Mean field evolution in an open quantum system

by

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Abstract

In this thesis, we consider N quantum particles coupled to collective thermal quantum environments. The coupling is energy conserving and scaled in the mean field way. There is no direct interaction between the particles, they only interact via the common reservoir. It is well known that an initially disentangled state of the N particles will remain disentangled at all times in the limit $N \to \infty$. In this thesis, we evaluate the n-body reduced density matrix (tracing over the reservoirs and the $N - n$ remaining particles). We identify the main, disentangled part of the reduced density matrix and obtain the first order correction term in $1/N$. We show that this correction term is entangled. We also estimate the speed of convergence of the reduced density matrix as $N \to \infty$. Our model is exactly solvable and it is not based on numerical approximation.

To my parents

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Chapter 1

Introduction

One of the important problems in quantum theory is the study of the effect of *noise* on quantum systems. There are many situations which have this behaviour. For instance, quantum information processing is based on manipulation of superposition and entanglement of basic quantum bits forming a quantum processor. The effect of noise can be due to interaction with thermal environment or another open system. This interaction will destroy phase coherence (decoherence) and quantum correlation (entanglement) [1–4]. To study this behaviour we need to model the system mathematically. In general, a collection of particles can interact directly or indirectly via common reservoirs. In direct interaction, there are mathematical difficulties to solve the problem exactly or numerically complicated models involving many particles. In indirect interaction, one can start with a system with disentangled states and create and control entanglement by coupling the system with a common thermal noise [1–4]. In this work we will consider indirect interaction.

Before studying a system with many particles interacting indirectly, first one should study the entanglement of two subsystems interacting indirectly via a common reservoir (quantum thermal noise)[4]. In [4], the authors obtain expressions for the characteristic time-scales for decoherence, relaxation, disentanglement, and for the evolution of observables, valid uniformly in time $t \geq 0$. In [5] the authors consider an open quantum system of N not directly interacting spins (qubits) in contact with both local and collective thermal environments and it is shown numerically that creation of two-spin entanglement is suppressed if N is large. Obviously, to study this problem analytically and solve it exactly we need some other restrictions and physical assumptions. Since the size of interaction energy will be proportional to the number of pairs of particles [6], for balanced competition between individual energy and interaction energy one can choose an appropriate scaling for interaction with negative power of N. In [6] it is shown the only scaling with non-trivial dynamics for large N is the so-called *mean field scaling.*

Note that the mean field models (theory) are very useful mathematical tools to study complicated systems. This approximation theory allows us to reduce a complex problem to a one particle problem. Mean field methods are deterministic methods, making use of tools such as Taylor expansions. They have many application in physics and information theory such as phase transition in Ising model. One of the main applications of mean field model is the study of N-body quantum system. In the classic paper of [7] mean field models of directly interacting systems have been studied. It has been proved that an entangled system will be disentangled after time evolution when we increase the number of particles. The evolution of the state of single particle turns out to be nonlinear in the state, the nonlinear Hartree equation.

In [6] the authors studied a large number N of particles interaction via thermal noise. They proved that common noise cannot create entanglement in the N particle system if N is large enough. Moreover, it is shown that dynamics of the system is very close to a product state and the difference is of order $1/N$. This behaviour has an explicit speed of convergence($N \to \infty$). This problem is very interesting since it is exactly solvable without any extra assumptions such as weak coupling or any Markovian approximations.

In reality, one does not have $N = \infty$, but only N very large, so it is important to know the corrections to the case $N = \infty$ for large but finite N only. In this thesis, we study this problem. We consider an open system of N particles interacting indirectly via a common thermal reservoir. Moreover, the interaction is energy preserving. To model the system and solve it exactly we follow these steps: First, we select appropriate Hilbert spaces for particles and Fock spaces for reservoirs. Second, we construct the Hamiltonian of the system as sum of the Hamiltonian of each particles and a mean-field scaled interaction Hamiltonian. Third, we find explicit expression for reduced n-body density matrix of the system. Then we calculate the leading order correction of the n body reduced density matrix (the one proportional to $1/N$). We show that the first order term ($\propto N^{-1}$) of the density matrix is entangled, while the one $\propto N^0$ is factorized. Moreover, we find the explicit expression for the entangled first order term and we compute the speed of this behaviour exactly.

This thesis is organized as follows. In Chapter 2 we review mathematical preliminaries and some basic concepts such as Hilbert spaces, linear operators, their spectra, and tensor products of Hilbert spaces and Fock spaces. Moreover, we continue with the concepts of dynamics of open and closed quantum systems, entanglement, and von Neumann entropy. In Chapter 3 we present the main structure of our model and give the main results. Finally, the proof of the main results are given in Chapter 4.

Chapter 2

Basic mathematical background

In different parts within this work, we will need some definitions and properties of the foundation of our study.

2.1 Hilbert Space

In this section we will define some basic definitions and theorems about Hilbert spaces [8–10].

Definition 1. *Let X be a vector space over either the scalar field $\mathbb{R}$ of real numbers or the scalar field $\mathbb{C}$ of complex numbers. Suppose we have a function $\|\cdot\| : X \to [0, \infty)$ such that*

- $\|x\| = 0$ *if and only if* $x = 0$ *for any* $x \in X$*,*
- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$*, and*
- $\|\alpha x\| = |\alpha| \, \|x\|$ *for all scalars* $\alpha \in \mathbb{R}$ *or* $\mathbb{C}$ *and vectors* $x \in X$*.*

We call $(X, \|\cdot\|)$ a normed linear space.

Example 2.1. *Let $X = \mathbb{C}^n = \{(z_1, z_2, \cdots, z_n) : z_j \in \mathbb{C}\}$ with $\|(z_1, z_2, \cdots, z_n)\| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}}$;*

this is called **Euclidean norm**.

Example 2.2. *The space $X = \mathbb{C}^n$ with $\|(z_1, z_2, \cdots, z_n)\| = \max\{|z_j| : 1 \leq j \leq n\}$ is a norm linear space.*

Definition 2. *A sequence $\{x_n\}$ in X is said to be a* Cauchy *sequence if it has the following property: Given any $\epsilon > 0$ there exists N such that if $n, m \geq N$, then $\|x_n - x_m\| < \epsilon$.*

Definition 3. *A space is said to be* complete *if every Cauchy sequence in X converges in X.*

Definition 4. *A Banach space X is a complete linear space with a norm $\|\cdot\|$.*

Example 2.3. *A basic example is the n-dimensional Euclidean space with the Euclidean norm. Usually, the notion of Banach space is only used in the infinite dimensional setting, typically as a vector space of functions. For example, the set of continuous functions on closed interval $I = [a, b]$ of the real line with the norm of a function f given by*

$$\|f\| = \sup_{x \in I} |f(x)| \tag{2.1}$$

is a Banach space.

Definition 5. *Let X be a vector space over $\mathbb{C}$. An inner product is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfying, for x, y and z in X and scalars $\alpha \in \mathbb{C}$*

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x, x \rangle \geq 0$, *with* $\langle x, x \rangle = 0$ *if and only if* $x = 0$,
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$.

An inner product on $L^2(X, \mu)$ is

$$\langle f, g \rangle = \int_X \bar{f} g d\mu. \tag{2.2}$$

This general framework includes, as special cases, the example $\mathbb{C}^n$ with

$$\langle (z_1, z_2, \cdots, z_n), (w_1, w_2, \cdots, w_n) \rangle = \sum_{j=1}^{n} \bar{z}_j w_j. \tag{2.3}$$

Proposition 1. *If $\langle \cdot, \cdot \rangle$ is an inner product on X then $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ is a norm on X.*

Definition 6. *A* **Hilbert space** *$\mathcal{H}$ is a vector space over $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{H}$ is complete in the norm obtained from the inner product*

$$\|\psi\|^2 = \langle \psi, \psi \rangle, \qquad \psi \in \mathcal{H}. \tag{2.4}$$

Any space $L^2(X, \mu)$ as described above is thus an example of a Hilbert space, since we have already observed that $L^2(X, \mu)$ is a Banach space under the norm $\|f\| = \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}}$ which we recognize as $\langle f, f \rangle^{\frac{1}{2}}$.

In quantum theory a normalized vector in a Hilbert space called *ket* and denoted by $|\psi\rangle$. Moreover, the dual element associated to $|\psi\rangle$ is called the bra $\langle\psi| = |\psi\rangle^*$, it is the linear functional on $\mathcal{H}$ defined by

$$\langle\psi|\,(\varphi) := \langle\psi, \varphi\rangle\,, \quad \forall\varphi \in \mathcal{H}. \tag{2.5}$$

Definition 7. *Given vectors f, g in a Hilbert space $\mathcal{H}$, we say that f is orthogonal to g, written $f \perp g$, if $\langle f, g\rangle = 0$. For $M \subseteq \mathcal{H}$, we define the orthogonal complement by $M^\perp = \{f \in \mathcal{H} \,:\, \langle f, g\rangle = 0$ for all $g \in M\}$.*

Theorem 1 (**Projection Theorem**)**.** *: Let M be a closed subspace of a Hilbert space $\mathcal{H}$. There is a unique pair of mappings $P : \mathcal{H} \to M$ and $Q : \mathcal{H} \to M^\perp$ such that $x = Px+Qx$ for all $x \in \mathcal{H}$. Furthermore, P and Q have the following additional properties:*

- *$x \in M \Rightarrow Px = x$ and $Qx = 0$.*
- *$x \in M^\perp \Rightarrow Px = 0$ and $Qx = x$.*
- *Px is the closest vector in M to x.*
- *Qx is the closest vector in $M^\perp$ to x.*
- *$\|Px\|^2 + \|Qx\|^2 = \|x\|^2$ for all x.*
- *P and Q are linear maps.*

Definition 8. *If X is a normed linear space over $\mathbb{C}$, then a linear functional on X is a map $\Lambda : X \to \mathbb{C}$ satisfying $\Lambda(\alpha x + \beta y) = \alpha\Lambda(x) + \beta\Lambda(y)$ for all vectors $x, y \in X$ and all scalars $\alpha, \beta \in \mathbb{C}$.*

Theorem 2 (**Riesz representation Theorem**)**.** *: Every bounded linear functional Λ on a Hilbert space $\mathcal{H}$ is given by taking the inner product with a (unique) fixed vector $h_0 \in \mathcal{H} : \Lambda(h) = \langle h_0, h\rangle$. Moreover, the norm of the linear functional Λ is $\|h_0\|$.*

Definition 9. *An orthonormal set in a Hilbert space $\mathcal{H}$ is a set S with the properties:*

- *for every $e \in S$, $\|e\| = 1$, and*
- *for distinct vectors e and f in S, $\langle e, f\rangle = 0$.*

For an easy example of an orthonormal set in the Hilbert space $L^2([0, 2\pi])$, with respect to normalized Lebesgue measure $dt/(2\pi)$, consider the collection of functions e^{int} for any integer n form an orthonormal set.

Definition 10. *An orthonormal basis B for a Hilbert space $\mathcal{H}$ is a maximal orthonormal set; that is, an orthonormal set that is not properly contained in any orthonormal set and every $x \in \mathcal{H}$ can be written as a linear combination of the elements of B.*

It is easy to see that the set $\{e^{int} : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 2\pi], dt/2\pi)$.

Definition 11. *Let X and Y be normed linear spaces, a map $T : X \to Y$ is linear if*

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 \tag{2.6}$$

for all $x_1, x_2 \in X$ and scalars $\alpha, \beta \in \mathbb{C}$. We say the linear map T is a bounded linear operator from X to Y if there is a finite constant C such that $\|Tx\|_Y \leq C \|x\|_X$ for all $x \in X$.

Proposition 2. *If $T : X \to Y$ is a linear map from a normed linear space X to a normed linear space Y, the following statements are equivalent:*

- *T is bounded.*
- *T is continuous.*
- *T is continuous at 0.*

Theorem 3. *Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a bounded linear map $A : \mathcal{H} \to \mathcal{K}$ there is a unique $A^* : \mathcal{K} \to \mathcal{H}$ such that*

$$\langle Ah, k \rangle = \langle h, A^* k \rangle \tag{2.7}$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

The operator A^* called the (Hilbert space) adjoint of A. In the case that $\mathcal{H} = \mathcal{K}$ and $A^* = A$ we say that A is *self-adjoint* or *Hermitian*.

Proposition 3. *For A and $B : \mathcal{H} \to \mathcal{K}$ we have*

- *$A^{**} = A$ where $A^{**} = (A^*)^*$.*
- *$(A + B)^* = A^* + B^*$.*
- *$(\alpha A)^* = \bar{\alpha} A^*$ for $\alpha \in \mathbb{C}$.*
- *$(AB)^* = B^* A^*$.*
- *$\|A\| = \|A^*\|$ and $\|A^* A\| = \|A\|^2$.*

Definition 12. *If $T : X \to X$ is a bounded linear operator on a Banach space X, then the set of complex numbers λ for which $T - \lambda I$ is not invertible is called the spectrum of T.*

We will denote the spectrum of T by σ(T). The collection of the eigenvalues of T is called the point spectrum of T; we denote it σ_p(T). If λ is in σ(T) but is not an eigenvalue, then the range of T $-\lambda I$ is a proper subset of a Hilbert space $\mathcal{H}$, and this can happen in two different ways: Either the range of T $-\lambda I$ is a proper, but dense, subset of $\mathcal{H}$, or the closure of the range of T $-\lambda I$ is a proper closed subspace of $\mathcal{H}$. This leads to a classification of $\sigma(\mathrm{T})\backslash\sigma_p(\mathrm{T})$ into two disjoint pieces: the *continuous spectrum*, where the range of T $-\lambda I$ is dense in, but not equal to, $\mathcal{H}$, and the *residual spectrum*, where the closure of the range of T $-\lambda I$ is a proper subset of $\mathcal{H}$. So now we have decomposed σ(T) into three disjoint pieces:

- point spectrum : $\{ \lambda : \mathrm{T} - \lambda$ is not one-to-one $\}$
- continuous spectrum : $\{ \lambda : \mathrm{T} - \lambda$ is one-to-one, $(\mathrm{T} - \lambda)\mathcal{H} \neq \mathcal{H}, \overline{(\mathrm{T} - \lambda)\mathcal{H}} = \mathcal{H}\}$
- residual spectrum : $\{ \lambda : \mathrm{T} - \lambda$ is one-to-one, $\overline{(\mathrm{T} - \lambda)\mathcal{H}} \neq \mathcal{H}\}$.

Definition 13. *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. For a fixed orthonormal basis $\{e_n\}$ of $\mathcal{H}$ we define trace of T by*

$$Tr(T) = \sum_{n\in\mathbb{N}} \langle e_n | T | e_n \rangle = \sum_{n\in\mathbb{N}} \langle e_n, Te_n \rangle \, . \tag{2.8}$$

Proposition 4. *The quantity $Tr(T)$ is independent of the choice of the orthonormal basis.*

Definition 14. *A bounded operator T on Hilbert space $\mathcal{H}$ is trace-class if*

$$Tr(T) < \infty \tag{2.9}$$

2.2 Tensor product of Hilbert space

Definition 15. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two (separable) Hilbert spaces with orthonormal bases $\{e_k\}_{k\geq 1}$ and $\{f_j\}_{j\geq 1}$. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is the separable Hilbert space with basis $\{e_k \otimes f_l\}_{k,l\geq 1}$. For $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$,*

$$\psi = \sum_{k,l=1}^{\infty} \alpha_{kl} e_k \otimes f_l \qquad \textit{where} \quad \alpha_{kl} = \langle e_k \otimes f_l, \psi \rangle \tag{2.10}$$

Note: $\langle e_k \otimes e_l, e_{k'} \otimes e_{l'} \rangle = \langle e_k \otimes e'_k \rangle \langle e_l \otimes e_{l'} \rangle = \delta_{kk}\delta_{ll'}$.

Theorem 4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (separable) Hilbert spaces. Let A and B be bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If A and B are trace-class operators then $A \otimes B$ is a trace-class operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, $Tr(A \otimes B) = Tr(A)\,Tr(B)$.*

Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces. The finite tensor product

$$\bigotimes_{1 \le i \le n} \mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \tag{2.11}$$

is well defined. But the infinite tensor product will be defined as follows.

Definition 16. *Let $\mathcal{H}^{\otimes^n}$ be a Tensor product of the Hilbert spaces, then $Sym(\mathcal{H}^{\otimes^n})$ is a Hilbert subspace of $\mathcal{H}^{\otimes^n}$ such that for all $T \in Sym(\mathcal{H}^{\otimes^n})$*

$$f_\sigma(T) = T \tag{2.12}$$

where σ is any permutation of $1, \cdots, n$ and

$$f_\sigma(l_1 \otimes \cdots \otimes l_n) = l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(n)}. \tag{2.13}$$

Definition 17. *Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces. Choose a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{H}_n$ and $\|u_n\| = 1$ for all $n \in \mathbb{N}$. This sequence is called a stabilizing sequence for $\bigotimes_{n\in\mathbb{N}} \mathcal{H}_n$. The space $\bigotimes_{n\in\mathbb{N}} \mathcal{H}_n$ is defined as the closure of the pre-Hilbert space of vectors of the form*

$$\bigotimes_{n\in\mathbb{N}} e_n \tag{2.14}$$

such that $e_n \in \mathcal{H}_n$ for all n and $e_n = u_n$ for all but a finite number of n. The scalar product space on the space being obviously defined by

$$\left\langle \bigotimes_{n\in\mathbb{N}} e_n, \bigotimes_{n\in\mathbb{N}} f_n \right\rangle = \prod_{n\in\mathbb{N}} \langle e_n, f_n \rangle\,. \tag{2.15}$$

Definition 18. *A (bosonic)* **Fock space** *is the direct sum of the symmetric tensor products of copies of a single-particle Hilbert space $\mathcal{H}$*

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n\ge 0} Sym\left(\mathcal{H}^{\otimes^n}\right) \tag{2.16}$$

Where $\mathcal{H}_0 = \mathbb{C}$. $\mathcal{F}(\mathcal{H})$ is called the Fock space over the Hilbert space $\mathcal{H}$. The Hilbert space $\mathcal{H}_n$ identified as a subspace of Fock space is called the n-sector (or the nth chaos, *in quantum probability). The zero-sector is also called the vacuum sector.*

Now we will define the following operators which we will use in the last chapter.

Definition 19. *Let R_j and L_j be the operators on $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ defined by*

$$R_j(W)X_1 \otimes \cdots \otimes X_n = X_1 \otimes \cdots \otimes X_{j-1} \otimes X_j W \otimes \cdots \otimes X_n \tag{2.17}$$

and

$$L_j(W)X_1 \otimes \cdots \otimes X_n = X_1 \otimes \cdots \otimes X_{j-1} \otimes W X_j \otimes \cdots \otimes X_n \tag{2.18}$$

Definition 20. *[11] Let H be an operator acting on $\mathcal{H}$. The second quantized operator $d\Gamma(H)$ acts on $\mathcal{F}$. Its action on the n-sector is defined by*

$$d\Gamma(H) = H \otimes \mathbb{1} \otimes \cdots \mathbb{1} + \mathbb{1} \otimes H \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H \tag{2.19}$$

2.3 Partial Trace

In this section we will define partial trace operator [12]

Definition 21. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $T = A_1 \otimes A_2 \in \mathcal{B}(\mathcal{H})$. The* **partial trace** *of T over $\mathcal{H}_2$ is given by*

$$Tr_2(T) := A_1 \cdot Tr_{\mathcal{H}_2}(A_2) \tag{2.20}$$

Tr_2 extends by linearity to a linear map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_1)$.

Example 2.4.

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad T = |0\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| \tag{2.21}$$

$$Tr_2(T) = |0\rangle\langle 1| \cdot 1 + |1\rangle\langle 0| \cdot 0 = |0\rangle\langle 1| \tag{2.22}$$

2.4 Density matrix

A **density matrix** ρ is a representation of a quantum state or a statistical ensemble of quantum states [13, 14]. A normalized vector in Hilbert space is called a **pure state**. The density matrix of a pure state $|\psi\rangle$ in Hilbert space $\mathbb{C}^2$ is given by the outer product of the state vector with itself:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}. \tag{2.23}$$

Where we have used the general two-dimensional state vector of $|\psi\rangle$. Density matrices can also describe **ensembles** of pure states $\{|\psi_i\rangle\}$ with probabilities $\{p_i\}$ by constructing a linear combination of pure states.

Definition 22. *Suppose $\{|\psi_i\rangle\}_i$ are (orthogonal) vectors in $\mathcal{H}$, and p_i are probabilities then the associated* **density matrix** *is*

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{2.24}$$

Here $0 \le p_i \le 1$ and $\sum_i p_i = 1$.

If $\rho = \frac{I}{n}$ where I is an identity matrix, then they are **maximally mixed**, without any quantum superpositions. We give some examples for density matrix.

Example 2.5. *Consider the mixed state $|0\rangle = (1,0)^t$ with probability of $1/4$ and $|1\rangle = (0,1)^t$ with probability $3/4$. Then*

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.25}$$

and

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.26}$$

Thus in this case

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}. \tag{2.27}$$

Example 2.6. *Now consider another mixed state, this time consisting of $|+\rangle = \frac{1}{\sqrt{2}}(1,1)^t$ with probability $1/2$ and $|-\rangle = \frac{1}{\sqrt{2}}(1,-1)^t$ with probability $1/2$. this time we have*

$$|+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.28}$$

and

$$|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.29}$$

Thus in this case the off-diagonals cancel, and we get

$$\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{2.30}$$

Note that the two density matrices we computed are identical.

2.4.1 Properties of density matrices

The definition of density matrix is (2.24). In an arbitrary orthonormal basis $\{|b_l\rangle\}$ each element of density matrix is

$$\rho_{kl} = \langle b_k|\rho|b_l\rangle = \sum_i p_i \langle b_k|\psi_i\rangle \langle\psi_i|b_l\rangle \,. \tag{2.31}$$

From this it follows that ρ has the following properties:

1. $\mathrm{Tr}\rho = 1$ (unit trace)
2. $\rho = \rho^*$ (Hermitian)
3. $\langle\psi|\rho|\psi\rangle \geq 0$ (positive definite)
4. $0 < \mathrm{Tr}(\rho^2) \leq \mathrm{Tr}(\rho) = 1$

Example 2.7. *The density operator of a pure state can be written* $\rho = |\psi\rangle\langle\psi|$*, so it's clear that* $\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$*, so* $Tr(\rho^2) = Tr(\rho) = 1$*. However, it can be shown that for a mixed state* $0 < Tr(\rho^2) < 1$*. We check this for the mixed state in example 2.5*

$$Tr(\rho^2) = Tr\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right) = Tr\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \frac{1}{2} \tag{2.32}$$

Definition 23. *Choosing an arbitrary orthonormal basis* $\{|b_l\rangle\}$ *and density matrix* $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$*, the expectation of an operator* T *with matrix elements* $T_{kl} = \langle b_k|T|b_l\rangle$ *is given by*

$$\langle T\rangle \equiv \sum_i p_i \langle\psi_i|T|\psi_i\rangle = \sum_{i,k,l} p_i \langle\psi_i|b_k\rangle\langle b_k|T|b_l\rangle\langle b_l|\psi_i\rangle = \sum_{k,l} \rho_{lk} T_{kl} = Tr(\rho T)\,. \tag{2.33}$$

The variance of T *in the state* ρ *is defined by*

$$var(T) \equiv \left\langle T^2\right\rangle - \langle T\rangle^2 = Tr\left(\rho T^2\right) - \{Tr(\rho T)\}^2\,. \tag{2.34}$$

Definition 24. *An observable is a self-adjoint operator having the following physical interpretation:*

- *Any physical quantity corresponds to an observable (e.g. the total energy is associated to the Hamiltonian).*
- *Eigenvalues are the possible outcomes of measurements of a given observable.*

2.5 Quantum system

Definition 25. *A* **closed quantum system** *is described by Hilbert space of (pure) states which does not interchange information with any other system with two cases. Pure states of the system are given by normalized vectors $\psi \in \mathcal{H}$, $\|\psi\| = 1$. Mixed states are density matrices $\rho \in B(\mathcal{H})$.*

Physics determines what the Hilbert space and Hamiltonian of a closed quantum system are, based on the physical system considered.

Example 2.8. *Assume there is a particle in a potential $V(x)$. Then the Hilbert space will be $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ with hamiltonian $H = -\Delta + V$ and $|\psi(t)\rangle \in \mathcal{H}$. Moreover, $|\psi(x,t)|^2$ is probability density of location of the particle.*

Example 2.9. *Assume an spin $\frac{1}{2}$ particle (e.g. a qubit). Then the Hilbert space will be $\mathcal{H} = \mathbb{C}^2$ with hamiltonian $H = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and states $|\psi_{up}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\psi_{down}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then every state vector can be written as linear combination of this basis*

$$|\psi(t)\rangle \in \mathcal{H} \qquad |\psi(t)\rangle = a(t)\,|\psi_{up}\rangle + b(t)\,|\psi_{down}\rangle$$

where $|a(t)|^2$ is probability of being in state "up" and $|b(t)|^2$ is probability of being in state "down".

2.5.1 Composite quantum system and reduced density matrices

Definition 26. *Let $(\mathcal{H}_S, H_S)$ and $(\mathcal{H}_B, H_B)$ be two quantum systems. The composite interacting system is $(\mathcal{H}, H)$ where*

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_{SB} \qquad \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$$

Now if we consider two orthonormal bases of states $\left\{|\psi_i\rangle^S\right\}_i$ and $\left\{|\psi_j\rangle^B\right\}_j$ in Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$ respectively, a general state in the tensor product space $\mathcal{H}$ may be written as

$$|\psi\rangle = \sum_{ij} \alpha_{ij}\, |\psi_i\rangle^S \otimes |\psi_j\rangle^B, \tag{2.35}$$

where α_{ij} are the coordinates of $|\psi\rangle$, with $\sum_{ij} |\alpha_{ij}|^2 = 1$. This means that $|\psi_i\rangle^S \otimes |\psi_j\rangle^B$ is an orthonormal basis for composite system $\mathcal{H}$. Now if A_S is an operator acting on $\mathcal{H}_S$ and A_B is an operator acting on $\mathcal{H}_B$, their tensor product is defined as

$$A_S \otimes A_B \left(|\psi_i\rangle^S \otimes |\psi_j\rangle^B\right) = \left(A_S\, |\psi_i\rangle^S\right) \otimes \left(A_B\, |\psi_j\rangle^B\right) \tag{2.36}$$

Then an operator on composite system $\mathcal{H}$ can be written as linear combination of tensor product of operators , i.e. $A = \sum_i {A_S}^i \otimes {A_B}^i$. A mixed state of the composite system is a density matrix ρ on $\mathcal{H}$. If ρ is of the form

$$\rho = \rho_S \otimes \rho_B \tag{2.37}$$

where $\rho_{S,B}$ are density matrices of the systems S and B, then ρ is called separable. For a separable state, expectation values of any tensor product of operators pertaining to the subsystems factorize, namely,

$$\langle A_S \otimes A_B \rangle = \mathrm{Tr}\left(A_S \otimes A_B(\rho)\right) = \mathrm{Tr}_S\left(A_S \rho_S\right) \cdot \mathrm{Tr}_B\left(A_B \rho_B\right) = \langle A_S \rangle \langle A_B \rangle \tag{2.38}$$

where Tr_S and Tr_B are the traces over Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively.

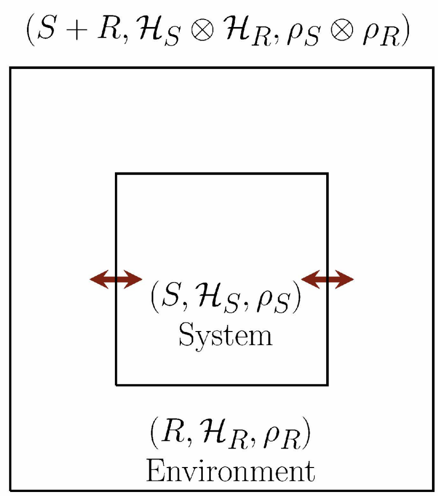

Figure 2.1: Open quantum system

Definition 27. *Let ρ be a state of the composite system. The reduced density matrix pertaining to the subsystems S and B are*

$$\rho_S = Tr_B\left(\rho\right) \qquad \textit{and} \qquad \rho_B = Tr_S\left(\rho\right), \tag{2.39}$$

respectively.

We can see in figure 2.1 an open quantum system, where a system S interacts with a reservoir R.

2.5.2 Dynamics of open and closed quantum systems

According to quantum mechanics the state vector $|\psi(t)\rangle$ evolves in time according to the *Schrödinger equation*

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{2.40}$$

where H is the Hamiltonian of the system and Planck's constant $\hbar$ has been set equal 1. The solution of the Schrödinger equation may be represented in terms of unitary time-evolution operator $U(t,t_0)$ which transform the state from t_0 to t by

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{2.41}$$

The time-evolution operator has the form

$$U(t,t_0) = e^{-i(t-t_0)H}.$$

and the time evolution of the system is

$$|\psi(t)\rangle = e^{-i(t-t_0)H}|\psi(t_0)\rangle \tag{2.42}$$

Similarly, an initial (time t_0) density

$$\rho(t_0) = \sum_i p_i |\psi_i(t_0)\rangle\langle\psi_i(t_0)| \tag{2.43}$$

evolves as

$$\rho(t) = e^{-i(t-t_0)H}\rho(t_0)e^{i(t-t_0)H} \tag{2.44}$$

The equation of motion for the density matrix obtained by differentiation of the previous equation and it is called the *von Neumann or Liouville-von Neumann equation.* It reads

$$\frac{d}{dt}\rho(t) = -i[H,\rho(t)] \tag{2.45}$$

There is another picture for the dynamics of a quantum system, called the *Heisenberg picture.* In this picture the dynamics is obtained by transferring the time dependence from the density matrix ρ to the observables A as follows. By the cyclicity of the trace (and setting $t_0 = 0$),

$$\mathrm{Tr}\big(\rho(t)A\big) = \mathrm{Tr}\big(e^{-itH}\rho(t_0)e^{itH}A\big) = \mathrm{Tr}\big(\rho(0)A(t)\big) \tag{2.46}$$

where

$$A(t) = e^{itH} A e^{-itH}.$$

The map $t \mapsto A(t)$ is called the *Heisenberg evolution* of the observable A. As (2.46) shows, the physical quantity (average of A in state ρ at time t) is expressed *equivalently* using the Schrödinger evolution of states ($\rho(t)$) or the Heisenberg evolution of observables ($A(t)$).

The case of open quantum system is more complicated. The time evolution of the reduced density matrix of an open quantum system is

$$\rho_S(t) = \mathrm{Tr}_R \{U(t,t_0)\rho(t_0)U^*(t,t_0)\} \tag{2.47}$$

where $U(t,t_0)$ is the time-evolution operator of the *total system*. The expectation of an observable A of the system alone is given by

$$\langle A(t)\rangle = \mathrm{Tr}_{S+R} \{\rho(t)(A \otimes \mathbb{1}_R)\} = \mathrm{Tr}_S \{\mathrm{Tr}_R(\rho(t))A\} = \mathrm{Tr}_S \{\rho_S A\} \tag{2.48}$$

In addition, the equation of motion of the open quantum system will be

$$\frac{d}{dt}\rho_S(t) = -i\mathrm{Tr}_R[H, \rho(t)] \tag{2.49}$$

We can distinguish three types of open quantum system models:

(1) System near equilibrium: e.g. a collection of spins in contact with an infinitely extended environment in thermal equilibrium

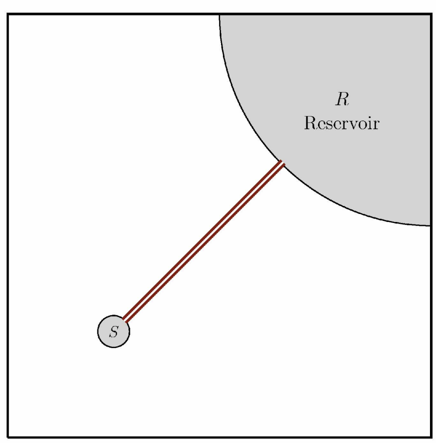

Figure 2.2: A system near to equilibrium

(2) Systems far from equilibrium: e.g. a collection of spins which has interaction with two (or more) infinite heat reservoirs.

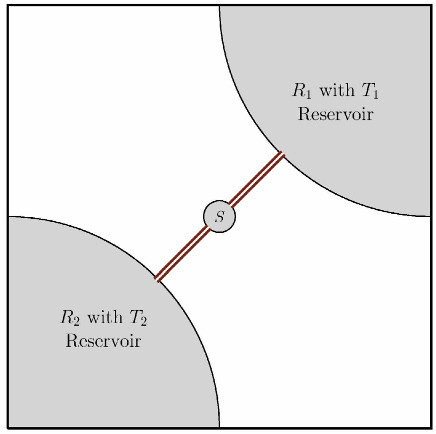

Figure 2.3: A system far from equilibrium

(3) Repeated interaction system: e.g. a collection of spin particles shooting inside of an open system which consists of a cylinder and a spin particle. Then they interact repeatedly.

2.6 Quantum entropy and entanglement

Quantum entropies play a crucial role in quantum statistical mechanics and quantum information theory. Von Neumann entropy provides an important entropy functional used in quantum statistical mechanics and thermodynamics.

Definition 28. *The von Neumann entropy of a density matrix ρ is $S(\rho) \equiv -Tr(\rho \ln \rho)$.*

Now we list properties of the von Neumann entropy

1. For all density matrices one has

$$S(\rho) \geq 0 \tag{2.50}$$

   where the equality sign holds if and only if ρ is a pure state.

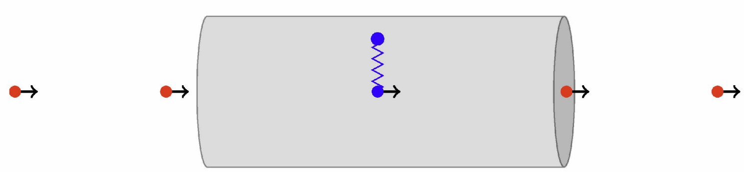

Figure 2.4: Particles shooting inside a cylinder and interact with the particle inside cylinder.

2. If the dimension of the Hilbert space is finite, $\dim\mathcal{H} = D < \infty$, the von Neumann entropy is bounded from above $S(\rho) \leq \ln D$, where the equality sign holds if and only if ρ is the completely mixed or infinite temperature state $\rho = \frac{I}{D}$.

3. The von Neumann entropy is invariant with respect to unitary transformations U of the Hilbert space, that is $S(U\rho U^*) = S(\rho)$.

4. The von Neumann entropy is a concave functional $\rho \to S(\rho)$ on the space of density matrices. This means that for any collection of densities ρ_i and numbers $\lambda_i \geq 0$ satisfying $\sum_i \lambda_i = 1$ one has the inequality

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i) \tag{2.51}$$

The equality sign in this relation holds if and only if all ρ_i with non-vanishing λ_i are equal to each other. This property is called *strict concavity* of the entropy functional.

5. Consider a composite system with Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$ and density matrix ρ and reduced density matrices ρ_S and ρ_R. Then the von Neumann entropy has the *sub-additiviy* property

$$S(\rho) \leq S(\rho_S) + S(\rho_R) \tag{2.52}$$

where the equality sign holds if and only if the total density matrix describes an uncorrelated or separable state, i.e. $\rho = \rho_S \otimes \rho_R$.

Definition 29. *Let $(\mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_R, H)$ be an open quantum system with density matrix ρ and*

$$\mathcal{R}(\rho) \equiv \left\{ (\psi_i, p_i) : \psi_i \in \mathcal{H}, \|\psi_i\| = 1, 0 \leq p_i \leq 1 \quad s.t. \quad \rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \right\}$$

The entanglement (of formation) of ρ is defined by

$$\mathcal{E}(\rho) := \inf_{\mathcal{R}(\rho)} \sum_i p_i\, S\left(Tr_R \left|\psi_i\right\rangle \left\langle\psi_i\right|\right) \tag{2.53}$$

The entanglement has the following properties

1. $\mathcal{E}(\rho) \geq 0$

2. $\mathcal{E}(\rho) = 0 \Longleftrightarrow \rho$ is separable, i.e

$$\rho = \sum_i p_i \left(\left|\psi_i^S\right\rangle \left\langle\psi_i^S\right| \otimes \left|\psi_i^R\right\rangle \left\langle\psi_i^R\right| \right) = \sum_i p_i \rho_S^i \otimes \rho_R^i$$

In other words for pure states, consider a state $|\psi_{AB}\rangle$ of a composite system. $|\psi_{AB}\rangle$ is *entangled* if and only if there do not exist $|\psi_A\rangle$, $|\psi_B\rangle$ such that

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \,.$$

Equivalently, we may say that the state is non-separable. If a state is not entangled, then it is called *separable* [15]

Example 2.10. *Consider* $|0\rangle = (1,0)^t$, $|1\rangle = (0,1)^t$, $|00\rangle = |0\rangle \otimes |0\rangle$, *and* $|11\rangle = |1\rangle \otimes |1\rangle$, *then the four Bell states*

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad \frac{|01\rangle + |10\rangle}{\sqrt{2}} \quad \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

are well known two-qubit entangled states.

Example 2.11. $|00\rangle + |11\rangle + |22\rangle$ *is an entangled state. In fact, this is a common form for writing entangled states, because, for example consider the case when Alice and Bob share two Bell states. They thus have*

$$\begin{aligned} (|00\rangle + |11\rangle)^{\otimes^2} &= (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \\ &= |0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle \end{aligned} \tag{2.54}$$

Rearranging the labels by grouping all of Alice's qubits together, followed by Bob's qubits gives us

$$|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle = |00\rangle\,|00\rangle + |01\rangle\,|01\rangle + |10\rangle\,|10\rangle + |11\rangle\,|11\rangle$$

In fact, most multi-qubit states are entangled.

Chapter 3

Statement of the problem and main result

3.1 Statement of the problem

We consider N quantum particles, each one coupled to its own *local reservoir* and all of them coupled to a *common reservoir*, as illustrated in Figure 3.1.

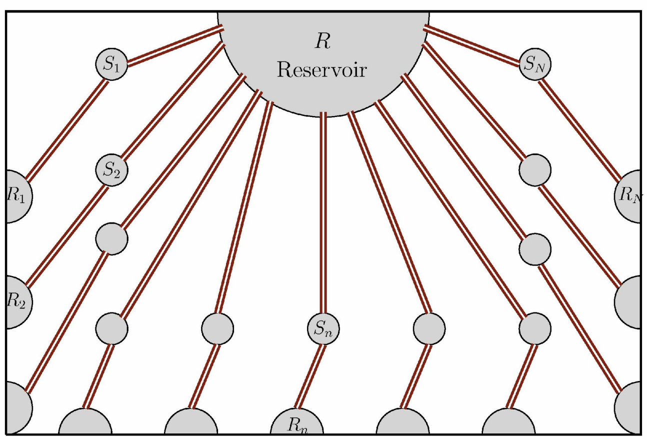

Figure 3.1: Interaction of N particles with common and local environments (reservoirs).

The associated Hilbert space is

$$\mathcal{H}_N = \bigotimes_{j=1}^{N} \mathcal{H}_{\mathrm{S}} \otimes \bigotimes_{j=1}^{N} \mathcal{H}_{\mathrm{R}} \otimes \mathcal{H}_{\mathrm{C}}, \tag{3.1}$$

where $\mathcal{H}_{\mathrm{S}}$, $\mathcal{H}_{\mathrm{R}}$ and $\mathcal{H}_{\mathrm{C}}$ are the Hilbert spaces of a single system, a single (local) reservoir and the collective reservoir, respectively. We assume that

$$\dim \mathcal{H}_{\mathrm{S}} = d < \infty.$$

In this sense, we can assume that $\mathcal{H}_{\mathrm{S}} = \mathbb{C}^d$. The Hilbert space of every reservoir is the *Fock space*

$$\mathcal{H}_{\mathrm{R}} = \mathcal{H}_{\mathrm{C}} = \mathcal{F} := \bigoplus_{n\geq 0} L^2_{\mathrm{symm}}(\mathbb{R}^3, d^{3n}k). \tag{3.2}$$

Here, $L^2_{\mathrm{symm}}(\mathbb{R}^3, d^{3n}k)$ is the space of square-integrable complex-valued functions which are *symmetric* in n arguments from $\mathbb{R}^3$. The direct summand for $n = 0$ is interpreted to be $\mathbb{C}$, it is called the *vacuum sector*. The one for $n \geq 1$ is called the n-particle sector.

A general element $\psi \in \mathcal{F}$ is a sequence $\psi = (\psi_0, \psi_1, \psi_2, \ldots)$, with $\psi_n \in L^2_{\mathrm{symm}}(\mathbb{R}^3, d^{3n}k)$, satisfying

$$\|\psi\| = \left(\sum_{n\geq 0} \|\psi_n\|^2_{L^2(\mathbb{R}^3, d^{3n}k)}\right)^{1/2}.$$

The last quantity defines the norm of $\mathcal{F}$. A $\psi \in \mathcal{F}$ with $\psi_{n_0} \neq 0$ for a single $n_0 \in \mathbb{N}$ is interpreted to be the *wave function* of a system having exactly n_0 particles. Fock space allows to describe creation and annihilation of particles (variable number of particles in a system under consideration), since it encompasses n-particle sectors for all n.

The *creation operator* $a^*(f)$, with $f \in L^2(\mathbb{R}^3, d^3k)$ (f is a single-particle wave function), is defined on $\mathcal{F}$ as follows. On the n-particle sector, for $\psi = (0, \ldots, \psi_n, 0, \ldots)$, its action is given by

$$\big(a^*(f)\psi\big)_k = \begin{cases} 0 & \text{if } k \neq n+1 \\ \sqrt{n+1}\, \mathcal{S} f \otimes \psi_n & \text{if } k = n+1. \end{cases} \tag{3.3}$$

Here, $\mathcal{S}$ is the symmetrization operator. In particular,

$$\big(\mathcal{S} f \otimes \psi_n\big)(k_1, \ldots, k_{n+1}) = \frac{1}{(n+1)!} \sum_{j=1}^{n+1} f(k_j) \psi_n(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{n+1}).$$

The action (3.3) is then extended to vectors in $\mathcal{F}$ by sector-wise linear action. Note that $a^*(f)$ maps the n-sector into the $(n+1)$-sector, hence the name of creation operator. The adjoint operator of $a^*(f)$, denoted by $a(f)$, maps the n-sector into the $(n-1)$-sector (and the vacuum sector onto zero). It acts, for $\psi = (0, \ldots, \psi_n, 0, \ldots)$, as

$$\big(a(f)\psi\big)_k = \begin{cases} 0 & \text{if } k \neq n-1 \\ \sqrt{n}\ \int_{\mathbb{R}^3} \overline{f(k_n)} \psi_n(k_1, \ldots, k_n) d^3k_n & \text{if } k = n-1. \end{cases} \tag{3.4}$$

The *field operator* is the self-adjoint operator

$$\varphi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f)). \tag{3.5}$$

The dynamics of the total system is generated by a self-adjoint Hamiltonian H_N, acting on $\mathcal{H}_N$, defined by

$$\begin{aligned} H_N &= \sum_{j=1}^{N} A_j + \sum_{j=1}^{N} K_j + K && (3.6)\\ &\quad + \sum_{j=1}^{N} \varkappa_j V_j \otimes \varphi_j(f_j) && (3.7)\\ &\quad + \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{N} W_j \otimes \varphi(f). && (3.8) \end{aligned}$$

Here, A_j is understood to act nontrivially only on the j-th factor $\mathcal{H}_{\mathrm{S}}$ in (3.1). There, it is a fixed (equal for all j) single-particle operator A on $\mathcal{H}_{\mathrm{S}}$ (represented by a $d \times d$ matrix). The operators K_j and K are the Hamiltonians of the j-th local and the collective reservoir, respectively. They are all given by

$$K_j = K = d\Gamma(|k|), \tag{3.9}$$

where $d\Gamma(X)$ is called the *second quantization of the operator* X. It is defined on the the n-sector of Fock space by

$$d\Gamma(X) = X \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes X \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes X,$$

then extended by linearity to $\mathcal{F}$, see also Definition 20. In (3.9), $X = |k|$ is the operator of multiplication by $|k|$, acting on $L^2(\mathbb{R}^3, d^3k)$.

The operators φ_j and φ in (3.7) and (3.8) are field operators, (3.5), of the j-th reservoir and the collective one, respectively. Also, $\varkappa_j$ and $\varkappa$ are *coupling constants.* The V_j is interaction operators, acting nontrivially on the j-th system factor as a fixed operator $V \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$. Similarly for W_j (which acts as a fixed $W \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$).

Note that the common interaction term (3.8) has the N-dependent scaling $\varkappa/\sqrt{N}$. This scaling is called the *mean field scaling.*

The dynamics of an initial (pure) state $\psi_0 \in \mathcal{H}_N$ is given by a path $\{\psi_t\}_{t\in\mathbb{R}}$, determined by the *Schrödinger equation*

$$i\frac{d}{dt}\psi_t = H_N\psi_t, \qquad \psi_t\big|_{t=0} = \psi_0, \tag{3.10}$$

equivalently expressed as

$$\psi_t = e^{-itH_N}\psi_0. \tag{3.11}$$

For initial mixed states, given by a density matrix ρ_0 (acting on $\mathcal{H}_N$), the dynamics is given by

$$\rho_t = e^{-itH_N}\rho_0 e^{itH_N}. \tag{3.12}$$

We are going to consider initial *product* states, namely, states of the form

$$\rho_N(0) = \rho_{\vec{S}} \otimes \rho_{\vec{R}}, \tag{3.13}$$

where

$$\rho_{\vec{S}} = \rho_0 \otimes \cdots \otimes \rho_0 \tag{3.14}$$

is the N-fold tensor product of a fixed initial single-system state ρ_0 and

$$\rho_{\vec{R}} = \rho_R \otimes \cdots \otimes \rho_R \otimes \rho_{\mathrm{C}} \tag{3.15}$$

is the N-fold tensor product of a fixed (local) initial reservoir state ρ_R times the initial state of the collective reservoir, ρ_{C}. Then the total density matrix at time t is given by

$$\rho_N(t) = e^{-itH_N}\rho_N(0)e^{itH_N}. \tag{3.16}$$

The state of the first n systems ($n \le N$) has the *reduced density matrix*

$$\rho_{n,N}(t) = \mathrm{Tr}_{[n+1,N],\vec{R}}\ \rho_N(t). \tag{3.17}$$

Here, the partial trace $\mathrm{Tr}_{[n+1,N]}$ (discussed in section 2.3) is taken over all local and the collective reservoir.

Assumption. We suppose that the interaction is *energy conserving*, meaning that all operators A, V and W commute. More precisely, there are rank-one spectral projections $P^{(1)},\ldots,P^{(d)}$ on $\mathcal{H}_{\rm S}$ such that

$$\begin{aligned} A &= \sum_{m=1}^{d} a^{(m)}P^{(m)} \\ V &= \sum_{m=1}^{d} v^{(m)}P^{(m)} \\ W &= \sum_{m=1}^{d} w^{(m)}P^{(m)}. \end{aligned}$$

Here, the $a^{(m)}$ are the d real eigenvalues of A (possibly repeated), and similarly for $v^{(m)}$ and $w^{(m)}$.

We introduce the quantities

$$S(t) = \frac{1}{2}\int_{\mathbb{R}^3} |f(k)|^2 \frac{|k|\,t - \sin(|k|\,t)}{|k|^2} d^3k \tag{3.18}$$

and

$$\Gamma(t) = \int_{\mathbb{R}^3} |f(k)|^2 \coth\left(\frac{\beta\,|k|}{2}\right)\frac{\sin^2(|k|\,t/2)}{|k|^2} d^3k. \tag{3.19}$$

The large N limit of the reduced density matrix has been investigated in [6].

Theorem 5 (Merkli & Berman 2012, [6])**.** *For any $t\in\mathbb{R}$ and $n\geq 1$ we have*

$$\lim_{N\to\infty} Tr|\rho_{n,N}(t) - \rho_{1,t}\otimes\rho_{2,t}\otimes\cdots\otimes\rho_{n,t}| = 0.$$

The single particle matrix $\rho_{j,t}$ satisfies the time-dependent non-linear Hartree-Lindblad equation

$$i\dot\rho_{j,t} = [A_j,\rho_{j,t}] + Tr_2[W_{\mathit{eff}}(t),\rho_{j,t}\otimes\rho_{j,t}] + \mathcal{L}_j(t)\rho_{j,t}.$$

Here, A_j is the single system Hamiltonian, $W_{\mathit{eff}}(t)$ is an effective two-body interaction given by

$$W_{\mathit{eff}} = 2\dot S(t) W\otimes W$$

and $\mathcal{L}_j(t)$ is a Lindblad operator, describing the effect of the local environment,

$$\mathcal{L}_j(t) = \dot S_j(t)[V_j^2,\rho] - i\dot\Gamma_j(t)[V_j,[V_j,\rho]].$$

Note that the theorem shows that for large N, the state of the n first systems is (nearly) of *product* form. However, the corresponding single-particle density matrix (of one factor) obeys a more complicated *non-linear* evolution equation.

Theorem 6 (Convergence Speed; Merkli & Berman 2012 [6]). *For $t \in \mathbb{R}$ and $1 \leq n \leq N$ such that $N > 4n \|W\|^2 |S(t)|$, we have*

$$Tr|\rho_{n,N}(t) - \rho_{1,t} \otimes \rho_{2,t} \otimes \cdots \otimes \rho_{n,t}| \leq \frac{C_n(t)}{N}, \tag{3.20}$$

where

$$C_n(t) = \eta_0 d^2 n \left((2n + 2\eta_0 |S(t)| + 1) e^{\eta_0 |S(t)| (n + 2\eta_0 |S(t)| + 1)} + n |\Gamma(t)| \right), \tag{3.21}$$

with $\eta_0 = 4\varkappa^2 n \|W\|^2$.

3.1.1 Main result

Theorems 5 and 6 show, formally, that

$$\rho_{n,N}(t) = \rho_{n,\infty}(t) + O(1/N), \tag{3.22}$$

where $\rho_{n,\infty}(t) = \rho_{1,t} \otimes \rho_{2,t} \otimes \cdots \otimes \rho_{n,t}$. In this thesis, we refine the result (3.22) and obtain an expansion

$$\rho_{n,N}(t) = \rho_{n,\infty}(t) + \frac{1}{N}\rho_n^{(1)}(t) + O(1/N^2), \tag{3.23}$$

with an explicit expression for the correction term $\rho_n^{(1)}(t)$. While the main term $\rho_{n,\infty}(t)$ is of product form, $\rho_n^{(1)}(t)$ is not. However, we show that $\rho_n^{(1)}(t)$ is obtained by applying an explicit operator to a product state. In this sense, we are calculating the lowest order correction to the mean field limit, which makes the density matrix *entangled* (for finite values of N).

In the following, we consider $A_j = 0$ and $\varkappa_j = 0$ in (3.6), for simplicity of notation. The main result, stating (3.23), is given in Theorem 7 below.

Recall that ρ_0 is the initial single-particle density matrix. We introduce the notation

$$\begin{aligned} \langle T \rangle &= \mathrm{Tr}\big(\rho_0 T\big) &(3.24)\\ \mathrm{var}(T) &= \left\langle T^2 \right\rangle - \langle T \rangle^2, &(3.25) \end{aligned}$$

for all $T \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$. Also, given any single-particle operator $T \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$, we define the operators T_L and T_R by

$$T_R \rho = \rho T, \quad T_L \rho = T\rho, \tag{3.26}$$

for any single-particle density matrix ρ. Finally, we define the time-dependent operator

$$\begin{aligned}\mathcal{E}_{n,t} \;:=\; & -\varkappa^2\Gamma(t)\big(d\Gamma(W_R)-d\Gamma(W_L)\big)^2 + i\varkappa^2 S(t)\big([d\Gamma(W_R)]^2-[d\Gamma(W_L)]^2\big)\\ & -2in\varkappa^2 S(t)\,\langle W\rangle\, d\Gamma(W_R-W_L) - 2\varkappa^4 S(t)^2\,\mathrm{var}(W)\,\big[d\Gamma(W_R-W_L)\big]^2,\end{aligned} \tag{3.27}$$

which acts on the n-fold tensor product of $\mathcal{H}_\mathrm{S}$ with itself. Recall that the functions $S(t)$ and $\Gamma(t)$ are defined in (3.18), (3.19).

Theorem 7 (Main result of thesis)**.** *For any $t\in\mathbb{R}$ and $n\geq 1$, we have*

$$\rho_{n,N}(t)=\rho_{n,\infty}(t)+\frac{1}{N}\rho_n^{(1)}(t)+R_{Main}, \tag{3.28}$$

where

$$\rho_n^{(1)}(t)=\mathcal{E}_{n,t}\;\tilde\rho_0\otimes\cdots\otimes\tilde\rho_0 \tag{3.29}$$

and

$$\tilde\rho_0 := e^{-2i\varkappa^2 S(t)\langle W\rangle W}\;\;\rho_0\;e^{2i\varkappa^2 S(t)\langle W\rangle W}. \tag{3.30}$$

The trace-norm of R_{Main} is bounded as

$$\|R_{Main}\|\leq\frac{C_n(t)}{N^2}. \tag{3.31}$$

The following result gives an upper bound on the constant $C_n(t)$, (3.31). Set

$$\xi := 4\varkappa^2 n\,\|W\|^2\,|S(t)|$$

and

$$\eta := 4n^2\,\|W\|^2\,(|S(t)|+|\Gamma(t)|).$$

Also, recall that $d=\dim\mathcal{H}_\mathrm{S}$.

Lemma 3.1. *For $N>2\xi$, the constant $C_n(t)$ in (3.31) satisfies the bound*

$$C_n(t)\leq d^{2n}\Big(M\xi^2+\varkappa^2\eta\xi\big(\tfrac12 M+n+\xi\big)+\varkappa^4\eta^2 e^{\varkappa^2\eta/n}\big(1+\tfrac32\xi+\tfrac14 M\big)\Big), \tag{3.32}$$

where

$$M:=\big(\tfrac{47}{12}\xi+\tfrac32 n\big)\;e^{\frac{\xi^2}{N^2}(\frac{47}{12}\xi+\frac32 n)}+\tfrac12\xi^2+ne^{\xi}(n+\xi^2)+\xi^2e^{\frac12\xi}(1+\xi+\xi e^{\xi}).$$

Note that the t-dependence on the right side of (3.32) is in $\eta=\eta(t)$ and $\xi=\xi(t)$.

Chapter 4

Proof of Results

4.1 Proof of Theorem 7

We start with the following expression for the reduced density matrix.

Lemma 4.1. *We have*

$$\begin{aligned}
\rho_{n,N}(t) &= e^{-it(A_1+\ldots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_n} \prod_{j=1}^{n} (P_j^{(m_j)} \rho_0 P_j^{(m'_j)}) \prod_{j=n+1}^{N} p_j \\
&\times \operatorname{Tr}_{\vec{R}} \left\{ e^{-it(K_R+I_l+I_c)} \rho_{\vec{R}}\, e^{it(K_R+I'_l+I'_c)} \right\} e^{it(A_1+\ldots+A_n)}, \qquad (4.1)
\end{aligned}$$

where

$$K_R = \sum_{j=1}^{N} K_j + K, \qquad (4.2)$$

$$I_l = \sum_{j=1}^{N} \varkappa_j v^{(m_j)} \varphi_j(f_j), \qquad (4.3)$$

$$I_c = \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{N} w^{(m_j)} \varphi(f) \qquad (4.4)$$

$$I'_l = \sum_{j=1}^{n} \varkappa_j v^{(m'_j)} \varphi_j(f_j) + \sum_{j=n+1}^{N} \varkappa_j v^{(m_j)} \varphi_j(f_j), \qquad (4.5)$$

$$I'_c = \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{n} w^{(m'_j)} \varphi(f) + \frac{\varkappa}{\sqrt{N}} \sum_{j=n+1}^{N} w^{(m_j)} \varphi(f). \qquad (4.6)$$

Proof. We know from the definition of the n-body reduced density matrix that

$$\rho_{n,N}(t) = \mathrm{Tr}_{[n+1,N],\vec{R}}\, e^{-itH_N} \rho_0 \otimes \rho_0 \otimes \cdots \otimes \rho_0 \otimes \rho_{\vec{R}}\, e^{itH_N} \tag{4.7}$$

By definition of H_N as (3.8) we have

$$\begin{aligned} e^{-itH_N} &= \sum_{m_1,\ldots,m_N} P_1^{(m_1)} \otimes \cdots \otimes P_N^{(m_N)} e^{-itH_N} \\ &= \sum_{m_1,\ldots,m_N} P_1^{(m_1)} \otimes \cdots \otimes P_N^{(m_N)} \\ &\times \; e^{-it(\sum_{j=1}^N A_j + K_R + \sum_{j=1}^N \varkappa_j V_j \otimes \varphi_j(f_j) + \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^N W_j \otimes \varphi(f))} \end{aligned} \tag{4.8}$$

Also we know $P^{(m)}V = v^{(m)}P^{(m)}$ and $P^{(m)}W = w^{(m)}P^{(m)}$, so

$$\begin{aligned} e^{-itH_N} &= \sum_{m_1,\ldots,m_N} P_1^{(m_1)} \otimes \cdots \otimes P_N^{(m_N)} \\ &\times \; e^{-it(\sum_{j=1}^N A_j + K_R + \sum_{j=1}^N \varkappa_j v^{(m_j)} \varphi_j(f_j) + \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^N w^{(m_j)} \varphi(f))} \end{aligned} \tag{4.9}$$

Thus, by the definition (4.3),(4.4)

$$e^{-itH_N} = \sum_{m_1,\ldots,m_N} P_1^{(m_1)} \otimes \cdots \otimes P_N^{(m_N)} e^{-it(\sum_{j=1}^N A_j + K_R + I_l + I_c)} \tag{4.10}$$

Similarly, if we proceed with similar steps as above for the conjugate we have

$$e^{itH_N} = \sum_{m'_1,\ldots,m'_N} P_1^{(m'_1)} \otimes \cdots \otimes P_N^{(m'_N)} e^{it(\sum_{j=1}^N A_j + K_R + J_l + J_c)} \tag{4.11}$$

where $J_l = \sum_{j=1}^N \varkappa_j v^{(m'_j)} \varphi_j(f_j)$ and $J_c = \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^N w^{(m'_j)} \varphi(f)$.

By equations (4.10) and (4.11) we have

$$\begin{aligned} \rho_{n,N}(t) &= \mathrm{Tr}_{[n+1,N],\vec{R}} \sum_{m_1,\ldots,m_N} P_1^{(m_1)} \otimes \cdots \otimes P_N^{(m_N)} e^{-it(\sum_{j=1}^N A_j + K_R + I_l + I_c)} \\ &\times \; (\rho_0 \otimes \rho_0 \otimes \cdots \otimes \rho_0 \otimes \rho_{\vec{R}}) \sum_{m'_1,\ldots,m'_N} P_1^{(m'_1)} \otimes \cdots \otimes P_N^{(m'_N)} e^{it(\sum_{j=1}^N A_j + K_R + J_l + J_c)} \\ &= \mathrm{Tr}_{[n+1,N],\vec{R}} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_N} e^{-it(\sum_{j=1}^N A_j + K_R + I_l + I_c)} P_1^{(m_1)} \rho_0 P_1^{(m'_1)} \otimes \cdots \\ &\otimes \; P_N^{(m_N)} \rho_0 P_N^{(m'_N)} \otimes \rho_{\vec{R}} e^{it(\sum_{j=1}^N A_j + K_R + J_l + J_c)} \end{aligned} \tag{4.12}$$

We know that K_R, I_l ,I_c,J_l and J_c are operators acting non-trivially on the space of the reservoirs, only. Therefore,

$$\begin{aligned}\rho_{n,N}(t) &= \mathrm{Tr}_{[n+1,N],\vec{R}}\ e^{-it\sum_{j=1}^{N} A_j} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_N} P_1^{(m_1)}\rho_0 P_1^{(m'_1)} \otimes \cdots \otimes P_N^{(m_N)}\rho_0 P_N^{(m'_N)} \\ &\quad \otimes e^{-it(K_R+I_l+I_c)}\rho_{\vec{R}} e^{it(K_R+J_l+J_c)} e^{it\sum_{j=1}^{N} A_j} \qquad (4.13)\end{aligned}$$

From trace properties we know $\mathrm{Tr}_{[n+1,N]}(N_1 \otimes \cdots \otimes N_N) = N_1 \otimes \cdots \otimes N_n\ \mathrm{Tr}(N_{n+1} \otimes \cdots \otimes N_N)$. Thus,

$$\begin{aligned}\rho_{n,N}(t) &= e^{-it(A_1+\ldots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_N} P_1^{(m_1)}\rho_0 P_1^{(m'_1)} \otimes \cdots \otimes P_n^{(m_n)}\rho_0 P_n^{(m'_n)} \\ &= \times\ \mathrm{Tr}_R\{e^{-it(K_R+I_l+I_c)}\rho_{\vec{R}} e^{it(K_R+J_l+J_c)}\} \qquad (4.14)\end{aligned}$$

But we know $\mathrm{Tr}(P^{(m_k)}\rho_0 P^{(m'_k)}) = \mathrm{Tr}(P^{(m_k)}P^{(m'_k)}\rho_0)$ and by projection properties we have

$$P^{(m_k)}P^{(m'_k)} = 0$$

unless $m_k = m'_k$. So, $\mathrm{Tr}(P^{(m_k)}\rho_0 P^{(m'_k)}) = \mathrm{Tr}(P^{(m_k)}P^{(m'_k)}\rho_0) = \mathrm{Tr}(\delta_{m_k,m'_k}\rho_0) = \delta_{m_k,m'_k} p_k$ for $n+1 \leq k \leq N$. Thus

$$\begin{aligned}\rho_{n,N}(t) &= e^{-it(A_1+\ldots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_n} \prod_{j=1}^{n} P_j^{(m_j)}\rho_0 P_j^{(m'_j)} \prod_{j=n+1}^{N} p_j \\ &\times\ \mathrm{Tr}_{\vec{R}}\{e^{-it(K_R+I_l+I_c)}\rho_{\vec{R}} e^{it(K_R+J_l+J_c)}\} e^{it(A_1+\ldots+A_n)} \qquad (4.15)\end{aligned}$$

We should note that since $m_j = m'_j$ for $n+1 \leq j \leq N$, we should define another version of J_c and J_l that divided summation by two parts, that are I'_c and I'_l. Therefore, we obtain the final result

$$\begin{aligned}\rho_{n,N}(t) &= e^{-it(A_1+\ldots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_n} \prod_{j=1}^{n} (P_j^{(m_j)}\rho_0 P_j^{(m'_j)}) \prod_{j=n+1}^{N} p_j \\ &\times\ \mathrm{Tr}_{\vec{R}}\{e^{-it(K_R+I_l+I_c)}\rho_{\vec{R}} e^{it(K_R+I'_l+I'_c)}\} e^{it(A_1+\ldots+A_n)} \qquad (4.16)\end{aligned}$$

This proves Lemma 4.1. $\square$

Now we give the prove of Theorem 7.

Proof. According Lemma 4.1, we know

$$\begin{aligned}\rho_{n,N}(t) &= e^{-it(A_1+\ldots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots,m'_n} \prod_{j=1}^{n} (P_j^{(m_j)} \rho_0 P_j^{(m'_j)}) \prod_{j=n+1}^{N} p_j \\ &\times \operatorname{Tr}_{\vec{R}} \{ e^{-it(K_R+I_l+I_c)} \rho_{\vec{R}} e^{it(K_R+I'_l+I'_c)} \} e^{it(A_1+\ldots+A_n)} \end{aligned} \tag{4.17}$$

Now we simplify the trace term in (4.17)

$$e^{-it(K_R+I_l+I_c)} = e^{-it[K+(\varkappa/\sqrt{N})\sum_{j=1}^{N} w^{(m_j)}\varphi(f)]} \prod_{j=1}^{n} e^{-it[K_j+\varkappa_j v^{(m_j)}\varphi_j(f_j)]} \tag{4.18}$$

and similarly for the second exponential in the trace in (4.17)

$$\begin{aligned} &\prod_{j=1}^{n} \left\langle e^{it[K_j+\varkappa_j v^{(m'_j)}\varphi_j(f_j)]} e^{-it[K_j+\varkappa_j v^{(m_j)}\varphi_j(f_j)]} \right\rangle_\beta \\ \times\ &\left\langle e^{it[K+(\varkappa/\sqrt{N})\{\sum_{j=1}^{n} w^{(m'_j)}+\sum_{j=n+1}^{N} w^{(m_j)}\}\varphi(f)]} e^{-it[K+(\varkappa/\sqrt{N})\sum_{j=1}^{N} w^{(m_j)}\varphi(f)]} \right\rangle_\beta \end{aligned} \tag{4.19}$$

where $\langle X \rangle_\beta$ is the average of an operator X in the Bosonic equilibrium state at temperature $1/\beta$.
By [16], for $x, y \in \mathbb{R}$

$$\left\langle e^{it(K+x\varphi(f))} e^{-it(K+y\varphi(f))} \right\rangle_\beta = e^{i(x-y)(x+y)S(t)} e^{-(x-y)^2\Gamma(t)}, \tag{4.20}$$

where $S(t)$ and $\Gamma(t)$ are defined in (3.18),(3.19). By (4.19),(4.20)

$$\begin{aligned} &\operatorname{Tr}_{\vec{R}} \left\{ e^{-it(K_R+I_l+I_c)} \rho_{\vec{R}} e^{it(K_R+I'_l+I'_c)} \right\} \\ =\ &\prod_{j=1}^{n} e^{i\varkappa_j^2[v^{(m'_j)}-v^{(m_j)}][v^{(m'_j)}+v^{(m_j)}]S_j(t)} e^{-\varkappa_j^2[v^{(m'_j)}-v^{(m_j)}]^2\Gamma_j(t)} \\ \times\ &e^{i(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})][\sum_{j=1}^{n}(w^{(m'_j)}+w^{(m_j)})+2\sum_{j=n+1}^{N} w^{(m_j)}]S(t)} \\ \times\ &e^{-(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})]^2\Gamma(t)}. \end{aligned} \tag{4.21}$$

Now we insert this expression into (4.17)

$$
\begin{aligned}
\rho_{n,N}(t) &= e^{-it(A_1+\cdots+A_n)} \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} (\prod_{j=1}^{n} P_j^{(m_j)} \rho_0 P_j^{(m'_j)}) \\
&\times \prod_{j=1}^{n} e^{i\varkappa_j^2[v^{(m'_j)}-v^{(m_j)}][v^{(m'_j)}+v^{(m_j)}]S_j(t)} e^{-\varkappa_j^2[v^{(m'_j)}-v^{(m_j)}]^2\Gamma_j(t)} \\
&\times e^{i(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})][\sum_{j=1}^{n}(w^{(m'_j)}+w^{(m_j)})]S(t)} \\
&\times e^{-(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})]^2\Gamma(t)} \\
&\times \left[\sum_m p_m e^{2i(\varkappa^2/N)w^{(m)}\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})S(t)}\right]^{N-n} e^{it(A_1+\cdots+A_n)} \qquad (4.22)
\end{aligned}
$$

The dependence on the uncoupled Hamiltonians A_j of the particles is trivial (a conjugacy by a unitary operator in (4.22)), so we consider just the case $A_j = 0$. Furthermore, we take $\varkappa_j = 0$, meaning that there is no coupling to local reservoirs. Then we have

$$
\begin{aligned}
\rho_{n,N}(t) &= \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} (\prod_{j=1}^{n} P_j^{(m_j)} \rho_0 P_j^{(m'_j)}) \\
&\times e^{i(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})][\sum_{j=1}^{n}(w^{(m'_j)}+w^{(m_j)})]S(t)} \\
&\times e^{-(\varkappa^2/N)[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})]^2\Gamma(t)} \\
&\times \left[\sum_m p_m e^{2i(\varkappa^2/N)w^{(m)}\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})S(t)}\right]^{N-n} \qquad (4.23)
\end{aligned}
$$

Now we define

$$
\begin{aligned}
\alpha &:= i[\sum_{j=1}^{n}(w^{(m'_j)} - w^{(m_j)})][\sum_{j=1}^{n}(w^{(m'_j)} + w^{(m_j)})]S(t) - [\sum_{j=1}^{n}(w^{(m'_j)} - w^{(m_j)})]^2\Gamma(t) \\
x_m &:= 2\varkappa^2 w^{(m)} \sum_{j=1}^{n}(w^{(m'_j)} - w^{(m_j)})S(t) \qquad (4.24)
\end{aligned}
$$

So (4.23) becomes

$$
\rho_{n,N}(t) = \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} (\prod_{j=1}^{n} P_j^{(m_j)} \rho_0 P_j^{(m'_j)}) \underbrace{e^{\frac{\varkappa^2}{N}\alpha}[\sum_m p_m e^{\frac{ix_m}{N}}]^{N-n}}_{I} \qquad (4.25)
$$

- Now we expand expression I. In the first step by Taylor expansion of the exponential function, we have

$$e^{\frac{\varkappa^2}{N}\alpha} = \sum_{r=0}^{\infty}\frac{(\alpha\varkappa^2)^r}{r!}(\frac{1}{N})^r \tag{4.26}$$

$$\sum_m p_m e^{\frac{ix_m}{N}} = \sum_{s=0}^{\infty}\frac{i^s}{s!}(\frac{1}{N})^s\sum_m p_m x_m^s \tag{4.27}$$

The second term in I is equation (4.27) with power $N-n$. To expand this we need a special trick by rewriting this term as the exponential of a logarithm and then expand the logarithm, that is

$$\begin{aligned}
[\sum_m p_m e^{\frac{ix_m}{N}}]^{N-n} &= e^{(N-n)\log\left(\sum_m p_m e^{\frac{ix_m}{N}}\right)} \\
&= e^{(N-n)\log\left(1+\sum_m p_m\left(e^{\frac{ix_m}{N}}-1\right)\right)} \\
&:= e^{(N-n)\Xi}
\end{aligned} \tag{4.28}$$

We start by expanding the power of the exponential function in (4.28) by the following expression

$$\log(1+x) = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}x^k \tag{4.29}$$

and since $e^{\frac{ix_m}{N}} - 1 = O(\frac{1}{N})$ we have

$$\begin{aligned}
\Xi &= \log\left(1 + \sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right)\right) \\
&= \sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right) - \frac{1}{2}\left[\sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right)\right]^2 \\
&+ \frac{1}{3}\left[\sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right)\right]^3 + O(\frac{1}{N^4}) \\
&= \sum_m p_m \sum_{j\geq 1} \frac{(ix_m)^j}{j!N^j} - \frac{1}{2}\left[\sum_m p_m \sum_{j\geq 1} \frac{(ix_m)^j}{j!N^j}\right]^2 \\
&+ \frac{1}{3}\left[\sum_m p_m \sum_{j\geq 1} \frac{(ix_m)^j}{j!N^j}\right]^3 + O(\frac{1}{N^4}) \\
&= \sum_m p_m \left(\frac{ix_m}{N} + \frac{(ix_m)^2}{2!N^2} + \frac{(ix_m)^3}{3!N^3} + O(\frac{1}{N^4})\right) \\
&- \frac{1}{2}\left[\sum_m p_m \left(\frac{ix_m}{N} + \frac{(ix_m)^2}{2!N^2} + O(\frac{1}{N^3})\right)\right]^2 \\
&+ \frac{1}{3}\left[\sum_m p_m \left(\frac{ix_m}{N} + O(\frac{1}{N^2})\right)\right]^3 + O(\frac{1}{N^4})
\end{aligned} \tag{4.30}$$

Therefore, we have

$$\begin{aligned}
\Xi &= \log\left(1 + \sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right)\right) \\
&= \frac{1}{N}\left(i\sum_m p_m x_m\right) + \frac{1}{N^2}\left(-\frac{1}{2}\sum_m p_m x_m^2 + \frac{1}{2}\left(\sum_m p_m x_m\right)^2\right) \\
&+ \frac{1}{N^3}\left(-\frac{i}{6}\sum_m p_m x_m^3 - \frac{i}{3}\left(\sum_m p_m x_m\right)^3 + \frac{i}{2}\langle x\rangle\langle x^2\rangle\right) + O(\frac{1}{N^4}) \\
&= \frac{i}{N}\langle x\rangle - \frac{1}{2N^2}\mathrm{var}(x) - \frac{i}{3N^3}\left(\frac{1}{2}\langle x^3\rangle + \langle x\rangle^3 + \frac{3}{2}\langle x\rangle\langle x^2\rangle\right) + O(\frac{1}{N^4})
\end{aligned} \tag{4.31}$$

Where $\langle x\rangle := \sum_m p_m x_m$ and $\mathrm{var}(x) := \langle x^2\rangle - \langle x\rangle^2$. Now we combine equations

(4.28) and (4.31)

$$
\begin{aligned}
[\sum_m p_m e^{\frac{ixm}{N}}]^{N-n} &= e^{(N-n)\log\left(\sum_m p_m e^{\frac{ixm}{N}}\right)} \\
&= e^{(N-n)\log\left(1+\sum_m p_m\left(e^{\frac{ixm}{N}}-1\right)\right)} \\
&= e^{i\frac{N-n}{N}\langle x\rangle-\frac{N-n}{2N^2}\mathrm{var}(x)-i\frac{N-n}{3N^3}\left(\frac{1}{2}\left\langle x^3\right\rangle+\langle x\rangle^3+\frac{3}{2}\langle x\rangle\left\langle x^2\right\rangle\right)+O(\frac{1}{N^3})} \\
&= e^{i\langle x\rangle}e^{-i\frac{n}{N}\langle x\rangle}e^{-\frac{N-n}{2N^2}\mathrm{var}(x)}e^{-i\frac{N-n}{3N^3}\left(\frac{1}{2}\left\langle x^3\right\rangle+\langle x\rangle^3+\frac{3}{2}\langle x\rangle\left\langle x^2\right\rangle\right)} \\
&+ O(\frac{1}{N^3}) \\
&= e^{i\langle x\rangle}\left(1-i\frac{n}{N}\langle x\rangle-\frac{n^2}{2N^2}\langle x\rangle^2\right) \\
&\times \left(1-\frac{1}{2N}\mathrm{var}(x)+\frac{1}{N^2}\left\{\frac{n}{2}\mathrm{var}(x)+\frac{1}{8}\mathrm{var}(x)^2\right\}\right) \\
&\times \left(1-\frac{i}{3N^2}\left(\frac{1}{2}\left\langle x^3\right\rangle+\langle x\rangle^3+\frac{3}{2}\langle x\rangle\left\langle x^2\right\rangle\right)\right)+O(\frac{1}{N^3})
\end{aligned} \tag{4.32}
$$

Now by combining equations (4.26) and (4.32) as I in equation (4.25) we have

$$
\begin{aligned}
\rho_{n,N}(t) &= \sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n}\left(\prod_{j=1}^{n}P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right)e^{i\langle x\rangle}\Bigg\{1 \\
&+ \frac{1}{N}\left[\varkappa^2\alpha-in\langle x\rangle-\frac{1}{2}\mathrm{var}(x)\right] \\
&+ \frac{1}{N^2}\Bigg[(\frac{\varkappa^2\alpha}{2})^2-\frac{1}{2}n^2\langle x\rangle^2+\frac{1}{2}n\mathrm{var}(x)+\frac{1}{8}(\mathrm{var}(x))^2+\varkappa^2\alpha(-in\langle x\rangle \\
&- \frac{1}{2}\mathrm{var}(x))+\frac{1}{2}in\langle x\rangle\mathrm{var}(x)-\frac{i}{3}(\frac{1}{2}\left\langle x^3\right\rangle+\langle x\rangle^3+\frac{3}{2}\langle x\rangle\left\langle x^2\right\rangle)\Bigg]+O(\frac{1}{N^3})\Bigg\}
\end{aligned} \tag{4.33}
$$

Now we consider the coefficient of $\frac{1}{N}$

$$
\text{coefficient of } \frac{1}{N} = \sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n}\left(\prod_{j=1}^{n}(P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right)e^{i\langle x\rangle}\left(\varkappa^2\alpha-in\langle x\rangle-\frac{1}{2}\mathrm{var}(x)\right) \tag{4.34}
$$

This equation includes three terms. The product of the first two terms is

$$\begin{aligned}
\left(\prod_{j=1}^{n} P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right) e^{i\langle x\rangle} &= \left(\prod_{j=1}^{n} P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right) e^{2i\varkappa^2 S(t)\sum_m p_m w^{(m)}\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})} \\
&= \prod_{j=1}^{n} P_j^{(m_j)} e^{-2i\varkappa^2 S(t)\langle W\rangle W}\rho_0 e^{2i\varkappa^2 S(t)\langle W\rangle W} P_j^{(m'_j)} \\
&= \prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)} \qquad (4.35)
\end{aligned}$$

Where $\tilde{\rho}_0$ is given in (3.30). Hence,

$$\text{coefficient of}\frac{1}{N} = \sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P^{(m'_j)}\right)\left(\varkappa^2\alpha - in\langle x\rangle - \frac{1}{2}\mathrm{var}(x)\right) \qquad (4.36)$$

Since this is a complicated formula we need to expand and simplify term by term. We compute this in three steps as follows

Step 1: Recall that α is given in (4.24)

$$\begin{aligned}
\sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)}\right)\varkappa^2\alpha &= \sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)}\right)\varkappa^2 \\
&\times \Bigg\{[iS(t)-\Gamma(t)][\sum_j w^{(m'_j)}]^2 + [-iS(t) \\
&- \Gamma(t)][\sum_j w^{(m_j)}]^2 + 2\Gamma(t)\sum_{j,k} w^{(m'_j)}w^{(m_k)}\Bigg\} \\
&= I + II + III \qquad (4.37)
\end{aligned}$$

Remark 4.1. *Let $W = \sum_m w^{(m)}P^{(m)}$ where $w^{(m)}$ are eigenvalue of matrix W. Then we have*

$$\begin{aligned}
P^{(m)}W &= P^{(m)}\sum_n w^{(n)}P^{(n)} \\
&= \sum_n w^{(n)}P^{(m)}P^{(n)} \\
&= w^{(m)}P^{(m)}P^{(m)} \\
&= w^{(m)}P^{(m)} \\
&= P^{(m)}w^{(m)} \qquad (4.38)
\end{aligned}$$

and

$$W^2 = \sum_n (w^{(n)})^2 P^{(n)} \tag{4.39}$$

Now we first compute I as following

$$\begin{aligned}
I &= \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} \left(\prod_{j=1}^{n} P_j^{(m_j)} \tilde{\rho}_0 P_j^{(m'_j)} \right) \varkappa^2 [iS(t) - \Gamma(t)] [\sum_j w^{(m'_j)}]^2 \\
&= \varkappa^2 (iS(t) - \Gamma(t)) \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} P_1^{(m_1)} \tilde{\rho}_0 P_1^{(m'_1)} \otimes \cdots \otimes P_n^{(m_n)} \tilde{\rho}_0 P_n^{(m'_n)} [\sum_j w^{(m'_j)}]^2 \\
&= \varkappa^2 (iS(t) - \Gamma(t)) \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} P_1^{(m_1)} \tilde{\rho}_0 P_1^{(m'_1)} \otimes \cdots \otimes P_n^{(m_n)} \tilde{\rho}_0 P_n^{(m'_n)} \\
&\times [\sum_j (w^{(m'_j)})^2 + 2 \sum_{j<k} w^{(m'_j)} w^{(m'_k)}] \\
&= \varkappa^2 (iS(t) - \Gamma(t)) \sum_j \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} P_1^{(m_1)} \tilde{\rho}_0 P_1^{(m'_1)} \otimes \cdots \otimes P_j^{(m_j)} \tilde{\rho}_0 P_j^{(m'_j)} (w^{(m'_j)})^2 \\
&\otimes \cdots \otimes P_n^{(m_n)} \tilde{\rho}_0 P_n^{(m'_n)} \\
&+ 2\varkappa^2 (iS(t) - \Gamma(t)) \sum_{j<k} \sum_{m_1,\ldots,m_n} \sum_{m'_1,\ldots,m'_n} P_1^{(m_1)} \tilde{\rho}_0 P_1^{(m'_1)} \otimes \cdots \otimes P_j^{(m_j)} \tilde{\rho}_0 P_j^{(m'_j)} w^{(m'_j)} \\
&\otimes \cdots \otimes P_k^{(m_k)} \tilde{\rho}_0 P_k^{(m'_k)} w^{(m'_k)} \otimes \cdots \otimes P_n^{(m_n)} \tilde{\rho}_0 P_n^{(m'_n)}
\end{aligned} \tag{4.40}$$

Since $\sum_{m_j=1}^{n} P_j^{(m_j)} = 1$ we obtain

$$\begin{aligned}
I &= \varkappa^2 (iS(t) - \Gamma(t)) \sum_j \tilde{\rho}_0 \otimes \cdots \otimes \underbrace{\tilde{\rho}_0 W^2}_{jth} \otimes \cdots \otimes \tilde{\rho}_0 \\
&+ 2\varkappa^2 (iS(t) - \Gamma(t)) \sum_{j<k} \tilde{\rho}_0 \otimes \cdots \otimes \underbrace{\tilde{\rho}_0 W}_{jth} \otimes \cdots \otimes \underbrace{\tilde{\rho}_0 W}_{kth} \otimes \cdots \otimes \tilde{\rho}_0 \\
&= \varkappa^2 (iS(t) - \Gamma(t)) \left(\sum_l R_l(W^2) \tilde{\rho}_0 \otimes \cdots \otimes \tilde{\rho}_0 + 2 \sum_{l<k} R_l(W) R_k(W) \tilde{\rho}_0 \otimes \cdots \otimes \tilde{\rho}_0 \right) \\
&= \varkappa^2 (iS(t) - \Gamma(t)) \sum_{l,k} R_l(W) R_k(W) \tilde{\rho}_0 \otimes \cdots \otimes \tilde{\rho}_0
\end{aligned} \tag{4.41}$$

We use here the notation introduced in definition 19. We carry out a similar argument for the term II in (4.37) and obtain

$$
\begin{aligned}
II &= \sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)}\right)\varkappa^2[-iS(t)-\Gamma(t)][\sum_j w^{(m_j)}]^2 \\
&= \varkappa^2(-iS(t)-\Gamma(t))\sum_{l,k} L_l(W)L_k(W)\tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0 \qquad (4.42)
\end{aligned}
$$

Next, we calculate

$$
\begin{aligned}
III &= \sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)}\right)2\varkappa^2\Gamma(t)\sum_{j,k} w^{(m'_j)}w^{(m_k)} \\
&= 2\varkappa^2\Gamma(t)\sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n} P_1^{(m_1)}\tilde{\rho}_0 P_1^{(m'_1)}\otimes\cdots\otimes P_n^{(m_n)}\tilde{\rho}_0 P_n^{(m'_n)}[\sum_{j,k} w^{(m'_j)}w^{(m_k)}] \\
&= 2\varkappa^2\Gamma(t)\sum_{j,k}\sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n} P_1^{(m_1)}\tilde{\rho}_0 P_1^{(m'_1)}\otimes\cdots\otimes P_j^{(m_j)}\tilde{\rho}_0 w^{(m'_j)}P_j^{(m'_j)}\otimes\cdots \\
&\otimes\; P_k^{(m_k)}w^{(m_k)}\tilde{\rho}_0 P_k^{(m'_k)}\otimes\cdots\otimes P_n^{(m_n)}\tilde{\rho}_0 P_n^{(m'_n)} \\
&= 2\varkappa^2\Gamma(t)\sum_{j,k}\tilde{\rho}_0\otimes\cdots\otimes\underbrace{\tilde{\rho}_0 W}_{jth}\otimes\cdots\otimes\underbrace{W\tilde{\rho}_0}_{kth}\otimes\cdots\otimes\tilde{\rho}_0 \\
&= 2\varkappa^2\Gamma(t)\sum_{j,k} R_j(W)L_k(W)\tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0 \qquad (4.43)
\end{aligned}
$$

Combining(4.37) with (4.41), (4.42), and (4.43), we obtain

$$
\begin{aligned}
&\sum_{m_1,\dots,m_n}\sum_{m'_1,\dots,m'_n}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m'_j)}\right)\varkappa^2\alpha \\
&= \varkappa^2\Bigg(-\Gamma(t)[\sum_l R_l(W)-L_l(W)]^2 + iS(t)\left[\left(\sum_l R_l(W)\right)^2-\left(\sum_l L_l(W)\right)^2\right]\Bigg) \\
&\times\; \tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0
\end{aligned}
$$

(4.44)

Step 2: We analyze the second term in (4.36). Proceeding above, we obtain

$$\begin{aligned}
& \sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right) - in\langle x\rangle \\
= & \sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right)\left\{ -2in\varkappa^2 S(t)\langle W\rangle \sum_j \left(w^{(m_j')} - w^{(m_j)}\right)\right\} \\
= & -2in\varkappa^2 S(t)\langle W\rangle \sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'} P_1^{(m_1)}\tilde{\rho}_0 P_1^{(m_1')}\otimes\cdots\otimes P_n^{(m_n)}\tilde{\rho}_0 P_n^{(m_n')} \\
\times & \sum_j \left(w^{(m_j')} - w^{(m_j)}\right) \\
= & -2in\varkappa^2 S(t)\langle W\rangle [\sum_l R_l(W) - L_l(W)]\tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0
\end{aligned} \tag{4.45}$$

Step 3: Similar to step 1 we can compute the last term in (4.36),

$$\begin{aligned}
& -\frac{1}{2}\sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right)\mathrm{var}(x) \\
= & -\frac{1}{2}\sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right)(\langle x^2\rangle - \langle x\rangle^2) \\
= & -2\varkappa^4 S(t)^2\mathrm{var}(w)\sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right)\left(\sum_j w^{(m_j')} + \sum_j w^{(m_j)}\right)^2 \\
= & -2\varkappa^4 S(t)^2\mathrm{var}(w)\left(\sum_l R_l(W) - L_l(W)\right)^2 \tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0
\end{aligned} \tag{4.46}$$

Thus, by equations (4.44), (4.45), (4.46) and Definition 20 we have

$$\begin{aligned}
& \sum_{m_1,\dots,m_n}\sum_{m_1',\dots,m_n'}\left(\prod_{j=1}^{n} P_j^{(m_j)}\tilde{\rho}_0 P_j^{(m_j')}\right)\left(\varkappa^2\alpha - in\langle x\rangle - \frac{1}{2}\mathrm{var}(x)\right) = \\
& \Big\{ - \varkappa^2\Gamma(t)\,[d\Gamma(W_R - W_L)]^2 + i\varkappa^2 S(t)\left([d\Gamma(W_R)]^2 - [d\Gamma(W_L)^2]\right) \\
& - 2in\varkappa^2 S(t)\langle W\rangle\, d\Gamma(W_R - W_L) - 2\varkappa^4 S(t)^2\mathrm{var}(W)[d\Gamma(W_R - W_L)]^2 \Big\} \\
& \times \tilde{\rho}_0\otimes\cdots\otimes\tilde{\rho}_0
\end{aligned} \tag{4.47}$$

where $W_R\tilde{\rho}_0 := \tilde{\rho}_0 W$ and $W_L\tilde{\rho}_0 := W\tilde{\rho}_0$. Therefore,according to (4.33),(4.34) and (4.47) the expansion of reduced density matrix is

$$
\begin{aligned}
\rho_{n,N}(t) = \rho_{n,\infty} + \frac{1}{N} \Bigg(& -\varkappa^2\Gamma(t)\left[d\Gamma(W_R - W_L)\right]^2 + i\varkappa^2 S(t)\left([d\Gamma(W_R)]^2 - [d\Gamma(W_L)]^2\right) \\
- & \ 2in\varkappa^2 S(t)\, \langle W \rangle\, d\Gamma(W_R - W_L) \\
- & \ 2\varkappa^4 S(t)^2 \mathrm{var}(W) [d\Gamma(W_R - W_L)]^2 \Bigg) \tilde{\rho}_0 \otimes \cdots \otimes \tilde{\rho}_0 \\
+ & \ R_{Main}
\end{aligned} \tag{4.48}
$$

This shows the expression (3.28) with $\rho_n^{(1)}(t)$ given in (3.29).

□

4.2 Proof of Lemma 3.1

From(4.25)

$$
\rho_{n,N}(t) = \sum_{m_1,\ldots,m_n} \sum_{m_1',\ldots,m_n'} \prod_{j=1}^{n} \left(P_j^{(m_j)} \rho_0 P_j^{(m_j')} \right) e^{\frac{\varkappa^2 \alpha}{N}} \left[\sum_m p_m e^{\frac{i x_m}{N}} \right]^{N-n} \tag{4.49}
$$

where α and x_m are given in (4.24). Now we expand

$$
\begin{aligned}
e^{\frac{\varkappa^2 \alpha}{N}} &= 1 + \frac{1}{N}\varkappa^2\alpha + \sum_{k=2}^{\infty} \frac{(\varkappa^2\alpha)^k}{N^k} \frac{1}{k!} \\
&= 1 + \frac{1}{N}\varkappa^2\alpha + \underbrace{\left(\frac{\varkappa^2\alpha}{N}\right)^2 \sum_{k=0}^{\infty} \frac{(\varkappa^2\alpha)^k}{N^k} \frac{1}{(k+2)!}}_{=:R}
\end{aligned} \tag{4.50}
$$

We have $|\alpha| < \eta$ where $\eta := 4n^2 \|W\|^2 (|S(t)| + |\Gamma(t)|)$

$$
\sum_{k=0}^{\infty} \left(\frac{\varkappa^2\alpha}{N}\right)^k \frac{1}{(k+2)!} < \sum_{k=0}^{\infty} \left(\frac{\varkappa^2\alpha}{N}\right)^k \frac{1}{k!} = e^{\frac{\varkappa^2\alpha}{N}} \tag{4.51}
$$

then we have

$$
|R| \le \frac{1}{N^2} \varkappa^4 \eta^2 e^{\frac{\varkappa^2}{N}\eta} \tag{4.52}
$$

Moreover,

$$
\left[\sum_m p_m e^{\frac{i x_m}{N}}\right]^{N-n} = e^{(N-n)\log \sum_m p_m e^{\frac{i x_m}{N}}} = e^{(N-n)\log(1+a)} \tag{4.53}
$$

Where

$$a := \sum_m p_m \left(e^{\frac{ix_m}{N}} - 1\right) \tag{4.54}$$

Expanding the logarithm we have (for $|a| < 1$)

$$\log(1+a) = a - \frac{a^2}{2} + a^3 \sum_{k=0}^{\infty} \frac{(-1)^k}{k+3} a^k \tag{4.55}$$

In addition, we know $\left|\sum_{k=0}^{\infty} \frac{(-1)^k}{k+3} a^k\right|$ is bounded above by the geometric series, $\frac{1}{1-|a|} < 2$. Thus

$$\log(1+a) = a - \frac{a^2}{2} + a^3 R_1 \tag{4.56}$$

with $|R_1| < 2$. Note that $a = \sum_{k=1}^{\infty} (\frac{i}{N})^k \frac{1}{k!} (\sum_m p_m x_m^k)$. Therefore, we have

$$(N-n)\log(1+a) = (N-n)\,(a - \frac{a^2}{2}) + R_2 \tag{4.57}$$

with $|R_2| \leq 2(N-n)|\sum_m p_m (e^{\frac{ix_m}{N}} - 1)|^3$and we know for x_m real, we have $|e^{\frac{ix_m}{N}} - 1| = |\int_0^{\frac{ix_m}{N}} e^{iy} dy| \leq \frac{|x_m|}{N} = \frac{\xi}{N}$ (by the definition of ξ), so

$$|R_2| \leq 2(N-n)|\sum_m p_m (e^{\frac{ix_m}{N}} - 1)|^3 \leq 2(N-n) \max_m |e^{\frac{ix_m}{N}} - 1|^3 \leq 2(N-n)\frac{\xi^3}{N^3} \leq 2\frac{\xi^3}{N^2} \tag{4.58}$$

Now we expand first term of (4.57) with 3 steps **Step 1:** First we multiply $N-n$ and a

$$\begin{aligned}
(N-n)a &= (N-n)\left(\sum_m p_m (e^{\frac{ix_m}{N}} - 1)\right) = (N-n)\left(\sum_m p_m \sum_{k\geq 1} \frac{(ix_m)^k}{k! N^k}\right) \\
&= i\sum_m p_m x_m - \frac{1}{N}\left(in \sum_m p_m x_m\right) \\
&- \frac{1}{N}\left(\frac{1}{2}\sum_m p_m x_m^2\right) + \frac{1}{N^2}\left(\frac{n}{2}\sum_m p_m x_m^2\right) \\
&+ \sum_m p_m (ix_m)^3 \frac{N-n}{N^3}\left(\frac{1}{3!} + \frac{ix_m}{N}\sum_{k\geq 0} (\frac{ix_m}{N})^k \frac{1}{(k+4)!}\right)
\end{aligned} \tag{4.59}$$

Step 2: second we multiply $N-n$ and $\frac{a^2}{2}$

$$
\begin{aligned}
(N-n)\frac{a^2}{2} &= \frac{1}{2}(N-n)\left(\sum_m p_m(e^{\frac{ix_m}{N}}-1)\right)^2 = \frac{1}{2}(N-n)\left(\sum_m p_m\sum_{k\geq 1}\frac{(ix_m)^k}{k!N^k}\right)^2 \\
&= \frac{1}{2}(N-n)\Bigg[-\frac{1}{N^2}\left(\sum_m p_m x_m\right)^2 + \left(\sum_m p_m\frac{(ix_m)^2}{N^2}\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)^2 \\
&+ 2\left(\frac{i}{N}\sum_{m'}p_{m'}x_{m'}\right)\left(-\frac{1}{N^2}\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)\Bigg] \\
&= -\frac{1}{2N}\left(\sum_m p_m x_m\right)^2 + \frac{1}{2N^2}\Bigg[-2i\left(\sum_{m'}p_{m'}x_{m'}\right) \\
&\times \left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right) + n\left(\sum_m p_m x_m\right)^2\Bigg] \\
&+ \frac{in}{N^3}\left(\sum_{m'}p_{m'}x_{m'}\right)\left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right) \\
&+ \frac{N-n}{2N^4}\left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)^2
\end{aligned}
\tag{4.60}
$$

Step 3: Then we add final terms from step 1 and 2

$$
(N-n)(a-\frac{a^2}{2}) = i\sum_m p_m x_m - \frac{1}{N}\left(in\sum_m p_m x_m + \frac{1}{2}\sum_m p_m x_m^2 - \frac{1}{2}(\sum_m p_m x_m)^2\right) + R_3
\tag{4.61}
$$

where

$$
\begin{aligned}
R_3 &= \frac{1}{N^2}\Bigg[\frac{n}{2}\left(\sum_m p_m x_m^2 - (\sum_m p_m x_m)^2\right) \\
&+ i\left(\sum_{m'}p_{m'}x_{m'}\right)\left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)\Bigg] \\
&- \frac{1}{N^3}\Bigg[in\left(\sum_{m'}p_{m'}x_{m'}\right)\left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)\Bigg] \\
&+ \frac{N-n}{N^3}\sum_m p_m(ix_m)^3\left(\frac{1}{3!}+\frac{ix_m}{N}\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+4)!}\right) \\
&+ \frac{N-n}{2N^4}\left(\sum_m p_m x_m^2\sum_{k\geq 0}(\frac{ix_m}{N})^k\frac{1}{(k+2)!}\right)^2
\end{aligned}
\tag{4.62}
$$

We know by assumption that $N > 2\xi \Rightarrow \frac{\xi}{N} < \frac{1}{2}$ and, since $|x_m| \leq \xi$

$$\left|\sum_{k\geq 0}\left(\frac{ix_m}{N}\right)^k \frac{1}{(k+2)!}\right| \leq \sum_{k\geq 0}\left|\frac{ix_m}{N}\right|^k \frac{1}{(k+2)!} \leq \sum_{k\geq 0}(\frac{1}{2})^k \frac{1}{(k+2)!} < 1 \tag{4.63}$$

Therefore,

$$\begin{aligned} |R_3| &\leq \frac{1}{N^2}\left(n\xi^2+\xi^3\right)+\frac{n\xi^3}{N^3}+\frac{2}{3}\frac{\xi^3}{N^2}+\frac{\xi^4}{2N^3} \\ &= \frac{1}{N^2}\left(n\xi^2+\xi^3+\frac{n\xi^3}{N}+\frac{2}{3}\xi^3+\frac{\xi^4}{2N}\right) \\ &\leq \frac{1}{N^2}\left(n\xi^2+\frac{n}{2}\xi^2+\frac{\xi^3}{4}+\frac{5}{3}\xi^3\right) \\ &= \frac{\xi^2}{N^2}\left(\frac{3}{2}n+\frac{23}{12}\xi\right) \end{aligned} \tag{4.64}$$

Now we add R_2 and R_3 and we call it R_4

$$\begin{aligned} (N-n)\log(1+a) &= i\sum_m p_m x_m - \frac{1}{N}\left(in\sum_m p_m x_m + \frac{1}{2}\sum_m p_m x_m^2 - \frac{1}{2}(\sum_m p_m x_m)^2\right) \\ &+ R_4 \end{aligned} \tag{4.65}$$

where

$$|R_4| \leq |R_2| + |R_3| \leq \frac{2\xi^3}{N^2} + \frac{\xi^2}{N^2}(\frac{3}{2}n + \frac{23}{12}\xi) = \frac{\xi^2}{N^2}(\frac{47}{12}\xi + \frac{3}{2}n). \tag{4.66}$$

Hence, by equations (4.53) and (4.66) we have

$$e^{(N-n)\log(1+a)} = e^{i\sum_m p_m x_m - \frac{1}{N}\left(in\sum_m p_m x_m + \frac{1}{2}\sum_m p_m x_m^2 - \frac{1}{2}(\sum_m p_m x_m)^2\right)} \times e^{R_4} \tag{4.67}$$

Also, since $\left|e^{R_4}-1\right| \leq |R_4|\, e^{|R_4|}$, we conclude that

$$(1+a)^{N-n} = e^{i\sum_m p_m x_m - \frac{1}{N}\left(in\sum_m p_m x_m + \frac{1}{2}\sum_m p_m x_m^2 - \frac{1}{2}(\sum_m p_m x_m)^2\right)} + R_5 \tag{4.68}$$

where R_5 has following bound

$$|R_5| \leq \frac{\xi^2}{N^2}\left(\frac{47}{12}\xi + \frac{3}{2}n\right) e^{\frac{\xi^2}{N^2}\left(\frac{47}{12}\xi + \frac{3}{2}n\right)} \tag{4.69}$$

We have used here that the first exponential on the right side of (4.67) has modulus bounded above by 1. Thus,

$$\begin{aligned}
(1+a)^{N-n} &= e^{i\langle x\rangle} e^{\frac{-in}{N}\langle x\rangle} e^{-\frac{1}{2}\frac{\mathrm{var}(x)}{N}} + R_5 \\
&= e^{i\langle x\rangle}\left(1 - \frac{in}{N}\langle x\rangle + \sum_{k=2}^{\infty}\left(\frac{in\langle x\rangle}{N}\right)^k \frac{(-1)^k}{k!}\right) \\
&\times \left(1 - \frac{1}{2}\frac{\mathrm{var}(x)}{N} + \sum_{k=2}^{\infty}\left(\frac{1}{2}\frac{\mathrm{var}(x)}{N}\right)^k \frac{(-1)^k}{k!}\right) + R_5 \\
&= e^{i\langle x\rangle}\left(1 - \frac{in}{N}\langle x\rangle + A\right)\left(1 - \frac{1}{2}\frac{\mathrm{var}(x)}{N} + B\right) + R_5 \\
&= e^{i\langle x\rangle}\left(1 - \frac{in}{N}\langle x\rangle - \frac{1}{2}\frac{\mathrm{var}(x)}{N}\right) + e^{i\langle x\rangle} R_6 + R_5 \qquad (4.70)
\end{aligned}$$

where

$$|R_6| \leq |A| + |B| + |A|\,|B| + \frac{1}{2}\left|\frac{n\langle x\rangle\,\mathrm{var}(x)}{N^2}\right| + \frac{1}{2}\left|\frac{\mathrm{var}(x)}{N}\right| |A| + \left|\frac{in\langle x\rangle}{N}\right| |B| \qquad (4.71)$$

where A and B have following bounds

$$|A| \leq \left|\frac{n\langle x\rangle}{N}\right|^2 \sum_{k=0}^{\infty}\left|\frac{in\langle x\rangle}{N}\right|^k \frac{1}{k!} \leq \frac{n^2\xi^2}{N^2} e^{\xi} \qquad (4.72)$$

$$|B| \leq \frac{1}{4}\left|\frac{\mathrm{var}(x)}{N}\right|^2 \sum_{k=0}^{\infty}\left|\frac{\mathrm{var}(x)}{2N}\right|^k \frac{1}{k!} \leq \frac{\xi^4}{N^2} e^{\frac{\xi}{2}} \qquad (4.73)$$

Therefore, we have

$$|R_6| \leq \frac{\xi^2}{N^2}\left\{\frac{1}{2}\xi^2 + ne^{\xi}(n+\xi^2) + \xi^2 e^{\frac{1}{2}\xi}(1+\xi+\xi e^{\xi})\right\} \qquad (4.74)$$

Now we add last two terms of equation (4.70) and call it R_7, then equation (4.70) will be

$$(1+a)^{(N-n)} = e^{i\langle x\rangle}\left(1 - \frac{in}{N}\langle x\rangle - \frac{1}{2}\frac{\mathrm{var}(x)}{N}\right) + R_7 \qquad (4.75)$$

To find bound of R_7 we need bounds of R_5 and R_6 which are equations (4.69) and (4.74), respectively. Then we have

$$\begin{aligned} |R_7| &\leq \left|e^{i\langle x\rangle}R_6\right| + |R_5| \\ &\leq \frac{\xi^2}{N^2}\left\{\left(\frac{47}{12}\xi+\frac{3}{2}n\right)e^{\frac{\xi^2}{N^2}\left(\frac{47}{12}\xi+\frac{3}{2}n\right)}+\frac{1}{2}\xi^2+ne^{\xi}(n+\xi^2)+\xi^2e^{\frac{1}{2}\xi}(1+\xi+\xi e^{\xi})\right\} \end{aligned} \tag{4.76}$$

Going back to (4.49), we multiply two terms as

$$e^{\frac{\varkappa^2\alpha}{N}}\left[\sum_m p_m e^{\frac{ix_m}{N}}\right]^{N-n} = \left\{1+\frac{1}{N}\varkappa^2\alpha+R\right\}\left\{e^{i\langle x\rangle}\left(1-\frac{in}{N}\langle x\rangle-\frac{1}{2}\frac{\mathrm{var}(x)}{N}\right)+R_7\right\} \tag{4.77}$$

where $|R| \leq \frac{\varkappa^4\eta^2}{N^2}e^{\frac{\varkappa^2\eta}{N}}$ and $\eta := 4n^2\left\|W\right\|^2(|S(t)|+|\Gamma(t)|)$ so we have

$$e^{\frac{\varkappa^2\alpha}{N}}\left[\sum_m p_m e^{\frac{ix_m}{N}}\right]^{N-n} = e^{i\langle x\rangle}+\frac{1}{N}\left\{-ie^{i\langle x\rangle}n\langle x\rangle - e^{i\langle x\rangle}\frac{\mathrm{var}(x)}{2}+e^{i\langle x\rangle}\varkappa^2\alpha\right\}+R_T \tag{4.78}$$

where

$$\begin{aligned} R_{\mathrm{T}} &= \frac{1}{N^2}\varkappa^2\alpha\left(-ine^{\langle x\rangle}\langle x\rangle - e^{i\langle x\rangle}\frac{\mathrm{var}(x)}{2}\right)+\frac{1}{N}\varkappa^2\alpha R_7 \\ &+ \frac{1}{N}R\left(-ine^{i\langle x\rangle}\langle x\rangle - e^{i\langle x\rangle}\frac{\mathrm{var}(x)}{2}\right)+R_7+Re^{i\langle x\rangle}+RR_7 \end{aligned} \tag{4.79}$$

If we define

$$M := \left(\frac{47}{12}\xi+\frac{3}{2}n\right)e^{\frac{\xi^2}{N^2}\left(\frac{47}{12}\xi+\frac{3}{2}n\right)}+\frac{1}{2}\xi^2+ne^{\xi}(n+\xi^2)+\xi^2e^{\frac{1}{2}\xi}(1+\xi+\xi e^{\xi}) \tag{4.80}$$

such that $|R_7| \leq \frac{\xi^2}{N^2}M$, then we have following bound

$$|R_T| \leq \frac{1}{N^2}\left\{\xi^2M+\varkappa^2\eta\xi\left(\frac{1}{2}M+n+\xi\right)+\varkappa^4\eta^2e^{\frac{\varkappa^2\eta}{n}}\left(1+\xi+\frac{1}{2}\xi+\frac{1}{4}M\right)\right\} \tag{4.81}$$

Thus, from (4.49)

$$\begin{aligned}
\rho_{n,N}(t) &= \sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\prod_{j=1}^{n}\left(P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right)e^{\frac{\varkappa^2\alpha}{N}}\left[\sum_m p_m e^{\frac{ix_m}{N}}\right]^{N-n} \\
&= \sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\prod_{j=1}^{n}\left(P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right) \\
&\times \left\{e^{i\langle x\rangle}+\frac{1}{N}\left\{\left(-ine^{i\langle x\rangle}\langle x\rangle - e^{i\langle x\rangle}\frac{\mathrm{var}(x)}{2}\right)+e^{i\langle x\rangle}\varkappa^2\alpha\right\}+R_T\right\} \\
&= \rho_{n,\infty}+\frac{1}{N}\rho_n^{(1)}(t)+R_{Main} \qquad (4.82)
\end{aligned}$$

Where $R_{Main}=\sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}\prod_{j=1}^{n}\left(P_j^{(m_j)}\rho_0 P_j^{(m'_j)}\right)R_T$. R_{Main} is an operator on the n-fold tensor product $\mathcal{H}_n := \mathcal{H}\otimes\cdots\otimes\mathcal{H}$ where $\mathcal{H}$ is the Hilbert space of a single particle. Since the space of bounded linear operators on $\mathcal{H}_n$, denoted by $\mathcal{B}(\mathcal{H}_n)$ is the dual space of the Banach space $L_1(\mathcal{H}_n)$ of trace-class operators on $\mathcal{H}_n$ (with norm $\|x\|_1 = \mathrm{Tr}|x|$), we have

$$\|R_{Main}\|_1 = \sup_{B\in(\mathcal{H}_n),\|B\|=1}|\mathrm{Tr}R_{Main}B| \qquad (4.83)$$

Let B be a bounded operator on $\mathcal{H}_n$. By cyclicity of the trace we have

$$\mathrm{Tr}R_{Main}B = \mathrm{Tr}\left(\rho_0\otimes\cdots\otimes\rho_0\sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}R_T\left(\prod_{j=1}^{n}P_j^{(m'_j)}\right)B\left(\prod_{j=1}^{n}P_j^{(m_j)}\right)\right) \qquad (4.84)$$

Since $|\mathrm{Tr}XY|\le\|Y\|\,\mathrm{Tr}|X|$ and because a density matrix has trace one, we obtain

$$\begin{aligned}
|\mathrm{Tr}R_{Main}B| &\le \left\|\sum_{m_1,\ldots,m_n}\sum_{m'_1,\ldots,m'_n}R_T\left(\prod_{j=1}^{n}P_j^{(m'_j)}\right)B\left(\prod_{j=1}^{n}P_j^{(m_j)}\right)\right\| \\
&\le \|B\|\,d^{2n}sup_{m_1,\ldots,m_n,m'_1,\ldots,m'_n}R_T \\
&\le \|B\|\,d^{2n}\,|R_T| \qquad (4.85)
\end{aligned}$$

where $d=\dim\mathcal{H}_S$. Thus, $\|R_{Main}\|\le d^{2n}\,|R_T|$. □

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