidirectional associative memory (BAM) neural network described by the system

$$\begin{cases} \dot{x}_i(t) &= -x_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\ \dot{y}_i(t) &= -y_i(t) + \sum_{j=1}^n b_{ij} g_j(x_j(t-r_{ij})) + J_i \end{cases} \tag{2.3.1}$$

and the delayed BAM neural network with self-connections described by

$$\begin{cases} \dot{x}_i(t) &= -x_i(t) + c_{ii}s_{1,i}(x_i(t-d_{ii})) + \sum_{j=1}^n a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\ \dot{y}_i(t) &= -y_i(t) + l_{ii}s_{2,i}(y_i(t-m_{ii})) + \sum_{j=1}^n b_{ij} g_j(x_j(t-r_{ij})) + J_i. \end{cases} \tag{2.3.2}$$

Here, c_{ii} and l_{ii} are the weights of self-connections; d_{ii} and m_{ii} are the associated self-connection delays; a_{ij}, b_{ij}, $i, j \in N(1, n)$ are the connection weights between the neurons in two layers: the $I-$layer and the $J-$layer. On the $I-$layer, the neurons with states denoted by $x_i(t)$ receive the inputs I_i from outside and the inputs outputted from those neurons in the $J-$layer via activation functions (input–output functions) f_i. While, on the $J-$layer, the neurons whose associated states denoted by $y_i(t)$ receive the exterior inputs J_i and the inputs outputted from those neurons in the $I-$layer via activation functions (output-input functions) g_i. The non-negative constants $\tau_{ij}, r_{ij}, i, j \in N(1, n)$ are the associated delays due to the finite transmission speed among neurons in different layers.

When there is no delay present, (2.3.1) reduces to a system of ordinary differential equations which was investigated by Kosko [59]–[61]. Although system (2.3.1) can be mathematically regarded as a Hopfield type neural network, which was extensively investigated recently (see, for example, [3], [12], [29], [31], [35], [38], [50], [69], [73],

[74], [95], [97] and [107]) with dimension $2n$, we will retain the model (2.3.1) as it stands since we do not want to alter the bidirectional interplay of the input-output nature of the two layers. Networks with such a bidirectional structure have practical applications in storing paired patterns or memories and the ability to search the desired patterns via both directions: forward and backward directions. See [36], [59]–[61] and [77] for details about the applications on learning and associative memories of neural networks.

As far as multiple delays are concerned, for the stability analysis, Gopalsamy and He [36] established some delay-independent stability criteria for (2.3.1) and Mohamad [77] addressed its global exponential stability. Note that in [96] and [97], as well as in Section 2.2 of this thesis, the stabilization role of self-inhibitory connections in Hopfield type neural networks has been revealed. It is natural to incorporate the self-inhibitory connections into (2.3.1) if we want to stabilize the BAM network. In this section, we therefore consider the stability of BAM neural network model with self-connections and we show the self-inhibitory connections do play a stabilization role in the BAM neural networks. On the other hand, not much work has been carried out in the literature for the bifurcation analysis of (2.3.1) and (2.3.2). Due to the complexity arising from the multiple delays and high dimension, even for the Hopfield type neural networks, very little work has been accomplished for the bifurcation analysis. For cases of general n but with only one single delay, we refer to Wu [107] and Wu and Zou [112] and for planar systems, i.e., the networks with two neurons, we refer to [28], [82], [86] and [105], where it was assumed that two delays are equal or are different but without self-connections (this is the case of $n = 1$ in (2.3.1)); For the networks with a special architecture, i.e. ring structure, Campbell [10], Campbell, Ruan and Wei [11], Ncube, Campbell and Wu [81] and Shayer and Campbell [90] obtained some nice results for the bifurcations. We may regard the BAM neural network models (2.3.1) and (2.3.2) as a Hopfield type neural networks with another special architecture, i.e. bidirectional two-layer structure.

2.3.1 BAMNNs without self–connections: model (2.3.1)

If we assume that in the BAM model (2.3.1), the activation functions $f_i,\ g_i,\ i \in N(1,n)$ are continuous and bounded, similar to Theorem 2.1.1, we can show that (2.3.1) has at least one equilibrium. Therefore, without loss of generality, we can place it at the origin and assume that $I_i = J_i = 0$ and $f_i(0) = g_i(0) = 0$ for $i \in N(1,n)$, and hence (2.3.1) reduces to

$$\left\{\begin{array}{lcl} \dot{x}_i(t) & = & -x_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t-\tau_{ij})) \\ \dot{y}_i(t) & = & -y_i(t) + \sum_{j=1}^n b_{ij} g_j(x_j(t-r_{ij})). \end{array}\right. \tag{2.3.3}$$

2.3.1.1 Global stability of (2.3.3)

We may establish the following global stability results by applying Theorems 2.2.1 and 2.2.2 to system (2.3.3):

Theorem 2.3.1. *Suppose f_i and g_i are Lipschitz continuous. If one of the following conditions holds for some positive real numbers $p_i,\ q_i,\ \xi_i,\ \eta_i,\ i \in N(1,n)$,*

$$\left\{\begin{array}{l} Lip(g_i) \sum_{j=1}^n |b_{ji}| q_j < p_i \\ Lip(f_i) \sum_{j=1}^n |a_{ji}| p_j < q_i, \end{array}\right. \tag{2.3.4}$$

$$\left\{\begin{array}{l} p_i \sum_{j=1}^n |a_{ij}| \xi_j + \frac{Lip^2(g_i)}{\eta_i} \sum_{j=1}^n |b_{ji}| q_j < 2p_i \\ q_i \sum_{j=1}^n |b_{ij}| \eta_j + \frac{Lip^2(f_i)}{\xi_i} \sum_{j=1}^n |a_{ji}| p_j < 2q_i, \end{array}\right. \tag{2.3.5}$$

$$\left\{\begin{array}{l} \frac{1}{2} \sum_{j=1}^n (p_i |a_{ij}| + q_j |b_{ji}|) < \frac{p_i}{Lip(g_i)} \\ \frac{1}{2} \sum_{j=1}^n (q_i |b_{ij}| + p_j |a_{ji}|) < \frac{q_i}{Lip(f_i)}, \end{array}\right. \tag{2.3.6}$$

then the system (2.3.3) is globally asymptotically stable (if (2.3.4) holds, we can get the global exponential stability).

Note that we can obtain some easily verified sufficient conditions to guarantee the global stability of (2.3.3) by varying the numbers p_i, q_i, ξ_i and η_i in the above inequalities. For example, letting $p_i = q_i = 1$ for $i \in N(1,n)$ in (2.3.4), we have

Corollary 2.3.1. *If*

$$Lip(g_i)\sum_{j=1}^{n}|b_{ji}| < 1, \text{ and } Lip(f_i)\sum_{j=1}^{n}|a_{ji}| < 1 \tag{2.3.7}$$

hold, then the system (2.3.3) is globally exponentially stable.

Remark 2.3.1. *When g_i and $f_i, i \in N(1,n)$ are smooth and satisfy*

$$g_i'(0) = \sup_{x\in\mathbb{R}} g_i'(x), \;\; f_i'(0) = \sup_{x\in\mathbb{R}} f_i'(x), \quad \text{for } i \in N(1,n),$$

Corollary 2.3.1 reproduces the main results of [35] and [77].

2.3.1.2 Local stability and Hopf bifurcation of (2.3.3)

In this subsection, we do some analysis on the local stability and Hopf bifurcation of (2.3.3). To this end, we assume that the activation functions f_i and g_i are differentiable. The existence of multiple delays makes such an analysis extremely hard, if not impossible. Thus, we just focus on a special case of (2.3.3): the delays in the same layer are identical, i.e., we consider

$$\begin{cases} \dot{x}_i(t) &=& -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_1)) \\ \dot{y}_i(t) &=& -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_2)). \end{cases} \tag{2.3.8}$$

Then the linearization of (2.3.8) at 0 gives

$$\begin{cases} \dot{x}_i(t) &=& -x_i(t) + \sum_{j=1}^{n} \alpha_{ij} y_j(t-\tau_1) \\ \dot{y}_i(t) &=& -y_i(t) + \sum_{j=1}^{n} \beta_{ij} x_j(t-\tau_2) \end{cases}, \tag{2.3.9}$$

where $\alpha_{ij} = a_{ij} f'_j(0)$, $\beta_{ij} = b_{ij} g'_j(0)$ for $i \in N(1,n)$. Denote the $n \times n$ identity matrix by E_n, and let $A = (\alpha_{ij})$, $B = (\beta_{ij})$ and $\tau = \tau_1 + \tau_2$. Let

$$W = \begin{pmatrix} (z+1)E_n & -e^{-z\tau_1}A \\ -e^{-z\tau_2}B & (z+1)E_n \end{pmatrix}.$$

Then the associated characteristic equation is

$$detW = 0. \tag{2.3.10}$$

If

$$d_0 := det \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \neq 0,$$

then $z = -1$ can not be a root of (2.3.10). In the sequel, we assume that $d_0 \neq 0$ and look for roots z of (2.3.10) satisfying $z \neq -1$, and thus $(z+1)E_n$ is nonsingular. It follows from Theorem 1.23 of [30] that

$$detW = det((z+1)E_n)det[W/(z+1)E_n],$$

where $[W/(z+1)E_n]$ is the Schur complement of the block $(z+1)E_n$ in W (for the definition of Schur complement, we refer to [30]). Therefore, (2.3.10) is equivalent to

$$det[(z+1)^2 E_n - e^{-z\tau} BA] = 0. \tag{2.3.11}$$

It is easy to see that z is a solution of (2.3.11) if and only if there is $\lambda \in \sigma(BA)$ such that

$$(z+1)^2 - \lambda e^{-z\tau} = 0. \tag{2.3.12}$$

Hence, if λ_j, $j \in N(1,n)$ are the eigenvalues of BA, then (2.3.11) is equivalent to the n scalar equations

$$(z+1)^2 - \lambda_j e^{-z\tau} = 0, \ j \in N(1,n). \tag{2.3.13}$$

Analyzing the distribution of roots of (2.3.13), we have

Theorem 2.3.2. *Let λ_j, $j \in N(1,n)$ be eigenvalues of BA. Then the following statements hold.*

(I) *The zero solution of system (2.3.8) is asymptotically stable when $\tau = 0$ if and only if*

$$|Re(\sqrt{\lambda_j})| < 1, \ j \in N(1,n); \tag{2.3.14}$$

(II) *The zero solution of system (2.3.8) is asymptotically stable for all non-negative τ if*

$$|\lambda_j| < 1, \ j \in N(1,n). \tag{2.3.15}$$

Proof. Note that the zero solution is asymptotically stable if and only if all roots of (2.3.11) have negative real parts. Case (I): $\tau = 0$, then (2.3.13) reads, for each $j \in N(1,n)$

$$(z+1)^2 - \lambda_j = 0, \tag{2.3.16}$$

which shows that

$$z = -1 \pm \sqrt{\lambda_j}.$$

It is easily seen that $Re(z) < 0$ if and only if (2.3.14) holds.

Case (II): Letting $\lambda_j = a + ib$ and $z = u + iv$ and substituting them to (2.3.13), we have

$$(1 + u + iv)^2 = (a + ib)e^{-(u+iv)\tau},$$

which gives

$$\begin{aligned} (1+u)^2 - v^2 &= e^{-u\tau}[a\cos(v\tau) + b\sin(v\tau)] \\ 2(1+u)v &= e^{-u\tau}[b\cos(v\tau) - a\sin(v\tau)]. \end{aligned}$$

Taking square on the both sides of the above two equations and summing them up, we get

$$[(1+u)^2+v^2]^2 = [e^{-u\tau}|\lambda_j|]^2,$$

or

$$(1+u)^2+v^2 = e^{-u\tau}|\lambda_j|. \tag{2.3.17}$$

Hence if $u \geq 0$, the left hand side of (2.3.17) will be greater than or equal to 1. However, the right hand side $e^{-u\tau}|\lambda_j| < 1$ due to $u > 0$ and (2.3.15). This shows if (2.3.15) holds, then (2.3.11) does not admit a root with nonnegative real part and thus the proof is complete. □

From the above theorem, we see that for positive τ, if (2.3.15) does not hold, then the stability of the zero solution of (2.3.8) may be destroyed. To check this point, in what follows we assume that

$$\max_{j\in N(1,n)} |Re\sqrt{\lambda_j}| < 1 < \max_{j\in N(1,n)} |\lambda_j|. \tag{2.3.18}$$

Here and in what follows, we will restrict our attention to the case that BA is a nonzero matrix having only real and purely imaginary eigenvalues. Since BA is a real matrix, its imaginary eigenvalues must appear in pairs. Thus we may assume that

$$\sigma(BA) = \{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_p,\ \pm i\delta_1,\ \pm i\delta_2,\ \cdots,\ \pm i\delta_q\}$$

with

$$p + 2q = n$$

and

$$\alpha_1 < \alpha_2 \le \cdots < \alpha_s < 0 \le \alpha_{s+1} \le \cdots \le \alpha_p,$$

and

$$0 \le \delta_q \le \delta_{q-1} \le \cdots < \delta_1.$$

In such a case (2.3.18) demands that

$$\alpha_p < 1, \tag{2.3.19}$$

and

$$\delta_1 < 2, \tag{2.3.20}$$

and

$$\alpha_1 < -1, \tag{2.3.21}$$

or

$$\delta_1 > 1. \tag{2.3.22}$$

Lemma 2.3.1. *Suppose that (2.3.22) holds. For each $\delta_j > 1$, define*

$$\tau(\delta_j) = \frac{1}{\omega_j} \arcsin \frac{1-\omega_j^2}{\delta_j}, \tag{2.3.23}$$

where $\omega_j = \sqrt{\delta_j - 1}$. Then we have

(a) *At $\tau(\delta_j)$, (2.3.12) with $\lambda = i\delta_j$ has a pair of purely imaginary simple roots $\pm i\omega_j$ and all other roots have negative real parts.*

(b) *$\tau \in [0, \tau(\delta_j))$, all roots of (2.3.12) with $\lambda = i\delta_j$ have negative real parts.*

(c)

$$Re \left. \frac{dz(\tau)}{d\tau} \right|_{\tau=\tau(\delta_j)} > 0.$$

(d) $\tau(\delta_j) > \tau(\delta_1)$, *if* $\delta_1 > \delta_j > 1$.

Proof. Suppose $z = i\omega$ is a root of (2.3.12) with $\lambda = i\delta_j$. Then we may get

$$1 - \omega^2 = \delta_j \sin \omega\tau,\ 2\omega = \delta_j \cos \omega\tau.$$

which shows that

$$\omega = \sqrt{\delta_j - 1}$$

and

$$\tau = \frac{1}{\omega}\left(\arcsin \frac{1-\omega^2}{\delta_j} + 2k\pi\right),$$

where $0 < \arcsin \frac{1-\omega^2}{\delta_j} < \pi$ and k is an integer. Clearly $\tau(\delta_j)$ is the least such positive τ. Hence at $\tau(\delta_j)$, (2.3.12) with $\lambda = i\delta_j$ has a pair of purely imaginary roots $\pm i\sqrt{\delta_j - 1}$. Let $H_{\delta_j}(z,\tau) = (z+1)^2 - i\delta_j e^{-z\tau}$. Then $\frac{\partial H_{\delta_j}}{\partial z} = 2(1+z) + i\tau\delta_j e^{-z\tau}$. Notice that $H_{\delta_j} = 0$ and $\frac{\partial H_{\delta_j}}{\partial z} = 0$ give $\tau = -\frac{2}{z+1}$, which implies z is real. This shows that the multiple zeros of $H_{\delta_j}(z,\tau)$ have to be real, and hence $i\sqrt{\delta_j - 1}$ is a simple purely imaginary root of (2.3.12) with $\lambda = i\delta_j$. Next we show that (2.3.12) with $\lambda = i\delta_j$ has no root with positive real part. Suppose, by way of contradiction, that $z = u + iv$ with $u > 0$ is a root of (2.3.11) with $\lambda = i\delta_j$. Since the roots of (2.3.12) continuously depend on the parameter τ, using Lemma 2.1 of [25], there exists $\hat{\tau} \in (0, \tau(\delta_j))$ such that (2.3.12) with $\lambda = i\delta_j$ has a purely imaginary root at $\tau = \hat{\tau}$, which contradicts the fact that $\tau(\delta_j)$ is the smallest such τ. Thirdly, we will show that **(c)** of this lemma is true. Differentiating both sides of (2.3.12) with respect to τ leads to

$$\left.\frac{dz(\tau)}{d\tau}\right|_{\tau=\tau(\delta_j)} = \left.\frac{-i\delta_j e^{-z\tau} z}{\tau + 2(z+1)}\right|_{\tau=\tau(\delta_j)}.$$

A straightforward calculation yields

$$Re\,\frac{dz(\tau)}{d\tau}\bigg|_{\tau=\tau(\delta_j)} = \frac{(\delta_j-1)[2+\delta_j+2\tau(\delta_j)]}{(\tau+2)^2+4(\delta_j+1)} > 0.$$

Let $\tau(x)$ be defined by

$$\tau(x) = \frac{1}{\sqrt{x-1}}\arcsin\frac{2-x}{x}, \quad \text{for } x > 1.$$

Then **(d)** of this lemma follows from the fact that $\tau(x)$ is decreasing for $x > 1$. Thus the proof is complete. □

Similarly, we have

Lemma 2.3.2. *Suppose that (2.3.21) holds. Then for each $\alpha_k < -1$, define*

$$\tau(\alpha_k) = \frac{1}{\omega_k}\arcsin\frac{2\omega_k}{-\alpha_k}, \tag{2.3.24}$$

where $\omega_k = \sqrt{|\alpha_k|-1}$, we have

(i) *At $\tau(\alpha_k)$, (2.3.12) with $\lambda = \alpha_k$ has a pair of purely imaginary simple roots $\pm i\omega_k$ and all other roots have negative real parts.*

(ii) *$\tau \in [0, \tau(\alpha_k))$, all roots of (2.3.12) with $\lambda = \alpha_k$ have negative real parts.*

(iii)

$$Re\frac{dz(\tau)}{d\tau}\bigg|_{\tau=\tau(\alpha_k)} > 0.$$

(iv) *$\tau(\alpha_k) > \tau(\alpha_1)$, if $\alpha_1 < \alpha_k < -1$.*

Proof. (i)–(iii) can be obtained from Lemma 5 of Wei and Ruan [105]. (iv) follows from the fact that the function

$$\tau(x) = \frac{1}{\sqrt{x-1}}\arcsin\frac{2\sqrt{x-1}}{x}, \quad \text{for } x > 1$$

is decreasing. $\square$

If both (2.3.21) and (2.3.22) are satisfied, then we may define

$$\tau^* = \min\{\tau(\delta_1),\ \tau(\alpha_1)\}, \tag{2.3.25}$$

which is the least value of τ destabilizing the trivial solution of (2.3.8). Let $x = x^*(\delta_1)$ be the unique solution of the equation

$$\frac{1}{\sqrt{x-1}} \arcsin \frac{2\sqrt{x-1}}{x} = \tau(\delta_1). \tag{2.3.26}$$

Then we have

$$\tau^* = \begin{cases} \tau(\delta_1) & \text{if } |\alpha_1| < x^*(\delta_1) \\ \tau(\alpha_1) & \text{if } |\alpha_1| \geq x^*(\delta_1) \end{cases}$$

In order to use the general Hopf bifurcation theory for functional differential equations developed in [43], we assume that

$$|\alpha_1| \neq x^*(\delta_1). \tag{2.3.27}$$

By the above lemmas, we immediately have the following result on local stability and bifurcation for system (2.3.8).

Theorem 2.3.3. *If (2.3.19)-(2.3.22) and (2.3.27) are satisfied, then we have*

(1) *If $\tau \in [0, \tau^*)$, then the zero solution of (2.3.8) is asymptotically stable;*

(2) *If $\tau > \tau^*$, then the zero solution of (2.3.8) is unstable;*

(3) *Hopf bifurcation occurs at $\tau = \tau^*$. That is, system (2.3.8) has a branch of periodic solutions bifurcating from the zero solution near $\tau = \tau^*$.*

2.3.1.3 Direction and stability of Hopf bifurcation

In the above subsection, we have shown that Hopf bifurcation occurs at some value $\tau^* = \tau_1^* + \tau_2^*$ for the BAMNNs without self–connections. In this subsection, by using the normal form method and the center manifold theory in [43], we will give an algorithm to determine the direction, stability and the period of the bifurcating periodic solutions. Usually, the direction and stability of Hopf bifurcation can be computed by the general algorithm developed in [43] (see also [57]). But in the practical application, it is not an easy job for high dimensional cases. We will give a specific algorithm for a special case. More precisely, we will consider the BAM neural networks with 2 neurons in each layer, that is $n = 2$ in (2.3.8). Moreover, noting that the most often used activation function $\tanh(x)$ has the property

$$\tanh'(0) \neq 0, \ \tanh''(0) = 0, \ \text{ and } \ \tanh'''(0) \neq 0,$$

we may assume that the activation functions in (2.3.8) satisfy:

(P) for $i \in N(1,n)$ $f_i'(0) \neq 0$, $f_i''(0) = 0$, $f_i'''(0) \neq 0$ and $g_i'(0) \neq 0$, $g_i''(0) = 0$, $g_i'''(0) \neq 0$.

Since when $n = 2$, the matrix BA is a 2×2 matrix, then based on the analysis in the previous subsection, we have two cases to consider: case (1) both eigenvalues of BA are real; case (2) the eigenvalues of BA are a pair of purely imaginary numbers. In what follows, we will deal with case (1). In this case, (2.3.19) and (2.3.21) require that the two real eigenvalues α_1 and α_2 of BA satisfy

$$\alpha_1 < \alpha_2 < 1, \ \alpha_1 < -1. \tag{2.3.28}$$

Obviously $\tau^* = \tau(\alpha_1)$ in this case. It is seen from the conclusions of Lemma 2.3.2 and Theorem 2.3.3 that all roots of (2.3.10) other than $\pm i\omega_0$ with $\omega_0 = \sqrt{|\alpha_1| - 1}$ have negative real parts, and the root of (2.3.10)

$$\lambda(\tau) := \alpha(\tau) + i\omega(\tau)$$

satisfying $\alpha(\tau^*) = 0$, $\omega(\tau^*) = \omega_0$ admits

$$\alpha'(\tau^*) := \frac{d\alpha(\tau^*)}{d\tau} > 0$$

and

$$\omega'(\tau^*) = \frac{d\omega(\tau^*)}{d\tau} = \frac{-\omega_0(2 + \tau^* + \tau^*\omega_0)}{(2+\tau^*)^2 + (\tau^*\omega_0)^2}.$$

Following the idea in [105], we let $\tau^* = \tau_1^* + \tau_2^*$ with $\tau_1^* < \tau_2^*$ and $\tau = \tau^* + \mu = (\tau_1^* + \mu) + \tau_2^*$, where $|\mu| \le \tau_2^* - \tau_1^*$. Then $\mu = 0$ is the Hopf bifurcation value for system (2.3.8). Choose the phase space as

$$C = C([-\tau_2^*, 0], \mathbb{C}^4),$$

where we use $\mathbb{C}^4$ instead of $\mathbb{R}^4$ for the convenience in the later computation. Now the system (2.3.8) can be rewritten as

$$\left\{\begin{array}{lcl}
\dot{x}_1(t) & = & -x_1(t) + \alpha_{11}y_1(t-\tau_1^*-\mu) + \alpha_{12}y_2(t-\tau_1^*-\mu) \\
 & & +\alpha_{11}^*y_1^3(t-\tau_1^*-\mu) + \alpha_{12}^*y_2^3(t-\tau_1^*-\mu) + O(y_1^4, y_2^4) \\
\dot{x}_2(t) & = & -x_2(t) + \alpha_{21}y_1(t-\tau_1^*-\mu) + \alpha_{22}y_2(t-\tau_1^*-\mu) \\
 & & +\alpha_{21}^*y_1^3(t-\tau_1^*-\mu) + \alpha_{22}^*y_2^3(t-\tau_1^*-\mu) + O(y_1^4, y_2^4) \\
\dot{y}_1(t) & = & -y_1(t) + \beta_{11}x_1(t-\tau_2^*) + \beta_{12}x_2(t-\tau_2^*) \\
 & & +\beta_{11}^*x_1^3(t-\tau_2^*) + \beta_{12}^*x_2^3(t-\tau_2^*) + O(x_1^4, x_2^4) \\
\dot{y}_2(t) & = & -y_2(t) + \beta_{21}x_1(t-\tau_2^*) + \beta_{22}x_2(t-\tau_2^*) \\
 & & +\beta_{21}^*x_1^3(t-\tau_2^*) + \beta_{22}^*x_2^3(t-\tau_2^*) + O(x_1^4, x_2^4)
\end{array}\right. \tag{2.3.29}$$

where $\alpha_{ij} = a_{ij}f_j'(0)$, $\alpha_{ij}^* = \frac{1}{6}a_{ij}f_j'''(0)$ and $\beta_{ij} = b_{ij}g_i'(0)$, $\beta_{ij}^* = \frac{1}{6}b_{ij}g_j'''(0)$ for $i, j \in N(1,2)$.

Let $U = (x_1(t), x_2(t), y_1(t), y_2(t))^T$, $B_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$, and

$$F(\mu,\phi) = \begin{pmatrix} \alpha_{11}^*\phi_3^3(-\tau_1^* - \mu) + \alpha_{12}^*\phi_4^3(-\tau_1^* - \mu) + O(\phi_3^4, \phi_4^4) \\ \alpha_{21}^*\phi_3^3(-\tau_1^* - \mu) + \alpha_{22}^*\phi_4^3(-\tau_1^* - \mu) + O(\phi_3^4, \phi_4^4) \\ \beta_{11}^*\phi_1^3(-\tau_2^*) + \beta_{12}^*\phi_2^3(-\tau_2^*) + O(\phi_1^4, \phi_2^4) \\ \beta_{21}^*\phi_1^3(-\tau_2^*) + \beta_{22}^*\phi_2^3(-\tau_2^*) + O(\phi_1^4, \phi_2^4) \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C$. Then system (2.3.29) can be rewritten as

$$\dot{U}_t = -U_t + B_1 U_{t-\tau_1^*-\mu} + B_2 U_{t-\tau_2^*} + F(\mu, U_t), \tag{2.3.30}$$

where $U_t(\theta) = U(t+\theta)$ for $\theta \in [-\tau_2^*, 0]$. Let

$$\eta(\theta,\mu) = \begin{cases} -Id, & \theta = 0 \\ B_1\delta(\theta + \tau_1^* + \mu), & \theta \in [-\tau_1^* - \mu, 0) \\ -B_2\delta(\theta + \tau_2^*), & \theta \in [-\tau_2^*, -\tau_1^* - \mu) \end{cases} \tag{2.3.31}$$

where Id is the identical matrix and δ is the usual Dirac function. For $\phi \in C$, define

$$A(\mu)\phi(\theta) = \begin{cases} \dot{\phi}, & \theta \in [-\tau_2^*, 0) \\ \int_{-\tau_2^*}^0 d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases} \tag{2.3.32}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-\tau_2^*, 0) \\ F(\mu,\phi), & \theta = 0 \end{cases} \tag{2.3.33}$$

Then (2.3.29) can be further rewritten as

$$\dot{U}_t = A(\mu)U_t + R(\mu)U_t. \tag{2.3.34}$$

Let $C^* = C^1([0, \tau_2^*], \mathbb{C}^4)$. For $\psi \in C^*$, we define, the adjoint operator $A^*(0)$ of $A(0)$ by

$$A^*(0)\psi(s) = \begin{cases} -\dot{\psi}, & s \in (0, \tau_2^*] \\ \int_{-\tau_2^*}^0 d\eta^T(\xi, 0)\psi(-\xi), & s = 0 \end{cases} \tag{2.3.35}$$

where η^T is the transpose of η. For $\phi \in C([-\tau_2^*, 0], \mathbb{C}^4)$ and $\psi \in C([0, \tau_2^*], \mathbb{C}^4)$, we define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0) \cdot \phi(0) - \int_{\theta=-\tau_2^*}^{0} \int_{\xi=0}^{\theta} \overline{\psi}^T(\xi - \theta) d\eta(\theta, 0)\phi(\xi)d\xi, \tag{2.3.36}$$

where $a \cdot b = \sum_{i=1}^{n} a_i b_i$ for $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$. As usual, we have

$$< \psi, A(0)\phi >=< A^*(0)\psi, \phi > .$$

It is easily seen that $\lambda(0) = i\omega_0$ is an eigenvalue of $A(0)$, and $-i\omega_0$ is an eigenvalue of $A^*(0)$. Denote their corresponding eigenfunctions by $q(\theta)$ and $q^*(s)$, respectively, namely,

$$A(0)q(\theta) = i\omega_0 q(\theta), \text{ and } A^*(0)q^*(s) = -i\omega_0 q^*(s).$$

We compute here with

$$q(\theta) = q(0)e^{i\omega_0\theta}, \text{ and } q^*(s) = q^*(0)e^{i\omega_0 s} \tag{2.3.37}$$

where

$$q(0) = (q_1(0), q_2(0), q_3(0), q_4(0))^T,$$

and

$$\begin{aligned} q^*(0) &= (q_1^*(0), q_2^*(0), q_3^*(0), q_4^*(0))^T \\ &= D(\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0))^T \end{aligned}$$

with

$$q_1(0) = 1,\ q_2(0) = \frac{\alpha_1\alpha_{21} + \beta_{21}detA}{\alpha_1\alpha 11 - \beta_{22}detA}$$

$$q_3(0) = \frac{e^{-i\omega_0\tau_2^*}}{1 + i\omega_0}P_3, q_4(0) = \frac{e^{-i\omega_0\tau_2^*}}{1 + i\omega_0}P_4$$

and

$$\psi_1(0) = 1,\ \psi_2(0) = \frac{\alpha_1\beta_{12} + \alpha_{12}detB}{\alpha_1\beta 11 - \alpha_{22}detA}$$

$$\psi_3(0) = \frac{e^{i\omega_0\tau_1^*}}{1 - i\omega_0}Q_3, \psi_4(0) = \frac{e^{i\omega_0\tau_1^*}}{1 - i\omega_0}Q_4$$

$$P_3 := \beta_{11} + \beta_{12}q_2(0),\ P_4 := \beta_{21} + \beta_{22}q_2(0)$$

$$Q_3 := \alpha_{11} + \alpha_{21}\psi_2(0),\ Q_4 := \alpha_{12} + \alpha_{22}\psi_2(0)$$

and

$$\begin{aligned} D &= \{1 + q_2(0)\psi_2(0) + \frac{1}{\alpha_1}(P_3Q_3 + P_4Q_4) + \frac{1}{\alpha_1}(1 - i\omega_0) \\ &\quad [\tau_1^*(P_3(\alpha 11 + \alpha 21\psi_2(0)) + P_4(\alpha 12 + \alpha 22\psi_2(0)) \\ &\quad + \tau_2^*(\beta_{11}Q_3 + \beta_{21}Q_4 + q_2(0)(\beta_{12}Q_3 + \beta_{22}Q_4)]\}^{-1}. \end{aligned}$$

Then q and q^* satisfy

$$< q^*, q >= 1, \quad \text{and} \quad < q^*, \overline{q} >= 0.$$

If U_t is a solution of (2.3.34), we define

$$z(t) =< q^*, U_t >,\ w(t,\theta) = w(z,\overline{z},\theta) = U_t(\theta) - 2Re\{z(t)q(\theta)\}.$$

Then on the center manifold for (2.3.34) at $\mu = 0$,

$$w(z,\overline{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots$$

holds. Therefore, at $\mu = 0$, (2.3.34) can be reduced to an ordinary differential equation

$$\dot{z}(t) =< q^*, A(0)U_t + RU_t >= i\omega_0 z(t) + \overline{q}^*(0) \cdot F_0, \qquad (2.3.38)$$

where

$$F_0 = F(0, w(z, \overline{z}, \theta) + 2Re\{z(t)q(\theta)\}).$$

We may rewrite (2.3.38) as

$$\dot{z}(t) = < q^*, A(0)U_t + RU_t >= i\omega_0 z(t) + g(z, \overline{z}), \tag{2.3.39}$$

where

$$\begin{aligned} g(z, \overline{z}) &= \overline{q}^*(0) \cdot F_0 && (2.3.40) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots && (2.3.41) \end{aligned}$$

We know from [43] and [57] that in order to study the stability and direction of the Hopf bifurcation, it is crucial to compute these coefficients g_{20}, g_{11}, g_{02} and g_{21}. Observing that

$$y_1(t - \tau_1^*) = w_3(t, -\tau_1^*) + z(t)q_3(-\tau_1^*) + \overline{z}(t)\overline{q}_3(-\tau_1^*),$$

$$y_2(t - \tau_1^*) = w_4(t, -\tau_1^*) + z(t)q_4(-\tau_1^*) + \overline{z}(t)\overline{q}_4(-\tau_1^*),$$

and

$$x_1(t - \tau_2^*) = w_1(t, -\tau_2^*) + z(t)q_1(-\tau_1^*) + \overline{z}(t)\overline{q}_1(-\tau_1^*),$$

$$x_2(t - \tau_2^*) = w_2(t, -\tau_2^*) + z(t)q_2(-\tau_1^*) + \overline{z}(t)\overline{q}_2(-\tau_1^*),$$

where

$$w_i(t, -\tau_j^*) = w_{20}^{(i)}(-\tau_j^*)\frac{z^2}{2} + w_{11}^{(i)}(-\tau_j^*)z\overline{z} + w_{02}^{(i)}(-\tau_j^*)\frac{\overline{z}^2}{2} + \cdots$$

for $i \in N(1,4)$ and $j = 1,2$, we then have

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)\cdot F_0 \\ &= \bar{q}_1^*(0)\left(\alpha_{11}^* y_1^3(t-\tau_1^*) + \alpha_{12}^* y_2^3(t-\tau_1^*)\right) \\ &\quad +\bar{q}_2^*(0)\left(\alpha_{21}^* y_1^3(t-\tau_1^*) + \alpha_{22}^* y_2^3(t-\tau_1^*)\right) \\ &\quad +\bar{q}_3^*(0)\left(\beta_{11}^* x_1^3(t-\tau_2^*) + \beta_{12}^* x_2^3(t-\tau_2^*)\right) \\ &\quad +\bar{q}_4^*(0)\left(\beta_{21}^* x_1^3(t-\tau_2^*) + \beta_{22}^* x_2^3(t-\tau_2^*)\right) \\ &\quad +O(U^4). \end{aligned}$$

Expanding the above and comparing the coefficients with (2.3.41), we have

$$g_{20} = g_{11} = g_{02} = 0$$

and

$$\begin{aligned} g_{21} &= \bar{q}_1^*(0)\left(\alpha_{11}^*|q_3(-\tau_1^*)|^2 q_3(-\tau_1^*) + \alpha_{12}^*|q_4(-\tau_1^*)|^2 q_4(-\tau_1^*)\right) \\ &\quad +\bar{q}_2^*(0)\left(\alpha_{21}^*|q_3(-\tau_1^*)|^2 q_3(-\tau_1^*) + \alpha_{22}^*|q_4(-\tau_1^*)|^2 q_4(-\tau_1^*)\right) \\ &\quad +\bar{q}_3^*(0)\left(\beta_{11}^*|q_1(-\tau_2^*)|^2 q_1(-\tau_1^*) + \beta_{12}^*|q_2(-\tau_2^*)|^2 q_2(-\tau_1^*)\right) \\ &\quad +\bar{q}_4^*(0)\left(\beta_{21}^*|q_1(-\tau_2^*)|^2 q_1(-\tau_1^*) + \beta_{22}^*|q_2(-\tau_2^*)|^2 q_2(-\tau_1^*)\right). \end{aligned}$$

Now we define

$$C_1(0) = \frac{1}{2} g_{21}$$

and

$$\mu_2 = -ReC_1(0),\ \beta_2 = 2ReC_1(0),\ T_2 = \frac{-1}{\omega_0}(ImC_1(0) + \mu_2\omega'(\tau^*)).$$

The above analysis, the general theory on Hopf bifurcation [43] and the fact that $\mu_2\beta_2 < 0$ immediately give

Theorem 2.3.4. *Assume that $n = 2$ in (2.3.8), the activation functions have the property (**P**) and both eigenvalues α_1, α_2 of BA are real satisfying (2.3.28). Then the direction and stability of Hopf bifurcation of (2.3.8) can be determined by the sign of μ_2. Indeed, if $\mu_2 > 0 (< 0)$, then the Hopf bifurcation of (2.3.8) at $\tau = \tau^*$ is supercritical (subcritical) and the periodic solution of (2.3.8) bifurcating from the Hopf bifurcation value $\tau = \tau^*$ is asymptotically orbitally stable (unstable). Moreover, the period of the bifurcated periodic solutions are given by*

$$P = \frac{2\pi}{\omega_0}(1 + T_2\epsilon + O(\epsilon^4))$$

with $\epsilon = \epsilon(\mu) = (\frac{|\mu|}{\mu_2})^{1/2}$.

Example 2.3.1. *Consider the following BAM neural network with two neurons on each layer*

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) + a_{11}\tanh(y_1(t-\tau_1)) + a_{12}\tanh(y_2(t-\tau_1)) \\ \dot{x}_2(t) &= -x_2(t) + a_{21}\tanh(y_1(t-\tau_1)) + a_{22}\tanh(y_2(t-\tau_1)) \\ \dot{y}_1(t) &= -y_1(t) + b_{11}\tanh(x_1(t-\tau_2)) + b_{12}\tanh(x_2(t-\tau_2)) \\ \dot{y}_2(t) &= -y_2(t) + b_{21}\tanh(x_1(t-\tau_2)) + b_{22}\tanh(x_2(t-\tau_2)). \end{cases} \tag{2.3.42}$$

Corollary 2.3.2. *If*

$$|a_{11}|p_1 + |a_{21}|p_2 < q_1, \; |a_{12}|p_1 + |a_{22}|p_2 < q_2 \tag{2.3.43}$$

and

$$|b_{11}|q_1 + |b_{21}|q_2 < p_1, \; |b_{12}|q_1 + |b_{22}|q_2 < p_2 \tag{2.3.44}$$

hold for some positive p_i, $q_i, i = 1, 2$, then the zero solution of (2.3.42) is globally asymptotically stable (exponentially) for any choice of τ_1 and τ_2.

Remark 2.3.2. *If $p_i = q_i = 1$, $i = 1, 2$ in (2.3.43) and (2.3.44), then Corollary 2.3.3 reproduces the main theorem in [77].*

If we take

$$a_{11} = 1,\ a_{12} = -1,\ a_{21} = -1,\ a_{22} = 1.2 \tag{2.3.45}$$

and

$$b_{11} = 0.8,\ b_{12} = 1,\ b_{21} = 1,\ b_{22} = -2 \tag{2.3.46}$$

and $\tau_1^* = 0.1$, then we have

$$\tau^* = 0.6568,\ \alpha_1 = -3.7391 < -1,\ \alpha_2 = 0.1391 < 1$$

and

$$\omega_0 = 1.6550,\ g_{21} = -4.4504 - 2.5948i,\ C_1(0) = -2.2252 - 1.2974i$$

and

$$\mu_2 = 2.2252,\ \beta_2 = -4.4504,\ T_2 = 1.9872.$$

This means the zero solution of system (2.3.42) with (2.3.45) and (2.3.46) is asymptotically stable if $\tau_1 + \tau_2 < \tau^* = 0.6568$, and the Hopf bifurcation occurs at $\tau_1 + \tau_2 = 0.6568$. Furthermore the Hopf bifurcation is supercritical and the bifurcating periodic solutions are asymptotically orbitally stable. Moreover, the period of the bifurcation periodic solutions can be estimated by

$$T = \frac{2\pi}{\omega_0}(1 + T_2\epsilon^2) = 3.7965 + 7.5443\epsilon^2$$

with $\epsilon = (\frac{|\mu|}{\mu_2})^{1/2}$. The numerical simulations, which are performed by the DDEs Solver developed by Shampine and Thompson [89], are given in Figs.2.3–2.4

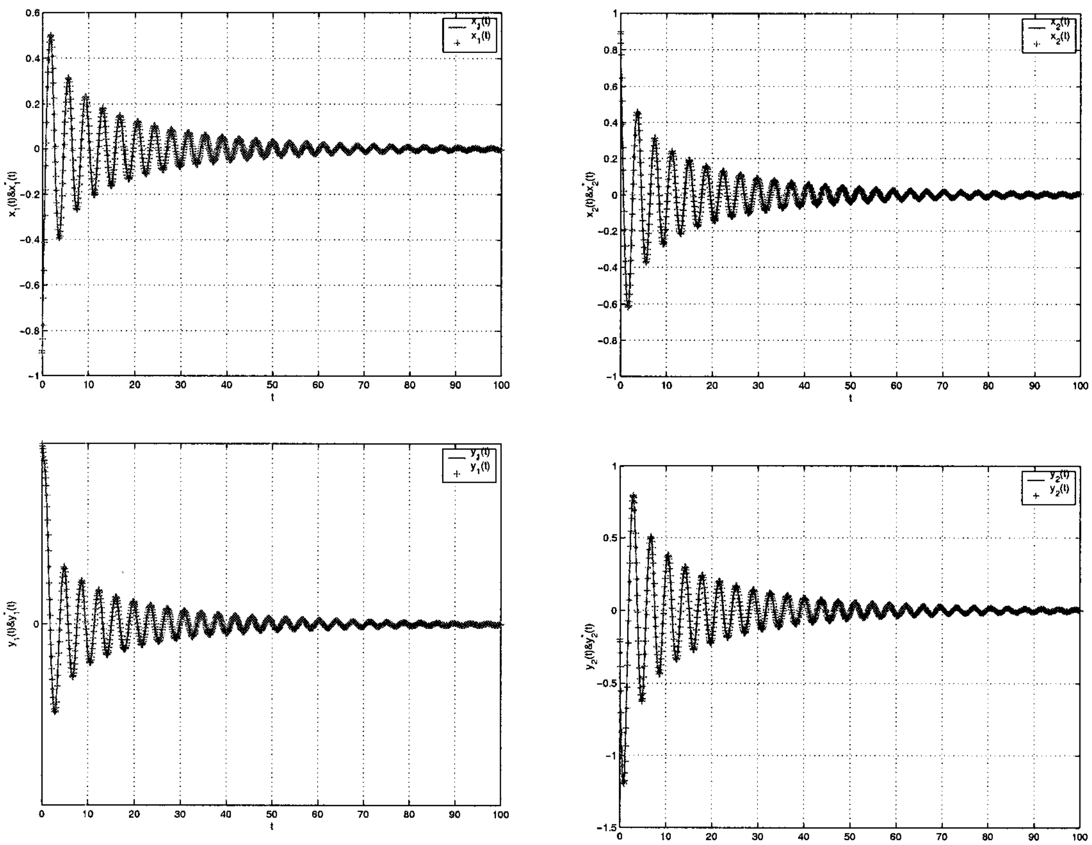

Figure 2.3: $\tau_1 = 0.1, \tau_2 = 0.5$ and thus $\tau_1 + \tau_2 < \tau^*$

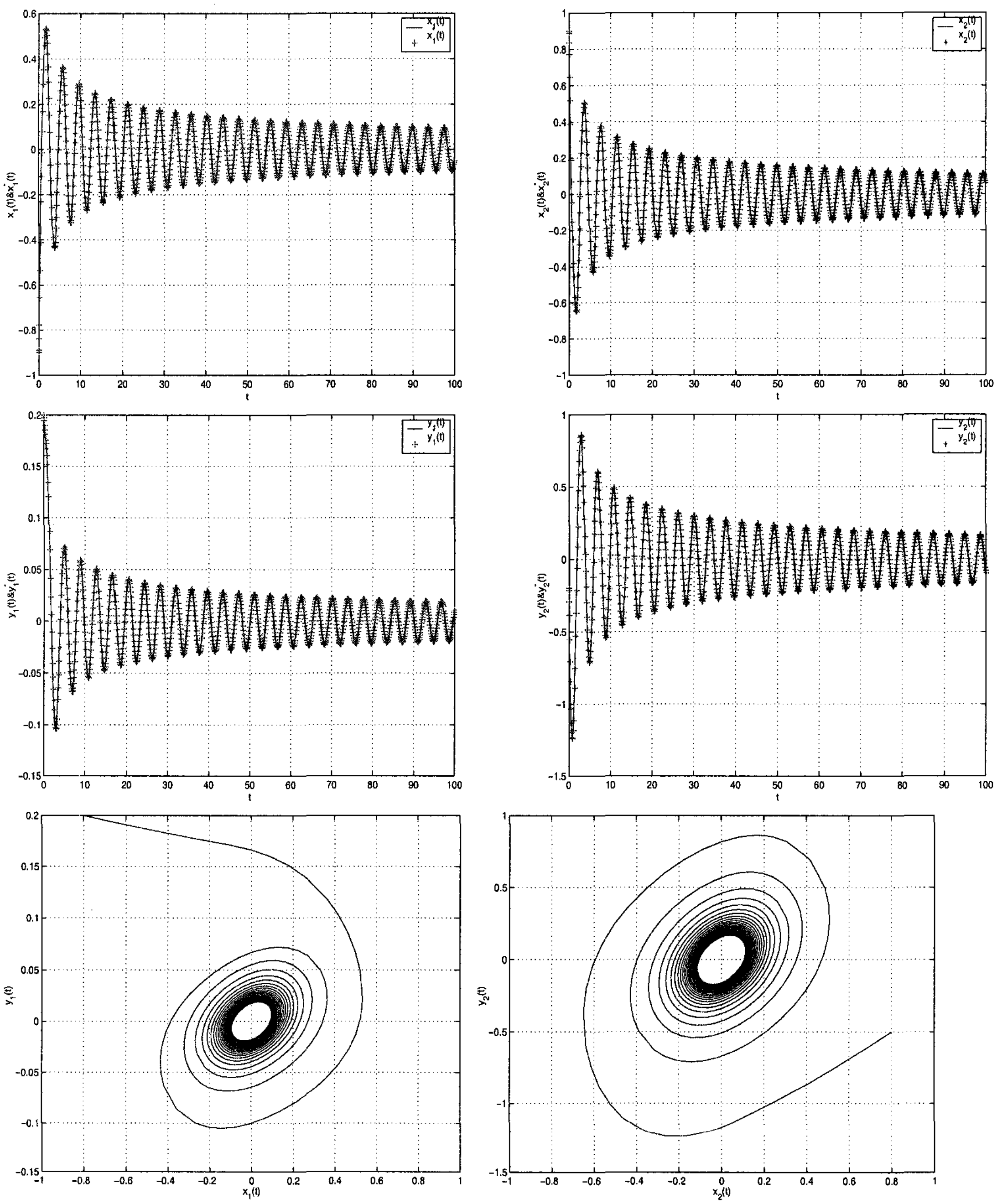

Figure 2.4: $\tau_1 + \tau_2 = 0.1 + 0.5569 > \tau^*$

2.3.2 BAMNNs with delayed self-connections

Now we turn to the stability and Hopf bifurcation for the BAM neural networks with *delayed self-connections*.

2.3.2.1 Global stability of (2.3.2)

We assume that (2.3.2) has at least one equilibrium. Indeed, by using the Brouwer fixed point theorem [26], we can establish

Lemma 2.3.3. *Suppose the activation functions f_i, g_i, $i \in N(1,n)$ are continuous and bounded, then (2.3.2) has at least one equilibrium.*

Hence we can always perform a transformation such that the origin is the equilibrium of the new system. Therefore, without loss of generality, in what follows, we assume that $I_i = J_i = 0$ and $f_i(0) = g_i(0) = s_{1,i}(0) = s_{2,i}(0) = 0$ for $i \in N(1,n)$. Then (2.3.2) reduces to

$$\begin{cases} \dot{x}_i(t) &= -x_i(t) + c_{ii}s_{1,i}(x_i(t-d_{ii})) + \sum_{j=1}^{n} a_{ij}f_j(y_j(t-\tau_{ij})) \\ \dot{y}_i(t) &= -y_i(t) + l_{ii}s_{2,i}(y_i(t-m_{ii})) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-r_{ij})). \end{cases} \tag{2.3.47}$$

Theorem 2.3.5. *If there exist some $p_i > 0$, $q_i > 0$, $i \in N(1,n)$, such that*

$$\begin{cases} |c_{ii}|Lip(s_{1,i})p_i + Lip(g_i)\sum_{j=1}^{n}|b_{ji}|q_j &< p_i \\ |l_{ii}|Lip(s_{2,i})q_i + Lip(f_i)\sum_{j=1}^{n}|a_{ji}|p_j &< q_i, \end{cases} \tag{2.3.48}$$

then the zero solution of (2.3.47) is globally exponentially stable.

Proof. The proof can be achieved by defining a Liapunov functional as

$$\begin{aligned} V(t) &= \sum_{i=1}^{n} p_i \Bigg(|x_i(t)| + |c_{ii}| Lip(s_{1,i}) \int_{t-d_{ii}}^{t} |x_i(s)| ds + \\ &\sum_{j=1}^{n} |a_{ij}| Lip(f_j) \int_{t-\tau_{ij}}^{t} |y_j(s)| ds \Bigg) + \\ &\sum_{i=1}^{n} q_i \Bigg(|y_i(t)| + |l_{ii}| Lip(s_{2,i}) \int_{t-m_{ii}}^{t} |y_i(s)| ds + \\ &\sum_{j=1}^{n} |b_{ij}| Lip(g_j) \int_{t-r_{ij}}^{t} |x_j(s)| ds \Bigg). \end{aligned}$$

□

Theorem 2.3.6. *If there are some real positive numbers p_i, q_i, ξ, η_i, $i \in N(1,n)$ such that*

$$p_i \left(|c_{ii}| + |c_{ii}| Lip^2(s_{1,i}) + \sum_{j=1}^{n} |a_{ij}| \xi_j \right) + \frac{Lip^2(g_i)}{\eta_i} \sum_{j=1}^{n} |b_{ji}| q_j < 2p_i, \tag{2.3.49}$$

and

$$q_i \left(|l_{ii}| + |l_{ii}| Lip^2(s_{2,i}) + \sum_{j=1}^{n} |b_{ij}| \eta_j \right) + \frac{Lip^2(f_i)}{\xi_i} \sum_{j=1}^{n} |a_{ji}|_j < 2q_i \tag{2.3.50}$$

hold, then system (2.3.47) is globally asymptotically stable.

Proof. The proof can be completed by using a different Liapunov functional defined by

$$\begin{aligned} V(t) &= \sum_{i=1}^{n} p_i \left(x_i^2(t) + |c_{ii}| \int_{t-d_{ii}}^{t} s_{1,i}^2(x_i(s)) ds + \sum_{j=1}^{n} \frac{|a_{ij}|}{\xi_j} \int_{t-\tau_{ij}}^{t} f_j^2(y_j(s)) ds \right) \\ &+ \sum_{i=1}^{n} q_i \left(y_i^2(t) + |l_{ii}| \int_{t-m_{ii}}^{t} s_{2,i}^2(y_i(s)) ds + \sum_{j=1}^{n} \frac{|b_{ij}|}{\eta_j} \int_{t-r_{ij}}^{t} g_j^2(x_j(s)) ds \right). \end{aligned}$$

□

Next we assume that $s_{1,i}$ is set to be g_i and $s_{2,i} = f_i$ for each $i \in N(1,n)$. Then we have

Lemma 2.3.4. *Assume that there are some positive real numbers p_i, q_i such that*

$$p_i c_{ii} + \frac{1}{2}\sum_{j=1}^{n} |p_i a_{ij} + q_j b_{ji}| < \frac{p_i}{Lip(g_i)},\ i \in N(1,n) \tag{2.3.51}$$

and

$$q_i l_{ii} + \frac{1}{2}\sum_{j=1}^{n} |q_i b_{ij} + p_j a_{ji}| < \frac{q_i}{Lip(f_i)},\ i \in N(1,n) \tag{2.3.52}$$

hold. Then system (2.3.47) admits a unique equilibrium which is globally asymptotically stable if no delay is present.

Proof. This lemma can be proved by using the main results of [31] and embedding this system to a single layer network with dimension $2n$. Also we can use the following Liapunov function to prove this lemma.

$$V(t) = \sum_{i=1}^{n}\left(p_i \int_0^{x_i} g_i(s)ds + q_i \int_0^{y_i} f_i(s)ds\right). \tag{2.3.53}$$

□

Using a similar argument to the one in [100], but taking

$$V(t) = \sum_{i=1}^{n} \left(p_i|x_i(t)| + q_i|y_i(t)|\right), \tag{2.3.54}$$

we can establish

Lemma 2.3.5. *If there are some positive constants p_i, q_i such that*

$$p_i c_{ii} + \sum_{j=1}^{n} q_j|b_{ji}| < \frac{p_i}{Lip(g_i)},\ i \in N(1,n) \tag{2.3.55}$$

and

$$q_i l_{ii} + \sum_{j=1}^{n} p_j |a_{ji}| < \frac{q_i}{Lip(f_i)}, \ i \in N(1,n) \tag{2.3.56}$$

hold and no delay is present in (2.3.47), then system (2.3.47) is globally exponentially stable.

Note that the delay dependent stability results in [96] and [97] are for smooth activation functions. In Section 2.2.3 (see also [100]), we have recently generalized the results in [96] and [97] to a general model with non-differentiable activation functions, where an even better estimation for the smallness of delays is given. Applying the delay dependent stability results established in Section 2.2.3 to (2.3.47), we have

Theorem 2.3.7. *Assume that the activation functions f_i, g_i, $i \in N(1,n)$ are non-decreasing and Lipschitz continuous, and the delays d_{ii}, m_{ii} corresponding to $c_{ii} < 0$, $l_{ii} < 0$ for $i \in N(1,n)$ satisfy*

$$d_{ii} \leq \frac{1}{d^*}, \ m_{ii} \leq \frac{1}{m^*} \tag{2.3.57}$$

where d^* and m^* are the unique positive roots of equations

$$1 + \frac{1}{d} - \ln \frac{d}{|c_{ii}| Lip(g_i)} = 0, \ 1 + \frac{1}{m} - \ln \frac{m}{|l_{ii}| Lip(f_i)} = 0, \tag{2.3.58}$$

respectively. If for some positive constants $p_i, q_i, i \in N(1,n)$, either (2.3.55) and (2.3.56) or

$$\begin{cases} p_i c_{ii} + \frac{1}{2}\sum_{j=1}^{n} (p_i |a_{ij}| + q_j |b_{ji}|) < \frac{p_i}{Lip(g_i)}, \\ q_i l_{ii} + \frac{1}{2}\sum_{j=1}^{n} (q_i |b_{ij}| + p_j |a_{ji}|) < \frac{q_i}{Lip(f_i)}, \end{cases} i \in N(1,n), \tag{2.3.59}$$

hold, then the trivial solution of system (2.3.47) is globally attractive.

2.3.2.2 Local stability and Hopf bifurcation

In this subsection, we focus on local stability and Hopf bifurcation for the BAM neural network with delayed self-connections (2.3.2) by viewing the self-connection delay as a parameter. We assume that $I_i = J_i = 0$ and $f_i(0) = g_i(0) = s_{1,i}(0) = s_{2,i}(0) = 0$ for $i \in N(1,n)$ so that the zero is a trivial solution of (2.3.2). In addition, since we will discuss the linear stability, we may assume that all the activation functions in (2.3.2) are differentiable with neuron gains 1, that is,

$$f_i'(0) = g_i'(0) = s_{1,i}'(0) = s_{2,i}'(0) = 1, i \in N(1,n).$$

From now on, we assume that

$$c_{ii} = l_{ii} = \beta; d_{ii} = m_{ii} = \sigma; \tau_{ij} = \tau_1; r_{ij} = \tau_2, \ i,j \in N(1,n).$$

The linearization of (2.3.2) at the origin is

$$\begin{cases} \dot{x}_i(t) &= -x_i(t) + \beta x_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} y_j(t-\tau_1) \\ \dot{y}_i(t) &= -y_i(t) + \beta y_i(t-\sigma) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau_2) \end{cases} \tag{2.3.60}$$

As in Section 2.3.1, denote the $n \times n$ identity matrix by E_n, $A = (a_{ij})$, $B = (b_{ij})$ and $\tau = (\tau_1 + \tau_2)/2$ and let

$$W = \begin{pmatrix} (z+1-\beta e^{-z\sigma})E_n & -e^{-z\tau_1}A \\ -e^{-z\tau_2}B & (z+1-\beta e^{-z\sigma})E_n \end{pmatrix}$$

and

$$W^* = \begin{pmatrix} -e^{-z\tau_2}B & (z+1-\beta e^{-z\sigma})E_n \\ (z+1-\beta e^{-z\sigma})E_n & -e^{-z\tau_1}A \end{pmatrix}.$$

Then the associated characteristic equation of (2.3.60) is given by

$$detW = 0. \tag{2.3.61}$$

Note that

$$detW = (-1)^n detW^*.$$

In what follows, we assume that

$$detB \neq 0, \tag{2.3.62}$$

which implies that $e^{-z\tau_2}B$ is nonsingular. Then from Theorem 1.23 of [30] we have

$$detW^* = det(e^{-z\tau_2}B)\, det[W^*/e^{-z\tau_2}B],$$

where $[W^*/e^{-z\tau_2}B]$ is the Schur complement of the block $e^{-z\tau_2}B$ in W^* (See, e.g.,[30]). Therefore, (2.3.61) is equivalent to

$$det[(z+1-\beta e^{-z\sigma})^2 E_n - e^{-2z\tau}BA] = 0. \tag{2.3.63}$$

It is easily seen that z is a solution of (2.3.63) if and only if there is a $\lambda \in \sigma(BA)$ such that

$$(z+1-\beta e^{-z\sigma})^2 - \lambda e^{-2z\tau} = 0. \tag{2.3.64}$$

Hence, if $\lambda_j,\ j \in N(1,n)$ are eigenvalues of BA, then (2.3.61) is equivalent to n scalar equations

$$(z+1-\beta e^{-z\sigma})^2 - \lambda_j e^{-2z\tau} = 0,\ j \in N(1,n). \tag{2.3.65}$$

For any $\lambda_j \in \sigma(BA), j \in N(1,n)$, we can write it as

$$\lambda_j = |\lambda_j| e^{i\theta_j}, \theta_j \in [0, 2\pi),$$

and then (2.3.65) is equivalent to

$$z+1-\beta e^{-z\sigma} \pm \sqrt{|\lambda_j|} e^{-z\tau} e^{i\frac{\theta_j}{2}} = 0. \tag{2.3.66}$$

Let $z = \mu + i\omega$, then (2.3.66) is equivalent to

$$\begin{cases} R(\mu,\omega) := \mu + 1 - \beta e^{-\mu\sigma}\cos(\omega\sigma) \pm \sqrt{|\lambda_j|}e^{-\mu\tau}\cos(\omega\tau - \theta_j/2) = 0 \\ I(\mu,\omega) := \omega + \beta e^{-\mu\sigma}\sin(\omega\sigma) \mp \sqrt{|\lambda_j|}e^{-\mu\tau}\sin(\omega\tau - \theta_j/2) = 0. \end{cases}$$

Noticing that

$$R(\mu,\omega) \geq 1 - |\beta| - \sqrt{|\lambda_j|}, \text{ for all } \mu \geq 0, \sigma \geq 0, \tau \geq 0,$$

we immediately have

Theorem 2.3.8. *Assume that (2.3.62) holds. If*

$$\sqrt{|\lambda|} + |\beta| < 1, \sigma \geq 0, \tau \geq 0, \tag{2.3.67}$$

where

$$|\lambda| := \max_{1 \leq i \leq n}\{|\lambda_j|, \lambda_j \in \sigma(BA)\},$$

then all roots of (2.3.65) have negative real parts, and hence the trivial solution of (2.3.2) is asymptotically stable.

From $R(\mu,\omega) = 0$ and $I(\mu,\omega) = 0$, we obtain

$$\mu = -1 + \beta e^{-\mu\sigma}\cos(\omega\sigma) \mp \sqrt{|\lambda_j|}e^{-\mu\tau}\cos(\omega\tau - \theta_j/2), \tag{2.3.68}$$

$$\omega = -\beta e^{-\mu\sigma}\sin(\omega\sigma) \pm \sqrt{|\lambda_j|}e^{-\mu\tau}\sin(\omega\tau - \theta_j/2). \tag{2.3.69}$$

and hence,

$$(\mu + 1 - \beta e^{-\mu\sigma}\cos(\omega\sigma))^2 + (\omega + \beta e^{-\mu\sigma}\sin(\omega\sigma))^2 = |\lambda_j|e^{-2\mu\tau},$$

or

$$(\mu+1)^2 + \omega^2 - 2\beta e^{-\mu\sigma}[(\mu+1)\cos(\omega\sigma) - \omega\sin(\omega\sigma)] + \beta^2 e^{-2\mu\sigma} - |\lambda_j|e^{-2\mu\tau} = 0. \tag{2.3.70}$$

If we assume that

$$\beta < 0, \text{ and } \sqrt{|\lambda|} < -\beta, \sigma \in [0, \frac{1}{-2\beta}], \tag{2.3.71}$$

then, it follows from (2.3.69) that

$$\omega < -2\beta \text{ for } \mu \geq 0, \tau \geq 0, \quad \text{and } \omega\sigma \in [0, 1].$$

Letting the left hand side of (2.3.70) be $M(\mu)$, we then have

$$\begin{aligned}
M(0) &= 1 + \omega^2 - 2\beta(\cos(\omega\sigma) - \omega\sin(\omega\sigma)) + \beta^2 - |\lambda_j| \\
&= 1 + \beta^2 - |\lambda_j| - 2\beta\cos(\omega\sigma) + \omega^2 + 2\beta\omega\sin(\omega\sigma) \\
&> \omega^2 + 2\beta\omega(\omega\sigma) \\
&= \omega^2(1 + 2\beta\sigma) \\
&\geq 0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\left.\frac{dM(\mu)}{d\mu}\right|_{\mu\geq 0} &= 2\left\{(\mu+1)[1 + \beta\sigma e^{-\mu\sigma}\cos(\omega\sigma)] - \beta e^{-\mu\sigma}[\cos(\omega\sigma) + \beta\sigma e^{-\mu\sigma}]\right. \\
&\quad \left. + |\lambda_j|\sigma e^{-2\mu\sigma} - \beta\sigma\omega e^{-\mu\sigma}\sin(\omega\sigma)\right\} \\
&\geq 0.
\end{aligned}$$

This shows that $M(\mu) > 0$ for all $\mu > 0$ and thus we have

Theorem 2.3.9. *If (2.3.62) and (2.3.71) hold, then for all $\tau \geq 0$, the trivial solution of (2.3.2) is asymptotically stable.*

In the following, we will regard σ as the parameter and try to find its critical value at which the bifurcation occurs.

Letting $\sigma = 0$ in (2.3.66), we have

$$R(\mu, \omega) = \mu + 1 - \beta \pm \sqrt{|\lambda_j|} e^{-\mu\tau} \cos(\omega\tau - \theta_j/2)$$

and hence

$$R(\mu, \omega) \geq 1 - \beta - \sqrt{|\lambda_j|} \text{ for all } \mu \geq 0,$$

which indicates

Lemma 2.3.6. *If*

$$\beta < 1 - \sqrt{|\lambda_j|}, \tag{2.3.72}$$

then all roots of (2.3.66) have negative real parts at $\sigma = 0$ for all $\tau \geq 0$.

Next we investigate if $\sigma > 0$ will destroy the stability. Theorem 2.3.2 and Lemma 2.3.6 suggest that in order to explore the possibility that $\sigma > 0$ destroys the stability, we need to assume that (2.3.72) and $|\beta| + \sqrt{\lambda_j} \geq 1$ hold, or equivalently,

$$\beta < -\left|1 - \sqrt{|\lambda_j|}\right|. \tag{2.3.73}$$

Under this assumption, we know for any fixed $\tau \geq 0$, all roots of (2.3.66) have negative real parts when $\sigma = 0$ and it is possible for some roots having non-negative real parts when $\sigma > 0$. It follows from [11] that the only way to achieve this is by way of crossing the imaginary axis.

Note that $z = 0$ can not be a root of (2.3.66) due to (2.3.73). If $z = i\omega$ with $\omega > 0$ is a root of (2.3.66) if and only if

$$\begin{cases} \beta\cos(\omega\sigma) & = 1 \pm \sqrt{|\lambda_j|} \cos(\omega\tau - \frac{\theta_j}{2}) \\ \beta\sin(\omega\sigma) & = -\omega \pm \sqrt{|\lambda_j|} \sin(\omega\tau - \frac{\theta_j}{2}) \end{cases} \tag{2.3.74}$$

which gives

$$\beta^2 = 1 + |\lambda_j| + \omega^2 \pm 2\sqrt{|\lambda_j|}\left(\cos(\omega\tau - \frac{\theta_j}{2}) - \omega\sin(\omega\tau - \frac{\theta_j}{2})\right). \qquad (2.3.75)$$

Eq.(2.3.75) can have either finitely many or no root for $\omega > 0$. In the case of finitely many roots, we denote them by $\omega_l^{\pm}(\lambda_j), l = 1, 2, \ldots, m$. It follows from (2.3.74) that

$$\sigma^{\pm} = \frac{1}{\omega_l^{\pm}(\lambda_j)}\left(\arccos\frac{1 \pm \sqrt{|\lambda_j|}\cos(\omega_l^{\pm}(\lambda_j)\tau - \frac{\theta_j}{2})}{\beta} + 2k\pi\right) =: \sigma_{l,j}^{\pm}(k), \qquad (2.3.76)$$

where $p \in N(0) = \mathbb{N}$. In the case where (2.3.75) has no root, we denote the corresponding $\sigma_{l,j}^{\pm}(0) = \infty$. The above analysis and a direct calculation give

Lemma 2.3.7. *Assume that (2.3.73) holds. Then*

(i) *all roots of (2.3.66) have negative real parts for any fixed $\tau \geq 0$ and for*

$$\sigma \in [0, \sigma(\lambda_j)); \qquad (2.3.77)$$

(ii) *Eq. (2.3.66) has a pair of simple purely imaginary roots and all other roots have negative real parts at $\sigma = \sigma(\lambda_j)$;*

(iii) *at least one root of (2.3.66) has positive real part if*

$$\sigma > \sigma(\lambda_j). \qquad (2.3.78)$$

Here $\sigma(\lambda_j) := \min\{\sigma_{l,j}^{+}(0), \sigma_{l,j}^{-}(0),\ l \in N(1, m)\}$. Moreover,

$$\left.\frac{dRe(z)}{d\sigma}\right|_{z=i\omega} \neq 0$$

if and only if

$$\tau \neq \tau^{\pm}(\lambda_j, \omega),$$

where $\tau^{\pm}(\lambda_j, \omega)$ *is the solution of*

$$\omega\left(1 \mp \tau\sqrt{|\lambda_j|}\cos(\omega\tau - \frac{\theta_j}{2})\right) = \pm\sqrt{|\lambda_j|}(1+\tau)\sin(\omega\tau - \frac{\theta_j}{2}), \tag{2.3.79}$$

and in the case that (2.3.79) has no solution, we denote $\tau^{\pm}(\lambda_j, \omega) = \infty$.

Let

$$\sigma^* = \min\{\sigma(\lambda_j), j \in N(1,n)\} = \sigma(\lambda_{j_0}), \text{ for some } j_0 \in N(1,n).$$

Then σ^* is the first critical value at which Hopf bifurcation possibly occurs. Corresponding to such value, we denote $i\omega$ by $i\omega_0$, λ_{j_0} by λ_0, and σ by σ_0. Summarizing the above analysis and applying the standard Hopf bifurcation Theorem in [43], we have

Theorem 2.3.10. *Assume that (2.3.62) holds. Let* $|\lambda| = \max\limits_{1 \leq i \leq n}\{|\lambda_j| : \lambda_j \in \sigma(BA)\}$.

(I) *If*

$$\beta < 1 - \sqrt{|\lambda|}, \tag{2.3.80}$$

then the trivial solution of (2.3.2) is asymptotically stable at $\sigma = 0$ *for all* $\tau \geq 0$;

(II) *If*

$$\beta < -\left|1 - \sqrt{|\lambda|}\right|, \tag{2.3.81}$$

then the trivial solution of (2.3.2) is asymptotically stable for $\sigma \in [0, \sigma_0)$ *and unstable if* $\sigma > \sigma_0$.

(III) *Hopf bifurcation occurs at $\sigma = \sigma_0$ provided*

$$m(\lambda_0) = 1, \ \tau \neq \tau(\lambda_0, \omega_0),$$

where $m(\lambda_0)$ is the multiplicity of λ_0 being an eigenvalue of the matrix BA.

Remark 2.3.3. *If $m(\lambda_0) = 2$ in* (III) *of the above theorem, Hopf-Hopf bifurcation occurs at $\sigma = \sigma_0$.*

2.3.2.3 Hopf bifurcation and its direction and stability for $\tau = 0$

The direction and stability of the Hopf bifurcation established in Section 2.3.2.2 is not easy to confirm and thus in this subsection, we will focus on a special case: $\tau = 0$, and give the Hopf bifurcation theorem and an algorithm for direction and stability. Note that (2.3.2) is now reduced to

$$\begin{cases} \dot{x}_i(t) &= -x_i(t) + \beta s_{1,i}(x_i(t-\sigma)) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) \\ \dot{y}_i(t) &= -y_i(t) + \beta s_{2,i}(y_i(t-\sigma)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) \end{cases} \tag{2.3.82}$$

Let

$$\omega_j^{\pm} = \sqrt{\beta^2 - (1 \pm \sqrt{|\lambda_j|} \cos \frac{\theta_j}{2})^2} \mp \sqrt{|\lambda_j|} \sin \frac{\theta_j}{2}$$

and

$$\sigma_j^{\pm}(0) = \frac{1}{\omega_j^{\pm}} \arccos \frac{1 \pm \sqrt{|\lambda_j|} \cos \frac{\theta_j}{2}}{\beta}.$$

If $\omega_j^{\pm} \notin \mathbb{R}^+$, we denote the corresponding $\sigma_j^{\pm}(0) = \infty$, where $\lambda_j = |\lambda_j| e^{i\theta_j}$ is the $j-$th eigenvalue of BA and $j = 1, 2, \ldots, n$. Let $\sigma(\lambda_j) = \min(\sigma_j^+(0), \sigma_j^-(0))$ and

$$\sigma_0 = \min_{1 \leq i \leq n} \{\sigma(\lambda_j)\} = \sigma(\lambda_{j_0}) \text{ for some } j_0 \in N(1, n).$$

For this special case, one can easily show that the condition $\tau \neq \tau^{\pm}(\lambda_j, \omega)$ for $\omega > 0$ holds. Thus, Theorem 2.3.10 reads in this case as following

Theorem 2.3.11. *Assume that (2.3.62) holds.*

(i) *If*

$$\beta < 1 - \sqrt{|\lambda|}, \tag{2.3.83}$$

then the trivial solution of (2.3.82) is asymptotically stable at $\sigma = 0$.

(ii) *If*

$$\beta < -\left|1 - \sqrt{|\lambda|}\right|, \tag{2.3.84}$$

then the trivial solution of (2.3.82) is asymptotically stable for $\sigma \in [0, \sigma_0)$ and unstable if $\sigma > \sigma_0$.

(iii) *Hopf bifurcation occurs at $\sigma = \sigma_0$ provided $m(\lambda_0) = 1$.*

We assume for simplicity that the activation functions in (2.3.82) satisfy

$$f_i^{''}(0) = g_i^{''}(0) = s_{1,i}^{''}(0) = s_{2,i}^{''}(0) = 0, \text{ for } i \in N(1, n),$$

as the prototype functions $\tanh(x)$ and $\arctan(x)$ do. Then the Taylor expansion of (2.3.82) at zero has the form

$$\left\{\begin{array}{rcl} \dot{x}_i(t) & = & -x_i(t) + \beta x_i(t-\sigma) + \sum_{j=1}^n a_{ij} y_j(t) \\ & & +\gamma_i x_i^3(t-\sigma) + \sum_{j=1}^n a_{ij}^* y_j^3(t) + h.o.t. \\ \dot{y}_i(t) & = & -y_i(t) + \beta y_i(t-\sigma) + \sum_{j=1}^n b_{ij} x_j(t) \\ & & +\alpha_i y_i^3(t-\sigma) + \sum_{j=1}^n b_{ij}^* x_j^3(t) + h.o.t. \end{array}\right. \tag{2.3.85}$$

where *h.o.t.* stands for the high order terms, $\gamma_i = \beta s_{1,i}^{'''}(0)/6, \alpha_i = \beta s_{2,i}^{'''}(0)/6, a_{ij}^* = a_{ij} f_j^{'''}(0)/6, b_{ij}^* = b_{ij} g_j^{'''}(0)/6, i, j \in N(1, n)$. Let $\sigma = \sigma_0 + \mu$, then Theorem 2.3.11 implies that Hopf bifurcation occurs at $\mu = 0$. By using the general method introduced in [43], we can give a specific algorithm to determine the direction and stability of such Hopf bifurcation as below. Note that a direct calculation shows that

$$\left.\frac{dRe(z)}{d\sigma}\right|_{z=i\omega_0} > 0.$$

Our algorithm is given as follows:

Algorithm

1. Put $\alpha_0 := 1 + i\omega_0 - \beta e^{-i\omega_0\sigma_0}$;

2. Find an eigenvector $Q = (q_1, q_2, \ldots, q_n)^T$ for matrix BA corresponding to its eigenvalue λ_0, i.e.,

$$(\lambda_0 E_n - BA)Q = 0;$$

3. Let

$$P = \alpha_0 B^{-1}Q, \quad P^* = \overline{\alpha}_0 B^{-1}Q,$$

where $P = (p_1, p_2, \ldots, p_n)^T, P^* = (p_1^*, p_2^*, \ldots, p_n^*)^T$;

4. Compute D, which is defined by

$$D = \frac{1}{(1 + \beta\sigma_0 e^{-i\omega_0\sigma_0}) \sum_{j=1}^n (\overline{q}_j p_j + \overline{p}_j^* q_j)}$$

5. Let

$$\begin{aligned} C_1(0) \;\; &= \;\; 3D\left\{\sum_{j=1}^n q_j \left(\gamma_j |p_j|^2 p_j e^{-i\omega_0\sigma_0} + \sum_{k=1}^n a_{jk}^* |q_k|^2 q_k\right)\right. \\ &\quad \left. + \sum_{j=1}^n p_j^* \left(\alpha_j |q_j|^2 q_j e^{-i\omega_0\sigma_0} + \sum_{k=1}^n b_{jk}^* |p_k|^2 p_k\right)\right\} \end{aligned}$$

6. Let

$$\mu_2 = -Re(C_1(0)).$$

Then we have

Theorem 2.3.12. *If $\mu_2 > 0$ (< 0), then the Hopf bifurcation of (2.3.82) occurred at $\sigma = \sigma_0$ is supercritical (subcritical) and the periodic solutions of (2.3.82) bifurcating from Hopf bifurcation value are asymptotically orbitally stable (unstable).*

2.3.2.4 Some examples and numerical simulations

Example 2.3.2. *A BAM neural model with three delays.*

Consider the following BAM neural network with two neurons on each layer

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & -x_1(t) + \beta f(x_1(t-\sigma)) + a_{11} f(y_1(t-\tau_1)) + a_{12} f(y_2(t-\tau_1)) \\ \dot{x}_2(t) & = & -x_2(t) + \beta f(x_2(t-\sigma)) + a_{21} f(y_1(t-\tau_1)) + a_{22} f(y_2(t-\tau_1)) \\ \dot{y}_1(t) & = & -y_1(t) + \beta f(y_1(t-\sigma)) + b_{11} f(x_1(t-\tau_2)) + b_{12} f(x_2(t-\tau_2)) \\ \dot{y}_2(t) & = & -y_2(t) + \beta f(y_2(t-\sigma)) + b_{21} f(x_1(t-\tau_2)) + b_{22} f(x_2(t-\tau_2)) \end{array}\right. \tag{2.3.86}$$

where $f(x) = \tanh x$.

Corollary 2.3.3. *If*

$$|\beta| q_1 + |a_{11}| p_1 + |a_{21}| p_2 < q_1, \; |\beta| q_2 + |a_{12}| p_1 + |a_{22}| p_2 < q_2 \tag{2.3.87}$$

and

$$|\beta| p_1 + |b_{11}| q_1 + |b_{21}| q_2 < p_1, \quad |\beta| p_2 + |b_{12}| q_1 + |b_{22}| q_2 < p_2 \tag{2.3.88}$$

hold for some positive p_i, $q_i, i = 1, 2$, then the zero solution of (2.3.86) is globally asymptotically stable for all $\sigma \geq 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

Take $\beta = -2$, and $A = \begin{pmatrix} 1.0 & -1.0 \\ -1.0 & 1.2 \end{pmatrix}$, $B = \begin{pmatrix} 0.8 & 1.0 \\ 1.0 & -2.0 \end{pmatrix}$. Then from Theorem 2.3.7 we know that the zero solution of (2.3.86) is globally attractive for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ provided $\sigma \leq 0.1572$. The eigenvalues of matrix BA are: $\lambda_1 = 0.1391, \lambda_2 = -3.7391$. If $\tau = (\tau_1 + \tau_2)/2 = 0$, then a direct calculation gives $\sigma_0 = 0.5598$ with the associated $\lambda_0 = 0.1391, \omega_0 = 1.4543$. This shows that the zero solution of (2.3.86) is asymptotically stable when $\sigma \in [0, 0.5598), \tau_1 = \tau_2 = 0$ and local periodic solutions appear via Hopf bifurcation near $\sigma = 0.5598$. The numerical simulations are shown in Figs.2.5–2.6. If $\tau_1 + \tau_2 = 0.02$, we can compute that $\sigma_0 = 0.5544$ and the associated $\lambda_0 = -3.7391$, $\omega_0 = 3.7038$. This implies that in this case the zero solution of (2.3.86) is asymptotically stable when $\sigma \in [0, 0.5544)$ and Hopf bifurcation occurs around $\sigma = 0.5544$. The numerical simulations, are given in Figs.2.7–2.8. We acknowledge that all numerical simulations presented here were performed by the DDE23 Solver developed by Shampine and Thompson [89].

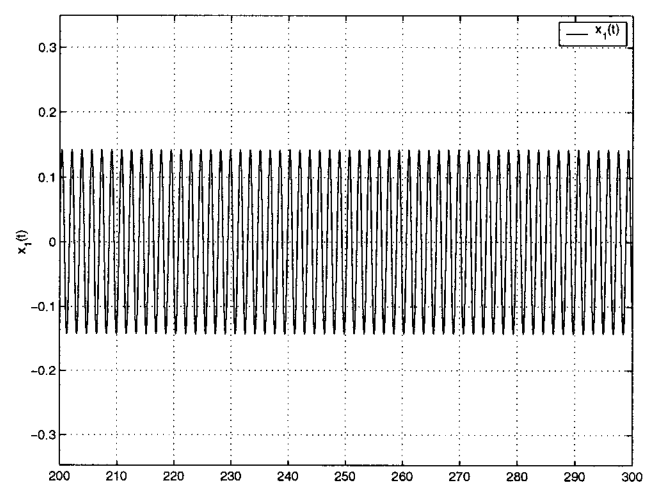

Figure 2.5: Hopf bifurcation occurs when σ is near the critical value σ_0, here we use $\tau = 0, \sigma = 0.58$ and just give the first component $x_1(t)$ vs t. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

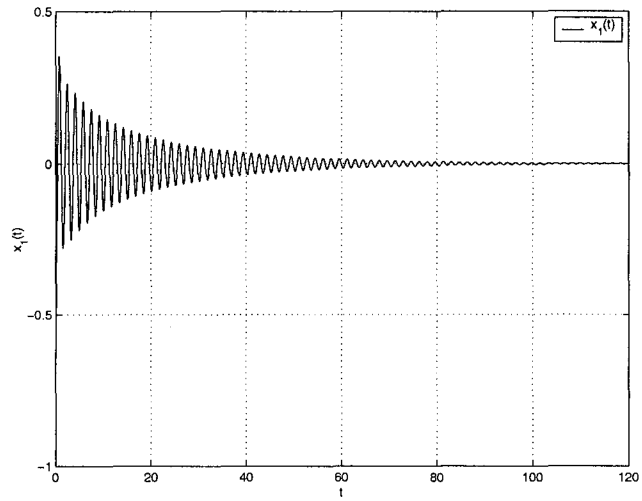

Figure 2.6: Locally stable solution of (2.3.86) is obtained when $\sigma < \sigma_0$, here $\tau_1 = \tau_2 = 0, \sigma = 0.55$ and $x_1(t)$ vs t is shown. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

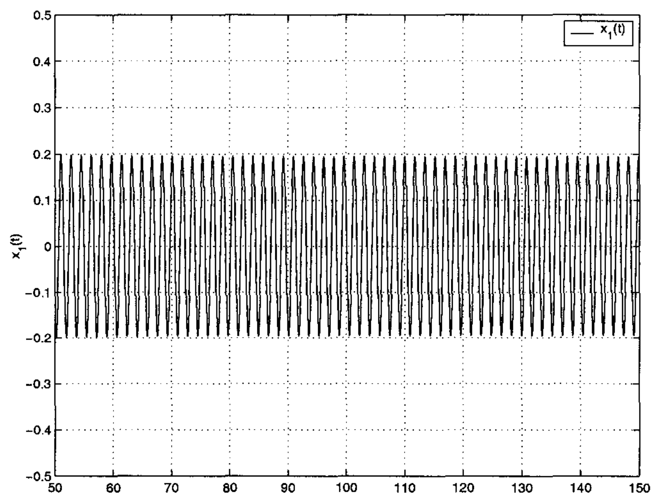

Figure 2.7: Long time behavior of solution of (2.3.86) which bifurcates from the zero solution when σ is near the critical value σ_0, here we use $\tau_1 = 0.008, \tau_2 = 0.012, \sigma = 0.57$. The component $x_1(t)$ is shown here and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

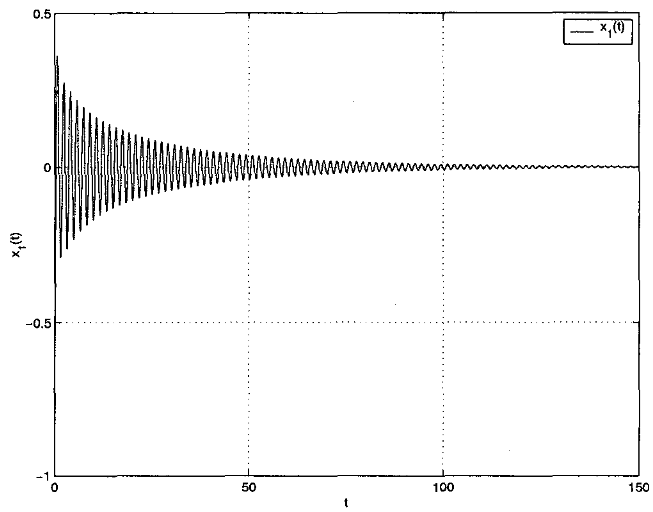

Figure 2.8: The zero solution of (2.3.86) is locally stable when $\sigma < \sigma_0$, here $\tau_1 = 0.008, \tau_2 = 0.012, \sigma = 0.54$. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

Example 2.3.3. *Ring structured neural network models.*

A general neural network model with a special connection architecture, i.e., ring structure, was investigated by Campbell in [10]. A simplified such model takes the form

$$\dot{u}_j(t) = -u_j(t) + s_j(u_j(t-\sigma)) + h_j(u_{j-1}(t-\tau)), \tag{2.3.89}$$

where $j = 1, 2, \ldots, k$ and $u_0 = u_k$. In the case where $k = 4$, (2.3.89) was discussed in [11] on the local stability and Hopf bifurcation. Note that we can topologically regard (2.3.89) as a simple BAM model when the number of neurons k is an even number. For example, a ring of 6 neurons shown in Fig.2.9 can be reorganized as a BAM neural model with $n = 3$ shown in Fig. 2.10.

For general even number $k = 2n$. Let

$$x_j(t) = u_{2j-1}(t), y_j(t) = u_{2j}(t), j = 1, 2, \ldots, n.$$

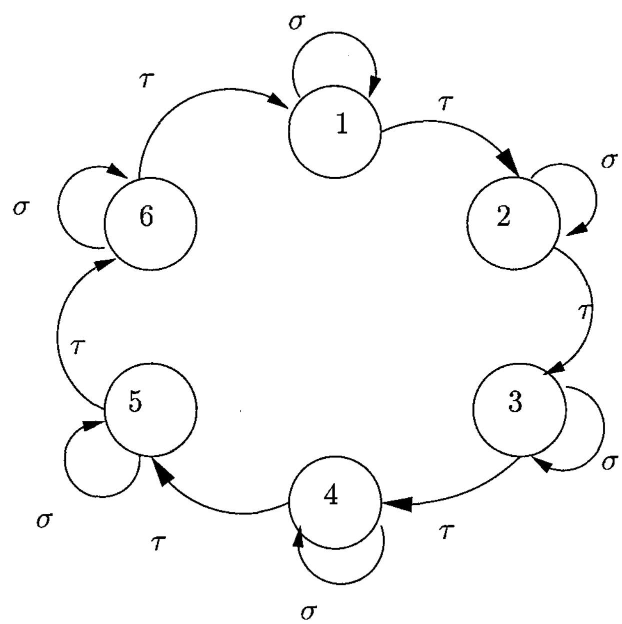

Figure 2.9: A ring of six neurons.

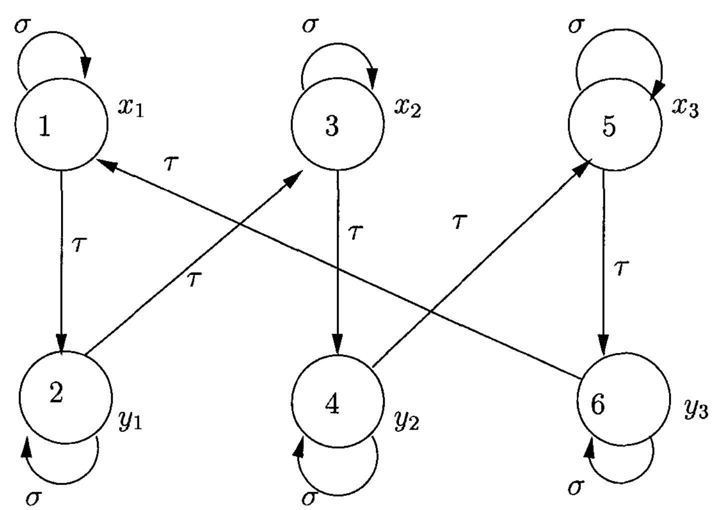

Figure 2.10: The BAM neural network obtained from the ring of six neurons.

Then we can rewrite (2.3.89) as

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + s_{2j-1}(x_j(t-\sigma)) + h_{2j-1}(y_{j-1}(t-\tau)) \\ \dot{y}_j(t) = -y_j(t) + s_{2j}(y_j(t-\sigma)) + h_{2j}(x_j(t-\tau)), \end{cases} \tag{2.3.90}$$

where $y_0(t) = y_n(t)$. For convenience, we may further rewrite (2.3.89) as

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + s_{1,j}(x_j(t-\sigma)) + f_{j-1}(y_{j-1}(t-\tau)) \\ \dot{y}_j(t) = -y_j(t) + s_{2,j}(y_j(t-\sigma)) + g_j(x_j(t-\tau)), \end{cases} \tag{2.3.91}$$

Without loss of generality, we can assume that zero is an equilibrium of (2.3.91), then its linearization at zero is

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + a_j x_j(t-\sigma) + b_{j-1}y_{j-1}(t-\tau) \\ \dot{y}_j(t) = -y_j(t) + a_{j+n}y_j(t-\sigma) + b_{j+n}x_j(t-\tau), \end{cases} \tag{2.3.92}$$

where $a_j = s'_{1,j}(0), a_{j+n} = s'_{2,j}(0), b_{j+n} = g'_j(0), j = 1, 2, \ldots, n$ and $b_j = f'_j(0), j = 1, 2, \ldots, n-1,\ b_0 = b_n = f'_0(0)$. If we let $a_j = \beta$ for $j = 1, 2, \ldots, 2n$, and denote $A = \begin{pmatrix} 0 & 0 & 0 & \ldots & b_n \\ b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \ldots & b_{n-1} & 0 \end{pmatrix}$, $B = \begin{pmatrix} b_{n+1} & 0 & \ldots & 0 \\ 0 & b_{n+2} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & b_{2n}. \end{pmatrix}$. Then we can apply our results to this model to discuss the local stability and Hopf bifurcation, regarding the self-connection delay σ as the parameter. Note that in [10], β works as the parameter, and in [11], τ does that job. Using our main results, we can obtain the bifurcation analysis by varying σ and this together with [10] and [11] can enrich the bifurcation analysis for the neural networks with ring structure.

In the following, we restrict our attention to a special case: $\tau = 0$ and $b_j = b$ for $j = 1, 2 \ldots, 2n$. We then have $BA = \begin{pmatrix} 0 & 0 & 0 & \ldots & b^2 \\ b^2 & 0 & 0 & \ldots & 0 \\ 0 & b^2 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \ldots & b^2 & 0 \end{pmatrix}_{n\times n}$, which implies that

$$\sigma(BA) = \{\lambda_j, j = 1, 2, \ldots, n\}$$

with $\lambda_j = b^2 e^{i\theta_j}, \theta_j = \frac{(j-1)2\pi}{n}$. (In particular, if $n = 2$, this corresponds to the model investigated in [11]) and we have $\lambda_1 = b^2$ and $\lambda_2 = -b^2$.) Let

$$\omega_j^1 = \sqrt{\beta^2 - (1 + |b| \cos\frac{\theta_j}{2})^2} - |b| \sin\frac{\theta_j}{2}, \omega_j^2 = \sqrt{\beta^2 - (1 - |b| \cos\frac{\theta_j}{2})^2} + |b| \sin\frac{\theta_j}{2},$$

and

$$\sigma_j^1 = \frac{1}{\omega_j^1} \arccos\frac{1 + |b| \cos\frac{\theta_j}{2}}{\beta}, \quad \sigma_j^2 = \frac{1}{\omega_j^2} \arccos\frac{1 - |b| \cos\frac{\theta_j}{2}}{\beta},$$

for $j \in N(1, n)$. If $\omega_j^s \notin \mathbb{R}^+, s = 1, 2, j \in N(1, n)$, we denote the corresponding $\sigma_j^s = +\infty$. Set $\sigma_0 = \min\{\sigma_j^s : j \in N(1, n), s = 1, 2\} = \sigma_{j_0}^{s_0}$ for some $j_0 \in N(1, n)$ and $s_0 \in \{1, 2\}$. We denote $\lambda_0 = \lambda_{j_0} = b^2 e^{i\theta_0}$ and $\omega_0 = \omega_{j_0}^{s_0}$. Letting

$$\alpha_0 = 1 + i\omega_0 - \beta e^{-i\omega_0\sigma_0}, \ q_j = e^{-i(j\theta_0)}, \ p_j = \frac{\alpha_0}{b} q_j, \ p_j^* = \frac{\overline{\alpha}_0}{b} q_j, \ j \in N(1, n),$$

and

$$D = \frac{b}{2n\alpha_0(1 + \beta\sigma_0 e^{-i\omega_0\sigma_0})}.$$

This gives

$$C_1(0) = 3D \sum_{j=1}^{n} e^{-i(2j\theta_0)} \left[\frac{e^{-i\omega_0\sigma_0}}{b} (\overline{\alpha}_0 \alpha_j + \alpha_0 \gamma_j) + d_j^* + d_{j-1} e^{i\theta_0} \right]$$

and

$$\mu_2 = -Re(C_1(0)), \tag{2.3.93}$$

where $\gamma_j = \frac{s_{1,j}'''(0)}{6}, \alpha_j = \frac{s_{2,j}'''(0)}{6}, d_j^* = \frac{g_j'''(0)}{6}$ and $d_{j-1} = \frac{f_{j-1}'''(0)}{6}, d_n = d_0$ for $j \in N(1, n)$.

Corollary 2.3.4. *Suppose that $b \neq 0$.*

(1) *If $\beta < 1 - |b|$, then the zero solution of (2.3.89) is asymptotically stable at $\sigma = 0$.*

(2) *If $\beta < -|1-|b||$, then the zero solution of (2.3.89) is asymptotically stable for $\sigma \in [0, \sigma_0)$ and unstable if $\sigma > \sigma_0$.*

(3) *Hopf bifurcation occurs at $\sigma = \sigma_0$ and its direction and stability are determined by μ_2 given by (2.3.93), namely, the Hopf bifurcation is supercritical (subcritical) and stable (unstable) if $\mu_2 > 0 (\mu_2 < 0)$.*

For example, taking $k = 4, \tau = 0$, $s_j(x) = -2\tanh(x)$ and $h_j(x) = 2\tanh(x)$ in (2.3.89), then we have $\sigma_0 = 0.5612, \lambda_0 = -4, \omega_0 = 3.7321, \theta_0 = \pi$, and $\mu 2 > 0$. This shows that in the case $k = 4, \beta = -2$, and $b = 2$, Hopf bifurcation occurs at $\sigma = 0.5612$, which is supercritical and the bifurcated periodic solutions are asymptotically orbitally stable. The corresponding numerical simulations are presented in Figs.2.11–2.12.

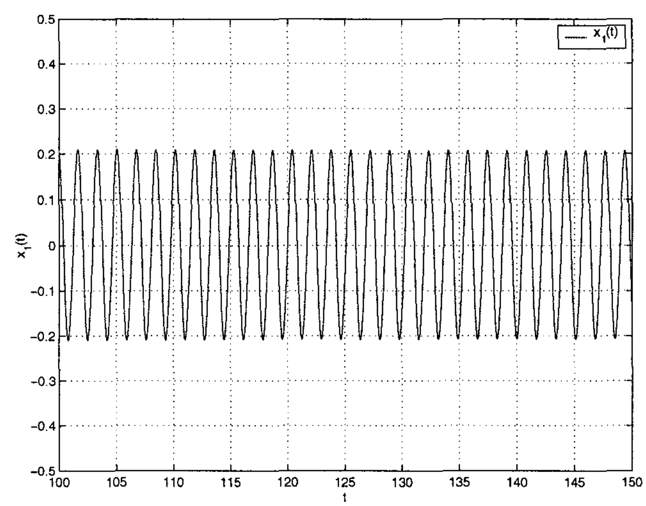

Figure 2.11: A periodic solution of (2.3.89) bifurcates from zero solution at $\sigma = 0.57$. Here $b = 2, \beta = -2, k = 4$, the component $x_1(t)$ is shown and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

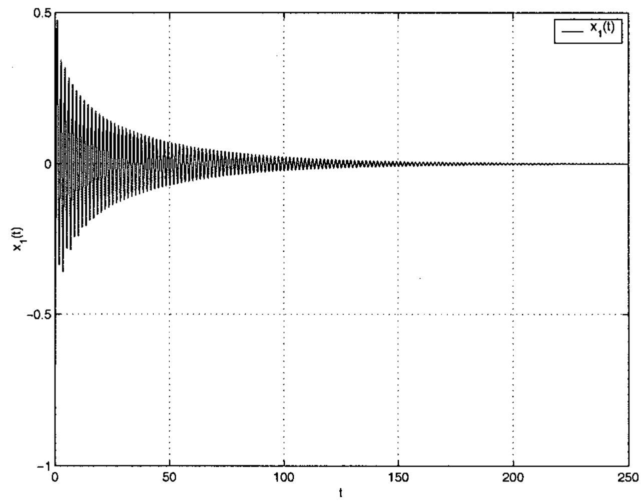

Figure 2.12: The zero solution of (2.3.89) is locally stable when $\sigma = 0.55 < \sigma_0$. Here $b = 2, \beta = -2, k = 4$, the component $x_1(t)$ is shown and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

If $k = 6$ and $s_j(x)$ and $h_j(x)$ remain the same, then we can compute that $\sigma_0 = 0.4209$, $\lambda_0 = -4(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = -4e^{i\theta_0}$, $\theta_0 = \frac{2\pi}{3}$ and $\mu_2 > 0$. This shows supercritical Hopf bifurcation occurs at $\sigma = 0.4209$ and the bifurcating periodic solutions are asymptotically orbitally stable.

Remark 2.3.4. *Our result works for (2.3.89) whenever k is an even number.*

2.3.2.5 Discussions

We have investigated the stability including both global and local stability and Hopf bifurcation for the BAM neural networks with delayed self–feedback. An effective algorithm to determine the direction and stability of the Hopf bifurcation is developed for a special case ($\tau = 0$). In the case $\tau \neq 0$, a similar algorithm can be expected. As an example, we showed that a ring structured neural network model with even number neurons could be reorganized as a BAM model and thus our results and algorithm obtained in this section are applicable to such a neural network. Indeed we successfully employed our results to consider the bifurcation of a simplified neural network model with ring structure (2.3.89) by viewing the self-connection delay as a parameter. Several specific examples and their numerical simulations were also presented to demonstrate our results.

Chapter 3

Dynamics of Discrete-time Neural Networks

In this chapter, we study the dynamics of delayed discrete-time neural networks. When a neural network is updated discretely, the model describing the network is in the form of a system of difference equations (See, e.g., Hopfield [50]). On the other hand, in numerical simulations and practical implementation of a continuous-time neural network, discretization is needed, leading again to a system of difference equations. Therefore, it is of both theoretical and practical importance to study the dynamics of discrete-time neural networks. For the same reasons as stated in the introduction of this thesis, we will incorporate time delays into the network models. More precisely, in Section 3.1, we will discuss the exponential stability of the discrete-time neural networks with variable delay and establish some criteria based on linear matrix inequalities (LMIs) to guarantee the global exponential stability and obtain some componentwise exponential stability results by using embedding technique. Section 3.2 is devoted to the study of the capacity for storing stable periodic solutions in the discrete-time BAM neural networks.

3.1 Exponential stability of discrete-time neural networks with variable delay

Consider the discrete-time neural network model with variable delay

$$x_i(n+1) = a_i x_i(n) + \sum_{j=1}^{m} w_{ij} g_j(x_j(n-k(n))) + I_i, n \in N(0),\ i \in N(1,m) \quad (3.1.1)$$

where $k(n)$ are positive integers with $0 \le k(n) \le k$, $a_i \in (0,1)$.

System (3.1.1) can be regarded as the discrete analog of the continuous-time Hopfield neural network model

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{m} w_{ij} g_j(x_j(t-\tau(t))) + I_i,\ i \in N(1,m). \quad (3.1.2)$$

However, generally speaking, the dynamics of discrete-time neural networks may be quite different from that of the continuous-time ones. For instance, the stability criteria established for system (3.1.2) may not be applicable to system (3.1.1). In the literature, there have been some papers (e.g., [23], [52], [53], [54], [55] and [68]) discussing the dynamics, including the stability, of some discrete-time neural networks. But for the *delayed* discrete-time neural networks, stability, especially exponential stability results are very few in the literature, in contrast to continuous-time neural networks with delays.

Our aim in this section is to investigate the exponential stability of system (3.1.1) by combining Liapunov function method, comparison method for monotone system and LMI approach. The latter approach has recently been used in [65] and [88]. Note that in terms of LMIs, our criteria can be tested by efficient and reliable algorithms [5].

3.1.1 LMI based criteria for exponential stability

We use the following notations: $Z^+ = N(0)$; $\lambda(W)$: the set of eigenvalues of the matrix W; $\lambda_M(W)$: the largest eigenvalue of the symmetric matrix W; $\lambda_m(W)$: the smallest eigenvalue of the symmetric matrix W; W^T: the transpose of the matrix W; W^{-1}: the inverse of the matrix W; $||x|| = (\sum_{i=1}^m x_i^2)^{\frac{1}{2}}$: the Euclidean norm of the vector $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m$ and $||W|| = ||W||_2$: the matrix norm induced by the Euclidean vector norm.

The initial conditions associated with (3.1.1) are of the form

$$x_i(s) = \phi_i(s), i = N(1, m), s \in N(-k, 0). \tag{3.1.3}$$

Throughout this subsection, we assume

(H) For each $i \in N(1, m)$, $g_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with

$$\sup_{u,v\in\mathbb{R},u\neq v} \frac{|g_i(u) - g_i(v)|}{|u - v|} = l_i,$$

and $|g_i(u)| \leq M_i, u \in \mathbb{R}, M_i > 0$.

If we let $x = (x_1, x_2, \ldots, x_m)^T$, $A = diag(a_1, a_2, \ldots, a_m)$, $W = (w_{ij})_{n\times n}$, $I = (I_1, I_2, \ldots, I_m)^T$, and $g(x(n)) = (g_1(x_1(n)), g_2(x_2(n)), \ldots, g_m(x_m(n)))^T$, then (3.1.1) can be written in matrix form:

$$x(n + 1) = Ax(n) + Wg(x(n - k(n)) + I, n \in N(0). \tag{3.1.4}$$

As usual, a vector $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ is said to be an equilibrium of (3.1.4) if it satisfies

$$x^* = Ax^* + Wg(x^*) + I.$$

Based on our assumption on the activation functions, it is easily seen that (3.1.4) admits at least one equilibrium.

In what follows, $S > (\geq)\ 0$ means the matrix S is symmetric and positive definite(semi-positive definite). From the theory of matrices, we have the following facts

Lemma 3.1.1. **(i)** *If $A > 0$, $B \geq 0$, $\alpha > 0$, then $A + B > 0$, $\alpha A > 0$;*

(ii) $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} > 0$ *if and only if $A_{11} > 0$ and $A_{22} - A_{21}A_{11}^{-1}A_{12} > 0$;*

(iii) *For any real matrices A, B, C and a scalar $\epsilon > 0$ with $C > 0$, the inequality*

$$A^T B + B^T A \leq \epsilon A^T C A + \epsilon^{-1} B^T C^{-1} B$$

holds.

Proof. (i), (ii) and (iii) can be found in [30] and [87], respectively. □

If we let $y(n) = x(n) - x^*$ and $f(y(n)) = g(x(n)) - g(x^*)$, then the stability of equilibrium x^* of (3.1.4) corresponds to that of the zero solution of the system

$$y(n+1) = Ay(n) + Wf(y(n - k(n))), \tag{3.1.5}$$

where f has the property:

$$||f(y)|| \leq ||L||||y||, \tag{3.1.6}$$

with $L = diag(l_1, l_2, \ldots, l_m)$.

We are now in a position to state our main results in this subsection, which are based on LMIs approach.

Theorem 3.1.1. *Assume that the variable delay $k(n)$ is bounded, satisfying $0 \leq k(n) \leq k$ and $\Delta k(n) = k(n+1) - k(n) < 1$. If there exist two scalars $q > 1, \epsilon > 0$ and two matrices $P > 0, R > 0$ such that*

$$\begin{pmatrix} R & W^T PA \\ \frac{q}{\epsilon} APW & P - qAPA - LQL \end{pmatrix} > 0, \tag{3.1.7}$$

then the equilibrium x^* of (3.1.4) is exponentially stable. More precisely, for any solution $x(n)$ of (3.1.4), the inequality

$$||x(n) - x^*||^2 \leq q^{-n} C_1 \sup_{s \in N(-k,0)} ||x(s) - x^*||^2, \tag{3.1.8}$$

holds, where

$$C_1 = \frac{\lambda_M(P) + \delta \lambda_M(Q)||L||^2}{\lambda_m(P)}, \text{ with } \delta = \frac{1 - (1/q)^k}{q - 1}$$

and

$$Q = q^{1+k}\epsilon R + q^{1+k} W^T PW > 0.$$

Proof. Define $V(n) = V(y(n))$ by

$$V(n) = q^n y^T(n) Py(n) + \sum_{s=n-k(n)}^{n-1} q^s f^T(y(s)) Qf(y(s)). \tag{3.1.9}$$

Then

$$\begin{aligned}
\Delta V(n) &= V(n+1) - V(n) \\
&= q^{n+1} y^T(n+1) Py(n+1) - q^n y^T(n) Py(n) \\
&\quad + \sum_{s=n+1-k(n+1)}^{n} q^s f^T(y(s)) Qf(y(s)) - \sum_{s=n-k(n)}^{n-1} q^s f^T(y(s)) Qf(y(s)) \\
&\leq q^{n+1} \left(Ay(n) + Wf(y(n-k(n)))\right)^T P \left(Ay(n) + Wf(y(n-k(n)))\right) \\
&\quad - q^n y^T(n) Py(n) + q^n f^T(y(n)) Qf(y(n)) \\
&\quad - q^{n-k(n)} f^T(y(n-k(n))) Qf(y(n-k(n))),
\end{aligned}$$

which further gives

$$\begin{aligned}
\Delta V(n) \;\le\;& q^{n+1}y^T(n)APAy(n) - q^n y^T(n)Py(n) + q^n f^T(y(n))Qf(y(n)) \\
&+q^{n+1}[y^T(n)APWf(y(n-k(n))) + f^T(y(n-k(n)))W^TPAy(n)] \\
&+f^T(y(n-k(n)))(q^{n+1}W^TPW - q^{n-k(n)}Q)f(y(n-k(n))).
\end{aligned}$$

From Lemma 3.1.1 (iii), we have

$$\begin{aligned}
&y^T(n)APWf(y(n-k(n))) + f^T(y(n-k(n)))W^TPAy(n) \\
&\le \epsilon f^T(y(n-k(n)))Rf(y(n-k(n))) + \tfrac{1}{\epsilon}y^T(n)APWR^{-1}W^TPAy(n).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Delta V(n) \;\le\;& -q^n y^T(n)\left(P - qAPA - LQL - \frac{q}{\epsilon}APWR^{-1}W^TPA\right)y(n) \\
&-q^{n-k(n)}f^T(y(n-k(n)))\left(Q - q^{1+k(n)}(\epsilon R + W^TPW)\right)f^T(y(n-k(n))).
\end{aligned}$$

Recalling that $Q = q^{1+k}(\epsilon R + W^TPW)$, we know from Lemma 3.1.1 that $Q > 0$ and $Q - q^{1+k(n)}(\epsilon R + W^TPW) > 0$. This shows that

$$\Delta V(n) \le -q^n y^T(n)\Omega y(n),$$

where $\Omega = P - qAPA - LQL - \frac{q}{\epsilon}APWR^{-1}W^TPA$. Condition (3.1.7) and Lemma 3.1.1–(ii) imply that $\Omega > 0$ and hence

$$\Delta V(n) \le 0.$$

Therefore, we have

$$\begin{aligned}
V(n) \le V(0) \;=\;& y^T(0)Py(0) + \sum_{s=-k(0)}^{-1} q^s f^T(y(s))Qf(y(s)) \\
\le\;& \lambda_M(P)||y(0)||^2 + \sum_{s=-k}^{-1}\lambda_M(Q)||L||^2||y(s)||^2 \\
=\;& \left(\lambda_M(P) + \delta\lambda_M(Q)||L||^2\right)\sup_{s\in N(-k,0)}||y(s)||^2.
\end{aligned}$$

On the other hand, from the definition of $V(n)$ that

$$V(y(n)) \geq q^n \lambda_m(P)||y(n)||^2.$$

We then obtain

$$||y(n)||^2 \leq q^{-n} \frac{\lambda_M(P) + \delta\lambda_M(Q)||L||^2}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||y(s)||^2,$$

which gives (3.1.8) and thus the proof is complete. □

Theorem 3.1.2. *Assume that there are two matrices $P > 0$, $\Sigma > 0$ and a scalar $\sigma \in (0,1)$ such that*

$$\begin{pmatrix} \Sigma & W^T PA \\ APW & \sigma P - APA \end{pmatrix} > 0 \tag{3.1.10}$$

and

$$\lambda_M(\Sigma + W^T PW)||L||^2 < \lambda_m(P)(1-\sigma). \tag{3.1.11}$$

Then every solution of (3.1.4) is exponentially stable with

$$||x(n) - x^*||^2 \leq C_2 \sigma^{\bar{\gamma} n} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2, \tag{3.1.12}$$

where

$$C_2 = \frac{\lambda_M(P)}{\lambda_m(P)(1 - C_3(\bar{\gamma}))}, \; C_3(\gamma) = \frac{\lambda_M(\Sigma + W^T PW)||L||^2}{\lambda_m(P)\sigma^{\gamma k}(\sigma^\gamma - \sigma)}$$

and

$$\bar{\gamma} = \sup\{\gamma \in (0,1) : 0 < C_3(\gamma) < 1\}.$$

Proof. Define $V(n) = V(y(n)) = y^T(n)Py(n)$, then we have

$$\lambda_m(P)||y(n)||^2 \leq V(n) \leq \lambda_M(P)||y(n)||^2 \tag{3.1.13}$$

and

$$\begin{aligned}
\Delta V(n) &= y^T(n+1)Py(n+1) - y^T(n)Py(n) \\
&= (Ay(n) + Wf(y(n-k(n))))^T P\,(Ay(n) + Wf(y(n-k(n)))) \\
&\quad -y^T(n)Py(n) \\
&= y^T(n)(APA - P)y(n) + f^T(y(n-k(n)))W^T PAy(n) \\
&\quad +y^T(n)APWf(y(n-k(n))) + f^T(y(n-k(n)))W^T PWf(y(n-k(n)))
\end{aligned}$$

Using Lemma 3.1.1–(iii), we can further have

$$\begin{aligned}
\Delta V(n) &\le y^T(n)[-P + APA + APW\Sigma^{-1}W^T PA]y(n) \\
&\quad +f^T(y(n-k(n)))[\Sigma + W^T PW]f(y(n-k(n))) \\
&\le y^T(n)[-P + APA + APW\Sigma^{-1}W^T PA]y(n) \\
&\quad +\lambda_M(\Sigma + W^T PW)||L||^2||y(n-k(n))||^2 \\
&= -(1-\sigma)y^T(n)Py(n) - y^T(n)[\sigma P - APA - APW\Sigma^{-1}W^T PA]y(n) \\
&\quad +\lambda_M(\Sigma + W^T PW)||L||^2||y(n-k(n))||^2
\end{aligned}$$

Notice that condition (3.1.10) and Lemma 3.1.1–(ii) imply that $\sigma P - APA - APW\Sigma^{-1}W^T PA > 0$. This shows that

$$\Delta V(n) \le -(1-\sigma)V(n) + \lambda_M(\Sigma + W^T PW)||L||^2||y(n-k(n))||^2, n \in N(1),$$

and hence we have

$$V(n) \le \sigma^n V(0) + \lambda_M(\Sigma + W^T PW)||L||^2 \sum_{s=0}^{n-1} \sigma^{n-1-s}||y(s-k(s))||^2. \tag{3.1.14}$$

From (3.1.13), it follows that

$$V(0) \le \lambda_M(P)||y(0)||^2 \le \lambda_M(P)||\sup_{s\in N(-k,0)}||x(s) - x^*||^2.$$

Thus, (3.1.13) together with (3.1.14) shows that

$$||y(n)||^2 \le \sigma^n \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_4 \sum_{s=0}^{n-1} \sigma^{n-1-s} ||y(s-k(s))||^2, \quad (3.1.15)$$

where

$$C_4 = \frac{\lambda_M(\Sigma + W^T P W)||L||^2}{\lambda_m(P)}.$$

Condition (3.1.11) guarantees that $\bar{\gamma} \in (0,1)$ exists. Multiplying both sides of (3.1.15) by $\sigma^{-\bar{\gamma}n}$, we have

$$\begin{aligned} \sigma^{-\bar{\gamma}n}||y(n)||^2 &\le \sigma^{(1-\bar{\gamma})n} \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 \\ &\quad + C_4 \sum_{s=0}^{n-1} \sigma^{n-1-s-\bar{\gamma}n} ||y(s-k(s))||^2 \\ &\le \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_4 \sigma^{-1+(1-\bar{\gamma})n} \times \\ &\quad \sum_{s=0}^{n-1} \sigma^{-(1-\bar{\gamma})s} \sigma^{-\bar{\gamma}k(s)} \sigma^{-\bar{\gamma}(s-k(s))} ||y(s-k(s))||^2. \end{aligned}$$

Letting

$$z(n) := \sup_{s\in[-k,n]} \sigma^{-\bar{\gamma}s} ||y(s)||^2 \quad (3.1.16)$$

and noticing that $k(n) \le k$ and $\sigma \in (0,1)$, we obtain

$$\begin{aligned} \sigma^{-\bar{\gamma}n}||y(n)||^2 &\le \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_4 \sigma^{-\bar{\gamma}k} \frac{1}{\sigma^{\bar{\gamma}} - \sigma} z(n) \\ &= \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_3(\bar{\gamma}) z(n), \end{aligned}$$

which shows that

$$
\begin{aligned}
z(n) &= \sup_{s\in[-k,n]} \sigma^{-\bar{\gamma}s}||y(s)||^2 \\
&\leq \sup_{s\in[-k,n]} \left(\frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_3(\bar{\gamma})z(s)\right) \\
&= \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2 + C_3(\bar{\gamma})z(n).
\end{aligned}
$$

Therefore,

$$
(1-C_3(\bar{\gamma}))z(n) \leq \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s\in N(-k,0)} ||x(s)-x^*||^2.
$$

This indicates that

$$
||y(n)||^2\sigma^{-\bar{\gamma}n} \leq z(n) \leq C_2 \sup_{s\in N(-k,0)} ||x(s)-x^*||^2,
$$

that is,

$$
||y(n)||^2 \leq \sigma^{\bar{\gamma}n} C_2 \sup_{s\in N(-k,0)} ||x(s)-x^*||^2.
$$

This shows the proof is complete. □

Remark 3.1.1. *Theorem 3.1.1 and Theorem 3.1.2 show that the equilibrium of (3.1.4) is unique under the hypotheses of these theorems.*

Remark 3.1.2. *In Theorem 3.1.1, we require $\Delta k(n) < 1$, while Theorem 3.1.2 only requires that $k(n)$ be bounded. In Theorem 3.1.1, the condition (3.1.7) is delay-dependent through the expression of Q, while the condition (3.1.10) in Theorem 3.1.2 is independent of the delay, even though the delay does have impact on the solution orbits which can be seen from (3.1.12).*

Remark 3.1.3. *Based on Theorem 3.1.1 and Theorem 3.1.2, we can determine an upper bound of q in (3.1.8) and a lower bound of σ in (3.1.12) so that the neural network (3.1.4) has rapid convergence. This requires us to solve the following optimization problems:*

$$\begin{cases} \textit{Max } q \\ \textit{Subject to } P > 0, R > 0 \textit{ and (3.1.7) is satisfied} \end{cases} \tag{3.1.17}$$

and

$$\begin{cases} \textit{Min } \sigma \\ \textit{Subject to } P > 0, \Sigma > 0 \textit{ and (3.1.10) and (3.1.11) are satisfied,} \end{cases} \tag{3.1.18}$$

*respectively. Note that (3.1.17) and (3.1.18) can be solved easily by the LMI Toolbox such as the Scilab developed by INRIA and ENPC in France, which is available at:*www-rocq.inria.fr/scilab/.

The following example demonstrates the feasibility of our main result.

Example 3.1.1. *Consider*

$$\begin{cases} x_1(n+1) = & 1/2x_1(n) + 1/4\tanh(x_1(n-1)) + 1/8\tanh(x_2(n-1)) \\ x_2(n+1) = & 1/2x_2(n) + 1/4\tanh(x_1(n-1)) + 1/16\tanh(x_2(n-1)). \end{cases} \tag{3.1.19}$$

In this example, $k(n) = k = 1$, $L = I$, $W = \begin{pmatrix} 1/4 & 1/8 \\ 1/4 & 1/16 \end{pmatrix}$, and if we take $R = I$, $P = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.8 \end{pmatrix}$, $\epsilon = 0.5$ and $q = 1.2$, we then find that $R > 0$, $P > 0$, $\epsilon > 0$, $q > 1$ and (3.1.7) holds. This shows, according to Theorem 3.1.1, that the zero solution of (3.1.19) is globally exponentially stable with the exponential decay rate less than $1/q = 5/6$.

3.1.2 Componentwise exponential stability

Under certain circumstances, one may wish to estimate the rate of convergence of *each or some* of the neurons in the network. This subsection deals with such componentwise convergence. To this end, we will employ the comparison method in monotone dynamical systems. Due to the variety of connections in a network, the network system may not be monotone and thus the comparison method can not be applied directly. Motivated by the work of Chu [23] and van den Driessche, Wu and Zou [96], we will first use an embedding technique to embed the model system into a monotone dynamical system with (double) higher dimension and then from the global componentwise convergence of the new system, we obtain that of the original system.

As is in Section 3.1.1, we only need to consider the stability of the zero solution of system (3.1.5), that is,

$$y(n+1) = Ay(n) + Wf(y(n-k(n))). \tag{3.1.20}$$

In this subsection, we assume that for each $i \in N(1,m)$, f_i satisfies

$$0 \leq \frac{f_i(u) - f_i(v)}{u - v} \leq l_i, \text{ for } u \neq v.$$

Denote $W^+ = (w_{ij}^+), W^- = (w_{ij}^-)$ with $w_{ij}^+ = \max\{w_{ij}, 0\}, w_{ij}^- = \max\{-w_{ij}, 0\}$ and $h(-s) = -f(s)$. It follows from $W = W^+ - W^-$ that (3.1.20) can be embedded into a $2m$-dimensional system

$$\begin{bmatrix} u(n+1) \\ v(n+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} u(n) \\ v(n) \end{bmatrix} + \begin{bmatrix} W^+ & W^- \\ W^- & W^+ \end{bmatrix} \begin{bmatrix} f(u(n-k(n)) \\ h(v(n-k(n)) \end{bmatrix}. \tag{3.1.21}$$

Let

$$z(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}, B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, C = \begin{bmatrix} W^+ & W^- \\ W^- & W^+ \end{bmatrix}, F(z(n)) = \begin{bmatrix} f(u(n)) \\ h(v(n)) \end{bmatrix},$$

then (3.1.21) can be rewritten as

$$z(n+1) = Bz(n) + CF(z(n-k(n))). \tag{3.1.22}$$

For system (3.1.22) we have the following comparison theorem.

Theorem 3.1.3. *Let $\phi(n)$ and $\psi(n)$ be two solutions of (3.1.22) with initial data $\phi(s), \psi(s), s \in N(-k,0)$. Then $\phi(n) \leq \psi(n)$ provided that $\phi(s) \leq \psi(s)$ for $s \in N(-k,0)$. Moreover, if $\phi(n)$ satisfies*

$$\phi(n+1) \geq B\phi(n) + CF(\phi(n-k(n))),\ n \geq 0,$$

and $z(n)$ is the solution of (3.1.22) with initial data $z(s), s \in N(-k,0)$, then $z(s) \leq \phi(s), s \in N(-k,0)$ implies $z(n) \leq \phi(n), n \geq 1$.

Proof. Taking advantage of the fact that both B and C are non-negative matrices, we can easily complete the proof by using the method of induction. □

A consequence of Theorem 3.1.3 is the following

Corollary 3.1.1. *Assume for system (3.1.20) and system (3.1.22) that initial data $y(s), \phi(s) = \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}, s \in N(-k,0)$ satisfy $-v(s) \leq y(s) \leq u(s), s \in N(-k,0)$, then the corresponding solutions $y(n)$ of (3.1.20) and $\phi(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}$ of (3.1.22)satisfy $-v(n) \leq y(n) \leq u(n)$ for $n \in N(1)$.*

In order to establish the componentwise exponential stability, we need to introduce the definition of *Class* K_0 and *Class* K for matrices.

Definition 3.1.1. *Let $A \in \{A = (a_{ik}), i,k = 1,\ldots,n;\ a_{ik} \leq 0, i \neq k\}$. The matrix A is said to be of class K_0 (respectively, K) if there is a vector $x > 0$ such that $Ax \geq 0$ (respectively, $Ax > 0$).*

Denoting $L = diag(l_1, \ldots, l_m)$, $D = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$ and the identity matrix with dimension m by I_m, and using the property of matrices of class K_0 and class K, we may establish our componentwise exponential stability result as follows.

Theorem 3.1.4. *Assume that there is a* $\sigma \in (0,1)$ *such that*

$$\Omega_1 := \sigma I_{2m} - B - \sigma^{-k} CD$$

is of class K_0. *Then the zero solution of (3.1.20) is componentwise (globally) exponentially stable in the sense that for every solution* $y(n)$ *of (3.1.20), there exist* ξ_0, $\eta_0 \in \mathbb{R}^m$ *with* $\xi_0 > 0$ *and* $\eta_0 > 0$ *such that*

$$-\sigma^n \eta_0 \le y(n) \le \sigma^n \xi_0.$$

Proof. $\Omega_1 \in K_0$ implies that there exists a vector $(\xi, \eta)^T \in \mathbb{R}^{2m}$ with $\xi \in \mathbb{R}^m, \eta \in \mathbb{R}^m$ and $\xi > 0, \eta > 0$ such that

$$\Omega_1 \begin{bmatrix} \xi \\ \eta \end{bmatrix} \ge 0.$$

Let $y(n)$ be a solution of (3.1.20) with given initial data $y(s), s \in N(-k, 0)$. We then can find a positive constant q such that $-q\eta \le y(s) \le q\xi$ for $s \in N(-k, 0)$. Denoting the solution of (3.1.22) with initial data $\phi(s) = \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = q \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, $s \in N(-k, 0)$, by $\phi(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}$, we then have $-v(n) \le y(n) \le u(n)$ for $n \in N(1)$. The fact that $\Omega_1 \begin{bmatrix} q\xi \\ q\eta \end{bmatrix} \ge 0$ implies that $z(n) = \begin{bmatrix} q\sigma^n \xi \\ q\sigma^n \eta \end{bmatrix}, n \in N(-k)$, satisfies

$$z(n+1) \ge Bz(n) + CDz(n - k(n)), \text{ for } n \ge 0,$$

which shows that $z(n) = \begin{bmatrix} q\sigma^n \xi \\ q\sigma^n \eta \end{bmatrix}, n \in N(1)$ is a solution of the following inequality

$$z(n+1) \ge Bz(n) + CFz(n - k(n)), \text{ for } n \ge 0,$$

with initial data $z(s) = \begin{bmatrix} q\sigma^s\xi \\ q\sigma^s\eta \end{bmatrix}$, $s \in N(-k, 0)$. The fact that $\phi(s) = q\begin{bmatrix} \xi \\ \eta \end{bmatrix} \leq \begin{bmatrix} q\sigma^s\xi \\ q\sigma^s\eta \end{bmatrix}$, $s \in N(-k, 0)$, and Theorem 3.1.3 imply that

$$\phi(n) \leq z(n) = \begin{bmatrix} q\sigma^n\xi \\ q\sigma^n\eta \end{bmatrix}, \ n \in N(1).$$

This indicates that

$$-\sigma^n\eta_0 := -q\sigma^n\eta \leq y(n) \leq q\sigma^n\xi =: \sigma^n\xi_0$$

for all $n \in N(1)$. Thus $y(n) \to 0$ exponentially and componentwise as $n \to \infty$ and the proof is complete. □

Corollary 3.1.2. *If*

$$\Omega_1' := I_{2m} - B - CD$$

is of class K, then the zero solution of (3.1.20) is componentwise (globally) exponentially stable in the sense of Theorem 3.1.4.

Proof. Ω_1' is of class K implies that there is a $\sigma \in (0,1)$ such that Ω_1 defined as in Theorem 3.1.4 is of class K_0. Therefore the proof follows from Theorem 3.1.4. □

Denoting $|W| = (|w_{ij}|)$, we have

Corollary 3.1.3. *If there exists a $\sigma \in (0,1)$ such $\Omega_2 := \sigma I_m - A - \sigma^{-k}|W|L$ is of class K_0 or equivalently if $\Omega_2' := I_m - A - |W|L$ is of class K, then the zero solution of (3.1.20) is componentwise (globally) exponentially stable in the sense of Theorem 3.1.4.*

Proof. Since $\Omega_2 := \sigma I_m - A - \sigma^{-k}|W|L$ is of class K_0, there exists a positive vector $\xi \in \mathbb{R}^m$ such that $\Omega_2 \xi \geq 0$. This implies that

$$\Omega_1 \begin{bmatrix} \xi \\ \xi \end{bmatrix} \geq 0,$$

which shows that Ω_1 is of class K_0 and then the conclusion follows from Theorem 3.1.4. □

Remark 3.1.4. *In [30], the matrices of class K_0 (K) are called $M-$matrices (non-singular $M-$matrices). Many other equivalent definitions are also available in [30]. For example, a matrix M is of class K if* **(a)** *all principal minors of M are positive; or* **(b)** *every real eigenvalue of M is positive.*

Remark 3.1.5. *The embedding technique used in the proofs of Theorem 2.2.9 and Theorem 3.1.4 was used by Chu [23] to get the specific performance for a class of discrete-time neural networks without delay; by van den Driessche, Wu and Zou [96] to obtain global attractivity for the continuous-time Hopfield neural networks with constant delays; by Wu and Zhao [111] for delayed differential systems and by Smith [92] for difference systems.*

Next, we give an example to demonstrate the componentwise exponential stability of a two-neuron network.

Example 3.1.2. *Consider*

$$\begin{cases} x_1(n+1) = & 1/2x_1(n) + 1/4\tanh(x_1(n-2)) - 1/4\tanh(x_2(n-2)) \\ x_2(n+1) = & 1/4x_2(n) - 1/8\tanh(x_1(n-2)) + 1/2\tanh(x_2(n-2)). \end{cases} \tag{3.1.23}$$

In this example, $m = 2$, $L = I_2$ and $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$, $W = \begin{pmatrix} 1/4 & -1/4 \\ -1/8 & 1/2 \end{pmatrix}$. It is easy to verify that $\Omega_2' = I_2 - A - |W|L = \begin{pmatrix} 1/4 & -1/4 \\ -1/8 & 1/4 \end{pmatrix}$ is of class K and thus, by Corollary 3.1.3, every component of each solution of (3.1.23) exponentially converges to zero.

3.1.3 Discussions

In this section, some LMI based criteria for the exponential stability and componentwise exponential stability are derived for the (autonomous) discrete-time neural networks with variable delay. The LMI based criteria have the advantages that they can be numerically verified by using LMI algorithms and the componentwise exponential stability can be obtained by examining if the related matrices are of class K or not, which admits many effective methods.

Note that the globally exponential stability achieved in this section shows that the equilibrium is unique under the stability conditions. Applying the theory in asymptotic autonomous systems, we may even establish some related convergence results for the asymptotic autonomous discrete-time neural networks. For instance, consider

$$x(n+1) = A(n)x(n) + W(n)g(x(n-k)) + I(n), n \in N(0), \tag{3.1.24}$$

where $A(n) \to A, W(n) \to W, I(n) \to I$ as $n \to \infty$. That is, the limiting system of (3.1.24) is system (3.1.4). Then we have

Theorem 3.1.5. *Assume that all conditions in Theorem 3.1.1 or in Theorem 3.1.2 or Theorem 3.1.4 are satisfied, then all solutions of (3.1.24) will converge to the unique equilibrium of the limiting system (3.1.4).*

Proof. By the variation of constants formula, it is easy to show that given any bounded initial data, the solution of (3.1.24) will be bounded. Under the assumptions, we know that the ω limit set of a solution sequence of (3.1.24) is an *internally chain transitive set* of its limiting system (3.1.4)(See [48] for the definition of internally chain transitive sets and their properties). On the other hand, by the *Strong Attractivity* Theorem (Theorem 1.2.1, [120]), under the given conditions, an internally chain transitive set of (3.1.4) is an equilibrium set. This shows that all solutions of (3.1.24) converge to the unique equilibrium of the limiting system (3.1.4). □

3.2 Capacity of Periodic Solutions in Discrete-Time Bidirectional Associative Memory Neural Networks

One of the main tasks that artificial neural networks can fulfil is associate memory. In associative memory neural network, the addressable memories or patterns are stored as stable equilibria or stable periodic solutions. Thus, for the purpose of the associate memories, it is desirable for the network to have as large capacity as possible for retrievable memories. In terms of the terminology of dynamical systems, this requires that the network admit as many as possible stable equilibria or stable periodic solutions.

In continuous-time models, the series papers [19]-[22] established the co-existence of multiple periodic solutions and described their domains of attraction. However, all these periodic solutions, except one, are unstable and they have large domains of attraction only in some sub-manifolds. On the contrary, for the discrete-time models, a large number of stable periodic solutions can possibly coexist. In this context, Zhou and Wu [121], [122] proved the existence of 2 stable periodic solutions with special periods for a class of discrete-time neural network model with two identical neurons. For this model, Zhang and Wu [109] recently explored the existence of periodic orbits with all possible periods and even provided a formula to compute the number of all possible stable periodic orbits. More recently, Wu, Zhang and Zou [110] extended the idea in [109] to a model with ring structure and showed that the number of neurons and the delays all have impacts on the periodic solutions capacity of the neural network model under certain conditions. One naturally wonders what would happen if the network has other types of connection structure. For general connection topology, it is very difficult, if not impossible, to answer this question. In this section, we will further consider a class of discrete-time neural network model with more trainable parameters and with another special connection topology: BAM models. As is seen in Chapter 2, a ring network with even number of neurons is a special case of BAM networks. More precisely, we study the delayed discrete-time BAM neural network model described by

$$\left\{\begin{array}{lll} x_i(n) & = & \beta_i x_i(n-1)+\sum_{j=1}^{m} a_{ij} f_j(y_j(n-k_j)) \\ y_i(n) & = & \alpha_i y_i(n-1)+\sum_{j=1}^{m} b_{ij} g_j(x_j(n-l_j)) \end{array}\right. \tag{3.2.1}$$

where $\beta_i, \alpha_i \in (0,1)$, $i \in N(1,m)$ are decay rates, a_{ij}, b_{ij}, $i,j \in N(1,m)$ are the connection weights between the neurons in two layers: $X-$layer with neurons whose states denoted by $x_i, i \in N(1,n)$ and $Y-$layer with neurons whose states denoted by $y_i, i \in N(1,n)$, and the positive integers k_i, l_i, $i \in N(1,m)$ are the associated delays due to the finite transmission speed among neurons in different layers in the network. The activation functions $f_i, g_i, i \in N(1,m)$ are of class $CL^{\epsilon}_{(r,R]}$, where

$$CL^{\epsilon}_{(r,R]} := \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \begin{array}{ll} |f(x) - 1| \leq \epsilon, & x \in (r, R], \\ |f(x) + 1| \leq \epsilon, & x \in [-R, -r) \end{array} \right\},$$

and the constants $\epsilon > 0, 0 \leq r \leq R$ as well as $\beta_i, \alpha_i \in (0,1)$, $i \in N(1,m)$ will be specified later. We will show that for this network, the delays, together with the size of the network, also have advantageous impact on the capacity of stable periodic solutions.

Note that the delays in (3.2.1) do not change the number of its equilibria. However, as we will show, they are related to the number of periodic solutions of (3.2.1) under certain assumptions and indeed the delayed discrete-time BAM neural networks can have large periodic solution capacity to store the paired patterns or memories.

3.2.1 Preliminaries

As usual, a solution of (3.2.1) is a sequence

$$\{(x_1(n), x_2(n), \ldots, x_m(n), y_1(n), y_2(n), \ldots, y_m(n))\}$$

of points in $\mathbb{R}^{2m}$ which is defined for every integer $n \geq -\max\{k_i, l_i, i \in N(1,m)\}$ and satisfies (3.2.1) for $n \geq 1$. In what follows, we denote

$$A = (a_{ij})_{m \times m}, \; B = (b_{ij})_{m \times m}, \; K = \sum_{i=1}^{m} k_i, \; L = \sum_{i=1}^{m} l_i$$

and suppose that: A is *strongly diagonally dominant*, that is,

$$a_{ii} > \sum_{j \neq i} |a_{ij}| =: \bar{A}_i, i \in N(1,m)$$

and B is *strongly quasi-diagonally dominant*, i.e.,

$$b_{ii+1} > \sum_{j \neq i+1} |b_{ij}| =: \bar{B}_i, i \in N(1,m), \text{ where } b_{mm+1} := b_{m1}.$$

Let

$$u_{i,j}(n) = x_i(n - l_i + j - 1), \ j = 1, 2, \ldots, l_i;$$

$$v_{i,j}(n) = y_i(n - k_i + j - 1), \ j = 1, 2, \ldots, k_i;$$

$$u_i(n) = (u_{i,1}(n), u_{i,2}(n), \ldots, u_{i,l_i}(n))^T \in \mathbb{R}^{l_i};$$

$$v_i(n) = (v_{i,1}(n), v_{i,2}(n), \ldots, v_{i,k_i}(n))^T \in \mathbb{R}^{k_i}$$

for $i \in N(1,m)$. Setting $\omega(n) := (u_1(n), v_1(n), u_2(n), v_2(n), \ldots, u_m(n), v_m(n))$ by

$$\omega(n) = (\omega_1(n), \omega_2(n), \ldots, \omega_{K+L}(n)) \in \mathbb{R}^{K+L},$$

and letting

$$\bar{k}_i := \sum_{j=1}^{i} k_j, \ \bar{k}_0 := 0, \ \bar{l}_i := \sum_{j=1}^{i} l_j, \ \bar{l}_0 := 0,$$

we then may rewrite (3.2.1) as

$$\omega(n+1) = F(\omega(n)), \tag{3.2.2}$$

where $F : \mathbb{R}^{K+L} \to \mathbb{R}^{K+L}$ is defined by

$$F_s(\omega) := \begin{cases} F_s(\omega), & s \in S := \{\bar{l}_i + \bar{k}_j, \ \ j = i-1, i, \ \text{ and } i \in N(1,m)\} \\ \omega_{s+1}, & s \in N(1, K+L) - S \end{cases} \tag{3.2.3}$$

with

$$F_{\bar{l}_i + \bar{k}_{i-1}}(\omega) := \beta_i \omega_{\bar{l}_i + \bar{k}_{i-1}} + \sum_{j=1}^{m} a_{ij} f_j(\omega_{\bar{l}_j + \bar{k}_{j-1} + 1})$$

and

$$F_{\bar{l}_i + \bar{k}_i}(\omega) := \alpha_i \omega_{\bar{l}_i + \bar{k}_i} + \sum_{j=1}^{m} b_{ij} g_j(\omega_{\bar{l}_{j-1} + \bar{k}_{j-1} + 1}).$$

We denote the solution of (3.2.2) with initial value $w(0)$ by $\omega(n, w(0)), n = 1, 2, \ldots$. For $\omega = (\omega_1, \ldots, \omega_{K+L}) \in \mathbb{R}^{K+L}$, its norm is defined by

$$||\omega|| = \max\{|\omega_j|, j \in N(1, K+L)\}.$$

Let

$$d = \max\left\{ \frac{\sum_{j=1}^{m} |a_{ij}|}{1-\beta_i},\ \frac{\sum_{j=1}^{m} |b_{ij}|}{1-\alpha_i},\ i \in N(1,m) \right\}.$$

We assume that the following holds:

$$(DH_1): \begin{cases} 0 < \beta_i < \frac{1}{2}(1 - \frac{\bar{A}_i}{a_{ii}}),\ \ 0 < \alpha_i < \frac{1}{2}(1 - \frac{\bar{B}_i}{b_{ii+1}}), \\ \epsilon < \min_{i \in N(1,m)} \left\{ \begin{array}{l} \frac{1-2\beta_i}{1-\beta_i} \frac{a_{ii}}{\bar{A}_i}[1 - (1-2\beta_i)\frac{\bar{A}_i}{a_{ii}}] \\ \frac{1-2\alpha_i}{1-\alpha_i} \frac{b_{ii+1}}{\bar{B}_i}[1 - (1-2\alpha_i)\frac{\bar{B}_i}{b_{ii+1}}] \end{array} \right\}, \\ R > d(1+\epsilon) =: b^*, \\ r < \min_{i \in N(1,m)} \left\{ \begin{array}{l} a_{ii} - \bar{A}_i - \beta_i b^* - \sum_{j=1}^{m} |a_{ij}|\epsilon \\ b_{ii+1} - \bar{B}_i - \alpha_i b^* - \sum_{j=1}^{m} |b_{ij}|\epsilon \end{array} \right\} =: a^*. \end{cases}$$

Let $r_* := \min\{R - b^*, a^* - r\}$ and define

$$a_c := a^* - c,\ b_c := b^* + c,\ \text{for } c \in [0, r_*).$$

In the sequel, we will use the following notations:

$$sgn(x) := \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \quad \text{for } x \in \mathbb{R}.$$

$$sgn(x) = (sgn(x_1), \ldots, sgn(x_m)),\ \text{for } x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

$$\Sigma := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{K+L}) \in \mathbb{R}^{K+L}; \sigma_j \in \{-1, 1\}, j \in N(1, K+L)\}$$

$$CL^{Lip}_{(r,R]} = \{f : \mathbb{R} \to \mathbb{R}; |f(x) - f(y)| \leq Lip(f)|x-y|, x, y \in [-R, -r) \cup (r, R]\}$$

$$\Omega := \{\omega \in \mathbb{R}^{K+L}; ||\omega|| \in (r, R)\}$$

$$\Omega(\sigma, c) := \{\omega \in \mathbb{R}^{K+L}; ||\omega|| \in [a_c, b_c], sgn(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}$$

$$\Omega(\sigma) := \{\omega \in \mathbb{R}^{K+L}; ||\omega|| \in (r, R), sgn(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}$$

$$\Omega^*(\sigma) := \{\omega \in \mathbb{R}^{K+L}; ||\omega|| \in (r, b^*), sgn(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}.$$

We point out that $CL^{\epsilon}_{(r,R]}$ and $CL^{Lip}_{(r,R]}$ include those frequently used sigmoid functions when r and R are properly chosen.

3.2.2 Multiplicity of stable periodic solutions

Define a mapping $\pi : \Sigma \to \Sigma$ by for any $\sigma \in \Sigma$

$$(\pi\sigma)_j = \begin{cases} \sigma_{j+1}, & \text{for } j \in N(1, K+L-1) \\ \sigma_1, & \text{for } j = K+L \end{cases} \tag{3.2.4}$$

For $p \geq 2$, the mapping $\pi^p : \Sigma \to \Sigma$ is given by

$$\pi^p \sigma = \pi(\pi^{p-1}\sigma)$$

and it follows that

$$(\pi^p\sigma)_j = \begin{cases} \sigma_{j+p}, & \text{for } j \in N(1, K+L-p) \\ \sigma_{j-K-L+p}, & \text{for } j = K+L-p+1, \ldots, K+L \end{cases}$$

and

$$\pi^{K+L}\sigma = \sigma, \quad \forall \sigma \in \Sigma.$$

We denote by

$$\Sigma_p := \{\sigma \in \Sigma : \pi^p\sigma = \sigma, \pi^q\sigma \neq \sigma, q \in \{1, 2, \ldots, p-1\}\}$$

the set of all $p-$periodic points of π in Σ for $p = 1, 2, \ldots$. Thus Σ_1 is the set of all fixed points of π in Σ.

The following lemma is needed in the proofs of our main results.

Lemma 3.2.1. *Assume that K and L are positive integers. Then*

(i) *for each $p \in N(1, K+L), \Sigma_p \neq \emptyset \Longleftrightarrow p \mid K+L$;*

(ii)

$$\Sigma = \bigcup_{p|K+L} \Sigma_p\,;$$

(iii) *for each* $p \in N(1, K+L)$*, the number of elements in* Σ_p*, denoted by* $N(\Sigma_p)$*, is given by*

$$N(\Sigma_p) = \begin{cases} 2, & p = 1, \\ 2^p - 2, & p \text{ is prime}, \\ 2^p - \sum_{q|p, q<p} N(\Sigma_q), & \text{otherwise}. \end{cases}$$

Proof. See [109] or [110] □

We next give an existence result for periodic solutions of system (3.2.2).

Theorem 3.2.1. *Assume that* (DH$_1$) is satisfied and $f_i, g_i \in CL^{\epsilon}_{(r,R]}$ for $i \in N(1, m)$. Then for any p and σ with $p \mid K + L$ and $\sigma \in \Sigma_p$, (3.2.2) has a $p-$periodic solution $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$.

Proof. We first show that for any $\sigma \in \Sigma$ and $c \in [0, r_*)$,

$$F : \Omega(\sigma, c) \to \Omega(\pi\sigma, c).$$

Define

$$h_1(z) := \beta_1 z_0 + \sum_{j=1}^{m} a_{ij} f_j(z_j)$$

where

$$z = (z_0, z_1, \ldots, z_m) \in \mathbb{R}^{m+1}, \text{ with } |z_j| \in [a_c, b_c], j = 0, 1, \ldots, m.$$

We claim that

$$sgn(h_1(z)) = sgn(z_1), \text{ and } |h_1(z)| \in [a_c, b_c].$$

To this end, we have two cases: 1)$z_1 \geq 0$; 2)$z_1 < 0$, to be considered. If $z_1 \geq 0$, we then have

$$h_1(z) \leq \beta_1 b_c + a_{11}(1+\epsilon) + \sum_{j\neq 1} |a_{1j}|(1+\epsilon) \leq b_c,$$

and

$$h_1(z) \geq -\beta_1 b_c + a_{11}(1-\epsilon) - \sum_{j\neq 1} |a_{1j}|(1+\epsilon) \geq a_c,$$

which are due to

$$(1-\beta_1)b_c \geq (1-\beta_1)b^* = (1-\beta_1)d(1+\epsilon^*) \geq \sum_{j=1}^{m} |a_{1j}|(1+\epsilon)$$

and

$$\begin{aligned} a_c + \beta_1 b_c &= a^* - c + \beta_1(b^* + c) = a^* + \beta_1 b^* - (1-\beta_1)c \\ &\leq a^* + \beta_1 b^* \leq a_{11} - \sum_{j\neq 1} |a_{1j}| - \sum_{j=1}^{m} |a_{1j}|\epsilon \\ &= a_{11}(1-\epsilon) - \sum_{j\neq 1} |a_{1j}|(1+\epsilon). \end{aligned}$$

Similarly, for case 2), we can show that

$$-b_c \leq h_1(z) \leq -a_c.$$

Therefore our claim is true. Using this argument and the definition of F, we can show that

$$|F_j(\omega)| \in [a_c, b_c], \text{ for } j = 1, 2, \ldots, K+L,$$

$$sgn(F_j(\omega)) = sgn(\omega_{j+1}) = \sigma_{j+1}, j = 1, 2, \ldots, K+L-1$$

and

$$sgn(F_{K+L}(\omega)) = sgn(\omega_1) = \sigma_1.$$

This shows that for any $\omega \in \Omega(\sigma, c)$,

$$F(\omega) \in \Omega(\pi\sigma, c).$$

Notice that $\Omega(\sigma, c)$ is convex and closed. Then for any $p \mid K + L$ and $\sigma \in \Sigma_p$, we have

$$F^p(\Omega(\sigma, c)) \subset \Omega(\pi^p\sigma, c) = \Omega(\sigma, c)$$

and hence the continuous mapping F^p admits a fixed point in $\Omega(\sigma, c)$, which is exactly a $p-$periodic solution, denoted by $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$, of (3.2.2) with initial value in $\Omega(\sigma, c)$. The proof is complete. □

Theorem 3.2.2. *In addition to the conditions in Theorem 3.2.1, assume that $f_i, g_i \in CL^{Lip}_{(r,R]}$ with*

$$J := \max_{i\in N(1,m)} \left\{ \beta_i + \sum_{j=1}^{m} |a_{ij}| Lip(f_j),\ \alpha_i + \sum_{j=1}^{m} |b_{ij}| Lip(g_j) \right\} < 1.$$

Then

(I) *For any $p \mid K + L$ and $\sigma \in \Sigma_p$, (3.2.2) has a unique $p-$periodic solution $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$ with $\omega^\sigma \in \Omega(\sigma, 0)$ and this solution is exponential stable in the sense that for any $\bar{\omega}^\sigma$ with $||\bar{\omega}^\sigma - \omega^\sigma|| < r(\sigma)$, we have*

$$||\omega(n, \bar{\omega}^\sigma) - \omega(n, \omega^\sigma)|| \leq C\xi^n ||\bar{\omega}^\sigma - \omega^\sigma||,$$

where

$$\xi := J^{\frac{1}{K+L}} < 1,\ C := \xi^{1-(K+L)} > 0$$

and

$$r(\sigma) := \min\{||\omega^\sigma|| - a^* + r_*, b^* + r_* - ||\omega^\sigma||\} > 0.$$

(II) *If* $\{\omega(n)\}_{n\in\mathbb{N}}$ *is a* p*−periodic solution of (3.2.2) in* Ω*, then* $p \mid K+L$ *and there exists a unique* $\sigma \in \Sigma_p$ *and some* $\omega^\sigma \in \Omega(\sigma, 0)$ *such that* $\omega(n) = \omega(n, \omega^\sigma)$.

(III) *For any solution* $\{\omega(n, \omega(0))\}_{n\in\mathbb{N}}$ *of (3.2.2) with* $||\omega(0)|| \in (a^* - r_*, b^* + r_*)$*, there exist a unique* $p \in \mathbb{N}$ *with* $p \mid K+L$ *and a unique* $\sigma \in \Sigma_p$ *such that*

$$|\omega(n, \omega(0)) - \omega(n, \omega^\sigma)|| \leq C\xi^n ||\omega(0) - \omega^\sigma||, \ n \in \mathbb{N}.$$

(IV) *For* $p \in \mathbb{N}$ *with* $p \mid K+L$*, (3.2.2) has* $N(\Sigma_p)$ p*−periodic solutions in* Ω*, which are all exponentially stable. If* $p \nmid K+L$*, (3.2.2) has no* p*−periodic solution in* Ω.

To prove this theorem, we first establish the following useful lemmas under the same assumptions.

Lemma 3.2.2. *For* $\omega', \omega'' \in \Omega(\sigma, c)(\Omega, \Omega^*(\sigma))$*, we have*

$$||F^{K+L}(\omega') - F^{K+L}(\omega'')|| \leq J||\omega' - \omega''||. \tag{3.2.5}$$

Proof. This can be easily proved by the fact that $J \in (0,1)$ and the definition of F. □

Lemma 3.2.3. *If* $\{\omega(n, \omega(0))\}_{n\in\mathbb{N}}$ *is a* p*−periodic solution of (3.2.2) in* Ω*, then* $||\omega(n, \omega(0))|| \leq b^*$.

Proof. Since $\{\omega(n, \omega(0))\}_{n\in\mathbb{N}}$ is a p−periodic solution of (3.2.2) in Ω, we can obtain a p−periodic solution $\{(x_1(n), \ldots, x_m(n), y_1(n), \ldots, y_m(n))\}_{n\in\mathbb{N}}$ for (3.2.1). We will show that

$$|x_i(n)| \leq d_i(1+\epsilon), \ \ |y_i(n)| \leq d_{i+m}(1+\epsilon),$$

where

$$d_i := \frac{\sum_{j=1}^m |a_{ij}|}{1-\beta_i}, \quad d_{i+m} := \frac{\sum_{j=1}^m |b_{ij}|}{1-\alpha_i}, \; i = 1, 2, \ldots, m.$$

By way of contradiction, suppose that for some i, there exists n_0 such that $|x_i(n_0)| > d_i(1+\epsilon)$, say, $x_i(n_0) = d_i(1+\epsilon) + \delta_0$ (the proof for the case $x_i(n_0) < -d_i(1+\epsilon)$ is similar) for some $\delta_0 > 0$. Then from (3.2.1), we have

$$\begin{aligned} x_i(n_0-1) &= \frac{1}{\beta_i}\left(x_i(n_0) - \sum_{j=1}^m a_{ij} f_j(y_j(n-k_j))\right) \\ &\geq \frac{1}{\beta_i}\left(d_i(1+\epsilon) + \delta_0 - \sum_{j=1}^m |a_{ij}|(1+\epsilon)\right) \\ &= \frac{1}{\beta_i}\delta_0 + d_i(1+\epsilon) \\ &> d_i(1+\epsilon) + \delta_0 \text{ (since } \beta_i < 1) \\ &= x_i(n_0). \end{aligned}$$

Repeating this procedure, we can show

$$x_i(n_0 - p) > x_i(n_0),$$

which is a contradiction. Thus we have shown that for all $n \in \mathbb{N}$,

$$|x_i(n)| \leq d_i(1+\epsilon), \quad |y_i(n)| \leq d_{i+m}(1+\epsilon),$$

which implies that

$$||\omega(n, \omega(0))|| \leq b^* := \max\{d_i(1+\epsilon), i \in \{1, 2, \ldots, 2m\}\}$$

and the proof is complete. □

Lemma 3.2.4. *If $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ is a $p-$periodic solution of (3.2.2) in Ω, then $p \mid K+L$ and $\omega(n,\omega(0)) = \omega(n,\omega^\sigma)$ for some $\sigma \in \Sigma_p$, and $\omega^\sigma \in \Omega(\sigma, c)$.*

Proof. Note that

$$\Omega = \bigcup_{q \mid K+L} \bigcup_{\sigma\in\Sigma_q} \Omega(\sigma). \tag{3.2.6}$$

Then there exist q and σ with $q \mid K+L$ and $\sigma \in \Sigma_q$ such that $\omega(0) \in \Omega(\sigma)$. From Lemma 3.2.3, we further know $\omega(n,\omega(0)) \in \Omega^*(\sigma)$. Moreover, for such q and σ, it follows from Theorem 3.2.1 and Lemma 3.2.3 that (3.2.2) has a $q-$periodic solution denoted by $\{\omega(n,\omega^\sigma))\}$ with $\omega(n,\omega^\sigma)) \in \Omega(\sigma, c)$ for $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, we have

$$\begin{aligned} ||\omega(n,\omega(0)) - \omega(n,\omega^\sigma)|| &= ||\omega(n+pq(K+L),\omega(0)) - \omega(n+pq(K+L),\omega^\sigma)|| \\ &= ||(F^{K+L})^{pq}(\omega(n,\omega(0))) - (F^{K+L})^{pq}(\omega(n,\omega^\sigma))|| \\ &\leq J^{pq}||\omega(n,\omega(0)) - \omega(n,\omega^\sigma)||, \end{aligned}$$

which shows that $\omega(n,\omega(0)) = \omega(n,\omega^\sigma)$ for $n \in \mathbb{N}$ and $q = p$ and hence $p \mid K+L$. $\square$

Now we are in the position to prove Theorem 3.2.2.

Proof of Theorem 3.2.2:

(I). The existence and the uniqueness follow from Theorem 3.2.1 and Lemma 3.2.4. We just need to show the exponential stability. For any $n \in \mathbb{N}$, we have $n = s(K+L) + q$ with $q \in \{1, 2, \ldots, K+L-1\}$, and then for any $\bar{\omega}^\sigma$ with

$||\bar{\omega}^{\sigma} - \omega^{\sigma}|| < r(\sigma)$, it follows from Lemma 3.2.2 that

$$\begin{aligned}
||\omega(n,\bar{\omega}^{\sigma}) - \omega(n,\omega^{\sigma})|| &= ||F^{s(K+L)+q}(\bar{\omega}^{\sigma}) - F^{s(K+L)+q}(\omega^{\sigma})|| \\
&\leq |F^{s(K+L)}(\bar{\omega}^{\sigma}) - F^{s(K+L)}(\omega^{\sigma})|| \\
&\leq J^{s}||\bar{\omega}^{\sigma} - \omega^{\sigma}|| \\
&= C\xi^{(s(K+L)+K+L-1}||\bar{\omega}^{\sigma} - \omega^{\sigma}|| \\
&\leq C\xi^{n}||\bar{\omega}^{\sigma} - \omega^{\sigma}||.
\end{aligned}$$

(II). The proof follows from Lemma 3.2.4.

(III). We may find a $c \in [0, r_*)$ such that $||\omega(0)|| \in [a_c, b_c]$. Now let $\sigma \in \Sigma$ with $\sigma = sgn(\omega(0))$, that is, $\omega^{\sigma} := \omega(0) \in \Omega(\sigma, c)$. Since $\Sigma = \bigcup_{p|K+L} \Sigma_p$, there must exist a unique $p \mid K + L$ such that $\sigma \in \Sigma_p$. For such σ and ω^{σ}, there exists a $p-$periodic solution $\{\omega(n,\omega^{\sigma})\}_{n\in\mathbb{N}}$. The rest of the proof follows from Lemma 3.2.2 and (I).

(IV). This follows from the definition of $N(\Sigma_p)$, (I) and Lemma 3.2.4.

Remark 3.2.1. *(I) gives a domain of attraction for each stable periodic solution of (3.2.2).*

It is possible for two periodic solutions to have the same orbit. To distinct orbits, we give a definition for equivalent periodic solutions:

Definition 3.2.1. *Two $p-$periodic solutions $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ and $\{\omega(n,\bar{\omega}(0))\}_{n\in\mathbb{N}}$ are said to be equivalent, denoted by*

$$\omega(n,\omega(0)) \sim \omega(n,\bar{\omega}(0)),$$

if there exists $q \in \{1, 2, \ldots, p-1\}$ such that

$$\omega(n,\omega(0)) = \omega(n+q,\bar{\omega}(0)).$$

In other words, two $p-$periodic solutions are equivalent if they generate the same orbit.

Lemma 3.2.5. *For any $p \mid K+L$ and any $\sigma, \bar{\sigma} \in \Sigma_p$ with $\sigma \neq \bar{\sigma}$, then the two $p-$periodic solutions $\{\omega(n, \omega^{\sigma})\}_{n\in\mathbb{N}}$ and $\{\omega(n, \omega^{\bar{\sigma}})\}_{n\in\mathbb{N}}$ generated by ω^{σ} and $\omega^{\bar{\sigma}}$ are equivalent if and only if there exists a $q \in \{1, 2, \ldots, p-1\}$ such that*

$$\bar{\sigma} = \pi^q \sigma, \ \ or \ \sigma = \pi^q \bar{\sigma}.$$

Proof. Suppose $\sigma, \bar{\sigma} \in \Sigma_p$ and $\bar{\sigma} = \pi^q \sigma$ for some $q \in \{1, 2, \ldots, p-1\}$. Note that $F^q : \Omega(\sigma, 0) \to \Omega(\pi^q \sigma, 0)$ and $\omega(n+q, \omega^{\sigma}) = \omega(n, F^q(\omega^{\sigma}))$, which implies that $\omega(n, F^q(\omega^{\sigma}))$ is a $p-$periodic solution with initial value $F^q(\omega^{\sigma}) \in \Omega(\pi^q\sigma, 0) = \Omega(\bar{\sigma}, 0)$. On the other hand, we know that $\omega(n, \omega^{\bar{\sigma}})$ is a $p-$periodic solution with initial value $\omega^{\bar{\sigma}} \in \Omega(\bar{\sigma}, 0)$ too. Therefore, we have

$$\omega(n, \omega^{\bar{\sigma}}) = \omega(n, F^q(\omega^{\sigma})) = \omega(n+q, \omega^{\sigma}), n = 0, 1, \ldots$$

That is, $\{\omega(n, \omega^{\sigma})\}_{n\in\mathbb{N}}$ and $\{\omega(n, \omega^{\bar{\sigma}})\}_{n\in\mathbb{N}}$ are equivalent. Next suppose

$$\omega(n, \omega^{\sigma}) \sim \omega(n, \omega^{\bar{\sigma}}),$$

and thus there exists $q \in \{1, 2, \ldots, p-1\}$ such that $\omega(0, \omega^{\sigma}) = \omega(q, \omega^{\bar{\sigma}})$. It follows that

$$\sigma = sgn(\omega(0, \omega^{\sigma})) = sgn(\omega(q, \omega^{\bar{\sigma}})) = sgn(F^q(\omega^{\bar{\sigma}})) = \pi^q \bar{\sigma}.$$

This completes the proof. $\square$

Consequently, we have

Corollary 3.2.1. *For any $p \mid K+L$ and any $\sigma \in \Sigma_p$, we have*

$$\omega(n, \omega^{\sigma}) \sim \omega(n, \omega^{\pi\sigma}) \sim \cdots \sim \omega(n, \omega^{\pi^{p-1}\sigma})$$

and

$$\omega(n, \omega^{\pi^i\sigma}) \nsim \omega(n, \omega^{\pi^j\sigma}), \text{ for } i, j \in \{1, 2, \ldots, p-1\} \text{ with } i \neq j.$$

If we use $n(p)$ to denote the number of all p–periodic orbits of (3.2.2) (and thus that of (3.2.1)), then we have

Theorem 3.2.3. $\forall p \in \mathbb{N}$ *with* $p \mid K+L$,

$$n(p) = \frac{N(\Sigma_p)}{p}.$$

Proof. The proof follows immediately from the definition of $N(\Sigma_p)$ and Corollary 3.2.1.

Remark 3.2.2. *The number of all periodic orbits of (3.2.1) is*

$$n(K+L) = \sum_{p \mid K+L} n(p).$$

The related numbers for $N(\Sigma_p)$, $n(p)$ and $n(K+L)$ are given in the following tables.

p	1	2	3	4	5	10	15	20
$N(\Sigma_p)$	2	2	6	12	30	990	32730	1047540
$n(p)$	2	1	2	3	6	99	2182	252377

Table 3.1: $N(\Sigma_p)$ and $n(p)$ for some p.

$K+L$	2	3	4	5	10	15	20
$p{:}p \mid K+L$	$1,2$	$1,3$	$1,2,4$	$1,5$	$1,2,5,10$	$1,3,5,15$	$1,2,4,5,10,20$
$n(K+L)$	3	4	6	11	108	2192	52488

Table 3.2: $n(K+L)$ for some $K+L$.

3.2.3 Discussions

We have shown that the delayed discrete–time bidirectional associative memory neural network (3.2.1) can admit $\sum_{p|K+L} n(p)$ stable periodic solutions. We have also investigated the relation between the number of periodic solutions and the sum of all delays $(K+L)$ and discussed the multi-stability of those periodic solutions. This shows that (3.2.1) is a network model admitting large capacity of stable periodic solutions and thus, has great potential for applications in associative memories of periodic patterns.

Note that for a simple two-neuron discrete-time neural network with delayed feedback, [109], [121] and [122] discussed the existence and stability of periodic solutions, and [109] also showed the large capacity of periodic solutions. However, in their models, there are just a few parameters, and thus as pointed out in [109], it is hard to train the network to store the large number of stable periodic solutions. In contrast, there are many parameters in (3.2.1), which can be used to train the network to have the ability to generate a large number of stable periodic solutions so that the network can serve the purpose storing large number of content-addressable memories or patterns.

Chapter 4

Dynamics of Stochastic Neural Networks

We have studied the dynamics of continuous-time neural networks and discrete-time neural networks in Chapter 2 and Chapter 3, where both type of neural network models are deterministic. However, in real nervous systems and in implementation of artificial neural networks, noise is unavoidable and should be taken into consideration in modelling the activation of neurons [44] . It is therefore important to study the stochastic neural networks. In this chapter, we will study the stability of the general stochastic Cohen-Grossberg neural networks and stochastic delayed Cohen-Grossberg neural networks by employing Liapunov methods, Razumikhin techniques and LMI approaches. Note that some stability results for the stochastic Hopfield neural networks with specific activation functions were established in [4], [66] and [67].

For the basic theory of stochastic differential equations and stochastic functional differential equations, we refer to [1], [71], [72] and [78].

The rest of this chapter is organized as follows. Section 4.1 is devoted to the study of stochastic Cohen–Grossberg neural networks. The almost sure exponential stability and mean square stability of stochastic Cohen–Grossberg neural networks with multiple delays are discussed in Section 4.2.

4.1 Stability of stochastic Cohen-Grossberg neural networks

Consider the stochastic Cohen-Grossberg neural network model described by

$$du(t) = -A(u(t))[b(u(t)) - Wg(u(t))]dt + \sigma(u(t))dB(t),\ t \geq 0, \tag{4.1.1}$$

where $u(t) = (u_1(t), \ldots, u_n(t))^T$ is the neuron states vector; $A(u(t)) = diag(a_i(u_i(t)))$; $b(u(t)) = (b_1(u(t)), \cdots, b_n(u(t)))^T$; $W = (w_{ij})_{n\times n}$ is the connection matrix; $g(u) = (g_1(u_1), \cdots, g_n(u_n))^T$ is the activation functions vector; $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix and $B(t) = (B_1(t), \ldots, B_n(t))^T$ is an $n-$dimensional Brownian motion.

From the standard textbook [71] on stochastic differential equation, we know that for any given initial data u_0 there is a unique solution denoted by $u(t; u_0)$ for the system (4.1.1) if we assume that $a_i(u), g_i(u), \sigma_{ij}(u)$ are locally Lipschitz and satisfy the linear growth condition.

In what follows, we assume that $b_i(0) = 0, g_i(0) = 0$ for $i \in N(1, n)$ and $\sigma(0) = \mathbf{0}$ so that $u = 0$ is a trivial (equilibrium) solution of (4.1.1).

For convenience, we introduce some definitions.

Definition 4.1.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, X and X_k, $k \geq 1$ the $\mathbb{R}^n$-valued random variables. If there exists a $P-$null set $\Omega_0 \in \mathcal{F}$ (meaning $P(\Omega_0) = 0$)*

such that for every $\omega \notin \Omega_0$, *the sequence* $\{X_k(\omega)\}$ *converges to* $X(\omega)$ *in the usual sense in* $\mathbb{R}^n$, *then* $\{X_k\}$ *is said to converge to* X *almost surely and we write*

$$\lim_{k\to\infty} X_k = X \quad a.s..$$

Definition 4.1.2. *The trivial solution of system (4.1.1) is said to be almost surely exponentially stable if*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|u(t;u_0)|) < 0 \quad a.s.$$

for all $u_0 \in \mathbb{R}^n$.

Definition 4.1.3. *The trivial solution of system (4.1.1) is said to be almost surely exponentially unstable if*

$$\liminf_{t\to\infty} \frac{1}{t}\log(|u(t;u_0)|) > 0 \quad a.s.$$

for all $u_0 \in \mathbb{R}^n$.

Throughout this chapter, we follow the standard notation to denote the mathematical expectation or mean of a random variable ξ by $E(\xi)$.

Definition 4.1.4. *The trivial solution of system (4.1.1) is said to be exponentially stable in mean square if*

$$\limsup_{t\to\infty} \frac{1}{t}\log(E|u(t;u_0)|^2) < 0$$

for all $u_0 \in \mathbb{R}^n$.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ denote the family of all nonnegative functions $V(u;t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are twice differentiable in x and once in t. For each such $V(u;t)$, we define an operator $\mathcal{L}V$ associated with (4.1.1) as

$$\begin{aligned}\mathcal{L}V(u;t) &= V_t(u;t) + V_u(u;t)(-A(u(t))[B(u(t)) - Wg(u(t))]) \\ &\quad +\tfrac{1}{2}trace[\sigma^T(u(t))V_{uu}(u;t)\sigma(u(t))],\end{aligned} \tag{4.1.2}$$

where

$$V_t(u;t) = \frac{\partial V(u;t)}{\partial t},\ V_u = \left(\frac{\partial V(u;t)}{\partial u_1}, \cdots, \frac{\partial V(u;t)}{\partial u_n}\right),\ V_{uu} = \left(\frac{\partial^2 V(u;t)}{\partial u_i \partial u_j}\right)_{n\times n}.$$

We will need the following assumptions:

(H1) for each $i \in N(1,n)$, there is $\gamma_i > 0$ such that $ub_i(u) \geq \gamma_i u^2$;

(H2) for each $i \in N(1,n)$, $|g_i(u)| \leq L_i|u|$;

(H3) for each $i \in N(1,n)$, $ug_i(u) > 0$ for $u \neq 0$;

(H4) for each $i \in N(1,n)$, $\alpha_i \leq a_i(u) \leq \bar{a}_i$;

(H5) $|\sigma(u)|^2 \leq k|u|^2$;

(H6) for each $i \in N(1,n)$, $0 \leq \frac{b_i(u)-b_i(v)}{u-v} \leq \beta_i\ \forall u, v \in \mathbb{R}$ with $u \neq v$.

Our special notations are as follows: $|u| = (\sum_{i=1}^n u_i^2)^{1/2}$ for $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$; for a matrix Q, $|Q| = \sqrt{trace(Q^TQ)}$ denotes its trace norm; $Q > 0 (\geq 0)$ means the matrix Q is symmetric positive (semi-positive) definite and $\lambda_m(Q)$ and $\lambda_M(Q)$ denote its smallest eigenvalue and largest eigenvalue, respectively.

The following lemma plays a crucial role in establishing our main results in this section.

Lemma 4.1.1. *Assume that there exists a symmetric positive definite matrix Q and two real numbers $\mu \in \mathbb{R}$ and $\rho \geq 0$ such that*

$$2u^TQ[-A(u)b(u) + A(u)Wg(u)] + trace(\sigma^TQ\sigma) \leq \mu u^TQu$$

and

$$|u^T Q\sigma(u)|^2 \geq \rho(u^T Qu)^2, \ \forall u \in \mathbb{R}^n. \tag{4.1.3}$$

Then we have

$$\limsup_{t\to\infty} \frac{1}{t}\log(|u(t;u_0)|) \leq -(\rho - \frac{\mu}{2}) \ \ a.s.$$

whenever $u_0 \neq 0$. *If* $\rho > \frac{\mu}{2}$, *then the trivial solution of (4.1.1) is almost surely exponentially stable.*

Proof. This lemma follows from Theorem 4.3.3 [71] by letting $V(u;t) = u^T Qu$. □

Based on this lemma, we establish our main results in this section as follows.

Theorem 4.1.1. *Suppose that* (H1), (H2), (H4) *and* (H5) *hold. Assume also that there are a matrix* $Q = diag(q_1, q_2, \ldots, q_n) > 0$, *a real number* $\rho \geq 0$ *and some positive constants* ξ_i, $i \in N(1,n)$ *such that (4.1.3) and*

$$\lambda := \min_{i\in N(1,n)} \left\{ 2\alpha_i q_i \gamma_i - \bar{a}_i q_i \sum_{j=1}^{n} |w_{ij}| L_j \xi_j - \frac{L_i}{\xi_i} \sum_{j=1}^{n} \bar{a}_j q_j |w_{ji}| \right\} > 0.$$

If $\rho > \frac{\mu}{2}$ *with*

$$\mu := k\frac{\max_{i\in N(1,n)} q_i}{\min_{i\in N(1,n)} q_i} - \frac{\lambda}{\max_{i\in N(1,n)} q_i},$$

then the trivial solution of (4.1.1) is almost surely exponentially stable, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t}\log(|u(t;u_0)|) \leq -\left(\rho - \frac{\mu}{2}\right) \ \ a.s.$$

whenever $u_0 \neq 0$.

Proof. Let $V(u,t)$ be $V(u,t) = u^T(t)Qu(t)$. Then, by (4.1.2),

$$\begin{aligned}
\mathcal{L}V &= 2u^T Q[-A(u)b(u) + A(u)Wg(u)] + trace(\sigma^T Q\sigma) \\
&= -2\sum_{i=1}^{n} a_i(u_i)q_i u_i b_i(u_i) + 2\sum_{j=1}^{n} a_i(u_i)q_i u_i \sum_{j=1}^{n} w_{ij}g_j(u_j) \\
&\quad + trace(\sigma^T(u)Q\sigma(u))
\end{aligned}$$

Following from the assumptions (H1), (H2), (H4) and (H5), we have

$$\begin{aligned}
\mathcal{L}V &\leq -2\sum_{i=1}^{n} \alpha_i q_i \gamma_i u_i^2 + 2\sum_{j=1}^{n} \bar{a}_i q_i u_i \sum_{j=1}^{n} |w_{ij}|L_j|u_j| \\
&\quad + trace(\sigma^T(u)Q\sigma(u)) \\
&\leq -2\sum_{i=1}^{n} \alpha_i q_i \gamma_i u_i^2 + \sum_{j=1}^{n} \bar{a}_i q_i \sum_{j=1}^{n} |w_{ij}|L_j(\xi_j u_i^2 + \frac{1}{\xi_j}u_j^2) \\
&\quad + \max_{i\in N(1,n)} q_i|\sigma(u)|^2 \\
&= -\sum_{i=1}^{n} \left[2\alpha_i q_i \gamma_i - \bar{a}_i q_i \sum_{j=1}^{n} |w_{ij}|L_j\xi_j - \frac{L_i}{\xi_i}\sum_{j=1}^{n} \bar{a}_j q_j |w_{ji}| \right] u_i^2 \\
&\quad + \max_{i\in N(1,n)} q_i|\sigma(u)|^2 \\
&\leq -\lambda \sum_{i=1}^{n} u_i^2 + k\max_{i\in N(1,n)} q_i \sum_{i=1}^{n} u_i^2 \\
&\leq \left(-\frac{\lambda}{\max_{i\in N(1,n)} q_i} + k\frac{\max_{i\in N(1,n)} q_i}{\min_{i\in N(1,n)} q_i} \right) \sum_{i=1}^{n} q_i u_i^2 \\
&= \mu u^T Q u
\end{aligned}$$

The the rest of the proof is a consequence of Lemma 4.1.1. □

Note that if $\mu < 0$, then one can take $\rho = 0$. Thus, we have

Corollary 4.1.1. *Suppose that* (H1), (H2), (H4) *and* (H5) *hold. Assume also that there are a matrix* $Q = diag(q_1, q_2, \ldots, q_n) > 0$ *and some positive constants* ξ_i, $i \in$

$N(1,n)$ such that

$$\lambda := \min_{i\in N(1,n)} \left\{ 2\alpha_i q_i \gamma_i - \bar{a}_i q_i \sum_{j=1}^{n} |w_{ij}| L_j \xi_j - \frac{L_i}{\xi_i} \sum_{j=1}^{n} \bar{a}_j q_j |w_{ji}| \right\} > 0.$$

If

$$\mu := k \frac{\max_{i\in N(1,n)} q_i}{\min_{i\in N(1,n)} q_i} - \frac{\lambda}{\max_{i\in N(1,n)} q_i} < 0,$$

then the trivial solution of (4.1.1) is almost sure exponential stable, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|u(t;u_0)|) \le \frac{\mu}{2} \quad a.s.$$

whenever $u_0 \neq 0$.

If we denote $[w]^+ = \max\{0, w\}$, then the same argument, together with assumption (H3), gives the following

Theorem 4.1.2. *Assume that that* (H1)-(H5) *hold. Assume also that there are a matrix $Q = diag(q_1, q_2, \ldots, q_n) > 0$ and a real number $\rho \ge 0$ such that (4.1.3) holds and*

$$\widehat{\lambda} := \min_{i\in N(1,n)} \left\{ 2\alpha_i q_i \gamma_i - 2\bar{a}_i q_i L_i [w_{ii}]^+ - \bar{a}_i q_i \sum_{j=1, j\neq i}^{n} |w_{ij}| L_j - L_i \sum_{j=1, j\neq i}^{n} \bar{a}_j q_j |w_{ji}| \right\} > 0.$$

If $\rho > \frac{\mu_1}{2}$ with

$$\mu_1 := k \frac{\max_{i\in N(1,n)} q_i}{\min_{i\in N(1,n)} q_i} - \frac{\widehat{\lambda}}{\max_{i\in N(1,n)} q_i},$$

then the trivial solution of (4.1.1) is almost surely exponentially stable, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|u(t;u_0)|) \le -\left(\rho - \frac{\mu_1}{2}\right) \quad a.s.$$

whenever $u_0 \neq 0$.

Corollary 4.1.2. *Suppose that* (H1)–(H5) *hold. Assume also that there is a matrix* $Q = diag(q_1, q_2, \ldots, q_n) > 0$ *such that*

$$\widehat{\lambda} := \min_{i \in N(1,n)} \left\{ 2\alpha_i q_i \gamma_i - 2\bar{a}_i q_i L_i [w_{ii}]^+ - \bar{a}_i q_i \sum_{j=1, j\neq i}^{n} |w_{ij}| L_j - L_i \sum_{j=1, j\neq i}^{n} \bar{a}_j q_j |w_{ji}| \right\} > 0.$$

If

$$\mu_1 := k \frac{\max_{i \in N(1,n)} q_i}{\min_{i \in N(1,n)} q_i} - \frac{\widehat{\lambda}}{\max_{i \in N(1,n)} q_i} < 0,$$

then the trivial solution of (4.1.1) is almost surely exponentially stable, i.e.,

$$\limsup_{t \to \infty} \frac{1}{t} \log(|u(t; u_0)|) \leq \frac{\mu_1}{2} \quad a.s.$$

whenever $u_0 \neq 0$.

Under other assumptions, we may obtain an instability result.

Theorem 4.1.3. *Assume that* (H2)-(H4) *and* (H6) *hold. If there are a matrix* $Q = diag(q_1, q_2, \ldots, q_n) > 0$ *and a real number* $\hat{\rho} > 0$ *such that*

$$|u^T Q \sigma(u)|^2 \leq \hat{\rho} (u^T Q u)^2 \tag{4.1.4}$$

holds for all $u \in \mathbb{R}^n$ *and* $|\sigma(u)|^2 \geq \bar{k}|u|^2$, *then the solution of (4.1.1) satisfies*

$$\liminf_{t \to \infty} \frac{1}{t} \log(|u(t; u_0)|) \geq \frac{\hat{\mu}}{2} - \hat{\rho} \quad a.s.$$

whenever $u_0 \neq 0$ *and* $\hat{\mu}$ *will be specified later in the proof. Particularly if* $\frac{\hat{\mu}}{2} - \hat{\rho} > 0$, *then (4.1.1) is almost surely exponentially unstable.*

Proof. Let $V = u^T Q u = \sum_{i=1}^n q_i u_i^2$, then

$$\begin{aligned}
\mathcal{L}V &= -2u^T QA(u)b(u) + 2u^T QA(u)Wg(u) + trace(\sigma^T(u)Q\sigma(u)) \\
&\geq -2\sum_{i=1}^n \bar{a}_i\beta_i q_i u_i^2 + 2\sum_{i=1}^n g_i(u_i)\sum_{j=1}^n a_j q_j w_{ji} u_j \\
&\quad + \min\{q_i, i \in N(1,n)\}|\sigma(u)|^2 \\
&\geq -2\sum_{i=1}^n \bar{a}_i\beta_i q_i u_i^2 + 2\sum_{i=1}^n [w_{ii}]^- L_i \bar{a}_i q_i u_i^2 \\
&\quad - \sum_{i=1}^n \left(\frac{L_i}{q_i} \sum_{j=1, j\neq i}^n \bar{a}_j q_j |w_{ji}| + \bar{a}_i \sum_{j=1, j\neq i}^n L_j |w_{ij}| \right) q_i u_i^2 \\
&\quad + \min\{q_i, i \in N(1,n)\}|\sigma(u)|^2 \\
&\geq -\nu \sum_{i=1}^n q_i u_i^2 + \min\{q_i, i \in N(1,n)\}\bar{k}|u|^2,
\end{aligned}$$

where

$$\nu := \max_{i \in N(1,n)} \left\{ 2\bar{a}_i\beta_i + \frac{L_i}{q_i}\sum_{j=1, j\neq i}^n \bar{a}_j q_j |w_{ji}| + \bar{a}_i \sum_{j=1, j\neq i}^n L_j |w_{ij}| - 2[w_{ii}]^- \bar{a}_i L_i \right\}$$

and $[w_{ii}]^- = \min(0, w_{ii})$. Then we have

$$\mathcal{L}V \geq \hat{\mu} \sum_{i=1}^n q_i u_i^2,$$

where

$$\hat{\mu} := \bar{k}\frac{\min\{q_i, i \in N(1,n)\}}{\max\{q_i, i \in N(1,n)\}} - \nu.$$

The rest of the proof follows from Theorem 4.3.5. of [71]. □

4.2 Stochastic Cohen-Grossberg neural networks with multiple delays

As stated in Chapter 1 of this thesis, time delays can not be avoided in many networks. Therefore, we will study the delayed stochastic neural networks in this section. Let us first consider the stochastic Cohen-Grossberg neural networks with constant delays modelled by

$$du_i(t) = -a_i(u(t))[b_i(u(t)) - \sum_{j=1}^{n} w_{ij}g_j(u_j(t-\tau_{ij}))]dt + \sum_{j=1}^{n} \sigma_{ij}(u_j(t))dB_i(t), \ t \geq 0, \tag{4.2.1}$$

where $\tau_{ij} \in [0, \tau]$ are associated delays and the other terms are the same as in Section 4.1. We assume for any given initial data $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in C([-\tau, 0], \mathbb{R}^n)$, system (4.2.1) admits a unique solution denoted by $u(t; \phi)$.

Theorem 4.2.1. *Assume that* (H1), (H2), (H4) *and* (H5) *hold. Assume also there are positive numbers* $p_i, \xi_i, i \in N(1,n)$ *with* $\max\{p_i, i \in N(1,n)\} = \hat{p}$ *and* $P = diag(p_1, p_2, \ldots, p_n)$ *such that*

$$\delta_0 = \min_{i \in N(1,n)} \left\{ 2\alpha_i p_i \gamma_i - \left(\sum_{j=1}^{n} q_{ji} + p_i \bar{a}_i \sum_{j=1}^{n} |w_{ij}| L_j \xi_j \right) \right\} > 0$$

and

$$\delta_1 = \delta_0 - \hat{p}k > 0,$$

where

$$q_{ij} = \frac{p_i \bar{a}_i |w_{ij}| L_j}{\xi_j}.$$

Then the trivial solution of (4.2.1) is exponentially stable the following mean square sense:

$$\limsup_{t \to \infty} \frac{1}{t} \log(E|u(t;\phi)|^2) \leq -\epsilon$$

with $\epsilon \in (0, \frac{\delta_1}{\hat{p}})$ *satisfying*

$$\max_{i \in N(1,n)} \left\{ \epsilon \sum_{j=1}^{n} q_{ji} \tau_{ji} e^{\epsilon \tau_{ji}} + \epsilon \hat{p} \right\} = \delta_1.$$

Proof. For any fixed ϕ, we denote $u(t;\phi) = u(t)$ and define

$$V(u,t) = \sum_{i=1}^{n} p_i u_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} \int_{t-\tau_{ij}}^{t} u_j^2(s)ds, \text{ for } (u,t) \in \mathbb{R}^n \times [0,\infty).$$

By Itô formula [71], we have

$$\begin{aligned} dV(u,t) &= \left(\sum_{i=1}^{n} -2p_i u_i a_i(u_i)[b_i(u_i) - \sum_{j=1}^{n} w_{ij} g_j(u_j(t-\tau_{ij}))] \right. \\ &\quad \left. + \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}(u_j(t)^2 - u_j^2(t-\tau_{ij})) + trace(\sigma^T(u)P\sigma(u)) \right) dt \\ &\quad + 2u^T(t)P\sigma(u(t))dB(t). \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{i=1}^{n} -2p_i u_i(t) a_i(u_i(t)) \left[b_i(u_i(t)) - \sum_{j=1}^{n} w_{ij} g_j(u_j(t-\tau_{ij})) \right] \\ &\leq \sum_{i=1}^{n} \left(-2\alpha_i p_i \gamma_i u_i^2(t) + 2p_i \bar{a}_i |u_i(t)| \sum_{j=1}^{n} |w_{ij}| L_j |u_j(t-\tau_{ij})| \right) \\ &\leq \sum_{i=1}^{n} \left(-2\alpha_i p_i \gamma_i u_i^2(t) + p_i \bar{a}_i \sum_{j=1}^{n} |w_{ij}| L_j |(\xi_j u_i^2 + \frac{1}{\xi_j} u_j^2(t-\tau_{ij})) \right) \\ &= \sum_{i=1}^{n} \left(-2\alpha_i p_i \gamma_i + p_i \bar{a}_i \sum_{j=1}^{n} |w_{ij}| L_j \xi_j \right) u_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} u_j^2(t-\tau_{ij}) \end{aligned}$$

and

$$trace(\sigma^t(u(t))P\sigma(u(t)) \leq \hat{p}|\sigma(u(t))|^2 \leq \hat{p}k|u(t)|^2.$$

We therefore have

$$dV(u,t) \leq -\delta_1 |u(t)|^2 dt + 2u^T(t)P\sigma(u(t))dB(t).$$

For $\epsilon \in (0, \frac{\delta_1}{\hat{p}})$,

$$\begin{aligned} d(e^{\epsilon t}V(u,t)) &= e^{\epsilon t}\left(\epsilon V(u,t)dt + dV(u,t)\right) \\ &\leq e^{\epsilon t}[-(\delta_1 - \epsilon\hat{p})|u(t)|^2 + \epsilon\sum_{i,j} q_{ij}\int_{t-\tau_{ij}}^{t} u_j^2(s)ds]dt \\ &\quad +2e^{\epsilon t}u^T(t)P\sigma(u(t))dB(t). \end{aligned}$$

Integrating the above inequality from 0 to T and then taking the expectation give

$$e^{\epsilon T}EV(u,T) \leq C_1 + E\int_0^T e^{\epsilon t}\left[\epsilon\sum_{i,j} q_{ij}\int_{t-\tau_{ij}}^{t} u_j^2(s)ds - (\delta_1 - \epsilon\hat{p})|u(t)|^2\right]dt$$

where

$$C_1 := EV(u,0) = \sum_{i=1}^{n}\left(p_i\phi_i^2(0) + \sum_{j=1}^{n} q_{ij}\int_{-\tau_{ij}}^{0}\phi_j^2(s)ds\right) < \infty.$$

It is seen that

$$\begin{aligned} \int_0^T e^{\epsilon t}\int_{t-\tau_{ij}}^{t} u_j^2(s)dsdt &\leq \int_{-\tau_{ij}}^{T} u_j^2(s)\int_{\max(0,s)}^{\min(T,s+\tau_{ij})} e^{\epsilon t}dt \\ &\leq \int_{-\tau_{ij}}^{T} \tau_{ij}e^{\epsilon(s+\tau_{ij})}u_j^2(s)ds \\ &\leq \tau_{ij}^2 e^{\epsilon\tau_{ij}}||\phi_j||^2 + \int_0^T \tau_{ij}e^{\epsilon\tau_{ij}}e^{\epsilon s}u_j^2(s)ds, \end{aligned}$$

where $||\phi_i|| = \max_{s\in[-\tau,0]}|\phi_i(s)|$. Hence,

$$\begin{aligned} e^{\epsilon T}EV(u,T) &\leq C_1 + C_2 + E\epsilon\sum_{i,j} q_{ij}\tau_{ij}e^{\epsilon\tau_{ij}}\int_0^T e^{\epsilon t}u_j^2(s)ds \\ &\quad -E\int_0^T\sum_{i=1}^{n}(\delta_1 - \epsilon\hat{p})e^{\epsilon t}u_i^2(t)dt \\ &\leq C_1 + C_2 + \epsilon C_3 E\int_0^T e^{\epsilon t}|u(t)|^2dt - E\int_0^T(\delta_1 - \epsilon\hat{p})e^{\epsilon t}|u(t)|^2dt \\ &= C_1 + C_2 + E\int_0^T\left[\epsilon C_3 + \epsilon\hat{p} - \delta_1\right]e^{\epsilon t}|u(t)|^2dt, \end{aligned}$$

where

$$C_2 := \epsilon \sum_{i,j}^{n} q_{ij}\tau_{ij}^2 e^{\epsilon\tau_{ij}}||\phi_j||^2,\ C_3 := \max_{i\in N(1,n)} \sum_{j=1}^{n} q_{ji}\tau_{ji}e^{\epsilon\tau_{ji}}.$$

By our choice of ϵ, we obtain

$$e^{\epsilon T}EV(u,T) \le C_1 + C_2 < \infty,$$

which implies

$$e^{\epsilon T}E(|u(T)|^2) < \infty$$

and hence

$$\limsup_{t\to\infty} \frac{1}{t}\log E|u(t)|^2 \le -\epsilon.$$

The proof is complete. □

Using Razumikhin technique [71], we may obtain

Theorem 4.2.2. *Assume that* (H1), (H2) *and* (H4) *hold. Let*

$$\lambda_1 = \min_{i\in N(1,n)} \left\{ 2\alpha_i\gamma_i - \bar{a}_i \sum_{j=1}^{n} |w_{ij}|L_j \right\},\ \ \lambda_2 = \max_{i\in N(1,n)} \left\{ L_i \sum_{j=1}^{n} |w_{ji}|\bar{a}_j \right\}.$$

If $\lambda_1 > \lambda_2$, then the trivial solution of (4.2.1) is exponentially stable in the following mean square sense:

$$E|u(t;\phi)|^2 \le E||\phi||^2 e^{-\eta t}, t \ge 0,$$

where $\eta \le \lambda_1 - q\lambda_2$ with $q > 0$ satisfying

$$e^{(\lambda_1 - q\lambda_2)\tau} = q.$$

In addition, if (H5) holds, then the trivial solution of (4.2.1) is almost surely exponentially stable, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t}\log|u(t;\phi)| \le -\frac{\eta}{2},\quad a.s..$$

Proof. The proof can be completed by letting

$$V(u,t) = \frac{1}{2}\sum_{i=1}^{n} u_i^2(t)$$

and applying Theorem 5.6.1 and Theorem 5.6.2 in [71]. $\square$

We next consider the stochastic Cohen-Grossberg neural network model with a variable delay described by

$$du(t) = -A(u(t))[b(u(t)) - Wg(u(t-\tau(t)))]dt + \sigma(u(t), u(t-\tau(t)), t)dB(t), \quad (4.2.2)$$

where $\tau(t) \in [0, \tau]$. We assume that $\sigma(0,0,t) \equiv 0$ so that $u = 0$ is the trivial solution of (4.2.2). Then we have

Theorem 4.2.3. *Assume that* (H1), (H2), (H4) *hold and there exist three matrices* $D_1 \geq 0, D_2 \geq 0, D_3 \geq 0$ *such that*

$$trace(\sigma^T(u,y,t)\sigma(u,y,t)) \leq u^T D_1 u + g^T(y) D_2 g(y) + y^T D_3 y$$

for all $(u,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. *Assume also that there exist* $M > 0$ *and* $P = diag(p_1, p_2, \cdots, p_n) > 0$ *with* $\hat{p} = \max_{i \in N(1,n)} p_i$ *such that*

$$\Omega := 2diag(\alpha_i \gamma_i)P - \hat{p}D_1 - P\bar{A}WM^{-1}W^T\bar{A}P > 0,$$

where $\bar{A} = diag(\bar{a}_i)$. *Let*

$$\lambda_1 = \lambda_m(\Omega), \quad \lambda_2 = \max_{i \in N(1,n)}[\lambda_M(M + \hat{p}D_2)L_i^2 + \lambda_M(\hat{p}D_3)].$$

If $\lambda_1 > \lambda_2 \geq 0$, *then the trivial solution of (4.2.2) is almost surely exponentially stable, i.e., the following holds:*

$$\limsup_{t\to\infty} \frac{1}{t} log|u(t;\phi)| \leq -\frac{r}{2} \quad a.s.,$$

where $r \in (0, \lambda_1 - \lambda_2)$ *is the unique root of*

$$\lambda_1 = r + \lambda_2 e^{r\tau}.$$

Proof. Let $V(u,t)$ be defined as

$$V(u,t) = \sum_{i=1}^{n} p_i u_i^2(t) = u^T(t) P u(t).$$

Then

$$\begin{aligned}
\mathcal{L}V(u,y,t) &\leq -\sum_{i=1}^{n} 2\alpha_i p_i \gamma_i u_i^2(t) + g^T(y) W^T A P u + u^T P A W g(y) \\
&\quad + trace(\sigma^T(u,y,t) P \sigma(u,y,t)) \\
&\leq -2u^T(t) diag(\alpha_i\gamma_i) P u(t) + u^T P A W M^{-1} W^T A P u \\
&\quad + g^T(y) M g(y) + \hat{p}\, trace(\sigma^T(u,y,t) P \sigma(u,y,t)) \\
&\leq -2u^T diag(\alpha_i\gamma_i) P u + u^T P \bar{A} W M^{-1} W^T \bar{A} P u \\
&\quad + g^T(y) M g(y) + \hat{p}(u^T D_1 u + g^T(y) D_2 g(y) + y^T D_3 y) \\
&= -u^T[2diag(\alpha_i\gamma_i)P - \hat{p}D_1 - P\bar{A}WM^{-1}W^T\bar{A}P]u \\
&\quad + g^T(y)(M + \hat{p}D_2) g(y) + y^T(\hat{p}D_3) y \\
&= -u^T \Omega u + g^T(y)(M + \hat{p}D_2) g(y) + y^T(\hat{p}D_3) y \\
&\leq -\lambda_1 \sum_{i=1}^{n} u_i^2 + \lambda_2 \sum_{i=1}^{n} y_i^2.
\end{aligned}$$

In the above, we have used Lemma 3.1.1 stated in Chapter 3. The rest of the proof follows from Theorem 2.1 of [4]. □

Remark 4.2.1. *In the above theorem, based on Schur complement* [30], $\Omega > 0$ *if and only if*

$$\begin{pmatrix} M & W^T \bar{A} P \\ P\bar{A}W & 2diag(\alpha_i\gamma_i)P - \hat{p}D_1 \end{pmatrix} > 0,$$

which can be easily verified by an LMI algorithm [5].

Example 4.2.1. *Consider*

$$d\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} g_1(x_1(t-\tau)) \\ g_2(x_2(t-\tau)) \end{pmatrix} dt$$
$$+G_1\begin{pmatrix} g_1(x_1(t-\tau)) \\ g_2(x_2(t-\tau)) \end{pmatrix} dB(t),$$

where $g_1(x) = g_2(x) = \tanh(x)$, $\tau = 1.2$, $G_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.25 \end{pmatrix}$ *and* $B(t)$ *is a 2-dimensional Brownian motion. It is easily seen that in this example* $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = 4$, $\gamma_2 = 2$, $D_1 = D_3 = 0$, $D_2 = G_1^T G_1 > 0$. *If we choose* $M = P = I$ *then* $\Omega = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} > 0$ *and* $\lambda_1 = 2$ *and* $\lambda_2 = 1.25$. *Therefore, from Theorem 4.2.3, the trivial solution in this example is almost surely exponentially stable with* $r = 0.2705 \in (0, \lambda_1 - \lambda_2) = (0, 0.75)$.

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