STRUCTURE AND UNIQUENESS OF SUMS OF SIMPLE LIE SUPERALGEBRAS

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Structure and uniqueness of sums of simple Lie superalgebras

by

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Abstract

In this thesis we consider decompositions of algebras and superalgebras into the sum of two subalgebras. The sum is understood in a sense of a vector space sum and not necessarily direct. The structure of these sums has attracted considerable attention for various types of algebras. Originally, this problem arises in the work of M. Goto (1963) where he studied the case of nilpotent Lie algebras. In 1969 A. Onishchik classified decompositions of simple complex Lie algebras into the sum of two reductive subalgebras. In 1999 Y. Bahturin and O. Kegel [1] proved that no simple associative algebra can be written as the sum of two simple subalgebras over an algebraically closed field. In the joint paper with M. Tvalavadze [24], we classify decompositions of simple Jordan algebras over an algebraically closed field of characteristic not two.

In the case of Lie superalgebras this problem was open until now. The main result of this thesis is a classification of all such decompositions in the case of basic non-exceptional Lie superalgebras, up to conjugation, over an algebraically closed field of characteristic zero. Moreover, we construct precise matrix realizations of each decomposition.

To prove this result we consider a Lie superalgebra as a module over its even component which is a Lie algebra. Using techniques of the representation theory of semisimple Lie algebras we present the precise description of such modules for each superalgebra in the sum. This research is significantly based on the result from [26] which extends Onishchik's Classification Theorem to an arbitrary algebraically closed field of characteristic zero.

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Chapter 1

Preliminaries

1.1 Decompositions of simple Lie algebras

In this section our main goal is to recall the decompositions of simple Lie algebras over an algebraically closed field $\mathbb{F}$ of zero characteristic as the sum of two reductive subalgebras.

The classification of simple decompositions over the field of complex numbers was obtained by A. Onishchik [16]. It is based on the following Lie Theory result.

Theorem 1.1.1 *Any non-trivial irreducible factorization $G = G'G''$ of a connected simple compact Lie group G into the product of two connected subgroup G' and G'' is equivalent to one of the following factorizations:*

$$\mathbf{SU}_{2n} = \mathbf{Sp}_n \cdot \mathbf{SU}_{2n-1},\ n \geq 2$$

$$\mathbf{SO}_7 = \mathbf{G}_2 \cdot \mathbf{SO}_6,$$

$$\mathbf{SO}_7 = \mathbf{G}_2 \cdot \mathbf{SO}_5,$$

$$\mathbf{SO}_{2n} = \mathbf{SO}_{2n-1} \cdot \mathbf{SU}_n,\ n \geq 4,$$

$$\mathbf{SO}_{4n} = \mathbf{SO}_{4n-1} \cdot \mathbf{Sp}_n,\ n \geq 2,$$

$$\mathbf{SO}_{16} = \mathbf{SO}_{15} \cdot \mathbf{Spin}_9,$$

$$\mathbf{SO}_8 = \mathbf{SO}_7 \cdot \mathbf{Spin}_7.$$

Next we formulate a theorem from [26] which extends Onishchik's Classification theorem for a simple Lie algebra to an arbitrary algebraically closed field of characteristic zero.

Theorem 1.1.2 *Any decomposition of a simple Lie algebra into the sum of two reductive subalgebras over an algebraically closed field of characteristic zero has up to conjugation one of the following forms:*

$$sl(2n) = sl(2n-1) + sp(2n),\ n \geq 2$$

$$o(2n) = o(2n-1) + sl(n),\ n \geq 4,$$

$$o(4n) = o(4n-1) + sp(2n),\ n \geq 2,$$

$$o(7) = G_2 + o(6),$$

$$o(7) = G_2 + o(5).$$

The following three lemmas produce decompositions of simple Lie algebras as the sum of simple subalgebras. The matrix forms of these decompositions have been constructed in [2].

Lemma 1.1.3 *There is a basis of* $\mathbb{F}^{2n}$ *such that the decomposition* $sl(2n) = sl(2n-1) + sp(2n)$ *takes the following matrix form:*

$$S = N + M, \tag{1.1}$$

where $S \cong sl(2n)$ *consists of all matrices of order* $2n$ *with zero trace. The first subalgebra* $N \cong sl(2n-1)$ *consists of matrices:*

$$\left\{ \left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & T & \\ 0 & & & \end{array} \right) \right\}$$

where T *is a matrix of order* $2n-1$ *with trace zero.*

Any element of the second subalgebra $M \cong sp(2n)$ *has the form:*

$$\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right)$$

where $C_{22} = -C_{11}^t$ *and* C_{12}, C_{21} *are symmetric matrices of order* n.

Lemma 1.1.4 *There is a basis of* $\mathbb{F}^{2n}$ *such that the decomposition* $o(2n) = o(2n-1) + sl(n)$ *takes the following matrix form:*

$$S = N + M, \tag{1.2}$$

where $S \cong o(2n)$ *consists of the matrices:*

$$\left\{ \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \right\} \tag{1.3}$$

where A_{12}, A_{21} *are skew-symmetric matrices of order* n *and* A_{11}, A_{22} *are matrices of order* n *such that* $A_{22} = -A_{11}^t$.

The first subalgebra $N \cong o(2n-1)$ consists of the matrices:

$$\left\{ \left(\begin{array}{c|ccc|c|ccc} 0 & y_1 & \dots & y_{n-1} & 0 & x_1 & \dots & x_{n-1} \\ \hline x_1 & & & & -x_1 & & & \\ \vdots & & A'_{11} & & \vdots & & A'_{12} & \\ x_{n-1} & & & & -x_{n-1} & & & \\ \hline 0 & -y_1 & \dots & -y_{n-1} & 0 & -x_1 & \dots & -x_{n-1} \\ \hline y_1 & & & & -y_1 & & & \\ \vdots & & A'_{21} & & \vdots & & A'_{22} & \\ y_{n-1} & & & & -y_{n-1} & & & \end{array} \right) \right\} \tag{1.4}$$

where A'_{12}, A'_{21} are skew-symmetric matrices of order $n-1$ and A'_{11}, A'_{22} are matrices of order $n-1$ such that $A'_{22} = -A'^t_{11}$.

Any element of the second subalgebra $M \cong sl(n)$ has the form:

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right)$$

where A_1, A_2 are matrices of order n with zero trace such that $A_2 = -A_1^t$.

Lemma 1.1.5 *There is a basis of $\mathbb{F}^{4n}$ such that the decomposition $o(4n) = o(4n-1) + sp(2n)$ takes the following matrix form:*

$$S = N + M, \tag{1.5}$$

where $S \cong o(4n)$ consists of the matrices of the form (1.3) where A_{11}, A_{12}, A_{21}, A_{22} are of the order $2n$. The first subalgebra $N \cong o(4n-1)$ has the form (1.4), where A'_{11}, A'_{12}, A'_{21} and A'_{22} are of the order $2n-1$. The second subalgebra $M \cong sp(2n)$ consists of the matrices:

$$\left\{ \left(\begin{array}{c|c} Y & 0 \\ \hline 0 & -Y^t \end{array} \right) \right\}$$

where Y is of the form:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where B, C are symmetric matrices of order $2n$ and $D = -A^t$ of order $2n$.

Remark 1.1.1 *Let χ be an automorphism of $gl(2k)$ such that $\chi(X) = Q_k X Q_k^{-1}$, where*

$$Q_k = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} I_k & I_k \\ \hline iI_k & -iI_k \end{array}\right) \tag{1.6}$$

where I_k is the identity matrix of order k.

We consider the decomposition $\chi(S) = \chi(N) + \chi(M)$ where S, N and M are from Lemma 1.1.4 (or 1.1.5). Using straightforward calculations we can show that $\chi(S)$ consists of all skew-symmetric matrices of order $2k$ (or $4k$). Besides, $\chi(N)$ consists of all skew-symmetric matrices of order $2k$ (or $4k$) with the first column and row zero. In particular $\chi(N)$ has a nontrivial annihilator in $gl(2k)$ (or $gl(4k)$).

1.2 Lie superalgebras: basic facts and definitions

In this section we formulate basic properties of Lie superalgebras ([13], [18]).

Let A be an algebra. We say that A is a $\mathbb{Z}_2$-graded algebra, if there is a vector space sum decomposition

$$A = \bigoplus_{g \in \mathbb{Z}_2} A_g$$

such that $A_g A_h \subset A_{gh}$ for all $g, h \in \mathbb{Z}_2$

Definition 1 *A Lie superalgebra S over a field $\mathbb{F}$ of characteristic zero is a $\mathbb{Z}_2$-graded algebra, that is the direct sum of two vector spaces S_0 and S_1, and is equipped* *with a Lie superbracket* $[\ ,\ ]$, *such that for any* $x \in S_\alpha$, $y \in S_\beta$ *and* $z \in S$ *the following identities hold:*

$$[x, y] = -(-1)^{\alpha\beta}[y, x] \tag{1.7}$$

$$[[x, y], z] = [x, [y, z]] - (-1)^{\alpha\beta}[y, [x, z]]. \tag{1.8}$$

The even subspace, i.e. the set of all even elements of a Lie superalgebra $S = S_0 \oplus S_1$ is a Lie algebra. Since $[S_0, S_1] \subseteq S_1$ and by (1.8), which with $\alpha = 0$, $\beta = 1$ and $z \in S_1$ takes the form

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]],$$

we observe that the commutator of S makes S_1 into an S_0-module. Furthermore, the restriction of the commutator to S_1 defines a bilinear symmetric mapping Φ: $S_1 \times S_1 \mapsto S_0$. Since S_0 is the adjoint S_0-module one may speak about the action of S_0 on the bilinear mapping from S_1 into S_0. Thus the following properties of a superalgebra $S = S_0 \oplus S_1$ hold:

- S_0 is a Lie algebra;
- S_1 is an S_0-module;
- the bilinear mapping $[\ ,\ ] : S_1 \times S_1 \mapsto S_0$ is symmetric and S_0-invariant;
- $[x, y] = -[y, x]$ for $x \in S_0$, $y \in S_1$.

If $A = A_0 \oplus A_1$ is an associative superalgebra ($\mathbb{Z}_2$-graded associative algebra) then, introducing a superbracket (supercommutator) on A by the formula

$$[x, y] = xy - (-1)^{\alpha\beta}yx \tag{1.9}$$

with $x \in A_\alpha$, $y \in A_\beta$, one turns A into a Lie superalgebra sometimes denoted by $[A]$.

We say that T is a homogeneous (or $\mathbb{Z}_2$-graded) subspace of S if T can be represented in the form

$$T = (T \cap S_0) \oplus (T \cap S_1).$$

If this holds we write $T_0 = (T \cap S_0)$ and $T_1 = (T \cap S_1)$. In addition, if T is a subalgebra (or an ideal) of S then we say that T is a subsuperalgebra (or $\mathbb{Z}_2$-graded ideal) of S. The quotient algebra S/T, where T is a $\mathbb{Z}_2$-graded ideal, can be naturally made into a Lie superalgebra if one sets

$$(S/T)_\alpha = (S_\alpha + T)/T.$$

We say that a Lie superalgebra S is simple if S has no $\mathbb{Z}_2$-graded ideals except itself and zero.

The classification of simple Lie superalgebras over an algebraically closed field was obtained by V. Kac in 1975. Among Lie superalgebras appearing in the classification of simple Lie superalgebras, one distinguishes two families: the classical Lie superalgebras in which the representation of the even subalgebra on the odd part is completely reducible, and the Cartan type superalgebras in which this property is no longer valid. Among the classical superalgebras, one naturally separates the basic series from strange ones.

The basic Lie superalgebras split into infinite families denoted by $sl(m,n)$ for $m > n \geq 1$ and $psl(n,n)$, $n \geq 2$, (special linear series), $osp(m,2n)$, n, $m \geq 1$, (orthosymplectic series) and three exceptional superalgebras $F(4)$, $G(3)$ and $D(2,1;\alpha)$, the last one being actually a one-parameter family of superalgebras. Two infinite families denoted by $P(n)$ and $Q(n)$, $n \geq 2$, constitute the strange superalgebras.

The classical Lie superalgebras can be described as matrix superalgebras as follows. Consider a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ with $\dim V_0 = m$ and $\dim V_1 = n$. Then the algebra $\operatorname{End} V$ acquires naturally a superalgebra structure by

$$\operatorname{End} V = \operatorname{End}_0 V \oplus \operatorname{End}_1 V$$

where

$$\operatorname{End}_j V = \{\phi \in \operatorname{End} V \mid \phi(V_i) \subseteq V_{i+j}\}$$

The Lie superalgebra $gl(m,n)$, m, $n > 0$, is defined as the superalgebra $\operatorname{End} V$ supplied with the Lie superbracket (1.9). Clearly, $gl(m,n)$ consists of all matrices of the form

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where $A \in gl(m)$, $D \in gl(n)$, B and C are $m \times n$ and $n \times m$ rectangular matrices.

One defines on $gl(m,n)$ the supertrace function denoted by str:

$$\operatorname{str}(M) = \operatorname{tr}(A) - \operatorname{tr}(D).$$

The superalgebra $sl(m,n)$, $m > n \geq 1$, consists of all matrices $M \in gl(m,n)$ satisfying the supertrace condition $\operatorname{str}(M) = 0$. The superalgebra $sl(n,n)$ has a one-dimensional center Z which is contained in the zero component. The simple algebra $psl(n,n)$, $n \geq 2$, is given by $psl(n,n) = sl(n,n)/Z$.

The orthosymplectic superalgebra $osp(m,2n)$, $m > n \geq 1$, is defined as the superalgebra of all matrices $M \in gl(m,2n)$ satisfying the conditions

$$A^t = -A, \qquad D^t J_n = -J_n D, \qquad C = J_n B^t$$

where ^{t} denotes the usual transpose, and the matrix J_n is given by

$$J_n = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

The strange superalgebra $P(n)$, $n \geq 2$, is defined as the superalgebra of matrices $M \in gl(n,n)$ satisfying the conditions

$$A^t = -D, \qquad B^t = B, \qquad C^t = -C, \qquad \mathrm{tr}\,(A) = 0.$$

The strange superalgebra $\tilde{Q}(n)$ is defined as the superalgebra of matrices $M \in gl(n,n)$ satisfying the conditions

$$A = D, \qquad B = C, \qquad \mathrm{tr}\,(B) = 0.$$

The superalgebra $\tilde{Q}(n)$ has a one-dimensional center Z which is contained in the zero component. The simple algebra $Q(n)$, $n \geq 2$, is given by $Q(n) = \tilde{Q}(n)/Z$.

Finally we cite two important lemmas which will be used later.

Lemma 1.2.1 *Let either $L \cong sl(m,n)$ where $m \neq n$ or $L \cong psl(n,n)$. Then $L = L_0 \oplus L_1$ where L_0 is the even part of L, L_1 is the odd part of L. The following conditions hold:*

(a) $L_0 = I_1 \oplus I_2 \oplus U$, where $I_1 \cong sl(m)$, $I_2 \cong sl(n)$ and U is either one dimensional Lie algebra if $m = n$ or zero.

(b) $I_1 \oplus I_2$-module L_1 is the direct sum of two simple $I_1 \oplus I_2$-modules of dimension mn with the highest weights (λ, μ^*) and (λ^*, μ) where $\lambda = (1, 0, \ldots, 0)$ and $\mu = (1, 0, \ldots, 0)$.

(c) $[L_1, L_1] = L_0$

(d) $[I_1, L_1] = L_1$ and $[I_2, L_1] = L_1$

(e) I_1-module L_1 is the direct sum of $2n$ simple I_1-modules of dimension m and I_2-module L_1 is the direct sum of $2m$ simple I_2-modules of dimension n.

Lemma 1.2.2 *Let $L \cong osp(m, 2n)$. Then*

(a) $L_0 = I_1 \oplus I_2$, where $I_1 \cong o(m)$, $I_2 \cong sp(2n)$

(b) L_1 is a simple $I_1 \oplus I_2$-module of dimension $2mn$

(c) $[L_1, L_1] = L_0$

(d) $[I_1, L_1] = L_1$ and $[I_2, L_1] = L_1$

(e) I_1-module L_1 is the direct sum of $2n$ simple I_1-modules of dimension m and I_2-module L_1 is the direct sum of m simple I_2-modules of dimension $2n$.

The proof of these lemmas is straightforward (see [13], [18]).

1.3 Description of some modules associated with decompositions

In this section we introduce three types of L_0-modules which will be repeatedly used throughout the thesis.

Let either $S \cong sl(m, n)$ or $S \cong osp(m, n)$, and $S \subsetneq gl(m, n)$. We consider the decomposition $S = K + L$ where K and L are two proper basic simple subsuperalgebras. If no confusion is likely, we will use the term subalgebra instead of subsuperalgebra. Since $L \subset S \subseteq gl(m, n)$, $L_0 \subset gl(m) \oplus gl(n)$. Hence we have two natural representations ρ_1 and ρ_2 of L_0 in vector spaces V and W where V is a vector column space of dimension m, and W is a vector column space of dimension n.

To define L_0-module structure on V and W we consider the following formulas:

$$xv = \rho_1(x)(v)$$

and

$$xw = \rho_2(x)(w),$$

for any $x \in L_0$, $v \in V$, $w \in W$.

Since L_0 is a direct sum of a semi-simple subalgebra and a one-dimensional center, according to [12], L_0-modules V and W are completely reducible. Let $V = V_1 \oplus \ldots \oplus V_r$ and $W = W_1 \oplus \ldots \oplus W_d$, where V_i, W_j are simple L_0-modules.

In the following definition we introduce three different types of L_0-module W_j.

Definition 2 *If I_1 and I_2 are ideals of L_0 defined in Lemmas 1.2.1 and 1.2.2, then L_0-module W_j can be of one of the following types:*

Type 1. I_2 acts trivially on W_j.

Type 2. I_2 acts nontrivially on W_j and I_1 acts nontrivially on W_j.

Type 3. I_2 acts nontrivially on W_j but I_1 acts trivially on W_j.

Next we look at the decomposition $S = K + L$ where $S \subseteq gl(m,n)$. Hence $S_0 = K_0 + L_0 \subset gl(m,n)_0$ and $S_1 = K_1 + L_1 \subset gl(m,n)_1$.

We consider $gl(m,n)$ in the following form: $(V \oplus W) \otimes (V \oplus W)^*$. Thus $gl(m,n)_0$ takes the form $(V \otimes V^*) \oplus (W \otimes W^*)$, and $gl(m,n)_1$ takes the form $(V \otimes W^*) \oplus (V^* \otimes W)$. As a result, L_0-module $gl(m,n)_1$ can be viewed as the direct sum of two L_0-modules $V \otimes W^*$ and $V^* \otimes W$ such that

$$x(v \otimes f) = \rho_1(x)(v) \otimes f + v \otimes \rho_2^*(x)(f)$$

and

$$x(g \otimes w) = \rho_1^*(x)(g) \otimes w + g \otimes \rho_2(x)(w),$$

for any $x \in L_0$, $v \in V$, $w \in W$, $g \in V^*$, $f \in W^*$, and ρ_1^*, ρ_2^*, are the dual representations for ρ_1, ρ_2.

Since $V = V_1 \oplus \ldots \oplus V_r$ and $W = W_1 \oplus \ldots \oplus W_d$ where V_i, W_j are simple L_0-modules, we can express L_0-module $V \otimes W^*$ as the direct sum of L_0-modules $V_i \otimes W_j^*$,

$$V \otimes W^* = \bigoplus_{i,j} (V_i \otimes W_j^*).$$

We denote the projection of $V \otimes W^*$ onto $V_i \otimes W_j{}^*$ by ϱ_{ij}.

1.4 Main result and general properties of decompositions

First we formulate the main result of this thesis

Theorem 1.4.1 *Any decomposition of a basic non-exceptional Lie superalgebra into the sum of two basic non-exceptional Lie subsuperalgebras over an algebraically closed field of characteristic zero has up to conjugation by a non-degenerate matrix one of the following forms:*

1. $sl(2k, n) = sl(2k-1, n) + osp(n, 2k)$,

2. $sl(n, 2k) = sl(n, 2k-1) + osp(n, 2k)$,

3. $osp(4k, 2n) = osp(4k-1, 2n) + osp(n, 2k)$,

4. $osp(2k, 2n) = osp(2k-1, 2n) + sl(k, n)$ *where* $k \geq 1$, $n \geq 1$.

In the case of decompositions of special linear superalgebras, the proof of this result is based on Theorems 2.1.8, 2.2.1 and 2.3.7. Examples 1 and 2 demonstrate the existence of these decompositions. The uniqueness of the decompositions was shown in Theorem 2.4.2. In the case of orthosymplectic superalgebras, the proof of this result is based on Theorems 3.1.1, 3.2.12 and 3.3.6. Examples 3 and 4 demonstrate the existence of these decompositions. Finally the uniqueness of the decompositions was shown in Theorem 3.4.2.

In both cases we will use the following definitions and lemmas.

Definition 3 *An $(n+m)$-dimensional column vector v is called a vector annihilator of L in $gl(m,n)$ if $v^t L = \{0\}$ and $Lv = \{0\}$.*

Lemma 1.4.2 *Let S be decomposable into the sum of two superalgebras K and L where $S \cong sl(m,n)$ (or $osp(m,n)$) and $S \subseteq gl(m,n)$. Then either K or L has a trivial vector annihilator in $gl(m,n)$.*

Proof.

Let $\langle S \rangle$ denote the associative enveloping algebra of S. By definition, $\langle S \rangle$ is a linear span in $Mat_{m\times n}(\mathbb{F})$ of $s_n s_{n-1} \dots s_1$ where $s_1, \dots, s_n \in S$. Since S is an irreducible subset of $Mat_{m\times n}(\mathbb{F})$, $\langle S \rangle$ coincides with $Mat_{m\times n}(\mathbb{F})$.

First we show that for any $l \in L$, the following inclusion holds

$$l\langle K \rangle \subseteq \langle K \rangle + \langle L \rangle + \langle K \rangle\langle L \rangle. \tag{1.10}$$

To prove this formula we use mathematical induction by the number of elements in the product $k_n k_{n-1} \dots k_1$ where $k_i \in K$.

Let $n = 1$. We are going to prove that $lk_1 \in \langle K \rangle + \langle L \rangle + \langle K \rangle\langle L \rangle$. By using the following formula for supercommutator in S:

$$[x, y] = xy - (-1)^{ij} yx$$

where $x \in S_i$, $y \in S_j$ and $i,j \in \{0, 1\}$, we have

$$lk_1 = [l, k_1] + (-1)^{ij} k_1 l \in S + \langle K \rangle\langle L \rangle.$$

Next we prove that $l(k_n k_{n-1} \dots k_1)$ has the form (1.10). We have that

$$l(k_n k_{n-1} \dots k_1) = (lk_n)k_{n-1} \dots k_1 = ([l, k_n] + (-1)^{ij} k_n l)k_{n-1} \dots k_1.$$

Notice that $[l, k_n] = k' + l'$ where $k' \in K$, $l' \in L$ since $[l, k_n] \in S$. It follows that

$$([l, k_n] + (-1)^{ij} k_n l)k_{n-1} \dots k_1 = (k' + l' + (-1)^{ij} k_n l)k_{n-1} \dots k_1.$$

This implies that

$$(k' + l' + (-1)^{ij} k_n l)k_{n-1} \dots k_1 = k'k_{n-1} \dots k_1 + l'k_{n-1} \dots k_1 + (-1)^{ij} k_n l k_{n-1} \dots k_1.$$

Clearly $k'k_{n-1} \dots k_1 \in \langle K \rangle$. By induction, both $l'k_{n-1} \dots k_1$ and $lk_{n-1} \dots k_1$ are of the form (1.10). Therefore

$$(-1)^{ij} k_n (lk_{n-1} \dots k_1) \in \langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle$$

since

$$K(\langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle) \subseteq \langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle.$$

Therefore we have proved (1.10).

Further, we want to prove that

$$\langle S \rangle = \langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle. \tag{1.11}$$

To prove this formula we use mathematical induction on the number of elements in the product $s_n s_{n-1} \dots s_1$.

If $n = 1$ then $s_1 \in S = K + L$.

Next we are going to show that $s_n(s_{n-1} \dots s_1)$ has the form $\langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle$. Let

$$s_n = k_n + l_n$$

where $k_n \in K$, $l_n \in L$ and

$$s_{n-1} \ldots s_1 = k + l + k'l'$$

where k, $k' \in \langle K \rangle$ and l, $l' \in \langle L \rangle$. So we obtain that

$$s_n(s_{n-1} \ldots s_1) = (k_n + l_n)(k + l + k'l') = k_n k + k_n l + k_n k'l' + l_n k + l_n l + l_n k'l'.$$

As was shown above, both $l_n k$ and $l_n k'l'$ have the form $\langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle$. Therefore we have proved (1.11).

Finally we prove that either K or L has a trivial vector annihilator in $gl(m, n)$. Let us assume the contrary, that is, there exists a pair of $(n+m)$-column-vectors v, u such that $v^t K = \{0\}$ and $Lu = \{0\}$. Then $v^t(\langle K \rangle) = \{0\}$ and $(\langle L \rangle)u = \{0\}$. This implies that

$$v^t(\langle K \rangle + \langle L \rangle + \langle K \rangle \langle L \rangle)u = \{0\}.$$

On the other hand, $\langle S \rangle$ coincides with $Mat_{m \times n}(\mathbb{F})$. Thus $v^t(Mat_{m \times n}(\mathbb{F}))u = \{0\}$, which is a contradiction.

In the following lemmas we are going to use notation from Section 1.3

Lemma 1.4.3 *Let I be a nontrivial ideal of L_0 where $L \cong sl(p, q)$ (or $osp(p, q)$) and $L \subseteq gl(m, n)$. If I acts trivially on V, and there exists $j_0 \in \{1, \ldots, d\}$ such that I acts trivially on W_{j_0}, then L has a vector annihilator in $gl(m, n)$, namely W_{j_0} is annihilated by L.*

Proof.

We choose a basis in $V \oplus W$ from elements of subspaces V_i, $i = 1 \dots r$, and W_j, $j = 1 \dots d$, respectively. Then L_0 takes the form

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{1.12}$$

where

$$D = diag(D_1, \dots, D_d),$$

$D_j \in M_{n_j}(\mathbb{F})$ such that $\Sigma_{j=1}^d n_j = n$. Besides, L_1 takes the form

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{1.13}$$

where

$$B = \left(\begin{array}{ccc} B_1 & \dots & B_d \end{array} \right),$$

$B_i \in M_{m \times n_i}(\mathbb{F})$ and

$$C = \left(\begin{array}{c} C_1 \\ \vdots \\ C_d \end{array} \right),$$

$C_i \in M_{n_i \times m}(\mathbb{F})$.

Therefore I takes the form (1.12) where $A = 0$ and $D_{j_0} = 0$. By Lemmas 1.2.1(d) and 1.2.2(d), $L_1 = [I, L_1]$. In matrix terms this formula takes the form

$$\left[\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D \end{array} \right) \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right] = \left(\begin{array}{c|c} 0 & -BD \\ \hline DC & 0 \end{array} \right)$$

where

$$BD = \left(\begin{array}{ccccc} B_1 D_1 & \dots & B_{j_0} D_{j_0} & \dots & B_d D_d \end{array} \right),$$

$$DC = \begin{pmatrix} D_1C_1 \\ \vdots \\ D_{j_0}C_{j_0} \\ \vdots \\ D_dC_d \end{pmatrix}$$

Since $B_{j_0}D_{j_0} = 0$ and $D_{j_0}C_{j_0} = 0$, any vector from W_{j_0} is annihilated by L. $\square$

Remark 1.4.1 *Similarly, if I acts trivially on W, and there exists $i_0 \in \{1, \ldots, r\}$ such that I acts trivially on V_{i_0}, then L also has a vector annihilator in $gl(m, n)$, namely V_{i_0} is annihilated by L.*

In this thesis we use the following

Corollary 1.4.4 *Let $L \cong osp(m-1, n) \subseteq gl(m, n)$, and $L_0 = I_1 \oplus I_2$ where $I_1 \cong o(m-1)$, $I_2 \cong sp(n)$. In addition assume I_1 has the form*

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array} \right) \right\} \tag{1.14}$$

where A is an arbitrary skew-symmetric matrix of order m with the first row and column zero. Then the first row and column of all matrices in L are also zero.

Proof. In Remark 1.4.1, set $I = I_2$ and $V_{i_0} = span(e_1)$.

Lemma 1.4.5 *Let $L \cong sl(p, q)$ (or $osp(p, q)$), $L \subseteq gl(m, n)$. In addition assume L_0 has the form (1.12), and L_1 has the form (1.13). If there exists a pair of indices j_1 and j_2, $j_1 \neq j_2$, such that D_{j_1} and D_{j_2} are not zero for some elements from L_0, then L_1 cannot be of the form (1.13) where $B_{i_1} = \lambda B_{i_2}$ for some fixed $\lambda \in \mathbb{F}$.*

Proof.

Without any loss of generality, $j_1 = 1$ and $j_1 = 2$. Assume the contrary, that is, any element from $\tilde{L}_1$ has the form (1.13) where $B_2 = \lambda B_1$.

The commutator of two arbitrary elements from L_1:

$$\left(\begin{array}{c|cc} 0 & B_1 & \lambda B_1 \\ \hline C_1 & 0 & \\ C_2 & & 0 \end{array}\right)$$

and

$$\left(\begin{array}{c|cc} 0 & B_1' & \lambda B_1' \\ \hline C_1'' & 0 & \\ C_2' & & 0 \end{array}\right)$$

has the following form

$$\left(\begin{array}{c|cc} * & 0 & 0 \\ \hline 0 & C_1B_1' + C_1'B_1 & \lambda(C_1B_1' + C_1'B_1) \\ 0 & C_2B_1' + C_2'B_1 & \lambda(C_2B_1' + C_2'B_1) \end{array}\right) \quad (1.15)$$

We know that there exists $x \in L_0$ of the form (1.12) such that $D_1 \neq 0$ and $D_2 \neq 0$. Since $L_0 = [L_1, L_1]$, x can be represented as a linear combination of commutators of elements from L_1. Hence there exists a commutator of the form (1.15) such that $\lambda(C_2B_1' + C_2'B_1) \neq 0$ since $D_2 \neq 0$. Thus $\lambda \neq 0$. Similarly, there exists a commutator of the form (1.15) such that $C_1B_1' + C_1'B_1 \neq 0$ since $D_1 \neq 0$. Therefore $\lambda(C_1B_1' + C_1'B_1) \neq 0$. This contradicts the fact that a commutator of two elements from L_1 belongs to L_0 of the form (1.12). □

Lemma 1.4.6 *Let W_{j_0}, $j_0 \in \{1 \dots d\}$ be a nontrivial L_0-module. Then there exist $i_0 \in \{1 \dots r\}$ such that $\varrho_{i_0 j_0}(L_1) \neq \{0\}$.*

Proof.

There is no loss in generality if we consider only the case where $j_0 = 1$. Let us assume the contrary, that is, for any $i \in \{1 \dots r\}$ we have that $\varrho_{ij_0}(L_1) = \{0\}$. Hence L_1 takes the following matrix form:

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\}$$

where

$$B = \begin{pmatrix} B_1 & \dots & B_d \end{pmatrix},$$

$B_i \in M_{m \times n_i}(\mathbb{F})$ and $B_1 = 0$.

On the other hand, by Lemmas 1.2.1(c) and 1.2.2(c), $L_0 = [L_1, L_1]$. Therefore L_0 takes the form

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\}$$

where

$$D = diag(D_1, \dots, D_d),$$

$D_i \in M_{n_i}(\mathbb{F})$ and $D_1 = 0$. This contradicts the fact that W_1 is a nontrivial L_0-module.

□

In this thesis we will employ the following construction. Let A, B be simple Lie algebras and A-module $V(\lambda)$ and B-module $V(\mu)$ be two simple modules with the highest weights λ and μ, respectively. Then one can define $A \oplus B$-module $V(\lambda) \otimes V(\mu)$ in the natural way

$$(X, Y)(v \otimes w) = X(v) \otimes w + v \otimes Y(w). \tag{1.16}$$

Taking into account this construction we can state the following lemma from [11]

Lemma 1.4.7 *If A-module $V(\lambda)$ and B-module $V(\mu)$ are two simple modules then $A \oplus B$-module $V(\lambda) \otimes V(\mu)$ is also simple with the highest weight (λ, μ).*

We will use the following simple lemma.

Lemma 1.4.8 *Let U be a simple $I_1 \oplus I_2$-module such that $I_1(U) \neq \{0\}$ and $I_2(U) \neq \{0\}$. Then there exist U', $U'' \subseteq U$ such that U' is a simple I_1-module and U'' is a simple I_2-module. Moreover, U is isomorphic to $U' \otimes U''$ as an $I_1 \oplus I_2$-module.*

Proof.

Let $\lambda = (\lambda', \lambda'')$ be the highest weight of an $I_1 \oplus I_2$-module U where λ' and λ'' correspond to I_1 and I_2, respectively. Next we can choose an I_1-module U_1 and an I_2-module U_2 with the highest weights λ' and λ'', respectively, and form an $I_1 \oplus I_2$-module $U_1 \otimes U_2$ as was shown above (1.16). By Lemma 1.4.7, an $I_1 \oplus I_2$-module $U_1 \otimes U_2$ is simple with the highest weight $(\lambda', \lambda'') = \lambda$. Therefore $I_1 \oplus I_2$-modules $U_1 \otimes U_2$ and U are isomorphic. Let ψ be an isomorphism between $U_1 \otimes U_2$ and U. Next we choose some non-zero $u_1 \in U_1$ and $u_2 \in U_2$. By (1.16), $U_1 \otimes u_2$ is an I_1-module and $u_1 \otimes U_2$ is an I_2-module. Moreover, $U_1 \otimes u_2$ is isomorphic to U_1 as an I_1-module and $u_1 \otimes U_2$ is isomorphic to U_2 as an I_2-module. Next, we define $U' = \psi(U_1 \otimes u_2)$ and $U'' = \psi(u_1 \otimes U_2)$. Since $U_1 \cong U'$ as an I_1-module and $U_2 \cong U''$ as an I_2-module, it follows that $U_1 \otimes U_2 \cong U' \otimes U''$ as an $I_1 \oplus I_2$-module. Therefore U is isomorphic to $U' \otimes U''$ as an $I_1 \oplus I_2$-module. □

Chapter 2

Decompositions of special linear superalgebras

2.1 Sums of two special linear superalgebras

In this section we consider decompositions of the form $S = K + L$ where S, K and L are special linear algebras.

Remark 2.1.1 *Since both K and L have the same type, by Lemma 1.4.2, we can assume that L has a trivial vector annihilator in $gl(m,n)$.*

Lemma 2.1.1 *Let $S = sl(m,n)$ (or $psl(n,n)$) be a Lie superalgebra, and S be decomposed as the sum of two proper special linear subalgebras K and L. Then $K \cong sl(p,n)$ (or $psl(n,n)$) and $L \cong sl(m,l)$ (or $psl(m,m)$).*

Proof.

By Lemma 1.2.1(a), either $S_0 = sl(m) \oplus sl(n) \oplus U$ or $S_0 = sl(n) \oplus sl(n)$. We define two projections π_1 and π_2 of S_0 onto the ideals $sl(m)$ and $sl(n)$, $\pi_1 : S_0 \longrightarrow sl(m)$ and

$\pi_2 : S_0 \to sl(n)$. We have that $K_0 \cong sl(p_1) \oplus sl(p_2) \oplus U$ and $L_0 \cong sl(l_1) \oplus sl(l_2) \oplus U$ since $K \cong sl(p_1, p_2)$ and $L \cong sl(l_1, l_2)$. Hence $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are reductive Lie algebras as homomorphic images of reductive algebras K_0 and L_0. Since $S = K + L$, S_0 is also decomposable into the sum of two subalgebras K_0 and L_0, $S_0 = K_0 + L_0$. Therefore, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$, where $\pi_1(S_0) = sl(m)$ and $\pi_2(S_0) = sl(n)$. Thus, we obtain two decompositions of simple Lie algebras $sl(m)$ and $sl(n)$ into the sum of two reductive subalgebras.

By Theorem 1.1.2, $sl(n)$ cannot be decomposed into the sum of two proper reductive subalgebras of any of the following types: $sl(k)$, $sl(k) \oplus sl(l)$ or $sl(k) \oplus sl(l) \oplus U$. Hence one of the subalgebras coincides with $sl(n)$.

Next we consider the following decomposition: $sl(m) = \pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$. Without any loss of generality, we assume that $\pi_1(L_0)$ coincides with $\pi_1(S_0)$. Then $\pi_1(L_0)$ is isomorphic to $sl(m)$. On the other hand, $\pi_1(L_0)$ is a homomorphic image of L_0 where $L_0 \cong sl(l_1) \oplus sl(l_2) \oplus U$. Therefore $sl(l_1)$, $sl(l_2)$ are the only possible simple homomorphic images of L_0. Thus either $l_1 = m$ or $l_2 = m$. Set $l = l_1$ if $l_2 = m$ and $l = l_2$ if $l_1 = m$. It follows that $L \cong sl(m, l)$.

Finally we consider the decomposition $sl(n) = \pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$. We want to prove that $\pi_2(L_0)$ does not coincide with $\pi_2(S_0) = sl(n)$. Assume the contrary, that is, $\pi_2(L_0) = sl(n)$. Let $L_0 = I_1 \oplus I_2 \oplus U$ where $I_1 \cong sl(m)$ and $I_2 \cong sl(l)$. Therefore we obtain that either $m = n$ or $l = n$ since $\pi_2(I_1 \oplus I_2 \oplus U) = sl(n)$.

Let $l \neq n$. This implies that $m = n$. Therefore $\pi_1(L_0) \cong sl(m)$ and $\pi_2(L_0) \cong sl(m)$. Since $L_0 \cong I_1 \oplus I_2 \oplus U$ where $I_1 \cong sl(m)$, $I_2 \cong sl(l)$ and $l \neq m$, we obtain that $\pi_1(I_1) = \pi_1(L_0)$ and $\pi_2(I_1) = \pi_2(L_0)$. However $[\pi_1(I_1), \pi_1(I_2)] = \{0\}$ and $[\pi_2(I_1), \pi_2(I_2)] = \{0\}$ since $[I_1, I_2] = \{0\}$. Therefore $\pi_1(I_2) = \{0\}$ and $\pi_2(I_2) = \{0\}$, which is wrong. Thus $l = n$ and $L \cong sl(m, n)$. This contradicts the fact that L is a proper simple subalgebra of S.

Thus we have proved that $\pi_2(L_0)$ does not coincide with $\pi_2(S_0) = sl(n)$. Therefore $\pi_2(K_0)$ coincides with $\pi_2(S_0)$. Thus, either $p_1 = n$ or $p_2 = n$ since $K_0 \cong sl(p_1) \oplus sl(p_2) \oplus U$. Set $p = p_1$ if $p_2 = n$ and $p = p_2$ if $p_1 = p$. It follows that $K \cong sl(p, n)$. □

Corollary 2.1.2 *Let $S = K + L$, $K \cong sl(p,n)$, $L \cong sl(m,l)$ and $I_1 \cong sl(m)$, $I_2 \cong sl(l)$ be two ideals of L_0. Then I_2 acts trivially on V. Moreover I_1-module V is standard.*

Proof. The proof follows from the fact that $\pi_1(I_1) = \pi_1(L_0) = sl(m)$ and $\pi_1(I_2) = \{0\}$ since $[\pi_1(I_1), \pi_1(I_2)] = \{0\}$. □

Lemma 2.1.3 *Let $S = K + L$ where $S \cong sl(m,n)$, $K \cong sl(p,n)$ and $L \cong sl(m,l)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 1.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 1. By Remark 2.1.1, L has a trivial vector annihilator in $gl(m,n)$. By Corollary 2.1.2, I_2 acts trivially on V. Moreover I_2 acts trivially on W_{j_0} since L_0-module W_{j_0} is of the type 1. Therefore, by Lemma 1.4.3, L has a vector annihilator in $gl(m,n)$, which is a contradiction. □

Lemma 2.1.4 *Let $S = K + L$ where $S \cong sl(m,n)$, $K \cong sl(p,n)$, $L \cong sl(m,l)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 2 .*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 2. By Lemma 1.4.8, there exist subspaces $W'_{j_0} \subseteq W_{j_0}$ and $W''_{j_0} \subseteq W_{j_0}$ such that W'_{j_0} is a simple I_1-module, W''_{j_0} is a simple I_2-module and $W_{j_0} \cong W'_{j_0} \otimes W''_{j_0}$ as $I_1 \oplus I_2$-modules.

First we show that dim $W'_{j_0} = m$ and dim $W''_{j_0} = l$. We have that W'_{j_0} is a simple $sl(m)$-module and W''_{j_0} is a simple $sl(l)$-module. Hence $\dim W'_{j_0} \geq m$ and $\dim W''_{j_0} \geq l$. Without any loss of generality, we assume that $\dim W'_{j_0} > m$. Therefore

$$n = \dim W \geq \dim W_{j_0} = \dim W'_{j_0} \dim W''_{j_0} > ml.$$

On the other hand,

$$\dim L_1 \geq \dim S_1 - \dim K_1 \geq 2mn - 2(m-1)(n) = 2n$$

since

$$\dim S_1 \leq \dim K_1 + \dim L_1.$$

It follows that $ml \geq n$ since $\dim L_1 = 2ml$. This contradicts the fact that $n > ml$. Therefore $\dim W'_{j_0} = m$, $\dim W''_{j_0} = l$ and $W = W_{j_0}$. If we denote W'_{j_0} and W''_{j_0} as W' and W'', then $W \cong W' \otimes W''$.

Let us fix the following basis for W: $\{e'_i \otimes e''_j\}$, where $\{e'_i\}$ is a basis of W' and $\{e''_j\}$ is a basis of W''. If we consider W as I_1-module then it can be expressed as the direct sum of I_1-modules:

$$W = (W' \otimes e''_1) \oplus \ldots \oplus (W' \otimes e''_l). \tag{2.1}$$

The next step is to prove that the projection π of L_1 onto $V \otimes W^*$ is not zero. Assume that $\pi(L_1) = \{0\}$. Then L_1 has the following matrix form:

$$\left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array} \right) \right\}.$$

It follows that $[L_1, L_1] = \{0\}$. However this contradicts the fact that, by Lemma 1.2.1(c), $[L_1, L_1] = L_0 \neq \{0\}$. Hence $\pi(L_1) \neq \{0\}$. Let us consider $V \otimes W^*$ as I_1-module. From (2.1) we obtain that

$$V \otimes W^* = (V \otimes (W' \otimes e''_1)^*) \oplus \ldots \oplus (V \otimes (W' \otimes e''_l)^*)$$

where all $V \otimes (W' \otimes e''_j)^*$ are also I_1-modules. There exists j_0 such that the projection of L_1 onto $V \otimes (W' \otimes e''_{j_0})^*$ is not zero since the projection of L_1 onto $V \otimes W^*$ is not zero.

We consider I_1-module $V \otimes (W' \otimes e''_{j_0})^*$. By Corollary 2.1.2, I_1-module V is standard. Besides, I_1-module W' is either standard or dual since $\dim W' = m$.

Next we apply Young tableaux technique (see [10]) to find irreducible submodules of I_1-module $(V \otimes W'^*) \otimes e''^*_{j_0}$. Let ϱ and ϱ' be either standard or dual representations of $sl(m)$. Then the tensor product $\varrho \otimes \varrho'$ is also a representation of $sl(m)$. Then Young tableaux technique shows that it can only contain irreducible subrepresentations with the highest weights: $(2, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(1, 0, \ldots, 0, 1)$ or a trivial representation.

Since I_1-modules V_{i_0} and W' are either standard or dual, we obtain that I_1-module $V \otimes (W' \otimes e''_{j_0})^*$ can only contain simple submodules with the highest weights listed above. On the other hand, by Lemma 1.2.1(e) I_1-module L_1 has only simple submodules of dimension m with the highest weight $(1, 0, \ldots, 0)$, which is a contradiction. $\square$

Lemma 2.1.5 *Let $L \cong sl(s, l) \subseteq gl(m, n)$, and I_1, I_2 be ideals of L_0. If I_1 acts trivially on W_{j_0} for some $j_0 \in \{1 \ldots d\}$, I_2 acts trivially on V and I_2 acts nontrivially on W_{j_0}, then I_2-module W_{j_0} is either standard or dual.*

Proof.

By Lemma 1.4.6, there exists i_0 such that $\varrho_{i_0j_0}(L_1) \neq \{0\}$. We consider $I_1 \oplus I_2$-module $V_{i_0} \otimes W_{j_0}^*$. By Lemma 1.4.7, $I_1 \oplus I_2$-module $V_{i_0} \otimes W_{j_0}^*$ is simple since I_1-module V_{i_0} and I_2-module W_{j_0} are both simple. Therefore $I_1 \oplus I_2$-module $\varrho_{i_0j_0}(L_1)$ coincides with $V_{i_0} \otimes W_{j_0}^*$ since $\varrho_{i_0j_0}(L_1) \neq \{0\}$. By Lemma 1.2.1(b), $I_1 \oplus I_2$-module L_1 is the direct sum of two simple $I_1 \oplus I_2$-submodules of dimension sl each. Since $\varrho_{i_0j_0}(L_1)$ is a simple $I_1 \oplus I_2$-module, the dimension of $\varrho_{i_0j_0}(L_1)$ is sl. On the other hand, we have

$$(\dim V_{i_0}) \cdot (\dim W_{j_0}) = \dim (V_{i_0} \otimes W_{j_0}^*) = \dim \varrho_{i_0j_0}(L_1) = sl.$$

Since V_{i_0} is a nontrivial $sl(s)$-module, and W_{j_0} is a nontrivial $sl(l)$-module, $\dim V_{i_0} \geq s$ and $\dim W_{j_0} \geq l$. Therefore, $\dim V_{i_0} = s$ and $\dim W_{j_0} = l$. Hence I_2-module W_{j_0} is either standard or dual. □

Lemma 2.1.6 *Let $S = K + L$ where $S \cong sl(m,n)$, $K \cong sl(p,n)$ and $L \cong sl(m,l)$. If L_0-module W_{j_0}, $j_0 \in \{1 \dots d\}$, is of the type 3 then W_{j_0} is a standard I_2-module.*

Proof.

First we are given that I_1 acts trivially on W_{j_0}. By Corollary 2.1.2, I_2 acts trivially on $V = V_1$. Hence, by Lemma 2.1.5, I_2-module W_{j_0} is either standard or dual.

Next we prove that W_{j_0} is not a dual I_2-module. Let us assume the contrary, that is, W_{j_0} is a dual I_2-module. Let $\lambda = (1, 0, \dots, 0)$ be the highest weight of I_1-module V, and $\mu^* = (0, \dots, 0, 1)$ be the highest weight of I_2-module W_{j_0}. Then, by Lemma 1.4.7, $I_1 \oplus I_2$-module $V \otimes W_{j_0}^*$ has the highest weight $(\lambda, \mu^{**}) = (\lambda, \mu)$.

By Lemma 1.2.1(b), $I_1 \oplus I_2$-module L_1 is the direct sum of two simple submodules with the highest weights (λ, μ^*) and (λ^*, μ). Hence the projection of L_1 onto $V \otimes W_{j_0}^*$

is zero since L_0-module L_1 contains no submodules with the highest weight (λ, μ). This contradicts the fact that, by Lemma 1.4.6, $\varrho_{1j_0}(L_1) \neq \{0\}$. So W_{j_0} is a standard I_2-module. $\square$

Lemma 2.1.7 *Let $S = K + L$ where $S \cong sl(m,n)$, $K \cong sl(p,n)$ and $L \cong sl(m,l)$. Then L_0-module W contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$ of the type 3.*

Proof.

Let us assume the contrary, that is, there exist two L_0-submodules W_1 and W_2 of the type 3. By Lemma 2.1.6, W_1 and W_2 are standard I_2-modules. Hence we can fix a basis in $V \oplus W$ of vectors of subspaces $V = V_1$ and W_j, $j \in \{1 \dots d\}$, respectively, such that L_0 takes the following form

$$\left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{2.2}$$

where $X \in sl(m)$ and $D = diag(D_1, \dots, D_d)$, $D_j \in M_{n_j}(\mathbb{F})$ such that $D_1 = D_2 = Y$, $Y \in sl(l)$.

Besides, L_1 has the following form

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{2.3}$$

where

$$B = \begin{pmatrix} B_1 & \dots & B_d \end{pmatrix}$$

and $B_j \in M_{m \times n_j}(\mathbb{F})$.

We consider $I_1 \oplus I_2$-modules $V_1 \otimes W_1^*$ and $V_1 \otimes W_2^*$. In matrix terms the first module consists of all $m \times l$ matrices, and the action of $I_1 \oplus I_2$ is given by

$$x(B_1) = XB_1 - B_1Y \tag{2.4}$$

where $x \in I_1 \oplus I_2$ of the form (2.2), and B_1 is an arbitrary $m \times l$ matrix. Similarly, the action of $I_1 \oplus I_2$ on $V_1 \otimes W_2^*$ is given by

$$x(B_2) = XB_2 - B_2Y$$

where $x \in I_1 \oplus I_2$ is of the form (2.2) and B_2 is an arbitrary $m \times l$ matrix.

Let $I_1 \oplus I_2$-module $\varrho_{11}(L_1)$ be an image of $I_1 \oplus I_2$-module L_1 under the projection ϱ_{11} onto $V \otimes W_1^*$. Likewise $\varrho_{12}(L_1)$ is an image of $I_1 \oplus I_2$-module L_1 under the projection ϱ_{12} onto $V \otimes W_2^*$. By Lemma 1.4.7, $I_1 \oplus I_2$-modules $V \otimes W_1^*$ and $V \otimes W_2^*$ are simple. Hence $I_1 \oplus I_2$-module $\varrho_{11}(L_1)$ coincides with $V \otimes W_1^*$, and $I_1 \oplus I_2$-module $\varrho_{12}(L_1)$ coincides with $V \otimes W_2^*$. Therefore both $I_1 \oplus I_2$-modules $V \otimes W_1^*$ and $V \otimes W_2^*$ have the same matrix form (2.4). On the other hand, $\varrho_{11}(L_1)$ and $\varrho_{12}(L_1)$ are isomorphic as $I_1 \oplus I_2$-modules since they are both simple and homomorphic images of $I_1 \oplus I_2$-module L_1. Hence, by Schur's Lemma, the only isomorphism between $I_1 \oplus I_2$-modules $\varrho_{11}(L_1)$ and $\varrho_{12}(L_1)$ is a scalar mapping. In matrix terms it means that for any matrices from L_1 of the form (2.3), $B_1 = \lambda B_2$, $\lambda \in \mathbb{F}$. This contradicts the fact that, by Lemma 1.4.5, L_1 cannot be of this form. □

Theorem 2.1.8 *A Lie superalgebra $S \cong sl(m,n)$, $m > n > 0$, cannot be decomposed into the sum of two proper special linear superalgebras.*

Proof.

Let us assume that this decomposition exists. Then, according to Lemma 2.1.1, $K \cong sl(p,n)$ and $L \cong sl(m,l)$. By Lemma 2.1.7, L_0-module contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$, of the type 3.

On the other hand, I_2 acts nontrivially on W since, by Corollary 2.1.2, I_2 acts trivially on V. Therefore W contains at least one L_0-submodule W_{j_0}. This implies that W_{j_0} coincides with W. According to Lemma 2.1.6, I_2-module W_{j_0} is standard. Hence $l = n$ since $\dim W_{j_0} = \dim W = n$. This contradicts the fact that $L \cong sl(m,l)$ is a proper subalgebra of $S \cong sl(m,n)$. □

2.2 Sum of two orthosymplectic superalgebras

In this section we study decompositions of $sl(m,n)$ as the sum of two proper simple orthosymplectic subalgebras.

Theorem 2.2.1 *A Lie superalgebra $S \cong sl(m,n)$, $m > n > 0$, cannot be decomposed into the sum of two proper orthosymplectic subalgebras K and L.*

Proof. By Lemma 1.2.1(a), $S_0 = sl(m) \oplus sl(n) \oplus U$. As above we define two projections π_1 and π_2 of S_0 onto the ideals $sl(m)$ and $sl(n)$, $\pi_1 : S_0 \to sl(m)$ and $\pi_2 : S_0 \to sl(n)$. We have that $K_0 \cong o(p) \oplus sp(2q)$ and $L_0 \cong o(s) \oplus sp(2l)$ since $K \cong osp(p, 2q)$ and $L \cong osp(s, 2l)$. Hence the projections $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are also reductive as homomorphic images of reductive algebras.

Since $S = K + L$, S_0 is also decomposable into the sum of two subalgebras K_0 and L_0, $S_0 = K_0 + L_0$. Therefore, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$, where $\pi_1(S_0) = sl(m)$ and $\pi_2(S_0) = sl(n)$. We have the decompositions of simple Lie algebras $sl(m)$ and $sl(n)$ into the sum of two reductive subalgebras.

By Theorem 1.1.2, $sl(n)$ cannot be decomposed into the sum of two subalgebras of these types. As a result, $S \cong sl(m,n)$ cannot be decomposed into the sum of $K \cong osp(p, 2q)$ and $L \cong osp(s, 2l)$. □

2.3 Sum of special linear and orthosymplectic superalgebras

In this section we consider the decomposition $S = K + L$ where $S \cong sl(m,n)$, $K \cong sl(p,q)$ and $L \cong osp(s,2l)$.

Lemma 2.3.1 *Let $S = sl(m,n)$ be a Lie superalgebra, and S be decomposed into the sum of a proper special linear subalgebras K and a proper orthosymplectic subalgebras L. Then only two cases are possible:*

1. *$m = 2k$, $K \cong sl(2k-1,n)$ and $L \cong osp(s,2k)$.*
2. *$n = 2k$, $K \cong sl(m,2k-1)$ and $L \cong osp(s,2k)$.*

Proof.

By Lemma 1.2.1(a), $S_0 = sl(m) \oplus sl(n) \oplus U$. As usual, we define two projections π_1 and π_2 of S_0 onto the ideals $sl(m)$ and $sl(n)$, $\pi_1 : S_0 \to sl(m)$ and $\pi_2 : S_0 \to sl(n)$.

We have that $K_0 \cong sl(p) \oplus sl(q) \oplus U$ and $L_0 \cong o(s) \oplus sp(2l)$ since $K \cong sl(p,q)$ and $L \cong osp(s,2l)$. Hence $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are reductive Lie algebras as homomorphic images of reductive Lie algebras K_0 and L_0.

The given decomposition induces the following representations of simple Lie algebras $sl(m)$ and $sl(n)$ as the sum of two reductive subalgebras:

$$sl(n) = \pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0), \tag{2.5}$$

$$sl(m) = \pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0). \tag{2.6}$$

By Theorem 1.1.2, the only possible decomposition of $sl(n)$ into the sum of two proper reductive subalgebras is

$$sl(2n) = \mathcal{A} + \mathcal{B}, \tag{2.7}$$

where $\mathcal{A} \cong sl(2n-1)$, $\mathcal{B} \cong sp(2n)$.

Notice that one of two decompositions (2.5) and (2.6) is nontrivial. Indeed, if both decompositions are trivial then $\pi_1(K_0) = \pi_1(S_0) \cong sl(m)$ and $\pi_2(K_0) = \pi_2(S_0) \cong sl(n)$. Acting in the same manner as in Lemma 2.1.1 we can prove that $p = m$, $q = n$. This contradicts the fact that $K \cong sl(p,q)$ is a proper subalgebra of $S \cong sl(m,n)$.

Therefore two cases are possible:

1. The first decomposition is nontrivial.
2. The second decomposition is nontrivial.

Let us consider the first case. Thus, according to (2.7), $\pi_1(K_0) \cong sl(2k-1)$ and $\pi_1(L_0) \cong sp(2k)$ where $m = 2k$. It follows that $p = 2k-1$ and $l = k$.

Further we want to prove that the decomposition (2.6) is trivial. Let us assume the contrary, that is, (2.6) is nontrivial and has the form (2.7). Thus $\pi_2(L_0)$ is isomorphic to $sp(2n)$. On the other hand, $\pi_1(L_0) \cong sp(2k)$. This contradicts the fact that $L_0 \cong o(s) \oplus sp(2l)$. Therefore the decomposition $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$ is trivial, and $\pi_2(K_0)$ coincides with $\pi_2(S_0) = sl(n)$. It follows that $q = n$ since $K_0 \cong sl(p) \oplus sl(q) \oplus U$. Thus $K \cong sl(2k-1,n)$ and $L \cong osp(s,2k)$.

The second case is similar, and acting as above, we can show that $K \cong sl(m, 2k-1)$ and $L \cong osp(s,2k)$. □

From now on, we will consider only the first case in Lemma 2.3.1 since the second case can be considered in a similar manner.

Corollary 2.3.2 *Let $S = K + L$, $S = sl(2k,n)$, $K \cong sl(2k-1,n)$, $L \cong osp(s,2k)$ and $I_1 \cong sp(2k)$ and $I_2 \cong o(s)$ be ideals of L_0. Then I_2 acts trivially on V. Moreover*

I_1-module $V_1 = V$ is standard.

Proof. The proof follows from the fact that $\pi_1(I_1) = \pi_1(L_0) = sp(2k)$ and $\pi_1(I_2) = \{0\}$ since $[\pi_1(I_1), \pi_1(I_2)] = \{0\}$. □

Lemma 2.3.3 *Let $S = K + L$ where $S \cong sl(2k, n)$, $K \cong sl(2k-1, n)$, $L \cong osp(s, 2k)$. Then for any $j \in \{1 \dots d\}$, L_0-module W_j is not of the type 1.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 1. First we prove that K has a nontrivial vector annihilator in $gl(m, n)$. Let $K = J_1 \oplus J_2$ where $J_1 \cong sl(2k-1)$ and $J_2 \cong sl(n)$. As was shown in Lemma 2.3.1, $\pi_2(S_0) = \pi_2(K_0) \cong sl(n)$. We are going to show that either $\pi_2(J_1) = \{0\}$ or $\pi_2(J_2) = \{0\}$. Indeed, if $\pi_2(J_2) \neq \{0\}$ then $\pi_2(J_2) = \pi_2(K_0) = sl(n)$ since $J_2 \cong sl(n)$. Howevere $[\pi_2(J_1), \pi_2(J_2)] = \{0\}$ since $[J_1, J_2] = \{0\}$. Therefore $\pi_2(J_1) = \{0\}$ since $\pi_2(J_1) \subseteq \pi_2(K_0) = \pi_2(J_2)$. So we have proved that either $\pi_2(J_1) = \{0\}$ or $\pi_2(J_2) = \{0\}$. Let J be either J_1 or J_2 such that $\pi_2(J) = \{0\}$.

By Lemma 2.3.1, the decomposition $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ has the form $sl(2k) = sl(2k-1) + sp(2k)$. Therefore, by Remark 1.1.1, $\pi_1(K_0)$ has a nontrivial annihilator in $gl(2k)$. Hence $\pi_1(J)$ also has a nontrivial annihilator in $gl(2k)$. So we obtain that J is an ideal of K_0, $K \subseteq gl(2k, n)$, and J acts trivially on W and on one-dimensional subspace of V. Hence, by Lemma 1.4.3, K has a nontrivial vector annihilator in $gl(2k, n)$.

Therefore, by Lemma 1.4.2, L has a trivial two-sided annihilator in $gl(2k, n)$ since K has a nontrivial vector annihilator in $gl(m, n)$. Let us consider $I_2 \subseteq L$. By Corollary 2.3.2, I_2 acts trivially on V. Moreover I_2 acts trivially on W_{j_0} since

L_0-module W_{j_0} is of the type 1. Therefore, by Lemma 1.4.3, L has a nontrivial vector annihilator in $gl(2k, n)$, which is a contradiction. □

Lemma 2.3.4 *Let $S = K + L$ where $S \cong sl(2k, n)$, $K \cong sl(2k-1, n)$, $L \cong osp(s, 2k)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 2.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 2. By Lemma 1.4.7, there exist subspaces $W'_{j_0} \subseteq W_{j_0}$ and $W''_{j_0} \subseteq W_{j_0}$ such that W'_{j_0} is a simple I_1-module, W''_{j_0} is a simple I_2-module and $W_{j_0} \cong W'_{j_0} \otimes W''_{j_0}$.

We have that $\dim W'_{j_0} \geq 2k$ and $\dim W''_{j_0} \geq s$ since W'_{j_0} is a simple $sp(2k)$-module and W''_{i_0} is a simple $o(s)$-module. Hence

$$n = \dim W \geq \dim W_{j_0} = \dim W'_{j_0} \dim W''_{j_0} \geq 2ks.$$

On the other hand

$$\dim L_1 \geq \dim S_1 - \dim K_1 \geq 2nm - 2n(m-1) = 2n$$

since $\dim S_1 \leq \dim K_1 + \dim L_1$. It follows that $2ks \geq 2n$ since $\dim L_1 = 2ks$. This contradicts the fact that $n \geq 2ks$ since $s, k > 0$. □

Lemma 2.3.5 *Let $L \cong osp(s, 2l) \subseteq gl(m, n)$ and $L_0 = I_1 \oplus I_2$. If I_1 acts trivially on W_{j_0} for some $j_0 \in \{1 \ldots d\}$, I_2 acts trivially on V and nontrivially on W_{j_0} then I_2-module W_{j_0} is standard.*

Proof.

We consider only the case where $I_1 \cong o(s)$ and $I_2 \cong sp(2l)$. The case when $I_1 \cong sp(2l)$ and $I_2 \cong o(s)$ can be treated in the similar way. Notice that, by Lemma

1.4.6, there exists i_0 such that $\varrho_{i_0 j_0}(L_1) \neq \{0\}$. We consider $I_1 \oplus I_2$-module $V_{i_0} \otimes W_{j_0}^*$. By Lemma 1.4.7, $I_1 \oplus I_2$-module $V_{i_0} \otimes W_{j_0}^*$ is simple since I_1-module V_{i_0} and I_2-module W_{j_0} are both simple.

Therefore $I_1 \oplus I_2$-module $\varrho_{i_0 j_0}(L_1)$ coincides with $V_{i_0} \otimes W_{j_0}^*$ since $\varrho_{i_0 j_0}(L_1) \neq \{0\}$. By Lemma 1.4.2(b), $I_1 \oplus I_2$-module L_1 is simple, and $\dim L_1 = 2sl$. Since $\varrho_{i_0 j_0}(L_1)$ is a simple $I_1 \oplus I_2$-module, the dimension of $\varrho_{i_0 j_0}(L_1)$ is $2sl$. Therefore

$$(\dim V_{i_0}) \cdot (\dim W_{j_0}) = \dim (V_{i_0} \otimes W_{j_0}^*) = \dim \varrho_{i_0 j_0}(L_1) = 2sl.$$

On the other hand, $\dim V_{i_0} \geq s$ and $\dim W_{j_0} \geq 2l$ since V_{i_0} is a nontrivial $o(s)$-module, and W_{j_0} is a nontrivial $sp(2l)$-module. This implies that $\dim V_{i_0} = s$ and $\dim W_{j_0} = 2l$. Hence I_2-module W_{j_0} is standard. □

Lemma 2.3.6 *Let $S = K + L$ where $S \cong sl(2k, n)$, $K \cong sl(2k-1, n)$ and $L \cong osp(s, 2k)$. Then L_0-module W contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$ of the type 3.*

Proof.

Let us assume the contrary, that is, there exist two L_0-submodules W_1 and W_2 of the type 3.

Notice that $I_1 \cong sp(2k)$ acts trivially on both W_1, W_2, and $I_2 \cong o(s)$ acts nontrivially on W_1, W_2. Moreover, by Corollary 2.3.2, $I_2 \cong o(s)$ acts trivially on V. Hence, by Lemma 2.3.5, I_2-modules W_1 and W_2 are standard. Hence we can fix a basis in $V \oplus W$ from vectors of subspaces $V = V_1$ and W_j, $j \in \{1, 2\}$, such that L_0 takes the following form

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{2.8}$$

where $A \in sp(2k)$ and $D = diag(D_1, \ldots, D_k)$, $D_j \in M_{n_j}(\mathbb{F})$ such that $D_1 = D_2 = Y$, $Y \in o(s)$. Besides, L_1 has the following form

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{2.9}$$

where

$$B = \begin{pmatrix} B_1 & \ldots & B_d \end{pmatrix}$$

and $B_j \in M_{m \times n_j}(\mathbb{F})$.

Next we consider $I_1 \oplus I_2$-modules $V_1 \otimes W_j^*$, $j \in \{1, 2\}$. In matrix terms $I_1 \oplus I_2$-modules $V_1 \otimes W_j^*$ consist of all $2k \times s$ matrices, and the action of $I_1 \oplus I_2$ is given by

$$x(B_j) = XB_j - B_jY$$

where $x \in I_1 \oplus I_2$ of the form (2.9) and B_j, $j \in \{1, 2\}$, are arbitrary $2k \times s$ matrices. Acting in the same manner as in Lemma 2.1.7, we prove that L_1 has the form (2.8) where $B_1 = \lambda B_2$, $\lambda \in \mathbb{F}$. This contradicts the fact that L_1 cannot be of this form (Lemma 1.4.5). □

Theorem 2.3.7 *Let $S = sl(m, n)$, $m > n > 0$, be decomposed into the sum of a special linear and orthosymplectic subalgebras. Then only two cases are possible:*

1. *$m = 2k$, $K \cong sl(2k - 1, n)$ and $L \cong osp(n, 2k)$.*
2. *$n = 2k$, $K \cong sl(m, 2k - 1)$ and $L \cong osp(m, 2k)$.*

Proof.

According to Lemma 2.3.1, only two cases are possible:

1. $m = 2k$, $K \cong sl(2k - 1, n)$ and $L \cong osp(s, 2k)$.
2. $n = 2k$, $K \cong sl(m, 2k - 1)$ and $L \cong osp(s, 2k)$.

We only consider the first case since the second case can be considered in the similar manner. Therefore, we only have to prove that $s = n$.

By Lemma 2.3.6, L_0-module W contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$ of the type 3. On the other hand, I_2 acts nontrivially on W since, by Corollary 2.3.2, I_2 acts trivially on V. Therefore W contains at least one L_0-submodule W_{j_0}. This implies that W_{j_0} coincides with W. By Lemma 2.3.5, $I_2 \cong o(s)$-module W_{j_0} is standard. Hence $s = n$ since $\dim W_{j_0} = \dim W = n$. □

Now we want to show that the decompositions as in Theorem 2.3.7 are possible.

Example 1 *There exists a decomposition of $S \cong sl(2k, n)$ of the form $S = K + L$ where S has the standard matrix realization. The first subalgebra K consists of all matrices in S of the form:*

$$\left\{ \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & X & \\ 0 & & & \end{array} \right) \right\}$$

where X is a matrix of order $(2k + n - 1) \times (2k + n - 1)$. The second subalgebra L consists of all matrices of the form:

$$\left\{ \left(\begin{array}{cc|c} E & F & C \\ H & -E^t & D \\ \hline -D^t & C^t & A \end{array} \right) \right\}$$

where A is a skew-symmetric matrix of order n, H and F are symmetric matrices of order $k \times k$, E is a matrix of order $k \times k$, and C, D are matrices of order $k \times n$.

In this decomposition, $K \cong sl(2k - 1, n)$ and $L \cong osp(n, 2k)$.

Example 2 *There exists a decomposition of $S \cong sl(m, 2k)$ of the form $S = K + L$ where S has the standard matrix realization. The first subalgebra K consists of all matrices in S of the form:*

$$\left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & X & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

where X is a matrix of order $(m+2k-1) \times (m+2k-1)$ with zero trace. The second subalgebra L consists of all matrices of the form:

$$\left\{ \left(\begin{array}{c|cc} A & C & D \\ \hline D^t & E & F \\ -C^t & H & -E^t \end{array} \right) \right\}$$

where A is a skew-symmetric matrix of order m, H and F are symmetric matrices of order $k \times k$, E is a matrix of order $k \times k$, and C, D are matrices of order $m \times k$.

In this decomposition, $K \cong sl(m, 2k-1)$, $L \cong osp(m, 2k)$.

2.4 Uniqueness of decompositions

Lemma 2.4.1 *Let $S \cong osp(m, 2n)$, $S \subseteq gl(m, 2n)$ and*

$$S_0 = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{2.10}$$

where $A \in o(m)$ and $D \in sp(2n)$.

Then there exists an inner automorphism ψ of $gl(m, 2n)$ of the form

$$\psi(X) = CXC^{-1} \tag{2.11}$$

where

$$C = \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & \lambda I_{2n} \end{array}\right)$$

where $\lambda \in \mathbb{F}$ such that $\psi(S)$ takes the standard matrix form.

Proof.

Let S_{st} be a standard realization of $osp(m, 2n)$. Then

$$(S_{st})_0 = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \right\} \tag{2.12}$$

where $A \in o(m)$, $D \in sp(2n)$ and

$$(S_{st})_1 = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \right\} \tag{2.13}$$

where $C = J_n B^t$.

Let φ be an isomorphism between S_{st} and S, $\varphi(S_{st}) = S$. Then $\varphi((S_{st})_0) = S_0$ and $\varphi((S_{st})_1) = S_1$. Notice that $(S_{st})_0 = S_0$ since S_0 is of the form (2.10).

Let π_1 be the projection of $V^* \otimes W \oplus V \otimes W^*$ onto $V \otimes W^*$. We consider S_0-modules $\pi(S_1)$ and $\pi((S_{st})_1)$. We have that S_0-module $V \otimes W^*$ is simple as a tensor product of the simple I_1-module V and the simple I_2-module W^*. Therefore both S_0-modules $\pi_1(S_1)$ and $\pi_1((S_{st})_1)$ coincide with S_0-module $V \otimes W^*$.

Notice that S_0-module S_1 has the following matrix form

$$\left\{ \left(\begin{array}{c|c} 0 & B' \\ \hline C' & 0 \end{array}\right) \right\} \tag{2.14}$$

where $B' \in M_{m\times 2n}(\mathbb{F})$ and $C' \in M_{2n\times m}(\mathbb{F})$. Hence S_0-module $\pi_1(S_1)$ consists of all $m \times 2n$ matrices under the action of S_0 given by

$$x(B') = AB' - B'D \tag{2.15}$$

where $x \in S_0$ is of the form (2.10) and B' is an arbitrary $m \times 2n$ matrix. Likewise, S_0-module $\pi_1((S_{st})_1)$ is the set of all $m \times 2n$ matrices under the following action of S_0:

$$x(B) = AB - BD$$

where $x \in S_0 = (S_{st})_0$ is of the form (2.10) and B is an arbitrary $m \times 2n$ matrix.

On the other hand, both S_0-modules $\pi_1(S_1)$ and $\pi_1((S_{st})_1)$ are isomorphic and have the same matrix form (2.15). Therefore, by Schur's Lemma, the only isomorphism between S_0-modules $\pi_1(S_1)$ and $\pi_1((S_{st})_1)$ is a scalar mapping. That is, there exists μ_1 such that for any $y \in (S_{st})_1$ of the form (2.13), $\varphi(y) \in S_1$ has the form (2.14) where $B' = \mu_1 B$. Similarly we can prove that there exists μ_2 such that $C' = \mu_2 C$.

Thus S_1 takes the form

$$\left\{ \left(\begin{array}{c|c} 0 & \mu_1 B \\ \hline \mu_2 C & 0 \end{array} \right) \right\}.$$

Let ψ be of the form (2.11) where $\lambda = \sqrt{\frac{\mu_1}{\mu_2}}$. Then for any $X \in S_1$

$$CXC^{-1} = \left(\begin{array}{c|c} 0 & \lambda^{-1}\mu_1 B \\ \hline \lambda\mu_2 C & 0 \end{array} \right) = \beta \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right)$$

where $\beta = \sqrt{\mu_1\mu_2}$. Hence $CS_1C^{-1} = (S_{st})_1$. Therefore, by an automorphism ψ, S can be brought to the standard matrix form.

Theorem 2.4.2 *Let $S = K + L$, $S \cong sl(2k, n)$, $K \cong sl(2k-1, n)$ and $L \cong osp(n, 2k)$. Then there exists a basis of $V \oplus W$ such that this decomposition takes the matrix form as in Example 1.*

Proof.

First we are going to prove that there exists a basis of $V \oplus W$ such that in this basis K consists of all matrices in $sl(2k, n)$ with the first row and column zero.

Let π_1, π_2 denote projections of S_0 onto the ideals $sl(2k)$ and $sl(n)$, respectively. These projections induces two decompositions: $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$. By Lemma 2.1.1, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ takes the form $sl(2k) = sl(2k-1) + sp(2k)$. Hence, by Lemma 1.1.3, there exists a basis of V such that this decomposition takes the form (1.1). This implies that K_0 has the form:

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{2.16}$$

where $A \in M_{2k}(\mathbb{F})$ with the first row and column zero, and $D \in M_n(\mathbb{F})$.

Let J_1, J_2 be ideals of K_0, $J_1 \cong sl(2k-1)$ and $J_2 \cong sl(n)$. Notice that $\pi_1(J_1) = \pi_1(K_0)$ and $\pi_2(J_2) = \pi_2(K_0)$ since $\pi_1(K_0) \cong sl(2k-1)$ and $\pi_2(K_0) \cong sl(2n)$. Next $[\pi_1(J_1), \pi_1(J_2)] = \{0\}$ and $[\pi_2(J_1), \pi_2(J_2)] = \{0\}$ since $[J_1, J_2] = \{0\}$. This implies that $\pi_2(J_1) = \{0\}$ and $\pi_1(J_2) = \{0\}$ since $[\pi_1(K_0), \pi_1(J_2)] = \{0\}$ and $[\pi_2(J_1), \pi_2(K_0)] = \{0\}$. Therefore J_1 consists of all matrices of the form (2.16) where $D = 0$, and J_2 consists of all matrices of the form (2.16) where $A = 0$. By Lemma 1.2.1(d), $K_1 = [K_1, J_1]$. This implies that K_1 takes the matrix form:

$$\left\{ \left(\begin{array}{c|c} 0 & AB \\ \hline -CA & D \end{array} \right) \right\}$$

Therefore the first rows and columns of matrices from K_1 are zero since the first row and column of A is zero. This implies that K consists of all matrices in S with the first row and column zero.

On the other hand, by Lemma 2.4.1, there exists an inner automorphism ψ of $gl(2k, n)$ such that $\psi(L)$ takes the standard matrix form. Clearly $\psi(K)$ takes the same matrix form as K. Notice that in this basis $S = sl(2k, n)$ since $S \subseteq gl(2k, n)$. Therefore we have proved that there exists a basis of $V \oplus W$ such that the decomposition $S = K + L$ where $S \cong sl(2k, n)$, $K \cong sl(2k-1, n)$ and $L \cong osp(n, 2k)$, takes the form as in Example 1.

Chapter 3

Decompositions of orthosymplectic superalgebras

3.1 Sum of two special linear superalgebras

In this section we study decompositions of $osp(m, 2n)$ into the sum of two special linear superalgebras.

Theorem 3.1.1 *A Lie superalgebra $S \cong osp(m, 2n)$, m, $n > 0$, cannot be decomposed into the sum of two proper special linear subalgebras.*

Proof. By Lemma 1.2.2(a), $S_0 = o(m) \oplus sp(2n)$. As above we define two projections π_1 and π_2 of S_0 onto the ideals $o(m)$ and $sp(2n)$, $\pi_1 : S_0 \to o(m)$ and $\pi_2 : S_0 \to sp(2n)$. We have that $K_0 \cong sl(p) \oplus sl(q) \oplus U$ and $L_0 \cong sl(s) \oplus sl(l) \oplus U$ since $K \cong sl(p, q)$ and $L \cong sl(s, l)$. Hence the projections $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are also reductive as homomorphic images of reductive algebras.

Since $S = K + L$, S_0 is also decomposable into the sum of two subalgebras K_0 and L_0, $S_0 = K_0 + L_0$. Therefore, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$, where $\pi_1(S_0) = o(m)$ and $\pi_2(S_0) = sp(2n)$. We have the decompositions of simple Lie algebras $o(m)$ and $sp(2n)$ into the sum of two reductive subalgebras.

By Theorem 1.1.2, $sp(2n)$ and $o(m)$ cannot be decomposed into the sum of two subalgebras of these types. As a result, $S \cong osp(m, 2n)$ cannot be decomposed into the sum of $K \cong sl(p, q)$ and $L \cong sl(s, l)$. □

3.2 Sum of two orthosymplectic superalgebras

In this section we study decompositions of $osp(m, 2n)$ into the sum of two proper simple subalgebras $K \cong osp(p, 2q)$ and $L \cong osp(s, 2l)$.

Lemma 3.2.1 *Let $S \cong osp(m, 2n)$, m, $n > 0$, be decomposed into the sum of two proper orthosymplectic subalgebras K and L, respectively. Then only two cases are possible:*

1. $m = 4k$, $K \cong osp(4k-1, 2n)$, $L \cong osp(s, 2k)$
2. $K \cong osp(p, 2n)$, $L \cong osp(m, 2l)$.

Proof.

By Lemma 1.2.2(a), $S_0 = o(m) \oplus sp(2n)$. We define two projections π_1 and π_2 of S_0 onto the ideals $o(m)$ and $sp(2n)$ as follows, $\pi_1 : S_0 \to o(m)$ and $\pi_2 : S_0 \to sp(2n)$. We have that $K_0 \cong o(p) \oplus sp(2q)$ and $L_0 \cong o(k) \oplus sp(2l)$ since $K \cong osp(p, 2q)$ and $L \cong osp(k, 2l)$. Hence $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are also reductive as homomorphic images of reductive algebras.

Since $S = K + L$, S_0 is decomposable into the sum of K_0 and L_0, $S_0 = K_0 + L_0$. Therefore, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$. Moreover, $\pi_1(S_0) = o(m)$ and $\pi_2(S_0) = sp(2n)$, and we have decompositions of simple Lie algebras of the types $o(m)$ and $sp(2n)$ into the sum of two reductive subalgebras. By

Theorem 1.1.2, $sp(2n)$ has no decompositions into the sum of two proper reductive subalgebras of these types. Hence $sp(2n) = \pi_2(K_0) + \pi_2(L_0)$ is a trivial decomposition and either $\pi_2(K_0) = sp(2n)$ or $\pi_2(L_0) = sp(2n)$. For clarity, we assume that $\pi_2(K_0) = sp(2n)$. Hence $q = n$.

Again, by Theorem 1.1.2, $o(m)$ has only two decompositions into the sum of two proper reductive subalgebras:

1. If $m = 2k$ then $o(2k) = o(2k-1) + sl(k)$,
2. If $m = 4k$ then $o(4k) = o(4k-1) + sp(2k)$.

Notice that $o(m) = \pi_1(K_0) + \pi_1(L_0)$ cannot be of the first type, because $\pi_1(K_0)$ and $\pi_1(L_0)$ are not isomorphic to $sl(k)$.

Next the two cases occur:

1. $o(m) = \pi_1(K_0) + \pi_1(L_0)$ has the second form.
2. $o(m) = \pi_1(K_0) + \pi_1(L_0)$ is trivial.

In the first case either $\pi_1(K_0) \cong o(4k-1)$ or $\pi_1(K_0) \cong sp(2k)$. Let $\pi_1(K_0) \cong sp(2k)$. Hence either $K_0 \cong sp(2k) \oplus sp(2n)$ or $K_0 \cong sp(2n)$ since $\pi_2(K_0) = sp(2n)$. This contradicts the fact that $K_0 \cong o(p) \oplus sp(2q)$. Therefore $\pi_1(K_0) \cong o(4k-1)$ and $\pi_1(L_0) \cong sp(2k)$. This implies that $p = 4k-1$ and $l = k$ since $K_0 \cong o(p) \oplus sp(2q)$ and $L_0 \cong o(s) \oplus sp(2l)$.

In the second case either $\pi_1(K_0) = o(m)$ or $\pi_1(L_0) = o(m)$. Let $\pi_1(K_0) = o(m)$. Therefore K_0 coincides with S_0 since $\pi_2(K_0) = sp(2n)$. This contradicts the fact that K is proper subalgebra of S. Therefore $\pi_1(L_0) = o(m)$. It follows that $s = m$ since $L_0 \cong o(s) \oplus sp(2l)$. □

Corollary 3.2.2 *Let $S = K + L$, $K \cong osp(4k-1, 2n)$, $L \cong osp(s, 2k)$ and $I_1 \cong sp(2k)$, $I_2 \cong o(s)$ be ideals of L_0. Then I_2 acts trivially on V. Moreover $V = V_1 \oplus V_2$,* *and both I_1-modules V_1 and V_2 are standard.*

Proof. The proof follows from the fact that $o(m) = \pi_1(K_0) + \pi_1(L_0)$ has the form(1.5) and $\pi_1(I_1) = \pi_1(L_0)$, $\pi_1(I_2) = \{0\}$ since $[\pi_1(I_1), \pi_1(I_2)] = \{0\}$. □

Corollary 3.2.3 *Let $S = K + L$ and $K \cong osp(p, 2n)$, $L \cong osp(m, 2l)$ and $I_1 \cong o(m)$, $I_2 \cong sp(2l)$ be ideals of L_0. Then I_2 acts trivially on V. Moreover I_1-module V is standard.*

Proof. The proof follows from the fact that $\pi_1(I_1) = \pi_1(L_0) = o(m)$ and $\pi_1(I_2) = \{0\}$ since $[\pi_1(I_1), \pi_1(I_2)] = \{0\}$. □

3.2.1 Sum of $osp(p, 2q)$ and $osp(m, 2l)$

In this section we consider the second type of the decomposition from Lemma 3.2.1.

Remark 3.2.1 *Since both superalgebras K and L have the same type, by Lemma 1.4.2, we can assume that L has a trivial vector annihilator in $gl(m, 2n)$.*

Lemma 3.2.4 *Let $S = K + L$ where $S \cong osp(m, 2n)$, $K \cong osp(p, 2n)$, $L \cong osp(m, 2l)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 1.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 1. By Remark 3.2.1, L has a trivial vector annihilator in $gl(m, 2n)$. Let us consider $I_2 \subseteq L$. By Corollary 3.2.3, I_2 acts trivially on V. Moreover I_2 acts trivially on W_{j_0} since L_0-module W_{j_0} is of the type 1. Therefore, by Lemma 1.4.3, L has a vector annihilator in $gl(m, 2n)$, which is a contradiction. □

Lemma 3.2.5 *Let $S = K + L$ where $S \cong osp(m, 2n)$, $K \cong osp(p, 2n)$ and $L \cong osp(m, 2l)$. Then for any $j \in \{1 \dots d\}$, L_0-module W_j is not of the type 2.*

Proof.

The proof of this lemma is similar to the proof of Lemma 2.1.4.

Lemma 3.2.6 *Let $S = K + L$ where $S \cong osp(m, 2n)$, $K \cong osp(p, 2n)$ and $L \cong osp(m, 2l)$. Then L_0-module W contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$ of the type 3.*

Proof. The proof of this lemma is similar to proof of Lemma 2.3.6. □

Lemma 3.2.7 *A Lie superalgebra $S \cong osp(m, 2n)$ cannot be decomposed into the sum of two proper simple subalgebras K and L of the type $osp(p, 2n)$ and $osp(m, 2l)$, respectively.*

Proof.

Let us assume that this decomposition exists. Then, by Lemma 3.2.6, L_0-module W contains at most one L_0-submodule W_j, $j \in \{1 \dots d\}$ of the type 3.

On the other hand, I_2 acts nontrivially on W since, by Corollary 3.2.3, I_2 acts trivial on V. Therefore W contains at least one L_0-submodule W_{j_0}. This implies that W_{j_0} coincides with W. By Lemma 2.3.5, $I_2 \cong sp(2l)$-module W_{j_0} is standard. Hence $2l = 2n$ since $\dim W_{j_0} = \dim W = 2n$. This contradicts the fact that $L \cong osp(m, 2l)$ is a proper subalgebra of $S \cong osp(m, 2n)$. □

3.2.2 Sum of $osp(4k-1, 2q)$ and $osp(s, 2l)$

In this section we consider the first type of the decomposition from Lemma 3.2.1

Lemma 3.2.8 *Let $S = K + L$ where $S \cong osp(4k, 2n)$, $K \cong osp(4k-1, 2n)$, $L \cong osp(s, 2k)$. Then L_0-module W_j, $j \in \{1 \dots d\}$ is not of the type 1.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.3.3.

Lemma 3.2.9 *Let $S = K + L$ where $S \cong osp(4k, 2n)$, $K \cong osp(4k-1, 2n)$, $L \cong osp(s, 2k)$. Then for any $j \in \{1 \dots d\}$, L_0-module W_j is not of the type 2.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 2. By Lemma 1.4.8, there exist subspaces $W'_{j_0} \subseteq W_{j_0}$ and $W''_{j_0} \subseteq W_{j_0}$ such that W'_{j_0} is a simple I_1-module, W''_{j_0} is a simple I_2-module and $W_{j_0} \cong W'_{j_0} \otimes W''_{j_0}$.

First we are going to show that $\dim W'_{j_0} = 2k$ and $\dim W''_{j_0} = s$. We have that $\dim W'_{j_0} \geq 2k$ and $\dim W''_{j_0} \geq s$, respectively, since W'_{j_0} is a simple $sp(2k)$-module, and W''_{i_0} is a simple $o(s)$-module. For clarity, we assume that $\dim W'_{j_0} > 2k$. Hence

$$2n = \dim W \geq \dim W_{j_0} = (\dim W'_{j_0}) \cdot (\dim W''_{j_0}) > 2ks.$$

On the other hand,

$$\dim L_1 \geq \dim S_1 - \dim K_1 \geq (4k)(2n) - (4k-1)(2n) = 2n$$

since $\dim S_1 \leq \dim K_1 + \dim L_1$. It follows that $2ks \geq 2n$ since $\dim L_1 = 2ks$. This contradicts the fact that $2n > 2ks$. Therefore $\dim W'_{j_0} = 2k$, $\dim W''_{j_0} = s$ and $W = W_{j_0}$. Let W' and W'' denote W'_{j_0} and W''_{j_0}, respectively. Thus $I_1 \oplus I_2$-modules W and $W' \otimes W''$ are isomorphic.

Let us fix the following basis for W: $\{e'_i \otimes e''_j\}$ where $\{e'_i\}$ is a basis of W' and $\{e''_j\}$ is a basis of W''. If we consider W as an I_1-module then it can be expressed as the direct sum of I_1-modules $W' \otimes e''_j$:

$$W = (W' \otimes e''_1) \oplus \dots \oplus (W' \otimes e''_s). \tag{3.1}$$

Clearly the projection of L_1 onto $V \otimes W^*$ is not zero. Therefore there exists $i_0 \in \{1, 2\}$ such that the projection of L_1 onto $V_{i_0} \otimes W^*$ is not zero. Let us consider $V_{i_0} \otimes W^*$ as an I_1-module. From (3.1) we obtain that

$$V_{i_0} \otimes W^* = (V_{i_0} \otimes (W' \otimes e''_1)^*) \oplus \ldots \oplus (V_{i_0} \otimes (W' \otimes e''_s)^*)$$

where $V_{i_0} \otimes (W' \otimes e''_j)^*$ are also I_1-modules. The projection of L_1 onto $V_{i_0} \otimes (W' \otimes e''_j)^*$ is not zero for some j_0 since the projection of L_1 onto $V^*_{i_0} \otimes W$ is not zero. We consider this I_1-module $V^*_{i_0} \otimes (W' \otimes e''_{j_0})$. By Corollary 3.2.2, I_1-module V_{i_0} is standard. We have already proved that I_1-module W' is standard with the highest weight $(1, 0, \ldots, 0)$.

Next we apply generalized Young tableaux technique (see [10]) to find simple submodules of I_1-module $(V_{i_0} \otimes W'^*) \otimes e''^*_{j_0}$.

If ϱ and ϱ' are standard representations of $sp(2k)$ $(o(k))$ with the same highest weight $(1, 0, \ldots, 0)$ then the tensor product $\varrho \otimes \varrho'$ is also a representation of $sp(2k)$ $(o(k))$. It can be decomposed into the direct sum of irreducible representations:

$$\varrho \otimes \varrho' = \varrho_1 \oplus \varrho_2 \oplus \varrho_3$$

where ϱ_1 has the highest weight $(2, 0, \ldots, 0)$, ϱ_2 has the highest weight $(0, 1, 0, \ldots, 0)$ and ϱ_3 is a trivial representation.

Therefore I_1-module $(V_{i_0} \otimes W'^*) \otimes e''^*_{j_0}$ contains only submodules with the highest weights $(2, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$. This contradicts the fact that, by Lemma 1.2.2(e), I_1-module L_1 has only simple submodules of dimension $2k$ with the highest weight $(1, 0, \ldots, 0)$. □

Lemma 3.2.10 *Let $S = K + L$ where $S \cong osp(4k, 2n)$, $K \cong osp(4k-1, 2n)$, $L \cong osp(s, 2k)$. Then L_0-module W contains at most two L_0-submodules W_j.*

Proof.

We have already proved that for any $j \in \{1 \dots d\}$ L_0-module W_j is of the type 3. Let us assume the contrary, that is, there exist three L_0-submodules of the type 3. Let W_1, W_2 and W_3 stand for these L_0-submodules.

Next we restrict our attention only to these submodules of W. By Lemma 2.3.5, W_1, W_2 and W_3 are standard $o(s)$-modules, and V_1, V_2 are standard $sp(2k)$-modules. Hence there exists a basis of W such that L_0 takes the following form

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{3.2}$$

where $A = diag(X, X)$, $X \in sp(2k)$ and $D = diag(Y, Y, Y)$, $Y \in o(s)$. This result follows from the fact that any automorphism of $sp(2k)$ and $o(s)$ is inner. Besides, L_1 has the following form

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{3.3}$$

where $B = (B_{ij})$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$.

Next we consider $I_1 \oplus I_2$-modules $V_i \otimes W_j^*$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. In matrix terms $I_1 \oplus I_2$-modules $V_i \otimes W_j^*$ consist of all $2k \times s$ matrices and the action of $I_1 \oplus I_2$ is given by the following formula:

$$x(M) = XM - MY$$

where $x \in I_1 \oplus I_2$ is of the form (3.2), and M is an arbitrary $2k \times s$ matrix. Next we consider $I_1 \oplus I_2$-modules $\varrho_{ij}(V_i \otimes W_j^*)$. Acting in the same manner as in Lemma 2.1.7, we can prove that $I_1 \oplus I_2$-modules $\varrho_{ij}(V_i \otimes W_j^*)$ are simple and homomorphic images of $I_1 \oplus I_2$-module L_1. Hence, by Schur's Lemma, the only isomorphism between these $I_1 \oplus I_2$-modules is a scalar mapping. This implies that for any matrix in L_1

of the form (3.3) we have that $B_{ij} = \omega_{ij} M$ where $M \in Mat_{k\times s}(\mathbb{F})$ and $\omega_{ij} \in \mathbb{F}$, $i \in \{1,2\}$, $j \in \{1,2,3\}$.

Let B_j denote $\begin{pmatrix} B_{1j} \\ B_{2j} \end{pmatrix}$. By Lemma 1.4.5, L_1 cannot be of the form (3.3) where $B_2 = \nu B_3$, $\nu \in \mathbb{F}$. Therefore vectors $\bar{\omega}_2 = \begin{pmatrix} \omega_{12} \\ \omega_{22} \end{pmatrix}$ and $\bar{\omega}_3 = \begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix}$ are linearly independent. Thus we can represent $\bar{\omega}_1 = \begin{pmatrix} \omega_{11} \\ \omega_{21} \end{pmatrix}$ as a linear combination of $\bar{\omega}_2$ and $\bar{\omega}_3$, i.e. $\bar{\omega}_1 = \lambda\bar{\omega}_2 + \mu\bar{\omega}_3$. It follows that for any element from L_1 of the form (3.3), we obtain that $B_1 = \lambda B_2 + \mu B_3$.

Next we consider a commutator of two arbitrary elements from L_1 of the form

$$\left(\begin{array}{c|ccc} 0 & B_1 & B_2 & B_3 \\ \hline C_1 & & & \\ C_2 & & 0 & \\ C_3 & & & \end{array}\right)$$

and

$$\left(\begin{array}{c|ccc} 0 & B_1' & B_2' & B_3' \\ \hline C_1' & & & \\ C_2' & & 0 & \\ C_3' & & & \end{array}\right).$$

In turn their commutator takes the following form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$$

where $D = (D_{ij})$ and $D_{ij} = C_iB_j' + C_i'B_j$, $i \in \{1,2\}$, $j \in \{1,2,3\}$.

We have that $B_1 = \lambda B_2 + \mu B_3$ and $B_1' = \lambda B_2' + \mu B_3'$. Therefore $D_{11} = C_1B_1' + C_1'B_1 = C_1(\lambda B_2' + \mu B_3') + C_1'(\lambda B_2 + \mu B_3) = \lambda(C_1B_2' + C_1'B_2) + \mu(C_1B_3' + C_1'B_3) =$

$\lambda D_{12} + \mu D_{13}$. Since $[L_1, L_1] \subseteq L_0$ has the form (3.2), $D_{12} = 0$ and $D_{13} = 0$. Thus $D_{11} = 0$.

On the other hand, L_0-module W_1 is not trivial. Therefore there exists an element from L_0 of the form (3.2) such that $D_{11} \neq 0$, which is a contradiction.

□

The following technical lemma will be used in our later discussion.

Lemma 3.2.11 *The Lie algebra $S = sp(2n)$, $n > 0$, does not contain a subalgebra $K \cong o(2n)$.*

Proof.

Let us fix an arbitrary basis in V, $\dim V = 2n$. Then S can be represented as the following set $S = \{X : CXC^{-1} = -X^t \text{ where } C = C^t,\ C \in M_{2n}(\mathbb{F})\}$. In this basis K can be represented as follows: $K = \{X : DXD^{-1} = -X^t \text{ where } D = -D^t,\ D \in M_{2n}(\mathbb{F})\}$. This implies that $CXC^{-1} = DXD^{-1}$ for any $X \in K$. Thus $XC^{-1}D = C^{-1}DX$ for any $X \in K$. Since $K \cong o(2n)$, K is an irreducible subset of $gl(2n)$. It follows that $C^{-1}D = \lambda I_n$ and $C = \lambda D$, $\lambda \in \mathbb{F}$. However, C is symmetric and D is skew-symmetric. Thus $sp(2n)$ does not contain a Lie subalgebra of the type $o(2n)$. □

Theorem 3.2.12 *Let $S = osp(4k, 2n)$, m, $n > 0$, be decomposed into the sum of two proper simple subalgebras K and L of the types $osp(4k-1, 2n)$ and $osp(s, 2k)$, respectively. Then $s = n$.*

Proof.

Let us consider L_0-modules $W = W_1 \oplus \ldots \oplus W_d$. For any $j \in \{1 \ldots d\}$ L_0-module W_j is of the type 3. Moreover, by Lemma 2.3.5, I_2-module W_j has dimension s.

Hence $\pi_2(I_2) \neq 0$, $\pi_2(I_2) \subseteq sp(2n)$ and $I_2 \cong o(s)$. It follows that, by Lemma 3.2.11, $s < 2n$. Therefore

$$\dim W_j = s < 2n = \dim W,$$

and W contains at least two L_0-modules W_1 and W_2 of type 3.

Next, by Lemma 3.2.10, $d = 2$. It follows that $s = \dim W_1 = \dim W/2 = n$. □

Example 3 *There exists a decomposition of $S \cong osp(4k, 2n)$ into the sum of two simple subalgebras K and L of the types $osp(4k-1, 2n)$ and $osp(n, 2k)$, respectively. Moreover, in this decomposition S is considered in the standard matrix realization*

$$\left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \right\}$$

where $A \in o(4k)$ and $D \in sp(2n)$ and $C = J_n B^t$, J_n given by

$$J_n = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right)$$

The first subalgebra $K \cong osp(4k-1, 2n)$ has the form:

$$\left\{ \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & X & \\ 0 & & & \end{array} \right) \right\}$$

where X is any $(4k + 2n - 1) \times (4k + 2n - 1)$ orthosymplectic matrix.

The second subalgebra $L \cong osp(n, 2k)$ consists of all matrices of the form:

$$\left\{ \left(\begin{array}{cc|cc} A - A^t & -i(A + A^t) & P & Q^t \\ i(A + A^t) & A - A^t & iP & -iQ^t \\ \hline Q & -iQ & D & 0 \\ -P^t & -iP^t & 0 & D \end{array} \right) \right\} \tag{3.4}$$

where $A \in sp(2k)$, $D \in o(n)$ and P is a matrix of order $2k \times n$, $Q = P^t J$.

Then $S = K + L$ is a decomposition of a simple Lie superalgebra into the sum of two simple subalgebras.

Proof.

First we prove that the set of matrices (3.4) is actually a subalgebra of the type $osp(n, 2k)$. The standard matrix realization of $osp(n, 2k)$ is

$$\left\{ \left(\begin{array}{c|c} A & P \\ \hline Q & D \end{array} \right) \right\}$$

where $A \in sp(2k)$, $D \in o(n)$ and P is a matrix of order $2k \times n$, $Q = P^t J$. It is easy to see that $osp(n, 2k)$ has another matrix realization:

$$\left(\begin{array}{c|c} -A^t & Q^t \\ \hline -P^t & -D^t \end{array} \right).$$

It follows that $L' \cong osp(n, 2k)$ can be imbedded into $gl(4k, 2n)$ as follows:

$$\left\{ \left(\begin{array}{cc|cc} A & 0 & P & 0 \\ 0 & -A^t & 0 & Q^t \\ \hline Q & 0 & D & 0 \\ 0 & -P^t & 0 & -D^t \end{array} \right) \right\}$$

Let $\bar{\chi}$ be an automorphism of $gl(4k, 2n)$ of the form

$$\bar{\chi}(X) = \bar{Q} X \bar{Q}^{-1} \tag{3.5}$$

where

$$\bar{Q} = \left(\begin{array}{c|c} Q_{2k} & 0 \\ \hline 0 & I_{2n} \end{array} \right)$$

where Q_{2k} has a form (1.6).

Using straightforward calculations we can show that $\bar{\chi}(L')$ has the form (3.4). Denote $\bar{\varphi}(L')$ as L. Therefore the set of matrices of the form (3.4) forms $osp(n, 2k)$. Clearly $L \cong osp(4k-1, 2n)$.

Next we will prove that the sum of two vector spaces K and L coincides with S. Set

$$P = \left(\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array} \right).$$

Then

$$Q^t = -J_n P = \left(\begin{array}{c|c} -P_3 & -P_4 \\ \hline P_1 & P_2 \end{array} \right).$$

Since P is an arbitrary matrix from $M_{k,n}(\mathbb{F})$, the first rows of matrices from L are arbitrary. Similarly the first column of matrices from L is also arbitrary. Therefore $S = K + L$. □

3.3 Sum of special linear and orthosymplectic superalgebras

Here we consider decompositions of the form $S = K + L$ where $S \cong osp(m, 2n)$, $K \cong osp(p, 2q)$ and $L \cong sl(s, l)$.

Lemma 3.3.1 *Let $S = osp(m, 2n)$ be a Lie superalgebra, and S be decomposed into the sum of a proper orthosymplectic subalgebra K and a special linear subalgebra L. Then m is even, $m = 2k$, $K \cong osp(2k-1, 2n)$ and $L \cong sl(k, l)$.*

Proof. By Lemma 1.2.2(a), $S_0 = o(m) \oplus sp(2n)$. Let π_1 and π_2 denote projections of S_0 onto the ideals $o(m)$ and $sp(2n)$, respectively. Since K is isomorphic to $osp(p, 2q)$,

K_0 is isomorphic to $o(p) \oplus sp(2q)$. By Lemma 1.2.1(a), L_0 is isomorphic to $sl(l_1) \oplus sl(l_2) \oplus U$. Since K_0 and L_0 are reductive subalgebras, the projections $\pi_1(K_0)$, $\pi_1(L_0)$, $\pi_2(K_0)$ and $\pi_2(L_0)$ are also reductive.

As usual, $S = K + L$ induces the decomposition of S_0 of the form $S_0 = K_0 + L_0$. Therefore, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$ where $\pi_1(S_0) = o(m)$ and $\pi_2(S_0) = sp(2n)$. By Theorem 1.1.2, $sp(2n)$ cannot be decomposed into the sum of two proper reductive subalgebras. Hence $sp(2n) = \pi_2(K_0) + \pi_2(L_0)$ is a trivial decomposition and $\pi_2(K_0) = sp(2n)$. It follows that $q = n$ since $K_0 = o(p) \oplus sp(2q)$.

Again, by Theorem 1.1.2, $o(m)$ only has the following nontrivial decompositions into the sum of two proper reductive subalgebras:

1. If $m = 2k$ then $o(2k) = o(2k - 1) + sl(k)$,
2. If $m = 4k$ then $o(4k) = o(4k - 1) + sp(2k)$.

Notice that $o(m) = \pi_1(K_0) + \pi_1(L_0)$ cannot be trivial. Indeed, assume that this decomposition is trivial. Hence $\pi_1(K_0) = \pi_1(S_0) \cong o(m)$ and $p = m$. Thus $K \cong osp(p, 2n)$ coincides with $S = osp(m, 2n)$, which is a contradiction. Moreover $o(m) = \pi_1(K_0) + \pi_1(L_0)$ cannot be of the second type, because $\pi_1(L_0)$ is not of the type $sp(2k)$. Therefore $o(m) = \pi_1(K_0) + \pi_1(L_0)$ is a decomposition of the first type and $m = 2k$, $\pi_1(K_0) \cong o(2k-1)$, $\pi_1(L_0) \cong sl(k)$. This implies that $p = 2k-1$, $q = n$, and either $l_1 = k$ or $l_2 = k$, since $K_0 \cong o(p) \oplus sp(2q)$ and $L_0 \cong sl(l_1) \oplus sl(l_2) \oplus U$. Set either $l = l_1$ if $l_2 = k$ or $l = l_2$ if $l_1 = k$. Therefore $L \cong sl(k, l)$. □

Corollary 3.3.2 *Let $S = K + L$, $K \cong osp(2k - 1, 2n)$, $L \cong sl(k, l)$ and $I_1 \cong sl(k)$, $I_2 \cong sl(l)$ be ideals of L_0. Then I_2 acts trivially on V. Moreover $V = V_1 \oplus V_2$ where I_1-module V_1 is standard, and I_1-module V_2 is dual.*

Lemma 3.3.3 *Let $S = K + L$ where $S \cong osp(2k, 2n)$, $K \cong osp(2k-1, 2n)$, $L \cong sl(k,l)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 1.*

Proof.

Let us assume the contrary, that is, there exists j_0 such that L_0-module W_{j_0} is of the type 1. First we prove that K has a nontrivial vector annihilator in $gl(m,n)$. Let $K = J_1 \oplus J_2$ where $J_1 \cong o(2k-1)$ and $J_2 \cong sp(2n)$. As was shown in Lemma 3.3.1, $\pi_2(S_0) = \pi_2(K_0) \cong sp(2n)$. Therefore $\pi_2(J_2) = \pi_2(K_0) \cong sp(2n)$ since $J_2 \cong sp(2n)$. However $[\pi_2(J_1), \pi_2(J_2)] = \{0\}$ since $[J_1, J_2] = \{0\}$. Therefore, $\pi_2(J_1) = \{0\}$ since $\pi_2(J_1) \subseteq \pi_2(K_0) = \pi_2(J_2)$.

By Lemma 3.3.1, $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ has the form $o(2k) = o(2k-1) + sl(k)$. Therefore, by Remark 1.1.1, $\pi_1(K_0)$ has a nontrivial annihilator in $gl(m)$. Hence $\pi_1(J_1)$ also has a nontrivial annihilator in $gl(m)$. So we obtain that J_1 is an ideal of K_0, $K \subseteq gl(m, 2n)$, and J_1 acts trivially on W and on a one-dimensional subspace of V. Hence, by Lemma 1.4.3, K has a nontrivial vector annihilator in $gl(m,n)$.

By Lemma 1.4.2, L has a trivial vector annihilator in $gl(m,n)$. Let us consider $I_2 \subseteq L$. By Corollary 3.3.2, I_2 acts trivially on V. Moreover I_2 acts trivially on W_{j_0} since L_0-module W_{j_0} is of the type 1. Therefore, by Lemma 1.4.3, L has a nontrivial vector annihilator in $gl(m,n)$, which is a contradiction. □

Lemma 3.3.4 *Let $S = K + L$ where $S \cong osp(2k, 2n)$, $K \cong osp(2k-1, 2n)$, $L \cong sl(k,l)$. Then for any $j \in \{1 \ldots d\}$, L_0-module W_j is not of the type 2 .*

Proof. The proof of this lemma is similar to the proof of Lemma 2.1.4.

Lemma 3.3.5 *Let $S = K + L$ where $S \cong osp(2k, 2n)$, $K \cong osp(2k-1, 2n)$, $L \cong sl(k,l)$. Then for any pairwise different $j_1, j_2 \in \{1 \ldots d\}$, I_2-module W_{j_1} is not isomorphic to I_2-module W_{j_2}.*

Proof.

Let us assume the contrary, that is, L_0-modules W_{j_1} and W_{j_2} are isomorphic. Any L_0-module W_j, $j \in \{1 \dots d\}$, is of the type 3. Moreover, by Lemma 2.1.5, I_2-module W_j is either standard or dual.

There is no loss in generality if we consider the case when I_2-module W_{j_1} is standard. Hence L_0-module W_{j_2} is also standard.

Let $\lambda = (1, 0, \dots, 0)$ be the highest weight of I_1-module V, and $\mu = (1, 0, \dots, 0)$ be the highest weight of I_2-modules W_{j_1} and W_{j_2}. Then, by Lemma 1.4.7, the following statements hold true:

1. $I_1 \oplus I_2$-modules $V_1 \otimes W^*_{j_1}$ and $V_1 \otimes W^*_{j_2}$ have the same highest weight (λ, μ^*), where $\mu^* = (1, 0, \dots, 0)$.

2. $I_1 \oplus I_2$-modules $V_2 \otimes W^*_{j_1}$ and $V_2 \otimes W^*_{j_2}$ have the same highest weight (λ^*, μ^*), where $\lambda^* = (0, \dots, 0, 1)$.

By Lemma 1.2.1(b), $I_1 \oplus I_2$-module L_1 is the direct sum of two simple submodules with the highest weights (λ, μ^*) and (λ^*, μ). Hence the projections of L_1 onto $V_2 \otimes W^*_{j_1}$ and $V_2 \otimes W^*_{j_2}$ are zero since L_0-module L_1 contains no submodules with the highest weight (λ^*, μ^*).

Next we fix a basis in $V \oplus W$ of vectors of subspaces V_i, $i \in \{1, 2\}$, and W_j, $j \in \{1 \dots d\}$ such that L_0 takes the following form

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \tag{3.6}$$

where $A = diag(X, -X^t)$, $X \in sl(k)$ and $D = diag(D_1, \dots, D_d)$, $D_j \in M_{n_j}(\mathbb{F})$ such that $D_1 = D_2 = Y$, $Y \in sl(l)$. Besides, L_1 has the following form

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{3.7}$$

where $B = (B_{ij})$, $i \in \{1, 2\}$, $j \in \{1 \dots d\}$ and $B_{ij} \in M_{2k,n_j}(\mathbb{F})$,

Next we look at a pair of $I_1 \oplus I_2$-modules $V_1 \otimes W_1^*$ and $V_1 \otimes W_2^*$. The matrix realization of the first module consists of all $k \times l$ matrices, and the action of $I_1 \oplus I_2$ is given by the following formula:

$$x(B_{11}) = XB_{11} - B_{11}Y \tag{3.8}$$

where $x \in I_1 \oplus I_2$ of the form (3.6) and B_{11} is an arbitrary $k \times l$ matrix. Similarly, the second module is the set of all $k \times l$ matrices under the following action:

$$x(B_{12}) = XB_{12} - B_{12}Y$$

where $x \in I_1 \oplus I_2$ of the form (3.6), and B_{12} is an arbitrary $k \times l$ matrix.

Let $I_1 \oplus I_2$-module $\varrho_{11}(L_1)$ be the projection of $I_1 \oplus I_2$-module L_1 onto $V_1 \otimes W_1^*$, and $\varrho_{12}(L_1)$ be the projection of $I_1 \oplus I_2$-module L_1 onto $V_1 \otimes W_2^*$. By Lemma 1.4.7, $I_1 \oplus I_2$-modules $V_1 \otimes W_1^*$ and $V_1 \otimes W_2^*$ are simple. Hence $I_1 \oplus I_2$-module $\varrho_{11}(L_1)$ coincides with $V_1 \otimes W_1^*$, and $I_1 \oplus I_2$-module $\varrho_{12}(L_1)$ coincides with $V_1 \otimes W_2^*$. Therefore $I_1 \oplus I_2$-modules $V_1 \otimes W_1^*$ and $V_1 \otimes W_2^*$ have the same matrix form (3.8). On the other hand, $\varrho_{11}(L_1)$ and $\varrho_{12}(L_1)$ are isomorphic as $I_1 \oplus I_2$-modules since they are both simple and homomorphic images of $I_1 \oplus I_2$-module L_1. Hence, by Schur's Lemma, any isomorphism between $I_1 \oplus I_2$-modules $\varrho_{11}(L_1)$ and $\varrho_{12}(L_1)$ is a scalar mapping. In matrix terms this implies that for any matrices from L_1 of the form (3.7), $B_{11} = \lambda B_{12}$, $\lambda \in \mathbb{F}$.

On the other hand, we have already proved that the projections of L_1 onto $V_2 \otimes W_{j1}^*$ and $V_2 \otimes W_{j2}^*$ are zero. Therefore for any matrices from L_1 of the form (3.7), we have that $B_{21} = B_{22} = 0$. However, by Lemma 1.4.5, L_1 cannot be of this form, a contradiction. □

Theorem 3.3.6 *Let $S = osp(m, 2n)$, m, $n > 0$, be decomposed into the sum of two proper simple subalgebras K and L of the type $osp(p, 2q)$ and $sl(s, l)$, respectively. Then m is even, $m = 2k$, $K \cong osp(2k-1, 2n)$ and $L \cong sl(k, n)$*

Proof.

By Lemma 3.3.1, $m = 2k$ and $K \cong osp(2k-1, 2n)$, $L \cong sl(k, l)$. Hence it remains to prove that $l = n$. Let us consider L_0-modules $W = W_1 \oplus \ldots \oplus W_d$. By Lemmas 3.3.3 and 3.3.4, for any $j \in \{1 \ldots d\}$ L_0-module W_j is not of the type 1 and 2. Hence any L_0-module W_j is of the type 3. Moreover, by Lemma 2.1.5, I_2-module W_j has dimension l.

Therefore $\pi_2(I_2) \neq 0$ and $\pi_2(I_2) \subseteq sp(2n)$ where $I_2 \cong sl(l)$. It follows that $l < 2n$. Hence $\dim W_j = l < 2n = \dim W$. Therefore W contains at least two L_0-modules W_1 and W_2 of type 3.

Next we show that $d = 2$. Let us assume the contrary, that is, there exists L_0-module W_3. Since L_0-module W_3 is of the type 3, it follows that L_0-module W_3 is either standard or dual. By Lemma 3.3.5, L_0-modules W_1 and W_2 are not isomorphic. Therefore L_0-module W_3 is isomorphic to either L_0-module W_1 or L_0-module W_2. However, this conflicts with Lemma 3.3.5. This implies that $d = 2$ and $l = \dim W_1 = (\dim W)/2 = n$. Therefore $L \cong sl(k, n)$. □

Corollary 3.3.7 *Let I_1 and I_2 be ideals of L_0 defined above. Then I_1 acts trivially on W, and I_2 acts trivially on V. Moreover $V = V_1 \oplus V_2$ where I_1-module V_1 is* *standard, I_1-module V_2 is dual, and $W = W_1 \oplus W_2$ where I_2-module W_1 is standard, I_2-module W_2 is dual.*

Now we want to show that the decompositions as in Theorem 3.3.6 are possible.

Example 4 *There exists a decomposition of $S \cong osp(2k, 2n)$ into the sum of two simple subalgebras K and L of the types $osp(2k-1, 2n)$ and $sl(k, n)$, respectively. Moreover, in this decomposition S is considered in the standard matrix realization*

$$\left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \right\}$$

where $A \in o(2k)$ and $D \in sp(2n)$, $C = J_n B^t$, J_n is given by

$$J_n = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right);$$

K is taken in the form:

$$\left\{ \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & X & \\ 0 & & & \end{array} \right) \right\}$$

where X is any $(2k+2n-1) \times (2k+2n-1)$ orthosymplectic matrix.

The second subalgebra $L \cong sl(k, n)$ consists of all matrices of the form:

$$\left\{ \left(\begin{array}{cc|cc} E & -F & P & Q^t \\ F & E & iP & -iQ^t \\ \hline Q & -iQ & D & 0 \\ -P^t & -iP^t & 0 & -D^t \end{array} \right) \right\} \tag{3.9}$$

where E is a skewsymmetric matrix of order k, F is a symmetric matrix of order k, P is a matrix of order $k \times n$, Q is a matrix of order $n \times k$ and D is a matrix of order n with zero trace.

Then $S = K + L$ is a decomposition of a simple Lie superalgebra as the sum of two simple subalgebras.

Proof.

First we prove that the set of matrices (3.9) is actually a subalgebra of the type $sl(k,n)$. The standard matrix realization of $sl(k,n)$ is the following:

$$\left\{ \left(\begin{array}{c|c} X & P \\ \hline Q & Y \end{array} \right) \right\}$$

where $X \in sl(k)$, $Y \in sl(n)$ and P is a matrix of order $k \times n$, Q is a matrix of order $n \times k$. Hence there is another matrix realization of $sl(k,n)$:

$$\left\{ \left(\begin{array}{c|c} -X^t & Q^t \\ \hline -P^t & -Y^t \end{array} \right) \right\}$$

It follows that $L' \cong sl(k,n)$ can be imbedded into $gl(2k,2n)$ as follows:

$$\left\{ \left(\begin{array}{cc|cc} X & 0 & P & 0 \\ 0 & -X^t & 0 & Q^t \\ \hline Q & 0 & Y & 0 \\ 0 & -P^t & 0 & -Y^t \end{array} \right) \right\}$$

Let $\bar{\chi}$ be an automorphism of $gl(2k,2m)$ of the form

$$\bar{\chi}(X) = \bar{Q}X\bar{Q}^{-1} \tag{3.10}$$

where

$$\bar{Q} = \left(\begin{array}{c|c} Q_k & 0 \\ \hline 0 & I_{2n} \end{array} \right)$$

where Q_k has a form (1.6). The direct calculation gives us that $\bar{\chi}(L')$ has the form (3.9) where $E = A - A^t$, $F = i(A + A^t)$. Therefore the set of matrices (3.9) forms $sl(k, n)$.

Next we prove that the sum of two vector spaces K and $L = \bar{\chi}(L')$ coincides with S. Set

$$B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right).$$

Then

$$C = JB^t = \left(\begin{array}{c|c} B_{12}^t & B_{22}^t \\ \hline -B_{11}^t & -B_{21}^t \end{array}\right)$$

We set $B_{11} = P$ and $B_{12} = Q^t$. Then $B_{12}^t = Q$ and $-B_{11}^t = -P^t$. Since P and Q are arbitrary matrices of order $k \times n$ and $n \times k$, respectively, the set of the first rows of matrices from L coincides with that of matrices from S. The same is true for the set of the first columns of matrices from L. Hence, $S = K + L$. □

3.4 Uniqueness of decompositions

First we prove the following technical lemma

Lemma 3.4.1 *Let $S \cong sp(2n)$, $S \subseteq gl(2n)$, and for any $X \in sl(n)$,*

$$\left(\begin{array}{c|c} X & 0 \\ \hline 0 & -X^t \end{array}\right) \in S.$$

Then S has the form

$$\left\{\left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array}\right)\right\} \tag{3.11}$$

where B, C are symmetric matrices of order n.

Proof.

Let L be the set of all matrices of the form

$$\left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & -X^t \end{array} \right) \right\}$$

where $X \in sl(n)$. Clearly, $L \subset S$ and $L \cong sl(n)$.

We are given that $gl(V) = V \otimes V^*$ where $V = V_1 \oplus V_2$ and both V_1, V_2 are vector column spaces of dimension n. Clearly, V_1 and V_2 are simple L-modules with the highest weights $\lambda = (1, 0, \ldots, 0)$ and $\lambda^* = (0, \ldots, 0, 1)$, respectively.

Next we consider L-module $V \otimes V^*$. Since $V = V_1 \oplus V_2$ and $V^* = V_1^* \oplus V_2^*$, we can express L-module $V \otimes V^*$ as the direct sum of L-modules $V_i \otimes V_j^*$,

$$V \otimes V^* = \bigoplus_{i,j}^{2} (V_i \otimes V_j^*).$$

According to [11], a tensor product of two standard $sl(n)$-modules is isomorphic to the direct sum of symmetric and skew-symmetric $sl(n)$-modules. That is, $sl(n)$-module $V(\lambda) \otimes V(\lambda^*)$ is isomorphic to the direct sum of two $sl(n)$-modules $V(\lambda_1)$ and $V(\lambda_2)$ where

$$\lambda_1 = (2, 0, \ldots, 0)$$

and

$$\lambda_2 = (1, 1, 0, \ldots, 0).$$

L-module V_2^* is standard since it has the highest weight $\lambda^{**} = \lambda$. Hence we obtain that a tensor product of two standard L-modules V_1 and V_2^* is isomorphic to L-module

$$V_1 \otimes V_2^* \cong V(\lambda_1) \oplus V(\lambda_2).$$

On the other hand, a tensor product of two dual $sl(n)$-modules $V(\lambda^*)$ and $V(\lambda^*)$ is isomorphic to the direct sum of two $sl(n)$-modules $V(\lambda_1^*)$ and $V(\lambda_2^*)$. Therefore a tensor product of two dual L-modules V_1^* and V_2 is isomorphic to L-module

$$V_1^* \otimes V_2 \cong V(\lambda_1^*) \oplus V(\lambda_2^*).$$

Acting in the same manner we obtain that a tensor product of a standard $sl(n)$-module V_1 and a dual $sl(n)$-module V_1^* is isomorphic to the direct sum of an adjoint $sl(n)$-module $V(\lambda_3)$ and a trivial $sl(n)$-module $I(V_1)$,

$$V_1 \otimes V_1^* \cong V(\lambda_3) \oplus I(V_1)$$

where $\lambda_3 = (1, 0, \dots, 0, 1)$.

Similarly a tensor product of a dual $sl(n)$-module V_2 and a standard $sl(n)$-module V_2^* is also isomorphic to the direct sum of an adjoint $sl(n)$-module $V(\lambda_3)$ and a trivial $sl(n)$-module $I(V_2)$

$$V_2 \otimes V_2^* \cong V(\lambda_3) \oplus I(V_2).$$

Let us denote $(V_1 \otimes V_1^*)$, $(V_2 \otimes V_2^*)$, $(V_1 \otimes V_2^*)$ and $(V_1^* \otimes V_2)$ as U_1, U_2, U_3 and U_4, respectively. Let U stand for $U_1 \oplus U_2$. Since L-modules U, U_3 and U_4 have pairwise different highest weights, any L-submodule M of $U \oplus U_3 \oplus U_4$ can be represented in the following form:

$$M = (M \cap U) \oplus (M \cap U_3) \oplus (M \cap U_4).$$

Next we consider S as an L-submodule of L-module $V \otimes V^* = U \oplus U_3 \oplus U_4$ and prove that L-module S does not contain two adjoint L-submodules.

Let us assume the contrary, that is, L-module S contains two adjoint L-submodules. Hence S contains a subspace T of the following form:

$$\left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array} \right) \right\}$$

where $X \in sl(n)$, $Y \in sl(n)$.

Notice that T is a Lie subalgebra of S, $T = T_1 \oplus T_2 \cong sl(n) \oplus sl(n)$.

We know that T_1-module V_1 and T_2-module V_2 are simple with the highest weight $\lambda = (1, 0, \ldots, 0)$. Hence T-module $V_1 \otimes V_2^*$ is simple as a tensor product of simple T_1-module V_1 and T_2-module V_1^*. Thus, by Lemma 1.4.7, T-module $V_1 \otimes V_2^*$ has the highest weight (λ, λ^*). Acting in the same way, we obtain that the highest weight of T-module $V_2^* \otimes V_1$ is (λ^*, λ). Therefore $U_3 = V_1 \otimes V_2^*$ and $U_4 = V_2^* \otimes V_1$ are not isomorphic as T-modules.

Since the projections of S on U_3 and U_4 are not zero, T-modules $S \cap U_3$ and $S \cap U_3$ are nontrivial. Thus T-module $S \cap U_3$ coincides with U_3, and T-module $S \cap U_4$ coincides with U_4. We have that

$$\dim S = \dim (S \cap U) \oplus \dim (S \cap U_3) \oplus \dim (S \cap U_4)$$

since $S = (S \cap U) \oplus (S \cap U_3) \oplus (S \cap U_4)$. Therefore

$$\dim S \geq 2(n^2 - 1) + n^2 + n^2 = 4n^2 - 2.$$

On the other hand, $\dim S = 2n^2 + n$ since $S \cong sp(2n)$. This contradicts the fact that $n > 1$ ($L \cong sl(n)$). Therefore L-module S does not contain two adjoint L-submodules.

Further the following cases are possible:

Case 1. L-module S contains two L-submodules isomorphic to $V(\lambda_2)$ and $V(\lambda_2^*)$, respectively.

Let us prove that this case is not possible. Notice that both U_3 and U_4 are direct sums of two $sl(n)$-modules of skew-symmetric and symplectic matrices. Since L-submodule S contains two L-submodules of skew-symmetric matrices, we have that

$S \cap (U_3 \oplus U_4)$ contains subspace $\tilde{L}$ of the following form:

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\}$$

where B, C are skewsymmetric matrices of order n. It is easy to check that $L + \tilde{L}$ forms a Lie subalgebra in S isomorphic to $o(2n)$.

On the other hand, by Lemma 3.2.11, $sp(2n)$ does not contain a Lie subalgebra of the type $o(2n)$. This contradict the fact that L-module S contains two L-submodules isomorphic to $V(\lambda_2)$ and $V(\lambda_2^*)$.

Case 2. L-module S contains two L-submodules isomorphic to $V(\lambda_1)$ and $V(\lambda_1^*)$, respectively.

Notice that $S \cap (U_3 \oplus U_4)$ contains subspace $\tilde{L}$ of the following form:

$$\left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\}$$

where B, C are symmetric matrices of order n. It is easily seen that $L + \tilde{L}$ forms a Lie subalgebra in S isomorphic to $sp(2n)$. This implies that S has the form (3.11). Hence the lemma is proved for this case.

Case 3. Both statements (1) and (2) are not true.

Let us prove that this case does not hold. We have that the dimension of L-modules $V(\lambda_1)$ and $V(\lambda_1^*)$ is equal to $n(n+1)/2$, and the dimension of L-modules $V(\lambda_2)$ and $V(\lambda_2^*)$ is equal to $n(n-1)/2$. Since L-module S does not contain both $V(\lambda_1)$, $V(\lambda_2^*)$, and L-module S does not contain both $V(\lambda_1)$ and $V(\lambda_1^*)$, we obtain the following inequality

$$\dim (S \cap U_3) + \dim (S \cap U_4) \leq n(n+1)/2 + n(n-1)/2 = n^2.$$

Since L-module $S \cap U$ contains only one adjoint L-submodule, we have that

$$\dim(S \cap U) \leq \dim V(\lambda_3) + \dim I(V_1) + \dim I(V_2) \leq (n^2 - 1) + 1 + 1 = n^2 + 1.$$

This implies that

$$\dim S = \dim(S \cap U) \oplus \dim(S \cap U_3) \oplus \dim(S \cap U_4) \leq n^2 + 1 + n^2 = 2n^2 + 1.$$

On the other hand, $\dim S = 2n^2 + n$ since $S \cong sp(2n)$. This contradicts the fact that $n > 1$ $(L \cong sl(n))$.

Lemma 3.4.2 *Let $S = K + L$, $S \cong osp(2k, 2n)$, $K \cong osp(2k-1, 2n)$ and $L \cong sl(k, n)$. Then there exists an automorphism φ of $gl(2k, 2n)$ such that $\varphi(S) = \varphi(K) + \varphi(L)$ has the form as in Example (4).*

Proof. First we consider $L \cong sl(k, n)$. By Corollary 3.3.7, there exists a homogeneous basis of $V \oplus W$ such that L_0 takes the form

$$\left\{ \left(\begin{array}{cc|cc} X & 0 & 0 & 0 \\ 0 & -X^t & 0 & 0 \\ \hline 0 & 0 & Y & 0 \\ 0 & 0 & 0 & -Y^t \end{array} \right) \right\} \tag{3.12}$$

where $X \in sl(k)$ and $Y \in sl(n)$.

Let π_1, π_2 denote projections of S_0 onto the ideals $o(2k)$ and $sp(2n)$, respectively. These projections induce two decompositions: $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ and $\pi_2(S_0) = \pi_2(K_0) + \pi_2(L_0)$. By Lemma 3.3.1, we have that $\pi_1(S_0) \cong o(2k)$, $\pi_1(K_0) \cong o(2k-1)$ and $\pi_1(L_0) \cong sl(k)$. By Lemma 1.1.4, there exist bases of V such that the decomposition $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ takes the form (1.2). Thus

$\pi_1(S_0)$ consists of all skew-symmetric matrices of order $2k$, i.e $\pi_1(S_0) = o(2k)$. Besides, $\pi_1(L_0)$ takes the form

$$\left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & -X^t \end{array} \right) \right\}$$

where $X \in sl(k)$.

Next we consider $\pi_2(S_0) \cong sp(2n)$. We are given that $\pi_2(L_0) \subset \pi_2(S_0)$, and $\pi_2(L_0)$ has the form:

$$\left\{ \left(\begin{array}{c|c} Y & 0 \\ \hline 0 & -Y^t \end{array} \right) \right\}$$

where $Y \in sl(n)$. Then, by Lemma 3.4.1, $\pi_2(S_0)$ takes the form

$$\left\{ \left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right) \right\}$$

where B, C are symmetric matrices.

By Lemma 2.4.1, there exists an automorphisms ψ of the form (2.11) such that $\psi(S)$ takes the standard form. Thus

$$\psi(S)_1 = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\} \tag{3.13}$$

where

$$B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array} \right)$$

and

$$C = \left(\begin{array}{c|c} B_{22}^t & B_{12}^t \\ \hline -B_{21}^t & -B_{11}^t \end{array} \right).$$

We are going to show that S uniquely defines L. Let $\lambda = (1, 0, \dots, 0)$ be the highest weight of I_1-module V_1, and $\mu = (1, 0, \dots, 0)$ be the highest weight of I_2-

module W_1. Then $I_1 \oplus I_2$-module $V_1 \otimes W_2^*$ has the highest weight (λ, μ) and $I_1 \oplus I_2$-module $V_2 \otimes W_1^*$ has the highest weight (λ^*, μ^*).

On the other hand, by Lemma 1.2.1(b), $I_1 \oplus I_2$-module L_1 is the direct sum of two simple submodules with the highest weights (λ, μ^*) and (λ^*, μ). Hence the projections of L_1 onto $V_1 \otimes W_2^*$ and $V_2 \otimes W_1^*$ are zero since L_0-module L_1 contains no submodules with the highest weights (λ, μ) and (λ^*, μ^*). Thus $L_1 \subset S_1$ is the subspace of the set of matrices of the form (3.13) where $B_{12} = 0$ and $B_{21} = 0$. The dimension of this set is $2kn$. Hence the dimension of L_1 is less than or equal to $2kn$. On the other hand, the dimension of L_1 is $2kn$ since $L_1 \cong sl(k, n)$. Thus L_1 coincides with the set of matrices of the form (3.13) where $B_{12} = 0$ and $B_{21} = 0$. Therefore we have proved that S uniquely defines L.

Finally we show that S uniquely defines K.

As was shown above, the decomposition $\pi_1(S_0) = \pi_1(K_0) + \pi_1(L_0)$ has the form (1.2). We consider the automorphisms $\bar{\chi}$ of the form (3.10). Let us denote $S' = \bar{\chi}(S)$, $K' = \bar{\chi}(K)$ and $L' = \bar{\chi}(L)$. According to Remark 1.1.1, $\pi_1(K_0')$ consists of all skew-symmetric matrices of order $2k$ with the first column and row zero. Therefore, by Remark 1.4.4, the first row and column of all matrices from K' are zero. Hence S' uniquely defines K' since K' consists of all matrices in S' with the first row and column zero. This implies that $S = \bar{\chi}^{-1}(S')$ uniquely defines $K = \bar{\chi}^{-1}(K'))$.

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