# SIMPLE SUBALGEBRAS OF SIMPLE JORDAN ALGEBRAS AND SIMPLE DECOMPOSITIONS OF SIMPLE JORDAN SUPERALGEBRAS

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# Simple subalgebras of simple Jordan algebras and simple decompositions of simple Jordan superalgebras

by

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# Abstract

In 1952 E. Dynkin classified semisimple subalgebras of semisimple Lie algebras over an algebraically closed field $F$ of zero characteristic. Until now there was no classification of simple (semisimple) subalgebras of simple finite-dimensional Jordan algebras. As a consequence the first problem of this thesis is a description of simple subalgebras in finite-dimensional special simple Jordan algebras over an algebraically closed field $F$ of characteristic not 2. Using a slightly generalized version of Malcev's Theorem, Racine's classification of maximal subalgebras and other techniques developed in the thesis we can show that each simple subalgebra of a simple Jordan algebra can be reduced to an appropriate canonical form. Besides we formulate necessary and sufficient conditions for conjugacy of simple subalgebras of simple special Jordan algebra $\mathcal{J}$. Therefore, in Jacobson's terminology we describe orbits of the algebra of symmetric matrices under $O(n)$ (the orthogonal group), orbits of the algebra of symplectic matrices under $Sp(n)$ ( the symplectic group) and orbits of full matrix algebra under $GL(n)$ (the general linear group).

The other problem considered in this thesis is the classification of simple decompositions that occur in simple Jordan superalgebras with semisimple even part over an algebraically closed field $F$ of characteristic not 2. By a *simple (semisimple)* decomposition of any algebra $\mathcal{J}$ (not necessarily simple) we understand any representation of $\mathcal{J}$ as vector sum space of two proper simple (semisimple) subalgebras. In general, the sum in this decomposition is not necessarily direct, and the subalgebras may not be ideals. The problem of finding simple decompositions has drawn

researchers' interest in late 60's after the pioneering works of O.Kegel, A. Onishchik and others. Given $\mathcal{J} = \mathcal{A} + \mathcal{B}$, the sum of two proper simple subalgebras $\mathcal{A}$ and $\mathcal{B}$, what abstract properties of $\mathcal{A}$ and $\mathcal{B}$ does $\mathcal{J}$ inherit? In addition, information about the structure of simple subalgebras can be used to describe the lattice properties of simple algebras. In this thesis we determined the conjugacy classes of simple decompositions of simple matrix Jordan superalgebras with semisimple even part over an algebraically closed field $F$ of characteristic not 2.

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# Chapter 1

# Basic facts and definitions

Let us begin with the following

**Definition 1.0.1.** *A Jordan algebra $\mathcal{J} = (V, p)$ over an arbitrary field $F$ of characteristic not two consists of a vector space $V$ over $F$ equipped with a bilinear product $p : V \times V \to V$ (usually abbreviated $p(x, y) = x \circ y$) satisfying the* **Commutative Law** *and the* **Jordan identity***:*

1. *$x \circ y = y \circ x$ ( Commutative Law)*
2. *$(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$ (Jordan Identity)*

Let $\mathcal{U}$ be an associative algebra over $F$ and $ab$ the associative product composition of $\mathcal{U}$. Then the vector space $\mathcal{U}$ is a Jordan algebra $\mathcal{U}^{(+)}$ relative to the composition $a \circ b = \frac{1}{2}(ab + ba)$, that is, this composition satisfies the defining identities 1 and 2. Next we define a *special Jordan algebra* to be a subspace $\mathcal{J}$ of an associative algebra over $F$ of characteristic not 2 which is closed under the composition $a \circ b = \frac{1}{2}(ab + ba)$. Jordan algebras which are not special will be called *exceptional.*

Further, let $(\mathcal{U}, J)$ be a pair consisting of an associative algebra $\mathcal{U}$ and an involution $J$. Then $H(\mathcal{U}, J)$ denotes the set of all elements of $\mathcal{U}$ symmetric with respect to $J$. It is easily seen that $H(\mathcal{U}, J)$ is always a subalgebra of $\mathcal{U}^{(+)}$.

Next we will consider a class of algebras $\mathcal{G}$, called *composition* algebras over an arbitrary field $F$ of characteristic not two. By definition $\mathcal{G}$ is not necessarily associative. It always has a unit 1, and $\mathcal{G}$ is the vector space direct sum: $\mathcal{G} = F \oplus \mathcal{G}_0$ where $\mathcal{G}_0$ is a subspace such that if $x$ in $\mathcal{G}_0$, then $x^2 = N(x)$ is in $F$. Here, $N(x)$ is a quadratic form on $\mathcal{G}_0$ whose associated symmetric bilinear form is non-singular. Moreover, the quadratic form $N(a)$ defined for every $a$ of $\mathcal{G}_0$ permits composition, that is, $N(ab) = N(a)N(b)$ where $ab$ is the product in $\mathcal{G}$. Composition algebras are alternative quadratic algebras. They have dimensions 1, 2, 4 and 8, and a canonical involution: $a \longrightarrow \bar{a}$ such that $N(a) = a\bar{a}$.

When $n = 1$ we know that $\mathcal{G} = F$. When $n = 2$ we use the notation $F[u]$ for $\mathcal{G}$ where $u$ is a non-zero element of $\mathcal{G}_0$ and $u^2 = \rho \neq 0$. If $a = \alpha + \beta u$, $\alpha, \beta$ in $F$, then $\bar{a} = \alpha - \beta u$ and

$$N(a) = \alpha^2 - \beta^2 \rho.$$

When $n = 4$ we will write $\mathcal{G} = \mathcal{Q}$, a (generalized) *quaternion* algebra. We can write $\mathcal{Q} = F[u] \oplus F[u]v$ where $F[u]$ is a quadratic subalgebra of dimension two containing $F$, and the product in $\mathcal{Q}$ is given by

$$(a + bv)(c + dv) = (ac + \sigma b\bar{d}) + (ad + b\bar{c})v,$$

for all $a, b, c, d$ in $F[u]$ with $v^2 = \sigma \neq 0$ in $F$. The involution in $\mathcal{Q}$ is $q = a + bv \longrightarrow$

$\bar{q} = \bar{a} - bv$, and

$$N(q) = N(a) - \sigma N(b).$$

The algebra $\mathcal{Q}$ is associative but not commutative.

The composition algebras of dimension eight are the (generalized) *octonion* algebras $\mathcal{O}$. Such an algebra is generated by a quaternion subalgebra containing $F$, and an element $w$ such that $\mathcal{O} = \mathcal{Q} \oplus \mathcal{Q}w$ with multiplication in $\mathcal{O}$ defined by

$$(q + rw)(s + tw) = (qs + \tau \bar{t} r) + (tq + r\bar{s})w$$

for $q, r, s, t$ in $\mathcal{Q}$. Thus $w^2 = \tau \neq 0$ in $F$ and this element together with $\mathcal{Q}$ determines the algebra $\mathcal{O}$. The involution in $\mathcal{O}$ is $x = q + rw \to \bar{x} = \bar{q} - rw$, and

$$N(x) = N(q) - \tau N(r).$$

Any composition algebra is either a division algebra or has a divisor of zero. It is easy to see that $\mathcal{G}$ is a division algebra if and only if $N(x) \neq 0$ for $x \neq 0$. If $\mathcal{G}$ with $N(x)$ contains divisors of zero, then we will call $\mathcal{G}$ a *split* composition algebra. For a fixed $F$ and a fixed dimension there is a unique split composition algebra: $F \oplus F$, $F_2$, Zorn vector matrices.

Now let $\mathcal{G}_n$ be the algebra of all $n \times n$ matrices with elements in a composition algebra $\mathcal{G}$. Then every element of $\mathcal{G}_n$ is a matrix $X = (x_{ij})$ with elements $x_{ij}$ in $\mathcal{G}$ for $i, j = 1, \ldots, n$, and we write

$$J(X) = \bar{X}^t = (y_{ij}), \quad y_{ji} = \bar{x}_{ij} \quad (i, j = 1, \ldots, n)$$

The mapping $J(X) = \bar{X}^t$ is an involution in $\mathcal{G}_n$ called a *standard* involution. If $\mathcal{G}$ is an associative algebra, then $H(\mathcal{G}_n, J)$ is a special Jordan algebra of dimension

$\frac{(n^2-n)}{2}d + n$ where $d = \dim \mathcal{G}$. If $\mathcal{G} = \mathcal{O}$, then $H(\mathcal{O}_n, J)$ is a Jordan algebra only if $n \leq 3$. If $n = 3$, then $H(\mathcal{O}_3, J)$ is an exceptional Jordan algebra (*Albert algebra*). When no confusion is likely, we will omit $J$ and write $H(\mathcal{G}_n)$ instead.

Next let $V$ be a finite-dimensional vector space equipped with a non-singular symmetric bilinear form $f : V \times V \to F$. Consider the direct sum of $F1$ and $V$, $\mathcal{J} = F1 \oplus V$ where 1 is the identity element, and determine multiplication according to

$$(\alpha 1 + v)(\beta 1 + w) = (\alpha\beta + f(v, w))1 + (\alpha w + \beta v).$$

Then $\mathcal{J}$ is a Jordan algebra of the type $J(V, f)$.

In the classification of finite-dimensional simple Jordan algebras, composition algebras play an important role. According to Albert's classification (1950), if $\mathcal{J}$ is a simple finite-dimensional Jordan algebra over an algebraically closed field $F$ of characteristic not 2, then we have the following possibilities for $\mathcal{J}$: (1) $\mathcal{J} = F$; (2) $\mathcal{J} = F \oplus V$ the Jordan algebra of a non-singular symmetric bilinear form $f$ in a finite-dimensional vector space $V$ such that $\dim V > 1$, (3) $H(\mathcal{G}_n, J)$, $n \geq 3$, where $\mathcal{G}$ is a composition algebra of dimension 1,2, or 4 if $n \geq 4$ and of dimensions 1,2,4, and 8 if $n = 3$, and $J$ is the standard involution. Therefore, we can conclude that the Albert algebra is the only exceptional simple Jordan algebra over algebraically closed $F$, char $F \neq 2$ in the sense of having no relizations in an associative algebra. In 1983 E. Zelmanov classified all possible simple Jordan algebras in arbitrary dimensions (*Zelmanov's Simplcity Theorem*). It appears that in arbitrary dimensions simple Jordan algebras also fall into *quadratic*, *hermitian* and *Albert* types as above.

Notice that the Jordan algebras $H(F[u]_n)$ and $H(\mathcal{Q}_n)$ also have other isomorphic realizations denoted as $F_n^{(+)}$ and $H(F_{2n}, j)$ where $j$ is a symplectic involution in $F_{2n}$ [9]. Further, we will frequently use these realizations in order to define canonical forms of simple subalgebras of $H(\mathcal{Q}_n)$ and $H(F[u]_n)$.

If $\mathcal{J}$ is a special Jordan algebra, then the concept of the universal algebra for the special representations has been defined in [9]. According to [9] the *special universal associative algebra* of $\mathcal{J}$ is the difference algebra $U(\mathcal{J}) = \mathcal{F}/\mathcal{R}^s$ where $\mathcal{F}$ is the free associative algebra based on the vector space $\mathcal{J}$, and $\mathcal{R}^s$ is the ideal generated by $a \times b + b \times a - ab$ ('$\times$' the product in $\mathcal{F}$). The special universal associative algebras of $H(F_n)$ and $H(F_{2n}, j)$, $n > 2$, are nothing but the matrix algebras $F_n$ with the canonical embedding as the set of symmetric matrices and $F_{2n}$ with the canonical embedding as the set of symplectic matrices. For $F_n^{(+)}$, the special universal associative algebra is $F_n \oplus F_n^0$ where $F_n^0$ is the opposite algebra. Next we introduce one more definition. Let $\mathcal{J}$ be a Jordan subalgebra of $\mathcal{A}^{(+)}$ where $\mathcal{A}$ is an associative algebra. Then $S(\mathcal{J})$ stands for the associative subalgebra generated by $\mathcal{J}$.

Since the description of simple subalgebras significantly relies on Racine's classification of maximal subalgebras [28, 29], we recall certain well-known classical theorems that we will use later. Before we state these theorems, we introduce some notation we are going to use in these theorems. Let $\mathcal{A}$ be a finite-dimensional central simple associative algebra of degree $n$. Then, we denote by $\mathcal{A}^0$ the opposite algebra of $\mathcal{A}$. Since we always deal with an algebraically closed basic field $F$, we formulate simplified versions of Racine's Theorems.

**Theorem** (Racine)*Let $\mathcal{A}$ be a finite-dimensional central associative algebra of degree greater than or equal to 3 over a field $F$ of characteristic not two. Any maximal subalgebra of $\mathcal{A}^{(+)} \cong H(\mathcal{A} \oplus \mathcal{A}^0, *)$ ( $*$ is the exchange involution) is of the form*

*(1) $J(V, f)$,*

*(2) $\mathcal{B}^{(+)} \cong H(\mathcal{B} \oplus \mathcal{B}^0, *)$, $\mathcal{B}$ a maximal subalgebra of $\mathcal{A}$, or*

*(3) $H(\mathcal{A}, -) \cong H(\mathcal{A} \oplus \mathcal{A}^0, *) \cap \{(a, \bar{a}) | a \in \mathcal{A}\}$ where $-$ denotes either the transpose involution or the symplectic involution.*

*$\mathcal{A}$ has maximal subalgebras of type (1) if and only if*

$$\mathcal{A} \cong \otimes_{i=1}^{m} \mathcal{Q}_i$$

*where $\mathcal{Q}_i$ is a quaternion algebra, $m = 2$ or $m$ odd, in which case* $\dim J(V, f) = 2(m+1)$.

**Theorem** (Racine)*Let $(\mathcal{A}, *)$ be a finite-dimensional central simple associative algebra with involution over $F$ a field of characteristic not 2, $\mathcal{A}$ central simple of degree $n$. If the degree of $H(\mathcal{A}, *) \geq 3$ then maximal subalgebras of $H(\mathcal{A}, *)$ are of the form*

*(1) $J(V, f)$, or*

*(2) $H(\mathcal{B}, *)$, $\mathcal{B}$ a maximal subalgebra of $(\mathcal{A}, *)$.*

*$H(\mathcal{A}, *)$ has maximal subalgebras of type (1) if and only if $(\mathcal{A}, *)$ is isomorphic to a Clifford algebra with the canonical involution. If $*$ is of the first kind then $n = 2^m$*

*and* $J(V, f) \subset H(\mathcal{A}, *)$ *is maximal if and only if* $\dim\, J(V, f) = 2m + 1$ *for* $m$ *odd,* $2(m + 1)$ *for* $m$ *even.*

# Chapter 2

# Subalgebras

The main focus of this chapter is a description of simple subalgebras in finite-dimensional special simple Jordan algebras over an algebraically closed field $F$ of characteristic not two. The problem of finding semisimple subalgebras in semisimple Lie algebras was fully solved by Dynkin [6]. A similar question for Jordan algebras arose in Jacobson's research. Namely, in [9] as an application of the general representation theory, he studied semisimple subalgebras of an arbitrary finite-dimensional Jordan algebra of characteristic zero. In this connection he obtained an analogoue of the results of Malcev and Harish-Chandra in the theory of the Levi decompositions of a Lie algebra [14, 7]. Then, in 1987 N. Jacobson determined the orbits under the orthogonal group $O(n)$ of the subalgebras of the Jordan algebra of $n \times n$ real symmetric matrices [8].

The description of simple subalgebras of simple Jordan algebras significantly relies on the classification of maximal subalgebras of finite-dimensional special simple

Jordan algebras obtained by M. Racine in 1974 ([28]). Three years later M. Racine published his paper [29] which completes the classification of maximal subalgebras in all types of simple finite-dimensional Jordan algebras.

The algebras we consider in this chapter will be assumed to be finite-dimensional special over an algebraically closed field $F$ of characteristic not 2. We will give a full classification of simple subalgebras in simple special Jordan algebras. Notice that the simple subalgebras of $J(V, f)$ have been studied in [33].

## 2.1 Matrix subalgebras

Let $\mathcal{J}$ be a simple Jordan algebra of the type $F_{\frac{n}{2}}^{(+)}$ where $n$ is even. Then, according to [8] it can always be presented as a subalgebra of $H(F_n)$ as follows

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\}, \tag{1}$$

where $A$ is any symmetric matrix of order $\frac{n}{2}$, and $B$ is any skewsymmetric matrix of order $\frac{n}{2}$.

The following lemma gives us an idea of the structure of the automorphism group of (1).

**Lemma 2.1.1.** *Any automorphism of a Jordan algebra of the form (1) is induced by an automorphism of $H(F_n)$.*

*Proof.* First, according to [9] any automorphism of $\mathcal{J}$ can be extended to an automorphism or antiautomorphism of a special universal enveloping algebra $U(\mathcal{J})$

which is isomorphic to $F_{\frac{n}{2}} \oplus F_{\frac{n}{2}}$. Notice that in this particular case the associative enveloping algebra $S(\mathcal{J})$ is isomorphic to $U(\mathcal{J})$ because from the explicit form (1) $S(\mathcal{J})$ consists of all matrices of the form:

$$\left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \right\},$$

where $X$ and $Y$ are any matrices of order $\frac{n}{2}$. Since any automorphism of $F_{\frac{n}{2}} \oplus F_{\frac{n}{2}}$ either induces non-trivial automorphisms of these ideals or sends one ideal onto another, it can be lifted up to an inner automorphism of the entire matrix algebra $F_n$. Consequently, for any antiautomorphism of $F_{\frac{n}{2}} \oplus F_{\frac{n}{2}}$ we can choose an automorphism (not necessarily non-trivial) of $F_{\frac{n}{2}} \oplus F_{\frac{n}{2}}$ such that their composition induces non-trivial antiautomorphisms of simple ideals. Therefore, any (Jordan) automorphism of $\mathcal{J}$ can be written as follows:

$$\varphi(X) = Q^{-1}XQ$$

or

$$\varphi(X) = Q^{-1}X^tQ, \qquad (2)$$

for some non-singular matrix $Q$.

The next step is to prove that $\varphi$ is orthogonal. In other words, all we have to show is that for any automorphism $\varphi$ of $\mathcal{J}$, we can choose $Q$ such that (2) holds and $Q^tQ = I$ where $I$ is the identity matrix. Since $\mathcal{J}$ is a subalgebra of $H(F_n)$, for each $X$ in $\mathcal{J}$, $(Q^{-1}XQ)^t = Q^{-1}XQ$, $Q^tX(Q^{-1})^t = Q^{-1}XQ$, $QQ^tX = XQQ^t$. Denote $B = QQ^t$. Next we are going to show that $B$ is actually a scalar multiple of the identity matrix. We are given that $BX = XB$ where $X$ is any matrix of the form (1). Let us write $B$ as follows:

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where $B_i$ are matrices of order $\frac{n}{2}$. By performing the matrix multiplication, we obtain $B_2 = B_3 = 0$, and $B_1 = B_4 = \alpha I$, for some non-zero $\alpha$. Since the ground field $F$ is algebraically closed, we can choose $\beta \in F$ such that $\alpha = \beta^2$. Set $Q' = \beta^{-1}Q$. Obviously, $Q'$ determines the same automorphism as $Q$ does, and $Q''Q' = I$. The lemma is proved. □

In the next lemma we state that subalgebras of $H(F_n)$ that have the type $F_{\frac{n}{2}}^{(+)}$ ($n$ even) are conjugate under an appropriate automorphism of $H(F_n)$.

**Lemma 2.1.2.** *Let $\mathcal{J} \cong F_{\frac{n}{2}}^{(+)}$ be a subalgebra of $H(F_n)$. Then, by an appropriate automorphism of $H(F_n)$, $\mathcal{J}$ can always be reduced to the form (1).*

*Proof.* At first we consider the enveloping algebra $S(\mathcal{J})$ of $\mathcal{J}$. It is known that $S(\mathcal{J})$ is either a simple associative algebra of degree $\frac{n}{2}$ or a direct sum of two isomorphic simple ideals of degree $\frac{n}{2}$. Hence, acting by an appropriate automorphism of $F_n$, $S(\mathcal{J})$ can be reduced to the following form:

$$\left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\}$$

where $X$, $Y$ are matrices of order $\frac{n}{2}$. It follows that $\mathcal{J}$ also takes the above block-diagonal form since $\mathcal{J} \subseteq S(\mathcal{J})$. If $S(\mathcal{J})$ is simple, then we can assume that $X = Y$.

If $S(\mathcal{J})$ is non-simple semisimple, then $\mathcal{J}$ can be brought to the form:

$$\left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \right\}. \tag{3}$$

In the case when $S(\mathcal{J})$ is simple, $\mathcal{J}$ is an associative subalgebra of $F_n$ that lies inside $H(F_n)$. According to [32], this is not possible. Then, it is easily seen that the automorphism of the form $\theta(Y) = S^{-1}YS$, where $S = \begin{pmatrix} I & iI \\ \frac{1}{2}I & -\frac{i}{2}I \end{pmatrix}$, $I$ is the identity matrix, $i^2 = -1$, sends each element of the form (3) into the algebra of the form (1). Therefore, by $\chi = \theta \circ \varphi$ we can bring $\mathcal{J}$ to the form (1).

Next we will show that $\chi$ is actually an orthogonal automorphism. Notice that $\chi$ sends $H(F_n)$ onto a Jordan subalgebra of $F_n^{(+)}$ which consists of all matrices symmetric with respect to the following involution: $j' = \chi \circ t \circ \chi^{-1}$ where $t$ is the standard transpose involution. This involution can be rewritten as follows $j'(X) = C^{-1}X^tC$ for some non-singular symmetric matrix $C$ of order $n$. It follows from the above considerations that any matrix of the form (1) is symmetric with respect to $j'$. Equivalently, for any $Y$ of the form (1), $C^{-1}Y^tC = Y$, $Y^tC = CY$, $YC = CY$ because $Y$ is symmetric. As was proved in the previous lemma, $C = \alpha I$ for some non-zero $\alpha$. Therefore, $j' = t$, and $\chi(H(F_n)) = H(F_n)$, and $\chi$ is actually an automorphism of $H(F_n)$. Hence, the lemma is proved. □

In [16] K. McCrimmon proves the following result: if $\mathcal{A}$ is a unital Jordan algebra over a field of characteristic not two with Wedderburn splitting $\mathcal{A} = \mathcal{S} \oplus \mathcal{R}$ for a solvable ideal $\mathcal{R}$ and $\mathcal{S} \cong \mathcal{A}/\mathcal{R}$ a separable subalgebra, then any other separable subalgebra $\mathcal{C}$ of $\mathcal{A}$ is conjugate under a generalized inner automorphism $T$ of $\mathcal{A}$ to some subalgebra of $\mathcal{S}$, $T(\mathcal{C}) \subset \mathcal{S}$. Here, we are going to use the following consequence of McCrimmon's Theorem.

**Lemma 2.1.3.** *Let $\mathcal{A}$ be a special simple matrix Jordan algebra, and $\mathcal{J}$ be a proper simple subalgebra of $\mathcal{A}$. Denote a maximal subalgebra which contains $\mathcal{J}$ as $\mathcal{M}$. Next, consider a Wedderburn splitting $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S}$ is a semisimple algebra, $\mathcal{R}$ is the radical. Then, there exists an automorphism $\varphi$ of $\mathcal{A}$ such that $\varphi(\mathcal{J}) \subseteq S$.*

*Proof.* Let 1 be the identity element of $\mathcal{A}$, and $1 \in \mathcal{J}$. According to [16], if $\mathcal{J}$ is special and the degree of $\mathcal{J}$ is not divisible by the characteristic, then $\mathcal{J}$ is conjugate under an inner automorphism $T$ of $\mathcal{M}$ to some subalgebra of $\mathcal{S}$, and $T$ is a composition of the standard automorphisms $T_{x,y}$ that can be represented in associative terms as follows

$$T_{x,y}(a) = tat^{-1}, \tag{4}$$

where $t = u^{-\frac{1}{2}}(1-xy)(1+yx)$, $u = (1-xy)(1+yx)(1+xy)(1-yx)$, $x, y \in M$.

Let $x$, $y$ be symmetric with respect to an involution $j$ of $\mathcal{A}$: $j(x) = x$, $j(y) = y$. Then, it is obvious that

$$j(u) = u, \quad \text{and} \quad j(t) = t^{-1}. \tag{5}$$

If $\mathcal{A} = F_n^{(+)}$, then from the explicit form (4) $T_{x,y}$ is easily extendable to $\mathcal{A}$. If $\mathcal{A} = H(F_n)$, then, because of (5), $T_{x,y}$ is orthogonal, therefore, extendable to $\mathcal{A}$. If $\mathcal{A} = H(F_{2n}, j)$, then, because of (5), $T_{x,y}$ is symplectic, therefore, extendable to $\mathcal{A}$.

If $\mathcal{J}$ is special and the degree of $\mathcal{J}$ is divisible by characteristic, then $T$ is a *generalized* inner automorphism [16], that is, $T$ is a composition of automorphisms

$T_{x_1,\ldots,x_n,m}$ of the form

$$T_{x_1,\ldots,x_n,m} = U_v^{-1}(I + V_{x_1,\ldots,x_n,m} + U_{x_1}\ldots U_{x_n}U_{-m})(I + V_{m,x_n\ldots,x_1} + U_m U_{x_n}\ldots U_{x_1})$$

where $v, x_i \in \mathcal{M}$, $m \in \mathcal{R}$. In associative terms quadratic operators take the form:

$$U_v(a) = vav,$$

$$U_{x_i}(a) = x_i a x_i,$$

$$V_{x_1,\ldots,x_n,m}(a) = x_1\ldots x_n ma + amx_n\ldots x_1.$$

Hence, if all $x_i$, $m$ and $a$ are symmetric with respect to an involution of $\mathcal{A}$, then $j(T_{x_1,\ldots,x_n,m}(a)) = T_{x_1,\ldots,x_n,m}(a)$. Therefore, $T_{x_1,\ldots,x_n,m}$ as well as $T$ is extendable to $\mathcal{A}$. The lemma is proved. □

The next lemma is an analogue of Lemma 2.1.2 for the case of symplectic matrices.

**Lemma 2.1.4.** *Let $\mathcal{J} \cong F_n^{(+)}$ be a subalgebra of $H(F_{2n}, j)$. Then, by an appropriate automorphism of $H(F_{2n}, j)$, $\mathcal{J}$ can always be reduced to the following form*

$$\left\{\begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix}\right\} \tag{6}$$

*Proof.* First of all, $\mathcal{J}$ of type $F_n^{(+)}$ has only two non-equivalent irreducible representations in an $n$-dimensional vector space [9]. By an appropriate automorphism $\varphi$ of $F_{2n}^{(+)}$, $\mathcal{J}$ can be brought to the form (6). Notice that $\varphi$ sends $H(F_{2n}, j)$ onto a Jordan subalgebra of $F_{2n}^{(+)}$ which consists of all matrices symmetric with respect to the following involution: $j' = \varphi \circ j \circ \varphi^{-1}$. This involution can be rewritten as follows

$j'(Y) = C^{-1}Y^tC$ for some non-singular skew-symmetric matrix $C$ of order $2n$. It follows from the above considerations that any matrix of the form (6) is symmetric with respect to $j'$. Equivalently, for any $Y$ of the form (6), $C^{-1}Y^tC = Y$, $Y^tC = CY$. Acting in the same manner as above, we can show that $C = \alpha \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ for some non-zero $\alpha$, where $I_n$ denotes the identity matrix of order $n$. Therefore, $\varphi(H(F_{2n}, j)) = H(F_{2n}, j)$, and $\varphi$ is an automorphism of $H(F_{2n}, j)$. Hence, the lemma is proved. $\square$

**Definition 2.1.5.** *Subalgebras $\mathcal{J}_1$ and $\mathcal{J}_2$ of a Jordan algebra $\mathcal{A}$ are said to be equivalent if there exists an automorphism $\varphi$ of $\mathcal{A}$ such that $\mathcal{J}_1 = \varphi(\mathcal{J}_2)$.*

**Definition 2.1.6.** *Let $\mathcal{J}$ be a subalgebra of $\mathcal{A}$. Then the set $C(\mathcal{J})$ of all subalgebras equivalent to $\mathcal{J}$ in $\mathcal{A}$ is said to be a conjugacy class of $\mathcal{J}$.*

### 2.1.1 Canonical realizations of simple subalgebras

Let $\mathcal{A}$ be a simple Jordan algebra, and $\mathcal{J}$ be a simple subalgebra of $\mathcal{A}$. All realizations listed below we will call *canonical.*

**1.** Let $\mathcal{A} = F_n^{(+)}$

*Type 1.* $\mathcal{J} \cong F_m^{(+)}$, $\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{X^t, \ldots, X^t}_{l}, \underbrace{0, \ldots, 0}_{s})\}$ where $X$ is any matrix of order $m$, $n = km + lm + s$.

*Type 2.* $\mathcal{J} \cong H(F_m)$, $\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0, \ldots, 0}_{l})\}$ where $X$ is any symmetric matrix of order $m$, $n = km + l$.

*Type 3.* $\mathcal{J} \cong H(F_{2m}, j)$, $\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0, \ldots, 0}_{l})\}$ where $X$ is any symplectic matrix of order $2m$, $2mk + l = n$.

**2**. Let $\mathcal{A} = H(F_n)$

*Type 4.* $\mathcal{J} \cong F_m^{(+)}$, $\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0 \ldots, 0}_{l})\}$ where $X$ is of the form (1) in which $A$ and $B$ are of order $m$, $n = km + l$.

*Type 5.* $\mathcal{J} \cong H(F_m)$, $\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0, \ldots, 0}_{l})\}$ where $X$ is any symmetric matrix of order $m$, $n = km + l$.

*Type 6.* $\mathcal{J} \cong H(F_{2m}, j)$, $\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0 \ldots, 0}_{l})\}$

$$X = \begin{pmatrix} A & -B & -C & D \\ B & A & D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}$$

where $A$ is a symmetric matrix of order $m$, $B$, $C$, $D$ are skew-symmetric matrices of order $m$, $n = 4mk + l$.

**3**. Let $\mathcal{A} = H(F_{2n}, j)$

*Type 7.* $\mathcal{J} \cong F_m^{(+)}$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{X^t, \ldots, X^t}_{l}, \underbrace{0, \ldots, 0}_{s}, \underbrace{X^t, \ldots, X^t}_{k}, \underbrace{X, \ldots, X}_{l}, \underbrace{0, \ldots, 0}_{s})\}$$

where $km + lm + s = n$, $X$ is any matrix of order $m$.

*Type 8.* $\mathcal{J} \cong H(F_m)$, $\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{0, \ldots, 0}_{l}, \underbrace{X, \ldots, X}_{k}, \underbrace{0, \ldots, 0}_{l})\}$ where $km + l = n$, $X$ is any symmetric matrix of order $m$.

*Type 9.* $\mathcal{J} \cong H(F_{2m}, j)$,

$$\mathcal{J} = \left\{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \right\}$$

where $A = \mathrm{diag}(\underbrace{X, \ldots, X}_{k}, \underbrace{\bar{X}, \ldots, \bar{X}}_{l}, \underbrace{0, \ldots, 0}_{s})$ and $\bar{X} = \begin{pmatrix} X & Y \\ Z & X^t \end{pmatrix}$,
$B = \mathrm{diag}(\underbrace{Y, \ldots, Y}_{k}, \underbrace{0, \ldots, 0}_{lm+s})$, $C = \mathrm{diag}(\underbrace{Z, \ldots, Z}_{k}, \underbrace{0 \ldots, 0}_{lm+s})$ for any $X$ of order $m$ and skew-symmetric $Y$,$Z$ of order $m$, $km + lm + s = n$.

**Definition 2.1.7.** *Let $\mathcal{J}$ and $\mathcal{J}'$ be two proper subalgebras of $\mathcal{A}$, and $\mathcal{J}'$ be given in the canonical realization. If $\mathcal{J}$ is equivalent to $\mathcal{J}'$, then $\mathcal{J}'$ is said to be the canonical form of $\mathcal{J}$.*

**Definition 2.1.8.** *Let $\mathcal{J} \cong F_m^{(+)}$ be a subalgebra of $\mathcal{A} \cong F_n^{(+)}$, and $\rho$ stand for the natural representation of $\mathcal{J}$ in $F^n$, $m < n$. Obviously, $\rho$ induces the representation of $S(\mathcal{J})$ in $F^n$. If $S(\mathcal{J})$ is a non-simple semisimple associative algebra, that is, $S(\mathcal{J}) = \mathcal{I}_1 \oplus \mathcal{I}_2$ where $\mathcal{I}_1$, $\mathcal{I}_2$ are isomorphic simple ideals, then $\rho = \rho_1 \oplus \rho_2$ where $\rho_i$ is a representation of $\mathcal{I}_i$ in the corresponding invariant subspace of $F^n$. Then $k_{\mathcal{A}}(\mathcal{J}) = |deg\, \rho_1(\mathcal{I}_1) - deg\, \rho_2(\mathcal{I}_2)|$. Otherwise, $k_{\mathcal{A}}(\mathcal{J}) = 0$.*

**Definition 2.1.9.** *Let $\mathcal{J}$ of the type $F_m^{(+)}$ be a subalgebra of $\mathcal{A} = H(F_{2n}, j)$. Then, $\mathcal{J}$ can be covered by a subalgebra $\mathcal{S}$ of the type $F_n^{(+)}$. By Lemma 2.1.4 all subalgebras of the type $F_n^{(+)}$ are conjugate under an automorphism of $H(F_{2n}, j)$ and can be reduced to (6). Hence $F^{2n} = V_1 \oplus V_2$ where $V_i$ is invariant under the action of $\mathcal{S}$,* $\dim V_i = n$. *Let $\rho$ denote the natural representation of $\mathcal{S}$ in $F^{2n}$. Since $\mathcal{S}$ is conjugate to (6),*

*$\rho$ is completely reducible, and $\rho = \rho_1 \oplus \rho_2$ where $\rho_i$ is a representation of $\mathcal{S}$ in $V_i$. Then $k_{\mathcal{A}}(\mathcal{J}) = k_{\rho_1(\mathcal{S})}(\rho_1(\mathcal{J}))$.*

**Definition 2.1.10.** *Let $\mathcal{J}$ of the type $H(F_{2m}, j)$ be a subalgebra of $\mathcal{A} = H(F_{2n}, j)$. Then, we can always choose a subalgebra $\mathcal{B}$ of $H(F_{2m}, j)$ of the type $F_m^{(+)}$ and define $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{B})$.*

**Theorem 2.1.11.** *Let $\mathcal{A}$ be a simple matrix Jordan algebra. Then, any simple matrix subalgebra of $\mathcal{A}$ has a unique canonical form as above.*

*Proof.* Let $\mathcal{J}$ be any proper simple matrix subalgebra of $\mathcal{A}$. In particular, the degree of $\mathcal{J} \geq 3$. Denote the identity of $\mathcal{A}$ as 1.

The proof of the theorem consists of three cases.

**Case 1** $\mathcal{A} = F_n^{(+)}$

**1.1** Let $\mathcal{J}$ be of the type $F_m^{(+)}$ for some $m < n$. Due to [9] any Jordan algebra of this type has precisely two non-equivalent irreducible representations both of which have degree $m$. Hence $\mathcal{J}$ is equivalent to the subalgebra in the canonical realization of type 1. If $1 \in \mathcal{J}$, then the last zeros in canonical form of type 1 are omitted. Since $k_{\mathcal{A}}(\mathcal{J})$ is invariant for $\mathcal{J}$, the canonical form of $\mathcal{J}$ is uniquely defined.

**1.2** Let $\mathcal{J}$ be of the type $H(F_m)$ for some $m \leq n$. Then, it follows from the uniqueness of the irreducible representation of $H(F_m)$ [9] that $\mathcal{J}$ is equivalent to the subalgebra in the canonical realization of type 2. If $1 \in \mathcal{J}$, then the last zeros in canonical form of type 2 are omitted.

**1.3** The proof of the case when $\mathcal{J} \cong H(F_{2m}, j)$, $2m < n$, is exactly the same as the previous proof. In particular, $\mathcal{J}$ of the type $H(F_{2m}, j)$ is equivalent to the subalgebra in the canonical realization of type 3. Obviously, the canonical form is unique.

**Case 2** $\mathcal{A} = H(F_n)$

Here, our main goal is to determine the canonical form of any simple matrix Jordan subalgebra of $H(F_n)$. Let $\mathcal{M}$ be a maximal subalgebra of $H(F_n)$. According to [28], $\mathcal{M}$ is isomorphic to one of the following:

1. $H(F_k) \oplus H(F_l)$, $k + l = n$,

2. $F_k^{(+)} \oplus H(F_l) \oplus \mathcal{R}$, $2k + l = n$, $\mathcal{R}$ is the radical (if $l = 0$, then $\mathcal{M} \cong F_{\frac{n}{2}}^{(+)} \oplus \mathcal{R}$)

3. $J(V, f)$ only if $n = 2^m$ and either $\dim J(V, f) = 2(m + 1)$, $m$ is even, or $\dim J(V, f) = 2m + 1$, $m$ is odd.

First, assume that $\mathcal{J}$ is a simple matrix subalgebra of $H(F_n)$ such that $1 \in \mathcal{J}$. There exists a maximal subalgebra $\mathcal{M}$ such that $\mathcal{J} \subset \mathcal{M}$. Since $\deg \mathcal{J} \geq 3$, $\mathcal{M}$ cannot be of the type 3. If $\mathcal{M}$ contains a non-zero radical, that is, $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$, where $\mathcal{S}$ a semisimple algebra, $\mathcal{R}$ the radical, then by Lemma 2.1.3 we can assume that $\mathcal{J} \subseteq \mathcal{S}$. If $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ where $\mathcal{S}_i$ non-trivial simple ideals, according to [29] we can choose three orthogonal idempotents: $e$, $e^t$, $ff^t$, $1 = e + e^t + ff^t$ such that

$$\mathcal{S}_1 = ff^t H(F_n) ff^t, \quad \mathcal{S}_2 = eF_n e + e^t F_n^t e^t \tag{7}$$

Since $ff^t$ is an element of $H(F_n)$, by an automorphism $\varphi$ of $H(F_n)$, it can be reduced to the following form:

$$\varphi(ff^t) = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$$

where $I_l$ is the identity matrix of order $l$. Since $e$ and $e^t$ are orthogonal to $ff^t$, they take the forms:

$$\varphi(e) = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}, \varphi(e^t) = \begin{pmatrix} 0 & 0 \\ 0 & K^t \end{pmatrix},$$

where $K$ is a matrix of order $n - l$. Therefore, according to (7),

$$\varphi(\mathcal{S}) = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\}, \varphi(\mathcal{S}_1) = \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{8}$$

$$\varphi(\mathcal{S}_2) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \right\},$$

where $X$ is any symmetric matrix of order $l$, $Y$ is a symmetric matrix of order $2k = n - l$ which is also an element of a subalgebra of the type $F_k^{(+)}$.

In the case when $\mathcal{M}$ is semisimple, that is, $\mathcal{M} = \mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, there exist two orthogonal idempotents such that

$$\mathcal{S} = eH(F_n)e + fH(F_n)f, \quad e + f = 1.$$

Acting in the same manner as above we can reduce $\mathcal{S}$ to (8).

Therefore, we can define two homomorphisms $\pi_1$, $\pi_2$ as projections on $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Since $1 \in \mathcal{J}$, $\pi_1(\mathcal{J}) \neq \{0\}$, $\pi_2(\mathcal{J}) \neq \{0\}$. This implies that $\mathcal{J} \cong \pi_1(\mathcal{J}) \subseteq H(F_l)$, $l < n$, and $\mathcal{J} \cong \pi_2(\mathcal{J}) \subseteq H(F_{2k})$, $2k < n$. Therefore, we can reduce the problem of finding the canonical form of $\mathcal{J}$ to the case of all symmetric matrices of order less than $n$. However, the above reduction does not work in the case when $\mathcal{S}$ is simple, that is, $\mathcal{M} = F_{\frac{r}{2}}^{(+)} \oplus \mathcal{R}$, $r \leq n$. Hence we can conclude that as soon as the given simple subalgebra $\mathcal{J}$ is in the maximal subalgebra $\mathcal{M}$ which has a non-simple semisimple factor $\mathcal{S}$, the problem can be reduced to the case of symmetric matrices of a lower order. This process stops only if at some step either $\pi_i(\mathcal{J}) \subseteq \mathcal{M} \cong F_{\frac{r}{2}}^{(+)} \oplus \mathcal{R}$, or $\pi_i(\mathcal{J})$ coincides with $\mathcal{S}_i$. Without any loss of generality, we can assume that $r = n$, that is, $\mathcal{J} \subseteq \mathcal{M} \cong F_{\frac{n}{2}}^{(+)} \oplus \mathcal{R}$.

All we need to reach our goal is to determine the canonical form of $\mathcal{J}$ which is covered by a maximal subalgebra of the type $F_{\frac{n}{2}}^{(+)} \oplus \mathcal{R}$. Notice that there is an isomorphic imbedding $\theta$ of $F_{\frac{n}{2}}^{(+)}$ into $H(F_n)$ such that $\theta(A+iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where $A$ is a symmetric matrix of order $\frac{n}{2}$, $B$ is a skew-symmetric matrix of order $\frac{n}{2}$, $i^2 = -1$.

**2.1**. Let us assume that $\mathcal{J}$ has the type $F_m^{(+)}$ where $n = 2ml$. We know that by an appropriate automorphism $\psi$ of $F_{\frac{n}{2}}^{(+)}$, we can reduce $\theta^{-1}(\mathcal{J})$ to the canonical form: $\psi(\theta^{-1}(\mathcal{J})) = \{\operatorname{diag}(X, \ldots, X, X^t, \ldots, X^t)\}$ where $X$ is any matrix of order $m$. Then, $X$ can be written as $A + iB$ for an appropriate symmetric $A$ and skew-symmetric $B$. Therefore, $\theta(\psi(\theta^{-1}(\mathcal{J})))$ has the following representation in $H(F_n)$:

$$\theta(\psi(\theta^{-1}(\mathcal{J}))) = \left\{ \begin{pmatrix} S & T \\ -T & S \end{pmatrix} \right\}$$

where $S = \operatorname{diag}(A, \ldots, A)$, $T = \operatorname{diag}(B, \ldots, B, -B, \ldots, -B)$. By Lemma 2.1.1, $\theta \circ \psi \circ \theta^{-1}$ (an automorphism of the algebra of the form (1)) can be extended to an automorphism of $H(F_n)$. Finally, by interchanging the $k$-th and $(\frac{n}{2}+k)$-th columns, and $k$-th and $(\frac{n}{2}+k)$-th rows, $1 \leq k \leq \frac{n}{2}$, and the columns and rows inside the block (if necessary), we can achieve the following block-diagonal canonical form of type 4. As a result any subalgebra of $H(F_n)$ of the type $F_m^{(+)}$ can be brought to the canonical form of type 4. This canonical form is obviously unique.

**2.2**. Let $\mathcal{J}$ be of the type $H(F_m)$. Acting in the same manner as before, $\mathcal{J}$ can be brought to the unique canonical form: $\theta(\psi(\theta^{-1}(\mathcal{J}))) = \{\mathrm{diag}(X, \ldots, X)\}$ where $X$ is a symmetric matrix of order $m$.

**2.3**. Let $\mathcal{J}$ be of the type $H(F_{2m}, j)$, $n = 4ml$. Like in the previous cases, by an appropriate automorphism $\psi$ of $F_{\frac{n}{2}}^{(+)}$, $\theta^{-1}(\mathcal{J})$ can be brought to the following block-diagonal form: $\psi(\theta^{-1}(\mathcal{J})) = \{\mathrm{diag}(X, \ldots, X)\}$, where $X$ is a symplectic matrix of order $2m$. If we represent $X$ as the sum of symmetric and skew-symmetric matrices as follows:

$$X = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} + \begin{pmatrix} -C & -D \\ -D & C \end{pmatrix}$$

where all matrices have order $m$; $A$ is symmetric, $B$, $C$,$D$ are skew-symmetric, then $\theta$ induces the following representation of $\mathcal{J}$ in $H(F_n)$

$$\theta(\psi(\theta^{-1}(\mathcal{J}))) = \left\{ \begin{pmatrix} S & T \\ -T & S \end{pmatrix} \right\}$$

where $S = \mathrm{diag}(X, \ldots, X)$, $T = \mathrm{diag}(Y, \ldots, Y, -Y, \ldots, -Y)$ and

$$X = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad Y = \begin{pmatrix} -C & -D \\ -D & C \end{pmatrix}.$$

Similarly, by Lemma 2.1.1, $\theta \circ \psi \circ \theta^{-1}$ (an automorphism of the algebra of the form (1)) can be extended to an automorphism of $H(F_n)$.

By interchanging appropriate blocks, we can reduce it to the canonical form of type 6. From the explicit form of type 6, the canonical form of $\mathcal{J}$ of the type $H(F_{2m}, j)$ is uniquely determined.

If $1 \notin \mathcal{J}$, then $\mathrm{rk}\,(e) = k < n$ where $e$ is the identity element of $\mathcal{J}$ and it is quite obvious that $\mathcal{J}$ can be covered by a subalgebra of $H(F_n)$ of the type $H(F_k)$. As was already shown, $\mathcal{J}$ can be reduced to the unique canonical form in $H(F_k)$, hence, in $H(F_n)$.

**Case 3** $\mathcal{A} = H(F_{2n}, j)$

Since the proof of this case is not much different from the proof of the case of $H(F_n)$, we will omit some details. According to [29], any maximal subalgebra $\mathcal{M}$ in $H(F_{2n}, j)$ is isomorphic to one of the following:

1. $H(F_{2k}, j) \oplus H(F_{2l}, j)$, $k + l = n$,

2. $H(F_{2k}, j) \oplus F_l^{(+)} \oplus \mathcal{R}$, $k + l = n$. If $k = 0$, then $\mathcal{M} = F_n^{(+)} \oplus \mathcal{R}$, $\mathcal{R}$ is the radical

3. $J(V, f)$ only if $n = 2^m$ and either $\dim J(V, f) = 2(m + 1)$, $m$ is even, or $\dim J(V, f) = 2m + 1$, $m$ is odd.

First we assume that $\mathcal{J}$ is a simple matrix subalgebra of $H(F_{2n}, j)$ such that $1 \in \mathcal{J}$. Let $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ be a maximal subalgebra which contains $\mathcal{J}$, $\mathcal{J} \subseteq \mathcal{M}$. By Lemma 2.1.3, $\mathcal{J} \subset \mathcal{S}$. If $\mathcal{S}$ is a non-simple semisimple algebra, then $\mathcal{J}$ can be projected into the simple components of $\mathcal{S}$. Hence, the problem will be reduced to the case of symplectic matrices of order less than $2n$. This reduction stops only when either the image of $\mathcal{J}$ can be covered by the maximal subalgebra with a simple Wedderburn factor $\mathcal{S}$ or the image of $\mathcal{J}$ coincides with one of the simple components of $\mathcal{S}$.

Next we look into the case when $\mathcal{J} \subset \mathcal{M}$, where $\mathcal{M}$ has a simple Wedderburn factor $\mathcal{S}$. There is no loss in generality if we assume that $\mathcal{M} = \mathcal{S} \ominus \mathcal{R}$, $\mathcal{S} \cong F_n^{(+)}$. By Lemma 2.1.4, $\mathcal{S}$ can be brought to the form (6). Notice that any automorphism of $F_n^{(+)}$ of the form $\varphi(X) = C^{-1}XC$ can be extended to an automorphism of $H(F_{2n}, j)$ in a natural way:

$$\bar{\varphi}(X) = \bar{C}^{-1}X\bar{C}, \quad \bar{C} = \begin{pmatrix} C & 0 \\ 0 & (C^{-1})^t \end{pmatrix} \tag{9}$$

**3.1** If $\mathcal{J} \cong H(F_m)$, $m \leq n$, then acting by some automorphism of the form (9), it can be reduced to canonical form of type 8. This canonical form is obviously uniquely determined.

**3.2** If $\mathcal{J} \cong F_m^{(+)}$, $m \leq n$, then by an automorphism of the form (9) it can be brought to $\{\mathrm{diag}\,(X, \ldots, X, X^t, \ldots, X^t)\}$ where $X$ is an arbitrary matrix of order $m$. This is the canonical form of type 7. With some effort it can be shown that in this case the canonical form is also unique. Namely all we have to show is that any two canonical forms $\mathcal{J}_1$ and $\mathcal{J}_2$ of the same type with $k_{\mathcal{A}}(\mathcal{J}_1) \neq k_{\mathcal{A}}(\mathcal{J}_2)$ are not conjugate under symplectic automorphism, or, equivalently, automorphism of $H(F_{2n}, j)$. For clarity, let $\mathcal{J}_1 = \mathrm{diag}\,\{\underbrace{X, \ldots, X}_{r}, \underbrace{X^t, \ldots, X^t}_{s}, \underbrace{X^t, \ldots, X^t}_{r}, \underbrace{X, \ldots, X}_{s}\}$, $rm + sm = n$, and $\mathcal{J}_2 = \mathrm{diag}\,\{\underbrace{Y, \ldots, Y}_{p}, \underbrace{Y^t, \ldots, Y^t}_{q}, \underbrace{Y^t, \ldots, Y^t}_{p}, \underbrace{Y, \ldots, Y}_{q}\}$, $pm + qm = n$, $p > r$, where $X$ and $Y$ are any matrices of order $m$, $k_{\mathcal{A}}(\mathcal{J}_1) \neq k_{\mathcal{A}}(\mathcal{J}_2)$. Next we assume the contrary, i.e. there exists a symplectic automorphism $\varphi$ such that $\varphi(\mathcal{J}_1) = \mathcal{J}_2$. Let $\mathcal{S}$ stand for the subalgebra of $H(F_{2n}, j)$ of the form (6). Obviously, $\mathcal{J}_1 \subseteq \mathcal{S}$, $\mathcal{J}_2 \subseteq \mathcal{S}$. Next we are going to show that for any automorphism $\varphi$ of $H(F_{2n}, j)$ such that $\varphi(\mathcal{J}_1) = \mathcal{J}_2$ we can always find a symplectic automorphism $\psi$ that can be restricted to $\mathcal{S}$ and $\varphi|_{\mathcal{J}_1} = \psi|_{\mathcal{J}_1}$. Let $C$ be a non-singular matrix that determines $\varphi$. Then, for any $A \in \mathcal{J}_1$ there exists $B \in \mathcal{J}_2$ such that

$$C^{-1}AC = B, \quad AC = CB. \tag{10}$$

Set $C = (C_{ij})_{i,j=1,s}$ where $C_{ij}$ is a square matrix of order $m$. By performing a matrix multiplication in (10) we obtain a series of equations:

$$XC_{ij} = C_{ij}Y, \quad X^tC_{kl} = C_{kl}Y$$

where $(i,j),(k,l) \in I \times I$, $I = \{1,\ldots,s\}$. Since $X$ and $Y$ can be any matrices of order $m$, $C_{ij}$ cannot be degenerate. Therefore, $Y = C_{ij}^{-1}XC_{ij}$, $Y = C_{kl}^{-1}X^tC_{kl}$. Hence, the matrix $\bar{C} = \operatorname{diag}\{\underbrace{C_{ij}, C_{kl}, \ldots, C_{ij}}_{n}, (C_{ij}^t)^{-1}, (C_{kl}^t)^{-1}, \ldots, (C_{ij}^t)^{-1}\}$ determines an automorphism $\psi$ of $H(F_{2n}, j)$ such that $\varphi|_{\mathcal{J}_1} = \psi|_{\mathcal{J}_1}$. In addition, $\psi$ can be restricted to $\mathcal{S}$, thereby inducing an automorphism of a subalgebra of the type $F_n^{(+)}$. However we have already shown (case 1.1) that the two canonical forms in $F_n^{(+)}$ with $k_{\mathcal{A}}(\mathcal{J}_1) \neq k_{\mathcal{A}}(\mathcal{J}_2)$ are not conjugate.

**3.3** If $\mathcal{J} \cong H(F_{2m}, j)$, $m \leq n$, then it can be reduced to canonical form of type 9. Next we are going to show that any two canonical forms $\mathcal{J}_1$ and $\mathcal{J}_2$ of the same type with $k_{\mathcal{A}}(\mathcal{J}_1) \neq k_{\mathcal{A}}(\mathcal{J}_2)$ are not conjugate under an automorphism of $H(F_{2n}, j)$. Assume the contrary, that is, there exists an automorphism $\varphi$ of $H(F_{2n}, j)$ such that $\varphi(\mathcal{J}_1) = \mathcal{J}_2$. Next we can choose $\mathcal{S}_1 \subseteq \mathcal{J}_1$, $\mathcal{S}_1 \cong F_m^{(+)}$ such that $k_{\mathcal{A}}(\mathcal{S}_1) = k_{\mathcal{A}}(\mathcal{J}_1)$. Similarly, we can select $\mathcal{S}_2 \subseteq \mathcal{J}_2$, $\mathcal{S}_2 \cong F_m^{(+)}$ such that $k_{\mathcal{A}}(\mathcal{S}_2) = k_{\mathcal{A}}(\mathcal{J}_2)$. By Lemma 2.1.4 there exists $\psi : \mathcal{J}_2 \to \mathcal{J}_2$, $\psi(\varphi(\mathcal{S}_1)) = \mathcal{S}_2$. From the explicit form of $\mathcal{J}_2$, $\psi$ can be extended to an automorphism $\psi \circ \varphi : H(F_{2m}, j) \to H(F_{2m}, j)$. It follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same canonical forms, in particular, $k_{\mathcal{A}}(\mathcal{S}_1) = k_{\mathcal{A}}(\mathcal{S}_2)$, a contradiction.

If $1 \notin \mathcal{J}$, then in order to find the canonical form of $\mathcal{J}$ we use the same approach as in the case of $H(F_n)$. The theorem is proved.

□

Before we state the following theorem we introduce one more notation. Let $\mathcal{J}$ be a simple Jordan subalgebra of $\mathcal{A} = F_n^{(+)}$, $H(F_n)$ or $H(F_{2n}, j)$, and $e$ be the identity of $\mathcal{J}$. Then $e$ can be decomposed into the sum of idempotents minimal in $\mathcal{A}$, $e = e_1 + \ldots + e_k$. Then $k = \deg_{\mathcal{A}}\mathcal{J}$.

**Theorem 2.1.12.** *Let $\mathcal{A}$ be a Jordan algebra of any of the following types: $F_n^{(+)}$, $H(F_n)$ or $H(F_{2n}, j)$, $n \geq 3$, and $\mathcal{J}$, $\mathcal{J}'$ be proper simple matrix subalgebras of $\mathcal{A}$. If $\mathcal{J}'$ has the same type as $\mathcal{J}$, then $\mathcal{J}' \in C(\mathcal{J})$ if and only if $deg_{\mathcal{A}}\mathcal{J} = deg_{\mathcal{A}}\mathcal{J}'$ in all cases except for $\mathcal{A} \cong H(F_{2n}, j)$, $\mathcal{J} \cong F_m^{(+)}$, $H(F_{2m}, j)$ and $\mathcal{A} \cong F_n^{(+)}$, $\mathcal{J} \cong F_m^{(+)}$. In these cases it is additionally required $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$.*

*Proof.* First it should be noted that the degree of $\mathcal{J} \geq 3$. The case of $\mathcal{J}$ of the degree 2 will be considered later in the text. Notice that in matrix terms $\deg_{\mathcal{A}}\mathcal{J} = \deg_{\mathcal{A}}\mathcal{J}'$ is equivalent to $\mathrm{rk}\,(e) = \mathrm{rk}\,(e')$ where $e$, $e'$ are the identity of $\mathcal{J}$, $\mathcal{J}'$.

**The case of $F_n^{(+)}$**

In this case we assume that $\mathcal{J}$ and $\mathcal{J}'$ are subalgebras of $F_n^{(+)}$ which is as usual the set of all matrices of order $n$ under Jordan multiplication. This case breaks into the following subcases.

**(1)** Let $\mathcal{J}$ be of the type $F_m^{(+)}$ for some $m < n$. First we assume that $S(\mathcal{J})$ is a simple algebra. Equivalently, $k_{\mathcal{A}}(\mathcal{J}) = \text{rk}(e)$. Let $\mathcal{J}'$ be as given in the hypothesis of the theorem. If $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$, then there exists an automorphism $\varphi$ of $F_n^{(+)}$ which maps $\mathcal{J}'$ onto $\mathcal{J}$. It follows that $\varphi(e') = e$, therefore, $\text{rk}\,(e') = \text{rk}\,(e)$. Besides, $\varphi(S(\mathcal{J}')) = S(\mathcal{J})$. Hence, $S(\mathcal{J}')$ is also simple, $k_{\mathcal{A}}(\mathcal{J}') = \text{rk}(e')$. It follows that $k_{\mathcal{A}}(\mathcal{J}') = \text{rk}\,(e') = \text{rk}\,(e) = k_{\mathcal{A}}(\mathcal{J})$.

Conversely, if $\text{rk}(e') = \text{rk}(e)$ and $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$, then $k_{\mathcal{A}}(\mathcal{J}') = k_{\mathcal{A}}(\mathcal{J}) = \text{rk}(e) = \text{rk}(e')$, because $k_{\mathcal{A}}(\mathcal{J}) = \text{rk}(e)$. Therefore, $k_{\mathcal{A}}(\mathcal{J}') = \text{rk}(e')$, that is, $S(\mathcal{J}')$ is also simple, and $\mathcal{J}$, $\mathcal{J}'$ have the same canonical forms. This implies that $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

Now we assume that $S(\mathcal{J})$ is a non-simple semisimple subalgebra. Let $\mathcal{J}'$ be another subalgebra which satisfies the hypothesis of the theorem. If $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$, then there exists an automorphism $\varphi$ of $F_n^{(+)}$ which maps $\mathcal{J}'$ onto $\mathcal{J}$. Therefore, $\mathcal{J}'$ and $\mathcal{J}$ have equivalent representations in $F^n$, and so do $S(\mathcal{J}')$ and $S(\mathcal{J})$. Consequently, either $\deg \rho_1(\mathcal{I}_1) = \deg \rho_1(\mathcal{I}_1')$ and $\deg \rho_2(\mathcal{I}_2) = \deg \rho_2(\mathcal{I}_2')$ or $\deg \rho_1(\mathcal{I}_1) = \deg \rho_2(\mathcal{I}_2')$ and $\deg \rho_2(\mathcal{I}_2) = \deg \rho_1(\mathcal{I}_1')$. Equivalently, $|\deg\rho_1(\mathcal{I}_1) - \deg\rho_2(\mathcal{I}_2)| = |\deg\rho_1(\mathcal{I}_1') - \deg\rho_2(\mathcal{I}_2')|$, that is, $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$.

Conversely, if $\text{rk}(e') = \text{rk}(e)$ and $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$, then $\mathcal{J}$ and $\mathcal{J}'$ have the same canonical forms. Therefore, these subalgebras are conjugate under some automorphism of $F_n^{(+)}$, and $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**(2)** Let $\mathcal{J}$ be of the type $H(F_m)$ for some $m \leq n$. Suppose that $\mathcal{J}'$ is another subalgebra of $F_n^{(+)}$ which has the type $H(F_m)$. If $\mathcal{J}'$ is conjugate to $\mathcal{J}$ under some automorphism $\varphi$ of $F_n^{(+)}$ then $\varphi(e') = e$ and $\text{rk}(e') = \text{rk}(e)$. In other words, the canonical form of $\mathcal{J}'$ is exactly the same as that of $\mathcal{J}$. Conversely, if $\mathrm{rk}(e') = \mathrm{rk}(e)$, then $\mathcal{J}$ and $\mathcal{J}'$ have the same canonical forms. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**(3)** Let $\mathcal{J}$ be of the type $H(F_{2m}, j)$ for some $m \leq n$. The proof of this case is exactly the same as the previous proof.

**The case of** $H(F_n)$

Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are two subalgebras of $H(F_n)$ that satisfy the hypothesis of the theorem.

**(1)** Let $\mathcal{J}$ as well as $\mathcal{J}'$ be of the type $F_m^{(+)}$. Assume that $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$. It follows that there exists an automorphism of $H(F_n)$ such that $\varphi(\mathcal{J}') = \mathcal{J}$. Hence, $\mathrm{rk}(e') = \mathrm{rk}(e)$. Conversely, if $\mathrm{rk}(e') = \mathrm{rk}(e)$, then $\mathcal{J}$ and $\mathcal{J}'$ have the same canonical form. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**(2)** Now let both $\mathcal{J}$ and $\mathcal{J}'$ have the type $H(F_m)$ (or $H(F_{2m}, j)$). If $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$, then there exists an automorphism $\varphi$ of $H(F_n)$ that sends $\mathcal{J}'$ onto $\mathcal{J}$, $\varphi(\mathcal{J}') = \mathcal{J}$. Consequently, $\mathrm{rk}(e') = \mathrm{rk}(e)$.

Conversely, if $\mathrm{rk}(e') = \mathrm{rk}(e)$, then they have the same canonical form. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**The case of** $H(F_{2n}, j)$

Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are two subalgebras of $H(F_{2n}, j)$ that satisfy the conditions of the theorem.

**(1)** Let $\mathcal{J}$ as well as $\mathcal{J}'$ be of the type $F_m^{(+)}$, $m < n$. Assume that $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$. It follows that there exists an automorphism of $H(F_{2n}, j)$ such that $\varphi(\mathcal{J}') = \mathcal{J}$. Hence, $\mathrm{rk}(e') = \mathrm{rk}(e)$. Since $\mathcal{J}'$ and $\mathcal{J}$ are conjugate in $H(F_{2n}, j)$, they have the same canonical forms in $H(F_{2n}, j)$. Therefore, $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$.

Conversely, if all conditions hold true, then $\mathcal{J}$ and $\mathcal{J}'$ have the same canonical forms. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**(2)** Now let both $\mathcal{J}$ and $\mathcal{J}'$ have the type $H(F_m)$, $m < n$. If $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$, then there exists an automorphism of $H(F_{2n}, j)$ that sends $\mathcal{J}'$ onto $\mathcal{J}$, $\varphi(\mathcal{J}') = \mathcal{J}$. Consequently, $\mathrm{rk}(e') = \mathrm{rk}(e)$.

Conversely, if $\mathrm{rk}(e') = \mathrm{rk}(e)$, then they have the same canonical forms. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$.

**(3)** Now let both $\mathcal{J}$ and $\mathcal{J}'$ have the type $H(F_{2m}, j)$, $m < n$. If $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$, then there exists an automorphism of $H(F_{2n}, j)$ that sends $\mathcal{J}'$ onto $\mathcal{J}$, $\varphi(\mathcal{J}') = \mathcal{J}$. Consequently, $\mathrm{rk}(e') = \mathrm{rk}(e)$, $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$.

Conversely, if $\mathrm{rk}(e') = \mathrm{rk}(e)$ and $k_{\mathcal{A}}(\mathcal{J}) = k_{\mathcal{A}}(\mathcal{J}')$, then they have the same canonical forms. Therefore, $\mathcal{J}' \in \mathcal{C}(\mathcal{J})$

The theorem is proved.

□

**Corollary 2.1.13.** *If $m$ is any number such that $m \leq n$, and $n = mk + r$, $0 \leq r < m$, then there exist subalgebras of $F_n^{(+)}$ of the type $H(F_m)$. Moreover, there are precisely $k$ conjugacy classes corresponding to $H(F_m)$. If $2m \leq n$, and $n = 2mk + r$, $0 \leq r < 2m$ then $F_n^{(+)}$ has subalgebras of the type $H(F_{2m}, j)$, and the number of conjugacy classes corresponding to $H(F_{2m}, j)$ is equal to $k$. Finally, if $m < n$, and $n = mk + r$, $0 \leq r < m$ then there exist subalgebras of $F_n^{(+)}$ of the type $F_m^{(+)}$, and, moreover, the number of conjugacy classes is given by $\sum_{j=1}^{k} [\frac{j}{2}]$.*

**Corollary 2.1.14.** *If $m$ is any number such that $m < n$, and $n = mk + r$, $0 \leq r < m$, then there exist subalgebras of $H(F_n)$ of the type $H(F_m)$. Moreover, there are precisely $k$ conjugacy classes corresponding to $H(F_m)$. If $2m \leq n$, and $n = 2mk + r$, $0 \leq r < m$ then $H(F_n)$ has subalgebras of the type $F_m^{(+)}$, and the number of conjugacy classes corresponding to $F_m^{(+)}$ is equal to $k$. Finally, if $4m \leq n$, and $n = 4mk + r$, $0 \leq r < 4m$ then there exist subalgebras of $H(F_n)$ of the type $H(F_{2m}, j)$, and, moreover, the number of conjugacy classes is $k$.*

**Corollary 2.1.15.** *If $m$ is any number such that $m \leq n$, and $n = mk + r$, $0 \leq r < m$, then there exist subalgebras of $H(F_{2n}, j)$ of the type $H(F_m)$. Moreover, there are precisely $k$ conjugacy classes corresponding to $H(F_m)$. If $m \leq n$, and $n = mk + r$, $0 \leq r < m$ then $H(F_{2n}, j)$ has subalgebras of the type $F_m^{(+)}$, and the number of conjugacy classes corresponding to $F_m^{(+)}$ is equal to $\sum_{j=1}^{k}[\frac{j}{2}]$. Finally, if $m < n$, and $n = mk+r$, $0 \leq r < m$ then there exist subalgebras of $H(F_n, j)$ of the type $H(F_m, j)$, and, moreover, the number of conjugacy classes is $\sum_{j=1}^{k}[\frac{j}{2}]$.*

## 2.2 Subalgebras of the type $J(V, f)$

First we recall a few facts from [29] concerning Clifford algebras over a field of characteristic not 2. Let $\mathcal{J} = F1 \oplus V$ where $V = \text{span}\langle x_1, \ldots, x_{2m}\rangle$, and $f$ a non-degenerate symmetric bilinear form on $V$. Then, $C(V, f)$ is a central simple associative algebra with a unique canonical involution '—' that fixes elements from $V$. In this case the imbedding of $\mathcal{J}$ into $C(V, f)^{(+)}$ we will call *canonical of the first type*. Next, let $\mathcal{J} = F1 \oplus V$ where $V = \text{span}\langle x_1, \ldots, x_{2m+1}\rangle$, and $V_0 = \mathrm{span}\langle x_1, \ldots, x_{2m}\rangle$. Then, $C(V, f)$ is isomorphic to a tensor product of $C(V_0, f)$ and the two-dimensional center $E$ of $C(V, f)$. Moreover, $E = F[z]$ where $z = x_1 x_2 \ldots x_{2m+1}$. In other words, $C(V, f) = \mathcal{I}_1 \oplus \mathcal{I}_2$, $\mathcal{I}_i \cong C(V_0, f)$. Note that $F1 \oplus V \cong \mathcal{J} + \mathcal{I}_i/\mathcal{I}_i \subseteq C(V, f)/\mathcal{I}_i \cong C(V_0, f)^{(+)}$. This imbedding of $\mathcal{J} = F1 \oplus V$ into $C(V_0, f)^{(+)}$ we will call *canonical of the second type.*

Let $\mathcal{A}$ be a simple matrix Jordan algebra, and $\mathcal{J}$ be a subalgebra of $\mathcal{A}$ of the type $J(V, f)$. According to [29], $\mathcal{J}$ of the type $J(V, f)$ is maximal in $\mathcal{A}$ if and only if one of the following cases holds

1. $\mathcal{A} = (C(V_0, f), -)$, $\mathcal{J} = F1 \oplus V$ where $\dim V = 2m + 1$ and $m$ is odd.
2. $\mathcal{A} = H(C(V_0, f), -)$, $\mathcal{J} = F1 \oplus V$ where $\dim V = 2m + 1$, $m$ is even.
3. $\mathcal{A} = H(C(V, f), -)$, $\mathcal{J} = F1 \oplus V$ where $\dim V = 2m$

Next we recall that if $\dim V = 2m$, and $m \equiv 0, 1 (\mathrm{mod}\, 4)$ then $\dim H(C(V, f), -) = 2^{m-1}(2^m + 1)$. If $\dim V = 2m$ and $m \equiv 2, 3 (\mathrm{mod}\, 4)$ then $\dim H(C(V, f), -) = 2^{m-1}(2^m - 1)$. If $\dim V = 2m + 1$ and $m \equiv 0 (\mathrm{mod}\, 4)$ then $\dim H(C(V_0, f), -) = 2^{m-1}(2^m + 1)$. If $\dim V = 2m + 1$ and $m \equiv 2 (\mathrm{mod}\, 4)$ then $\dim H(C(V_0, f), -) = 2^{m-1}(2^m - 1)$.

### 2.2.1 Canonical realizations of $J(V, f)$

Let $\mathcal{A}$ be a simple matrix Jordan algebra, and $\mathcal{J} = F1 \oplus V$ be a subalgebra of $\mathcal{A}$. Then all realizations of $\mathcal{J}$ in $\mathcal{A}$ listed below we will call *canonical.*

*Type 10.* $\mathcal{A} = F_n^{(+)}$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{l}, 0, \dots, 0)\}$$

where $X$ is a matrix of order $2^m$, and if $\pi_i$ denotes the projection on the $i$th non-zero block, then $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the first type.

*Type 11.* $\mathcal{A} = F_n^{(+)}$, $n = 2^m l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{l}, 0, \dots, 0)\}$$

where $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

*Type 12.* $\mathcal{A} = F_n^{(+)}$, $n = 2^m l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{s}, \underbrace{X^t, \dots, X^t}_{k}, 0, \dots, 0)\}$$

where $s + k = l$, $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

*Type 13.* $\mathcal{A} = H(F_n)$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{l}, 0, \dots, 0)\}$$

where $X$ is a symmetric matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the first type.

*Type 14.* $\mathcal{A} = H(F_n)$, $n = 2^{m+1} l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{l}, 0, \dots, 0)\}$$

where $X$ is of the form (1) in which $A$ and $B$ are of order $2^m$. If $\mathcal{S}$ denotes the algebra of the form (1), then $\pi_i(\mathcal{J}) \subseteq \mathcal{S}$ is a canonical imbedding of the first type.

*Type 15.* $\mathcal{A} = H(F_n)$, $n = 2^{m+1}l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_{l}, 0, \ldots, 0)\}$$

where $X$ is of the form (1) in which $A$ and $B$ are of order $2^m$. If $\mathcal{S}$ denotes the entire algebra of the form (1), then $\pi_i(\mathcal{J}) \subseteq \mathcal{S}$ is a canonical imbedding of the second type.

*Type 16.* $\mathcal{A} = H(F_n)$, $n = 2^m l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_{l}, 0, \ldots, 0)\}$$

where $X$ is a symmetric matrix, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

*Type 17.* $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_{l}, \underbrace{0, \ldots, 0}_{k}, \underbrace{X, \ldots, X}_{l}, \underbrace{0, \ldots, 0}_{k}), \}$$

where $k + l2^m = n$, $X$ is a symmetric matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the first type.

*Type 18.* $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m$, $\mathcal{J}$ has a canonical form (3.3), and if $\pi_i$ denotes the projection of $\mathcal{J}$ into $i$th simple component (of the type $H(F_{2^m}, j)$) of (3.3), then $\pi_i(\mathcal{J}) \subseteq H(F_{2^m}, j)$ is a canonical imbedding of the first type.

*Type 19.* $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m + 1$, $\mathcal{J}$ has a canonical form (3.3) where $\pi_i(\mathcal{J}) \subseteq H(F_{2^m}, j)$ is a canonical imbedding of the second type.

*Type 20.* $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, dim $V = 2m + 1$,

$$\mathcal{J} = \{\mathrm{diag}(\underbrace{X, \dots, X}_{s}, \underbrace{X^t, \dots, X^t}_{k} 0, \dots, 0, \underbrace{X^t, \dots, X^t}_{s}, \underbrace{X, \dots, X}_{k}, 0, \dots, 0)\}$$

where $s + k = l$, $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

**Theorem 2.2.1.** *Let $\mathcal{A}$ be a simple matrix Jordan algebra, and $\mathcal{J}$ be a subalgebra of $\mathcal{A}$ of the type $J(V, f)$. Then, $\mathcal{J}$ has a unique canonical form as above.*

*Proof.* Let $\mathcal{J} = F1 \oplus V$. Then the following cases occur.

**Case** $\mathcal{A} = F_n^{(+)}$

**1.1** Let dim $V = 2m$. Then $U(\mathcal{J}) \cong C(V, f)$ is a simple algebra. In particular, $S(\mathcal{J}) \cong U(\mathcal{J})$.

If $S(\mathcal{J}) = \mathcal{A}$, then $n = 2^m$, $\mathcal{A} \cong U(\mathcal{J})$. Therefore, the imbedding of $\mathcal{J}$ into $\mathcal{A}$ is equivalent to the imbedding of $F1 \oplus V$ into $C(V, f)^{(+)}$. Therefore, this is a canonical imbedding of the first type.

If $S(\mathcal{J}) \subset \mathcal{A}$, then $S(\mathcal{J})$ is a proper simple associative subalgebra of $F_n$. Therefore, $S(\mathcal{J})$ can be reduced to

$$\{\mathrm{diag}(\underbrace{Y, \dots, Y}_{l}, 0, \dots, 0)\} \tag{11}$$

where the order of $Y$ is $2^m$, and $n = 2^m l + r$. As a result, $\mathcal{J}$ also takes the canonical form of type 10.

**1.2** Let dim $V = 2m + 1$. Then $U(\mathcal{J}) \cong C(V, f)$, and $U(\mathcal{J}) = \mathcal{I}_1 \oplus \mathcal{I}_2$, $\mathcal{I}_i \cong C(V_0, f)$. Hence $S(\mathcal{J})$ is isomorphic to either $C(V, f)$ or $C(V_0, f)$.

If $S(\mathcal{J}) = \mathcal{A}$, then the imbedding of $\mathcal{J}$ into $\mathcal{A}$ is canonical of the second type.

If $S(\mathcal{J}) \cong \mathcal{I}_i$, and $S(\mathcal{J}) \subset \mathcal{A}$, then $S(\mathcal{J})$ is a proper simple associative subalgebra of $F_n$. Therefore, $S(\mathcal{J})$ can be reduced to (11). As a result, $\mathcal{J}$ takes the canonical form of type 11.

Finally, if $S(\mathcal{J}) = \mathcal{I}_1 \oplus \mathcal{I}_2$, then $\mathcal{J}$ takes the canonical form of type 12.

**Case** $\mathcal{A} = H(F_n)$

Let $M$ be the maximal subalgebra of $H(F_n)$ such that $\mathcal{J} \subseteq \mathcal{M} \subseteq H(F_n)$. Then, the following cases occur.

1. $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ a semisimple factor, $\mathcal{R}$ the radical. Then, we reduce the problem to the case of symmetric matrices of a lower dimension (see section 2.1).

2. $\mathcal{M} = \mathcal{S}$ where $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$. Like in the previous case we can reduce the problem to the case of symmetric matrices of a lower dimension.

3. $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_{\frac{n}{2}}^{(+)}$, $\mathcal{R}$ the radical.

4. $\mathcal{M} = F1 \oplus W$ where $W$ is a finite-dimensional vector space.

After a series of reductions of the form 1 and 2, the image of $\mathcal{J}$ becomes a subalgebra of

$$\begin{pmatrix} \mathcal{A}_1 & & & & 0 \\ & \ddots & & & \\ & & \mathcal{A}_i & & \\ & & & \ddots & \\ 0 & & & & \mathcal{A}_k \end{pmatrix}$$

where $\mathcal{A}_i \cong H(F_{n_i})$. Let $\pi_i$ be the projection of $\mathcal{J}$ into $\mathcal{A}_i$. To simplify our notation we denote $\pi_i(\mathcal{J})$ as $\mathcal{J}'$, and the maximal subalgebra of $\mathcal{A}_i$ which covers $\mathcal{J}'$ as $\mathcal{M}_i$, $\mathcal{J}' \subseteq \mathcal{M}_i \subsetneq \mathcal{A}_i = H(F_{n_i})$.

**Case 1.** Let dim $V = 2m$ and $m \equiv 0, 1 (\mathrm{mod}\, 4)$. Then we have the following cases:

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{n_i}$, then $n_i = 2^m$, $F_{n_i} \cong C(V, f)$, and the imbedding of $\mathcal{J}'$ into $F_{n_i}^{(+)}$ is equivalent to the imbedding of $F1 \oplus V$ into $C(V, f)^{(+)}$, that is, canonical of the first type. If $S(\mathcal{J}') \subset F_{n_i}$, then $H(S(\mathcal{J}')) \subsetneq H(F_{n_i})$ is a proper simple subalgebra of $H(F_{n_i})$. Hence, $n_i = 2^m l + r$, and $H(S(\mathcal{J}'))$ can be reduced to (11) in which $X$ denotes a symmetric matrix of order $2^m$. Then, $\mathcal{J}'$ takes the canonical form of type 13.

(b) Let $\mathcal{M}_i = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_{\frac{n_i}{2}}^{(+)}$. By using $\theta$-isomorphism defined in the proof of Theorem 2.1.11 we obtain that $\theta^{-1}(\mathcal{J}') \subseteq F_{\frac{n_i}{2}}^{(+)}$. If $S(\theta^{-1}(\mathcal{J}')) = F_{\frac{n_i}{2}}^{(+)}$, then $n_i = 2^{m+1}$ and the imbedding of $\theta^{-1}(\mathcal{J}')$ into $F_{\frac{n_i}{2}}^{(+)}$ is the canonical imbedding of the first type. In particular, $\theta^{-1}(\mathcal{J}') \subseteq H(F_{\frac{n}{2}})$. As a result $\mathcal{J}'$ takes the canonical form of type 13 in which $l = 2$ and there are no zeros. If $S(\mathcal{J}') \subset F_{\frac{n_i}{2}}^{(+)}$, then $S(\theta^{-1}(\mathcal{J}'))$ is a proper simple subalgebra of $F_{\frac{n_i}{2}}$, therefore, takes the form (11) and $n_i = 2^{m+1}l + r$.

Hence $\mathcal{J}$ takes the canonical form 2.1.

**Case 2** Let dim $V = 2m$, $m \equiv 2, 3 (\mathrm{mod}\, 4)$.

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{n_i}$, then $n_i = 2^m$, $F_{n_i} \cong C(V, f)$, $\mathcal{J}' \subseteq H(F_{n_i})$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{J}' = F1 \oplus V & \xrightarrow{id} & \mathcal{J}' = F1 \oplus V \\ \downarrow \sigma & & \downarrow \eta \\ U(\mathcal{J}') & \xleftarrow{\varphi} & F_{n_i}^{(+)} \end{array}$$

where $\sigma = \varphi \circ \eta$. Therefore, $\sigma(\mathcal{J}') = \varphi(\eta(\mathcal{J}'))$ is symmetric with respect to the canonical involution "—" which is symplectic in this particular case. On the other hand, $\sigma(\mathcal{J}')$ is also symmetric with respect to $j' = \varphi \circ \iota \circ \varphi^{-1}$. By the uniqueness of "—", $j'$ equals to "—". However this is not possible because dim $H(C(V, f), -) = \frac{2^m(2^m-1)}{2} \neq \frac{2^m(2^m+1)}{2} =$ dim $H(C(V, f), j')$. If $S(\mathcal{J}') \subset F_{n_i}$, then $H(S(\mathcal{J}')) \subseteq H(F_{n_i})$ is a proper subalgebra of $H(F_{n_i})$. Hence $n_i = 2^m l + r$, and $H(S(\mathcal{J}'))$ can be reduced to (11) where $X$ denotes a symmetric matrix of order $2^m$. Let $\pi_{ij}$ denote the projection on $j$th non-zero block of (11). Then the imbedding $\pi_{ij}(\mathcal{J}') \subseteq \pi_{ij}(H(S(\mathcal{J}')))$ is similar to the above imbedding, which is not possible.

(b) Let $\mathcal{M}_i = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_{\frac{n_i}{2}}^{(+)}$. Then $\theta^{-1}(\mathcal{J}') \subseteq F_{\frac{n_i}{2}}^{(+)}$. Since $S(\theta^{-1}(\mathcal{J}')) \cong U(\mathcal{J}')$, then $n_i = 2^{m+1} l + r$, and $S(\theta^{-1}(\mathcal{J}'))$ can be reduced to (11) in which $X$ is any matrix of order $2^m$. Hence $\pi_{ij}(\theta^{-1}(\mathcal{J}')) \subset F_{2^m}^{(+)}$ is a canonical imbedding of the first type, and $\mathcal{J}$ has the canonical form of type 14.

**Case 3** Let dim $V = 2m + 1$ where $m$ is odd.

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{n_i}$, then $n_i = 2^m$, $F_{n_i} \cong C(V_0, f)$. Therefore, the imbedding of $\mathcal{J}'$ into $F_{n_i}^{(+)}$ is equivalent to the imbedding of $F1 \oplus V$ into $C(V_0, f)^{(+)}$ which is a canonical imbedding of the second type. Since $m$ is odd, $\mathcal{J}'$ is a maximal subalgebra in $F_{n_i}^{(+)}$. However, $\mathcal{J}' \subseteq H(F_{n_i})$, hence, $\mathcal{J}'$ cannot be maximal. This case is not possible. If $S(\mathcal{J}') \subseteq F_{n_i}$, then $H(S(\mathcal{J}')) \subseteq H(F_{n_i})$ is a proper subalgebra of $H(F_{n_i})$, and, therefore, can be reduced to (11). However, the imbedding of $\pi_{ij}(\mathcal{J}')$ into $F_{2^m}^{(+)}$ is as shown above. Hence this case is also not possible.

(b)Let $\mathcal{M}_i = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_{\frac{n_i}{2}}^{(+)}$. Acting in the same manner as in case 2(b) we will come to the canonical form of type 15.

**Case 4.** Let dim $V = 2m + 1$ and $m \equiv 0 (\mathrm{mod}\, 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form of type 16.

**Case 5.** Let dim $V = 2m + 1$ and $m \equiv 2 (\mathrm{mod}\, 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form of type 15.

**Case** $\mathcal{A} = H(F_{2n}, j)$

Let $\mathcal{M}$ be the maximal subalgebra of $H(F_{2n}, j)$ such that $\mathcal{J} \subseteq \mathcal{M} \subseteq H(F_{2n}, j)$. Then, the following cases occur.

1. $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ a semisimple factor, $\mathcal{R}$ the radical. Then, we reduce the problem to the case of symplectic matrices of a lower dimension (see section 2.1).

2. $\mathcal{M} = \mathcal{S}$ where $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$. Like in the previous case we can reduce the problem to the case of symplectic matrices of a lower dimension.

3. $\mathcal{M} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_n^{(+)}$, $\mathcal{R}$ the radical.

4. $\mathcal{M} = F1 \oplus W$ where $W$ is a finite-dimensional vector space.

After a series of reductions of the form 1 and 2, the image of $\mathcal{J}$ becomes a subalgebra of the algebra in the canonical form of type 9 in which the $i$th component has order $2n_i$. Let $\pi_i$ denote the projection of $\mathcal{J}$ into the $i$th simple component of canonical form of type 9.

**Case 1.** Let dim $V = 2m$ and $m \equiv 0, 1 (\bmod 4)$.

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, $F_{2n_i} \cong C(V, f)$, $2n_i = 2^m$. Acting in the same manner as in case 2(a), we can show that this situation is not possible. Likewise if $S(\mathcal{J}') \subset F_{2n_i}$ then we can reduce this case to the case just considered. Therefore, it also never occurs.

(b) Let $\mathcal{M}_i = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S} \cong F_{n_i}^{(+)}$. Then $\mathcal{J}' \subset F_{n_i}^{(+)}$, therefore, $S(\mathcal{J}')$ can be brought to (11), and $\pi_{ij}(\mathcal{J}') \subset F_{2^m}^{(+)}$ is the canonical imbedding of the first type. Finally the original subalgebra takes the form of type 17.

**Case 2** Let dim $V = 2m$, $m \equiv 2, 3 (\bmod 4)$.

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, then $2n_i = 2^m$, $F_{2n_i} \cong C(V, f)$, $\mathcal{J}' \subseteq F_{2n_i}^{(+)}$ is the canonical imbedding of the first type. If $S(\mathcal{J}') \subset F_{2n_i}$, then $H(S(\mathcal{J}'), j) \subseteq H(F_{2n_i}, j)$ is a proper subalgebra of $H(F_{2n_i}, j)$, that is, $n_i = 2^m l + r$, and $H(S(\mathcal{J}'), j)$ can be reduced to canonical form of type 9 in which each component has order $2^m$. Then, $\mathcal{J}$ takes the canonical form of type 18.

(b) Let $\mathcal{M}_i = \mathcal{S} \ominus \mathcal{R}$ where $\mathcal{S} \cong F_{n_i}^{(+)}$. This case also leads us to the canonical form of type 18.

**Case 3** Let dim $V = 2m + 1$ where $m$ is odd.

(a) Let $\mathcal{M}_i = F1 \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, then $2n_i = 2^m$, $F_{2n_i} \cong C(V_0, f)$. Therefore, the imbedding of $\mathcal{J}'$ into $F_{2n_i}^{(+)}$ is equivalent to the imbedding of $F1 \oplus V$ into $C(V_0, f)^{(+)}$ which is canonical imbedding of the second type. Since $m$ is odd, $\mathcal{J}'$ is a maximal subalgebra in $F_{2n_i}^{(+)}$. However, $\mathcal{J}' \subseteq H(F_{2n_i}, j)$, hence, $\mathcal{J}'$ cannot be maximal. This case is not possible. If $S(\mathcal{J}') \subset F_{2n_i}$, then $H(S(\mathcal{J}'), j) \subset H(F_{2n_i}, j)$ is a proper subalgebra of $H(F_{2n_i}, j)$, therefore, can be reduced to canonical form of type 9. Let $\pi_{ij}$ denote the projection of $\mathcal{J}'$ into the $j$th simple component of canonical form of type 9. However, the imbedding of $\pi_{ij}(\mathcal{J}')$ into $F_{2^m}^{(+)}$ is as shown above. Hence this case is also not possible.

(b) Let $\mathcal{M}_i = \mathcal{S} \oplus \mathcal{R}$ where $S \cong F_{n_i}^{(+)}$. Then $\mathcal{J}' \subset F_{n_i}^{(+)}$, therefore, $S(\mathcal{J}')$ can be brought to (11), and $\pi_{ij}(\mathcal{J}') \subset F_{2^m}^{(+)}$ is the canonical imbedding of the second type. Finally the original subalgebra takes the form of type 20

**Case 4.** Let $\dim V = 2m + 1$ and $m \equiv 0 (\mathrm{mod}\, 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form of type 17.

**Case 5.** Let $\dim V = 2m + 1$ and $m \equiv 2 (\mathrm{mod}\, 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form of type 18.

**Corollary 2.2.2.** *Let $\mathcal{A}$ be a Jordan algebra of either type $H(F_n)$ or $H(F_{2n}, j)$, and $\mathcal{J}$ be of type $J(f, 1)$, $\dim \mathcal{J} = r$. If $2^{2+[\frac{r}{2}]} \leq n$, then $\mathcal{J}$ can be imbedded in $\mathcal{A}$ of type $H(F_n)$. If $2^{1+[\frac{r}{2}]} \leq n$, then $\mathcal{J}$ can be imbedded in $\mathcal{A}$ of type $H(F_{2n}, j)$.*

□

# Chapter 3

# Simple decompositions of simple Jordan superalgebras with semisimple even part

Jordan superalgebras were first studied by V.Kac [10] and I.Kaplansky [13]. In [13] V.Kac classified simple finite dimensional Jordan superalgebras over an algebraically closed field of zero characteristic. Let us introduce the definition of a Jordan superalgebra.

**Definition 3.0.3.** *A Jordan superalgebra is a $Z_2$-graded algebra of the form: $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$ that satisfies supercommutativity*

$$a_\alpha R_{b_\beta} = (-1)^{\alpha\beta} b_\beta R_{a_\alpha}$$

*where $a_\alpha \in \mathcal{J}_\alpha$, $\alpha, \beta \in Z_2$ and $R$ denotes multiplication on the right, and the lin-*

*earized Jordan identity in operator form*

$$R_{a_\alpha}R_{b_\beta}R_{c_\gamma} + (-1)^{\alpha\beta+\alpha\gamma+\beta\gamma}R_{c_\gamma}R_{b_\beta}R_{a_\alpha} + (-1)^{\beta\gamma}R_{(a_\alpha c_\gamma)b_\beta}$$

$$= R_{a_\alpha b_\beta}R_{c_\gamma} + (-1)^{\beta\gamma}R_{a_\alpha c_\gamma}R_{b_\beta} + (-1)^{\alpha\beta+\alpha\gamma}R_{b_\beta c_\gamma}R_{a_\alpha}$$

$$= (-1)^{\alpha\beta}R_{b_\beta}R_{a_\alpha}R_{c_\gamma} + (-1)^{\alpha\gamma+\beta\gamma}R_{c_\gamma}R_{a_\alpha}R_{b_\beta} + R_{a_\alpha(b_\beta c_\gamma)}$$

$$= (-1)^{\alpha\gamma+\beta\gamma}R_{c_\gamma}R_{a_\alpha b_\beta} + (-1)^{\alpha\beta}R_{b_\beta}R_{a_\alpha c_\gamma} + R_{a_\alpha}R_{b_\beta c_\gamma},$$

*where* $a_\alpha \in \mathcal{J}_\alpha$*,* $b_\beta \in \mathcal{J}_\beta$ *and* $c_\gamma \in \mathcal{J}_\gamma$*.*

**Definition 3.0.4.** *A superinvolution of an associative superalgebra* $\mathcal{B}$ *is a graded linear map* $* : \mathcal{B} \to \mathcal{B}$ *such that*

$$a^{**} = a, \qquad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta}b_\beta^* a_\alpha^*.$$

Finite-dimensional simple Jordan superalgebras with semisimple even part over a field of characteristic not two have been classified by M.L. Racine and E.I. Zelmanov [27]. To begin we briefly recall this classification. If $\mathcal{J}$ is a simple finite-dimensional Jordan superalgebra with semisimple even part over an algebraically closed field $F$ of characteristic not 2, then $\mathcal{J}$ is isomorphic to one of the following superalgebras:

**(1)** $M_{n,m}(F)^{(+)}$, the set of all matrices of order $n+m$. Let

$$M_{n,m}(F)_0^{(+)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in M_n(F),\ B \in M_m(F) \right\}$$

and

$$M_{n,m}(F)_1^{(+)} = \left\{ \begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}, C \in M_{m\times n}(F),\ D \in M_{n\times m}(F) \right\}.$$

Then $M_{n,m}(F)^{(+)}$ becomes a Jordan superalgebra with respect to the above $Z_2$-grading under the Jordan supermultiplication;

**(2)** $osp(n,m)$, the set of all matrices of order $n+2m$ symmetric with respect to the orthosymplectic superinvolution. The superalgebra consists of matrices $\left\{\begin{pmatrix} A & B \\ S^{-1}B^t & D \end{pmatrix}\right\}$ where $A^t = A$, $D$ is symplectic, $B$ is any $n \times 2m$ matrix, $S = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$;

**(3)** $P(n) = \left\{\begin{pmatrix} A & B \\ C & A^t \end{pmatrix}, A \in M_n(F), B^t = B, C^t = -C \in M_n(F)\right\}$;

**(4)** $Q(n)^{(+)} = \left\{\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right\}$ where $A$ and $B$ are any square matrices of order $n$.

**(5)** Let $V = V_0 + V_1$ be a $Z_2$-graded vector space with a non-singular symmetric bilinear superform $f : V \times V \to F$. Consider the direct sum of $F1$ and $V$, $\mathcal{J} = F1 \oplus V$ where 1 is the identity element, and determine multiplication according to

$$(\alpha 1 + v)(\beta 1 + w) = (\alpha\beta 1 + f(v,w)) + (\alpha w + \beta v).$$

Then $\mathcal{J}$ becomes a Jordan superalgebra of the type $J(V,f)$ with respect to the following $Z_2$-grading: $\mathcal{J}_0 = F + V_0$, $\mathcal{J}_1 = V_1$.

**(6)** The 3-dimensional Kaplansky superalgebra $K_3$, $(K_3)_0 = Fe$, $(K_3)_1 = Fx + Fy$, the multiplication $e^2 = e$, $e \cdot x = \frac{1}{2}x$, $e \cdot y = \frac{1}{2}y$, $[x,y] = e$.

**(7)** The 1-parametric family of 4-dimensional superalgebras $D_t$, $D_t = (D_t)_0 +$

$(D_t)_1$, where $(D_t)_0 = Fe_1 + Fe_2$, $(D_t)_1 = Fx + Fy$, where $e_i^2 = e_i$, $e_1 \cdot e_2 = 0$, $e_i \cdot x = \frac{1}{2}x$, $e_i \cdot y = \frac{1}{2}y$, $[x, y] = e_1 + te_2$, $i = 1, 2$. A superalgebra $D_t$ is simple only if $t \neq 0$. If $t = -1$, then $D_{-1}$ isomorphic $M_{1,1}(F)$.

**(8)** $K_{10}$, the *Kac superalgebra*:

$$\mathcal{J}_0 = (Ke + \sum_{1 \leq i \leq 4} Kv_i) \oplus Kf,$$

$$\mathcal{J}_1 = \sum_{i=1,2} (Kx_i + Ky_i),$$

where

$$e^2 = e, \quad e \cdot v_i = v_i, \quad v_1 \cdot v_2 = 2e = v_3 \cdot v_4,$$

$$f^2 = f, \quad f \cdot x_j = \frac{1}{2}x_j, \quad f \cdot y_j = \frac{1}{2}y_j,$$

$$e \cdot x_j = \frac{1}{2}x_j, \quad y_1 \cdot v_1 = x_2, \quad y_2 \cdot v_1 = -x_1, \quad x_1 \cdot v_2 = -y_2, \quad x_2 \cdot v_2 = y_1,$$

$$e \cdot y_j = \frac{1}{2}y_j, \quad x_2 \cdot v_3 = x_1, \quad y_1 \cdot v_3 = y_2, \quad x_1 \cdot v_4 = x_2, \quad y_2 \cdot v_4 = y_1,$$

$$[x_i, y_i] = e - 3f, \quad [x_1, x_2] = v_1, \quad [x_1, y_2] = v_3, \quad [x_2, y_1] = v_4, \quad [y_1, y_2] = v_2$$

and every other product is zero or obtained by the symmetry or skew-symmetry of one of the above products. If the characteristic of $F$ is not 3, then $K_{10}$ is simple. If the characteristic of $F$ is 3, it possesses a simple subsuperalgebra of dimension 9 spanned by $e$, $v_i$, $1 \leq i \leq 4$, $x_i$, $y_j$, $1 \leq j \leq 2$. We denote this superalgebra by $K_9$ and refer to it as the *degenerate Kac superalgebra.*

**(9)** Denote by $H_n(F)$ and $S_n(F)$ the symmetric and skew-symmetric $n \times n$ matrices. For $F$ a field of characteristic 3, let $A = H_3(F)$ and $M = S_3(F) \oplus \bar{S}_3(F)$, two isomorphic copies of $S_3(F)$. To extend the Jordan algebra structure on $A$ and $A$-bimodule structure on $M$ to a Jordan structure on $J = A + M$ one defines

$$[S_3(F), S_3(F)] = [\bar{S}_3(F), \bar{S}_3(F)] = \{0\},$$

and for any $a, b \in S_3(F)$,

$$[a, \bar{b}] = ab + ba.$$

**(10)** Let $B = B_0 + B_1$ with $B_0 = M_2(F)$, $B_1 = Fm_1 + Fm_2$ where $F$ is a field of characteristic 3. If we define a $B_0$-bimodule structure on $B_1$ by

$$e_{11}m_1 = m_1,\ e_{22}m_1 = 0,\ e_{12}m_1 = m_2,\ e_{21}m_1 = 0;$$

$$m_1e_{11} = 0,\ m_1e_{22} = m_1,\ m_1e_{12} = -m_2,\ m_1e_{21} = 0;$$

$$e_{11}m_2 = 0,\ e_{22}m_2 = m_2,\ e_{12}m_2 = 0,\ e_{21}m_2 = m_1;$$

$$m_2e_{11} = m_2,\ m_2e_{22} = 0,\ m_2e_{12} = 0, m_2e_{21} = -m_1,$$

and a multiplication from $B_1 \times B_1$ to $B_0$ by

$$m_1^2 = -e_{21},\ m_2^2 = e_{12},\ m_1m_2 = e_{11},\ m_2m_1 = -e_{22},$$

then $B$ is a superalternative algebra with superinvolution $(a + m)^* = \bar{a} - m$, where $-$ is the symplectic involution of $B_0$. Then $H_3(B)$, the symmetric matrices with respect to the $*$-transpose superinvolution, forms a simple Jordan superalgebra.

Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$ be the sum of two proper simple subalgebras $\mathcal{A}$ and $\mathcal{B}$. The structure of these sums has first attracted the considerable attention in the cases of Lie and associative algebras. The first case arises in Onichshik's paper [23]. If one considers a compact connected Lie group $G$ acting transitively on the manifold $G/G''$ where $G''$ is a closed subgroup of $G$. A connected closed subgroup $G' \subset G$ acts transitively on $G/G''$ if and only if $G = G'G''$. The triple $(G, G', G'')$ is called a decomposition and it entails a Lie algebra decomposition, $L = L' + L''$ where $L, L', L''$ are Lie algebras of $G$, $G'$, $G''$, respectively. Using topological methods Onishchik has determined all decompositions $L = L' + L''$ where $L$, $L'$, $L''$ are real or complex finite-dimensional semisimple Lie algebras.

For associative matrix algebras over an arbitrary field Y.Bahturin and O.Kegel [4] proved that a matrix algebra of finite order cannot be written as a sum of two proper subalgebras which are also matrix algebras.

It is worth noting that the problem of finding simple decompositions has seen some interest from researches in the case of simple Lie superalgebras and Jordan algebras. In the joint paper with T.Tvalavadze [32] we obtained complete classification of simple decompositions for special simple Jordan algebras over an algebraically closed field $F$ of characteristic not two. Besides, in [36] the case of a simple Lie superalgebra of the type $sl(n, m)$ was considered by T.Tvalavadze. In [31] A. Sudarkin classified simple decompositions Lie superalgebras of the types $P(n)$ and $Q(n)$.

In [32] we considered special simple finite-dimensional Jordan algebras decomposable as the sum of two proper simple subalgebras over an algebraically closed field of characteristic not 2. The main result in [32] is the following.

**Theorem.** *Let $\mathcal{J}$ be a finite-dimensional special simple Jordan algebra over an algebraically closed field $F$ of characteristic not two. The only possible decomposi-*

*tions of $\mathcal{J}$ as the sum of two simple subalgebras $\mathcal{J}_1$ and $\mathcal{J}_2$ are the following:*

**1.** *$\mathcal{J} \cong F \oplus V$ and $\mathcal{J}_1 \cong F1 \oplus V_1$, $\mathcal{J}_2 \cong F1 \oplus V_2$, where $V$, $V_1$, $V_2$ are vector spaces.*

**2.** *Either $\mathcal{J} \cong F_3^{(+)}$ and $\mathcal{J}_1 \cong H(F_3)$, $\mathcal{J}_2 \cong F1 \oplus V$, or $\mathcal{J} \cong F_n^{(+)}$, $n \geq 3$, $\mathcal{J}_1 \cong H(F_n)$ and $\mathcal{J}_2$ is isomorphic to one of the following algebras: $H(F_{n-1})$, $H(F_n)$ or $F_{n-1}^{(+)}$.*

**3.** *$\mathcal{J} \cong H(\mathcal{Q}_n)$ and $\mathcal{J}_1$, $\mathcal{J}_2 \cong F_n^{(+)}$.*

Notice that the above theorem describes all simple decompositions in simple Jordan algebras in terms of the types of simple subalgebras. Our purpose here is to obtain a classification of conjugacy classes of simple decompositions of simple Jordan matrix superalgebras with semisimple even part over an algebraically closed field of characteristic not 2. Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$ and $\mathcal{J} = \mathcal{A}' + \mathcal{B}'$ be two simple decompositions of $\mathcal{J}$. These decompositions are said to be *conjugate* if there exists an automorphism of $\mathcal{J}$ such that $\varphi(\mathcal{A}) = \mathcal{A}'$, $\varphi(\mathcal{B}) = \mathcal{B}'$. In this chapter we will look at conjugacy classes of simple decompositions in all types of simple matrix Jordan superalgebras with semisimple even part.

Let $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$ be a Jordan superalgebra. Then $\mathcal{J}$ is said to be *nontrivial* if $\mathcal{J}_1 \neq \{0\}$. All superalgebras considered in this paper are supposed to be nontrivial.

Next we cite some important lemmas and theorems from [32] which will be repeatedly used later.

**Lemma 3.0.5.** *Let a Jordan algebra $\mathcal{J}$ of the type $H(\mathcal{D}'_m)$ be a proper subalgebra of $H(\mathcal{D}_n)$ such that the identity of $H(\mathcal{D}_n)$ is an element of this subalgebra. If either*

*1.* $\mathcal{D}' = F$ *and* $\mathcal{D} = F$*, or*

*2.* $\mathcal{D}' = F[q]$*, or* $\mathcal{Q}$*, and* $\mathcal{D} = F[q]$*,*

*then* $m \leq \frac{n}{2}$.

**Lemma 3.0.6.** *Let $V$ be a finite-dimensional vector space with a non-singular symmetric bilinear form $f$, and $v_0$ a fixed non-trivial vector in $V$. Let $\mathcal{S}$ be the set of all linear operators which are symmetric with respect to $f$. Then, $\mathcal{S}v_0 = V$.*

**Theorem 3.0.7.** *Let $\mathcal{J}$ be a simple Jordan algebra of the type $H(F_n)$, and $\mathcal{A}$, $\mathcal{B}$ proper simple Jordan subalgebras of $\mathcal{J}$. Then $\mathcal{J} \neq \mathcal{A} + \mathcal{B}$.*

**Lemma 3.0.8.** *Let $W$ be the natural $2m$-dimensional module for $H(\mathcal{Q}_m)$, and $v$ be an arbitrary non-zero vector in $W$. Then,* $\dim H(\mathcal{Q}_m)v = 2m - 1$.

## 3.1 Decompositions of superalgebras of the type $M_{n,m}(F)^{(+)}$

Our main goal is to prove the following.

**Theorem 3.1.1.** *Let $\mathcal{A}$ be a superalgebra of the type $M_{n,m}(F)^{(+)}$ where $n, m > 0$ over an algebraically closed field of characteristic not 2. If both $n, m$ are odd, $(n, m) \neq (1, 1)$, then $\mathcal{A}$ has no decompositions into the sum of two proper nontrivial simple subsuperalgebras. If one of the indices, for example $m$, is an even number and the other is odd, then there are two conjugacy classes of simple decomposition of the form: $\mathcal{A} = \mathcal{B}+\mathcal{C}$ where $\mathcal{B}$ and $\mathcal{C}$ have types $osp(n, \frac{m}{2})$ and $M_{n-1,m}(F)^{(+)}$, respectively.*

*If both indices are even, then $\mathcal{A}$ admits two types of simple decompositions of the following forms:*

1. *$\mathcal{A} = \mathcal{B}_1 + \mathcal{C}_1$ where $\mathcal{B}_1$ and $\mathcal{C}_1$ have types $osp(n, \frac{m}{2})$ and $M_{n-1,m}(F)^{(+)}$,*

2. *$\mathcal{A} = \mathcal{B}_2 + \mathcal{C}_2$ where $\mathcal{B}_2$ and $\mathcal{C}_2$ have types $osp(m, \frac{n}{2})$ and $M_{m-1,n}(F)^{(+)}$. Besides there are exactly two conjugacy classes of simple decompositions corresponding to each type.*

**Remark 1** *If $(n, m) = (1, 1)$, then $M_{1,1}(F)^{(+)}$ is a 4-dimensional superalgebra of a bilinear superform. Its decompositions will be considered later in Section 3.4.*

Before the discussion of various properties of $M_{n,m}(F)^{(+)}$ we recall a definition of the universal associative enveloping superalgebra of a Jordan superalgebra which will be frequently used later [9].

An associative specialization $u : \mathcal{J} \to U(\mathcal{J})$ where $U(\mathcal{J})$ is an associative superalgebra is said to be universal if $U(\mathcal{J})$ is generated by $u(\mathcal{J})$, and for any other specialization $\varphi : \mathcal{J} \to \mathcal{A}$ where $\mathcal{A}$ is an associative superalgebra there exists a homomorphism $\psi : U(\mathcal{J}) \to \mathcal{A}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\varphi} & \mathcal{A} \\ \downarrow id & & \uparrow \psi \\ \mathcal{J} & \xrightarrow{u} & U(\mathcal{J}) \end{array} .$$

Then $U(\mathcal{J})$ is called a universal associative enveloping superalgebra of $\mathcal{J}$. It is worth noting that an associative superalgebra can be considered as an associative algebra. The following theorem [19] plays a key role in the later discussion.

**Theorem 3.1.2.** *Let $U(\mathcal{J})$ denote a universal associative enveloping superalgebra for a Jordan superalgebra $\mathcal{J}$. Then $U(M_{k,l}^{(+)}) \cong M_{k,l}(F) \oplus M_{k,l}(F)$ where $(k,l) \neq (1,1)$; $U(Q(k)) \cong Q(k) \oplus Q(k)$, $k \geq 2$; $U(osp(m,n)) \cong M_{m,2n}(F)$, $(m,n) \neq (1,1)$; $U(P(n)) \cong M_{n,n}(F)$, $n \geq 3$.*

**Remark 2** *In the case when $\mathcal{J} \cong M_{1,1}(F)^{(+)}$, $P(2)$, $osp(1,1)$, $K_3$ or $D_t$ the universal enveloping superalgebras have more complicated structure. Indeed, the universal associative enveloping superalgebras of the above Jordan superalgebras are no more finite-dimensional. Also we note that if the characteristic of the base field $F$ equals zero, then $K_3$ has no non-zero finite-dimensional associative specializations.* [19]

Now we look at the case when $\mathcal{J} \cong J(V, f)$. Let $V = V_0 + V_1$ be a $Z_2$-graded vector space, dim $V_0 = m$, dim $V_1 = 2n$. Let $f : V \times V \to F$ be a supersymmetric bilinear form on $V$. The universal associative enveloping algebra of the Jordan algebra $F1 \oplus V_0$ is the Clifford algebra $C(V_0, f) = \langle 1, e_1, \ldots, e_m | e_ie_j + e_je_i = 0, i \neq j, e_i^2 = 1\rangle$. In $V_1$ we can find a basis $v_1, w_1, \ldots, v_n, w_n$ such that $f(v_i, w_j) = \delta_{ij}$, $f(v_i, v_j) = f(w_i, w_j) = 0$. Consider the Weyl algebra $W_n = \langle 1, v_i, w_i, 1 \leq i \leq n, [v_i, w_j] = \delta_{ij}, [v_i, v_j] = [w_i, w_j] = 0\rangle$. According to [19], the universal associative enveloping algebra of $F1 \oplus V_0$ is isomorphic to the (super)tensor product $C(V_0, f) \otimes_F W_n$. We will utilize this fact in the following lemma.

**Lemma 3.1.3.** *There are no nontrivial subsuperalgebras of the type $J(V, f)$ in $\mathcal{A}^{(+)}$, where $\mathcal{A}$ is a finite-dimensional associative superalgebra.*

*Proof.* We assume the contrary. Let $\mathcal{B}$ of the type $J(V, f)$ be a subsuperalgebra of $\mathcal{A}^{(+)}$. For $\mathcal{B}$, we consider the universal associative enveloping superalgebra $U(\mathcal{B})$. As was mentioned above, $U(\mathcal{B}) = C(V_0, f) \otimes_F W_n$ where $C(V_0, f)$ is a Clifford algebra for $V_0$, $f$ is a bilinear form on $V_0$, $W_n$ is a Weyl algebra, $n = \frac{1}{2} \dim V_1$. Let $\varphi$ denote the embedding of $\mathcal{B}$ in $\mathcal{A}$, i.e. $\varphi(x) = x$ for any $x \in \mathcal{B}$. If follows from universal properties of $U(\mathcal{B})$ , $\varphi$ can be uniquely extended to a homomorphism $\bar{\varphi} : U(\mathcal{B}) \to \mathcal{A}$. Since $\bar{\varphi}(x) = \varphi(x) = x$ for $x \in V_1$, $\bar{\varphi}(V_1) \neq 0$. However, since $V_1$ generates $W_n$, $\bar{\varphi}(W_n) \neq 0$. It follows from simplicity of $W_n$ that $\bar{\varphi}(W_n) \cong W_n$. Therefore, $\mathcal{A}$ has an infinite-dimensional subsuperalgebra. This contradicts our assumptions. $\square$

**Lemma 3.1.4.** *Neither $K_3$ nor $D_t$ can be imbedded into $M_{2,2}(F)^{(+)}$.*

*Proof.* Let $\mathcal{A}$ of the type $K_3$ be a subsuperalgebra of $M_{2,2}(F)^{(+)}$. We may assume that $\mathcal{A}$ has a basis $e$, $x$, $y$ satisfying (i) $e \cdot x = \frac{1}{2}x$, (ii) $e \cdot y = \frac{1}{2}y$, (iii) $[x, y] = e$, (iv) $e^2 = e$. Since $e$ is an idempotent, it can be reduced to one of the following forms: (1) $\mathrm{diag}(1, 0, 0, 0)$, (2) $\mathrm{diag}(1, 1, 0, 0)$, (3) $\mathrm{diag}(1, 1, 1, 0)$, (4) $\mathrm{diag}(1, 1, 1, 1)$. Because of (i) and (ii), case (4) is not possible. In the first and third cases, directly performing the multiplications in (i), (ii) and (iii), we can show that there are no such $x$ and $y$ in $M_{2,2}(F)$. Finally we assume that $e = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ where $I$ is the $2 \times 2$ identity matrix. Then $x = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$, for some $A$, $B$, $C$, $D$, and (iii) can be rewritten as follows: $AD - CB = I$, and $BC - DA = 0$. Adding these two equations gives us $[A, D] + [B, C] = I$. The trace-zero matrix on the left-hand side is equal to the identity matrix on the right-hand side, which is a contradiction.

Let $\mathcal{A}$ of the type $D_t$ be a subsuperalgebra of $M_{2,2}(F)^{(+)}$. We may assume that $\mathcal{A}$ has a basis $e_1$, $e_2$, $x$, $y$ satisfying (i) $e_i \cdot x = \frac{1}{2}x$, (ii) $e_i \cdot y = \frac{1}{2}y$, (iii) $[x, y] = e_1 + te_2$, (iv) $e_i^2 = e_i$. Set $e = e_1 + e_2$. Since $e$ is an idempotent of rank $\geq 2$, $e$ can be reduced to

(1) $\mathrm{diag}(1, 1, 0, 0)$, (2) $\mathrm{diag}(1, 1, 1, 0)$, (3) $\mathrm{diag}(1, 1, 1, 1)$. In the second and third cases, by performing direct multiplications we can show that these cases are not possible. If $e = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ where $I$ is the $2 \times 2$ identity matrix, then the same argument as in case $\mathcal{A} \cong K_3$ works for $\mathcal{A} \cong D_t$. The lemma is proved.

□

**Corollary 3.1.5.** *Neither $K_3$ nor $D_t$ can be imbedded into $M_{1,1}(F)^{(+)}$, $M_{2,1}(F)^{(+)}$, $osp(1, 1)$, $osp(2, 1)$, $P(2)$ and $Q(2)^{(+)}$.*

*Proof.* The proof is a direct consequence of the fact that there are subsuperalgebras of the types $M_{1,1}(F)^{(+)}$, $M_{2,1}(F)^{(+)}$, $osp(1, 1)$, $osp(2, 1)$, $P(2)$ and $Q(2)^{(+)}$ in $M_{2,2}(F)^{(+)}$. □

**Lemma 3.1.6.** *Let $M_{n,m}(F)^{(+)} = \mathcal{A}+\mathcal{B}$ where $n, m > 0$, and $\mathcal{A}$, $\mathcal{B}$ be proper simple subsuperalgebras of $M_{n,m}(F)^{(+)}$ . Then neither $\mathcal{A}$ nor $\mathcal{B}$ has any of the following types: $K_3$, $D_t$, $M_{1,1}(F)^{(+)}$, $osp(1, 1)$.*

*Proof.* First of all, we note that if one of the subsuperalgebras, for example $\mathcal{A}$, has any of the types listed above, then either $\mathcal{A}_0 \cong Fe$ or $\mathcal{A}_0 \cong Fe_1 \oplus Fe_2$ where $e$,

$e_1$, $e_2$ are idempotents. Next we assume the contrary, i.e. $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$, and $\mathcal{A}$ has one of the types listed in the hypothesis of the lemma. If we define a pair of homomorphisms denoted as $\pi_1$, $\pi_2$ which are the projections on the ideals of $M_{n,m}(F)_0^{(+)}$, then the decomposition can be rewritten as follows:

$$F_n^{(+)} = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0),$$

$$F_m^{(+)} = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0).$$

Any proper simple subalgebra of $F_n^{(+)}$ (or $F_m^{(+)}$) has dimension less than or equal to $(n-1)^2$ (or $(m-1)^2$). Any proper non-simple semisimple subalgebra of $F_n^{(+)}$ (or $F_m^{(+)}$) has a dimension less than or equal to $(n-1)^2+1 = n^2-2n+2$ (or $m^2-2m+2$). If $\pi_1(\mathcal{B}_0)$ is a proper semisimple, then $\dim F_n^{(+)} = n^2 \leq 2 + n^2 - 2n + 2$, $n \leq 2$. Similarly, if $\pi_2(\mathcal{B}_0)$ is a proper semisimple subalgebra, then $m \leq 2$. Since both projections cannot be improper simultaneously, one of them, say $\pi_1(\mathcal{B}_0)$, is proper, hence, $n \leq 2$. If $\pi_2(\mathcal{B}_0) = F_m^{(+)}$, then $\mathcal{B}_0 \cong F_1^{(+)} \oplus F_m^{(+)}$ while $M_{n,m}(F) = M_{2,m}(F)$. However, in this case $\dim F_2 > \dim F_1 + 2$.

Next we are going to consider two remaining cases.

**1.** $M_{2,1}(F)^{(+)} = \mathcal{A} + \mathcal{B}$. This decomposition induces the following:

$$F_2^{(+)} = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0),$$

$$F_1^{(+)} = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0).$$

Notice that $\pi_1(\mathcal{B}_0)$ is necessarily proper, otherwise either $\mathcal{B} = M_{2,1}(F)^{(+)}$ (an improper subsuperalgebra) or $\mathcal{B} = M_2(F)^{(+)}$ (a trivial superalgebra). If $\pi_1(\mathcal{B}_0)$ is simple, then $\pi_1(\mathcal{B}_0) \cong F$ (a one-dimensional algebra). If $\pi_1(\mathcal{B}_0)$ is semisimple, then

$\pi_1(\mathcal{B}_0) \cong Fe'_1 \oplus Fe'_2$ where $e'_1$, $e'_2$ are idempotents. Besides, $\pi_2(\mathcal{B}_0)$ is either $F$ or $\{0\}$. Hence $\mathcal{B}_0$ is either $F$ or $Fe'_1 \oplus Fe'_2$. Therefore, $5 = \dim M_{2,1}(F)_0^{(+)} \leq \dim \mathcal{A}_0 + \dim \mathcal{B}_0 \leq 4$, a contradiction.

**2.** $M_{2,2}(F)^{(+)} = \mathcal{A} + \mathcal{B}$. The same argument as in case 1 works for $M_{2,2}(F)^{(+)}$, i.e. $\dim \mathcal{A}_0 \leq 2$ and $\dim \mathcal{B}_0 \leq 5$ while $\dim F_2^{(+)} \oplus F_2^{(+)} = 8$. This proves our lemma.

□

Next taking into account all previous lemmas we list simple decompositions that might exist in $M_{n,m}(F)^{(+)}$. Let $\mathcal{A}$ and $\mathcal{B}$ stand for simple non-trivial Jordan subsuperalgebras of $M_{n,m}(F)^{(+)}$. Then, $M_{n,m}(F)^{(+)}$ might be expressed as the sum of $\mathcal{A}$ and $\mathcal{B}$ where

| | $\mathcal{A}$ | $\mathcal{B}$ |
|---|---|---|
| 1 | $M_{k,l}(F)^{(+)}$ | $M_{p,q}(F)^{(+)}$ |
| 2 | $M_{k,l}(F)^{(+)}$ | $P(q)$ |
| 3 | $M_{k,l}(F)^{(+)}$ | $Q(p)^{(+)}$ |
| 4 | $P(k)$ | $Q(l)^{(+)}$ |
| 5 | $P(k)$ | $P(l)$ |
| 6 | $Q(k)^{(+)}$ | $Q(l)^{(+)}$ |
| 7 | $osp(k,l)$ | $M_{p,q}(F)^{(+)}$ |
| 8 | $osp(k,l)$ | $Q(p)^{(+)}$ |
| 9 | $osp(k,l)$ | $P(q)$ |
| 10 | $osp(k,l)$ | $osp(p,q)$ |

Let $S(\mathcal{A})$ and $S(\mathcal{B})$ denote associative subalgebras generated by $\mathcal{A}$ and $\mathcal{B}$ in $M_{n+m}(F)$. Any decomposition of the form $M_{n,m}(F)^{(+)} = \mathcal{A}+\mathcal{B}$ produces a new one of the form $M_{n+m}(F) = S(\mathcal{A}) + S(\mathcal{B})$. Note that $S(\mathcal{A})$ is a homomorphic image of $U(\mathcal{A})$. As a direct consequence of Theorem 3.1.2, $U(\mathcal{A})$ is either a simple associative algebra or a direct sum of two or more simple pairwise isomorphic associative algebras.

**Lemma 3.1.7.** *Let $M_{n,m}^{(+)} = \mathcal{A}+\mathcal{B}$ where $\mathcal{A}$, $\mathcal{B}$ are two proper non-trivial simple subsuperalgebras in $M_{n,m}(F)^{(+)}$ where $n, m > 0$. Then $S(\mathcal{A})$ coincides with $M_{n+m}(F)$ if and only if one of the following conditions holds*

*(1) Either $\mathcal{A} \cong osp(p,q)$, $p + 2q = n + m$, or*

*(2) $\mathcal{A} \cong P(n)$ for the case when $n = m$.*

*Proof.* First, we note that if conditions (1) and (2) are met, then $S(\mathcal{A}) = M_{n,m}(F)$. Let $S(\mathcal{A}) = M_{n,m}(F)$. First we show that $\mathcal{A}$ cannot be of the type either $M_{k,l}(F)^{(+)}$ or $Q(p)^{(+)}$. If $\mathcal{A}$ has the type $M_{k,l}(F)^{(+)}$, then $k + l < n + m$ because $\mathcal{A}$ is a proper subsuperalgebra. By Theorem 3.1.2, $S(\mathcal{A})$ is either a simple subalgebra of the type $M_{k+l}(F)$ or a non-simple semisimple subalgebra of the type $M_{k+l}(F) \oplus M_{k+l}(F)$. In both cases, $S(\mathcal{A}) \neq M_{n+m}(F)$.

If $\mathcal{A} \cong Q(k)^{(+)}$, then its associative enveloping algebra is always a non-simple semisimple subalgebra which is a direct sum of at least two simple ideals of the type $M_k(F)$. Therefore, $S(\mathcal{A}) \neq M_{n+m}(F)$.

For the other cases, $\mathcal{A}$ can either have the type $osp(p,q)$ or $P(k)$. If $\mathcal{A} \cong osp(p,q)$, then $S(\mathcal{A}) \cong M_{p+2q}(F)$. Hence $S(\mathcal{A}) = M_{n+m}(F)$ if and only if $p + 2q = n + m$.

This yields (1).

Next we continue our proof by assuming that $n \neq m$. For clarity, let $n < m$, and $\mathcal{A} \cong P(k)$. Since $U(\mathcal{A}) \cong S(\mathcal{A}) \cong M_{k,k}(F)$, $S(\mathcal{A}) = M_{n+m}(F)$ implies that $k = \frac{n+m}{2}$. In turn $\mathcal{A}_0 \cong F_k^{(+)} \subset (M_{n,m}(F)^{(+)})_0 \cong F_n^{(+)} \oplus F_m^{(+)}$ implies that $\frac{n+m}{2} \leq n$ and $\frac{n+m}{2} \leq m$, that is, $n = m$. However, this contradicts our assumption.

In conclusion, it remains to consider the case when $n = m$ and $\mathcal{A} \cong P(n)$. However, it is obvious that $S(\mathcal{A}) \cong M_{2k}(F)$, and $S(\mathcal{A}) = M_{2n}(F)$ if and only if $k = n$. This completes our proof. $\square$

**Lemma 3.1.8.** *Let $M_{n,m}(F)^{(+)} = \mathcal{A}{+}\mathcal{B}$, $n, m > 0$. Then one of the subsuperalgebras in the given decomposition has either the type $osp(p, q)$ where $p + 2q = n + m$ or $P(n)$ (only if $n = m$).*

*Proof.* Let us assume the contrary, that is, neither $\mathcal{A}$ nor $\mathcal{B}$ is a subsuperalgebra of any of the above types. Then, by Lemma 3.1.7, $S(\mathcal{A})$ and $S(\mathcal{B})$ are proper associative subalgebras in $M_{n+m}(F)$. Theorem 3.1.2 states that both $S(\mathcal{A})$ and $S(\mathcal{B})$ are either simple associative algebras or non-simple semisimple associative algebras decomposable into the sum of at least two pairwise isomorphic simple ideals. Therefore, $\dim S(\mathcal{A}) \leq k(\frac{n+m}{k})^2$ where $k$ is the number of simple ideals, $k \geq 1$. If one of the subsuperalgebras in the decomposition of $M_{n+m}(F)$ has a non-zero annihilator then by Proposition 1 in [4] no such decomposition exists. Therefore, both $S(\mathcal{A})$ and $S(\mathcal{B})$ contain the identity element of the whole superalgebra. Hence $S(\mathcal{A}) \cap S(\mathcal{B})$ contains the identity element as well. If $S(\mathcal{A})$ (or $S(\mathcal{B})$) is simple, then $\dim S(\mathcal{A}) \leq \frac{(n+m)^2}{4}$. If $S(\mathcal{A})$ (or $S(\mathcal{B})$) is non-simple semisimple, $\dim S(\mathcal{A}) \leq \frac{(n+m)^2}{2}$.

Thus, $\dim(S(\mathcal{A})+S(\mathcal{B})) < \dim S(\mathcal{A}) + \dim S(\mathcal{B}) \le (n+m)^2$. Therefore, $M_{n+m}(F) \ne S(\mathcal{A})+S(\mathcal{B})$. This implies that our assumption was wrong. The lemma is proved. $\square$

**Lemma 3.1.9.** *Let $M_{n,m}(F)^{(+)} = \mathcal{A}+\mathcal{B}$, $n,m>0$, $(n,m)\ne(1,1)$. If $m$ is even, and $n$ is odd, then one of the subsuperalgebras is of the type $osp(n,\frac{m}{2})$, and the other of the type $M_{k,l}(F)^{(+)}$ where either $k=n-1,n$ or $l=m$. On the contrary, if $m$ is odd, and $n$ is even, then one of the subsuperalgebras is of the type $osp(m,\frac{n}{2})$, and the other of the type $M_{k,l}(F)^{(+)}$ where either $k=m-1,m$ or $l=n$.*

*Proof.* Since the proof remains the same for both cases, we consider only the first case. First, let $n\ne m$. In view of Lemma 3.1.8, one of the subsuperalgebras in $M_{n,m}(F)^{(+)} = \mathcal{A}+\mathcal{B}$, for example $\mathcal{A}$, is isomorphic to $osp(p,q)$ where

$$p+2q=n+m. \tag{1}$$

The decomposition of $M_{n,m}(F)^{(+)}$ given above induces the following representation of the even component $M_{n,m}(F)^{(+)}_0 = \mathcal{A}_0+\mathcal{B}_0$ where $M_{n,m}(F)^{(+)}_0 = F_n^{(+)} \oplus F_m^{(+)}$, $\mathcal{A}_0 \cong H(F_p)\oplus H(\mathcal{Q}_q)$. If for some $i$, $\pi_i(\mathcal{A}_0)\cong H(F_p)\oplus H(\mathcal{Q}_q)$, then either $p+2q\le n$ or $p+2q\le m$. However these inequalities conflict with condition (1). Hence either $\pi_1(\mathcal{A}_0)\cong H(F_p)$, $\pi_2(\mathcal{A}_0)\cong H(\mathcal{Q}_q)$ or $\pi_1(\mathcal{A}_0)\cong H(\mathcal{Q}_q)$, $\pi_2(\mathcal{A}_0)\cong H(F_p)$. If the first possibility holds true, then

1. $F_n^{(+)} = \pi_1(\mathcal{A}_0)+\pi_1(\mathcal{B}_0), \quad \pi_1(\mathcal{A}_0)\cong H(F_p), \quad p\le n,$
2. $F_m^{(+)} = \pi_2(\mathcal{A}_0)+\pi_2(\mathcal{B}_0), \quad \pi_2(\mathcal{A}_0)\cong H(\mathcal{Q}_q), \quad q\le \frac{m}{2}.$

Since $p\le n$, $q\le\frac{m}{2}$ and $p+2q=n+m$, it follows that $p=n$ and $q=\frac{m}{2}$. Clearly, $\mathcal{A}$ has the type $osp(n,\frac{m}{2})$. If the second possibility holds true, then acting in the same manner, we can show that $p = m$, $q = \frac{n}{2}$. However, we assumed that $n$ is odd. Hence it remains to show that $\mathcal{B} \cong M_{k,l}(F)^{(+)}$ where a pair of indices $k, l$ satisfies the hypothesis of the lemma. To prove this, we consider all possible cases for $\mathcal{B}$ in a step-by-step manner.

If $\mathcal{B} \cong P(k)$, then the decomposition induces the following representation of the odd part: $M_{n,m}(F)_1^{(+)} = \mathcal{A}_1 + \mathcal{B}_1$ where $\dim \mathcal{A}_1 = nm$, $\dim \mathcal{B}_1 = k^2$, that is, $2nm \leq nm + k^2$, $nm \leq k^2$. Conversely, $k \leq n$, $k \leq m$ since both projections $\pi_1(\mathcal{B}_0)$, $\pi_2(\mathcal{B}_0)$ are non-zero. Moreover, one of the inequalities should be strict since $n \neq m$. Therefore, $k^2 < nm$, which is a contradiction.

If $\mathcal{B} \cong Q(k)^{(+)}$, then, acting in the same manner as in the previous case, we can prove that $M_{n,m}(F)^{(+)} \neq \mathcal{A} + \mathcal{B}$.

If $\mathcal{B} \cong osp(p, q)$, then $2mn = \dim (M_{n,m}(F)^{(+)})_1 \leq \dim \mathcal{A}_1 + \dim \mathcal{B}_1 \leq 2nm$ since $p \leq n$, $2q \leq m$. Hence, $\dim \mathcal{B}_1 = nm$. It follows that $p = n$, $q = \frac{m}{2}$. The original decomposition induces the representation of $F_m^{(+)}$ as the sum of two proper subalgebras one of which has the type $H(\mathcal{Q}_m)$, which is not possible [32].

Overall, it remains to consider the case when $\mathcal{A} \cong osp(n, \frac{m}{2})$, $\mathcal{B} \cong M_{k,l}(F)^{(+)}$.

Again the decomposition of $M_{n,m}(F)^{(+)}$ induces that of $M_{n,m}(F)_0^{(+)}$ as follows: $M_{n,m}(F)_0^{(+)} = \mathcal{A}_0 + \mathcal{B}_0$. Moreover, $M_{n,m}(F)_0^{(+)} = F_n^{(+)} \oplus F_m^{(+)}$, $\mathcal{A}_0 \cong H(F_n) \oplus H(\mathcal{Q}_{\frac{m}{2}})$, $\mathcal{B}_0 \cong F_k^{(+)} \oplus F_l^{(+)}$. If both $\pi_1(\mathcal{B}_0)$ and $\pi_2(\mathcal{B}_0)$ are non-simple semisimple, that is, $\pi_1(\mathcal{B}_0) \cong \mathcal{B}_0$ and $\pi_2(\mathcal{B}_0) \cong \mathcal{B}_0$, then we have the following restrictions: $k + l \leq n$ and $k + l \leq m$. For clarity, let $n < m$. Hence the dimension of $\pi_i(\mathcal{B}_0)$, $i = 1, 2$, is less than $n^2 - 2n + 2$. Without any loss of generality $\pi_1(\mathcal{B}_0) \cong \mathcal{B}_0$ that $\dim \mathcal{B}_0 \leq n^2 - 2n + 2$.

As a result, $\dim M_{n,m}(F)_0^{(+)} = n^2 + m^2 \le \frac{n(n+1)}{2} + \frac{m(m-1)}{2} + n^2 - 2n + 2$, so $\frac{m(m+1)}{2} \le \frac{n(n+1)}{2} + 2 - 2n$, which is wrong. Therefore, we have only two possibilities: either $\pi_1(\mathcal{B}_0)$ or $\pi_2(\mathcal{B}_0)$ is a simple algebra. According to [32], in the first case, $k = n - 1$, or $n$ and, in the second case, $l = m$. Thus the lemma is proved for the case when $n \neq m$.

To complete our proof we consider the case when $n = m$. First, we assume that neither $\mathcal{A}$ nor $\mathcal{B}$ has the type $P(n)$. By Lemma 3.1.8 one of the subsuperalgebras, for example $\mathcal{A}$, is isomorphic to $osp(p, q)$, $p + 2q = 2n$, that is, $p = n$, $q = \frac{n}{2}$. Then, we obtain two decompositions of the form

$$F_n^{(+)} = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0), \quad \pi_1(\mathcal{A}_0) \cong H(F_n),$$

$$F_n^{(+)} = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0), \quad \pi_2(\mathcal{A}_0) \cong H(Q_{\frac{n}{2}}).$$

For some $i$, let $\pi_i(\mathcal{B}_0)$ be a non-simple semisimple subalgebra, then

$$\pi_i(\mathcal{B}_0) \cong \begin{cases} F_k^{(+)} \oplus F_l^{(+)}, \ k + l \le n \quad \text{or} \\ H(F_k) \oplus H(Q_l), \ k + 2l \le n \end{cases}$$

Therefore, $\dim \pi_i(\mathcal{B}_0)$ and $\dim \mathcal{B}_0 \le n^2 - 2n + 2$. However $\dim M_{n,n}(F)_0^{(+)} \le \dim \mathcal{A}_0 + \dim \mathcal{B}_0$, $2n^2 \le n^2 - 2n + 2 + \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = 2n^2 - 2n + 2$, that is, $n \le 1$, and we assumed that $(n, m) \neq (1, 1)$. Hence both $\pi_1(\mathcal{B}_0)$ and $\pi_2(\mathcal{B}_0)$ are simple. It follows that $\pi_1(\mathcal{B}_0) \cong F_{n-1}^{(+)}$, $\pi_2(\mathcal{B}_0) \cong F_n^{(+)}$, that is, $\mathcal{B}_0 \cong M_{n-1,n}(F)$.

Next we let $\mathcal{A}$ be of the type $P(n)$. Then $\mathcal{B}$ can be isomorphic to any of the following superalgebras: $P(k)$, $Q(k)^{(+)}$, $osp(k, l)$, $M_{k,l}(F)$, for some integers $k$ and $l$.

**1.** $\mathcal{B} \cong P(k)$, hence dim $\mathcal{B} = 2k^2$, $k \leq n$. From dim $M_{n,n}(F)^{(+)} \leq \dim \mathcal{A} + \dim \mathcal{B}$, it is clear that $k = n$, and the sum in the decomposition is direct. However since both subsuperalgebras have the type $P(n)$, they contain the identity of $M_{n,n}(F)^{(+)}$, a contradiction.

**2.** $\mathcal{B} \cong Q(k)^{(+)}$. In this case the proof is the same as in previous case.

**3.** $\mathcal{B} \cong osp(k,l)$. Since $2n^2 = \dim (M_{n,n}(F))_0 \leq \dim \mathcal{A}_0 + \dim \mathcal{B}_0 \leq n^2 + \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = 2n^2$, it follows that $k = n$, $l = \frac{n}{2}$. However, in the decomposition of $(M_{n,n}(F))_0$ both subalgebras contain the identity element 1, a contradiction.

**4.** $\mathcal{B} \cong M_{k,l}(F)^{(+)}$, $k+l < 2n$. The even part of $M_{n,n}(F)^{(+)}$, that is, $M_{n,n}(F)_0^{(+)}$ is the sum of two orthogonal ideals denoted as $I_1$ and $I_2$, and both ideals are isomorphic to $F_n^{(+)}$. By the dimension argument, $\dim M_{n,n}(F)^{(+)} \leq 2n^2 + (k+l)^2$, $4n^2 \leq 2n^2 + (k+l)^2$, so $k + l \geq \sqrt{2}n$.

If $1 \in \mathcal{B}$, then $1 \in \mathcal{B}_0$. Notice that $\mathcal{B}_0$ is the sum of two orthogonal ideals denoted as $J_1$ and $J_2$ where $J_1 \cong F_k^{(+)}$ and $J_2 \cong F_l^{(+)}$. Since $k + l \geq \sqrt{2}n$, $J_1 \subset I_1$, $J_2 \subset I_2$, and both ideals contain the identity elements of $I_1$ and $I_2$, respectively. By Lemma 3.0.15, $k \leq \frac{n}{2}$, $l \leq \frac{n}{2}$, and $k + l \leq n$, a contradiction.

It follows that $1 \notin \mathcal{B}$. Equivalently, $\mathcal{B}$ has a non-zero two-sided annihilator. Let $V = V_0 + V_1$ be $Z_2$-graded vector space where dim $V_0 = n$ and dim $V_1 = n$. Let $\rho$ stand for the natural representation of $M_{n,n}(F)^{(+)}$ in $V$. Then there exists $v_0 \in V_0$ annihilated by $\rho(\mathcal{B})$. Since $M_{n,n}(F)^{(+)} = \mathcal{A} + \mathcal{B}$, $\rho(M_{n,n}(F)^{(+)})v_0 = \rho(\mathcal{A})v_0$. According to the structure theory, $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ where $\mathcal{A}_1$ is the sum of two irreducible unital $\mathcal{A}_0$-bimodules: the skew-symmetric matrices $M_1$ and the symmetric matrices $M_2$. Besides, $\rho(M_1 \oplus M_2)(V_0 + V_1) = \rho(M_1)V_0 + \rho(M_2)V_1$. Hence, $\rho(\mathcal{A})v_0 = \rho(\mathcal{A}_0)v_0 + \rho(\mathcal{A}_1)v_0 = \rho(\mathcal{A}_0)v_0 + \rho(M_1)v_0$. Since $M_1$ consists of skew-symmetric matrices, $\dim(\rho(M_1)v_0) = n-1$. It follows that $\dim \rho(\mathcal{A})v_0 = n + n - 1 < 2n$, a contradiction. The lemma is proved. □

**Example 1.** *A Jordan superalgebra of the type $M_{n,m}(F)^{(+)}$ where $m$ is even can be represented as the sum of two proper simple subsuperalgebras $\mathcal{A}$ and $\mathcal{B}$ which have types $osp(n, \frac{m}{2})$ and $M_{n-1,m}(F)^{(+)}$, respectively.*

*Proof.* To prove, we consider the first subsuperalgebra in the standard realization:

$$\left\{ \left( \begin{array}{c|c} A & C \\ \hline S^{-1}C^t & B \end{array} \right) \right\}$$

where $A$ is a symmetric matrix of order $n$, $B$ is a symplectic matrix of order $m$, $C$ is any matrix of order $n \times m$, $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $I$ is identity matrix of order $\frac{m}{2}$. The second subalgebra can be viewed in the following form:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\} \tag{1'}$$

where $A$ of order $n \times n$ has the last two columns equal and the last row zero; $B$ of order $n \times m$ has the last row zero; $C$ of order $m \times n$ has the last two columns equal and, finally, $D$ of order $m \times m$ is arbitrary. By straightforward calculations $\dim(\mathcal{A}_1 + \mathcal{B}_1) = \dim \mathcal{A}_1 + \dim \mathcal{B}_1 - \dim(\mathcal{A}_1 \cap \mathcal{B}_1) = mn + 2m(n-1) - m(n-2) = 2mn$. □

**Example 2.** *A Jordan superalgebra of the type $M_{n,m}(F)^{(+)}$ where $n$ is even can also be decomposed into the sum of $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} \cong osp(m, \frac{n}{2})$ and $\mathcal{B} \cong M_{m-1,n}(F)^{(+)}$.*

*Proof.* As above we consider the first subsuperalgebra in the standard realization:

$$\left\{ \left( \begin{array}{c|c} A & C \\ \hline S^{-1}C^t & B \end{array} \right) \right\}$$

where $A$ is a symmetric matrix of order $m$, $B$ is a symplectic matrix of order $n$, $C$ is any matrix of order $m \times n$, $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $I$ is identity matrix of order $\frac{n}{2}$. The second subalgebra can be viewed in the following form:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

where $A$ of order $m \times m$ has the last two columns equal and the last row zero; $B$ of order $m \times n$ has the last row zero; $C$ of order $n \times m$ has the last two columns equal and, finally, $D$ of order $n \times n$ is arbitrary.

By straightforward calculations $\dim(\mathcal{A}_1 + \mathcal{B}_1) = \dim \mathcal{A}_1 + \dim \mathcal{B}_1 - \dim(\mathcal{A}_1 \cap \mathcal{B}_1) = mn + 2n(m-1) - n(m-2) = 2mn$.

□

**Proposition 3.1.1.** *In terms of the types of simple subsuperalgebras Examples 1 and 2 are the only possible simple decompositions of $M_{n,m}(F)^{(+)}$, $n, m > 0$ for appropriate values of $n$, $m$.*

*Proof.* As usual, we assume the contrary, that is, there exist other types of simple decompositions of $M_{n,m}(F)^{(+)}$ different from ones in Examples 1 and 2. By Lemma 3.1.9, this decomposition takes the following form:

1. If $m$ is even, then $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \cong osp(n, \frac{m}{2})$, $\mathcal{B} \cong M_{l,k}(F)^{(+)}$ where either $l = n-1$, or $n$ or $k = m$.

2. If $n$ is even, then $M_{n,m}(F)^{(+)} = \mathcal{A}+\mathcal{B}$, $\mathcal{A} \cong osp(m, \frac{n}{2})$, $\mathcal{B} \cong M_{k,l}(F)^{(+)}$ where either $k = m-1$, or $m$ or $l = n$.

Then $M_{n,m}(F)_1 = \mathcal{A}_1 + \mathcal{B}_1$. It follows that $\dim M_{n,m}(F)_1 \leq \dim \mathcal{A}_1 + \dim \mathcal{B}_1$, that is, $2nm \leq nm + 2lk$, $nm \leq 2lk$. Hence, for even $m$, $l \geq \frac{n}{2}$, in the case $k = m$, and $k \geq \frac{m}{2}$, in the case $l = n-1$ or $n$. Likewise, if $n$ is even, then $k \geq \frac{m}{2}$, in the case $l = n$, and $l \geq \frac{n}{2}$, in the case $k = m-1$ or $m$. For clarity, we consider the case when $m$ is even, and $l = n-1$ because the proof remains the same for all other cases.

Let $V = V_0 + V_1$ denote a $Z_2$-graded vector space where dim $V_0 = n$ and dim $V_1 = m$. Fixing any homogenous basis of $V$, we get an isomorphism between $\operatorname{End} V$ and $M_{n,m}(F)$ as $Z_2$-graded algebras. Then let $\rho$ stand for the natural representation of $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$ in $V$. It follows from the definition of this action that $\rho(\mathcal{B}_0)(V_0) \subseteq V_0$, $\rho(\mathcal{B}_0)(V_1) \subseteq V_1$, $\rho(\mathcal{B}_1)(V_0) \subseteq V_1$, $\rho(\mathcal{B}_1)(V_1) \subseteq V_0$. Since $\mathcal{B}_0$ is a non-simple semisimple Jordan algebra it acts completely reducibly in $V$. Next we describe this action in more details. For this, we identify $V$ with a $Z_2$-graded vector space of the form $W = \langle v_0 \rangle \oplus (V_0' \otimes F^{r+1}) \oplus V_1'$, $r \geq 1$ where $v_0$ is a vector in $V_0$ annihilated by $\mathcal{B}_0$ ( in the case $l = n$ we omit this vector), $V_0'$ is an invariant complementary subspace of $\langle v_0 \rangle$, $\rho(\mathcal{B}_0)|_{V_0'} \cong F_{n-1}^{(+)}$, $V_1'$ is an invariant subspace of $V_1$ such that $\rho(\mathcal{B}_0)|_{V_1'} \cong F_k^{(+)}$. Moreover, $W_0 = \langle v_0 \rangle \oplus V_0' \otimes e_0$, $W_1 = V_0' \otimes \langle e_1, \ldots, e_r \rangle \oplus V_1'$ where $\langle e_0, e_1, \ldots . e_r \rangle$ is a basis for $F^{r+1}$. Then, $\rho(\mathcal{B}_0) = \rho(\mathcal{B}_0)|_{\langle v_0 \rangle} \oplus \rho(\mathcal{B}_0)|_{V_0'} \otimes Id_{r+1} \oplus \rho(\mathcal{B}_0)|_{V_1'}$. Note that $\rho(\mathcal{B}_0)|_{\langle v_0 \rangle} = 0$. In other words, by choosing an appropriate basis in $V_0$ and $V_1$, $\rho(\mathcal{B}_0)$ can be written in a block-diagonal form in which the first block of order 1 is zero, the last block has order $k$, and the other blocks have order $r+1$. Next we consider the representation of $\mathcal{B}_1$. For this, we choose any $a \in \mathcal{B}_0$ such that

$$\rho(a)(V_0' \otimes F^{r+1}) = 0, \quad \rho(a)(V_1') \neq 0. \tag{2}$$

All such elements form an ideal of $\mathcal{B}_0$ isomorphic to $F_k^{(+)}$. Then we choose any non-zero $x$ in $\mathcal{B}_1$. Let $e$ denote the identity of $\mathcal{B}$, $e \in \mathcal{B}_0$. If $\odot$ denotes the Jordan multiplication in $M_{n,m}(F)^{(+)}$, then $\rho(x)v_0 = \rho(x \odot e)v_0 = \rho(\frac{xe+ex}{2})v_0 = \frac{1}{2}(\rho(x)\rho(e)v_0 + \rho(e)\rho(x)v_0) = \frac{1}{2}\rho(x)v_0$, that is, $\rho(x)v_0 = 0$, for any $x \in \mathcal{B}_1$. Next we find the representation of $a \odot x \in \mathcal{B}_1$. As was mentioned above, $\rho(a \odot x)(v_0) = 0$. Besides, $2\rho(a \odot x)(V_0' \otimes e_0) = \rho(a)\rho(x)(V_0' \otimes e_0) + \rho(x)\rho(a)(V_0' \otimes e_0) \subseteq V_1'$, $\rho(a \odot x)(V_0' \otimes \langle e_1, \ldots, e_r\rangle) = 0$, $\rho(a \odot x)(V_1') \subseteq V_0' \otimes e_0$. Clearly, we can find $c \in \mathcal{B}_0$ whose action satisfies the following formula:

$$\rho(c)(V_0' \otimes F^{r+1}) \neq 0, \quad \rho(c)(V_1') = 0. \tag{3}$$

Now we need to determine the action of

$$c \odot (x \odot a). \tag{4}$$

Since $2\rho(c \odot (a \odot x)) = \rho(c)\rho(a \odot x) + \rho(a \odot x)\rho(c)$, we have the following: $\rho(c \odot (a \odot x))(v_0) = 0$, $\rho(c \odot (a \odot x))(V_0' \otimes \langle e_1, \ldots, e_r\rangle) = 0$. Besides,

$$\rho(c \odot (x \odot a))(V_0' \otimes e_0) = \rho(c)\rho(x)\rho(a)(V_0' \otimes e_0) + \rho(x)\rho(c)\rho(a)(V_0' \otimes e_0)+$$

$$\rho(a)\rho(c)\rho(x)(V_0' \otimes e_0) + \rho(a)\rho(x)\rho(c)(V_0' \otimes e_0) =$$

$$\rho(a)\rho(x)\rho(c)(V_0' \otimes e_0) \subseteq V_1'. \tag{5}$$

Similarly,

$$\rho(c \odot (x \odot a))(V_1') = \rho(c)\rho(x)\rho(a)(V_1') + \rho(x)\rho(c)\rho(a)(V_1')+$$

$$\rho(a)\rho(c)\rho(x)(V_1') + \rho(a)\rho(x)\rho(c)(V_1') =$$

$$\rho(c)\rho(x)\rho(a)(V_1') \subseteq V_0' \otimes e_0. \tag{6}$$

Assume that $\rho(x)(V_1') \neq 0$, $\rho(x)(V_0' \otimes e_0) \neq 0(\operatorname{mod} V_0' \otimes \langle e_1, \ldots, e_r\rangle)$. Then $\rho(c \odot (x \odot a))$ restricted to $V_1'$ represents all linear transformations from $V_1'$ to $V_0' \otimes e_0$, and $\rho(c \odot (x \odot a))$ restricted to $V_0' \otimes e_0$ represents all linear transformation from $V_0' \otimes e_0$ to $V_1'$. Next we choose any $y \in \mathcal{B}_1$. We have seen that there exists an element $y$ of form (4) such that either $\rho(y - a \odot (x \odot c))(V_1') = 0$ or $\rho(y - a \odot (x \odot c))(V_0' \otimes e_0) = 0(\operatorname{mod} V_0' \otimes \langle e_1, \ldots, e_r\rangle)$. Suppose that one of the above equations does not hold. Without any loss of generality we let $\rho(y')(V_0' \otimes e_0) \neq 0(\operatorname{mod} V_0' \otimes \langle e_1, \ldots, e_r\rangle)$, where $y' = y - a \odot (x \odot c)$. Multiplying $y'$ by the elements of the form (2) and then (3) we obtain $a' \odot (y' \odot c') \in \mathcal{B}_1$ and $\rho(a' \odot (y' \odot c'))(V_0' \otimes F^{r+1}) = 0$, $\rho(a' \odot (y' \odot c'))V_1' = \rho(a')\rho(y')\rho(c')V_1' \subseteq V_0' \otimes e_0$. Moreover, $\rho(a' \odot (y' \odot c')) : V_1' \to V_0' \otimes e_0$ represents all linear transformations from $k$-dimensional vector space into $(n-1)$-dimensional vector space. Besides, all such elements are linearly independent from all the elements of the form (4). Therefore, we found $2(n-1)k$ linearly independent elements of $\mathcal{B}_1$, $(\dim \mathcal{B}_1 = 2(n-1)k)$. If there is at least one element $\bar{y} \in \mathcal{B}_1$ such that either $\rho(\bar{y})(V_0' \otimes e_0) \neq 0(\operatorname{mod} V_1')$ or $\rho(\bar{y})(V_0' \otimes \langle e_1, \ldots, e_r\rangle) \neq 0$, then it will also be linearly independent with all above elements. Hence, by dimension arguments, there is no $\bar{y}$ satisfying the above conditions. Consequently, for all elements in $\mathcal{A}_1$, the following $\rho(\bar{y})(V_0') = 0(\operatorname{mod} V_1')$, $\rho(\bar{y})(V_0' \otimes \langle e_1, \ldots, e_r\rangle) = 0$, $\rho(\bar{y})(V_1') \subseteq V_0' \otimes e_0$ hold true. Hence, the odd part consists of all linear transformations $\varphi$ such that $\varphi(V_0' \otimes e_0) = V_1'$, $\varphi(V_0' \otimes \langle e_1, \ldots, e_r\rangle) = 0$, $\varphi(V_1') = V_0' \otimes e_0$. Then it follows from

$\mathcal{B}_1 \odot \mathcal{B}_1 \subseteq \mathcal{B}_0$ that $\mathcal{B}_1 = 0$, a contradiction.

We henceforth assume that the equations $\rho(y - a \odot (x \odot c))(V_1') = 0$ and $\rho(y - a \odot (x \odot c))(V_0' \otimes e_0) = 0 \,(\mathrm{mod}\, V_0' \otimes \langle e_1, \ldots, e_r \rangle)$ hold true simultaneously. Then multiplying $y - a \odot (x \odot c)$ by the elements (4), we obtain some elements of $\mathcal{B}_0$ which act on $V_1'$ and $V_0' \otimes \langle e_1, \ldots, e_r \rangle$ non-invariantly. Hence, $y - a \odot (x \odot c) = 0$. Therefore, the odd component of $\mathcal{A}_1$ has form (7). As proved before, this is not possible.

Next we assume that $\rho(x)(V_0' \otimes e_0) = 0(\mathrm{mod} V_0' \otimes \langle e_1, \ldots, e_r \rangle)$, for all $x \in \mathcal{B}_1$, and for at least one element $x' \in \mathcal{B}_1$, $\rho(x')(V_1') \neq 0$.

Acting in the same manner as before, we obtain $a' \odot (x' \odot c') \in \mathcal{B}_1$ which acts trivially on all subspaces except for $V_1'$, which it carries into $V_0' \otimes e_0$. Considering the difference between an arbitrary element $y \in \mathcal{B}_1$ and a corresponding element $a'' \odot (x'' \odot c'')$, we can show that $\rho(y - a'' \odot (x'' \odot c''))(V_0' \otimes e_0) = 0(\mathrm{mod}\, V_0' \otimes \langle e_1, \ldots, e_r \rangle \oplus V_1')$, $\rho(y - a'' \odot (x'' \odot c''))(V_1') = 0$. Again multiplying $a' \odot (x' \odot c')$ and $y - a'' \odot (x'' \odot c'')$, we obtain some elements from $\mathcal{B}_0$ acting on $V_1'$ non-trivially. Then we conclude that $\mathcal{B}_1$ consists of all elements which act on $V_0' \otimes e_0$ trivially and carry the other subspaces into $V_0' \otimes e_0$. Hence $\mathcal{B}_1 \odot \mathcal{B}_1 = 0$, a contradiction.

Finally, if $\rho(x)(V_1') = 0$, $\rho(x)(V_0' \otimes e_0) = 0(\mathrm{mod}\,(V_0' \otimes F^{r+1}))$, then it follows that $\mathcal{B}_1 \odot \mathcal{B}_0 = 0$, which is clearly a wrong statement. The proposition is proved. □

Now we are ready to explicitly describe all conjugacy classes of simple decompositions of $M_{n,m}(F)^{(+)}$. We start with the following lemma.

**Lemma 3.1.10.** *Let $F_n^{(+)} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \cong F_{n-1}^{(+)}$, $\mathcal{B} \cong H(F_n)$. Then there are precisely two conjugacy classes corresponding to the above decomposition.*

*Proof.* Let $V$ be a $n$-dimensional vector space with a non-singular symmetric bilinear form $f$. Next we consider the set $\mathcal{L}$ consisting of all $\varphi \in \operatorname{End} V$ such that $\varphi \circ \varphi_0 = \varphi_0 \circ \varphi = 0$ for some non-zero fixed $\varphi_0 \in \operatorname{End} V$. Further we will be interested only in $\varphi_0$ such that $\varphi_0^2 = \varphi_0$ and $\dim \varphi_0(V) = 1$, and $\mathcal{L}$ is a simple algebra of the type $F_{n-1}$. Let $\varphi_0(V) = \operatorname{span}\langle x \rangle$ where $x \in V$. Let $\{e_1, \ldots, e_n\}$ be a basis orthonormal with respect to $f$. Next the proof is divided into two cases.

**Case 1.** Let $f(x,x) \neq 0$. Therefore, we can normalize $\varphi_0$ in such a way that $f(x,x) = \sqrt{2}$. Hence, there exists an orthogonal linear mapping $\psi$ that sends $x$ to $e_n - e_{n-1}$ (Witt's Theorem [5]). Denote $\varphi_0' = \psi \circ \varphi_0 \circ \psi^{-1}$. Notice that $\varphi_0'(V) = \operatorname{span}\langle e_n - e_{n-1} \rangle$. Then, in an appropriate basis $\mathcal{L}' = \psi \mathcal{L} \psi^{-1}$ takes the form:

$$\left\{ \left( \begin{array}{ccc|c|c} & Y & & z & z \\ \hline x_{n,1} & \cdots & x_{n,n-2} & x_{n,n-1} & x_{n,n-1} \end{array} \right) \right\} \tag{8'}$$

where $Y = (y_{ji})$, $j = 1, \ldots, n-1$, $i = 1, \ldots, n-2$, $z$ is a $(n-1)$-dimensional column, $x_{n,i} = \sum_{j=1}^{n-1} \alpha_j y_{ji}$, $i = 1, \ldots, n-2$, $x_{n,n-1} = \sum_{j=1}^{n-1} \alpha_j z_j$ for some $\alpha_i$. Moreover, (8′) has the identity element of the form:

$$e = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & \alpha_{n-1} \end{pmatrix}.$$

However, it is easily seen that $e^2 = e$ if and only if $\alpha_1 = \ldots = \alpha_{n-1} = 0$. It follows that $x_{n,1} = \ldots = x_{n,n-1} = 0$. Hence, $\mathcal{L}'$ takes the form:

$$\left\{ \left( \begin{array}{ccc|c|c} & Y & & z & z \\ \hline 0 & \ldots & 0 & 0 & 0 \end{array} \right) \right\} \tag{8}$$

where $Y$ is any matrix of order $(n-1) \times (n-2)$, $z$ is a $(n-1)$-dimensional column.

**Case 2.** Let $f(x, x) = 0$. It is easily seen that in this case $x$ can be mapped to $e_{n-1} + ie_n$, $i^2 = -1$, by some orthogonal linear mapping $\psi$ (Witt's Theorem, [5]). Like in case 1, we denote $\varphi_0' = \psi \circ \varphi_0 \circ \psi^{-1}$. Notice that $\varphi_0'(V) = \operatorname{span}\langle e_{n-1} + ie_n \rangle$. Then, in an appropriate basis $\mathcal{L}' = \psi \mathcal{L} \psi^{-1}$ takes the form:

$$\left\{ \left( \begin{array}{ccc|c|c} & Y & & z & iz \\ \hline x_{n,1} & \ldots & x_{n,n-2} & x_{n,n-1} & ix_{n,n-1} \end{array} \right) \right\} \tag{9'}$$

where $Y = (y_{ji})$, $j = 1, \ldots, n-1$, $i = 1, \ldots, n-2$, $z$ is a $(n-1)$-dimensional column, $x_{n,i} = \sum_{j=1}^{n-1} \alpha_j y_{ji}$, $i = 1, \ldots, n-2$, $x_{n,n-1} = \sum_{j=1}^{n-1} \alpha_j z_j$ for some $\alpha_i$. Moreover, (9′) has the identity element of the form:

$$e = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & i \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & i\alpha_{n-1} \end{pmatrix}.$$

However, it is easily seen that $e^2 = e$ if and only if $\alpha_1 = \ldots = \alpha_{n-1} = 0$. It follows that $x_{n,1} = \ldots = x_{n,n-1} = 0$. It follows that in an appropriate basis $\mathcal{L}' = \psi \mathcal{L} \psi^{-1}$ takes the form:

$$\left\{ \left( \begin{array}{ccc|c|c} & Y & & z & iz \\ \hline 0 & \ldots & 0 & 0 & 0 \end{array} \right) \right\} \tag{9}$$

where $Y$ is any matrix of order $(n-1) \times (n-2)$, $z$ is a $(n-1)$-dimensional column.

Finally, according to above considerations any simple decomposition of $F_n^{(+)}$ into $\mathcal{A} \cong F_{n-1}^{(+)}$ and $\mathcal{B} \cong H(F_n)$ takes one of the following forms:

**(1)** $F_n^{(+)} = \mathcal{A}_1 + \mathcal{B}$ where $\mathcal{A}_1$ has the form (8), and $\mathcal{B}$ is taken in the canonical form, i.e. the set of all matrices symmetric with respect to the involution generated by $f$.

**(2)** $F_n^{(+)} = \mathcal{A}_2 + \mathcal{B}$ where $\mathcal{A}_2$ has the form (9), and $\mathcal{B}$ is taken in the canonical form as in (**1**).

Assume that there exists an automorphism $\varphi$ of $F_n^{(+)}$ such that $\varphi(\mathcal{A}_1) = \mathcal{A}_2$ and $\varphi(\mathcal{B}) = \mathcal{B}$. The latter implies that $\varphi$ can be restricted to the set of all matrices symmetric under the ordinary transpose involution. Let $V$ be a $n$-dimensional vector column space, and $f$ be a non-singular symmetric bilinear form that generates the ordinary transpose involution. Then $\varphi$ is induced by an orthogonal linear mapping $\psi$ of $V$. Recall that $\mathcal{A}_1$ annihilates the one-dimensional subspace $\langle x_1 \rangle \subset V$ and $f(x_1, x_1) \neq 0$. Then $\mathcal{A}_2$ also annihilates the one-dimensional subspace $\langle x_2 \rangle \subset V$ and $f(x_2, x_2) = 0$. Since $\varphi(\mathcal{A}_1) = \mathcal{A}_2$, $\psi(x_1) = \alpha x_2$ where $\alpha \in F$, $\psi$ is orthogonal, a contradiction. Therefore these decompositions cannot be conjugate. The lemma is proved.

$\square$

Now we are able to determine conjugacy classes of simple decompositions of $M_{n,m}(F)^{(+)}$. Recall that in terms of the types of simple subalgebras only the following decompositions are possible:

**1.** If $m$ is even, then $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \cong M_{n-1,m}(F)^{(+)}$ and $\mathcal{B} \cong osp(n, \frac{m}{2})$.

**2.** If $n$ is even, then $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A} \cong M_{m-1,n}(F)^{(+)}$ and $\mathcal{B} \cong osp(m, \frac{n}{2})$.

If both $n$, $m$ are even, then both decompositions are possible. Notice that since the associative universal enveloping superalgebra of any superalgebra of the type $osp(k,l)$ is $M_{k,2l}(F)$, any isomorphism between two subsuperalgebras of $M_{k,2l}(F)$ of the type $osp(k,l)$ can be extended to an automorphism of $M_{k,2l}(F)$. Therefore the second subsuperalgebra in decompositions 1 and 2 can be considered in the canonical form.

Examples 1 and 2 show us how the simple decomposition that occurs in the first case of Lemma 3.1.10 can be lifted up to the decompositions of $M_{n,m}(F)^{(+)}$. Next we are going to show that the simple decomposition in the second case of Lemma 3.1.10 can also be extended to the decompositions of $M_{n,m}(F)^{(+)}$.

**Example 3.** *There exists a simple decomposition of $M_{n,m}(F)^{(+)}$ ($m$ is even) of the form $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is taken in the canonical form, and $\mathcal{B}$ has the following realization:*

$$\left\{ \left( \begin{array}{c|c|c|c} A & x & ix & B \\ \hline 0 \ \ldots \ 0 & 0 & 0 & 0 \ \ldots \ 0 \\ \hline C & y & iy & D \end{array} \right) \right\} \tag{9''}$$

*where $A$ and $C$ are any matrices of orders $(n-1)\times(n-2)$ and $m\times(n-2)$, respectively, $x$ and $y$ are $(n-1)$-dimensional and $m$-dimensional columns, $B$, $D$ are matrices of orders $(n-1)\times m$ and $m\times m$, respectively. In this decomposition $\mathcal{A}\cong osp(n,\frac{m}{2})$ and $\mathcal{B}\cong M_{n-1,m}(F)^{(+)}$.*

**Example 4.** *There exists a simple decomposition of $M_{n,m}(F)^{(+)}$ ($n$ is even) of the form $M_{n,m}(F)^{(+)}=\mathcal{A}+\mathcal{B}$ where $\mathcal{A}$ is taken in the canonical form, and $\mathcal{B}$ has the following realization:*

$$\left\{\left(\begin{array}{c|c|c|c} A & x & ix & B \\ \hline 0 \ \ldots \ 0 & 0 & 0 & 0 \ \ldots \ 0 \\ \hline C & y & iy & D \end{array}\right)\right\}$$

*where $A$ and $C$ are any matrices of orders $(m-1)\times(m-2)$ and $n\times(m-2)$, respectively, $x$ and $y$ are $(m-1)$-dimensional and $n$-dimensional columns, $B$, $D$ are matrices of orders $(m-1)\times n$ and $n\times n$, respectively. In this decomposition $\mathcal{A}\cong osp(m,\frac{n}{2})$ and $\mathcal{B}\cong M_{m-1,n}(F)^{(+)}$.*

**Lemma 3.1.11.** *Let $M_{n,m}(F)^{(+)}=\mathcal{A}+\mathcal{B}$ where $\mathcal{A}\cong M_{n-1,m}(F)^{(+)}$, $\mathcal{B}\cong osp(n,\frac{m}{2})$. Then there are precisely two conjugacy classes corresponding to this decomposition.*

*Proof.* In order to reach our goal we first show that the decompositions in Example 1 and 3 are not conjugate under an automorphism of $M_{n,m}(F)^{(+)}$. Assume the contrary, i.e. $M_{n,m}(F)^{(+)}=\mathcal{A}_1+\mathcal{B}_1=\mathcal{A}_1'+\mathcal{B}_1'$, where $\mathcal{A}_1$ as in Example 1, $\mathcal{A}_1'$ as in Example 3, and both $\mathcal{B}_1$ and $\mathcal{B}_1'$ have the standard realizations, and there exists an automorphism $\varphi$ of $M_{n,m}(F)^{(+)}$ such that $\varphi(\mathcal{A}_1)=\mathcal{A}_1'$ and $\varphi(\mathcal{B}_1)=\mathcal{B}_1'$.

Notice that $(\mathcal{A}_1)_0 = J_1 \oplus J_2$, $J_1 \cong F_{n-1}^{(+)}$, $J_2 \cong F_m^{(+)}$, and $(\mathcal{B}_1)_0 = T_1 \oplus T_2$, $T_1 \cong H(F_n)$, $T_2 \cong H(Q_{\frac{m}{2}})$. Similarly, $(\mathcal{A}'_1)_0 = J'_1 \oplus J'_2$, $J'_1 \cong F_{n-1}^{(+)}$, $J'_2 \cong F_m^{(+)}$, and $(\mathcal{B}'_1)_0 = T'_1 \oplus T'_2$, $T'_1 \cong H(F_n)$, $T'_2 \cong H(Q_{\frac{m}{2}})$.

Since $\varphi$ is an automorphism of superalgebras, $\varphi = \varphi_0 + \varphi_1$ where $\varphi_0 : M_{n,m}(F)_0^{(+)} \to M_{n,m}(F)_0^{(+)}$, $\varphi_1 : M_{n,m}(F)_1^{(+)} \to M_{n,m}(F)_1^{(+)}$. In particular, $\varphi_0$ is an automorphism of the even part. Hence, $\varphi_0((\mathcal{A}_1)_0) = (\mathcal{A}'_1)_0$, $\varphi_0((\mathcal{B}_1)_0) = (\mathcal{B}'_1)_0$. As a consequence, $M_{n,m}(F)_0^{(+)} = (\mathcal{A}_1)_0 + (\mathcal{B}_1)_0 = (\mathcal{A}'_1)_0 + (\mathcal{B}'_1)_0$. Notice that $(M_{n,m}(F)^{(+)})_0 = I_1 \oplus I_2$, $I_1 \cong F_n^{(+)}$, $I_2 \cong F_m^{(+)}$. If $\varphi_0 : M_{n,m}(F)_0^{(+)} \to M_{n,m}(F)_0^{(+)}$ is an automorphism, then either $\varphi_0(I_1) = I_1$, $\varphi_0(I_2) = I_2$ or $\varphi_0(I_1) = I_2$, $\varphi_0(I_2) = I_1$. Notice that $T_1 \subseteq I_1$, $J_1 \subseteq I_1$, $T_2 \subseteq I_2$ , $J_2 \subseteq I_2$. Similarly, $T'_1 \subseteq I_1$, $J'_1 \subseteq I_1$, $T'_2 \subseteq I_2$, $J'_2 \subseteq I_2$. If $\varphi_0(I_1) = I_1$, $\varphi_0(I_2) = I_2$, then $\varphi_0(J_1 \oplus J_2) = \varphi_0(J_1) \oplus \varphi_0(J_2) = J'_1 \oplus J'_2$. Besides, $\varphi_0(J_1) \subseteq I_1$ and $\varphi_0(J_2) \subseteq I_2$. Hence $\varphi_0(J_1) = J'_1$, $\varphi_0(J_2) = J'_2$. Similarly, $\varphi_0(T_1) = T'_1$, $\varphi_0(T_2) = T'_2$.

Therefore, $F_n^{(+)} = J_1 + T_1 = J'_1 + T'_1$ and these decompositions are conjugate, which is a contradiction (see Lemma 3.1.10).

If $\varphi_0(I_1) = I_2$, $\varphi_0(I_2) = I_1$, then $\varphi_0(T_1 \oplus T_2) = \varphi(T_1) \oplus \varphi(T_2) = T'_1 \oplus T'_2$. Since $\varphi(T_1) \subseteq I_2$, $\varphi(T_2) \subseteq I_1$, $\varphi(T_1) = T'_2$, $\varphi(T_2) = T'_1$. However, $T_1$ and $T'_2$ are non-isomorphic, a contradiction.

Next we will prove that any simple decomposition of the form $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \cong M_{n-1,m}(F)^{(+)}$, $\mathcal{B} \cong osp(n, \frac{m}{2})$ is conjugate to the decomposition in either Example 1 or Example 3. In terms of the types of simple subsuperalgebras there is only one decomposition, that is, $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$ where

$\mathcal{A} \cong M_{n-1,m}(F)^{(+)}$, $\mathcal{B} \cong osp(n, \frac{m}{2})$. Moreover, $\mathcal{B}$ can be taken in the canonical form. As usual, the original decomposition induces the following decompositions of $I_1$ and $I_2$,

$$I_1 = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0),$$

$$I_2 = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0),$$

where $\pi_1(\mathcal{A}_0) \cong F_{n-1}^{(+)}$, $\pi_2(\mathcal{A}_0) \cong F_m^{(+)}$. Since all conjugacy classes of a decomposition of $F_n^{(+)}$ into the sum of $H(F_n)$ and $F_{n-1}^{(+)}$ are found, there exists an orthogonal automorphism $\varphi$ of $F_n$ of the form $\varphi(X) = C^{-1}XC$ that reduces the first subalgebra to the form either (8) or (9). Then, acting by an automorphism $\psi$ of $M_{n,m}(F)$ of the form $\psi(Y) = C'^{-1}YC'$ where $C' = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$ we can bring $\mathcal{A}$ to either form (1′) or (9″) while $\mathcal{B}$ does not change. Hence we can reduce the decomposition to either the first form or the second form. The lemma is proved.

□

Based on all above lemmas and Proposition 3.1.1, we conclude that Theorem 3.1.1 is true.

## 3.2 Decompositions of superalgebras of the type $osp(n, m)$

In this section we study simple decompositions of $osp(n, m)$ where $n, m > 0$. Actually, we will show that there are no such decompositions over an algebraically closed field $F$ of characteristic not 2. Our main purpose is to prove the following.

**Theorem 3.2.1.** *Let $\mathcal{J}$ be a superalgebra of the type $osp(n, m)$ where $n, m > 0$. Then $\mathcal{J}$ cannot be written as the sum of two proper nontrivial simple subsuperalgebras.*

The proof of this theorem is based on the following lemmas.

**Lemma 3.2.2.** *Let $\mathcal{J}$ be a superalgebra of type $osp(n, m)$ where $n, m > 0$, and $\mathcal{A}$, $\mathcal{B}$ be two proper simple subsuperalgebras none of which has any of the types $K_3$ or $D_t$. Then $\mathcal{J}$ cannot be represented as the sum of $\mathcal{A}$ and $\mathcal{B}$.*

*Proof.* First we identify $\mathcal{J}$ with $osp(n, m)$ that can be considered in the canonical form. Next we assume the contrary, that is,

$$osp(n, m) = \mathcal{A} + \mathcal{B}. \tag{8}$$

The decomposition (8) generates the following decomposition of the associative enveloping algebra into the sum of three non-zero subspaces.

$$M_{n+2m}(F) = S(osp(n, m)) = S(\mathcal{A}) + S(\mathcal{B}) + S(\mathcal{A})S(\mathcal{B}), \tag{9}$$

where $S(\mathcal{A})$, $S(\mathcal{B})$ denote the associative enveloping algebras of $\mathcal{A}$, $\mathcal{B}$, respectively. Let 1 denote the identity of $osp(n, m)$. Then we consider the following cases.

**Case 1.** Let $1 \notin \mathcal{A}$, $1 \notin \mathcal{B}$. This implies that there exist non-zero $a_0$ and $b_0$ in $\mathrm{Ann}(\mathcal{A})$ and $\mathrm{Ann}(\mathcal{B})$, respectively. Then multiplying every term of (9) by $a_0$ on the left and $b_0$ on the right, the following equation $a_0 M_{n+2m}(F) b_0 = 0$ takes place, which is clearly wrong.

**Case 2.** $1 \in \mathcal{A}$, $1 \in \mathcal{B}$. The following six cases arise:

**(a)** $\mathcal{A} \cong M_{k,l}(F)^{(+)}$, $\mathcal{B} \cong M_{p,q}(F)^{(+)}$. The given decomposition induces the decomposition of the even part $osp(n,m)_0 = \mathcal{A}_0 + \mathcal{B}_0$ which in turn can be written as

$$H(F_n) = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0),$$

$$H(\mathcal{Q}_m) = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0).$$

By Theorem 3.0.7, one of the projections in decomposition $H(F_n)$ must be non-simple semisimple, for example, $\pi_1(\mathcal{A}_0) \cong F_k^{(+)} \oplus F_l^{(+)}$. By Lemma 3.0.15, $k+l \leq \frac{n}{2}$. On the other hand, since $\mathcal{A}$ is proper, $S(\mathcal{A})$ is isomorphic to either $M_{k,l}(F)$ or $M_{k,l}(F) \oplus M_{k,l}(F)$. Besides, $S(M_{k,l}(F))$ contains the identity 1. This implies that $r(k+l) = n+2m$ where $r \geq 2$. Similarly, since $S(\mathcal{B})$ is isomorphic to either $M_{p,q}(F)$ or $M_{p,q}(F) \oplus M_{p,q}(F)$ $s(p+q) = n+2m$ where $s \geq 2$. Then, dim $osp(n,m) = \frac{n(n+1)}{2} + m(2m-1) + 2nm \leq (p+q)^2 + (k+l)^2$. Let $k+l = \frac{n+2m}{2}$ and $p+q = \frac{n+2m}{3}$. Then the above inequality is equivalent to $\frac{5}{36}n^2 + \frac{5}{9}m^2 + \frac{n}{2} + \frac{5}{9}nm \leq m$. Consequently, $\frac{5}{9}m^2 \leq m$, $m \leq 1$. However if $m = 1$, then this inequality does not hold no matter what non-zero value $n$ takes. It follows that the inequality is true only if both $k+l$ and $p+q$ are equal to $\frac{n+2m}{2} = \frac{n}{2} + m$. However, we know that $k+l \leq \frac{n}{2}$, a contradiction.

**(b)** $\mathcal{A} \cong M_{k,l}(F)^{(+)}$, $\mathcal{B} \cong P(q)$ or $Q(q)^{(+)}$ $(q > 1)$. The proof of this case is similar to $(a)$.

**(c)** $\mathcal{A}$, $\mathcal{B} \cong P(q)$ or $Q(q)^{(+)}$. Then the decomposition leads to the decomposition of $H(F_n)$ into the sum of two proper simple subalgebras, which does not exist (see

Theorem 3.0.7).

**(d)** $\mathcal{A} \cong osp(k,l)$, $\mathcal{B} \cong M_{p,q}(F)^{(+)}$. Since $S(\mathcal{A}) \cong M_{k+2l}(F)$ contains the identity of the whole superalgebra, $k+2l \leq \frac{n+2m}{2}$. Similarly, $p+q \leq \frac{n+2m}{2}$. Since $\dim osp(n,m) \leq \dim \mathcal{A} + \dim \mathcal{B}$, $\frac{n^2+n}{2} + m(2m-1) + 2nm \leq \frac{k^2+k}{2} + l(2l-1) + 2kl + \frac{(n+2m)^2}{4}$. Since $k+2l \leq \frac{n+2m}{2}$, dim $osp(k,l) \leq \frac{(n+2m)^2}{8} + \frac{n+2m}{4}$. By straightforward calculations we obtain $\frac{n^2}{4} + \frac{n}{2} + m^2 + nm \leq 3m$, which is true if and only if $m = n = 1$. Obviously, $osp(1,1)$ has no simple decompositions.

**(e)** $\mathcal{A} \cong osp(k,l)$, $\mathcal{B} \cong P(q)$. Then, we have $k+2l \leq \frac{n+2m}{2}$, $2q \leq \frac{n+2m}{2}$. Therefore, $\dim \mathcal{B} = 2q^2 \leq 2(\frac{n+2m}{4})^2$. Again, by the dimension argument, this decomposition is not possible.

**(f)** $\mathcal{A} \cong osp(k,l)$, $\mathcal{B} \cong osp(p,q)$. Then $k+2l \leq \frac{n+2m}{2}$, $p+2q \leq \frac{n+2m}{2}$. Hence dim $\mathcal{A}$, dim $\mathcal{B} \leq \frac{(n+2m)^2}{8} + \frac{n+2m}{4}$. Comparing $\dim osp(n,m)$ with dim $\mathcal{A}$ + dim $\mathcal{B}$ we have $\frac{n^2}{2} + 2nm + 2m^2 \leq 4m$, a contradiction. Therefore, in this case dim $osp(n,m) >$ dim $\mathcal{A}$ + dim $\mathcal{B}$.

**Case 3** Let $1 \in \mathcal{A}$, $1 \notin \mathcal{B}$. As mentioned above, the given decomposition induces the following decompositions of the ideals of the even component:

$$H(F_n) = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0), \tag{10}$$

$$H(\mathcal{Q}_m) = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0). \tag{11}$$

If either $\pi_1(\mathcal{A}_0)$, $\pi_2(\mathcal{B}_0)$ or $\pi_1(\mathcal{B}_0)$, $\pi_2(\mathcal{A}_0)$ are non-simple semisimple, then $\dim \mathcal{A}_0 = \dim \pi_1(\mathcal{A}_0) < \dim H(F_n)$, $\dim \mathcal{B}_0 = \dim \pi_2(\mathcal{B}_0) < \dim H(\mathcal{Q}_m)$. This implies that $\dim \mathcal{A}_0 + \dim \mathcal{B}_0 < \dim(H(F_n) \oplus H(\mathcal{Q}_m))$, which is wrong. Likewise we have a contradiction in the second case. Therefore, there is a simple algebra in each pair:($\pi_1(\mathcal{A}_0)$,

$\pi_2(\mathcal{B}_0)), (\pi_1(\mathcal{B}_0), \pi_2(\mathcal{A}_0))$. Since 1 is not an element of $\mathcal{B}$, $\mathcal{B}$ has a non-zero two-sided annihilator, and so does $\mathcal{B}_0$. It follows that one of $\pi_1(\mathcal{B}_0)$, $\pi_2(\mathcal{B}_0)$ has a non-zero two-sided annihilator. Let us assume the first possibility. Then $\pi_1(\mathcal{B}_0)$ can be embedded in a simple subalgebra which also has a non-zero annihilator. Since $H(F_n)$ cannot be written as the sum of two proper simple subalgebras, $\pi_1(\mathcal{A}_0)$ should be either a non-simple semisimple algebra or the whole algebra $H(F_n)$. If $\pi_1(\mathcal{A}_0)$ is non-simple semisimple, then

$$\pi_1(\mathcal{A}_0) \cong H(F_k) \oplus H(\mathcal{Q}_l), \quad \text{or} \quad F_k^{(+)} \oplus F_l^{(+)} \tag{12}$$

In other words, we represent $H(F_n)$ as the sum of a non-simple semisimple subalgebra of form (12) and a subalgebra which has a non-zero two-sided annihilator. Let $V$ denote the $n$-dimensional vector space with vectors written as columns. Then, there exists a non-zero vector $v \in V$ annihilated by the second subalgebra. By Lemma 3.0.6, dim $H(F_n)v = n$. It follows from (10) that $\dim \pi_1(\mathcal{A}_0)v = n$.

If $\pi_1(\mathcal{A}_0) \cong H(F_k) \oplus H(\mathcal{Q}_l)$, then by an automorphism of $F_n$ it can be reduced to the following form: $\{\operatorname{diag}(X, \ldots, X, Y \ldots, Y)\}$ where $X$ is a symmetric matrix of order $k$, $Y$ is a symplectic matrix of order $2l$. Next we represent $v$ as $(v_{11}, \ldots, v_{1k_1}, v_{21}, \ldots, v_{2l_1})^t$ where $v_{i1}$ is a vector of dimension $k$, $i = 1, \ldots, k_1$, $v_{2j}$ is a vector of dimension $2l$, $j = 1, \ldots, l_1$. Since $\pi_1(\mathcal{A}_0)$ contains 1, $kk_1 + 2ll_1 = n$. Then, $\dim\{Xv_{1i} |\, X \in H(F_k)\} = k$, $\dim\{Yv_{2j} |\, Y \in H(\mathcal{Q}_l)\} = 2l - 1$ (see Lemma 3.0.8). Therefore, $\dim \pi_1(\mathcal{A}_0)v = kk_1 + (2l - 1)l_1 < n$, a contradiction. If $\pi_1(\mathcal{A}_0) \cong F_k^{(+)} \oplus F_l^{(+)}$, then by some automorphism of $F_n$ it can be reduced to $\mathcal{T} = \{\operatorname{diag}(X, \ldots, X, X^t, \ldots, X^t, Y, \ldots, Y, Y^t, \ldots, Y^t)\}$ where $X$ and $Y$ are any matrices of orders $k$ and $l$, respectively. Then dimension of $\mathcal{T}v$ is less than $n$ [32]. If $\pi_1(\mathcal{A}_0) = H(F_n)$, then (10) becomes a trivial decomposition, and $\mathcal{A} \cong osp(n, s)$ for some integer $s$. Then $S(\mathcal{A}) \cong M_{n,2s}(F)$, $sr = m$, $r \geq 2$, is a subsuperalgebra of $M_{n,2m}(F)$, which is not possible (Proposition 3.1.1), a contradiction.

Hence, the second possibility holds, that is, $\pi_2(\mathcal{B}_0)$ has a non-trivial two-sided annihilator, that is, can be embedded in the simple algebra with a non-zero annihilator. Therefore, $\pi_2(\mathcal{A}_0)$ is non-simple semisimple because $H(\mathcal{Q}_m)$ cannot be written as the sum of two proper simple subalgebras one of which has a non-zero two-sided annihilator [32]. As a result, we have the decomposition of the form: $H(\mathcal{Q}_m) = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0)$ which in turn induces the following (the detailed proof of this fact can be found in [32])

$$F_{2m} = F_{2m-1} + \langle \pi_2(\mathcal{A}_0) \rangle,$$

in which the first subalgebra clearly has a non-zero two-sided annihilator, and the second is non-simple semisimple. According to [4], such decomposition cannot exist. The lemma is proved. □

**Lemma 3.2.3.** *A superalgebra $\mathcal{J}$ of the type $osp(n, m)$ where $n, m > 0$ cannot be decomposed into the sum of two proper simple subsuperalgebras one of which has either the type $K_3$ or $D_t$.*

*Proof.* First we identify $\mathcal{J}$ with $osp(n, m)$. Assume that $osp(n, m) = \mathcal{A} + \mathcal{B}$ where, for example, $\mathcal{A}$ is either of the type $K_3$ or $D_t$. Then, the above decomposition induces the following:

$$H(F_n) = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0).$$

$$H(\mathcal{Q}_m) = \pi_2(\mathcal{A}_0) + \pi_2(\mathcal{B}_0).$$

Let us note that $\dim \pi_1(\mathcal{B}_0) \leq \frac{n(n-1)}{2}$ if it is a simple subalgebra and $\dim \pi_1(\mathcal{B}_0) \leq \frac{n^2-3n+2}{2} + 2$ if it is a non-simple semisimple subalgebra. This implies that $\dim H(F_n) = \frac{n(n+1)}{2} \leq 2 + \frac{n^2}{2} - \frac{3n}{2} + 3$, so $2n \leq 5$ and $n \leq 2$. Similarly, $\dim \pi_2(\mathcal{B}_0) \leq 2m^2 - 5m + 3$ if $\pi_2(\mathcal{B}_0)$ is a simple subalgebra, and $\dim \pi_2(\mathcal{B}_0) \leq 2m^2 - 5m + 4$ if $\pi_2(\mathcal{B}_0)$ is a non-simple semisimple subalgebra. Thus $\dim H(\mathcal{Q}_m) = 2m^2 - m \leq 2 + 2m^2 - 5m + 4$, so $4m \leq 6$ and $m \leq \frac{3}{2}$. Therefore, either $\mathcal{J} \cong osp(1,1)$ or $\mathcal{J} \cong osp(2,1)$. By Corollary 3.1.5, these decompositions are not possible. The lemma is proved. □

## 3.3 Decompositions of superalgebras of types $Q(n)^{(+)}$ and $P(n)$

In several steps, we will prove that no Jordan superalgebra of the type either $P(n)$ or $Q(n)^{(+)}$ can be represented as the sum of two proper simple subsuperalgebras.

**Lemma 3.3.1.** *Let $\mathcal{A}$ of the type $osp(p,q)$ be a proper subsuperalgebra of $\mathcal{J}$ which has the type either $P(n)$ or $Q(n)^{(+)}$. Then* $\dim \mathcal{A} \leq \frac{n^2+n}{2}$.

*Proof.* Let $\mathcal{A}_0 \cong H(F_p) \oplus H(\mathcal{Q}_q)$ be a proper subalgebra of $\mathcal{J}_0$ which is isomorphic to $F_n^{(+)}$. Therefore, $p + 2q \leq n$, $p, q > 0$. Hence, $\dim \mathcal{A} = \frac{p^2+p}{2} + q(2q-1) + 2pq = \frac{1}{2}(p+2q)^2 + \frac{p-2q}{2} < \frac{1}{2}(p+2q)^2 + \frac{p+2q}{2} \leq \frac{n^2+n}{2}$. The lemma is proved. □

**Lemma 3.3.2.** *Let $\mathcal{A}$ of the type $M_{k,l}(F)^{(+)}$ where $k, l > 0$ be a proper subsuperalgebra of $\mathcal{J}$ of the type either $P(n)$ or $Q(n)^{(+)}$. Then* $\dim \mathcal{A} \leq n^2$.

*Proof.* Since $\mathcal{A}$ is proper, $k + l \leq n$, hence $(k + l)^2 \leq n^2$. $\square$

**Lemma 3.3.3.** *A superalgebra $\mathcal{J}$ of the type either $P(n)$ or $Q(n)^{(+)}$, $n > 1$, cannot be represented as the sum of two proper nontrivial subsuperalgebras one of which has either the type $K_3$ or $D_t$.*

*Proof.* Let $\mathcal{A}$ be of the type either $K_3$ or $D_t$. The given decomposition of $\mathcal{J}$ induces that of the form: $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$ where either $\mathcal{A}_0 = Fe$ or $\mathcal{A}_0 = Fe_1 \oplus Fe_2$, where $e$, $e_1$ and $e_2$ are idempotents. Next we estimate the dimension of $\mathcal{B}_0$. If $\mathcal{B}_0$ is simple, then $\dim \mathcal{B}_0 \leq n^2 - 2n + 1$. If $\mathcal{B}_0$ is non-simple semisimple, then $\dim \mathcal{B}_0 \leq n^2 - 2n + 2$. As a result, $\dim \mathcal{J}_0 = n^2 \leq 2 + n^2 - 2n + 2$, so $n \leq 2$. The only case that remains to be proven is the case $n = 2$. By Corollary 3.1.5, these decompositions are not possible. The lemma is proved. $\square$

**Lemma 3.3.4.** *Let $\mathcal{J}$ of the type either $P(n)$ or $Q(n)^{(+)}$ be represented as the sum of two proper non-trivial subsuperalgebras $\mathcal{A}$ and $\mathcal{B}$ whose even components are semisimple Jordan algebras such that $\langle \mathcal{A}_0 \rangle$, $\langle \mathcal{B}_0 \rangle$ are proper, and one of them has a non-zero two-sided annihilator. Then $\mathcal{J} \neq \mathcal{A} + \mathcal{B}$.*

*Proof.* Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$, and $\mathcal{A}_0$ have a non-zero two-sided annihilator. Then $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$ where $\mathcal{A}_0$, $\mathcal{B}_0$ are semisimple Jordan subalgebras, $\operatorname{Ann} \mathcal{A}_0 \neq \{0\}$. Since $\mathcal{J}_0 \cong F_n^{(+)}$, $S(\mathcal{J}_0) = F_n$. Obviously, $F_n = \langle \mathcal{A}_0 \rangle + \langle \mathcal{B}_0 \rangle$ where $\langle \mathcal{A}_0 \rangle$ and $\langle \mathcal{B}_0 \rangle$ denote associative enveloping algebras for $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively. This implies that $F_n$ can be written as the sum of two semisimple subalgebras $\langle \mathcal{A}_0 \rangle$ and $\langle \mathcal{B}_0 \rangle$ one of which has a non-zero two-sided annihilator. This contradicts Proposition 1 in [4]. The lemma is proved. $\square$

The following table summarizes all the information obtained above.

| | $\mathcal{A}$ | $Max\ dim$ |
|---|---|---|
| 1 | $M_{k,l}(F)^{(+)}$ | $n^2$ |
| 2 | $osp(p,q)$ | $\frac{n^2+n}{2}$ |
| 3 | $Q(k)^{(+)}$ | $2(n-1)^2$ |
| 4 | $P(k)$ | $2(n-1)^2$ |

In the second column we list all possible types that subsuperalgebras of $P(n)$ and $Q(n)^{(+)}$ can have. In the third column we point out the maximal dimension corresponding to each subsuperalgera.

**Theorem 3.3.5.** *Let $\mathcal{J}$ have the type either $Q(n)^{(+)}$ or $P(n)$, where $n > 1$. Then $\mathcal{J}$ cannot be represented as the sum of two proper simple non-trivial subsuperalgebras.*

*Proof.* Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$. Then the following cases occur.

**Case 1.** $\mathcal{A} \cong M_{k,l}(F)^{(+)}$, $\mathcal{B} \cong M_{s,t}(F)^{(+)}$. By Lemma 3.3.2, dim $\mathcal{A} \leq n^2$, dim $\mathcal{B} \leq n^2$, therefore, $\mathcal{J} = \mathcal{A} \oplus \mathcal{B}$. In particular, $\mathcal{J}_0 = \mathcal{A}_0 \oplus \mathcal{B}_0$. As a consequence, one of the subalgebras, for example $\mathcal{A}_0$, does not contain the identity of the whole superalgebra or, equivalently,$\langle \mathcal{A}_0 \rangle$ has a non-trivial two-sided annihilator. By Lemma 3.3.4, no such decomposition exists.

**Case 2.** $\mathcal{A} \cong osp(p,q)$, $\mathcal{B} \cong osp(k,l)$, $M_{k,l}(F)^{(+)}$, $Q(k)^{(+)}$ or $P(k)$. Taking into account Lemma 3.3.1, we can conclude that no decomposition into the sum of two subsuperalgebras of the type *osp* is possible. Assume that $\mathcal{B} \cong M_{k,l}(F)^{(+)}$. By Lemmas 3.3.1 and 3.3.2, dim $\mathcal{A} \leq \frac{n^2+n}{2}$ and dim $\mathcal{B} \leq n^2$, respectively. Hence, by the dimension argument, no such decompositions exist. Finally, if $\mathcal{B} \cong P(k)$ or $Q(k)^{(+)}$, then $\mathcal{B}_0 \cong F_k^{(+)}$. By Lemma 3.3.4, $1 \in \mathcal{B}_0$. By Lemma 3.0.5, dim $\mathcal{B}_0 \leq \frac{n^2}{4}$. However, $\dim(\mathcal{A}+\mathcal{B}) \leq \frac{n^2+n}{2} + \frac{n^2}{2} < 2n^2$, a contradiction.

**Case 3.** $\mathcal{A} \cong M_{k,l}(F)^{(+)}$, $\mathcal{B} \cong P(q)$, or $Q(q)^{(+)}$. This decomposition induces $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$, $\mathcal{A}_0 \cong F_k^{(+)} \oplus F_l^{(+)}$, $\mathcal{B}_0 \cong F_q^{(+)}$. By Lemma 3.3.4, $1 \in \mathcal{B}_0$. By Lemma 3.0.5, dim $\mathcal{B}_0 \leq \frac{n^2}{4}$. However, $\dim(\mathcal{A}+\mathcal{B}) \leq \frac{n^2+n}{2} + \frac{n^2}{2} < 2n^2$, a contradiction.

**Case 4.** $\mathcal{A} \cong P(k)$, $\mathcal{B} \cong Q(l)^{(+)}$, $k, l < n$. As above, this decomposition induces the decomposition of the even part into the sum of two subalgebras of the types $F_k^{(+)}$ and $F_l^{(+)}$, respectively. However it follows from the classification of simple decompositions of simple Jordan algebras [32] that no such decomposition exists. The theorem is proved. □

## 3.4 Decompositions of superalgebras of the types $J(V,f)$, $K_3$, $D_t$, $H_3(F) \oplus S_3(F) \oplus \bar{S}_3(F)$, $K_{10}$, $K_9$, $H_3(B)$

**Theorem 3.4.1.** *Let $\mathcal{J} = (F1 + V_0) + V_1$ where $V_1 \neq \{0\}$, and $\mathcal{A}$, $\mathcal{B}$ be proper simple non-trivial subsuperalgebras of $\mathcal{J}$. Then $\mathcal{J} = \mathcal{A} + \mathcal{B}$ if and only if one of the following cases holds:*

*(1) $\mathcal{A} = (F1+W_0)+W_1$, $\mathcal{B} = (F1+M_0)+M_1$ where $V_0 = W_0+M_0$, $V_1 = W_1+M_1$, $f|_{W_0}$, $f|_{W_1}$, $f|_{M_0}$, $f|_{M_1}$ are non-singular.*

*(2) $\mathcal{A} = (F1 + W_0) + W_1$, $\mathcal{B} = \langle \frac{1}{2} + v\rangle + M_1$ where $F1 + V_0 = W_0 \oplus \langle \frac{1}{2} + v\rangle$,*

*$V_1 = W_1 + M_1$, $f|_{W_0}$, $f|_{W_1}$, $f|_{M_1}$ are non-singular, $f(v,v) = \frac{1}{4}$.*

*(3) $\mathcal{A} = \langle\frac{1}{2}+v\rangle + W_1$, $\mathcal{B} = \langle\frac{1}{2}-v\rangle + M_1$, $F1+V_0 = \langle\frac{1}{2}+v\rangle + \langle\frac{1}{2}-v\rangle$, $V_1 = W_1 + M_1$, $f|_{W_1}$, $f|_{M_1}$ are non-singular, $f(v,v) = \frac{1}{4}$.*

*Proof.* Given that $\mathcal{J} = (F1 + V_0) + V_1$ where $\mathcal{J}_0 = F1 + V_0$, $\mathcal{J}_1 = V_1$. Notice that $\mathcal{J}_1 \cdot \mathcal{J}_1 = F1$, where 1 denotes the identity of $\mathcal{J}$. In particular, for any subsuperalgebra $\mathcal{A}$ of $\mathcal{J}$, $\mathcal{A}_1 \cdot \mathcal{A}_1 \subseteq F1$. Note that the idempotents in $\mathcal{J}_0$ have the form: either 1 or $\frac{1}{2} + v$ where $f(v,v) = \frac{1}{4}$, $v \in V_0$. In particular, if $v_1$ and $v_2$ are pairwise orthogonal idempotents in $\mathcal{J}_0$, then $e_1 = \frac{1}{2} + v$, $e_2 = \frac{1}{2} - v$ where $v \in V_0$. For any simple subsuperalgebra $\mathcal{A}$ of $\mathcal{J}$, $\mathcal{A}_0$ is a semisimple subalgebra of $\mathcal{J}_0 = F1 + V_0$. Hence, from Racine-Zelmanov classification, either $\mathcal{A}_0 = J(V', f')$ or $\mathcal{A}_0 = J(V', f') \oplus Fe$ where $e$ is an idempotent of $\mathcal{J}_0$. Since $J(V', f')$ and $Fe$ are orthogonal, $e = \frac{1}{2} + v$, $v \in V_0$, and for any $\alpha + w \in J(V', f')$: $(\frac{1}{2} + v) \cdot (\alpha + w) = 0$, $w = -2\alpha v$. As a consequence, dim $J(V', f') = 1$ and $\mathcal{A}_0 = Fe_1 \oplus Fe_2$ where $e_1$, $e_2$ are pairwise orthogonal idempotents. On the other hand, according to Racine-Zelmanov classification, if $\mathcal{A}_0 \cong J(V', f') \oplus Fe$, then $\mathcal{A}$ is isomorphic to one of the following: $osp_{2,2}(F)$, $M_{2,1}(F)^{(+)}$, $K_{10}$, $osp_{1,4}(F)$. However, for any of these superalgebras the dimension of the even part is greater than 2. Let $\mathcal{A}_0 = J(V', f')$. It follows from Racine-Zelmanov classification that

$(i)$ $\mathcal{A}$ is a superalgebra of a bilinear superform.

$(ii)$ $\mathcal{A} \cong Q(2)$.

$(iii)$ $\mathcal{A} \cong P(2)$.

However, for $(ii)$ and $(iii)$ the inclusion $\mathcal{A}_1 \cdot \mathcal{A}_1 \subseteq F1$ does not hold. Hence, if $\mathcal{A}$ is a simple subsuperalgebra of $\mathcal{J}$, then $\mathcal{A}$ has the type $J(V', f')$. Further, if $1 \in \mathcal{A}$, $1 \in \mathcal{B}$, then $\mathcal{A} = F1 + W_0 + W_1$, $\mathcal{B} = F1 + M_0 + M_1$. Hence, $V_0 = W_0 + M_0$, $V_1 = W_1 + M_1$. If $1 \in \mathcal{A}$, $1 \notin \mathcal{B}$, then $\mathcal{A} = F1 + W_0 + W_1$, $\mathcal{B} = \langle \frac{1}{2} + v \rangle + M_1$. If $1 \notin \mathcal{A}$, $1 \notin \mathcal{B}$, then $\mathcal{A}_0 = \langle \frac{1}{2} + v \rangle$, $\mathcal{B}_0 = \langle \frac{1}{2} - v \rangle$ [33]. The fact that (1),(2),(3) are decompositions is obvious. The theorem is proved. $\square$

### Decompositions of $K_3$

Let $\mathcal{A}$ be a proper subsuperalgebra of $K_3$. Then we have the following restrictions: $\dim \mathcal{A} < 3$ and $\deg \mathcal{A}_0 = 1$. Considering all cases one after another, we obtain the following.

**Theorem 3.4.2.** *A Jordan superalgebra of the type $K_3$ has no decompositions into the sum of two proper simple non-trivial subsuperalgebras.*

### Decompositions of $D_t$

Let $\mathcal{J}$ be of the type $D_t$. Since $\deg \mathcal{J} = 2$, $\dim \mathcal{J} = 4$, any proper simple subsuperalgebra of $\mathcal{J}$ has either the type $K_3$ or $J(V, f)$. Let $\mathcal{A} \cong K_3$ be a subsuperalgebra of $\mathcal{J}$. Then $\mathcal{A}_1 = (\mathcal{J})_1$, and $\mathcal{A}_0 = \langle e \rangle$ where $e^2 = e$, $e \in (\mathcal{J})_0$. If $(D_t)_0 = Fe_1 + Fe_2$, then either $e = e_i$, $i = 1, 2$, or $e = e_1 + e_2$. In the last case, we have $e(\alpha x + \beta y) = (e_1 + e_2)(\alpha x + \beta y) = (\alpha x + \beta y) \neq \frac{(\alpha x + \beta y)}{2}$. Hence $e = e_i$, $i = 1, 2$. On the other hand, $[(\alpha x + \beta y), (\alpha' x + \beta' y)] = (\alpha\beta' - \beta\alpha')(e_1 + te_2) \neq e = e_i$, $i = 1, 2$. This implies that $\mathcal{A}$ of the type $K_3$ cannot be a subsuperalgebra of $\mathcal{J}$.

Let $\mathcal{A} \cong J(V, f)$ be a subsuperalgebra of $\mathcal{J}$. Then $\mathcal{A}_0 \subseteq (\mathcal{J})_0 = \langle e_1, e_2 \rangle$, $\mathcal{A}_1 \subseteq (\mathcal{J})_1 = \langle x, y \rangle$. Let $e$ be the identity of $\mathcal{A}$, $e \in \mathcal{J}_0$, $e = e_1 + e_2$. If $\dim \mathcal{A}_0 > 1$, then we can always choose some element of the form $\alpha e_1 + \beta e_2$ which is linearly independent with $e_1 + e_2$, and $(\alpha e_1 + \beta e_2)^2$ is proportionate to $e_1 + e_2$. This implies that $\alpha = \beta$ and $\dim \mathcal{A}_0 = 1$. However if $\mathcal{J} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \cong J(V_1, f_1)$, $\mathcal{B} \cong J(V_2, f_2)$, then $\mathcal{A}_0 = \mathcal{B}_0 = \langle e_1 + e_2 \rangle$, that is, $(\mathcal{J})_0 \neq \mathcal{A}_0 + \mathcal{B}_0$.

**Theorem 3.4.3.** *A Jordan superalgebra of the type $D_t$ has no simple decompositions into the sum of two proper simple non-trivial subsuperalgebras.*

### Decompositions of $H_3(F) \oplus S_3(F) \oplus \bar{S}_3(F)$

Actually we are going to prove that there are no simple decompositions of $\mathcal{J}$ of the type $H_3(F) \oplus S_3(F) \oplus \bar{S}_3(F)$. Assume the contrary that $\mathcal{J} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are proper simple subsuperalgebras. Then, $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$ where $\mathcal{J}_0 \cong H(F_3)$. It follows from Theorem 3.0.7 that either $\mathcal{A}_0$ (or $\mathcal{B}_0$) coincides with $\mathcal{J}_0$ or $\mathcal{A}_0$ (or $\mathcal{B}_0$) is non-simple semisimple. According to Racine-Zelmanov classification if $\mathcal{A}_0$ (or $\mathcal{B}_0$) is $\mathcal{J}_0$ than $\mathcal{A}$ (or $\mathcal{B}$) is $\mathcal{J}$, a contradiction. Let $\mathcal{A}$ be non-simple semisimple. Then either $\mathcal{A}_0 \cong F_k \oplus F_l$, $k + l \leq 3$ or $\mathcal{A}_0 \cong H(F_k) \oplus H(\mathcal{Q}_l)$, $k + 2l \leq 3$. If $\mathcal{A}_0 \cong F_k \oplus F_l$, then by Lemma 3.0.5 $k + l \leq \frac{3}{2}$, so $k + l = 1$. If $\mathcal{A}_0 \cong H(F_k) \oplus H(\mathcal{Q}_l)$, then $k = 1$, $l = 1$. By dimension argument no such decomposition exists.

**Theorem 3.4.4.** *A Jordan superalgebra of the type $H_3(F) \oplus S_3(F) \ominus \bar{S}_3(F)$ has no simple decompositions into the sum of two proper simple non-trivial subsuperalgebras.*

### Decompositions of $K_{10}$ and $K_9$

If characteristic of $F$ is 3, then $K_{10}$ is not simple and possesses a simple subalgebra $K_9$. The even part of $K_9$ is of the type $J(V, f)$, and the odd part of $K_9$ is the same as the odd part of $K_{10}$. Since only these two properties will be primarily used, we consider only the case of $K_{10}$. Notice that the identity $[w, w] = 0$ holds for any $w \in \mathcal{J}_1$. On the other hand, for $v \cdot v = f(v, v)e'$, $v \in \mathcal{A}_1$. Since $f|_{\mathcal{A}_1}$ is non-singular, there exists $v' \in \mathcal{A}_1$ such that $f(v', v') \neq 0$. However, it conflicts with the above identity. Therefore, there are no subsuperalgebras of the type $J(V, f)$ in $\mathcal{J}$.

Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$ where $\mathcal{J} \cong K_{10}$. Then $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$. Recall that $\mathcal{J}_0 = I_1 \oplus I_2$ where $I_1 = Fe + \sum_{1 \leq i \leq 4} Fv_i$, $I_2 = Ff$. Hence, we can introduce the projection $\pi_1$ onto $I_1$. This projection induces the simple decompositions of $I_1$, $I_1 = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0)$.

Since $I_1$ is an algebra of a bilinear form, it follows from [32] that it can be decomposed only into the sum of simple subalgebras of bilinear forms. Hence, $\pi_1(\mathcal{A}_0) \cong J(V_1, f_1)$ and $\pi_1(\mathcal{B}_0) \cong J(V_2, f_2)$. According to Racine-Zelmanov classification there are only the following possibilities for $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A}, \mathcal{B} \cong osp_{2,2}(F)$, $M_{2,1}(F)$, $osp_{1,4}(F)$, $P(2)$, $Q(2)$.

| dimension | even part | odd part | total |
|---|---|---|---|
| $osp_{2,2}(F)$ | 4 | 4 | 8 |
| $M_{2,1}(F)$ | 5 | 4 | 9 |
| $P(2)$ | 4 | 4 | 8 |
| $Q(2)$ | 4 | 4 | 8 |
| $osp_{1,4}(F)$ | 7 | 4 | 11 |

The fifth case is obviously not possible since $\dim osp_{1,4}(F) > \dim \mathcal{J}$. In the first four cases, $\dim (osp_{2,2}(F))_1 = \dim (M_{2,1}(F))_1 = \dim (P(2))_1 = \dim (Q(2))_1 =$

dim $\mathcal{J}_1$. From the multiplication table for $\mathcal{J}$ we can choose a basis $\{x_1, y_1, x_2, y_2\}$ of $\mathcal{J}_1$ such that $[x_i, y_i] = e - 3f$, $[x_1, x_2] = v_1$, $[x_1, y_2] = v_3$, $[x_2, y_1] = v_4$, $[y_1, y_2] = v_2$. Hence $\mathcal{A}_0$ (or $\mathcal{B}_0$) contains $e - 3f$, $v_1$, $v_2$, $v_3$, $v_4$. Besides, $v_1 \cdot v_2 = 2e$. Since $[\mathcal{J}_1, \mathcal{J}_1] = \mathcal{J}_0]$, $\mathcal{A}_0 = \mathcal{J}_0$, hence, $\mathcal{A} = \mathcal{J}$, a contradiction.

**Theorem 3.4.5.** *A Jordan superalgebra of the type $K_{10}$ has no simple decompositions into the sum of two proper simple non-trivial subsuperalgebras.*

**Decompositions of $H_3(B)$**

Let $\mathcal{J}$ be of the type $H_3(B)$. To find simple decompositions of $\mathcal{J}$ we notice that $\mathcal{J}_0 \cong H(\mathcal{Q}_3)$ and dim $\mathcal{J} = 21$. Let $\mathcal{J} = \mathcal{A} + \mathcal{B}$. Hence $\mathcal{J}_0 = \mathcal{A}_0 + \mathcal{B}_0$. It is known that $H(\mathcal{Q}_3)$ can be decomposed only into the sum of subalgebras of the type $F_3^{(+)}$ [32]) Notice that if $\mathcal{A}_0$ (or $\mathcal{B}_0$) coincides with $\mathcal{J}_0$, then $\mathcal{A}$ (or $\mathcal{B}$) coincides with $\mathcal{J}$. It follows from [32] that either $\mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic to $F_3^{(+)}$ or one of them is non-simple semisimple. If $\mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic to $F_3^{(+)}$, than it follows from Racine-Zelmanov classification that $\mathcal{A}$ and $\mathcal{B}$ have the type either $P(3)$ or $Q(3)$. However dim $(P(3))_1 =$ dim $(Q(3))_1 = 9 >$ dim $H_3(B)_1 = 6$. Hence, a subsuperalgebra of the type $P(3)$ or $Q(3)$ cannot be imbedded into $H_3(B)$. If one of subalgebras, for example $\mathcal{A}_0$ is non-simple semisimple, than either $\mathcal{A}_0 \cong F_k^{(+)} \oplus F_l^{(+)}$, $k + l \leq 3$ or $\mathcal{A}_0 \cong H(F_k) \oplus H(\mathcal{Q}_l)$, $k + l \leq 3$. Again by dimension argument no such decomposition exists.

**Theorem 3.4.6.** *A Jordan superalgebra of the type $H_3(B)$ has no simple decompositions into the sum of two proper simple non-trivial subsuperalgebras.*

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