# SOME REGULARITY ESTIMATES FOR MILD SOLUTIONS TO FRACTIONAL HEAT-TYPE AND NAVIER-STOKES EQUATIONS

ZHICHUN ZHAI

# Some Regularity Estimates for Mild Solutions to Fractional Heat-type and Navier-Stokes Equations

by

©Zhichun Zhai, B.S, M.S.

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# Abstract


This work studies some regularity estimates for mild solutions to fractional heat-type and Navier-Stokes equations. Indeed, we mainly focus on the following topics: Carleson measures for homogenous fractional heat-type equations and their Cauchy problems, Strichartz type estimates for the inhomogeneous initial value problems of fractional heat-type equations, and the well-posedness and regularity for fractional Navier-Stokes equations.

Firstly, we characterize nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ having the property $\|u\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}$, $1 \le p \le q < \infty$, whenever $u(t,x) \in b^p_\beta(\mathbb{R}^{1+n}_+) \cap \dot{W}^{1,p}(\mathbb{R}^{1+n}_+)$. Here $b^p_\beta(\mathbb{R}^{1+n}_+)$ (see [57]) is the space of solutions having finite $L^p(\mathbb{R}^{1+n}_+)$ norm to the homogeneous fractional heat-type equations

$$\partial_t v(t,x) + (-\triangle)^\beta v(t,x) = 0,\ (t,x) \in \mathbb{R}^{1+n}_+$$

with $\beta \in (0,1]$. Denote the solution of the above equations with Cauchy data $v_0(x)$ by $v(t,x)$. Then, we give a characterization of nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ satisfying $\|v(t^{2\beta},x)\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}$, $s \in (0,n)$, $p \in [1, n/s]$ and $q \in (0,\infty)$.

For the inhomogeneous initial value problems of fractional heat-type equations, we obtain Strichartz estimates involving norms in Lebesgue spaces by using both the abstract Strichartz estimates of Keel-Tao and the Hardy-Littlewood-Sobolev inequality. Meanwhile, Strichartz type estimates involving norms in $BMO(\mathbb{R}^n)$, Sobolev and Besov spaces are established.

We introduce a new critical space $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$ which is useful for studying fractional Navier-Stokes equations. First, we give a Carleson measure characterization of $Q^\beta_\alpha(\mathbb{R}^n)$ by investigating a new type of tent spaces and an atomic decomposition of the predual for $Q^\beta_\alpha(\mathbb{R}^n)$. Then, via the Carleson measure characterization of $Q^\beta_\alpha(\mathbb{R}^n)$, we define $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$ as the derivative space of $Q^\beta_\alpha(\mathbb{R}^n)$ and study some properties of $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. In addition, we establish the mean oscillation characterization of $Q^\beta_\alpha(\mathbb{R}^n)$, John-Nirenberg and Gagliado-Nirenberg type inequalities in $Q^\beta_\alpha(\mathbb{R}^n)$.

Finally, using our results about Strichartz estimates and $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$, we prove the well-posedness and regularity for fractional Navier-Stokes equations in some Lebesgue spaces and $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$, $1/2 < \beta \le 1$. Especially, when $\beta = 1$, the well-posedness for incompressible Navier-Stokes equations in $Q^{1,-1}_{\alpha;\infty}(\mathbb{R}^n)$ was established by Xiao in [78].

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# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of positive integers |
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{R}^n$ | n dimensional Euclidean Space |
| $\mathbb{R}_+^{1+n}$ | n+1 dimensional upper half Space $(0, \infty) \times \mathbb{R}^n$ |
| $\mathbb{C}$ | field of complex numbers |
| $U \lesssim V$ | $U \leq CV$ for some positive number $C$ |
| $U \approx V$ | $U \lesssim V$ and $V \lesssim U$ |
| $C_0^\infty(\Omega)$ | the space of all smooth functions with compact support in $\Omega$. |
| $\mathcal{S}(\mathbb{R}^n)$ | the Schwartz class of rapidly decreasing functions |
| $\mathcal{S}'(\mathbb{R}^n)$ | the dual of $\mathcal{S}(\mathbb{R}^n)$ |

# Chapter 1

# Introduction

This work studies regularity estimates for mild solutions to fractional heat-type and Navier-Stokes equations. We mainly focus on Carleson measures and Strichartz type estimates for fractional heat-type equations, the existence and regularity of mild solutions to fractional Navier-Stokes equations. These two types of equations share a similar form as follows:

$$\begin{cases} \partial_t v(t,x) + Av(t,x) = F(t,x), & (t,x) \in \mathbb{R}^{1+n}_+; \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.0.1}$$

with operator $A$. For fractional Navier-Stokes equations, we assume that $\nabla \cdot v(t,x) = 0$ for vector $v(t,x)$, where $\nabla$ is the gradient with respect to the spatial variables. A common approach to find a solution $v(t,x)$ of (1.0.1) is to prove the existence of a fixed point $v(t,x)$ of the operator $T$ defined by

$$T(v)(t,x) = S(t)v_0(x) + \int_0^t S(t-s)F(s,x)ds$$

on an appropriate space. Here $S(t)$ is a semi-group with parameter $t$ generalized by $A$. This fixed point is called a mild solution of (1.0.1). For evolution equations similar to (1.0.1), we hope that a mild solution is smooth enough to qualify as a classical solution. However, for many evolution equations, we can not obtain the smoothness but estimates of a mild solution in some function spaces such as Lebesgue spaces. These size estimates are referred to regularity size estimates.

Carleson measure was firstly introduced (see Carleson [15] and Johnson [34]) as a means of describing measures for which solutions of the Dirichlet problem satisfied particular a priori estimates. As one of the most important concepts in modern analysis, Carleson measure has been applied in many areas, such as theory of partial differential equations, see Hastings [32], Johnson [34] and Xiao [76]-[77].

In Chapter 2, we consider Carleson measures for the homogeneous heat-type equations

$$\partial_t v(t,x) + (-\triangle)^\beta v(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+n}_+ \tag{1.0.2}$$

and their Cauchy problems

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) = 0, & (t,x) \in \mathbb{R}^{1+n}_+; \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.0.3}$$

with $\beta \in (0,1)$, where $\triangle$ is the Laplacian with respect to $x$ and

$$(-\triangle)^\beta u(t,x) = \mathcal{F}^{-1}(|\xi|^{2\beta}\mathcal{F}(u(t,\xi)))(x)$$

with $\mathcal{F}$ and $\mathcal{F}^{-1}$ being the Fourier transform and the inverse Fourier transform, respectively. Indeed, we are motivated by the case $\beta = 1$ which was studied by Xiao in [76]. The mild (classical) solution of equations (1.0.3) is given by

$$v(t,x) = e^{-t(-\triangle)^\beta} v_0(x) = S_\beta(t)v_0(x) := K_t^\beta(x) * v_0(x),$$

where

$$K_t^\beta(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^{2\beta}} d\xi \geq 0, \ \forall (t,x) \in \mathbb{R}^{1+n}_+ \tag{1.0.4}$$

and $g(x) * h(x)$ denotes the convolution between $g(x)$ and $f(x)$ on the spatial variables. More specifically, we characterize nonnegative Radon measures $\mu$ on $\mathbb{R}^{1+n}_+$ having either the property

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}, \quad \forall u(t,x) \in b^p_\beta(\mathbb{R}^{1+n}_+) \cap \dot{W}^{1,p}(\mathbb{R}^{1+n}_+),$$

for $1 \leq p \leq q < \infty$, where $b^p_\beta(\mathbb{R}^{1+n}_+)$ (see [57]) is the set of all solutions to (1.0.2) with finite $L^p(\mathbb{R}^{1+n}_+)$ norm, or the property

$$\|v(t^{2\beta},x)\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n),$$

for $s \in (0,n)$, $p \in [1, n/s]$ and $q \in (0,\infty)$, where $v(t,x)$ solves (1.0.3).

Our main results in Chapter 2 imply the following weighted mixed norm estimate for solution to (1.0.3): for $\beta \in (0,1)$, $s \in (0,n)$, $1 \leq p < n/s$ and $\gamma \in (-1,\infty)$,

$$\left(\int_{\mathbb{R}^{1+n}_+} |v(t,x)|^{\frac{p(1+n+\gamma)}{n-ps}} t^{\frac{\gamma+1-2\beta}{2\beta}} dtdx\right)^{\frac{n-ps}{p(1+n+\gamma)}} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

Such estimates are efficient to control the size of solutions to the linear problems in terms of the size of the initial datum when we perturb fractional heat-type equations. It turns out that for semi-linear equations such as nonlinear Schrödinger equations, nonlinear wave equations and nonlinear fractional heat-type equations, the estimates of solutions in mixed Lebesgue norms $L^q_t L^r_x(I \times \mathbb{R}^n)$, more generally mixed Sobolev norms $L^q_t W^{s,r}_x(I \times \mathbb{R}^n)$ and mixed Besov norms $L^q_t B^{s,r}_{x,p}(I \times \mathbb{R}^n)$, are called Strichartz estimates and particularly useful, see Tao [67].

In Chapter 3, we study Strichartz type estimates for the inhomogeneous initial problems associated with the fractional heat-type equations

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) = F(t,x), & (t,x) \in \mathbb{R}^{1+n}_+; \\ v(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.0.5}$$

where $\beta \in (0,\infty)$ and $n \in \mathbb{N}$. The work in this chapter is the main content of [82] which will appear in Journal of Mathematical Analysis and Applications.

By the Fourier transform and Duhamel's principle, the mild (classical) solution of (1.0.5) can be written as

$$v(t,x) = e^{-t(-\triangle)^\beta} f(x) + \int_0^t e^{-(t-s)(-\triangle)^\beta} F(s,x) ds.$$

The main goal of Chapter 3 is to determine pairs $(q,p)$ and $(q_1,p_1)$ ensuring

$$\|e^{-t(-\triangle)^\beta} f\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{1.0.6}$$

$$\left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} F(s,x) ds \right\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|F\|_{L^{q_1'}_t(I;L^{p_1'}_x(\mathbb{R}^n))}, \tag{1.0.7}$$

where $I$ is either $[0,\infty)$ or $[0,T]$ for some $0 < T < \infty$, and $p_1' = \frac{p_1}{p_1-1}$ is the conjugate of a given number $p_1 \geq 1$.

The Strichartz types estimates for (1.0.5) have just been studied by a few experts. Pierfelice [60] considered such estimates for (1.0.5) with $\beta = 1$ and small potentials of very low regularity. Miao-Yuan-Zhang in [53] studied the non-endpoint case of (1.0.6) for (1.0.5).

For the Schrödinger and wave equations, the Strichartz estimates have been well studied in recent years, see, for example, [62], [16], [38], [61], [10], [9], [36], [44], [55] and [79]. These estimates play an important role in the study of local and global existence for nonlinear equations, well posedness in Sobolev spaces with low order, scattering theory and many others, see, for example, [39], [40] and [24]. The Strichartz estimates for the Schrödinger and wave equations can be directly derived from the abstract Strichartz estimates of Keel-Tao [38] since the solution groups of these two equations act as unitary operators on $L^2(\mathbb{R}^n)$ and such operators obey both the energy estimate and the untruncated decay estimate. While, since $\{e^{-t(-\triangle)^\beta}\}_{t\geq 0}$ is a semigroup and acts as a self-adjoint operator on $L^2(\mathbb{R}^n)$ —see Lemma 3.2.1, we can only apply the abstract Strichartz estimates of Keel-Tao directly to obtain (1.0.6) if we have the energy estimate and untruncated decay estimate. But for (1.0.7), we can make use of the $L^p$–decay estimates and the Hardy-Littlewood-Sobolev inequality.

Moreover, if (1.0.5) has a time dependent potential $V(t,x)$, then it becomes

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) + V(t,x)v(t,x) = F(t,x), & (t,x) \in \mathbb{R}^{1+n}_+; \\ v(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.0.8}$$

We can obtain Strichartz estimates for (1.0.8) by using the Banach contraction mapping principle and assuming an appropriate integrability condition in space and time on $V(t,x)$.

A similar idea was used by D'Ancona-Pierfelice-Visciglia in [24] to get analogous estimates for the Schrödinger equations.

In addition, we also establish an endpoint case of (1.0.6) by replacing $L_x^\infty(\mathbb{R}^n)$ with the space of functions of bounded mean oscillation ($BMO_x(\mathbb{R}^n)$). Meanwhile, we obtain a parabolic homogeneous Strichartz estimate for (1.0.5), the two dimensional case of which is very useful for dealing with the global regularity of wave maps when combined with Lemma 3.2.2 for $\beta = 1$ and the comparison principle for heat equation, see Tao [68]. Moreover, we generalize (1.0.6) and (1.0.7) via replacing $L^p(\mathbb{R}^n)$ with either a Besov space or a Sobolev space. These function spaces will be made precise later.

In Chapter 4, we study $Q_\alpha^\beta(\mathbb{R}^n)$ which is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ with

$$\|f\|_{Q_\alpha^\beta(\mathbb{R}^n)} = \sup_I \left( (l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dxdy \right)^{1/2} < \infty$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

When $\beta = 1$, $Q_\alpha^\beta(\mathbb{R}^n)$ becomes $Q_\alpha(\mathbb{R}^n)$ which was introduced by Essen-Janson-Peng-Xiao in [25]. Xiao in [78] characterized $Q_\alpha(\mathbb{R}^n)$ equivalently as

$$\|f\|^2_{Q_\alpha(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r\in(0,\infty)} r^{2\alpha-n} \int_0^{r^2} \int_{|y-x|<r} |\nabla e^{t\triangle} f(y)|^2 t^{-\alpha} dtdy < \infty$$

and by this characterization he studied the classical Navier-Stokes equations in $Q^{-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla\cdot(Q_\alpha(\mathbb{R}^n))^n$ (Xiao [78]). The advantage of this equivalent characterization is the occurrence of $e^{t\triangle}$ which generates the mild solutions for the classical Navier-Stokes equations. Motivated by Xiao [78] and noting that the mild solutions for the fractional Navier-Stokes equations can be generated by $e^{-t(-\triangle)^\beta}$, to study the fractional Navier-Stokes equations in a space similar to $Q^{-1}_{\alpha;\infty}(\mathbb{R}^n)$, it is natural to introduce a new space $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ and characterize $Q_\alpha^\beta(\mathbb{R}^n)$ by the operator $e^{-t(-\triangle)^\beta}$. In fact, we should prove that $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if

$$\sup_{x\in\mathbb{R}^n, r\in(0,\infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |\nabla e^{-t(-\triangle)^\beta} f(y)|^2 t^{-\frac{\alpha}{\beta}} dydt < \infty. \tag{1.0.9}$$

Obviously, for each $j = 1, \cdots, n$, $\partial_j K_1^\beta(x) := \phi_j(x)$ is real $C^\infty(\mathbb{R}^n)$ satisfying:

$$\phi_j \in L^1(\mathbb{R}^n),\ |\phi_j(x)| \lesssim (1+|x|)^{-(n+1)},\ \int_{\mathbb{R}^n} \phi_j(x)dx = 0 \text{ and } (\phi_j)_t(x) = t^{-n}\phi_j\left(\frac{x}{t}\right). \tag{1.0.10}$$

Here $K_1^\beta(x)$ is defined in (1.0.4) when $t = 1$. This observation leads us to characterize $Q_\alpha^\beta(\mathbb{R}^n)$ more generally as

$$f \in Q_\alpha^\beta(\mathbb{R}^n) \Longleftrightarrow \sup_{x\in\mathbb{R}^n, r\in(0,\infty)} r^{2\alpha-n+2\beta-2} \int_0^r \int_{|y-x|<r} |f*\phi_t(y)|^2 t^{-(1+2(\alpha-\beta+1))} dtdy < \infty \tag{1.0.11}$$

by a general $C^\infty$ real-valued function $\phi$ on $\mathbb{R}^n$ with the property (1.0.10).

In order to get (1.0.11), inspired by Coifman-Meyer-Stein [19] and Dafni-Xiao [22], we introduce new tent spaces $T^1_{\alpha,\beta}$ and $T^\infty_{\alpha,\beta}$, then define a space $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ as a subspace of distributions in homogeneous Sobolev space $\dot{H}^{-\frac{n}{2}+2(\beta-1),2}$. Finally, we identify $Q^\beta_\alpha(\mathbb{R}^n)$ with the dual space of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. Then by (1.0.11), $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ is defined as the derivative space of $Q^\beta_\alpha(\mathbb{R}^n)$ and will be useful in studying the fractional Navier-Stokes equations in next chapter.

In addition, we establish the mean oscillation characterization of $Q^\beta_\alpha(\mathbb{R}^n)$. Using this characterization, we obtain John-Nirenberg type inequalities in $Q^\beta_\alpha(\mathbb{R}^n)$. Then, from a special John-Nirenberg type inequality, we get Gagliardo-Nirenberg type inequalities in $L^r(\mathbb{R}^n)\cap Q_\alpha(\mathbb{R}^n)$. Moreover, we deduce Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities from Gagliado-Nirenberg type inequalities in $Q_\alpha(\mathbb{R}^n)$.

John-Nirenberg inequality and Gagliardo-Nirenberg inequality are two classical inequalities in modern analysis and widely applied in theory of partial differential equations. There are many similar John-Nirenberg inequalities in different function spaces and many generalization of Gagliardo-Nirenberg inequalities which are very important for dealing with a prior estimates of various partial differential equations. In $Q_\alpha(\mathbb{R}^n)$, an analogue of the John-Nirenberg inequality was conjectured by Essen-Janson-Peng-Xiao in [25] and finally an modified version was established by Yue-Dafni [81]. Recently, in [18], Chen-Zhu established a Gagliardo-Nirenberg inequality for function in $L^r(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)$ via a John-Nirenberg inequality. In [43], Kozono-Wadade proved the generalized Gagliardo-Nirenberg inequalities in critical homogeneous Sobolev space and in $BMO(\mathbb{R}^n)$. From the generalized Gagliardo-Nirenberg inequality in $BMO(\mathbb{R}^n)$, they also get a John-Nirenberg inequality in $BMO(\mathbb{R}^n)$ used by Chen-Zhu in [18]. Obviously, in $BMO(\mathbb{R}^n)$, these two kinds of inequalities are equivalent.

In Chapter 5, by using the results obtained in previous chapters, we study the well-posedness and regularity of fractional Navier-Stokes system on the half-space $\mathbb{R}^{1+n}_+$, $n \geq 2$ :

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) + (v\cdot\nabla)v(t,x) - \nabla p(t,x) = h(t,x), & (t,x)\in\mathbb{R}^{1+n}_+; \\ \nabla\cdot v(t,x) = 0, & (t,x)\in\mathbb{R}^{1+n}_+; \\ v(0,x) = g(x), & x\in\mathbb{R}^n \end{cases} \tag{1.0.12}$$

with $\beta\in(1/2,1]$ and $\nabla$ is the gradient with respect to the spatial variables. The system (1.0.12) is very important since it becomes the classical Navier-Stokes system when $\beta = 1$ which is a celebrated nonlinear partial differential system. The mild solution for system (1.0.12) is

$$v(t,x) = e^{-t(-\triangle)^\beta}g(x) + \int_0^t e^{-(t-s)(-\triangle)^\beta}P(h - \nabla(v\otimes v))ds,$$

where $P$ is the Helmboltz-Weyl projection:

$$P = \{P_{j,k}\}_{j,k=1,\cdots,n} = \{\delta_{j,k} + R_j R_k\}_{j,k=1,\cdots,n}$$

with $\delta_{j,k}$ being the Kronecker symbol and $R_j = \partial_j(-\triangle)^{-1/2}$ being the Riesz transform.

In some Lebesgue spaces, we obtain the well-posedness and regularity for system (1.0.12) by applying the Strichartz type estimates established in Chapter 3. The well-posedness and regularity for system (1.0.12) with $h = 0$ are also established in the critical spaces $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla \cdot (Q^{\beta}_{\alpha}(\mathbb{R}^n))^n$ defined in Chapter 4. In addition, our regularity results apply to the classical incompressible Navier-Stokes equations with initial data in $Q^{1;-1}_{\alpha;\infty}(\mathbb{R}^n)$ where the well-posedness was established by Xiao in [78].

For equations (1.0.12) with $h = 0$, J. L. Lions [45] proved the global existence of the classical solutions when $\beta \geq \frac{5}{4}$ in dimension 3. Similar result holds for general dimension $n$ if $\beta \geq \frac{1}{2} + \frac{n}{4}$, see Wu [71]. For the important case $\beta < \frac{1}{2} + \frac{n}{4}$, Wu in [72]-[73] established the global existence for equations (1.0.12) with $h = 0$ in the homogeneous Besov spaces $\dot{B}^{r}_{2,\infty}(\mathbb{R}^n)$ with $r > \max\{1, 1 + \frac{n}{p} - 2\beta\}$, and $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n)$, where $1 \leq q \leq \infty$, and either $1/2 < \beta$ when $p = 2$ or $1/2 < \beta \leq 1$ when $2 < p < \infty$. For the corresponding regularity criteria, we refer the readers to [74]. The function spaces listed above are critical spaces. A space is called critical for equations (1.0.12) with $h = 0$ if it is invariant under the scaling

$$f_{\lambda}(x) = \lambda^{2\beta-1} f(\lambda x). \tag{1.0.13}$$

Note that, for $1 \leq q \leq \infty$ and $2 \leq p < \infty$, $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n)$ are continuously embedded in $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ which is also a critical space for (1.0.12) with $h = 0$. Thus, our well-posedness and regularity results in $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ generalize known results in $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n)$.

# Chapter 2

# Carleson Measures for Fractional Heat-Type Equations

This chapter studies the Carleson measures for fractional heat-type equations. We divide our discussion into two parts: Carleson measures for the fractional heat-type equations

$$\partial_t v(t,x) + (-\triangle)^\beta v(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+n}_+ \tag{2.0.1}$$

with $\beta \in (0,1]$ (called $\beta-$parabolic equations, see [57]), and the associated Cauchy problems

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) = 0, & (t,x) \in \mathbb{R}^{1+n}_+; \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.0.2}$$

Before stating our lemmas and main results, let us agree to some conventions in next section.

## 2.1 Notations and Preliminaries

In this chapter, we always assume that $s \in (0,n)\backslash\mathbb{N}$ when $p = 1$ or $n/s$. $\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)$ is the completion of $C_0^\infty(\mathbb{R}^{1+n}_+)$ with respect to the norm

$$\|f\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)} = \left(\int_{\mathbb{R}^{1+n}_+} |\nabla_{(t,x)} f|^p dt dx\right)^{1/p}.$$

$b^p_\beta(\mathbb{R}^{1+n}_+)(\beta \in (0,1])$ introduced by Nishio-Shimomura-Suzuki [57] is the parabolic Bergman space on $\mathbb{R}^{1+n}_+$, which is the set of all solutions of the parabolic equation (2.0.1) having finite $L^p(\mathbb{R}^{1+n}_+)$ norm. $\dot{W}^{s,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space which is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \begin{cases} \|(-\triangle)^{s/2} f\|_{L^p}, & p \in (1, n/s), \\ \left(\int\limits_{\mathbb{R}^n} \frac{\|\triangle_h^k f\|^p_{L^p}}{|h|^{n+ps}} dh\right)^{1/p}, & p = 1 \text{ or } p = n/s\ , s \in (0,n)\backslash\mathbb{N}, \end{cases}$$

where

$$\triangle_h^k f(x) = \begin{cases} \triangle_h^1 \triangle_h^{k-1} f(x), & k > 1, \\ f(x+h) - f(x), & k = 1, \end{cases}$$

$k = 1 + [\beta]$, $s = [s] + \{s\}$ with $\{s\} \in (0, 1)$.

If $X = \mathbb{R}_+^{1+n}$, $s = 1$ and $p \geq 1$, or $X = \mathbb{R}^n$, $s \in (0, n)$ and $p \in [1, n/s]$, $cap_{\dot{W}^{s,p}(X)}(S)$ (see Maz'ya [49]) is the variational capacity of an arbitrary set $S \subseteq X$ :

$$cap_{\dot{W}^{s,p}(X)}(S) = \inf\left\{\|f\|^p_{\dot{W}^{s,p}(X)} : f \in V_X(S)\right\}.$$

Here

$$V_{\mathbb{R}_+^{1+n}}(S) = \{f \in \dot{W}^{1,p}(\mathbb{R}_+^{1+n}) : S \subseteq \text{Int}(\{x \in \mathbb{R}_+^{1+n} : f \geq 1\})\}$$

and

$$V_{\mathbb{R}^n}(S) = \{f \in \dot{W}^{s,p}(\mathbb{R}^n) : f \geq 0, S \subseteq \text{Int}(\{x \in \mathbb{R}^n : f \geq 1\})\}$$

with $\text{Int}(E)$ be the interior of a set $E \subseteq X$. For $t \in (0, \infty)$, $c_p^\beta(\mu; t)$ is the $(p, \beta)-$variational capacity minimizing function associated with both $\dot{W}^{\beta,p}(\mathbb{R}^n)$ and a nonnegative measure $\mu$ on $\mathbb{R}_+^{1+n}$ defined by

$$c_p^s(\mu; t) = \inf\{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O) : \text{ bouded open } O \subseteq \mathbb{R}^n, \mu(T(O)) > t\},$$

where $T(O)$ is the tent based on an open subset $O$ of $\mathbb{R}^n$ :

$$T(O) = \{(r, x) \in \mathbb{R}_+^{1+n} : B(x, r) \subseteq O\},$$

with $B(x, r)$ be the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$.

For handling the endpoint case $p = n/s$ we also need the definition of the Riesz potentials (see Adams-Xiao [7] and Adams [2]) on $\mathbb{R}^{2n}$ as follows. The Riesz potential of order $\gamma \in (0, 2n)$ is defined by

$$I_\gamma^{(2n)} * f(z) = \int_{\mathbb{R}^{2n}} |z - y|^{\gamma - 2n} f(y) dy, \ z \in \mathbb{R}^{2n}.$$

From Adams [2, Theorem 5.1], we have that if $u(x)$ and $I_\gamma^{(2n)} * |f|(x, 0)$ are both in $L^1_{loc}(\mathbb{R}^n)$ with

$$f(x, h) = |h|^{-\gamma} \triangle_h^k u(x), \tag{2.1.1}$$

then $u(x) = CI_\gamma^{(2n)} * f(x, 0)$, for a.e. $x \in \mathbb{R}^n$ and some $C > 0$. Note that if $u \in \dot{W}^{s,n/s}(\mathbb{R}^n)$ and $\gamma = 2s \in (0, 2n)$ then the function $f(\cdot, \cdot)$ in (2.1.1) belongs to the space $L^{n/s}(\mathbb{R}^{2n})$. For any $\gamma \in (0, 2n)$, $\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n}) = I_\gamma^{2n} * L^p(\mathbb{R}^{2n})$ defined by $\|I_\gamma^{2n} * f\|_{\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n})} = \|f\|_{L^p_{2n}}$.

For $0 < p, q < \infty$ and a nonnegative Radon measure $\mu$ on $X = \mathbb{R}_+^{1+n}$ or $\mathbb{R}^n$, $L^{q,p}(X, \mu)$ and $L^q(X, \mu)$ denote the Lorentz space and the Lebesgue space of all functions $f$ on $X$ for which

$$\|f\|_{L^{q,p}(X,\mu)} = \left(\int_0^\infty (\mu(\{x \in X : |f(x)| > \lambda\}))^{p/q} d\lambda^p\right)^{1/p} < \infty$$

and

$$\|f\|_{L^q(X,\mu)} = \left(\int_X |f(x)|^q d\mu\right)^{1/q} < \infty,$$

respectively. Moreover, we use $L^{q,\infty}(X,\mu)$ as the set of all $\mu-$measurable functions $f$ on $X$ with

$$\|f\|_{L^{q,\infty}(X,\mu)} = \sup_{\lambda>0} \lambda(\mu(\{x \in X : |f(x)| > \lambda\}))^{1/q} < \infty.$$

## 2.2 Carleson Measures for $\beta-$Parabolic Equations

In this section, we establish our main results about Carleson measures for $\beta-$parabolic equations. We need the following lemma which studies the capacity strong-type inequalities for $f \in \dot{W}^{s,p}(\mathbb{R}^n)$ and its Hardy-Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|\mathrm{d}y, \ x \in \mathbb{R}^n.$$

**Lemma 2.2.1** *The following three inequalities hold:*
*(a) If $s \in (0,n)$ and $p \in [1, n/s]$, then, $\forall f \in \dot{W}^{s,p}(\mathbb{R}^n)$,*

$$\int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\})d\lambda^p \lesssim \|f\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)};$$

*If $1 \le p < \infty$, then, $\forall f \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+)$,*

$$\int_0^\infty cap_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}(\{(t,x) \in \mathbb{R}^{1+n}_+ : |f(t,x)| \geq \lambda\})d\lambda^p \lesssim \|f\|^p_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}.$$

*(b) If $s \in (0,n)$ and $p \in [1, n/s]$, then, $\forall f \in \dot{W}^{s,p}(\mathbb{R}^n)$,*

$$\int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\{x \in \mathbb{R}^n : |\mathcal{M}f(x)| \geq \lambda\})d\lambda^p \lesssim \|f\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

**Proof.** (a) *Part 1*, $f \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_n)$ : This assertion is due to Maz'ya [49, Section 2.3.1] or his another work [46]. *Part 2*, $f \in \dot{W}^{s,p}(\mathbb{R}^n)$ : *Case 1*, $p \in (1, n/s)$ : This case is due to Maz'ya [48, Proposition 4.1] or Maz'ya [49, p. 368 Theorem]. *Case 2*, $p = 1$ : This case is essentially proved by Wu [75] when $s \in (0,1)$ and Xiao [77] when $s \in (0,n)$. *Case 3*, $p = n/s$ : It can be found in Maz'ya [47] or Adams-Xiao [7].
(b) If $f \in \dot{W}^{s,p}(\mathbb{R}^n)$ : We divide the proof into three cases.
*Case 1*, $p = 1$ : It is due to Xiao [77]. *Case 2*, $p = n/s$ : This is proved by Adams-Xiao [7]. *Case 3*, $p \in (1, n/s)$ : It follows from Maz'ya [49, p. 347, Theorem 2] or his earlier work [51] that for $1 < p < n/s$, $f \in \dot{W}^{s,p}(\mathbb{R}^n)$ if and only if

$$f = (-\triangle)^{-s/2} g = I_s * g(x) \ \text{ and } \ \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)},$$

for some $g \in L^p(\mathbb{R}^n)$, where

$$I_s * g(x) = \frac{1}{\gamma_s}\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{n-s}}dy$$

with $\gamma_s = \pi^{n/2}2^s\Gamma(s/2)/\Gamma(\frac{n-s}{2})$. Then for fixed $f \in \dot{W}^{s,p}(\mathbb{R}^n)$, and $g \in L^p(\mathbb{R}^n)$ with $f(x) = I_s * g(x)$, according to Johnson [34, p. 33, Proof of Theorem 1.9], we have

$$\mathcal{M}(I_s * g) \le I_s * (\mathcal{M}g)$$

and

$$M_\lambda(\mathcal{M}f(x)) \subseteq M_\lambda(I_s * (\mathcal{M}(g)).$$

It follows from Maximal Theorem Stein [64, p. 13, Theorem 1] that

$$\mathcal{M}(g) \in L^p(\mathbb{R}^n) \text{ and } \|\mathcal{M}(g)\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}.$$

Thus (a) implies (b). □

**Theorem 2.2.2** *Let $1 \le p \le q < \infty$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then the following statements are equivalent:*

*(a)*

$$\|u\|_{L^{q,p}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}, \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+) \cap b^p_\beta(\mathbb{R}^{1+n}_+),$$

*(b)*

$$\|u\|_{L^{q}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}, \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+) \cap b^p_\beta(\mathbb{R}^{1+n}_+),$$

*(c)*

$$\|u\|_{L^{q,\infty}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u\|_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}, \quad u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+) \cap b^p_\beta(\mathbb{R}^{1+n}_+),$$

*(d)*

$$(\mu(O))^{p/q} \lesssim cap_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}(O), \quad \text{open } O \subseteq R^{1+n}_+.$$

*If $0 < q < p = 1$, then (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a).*

**Proof**. Assume that $1 \le p \le q < \infty$. In what follows, for $\lambda > 0$ and $u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+) \cap b^p_\beta(\mathbb{R}^{1+n}_+)$, let

$$M_\lambda(u) = \{(t,x) \in \mathbb{R}^{1+n}_+ : |u(t,x)| \ge \lambda\}.$$

(a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Since $0 < \lambda_1 < \lambda_2$ implies $\mu(M_{\lambda_2}(u)) \le \mu(M_{\lambda_1}(u))$, we can conclude

$$q\mu(M_\lambda(u))\lambda^{q-1} \le \frac{d}{d\lambda}\left(\int_0^\lambda (\mu(M_s(u)))^{p/q}ds^p\right)^{q/p}.$$

This implies

$$(s^q\mu(M_s(u)))^{p/q} \le \left(q\int_0^\infty \mu(M_\lambda(u))\lambda^{q-1}d\lambda\right)^{p/q} \le \int_0^\infty (\mu(M_\lambda(u)))^{p/q}d\lambda^p, \quad s > 0,$$

and obtains the desired implications.
$(c) \Longrightarrow (d)$. Let (c) be true. For an given open set $O \subseteq \mathbb{R}^{1+n}_+$, and any function $u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+) \cap b^p_\beta(\mathbb{R}^{1+n}_+)$ with

$$O \subseteq \mathrm{Int}(\{(t,x) \in \mathbb{R}^{1+n}_+ : u(t,x) \geq 1\}),$$

we have $\mu(O) \leq \mu(M_1(u)) \lesssim \|u\|^q_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}$. This derives (d).
$(d) \Longrightarrow (a)$. If (d) is true, then for $u \in \dot{W}^{1,p}(\mathbb{R}^{1+n}_+)$, $k \in \mathbb{N}$ and $B(0,k) \subseteq \mathbb{R}^n$, Lemma 2.2.1 (a) implies

$$\begin{aligned}
&\int_0^\infty (\mu(M_\lambda(u) \cap ((0,k) \times B(0,k))))^{p/q} d\lambda^p \\
\lesssim\ & \int_0^\infty cap_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}(M_\lambda(u) \cap ((0,k) \times B(0,k))))d\lambda^p \\
\lesssim\ & \int_0^\infty cap_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}(M_\lambda(u))d\lambda^p \lesssim \|u\|^p_{\dot{W}^{1,p}(\mathbb{R}^{1+n}_+)}.
\end{aligned}$$

Letting $k \longrightarrow \infty$ we see that (a) holds. When $0 < q < p = 1$, the implications are obviously. □

Nishio-Yamada [58] gave a characterization for a nonnegative Radon measure $\mu$ on $\mathbb{R}^{1+n}_+$ to be a Carleson type measure on $b^p_\beta(\mathbb{R}^{1+n}_+)$, which is called (0,1)-type Carleson measure and means that $|\nabla_{(t,x)}u(t,x)| \in L^p(\mathbb{R}^{1+n}_+, \mu)$, that is,

$$\|\nabla_{(t,x)}u(t,x)\|_{L^p(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|u(t,x)\|_{L^p(\mathbb{R}^{1+n}_+)}, \ \forall u \in b^p_\beta(\mathbb{R}^{1+n}_+).$$

We find a sufficient condition for a nonnegative Radon measure $\mu$ on $\mathbb{R}^{1+n}_+$ to be a Carleson type measure on $b^p_{1/2}(\mathbb{R}^{1+n}_+)$.

**Theorem 2.2.3** *If $\mu$ is a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$ satisfying the property*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{(\mu(T(B(x,r))))^{p/q}}{cap_{\dot{W}^{1/2,p}}(B(x,r))} < \infty$$

*for $1 \leq p < 2n$ and $\frac{4pn+4p}{2n-p} \leq q < \infty$, then $\mu$ is a (0,1)-type Carleson measure on $b^{p_1}_{1/2}(\mathbb{R}^{1+n}_+)$ for $p_1 = \frac{q(2n-p)}{2p(n+1)} - 1$.*

**Proof**. Assume that $\mu$ is a nonnegative Radon measure such that

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{(\mu(T(B(x,r))))^{p/q}}{cap_{\dot{W}^{1/2,p}}(B(x,r))} < \infty$$

for $1 \le p < 2n$ and $\frac{4pn+4p}{2n-p} \le q < \infty$. According to the definition of $1/2-$parabolic rectangle (see Nishio-Yamada [58])

$$Q^{1/2}(s,y) = \{(s,y) \in \mathbb{R}^{1+n}_+ : |x_j - y_j| < s/2, 1 \le j \le n, s \le t \le 2s\}$$

with center $(s,y)$, we have

$$Q^{1/2}(s,y) = [s,2s] \times B(y, \sqrt{n}s/2).$$

The definition of $T(B(y,r))$ implies that there is a dimensional constant $c(n)$ such that

$$Q^{1/2}(s,y) \subseteq T(B(y,c(n)s)),$$

for each $(s,y) \in \mathbb{R}^{1+n}_+$, so

$$\mu(Q^{1/2}(s,y)) \le \mu(T(B(y,c(n)s))) \lesssim s^{q(n-p/2)/p}.$$

If $p_1 = \frac{q(2n-p)}{2p(n+1)} - 1$, then for each $(s,y) \in \mathbb{R}^{1+n}_+$,

$$\mu(Q^{1/2}(s,y)) \lesssim s^{(n+1)(1+p_1)}.$$

Note that $p_1 \ge 1$ since $q \ge \frac{4p(n+1)}{2n-p}$ and $p < 2n$. It follows from Nishio-Yamada [58, p. 91 Theorem 2] that $\nu$ is a (0,1)-type Carleson measure on $b^q_{1/2}(\mathbb{R}^{1+n}_+)$ $(q \ge 1)$ if and only if $\nu(Q^{1/2}(s,y)) \lesssim s^{(n+1)(1+q)}$, for each $(s,y) \in \mathbb{R}^{1+n}_+$. Thus $\mu$ is a (0,1)-type Carleson measure on $b^{p_1}_{1/2}(\mathbb{R}^{1+n}_+)$. □

## 2.3 Carleson Measures for Cauchy Problems

In this section, we study Carleson measures for the Cauchy problems of fractional heat equations. We divide this section into several parts as follows.

### 2.3.1 Case: $0 < q < p$ and $1 < p \le n/s$

We need the following lemmas which are useful in this chapter.

**Lemma 2.3.1** *[58] For $\beta \in (0,1]$, there are positive constants $\sigma$ and $C$ such that*

$$\inf\{|K^\beta_t(x)| : |x| \le \sigma t^{\frac{1}{2\beta}}\} \ge C t^{-\frac{n}{2\beta}},$$

*where $\sigma$ and $C$ depend only on $n, \beta$.*

**Lemma 2.3.2** *Let $\beta \in (0,1]$ and $s \in (0,n)$. Given $f \in \dot{W}^{s,p}(\mathbb{R}^n)$, $\lambda > 0$, and a nonnegative measure $\mu$ on $\mathbb{R}^{1+n}_+$, let*

$$E^{\beta,s}_\lambda(f) = \{(t,x) \in \mathbb{R}^{1+n}_+ : |S_\beta(t^{2\beta})f(x)| > \lambda\}$$

*and*

$$O_\lambda^{\beta,s}(f) = \{y \in \mathbb{R}^n : \sup_{|y-x|<t} |S_\beta(t^{2\beta})f(x)| > \lambda\}.$$

*Then the following four statements are true:*
*(a) For any natural number $k$,*

$$\mu\left(E_\lambda^{\beta,s}(f) \cap T(B(0,k))\right) \le \mu\left(T(O_\lambda^{\beta,s}(f) \cap B(0,k))\right).$$

*(b) For any natural number $k$,*

$$cap_{\dot{W}^{s,p}(\mathbb{R}^n)}\left(O_\lambda^{\beta,s}(f) \cap B(0,k)\right) \ge c_p^s\left(\mu; \mu\left(T(O_\lambda^{\beta,s}(f) \cap B(0,k))\right)\right).$$

*(c) There exists a dimensional constant $\theta_1 > 0$ such that*

$$\sup_{|y-x|<t} |S_\beta(t^{2\beta})f(y)| \le \theta_1 \mathcal{M}f(x), x \in \mathbb{R}^n.$$

*(d) There exists a dimensional constant $\theta_2 > 0$ such that*

$$(t,x) \in T(O) \Longrightarrow S_\beta(t^{2\beta})|f|(x) \ge \theta_2,$$

*where $O$ is a bounded open set contained in Int$(\{x \in \mathbb{R}^n : f(x) \ge 1\})$.*

**Proof.** (a) Since $\sup_{|y-x|<t} |S_\beta(t^{2\beta})f(x)|$ is lower semicontinuous on $\mathbb{R}^n$, $O_\lambda^{\beta,s}(f)$ is an open subset of $\mathbb{R}^n$. By the definition of $E_\lambda^{\beta,s}(f)$ and $O_\lambda^{\beta,s}(f)$, we have

$$E_\lambda^{\beta,s}(f) \subseteq T(O_\lambda^{\beta,s}(f)) \text{ and } \mu(E_\lambda^{\beta,s}(f)) \le T(\mu(O_\lambda^{\beta,s}(f))).$$

Then

$$\mu\left(E_\lambda^{\beta,s}(f) \cap T(B(0,k))\right) \le \mu(T(O_\lambda^{\beta,s}(f) \cap T(B(0,k)))) = \mu\left(T(O_\lambda^{\beta,s}(f) \cap B(0,k))\right).$$

(b) It follows from the definition of $c_p^s(\mu; t)$.
(c) By (2.3.4), we have

$$|S_\beta(t^{2\beta})f(x)| = |K_{t^{2\beta}}^\beta(x) * f(x)| \le \int_{\mathbb{R}^n} \frac{Ct^{2\beta}}{(t+|x-y|)^{n+2\beta}}|f(y)|\mathrm{d}y := H_t(x) * |f(x)|.$$

Thus

$$\sup_{|y-x|<t} |S_\beta(t^{2\beta})f(y)| \le \sup_{|y-x|<t} H_t(y) * |f(y)| \le \theta_1 \mathcal{M}f(x).$$

The last inequality follows from Stein [64, p. 57, Proposition].
(d) For any $(t,x) \in T(O)$, we have

$$B(x,t) \subseteq O \subseteq \mathrm{Int}(\{x : f(x) \ge 1\}).$$

It follows from Lemma 2.3.1 that there exist $\sigma$ and $C$ which are only depending on $n$ and $\beta$ such that

$$\inf\{K_t^\beta(x) : |x| \le \sigma t^{\frac{1}{2\beta}}\} \ge Ct^{-\frac{n}{2\beta}}.$$

Then

$$\begin{aligned} S_\beta(t^{2\beta})|f|(x) &= \int_{\mathbb{R}^n} K^\beta_{t^{2\beta}}(x-y)|f|(y)\mathrm{d}y \\ &\ge Ct^{-n}\int_{B(x,\sigma t)\cap \mathrm{Int}(\{x:f(x)\ge 1\})} |f|(y)\mathrm{d}y. \end{aligned}$$

If $\sigma > 1$, then

$$B(x,\sigma t)\cap \mathrm{Int}(\{x : f(x)\ge 1\}) \supseteq B(x,t)\cap \mathrm{Int}(\{x : f(x)\ge 1\}) = B(x,t);$$

if $\sigma \le 1$ then

$$B(x,\sigma t)\cap \mathrm{Int}(\{x : f(x)\ge 1\}) = B(x,\sigma t).$$

Thus $S_\beta(t^{2\beta})|f|(x) \ge \theta_2$ for some dimensional constant $\theta_2 > 0$. □

The following result is a special case of Adams [2, Theorem 5.2] or Adams-Xiao [7, Theorem A].

**Lemma 2.3.3** *Let $s\in(0,n)$. Then there are a linear extension operator*

$$\mathcal{E} : \dot{W}^{s,n/s}(\mathbb{R}^n) \longrightarrow \dot{\mathcal{L}}^{n/s}_{2s}(\mathbb{R}^{2n})$$

*and a linear restriction operator*

$$\mathcal{R} : \dot{\mathcal{L}}^{n/s}_{2s}(\mathbb{R}^{2n}) \longrightarrow \dot{W}^{s,n/s}(\mathbb{R}^n)$$

*such that $\mathcal{R}\mathcal{E}$ is the identity, and*

*(a)*

$$\|\mathcal{E}f\|_{\dot{\mathcal{L}}^{n/s}_{2s}(\mathbb{R}^{2n})} \lesssim \|f\|_{\dot{W}^{s,n/s}(\mathbb{R}^n)}, \ \forall f \in \dot{W}^{s,n/s}(\mathbb{R}^n);$$

*(b)*

$$\|\mathcal{R}g\|_{\dot{W}^{s,n/s}(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\mathcal{L}}^{n/s}_{2s}(\mathbb{R}^{2n})}, \ \ \forall g \in \dot{\mathcal{L}}^{n/s}_{2s}(\mathbb{R}^{2n}).$$

In the rest of this chapter, $v(t,x)$ is the solution of equation (2.0.2) with data $v_0(x)$.

**Theorem 2.3.4** *Let $s\in(0,n)$, $0<q<p$, $1<p\le n/s$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then the following two conditions are equivalent:*

*(a)*

$$\|v(t^{2\beta},x)\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \ \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

*(b)*

$$\int_0^\infty \left(\frac{t^{p/q}}{c_p^s(\mu;t)}\right)^{q/(p-q)} \frac{dt}{t} < \infty.$$

**Proof.** Let $0 < q < p$. Then we finish the proof in two steps.

*Part 1*: $(b) \Longrightarrow (a)$. If

$$I_{p,q}(\mu) = \int_0^\infty \left( \frac{t^{p/q}}{c_p^s(\mu; t)} \right)^{\frac{q}{p-q}} \frac{dt}{t} < \infty,$$

then for each $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$, each $j = 0, \pm 1, \pm 2, \cdots$ and each natural number $k$, Lemma 2.3.2 (c) implies

$$cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0,k)) \le cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\{x \in \mathbb{R}^n : \theta_1 \mathcal{M} v_0(x) > 2^j\} \cap B(0,k)).$$

Let $\mu_{j,k}(v_0) = \mu(T(O_{2^j}(v_0) \cap B(0,k))$, and

$$S_{p,q,k}(\mu; v_0) = \sum_{j=-\infty}^{\infty} \frac{(\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0))^{\frac{p}{p-q}}}{\left(cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0,k))\right)^{\frac{q}{p-q}}}.$$

Lemma 2.3.2 (b) implies that

$$\begin{aligned}
(S_{p,q,k}(\mu; v_0))^{\frac{p-q}{p}} &= \left( \sum_{j=-\infty}^{\infty} \frac{(\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0))^{\frac{p}{p-q}}}{\left(cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_{2^j}(v_0) \cap B(0,k))\right)^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\lesssim \left( \sum_{j=-\infty}^{\infty} \frac{(\mu_{j,k}(v_0) - \mu_{j+1,k}(v_0))^{\frac{p}{p-q}}}{(c_p^s(\mu; \mu_{j,k}(v_0)))^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\lesssim \left( \sum_{j=-\infty}^{\infty} \frac{((\mu_{j,k}(v_0))^{\frac{p}{p-q}} - (\mu_{j+1,k}(v_0))^{\frac{p}{p-q}}}{(c_p^s(\mu; \mu_{j,k}(v_0)))^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\lesssim \left( \int_0^\infty \frac{dr^{\frac{p}{p-q}}}{(c_p^s(\mu; r))^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\approx (I_{p,q}(\mu))^{\frac{p-q}{p}}.
\end{aligned}$$

On the other hand, using Hölder's inequality and Lemmas 2.2.1 (b) and 2.3.2 (b)–(c), we have

$$\begin{aligned}
&\int_{T(B(0,k))} |v(t^{2\beta},x)|^q \mathrm{d}\mu(t,x) \\
= \;& \int_0^\infty \mu\left(E_\lambda^{\beta,s}(v_0)\cap T(B(0,k))\right)\mathrm{d}\lambda^q \\
\lesssim \;& \sum_{j=-\infty}^{\infty} (\mu_{j,k}(v_0)-\mu_{j+1,k}(v_0))2^{jq} \\
\lesssim \;& (S_{p,q,k}(\mu;v_0))^{\frac{p-q}{p}} \left(\sum_{j=-\infty}^{\infty} 2^{jp} cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_{2^j}(v_0)\cap B(0,k))\right)^{q/p} \\
\lesssim \;& (S_{p,q,k}(\mu;v_0))^{\frac{p-q}{p}} \left(\sum_{j=-\infty}^{\infty} 2^{jp} cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\{x\in\mathbb{R}^n:\theta_1\mathcal{M}v_0(x)>2^j\}\cap B(0,k))\right)^{\frac{q}{p}} \\
\lesssim \;& (S_{p,q,k}(\mu;v_0))^{\frac{p-q}{p}} \left(\int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\{x\in\mathbb{R}^n:\theta_1\mathcal{M}v_0(x)>\lambda\})\mathrm{d}\lambda^p\right)^{q/p} \\
\lesssim \;& (S_{p,q,k}(\mu;v_0))^{\frac{p-q}{p}} \|v_0\|^q_{\dot{W}^{s,p}(\mathbb{R}^n)}.
\end{aligned}$$

Hence

$$\left(\int_{T(B(0,k))} |v(t^{2\beta},x)|^q \mathrm{d}\mu(t,x)\right)^{1/q} \lesssim (I_{p,q}(\mu))^{\frac{p-q}{pq}} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

Letting $k\longrightarrow\infty$ in the left side of the above estimate, we have

$$\left(\int_{\mathbb{R}^{1+n}_+} |v(t^{2\beta},x)|^q \mathrm{d}\mu(t,x)\right)^{1/q} \lesssim (I_{p,q}(\mu))^{\frac{p-q}{pq}} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

*Part 2*: $(a)\Longrightarrow(b)$.

If (a) is true, then

$$J_{p,q}(\mu)=\sup_{v_0\in\dot{W}^{s,p}(\mathbb{R}^n),\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}>0} \frac{\left(\int_{\mathbb{R}^{1+n}_+}|v(t^{2s},x)|^q\mathrm{d}\mu(t,x)\right)^{1/q}}{\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}}<\infty.$$

Thus for each $v_0\in\dot{W}^{s,p}(\mathbb{R}^n)$, with $\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}>0$, we have

$$\left(\int_{\mathbb{R}^{1+n}_+} |v(t^{2\beta},x)|^q \mathrm{d}\mu(t,x)\right)^{1/q} \le J_{p,q}(\mu)\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

Since $\mu(E_\lambda^{\beta,s}(v_0))$ is nonincreasing in $\lambda$, we have

$$\sup_{\lambda>0}\lambda\left(\mu(E_\lambda^{\beta,s}(v_0))^{1/q}\right)\lesssim J_{p,q}(\mu)\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}. \tag{2.3.1}$$

For fixed positive $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$, and a bounded open set $O \subseteq \mathrm{Int}(\{x \in \mathbb{R}^n : v_0(x) \geq 1\})$, then (2.3.1) and Lemma 2.3.2 (d) imply that

$$\mu(T(O)) \leq \mu(E^{\beta,s}_{\frac{\theta_2}{2}}(v_0)) \lesssim (J_{p,q}(\mu))^q \|v_0\|^q_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

This along with the definition of $cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(\cdot)$ give

$$\mu(T(O)) \lesssim (J_{p,q}(\mu))^q \left(cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O)\right)^{q/p}. \tag{2.3.2}$$

It follows from (2.3.2) and the definition of $c^s_p(\mu;t)$ that for $0 < t < \infty$, $c^s_p(\mu;t) > 0$. The definition of $c^s_p(\mu;t)$ implies that for every integer $j$ there exists a bounded open set $O_j \subseteq \mathbb{R}^n$ such that

$$cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_j) \leq 2\ c^s_p(\mu; 2^j) \text{ and } \mu(T(O_j)) > 2^j.$$

We divide the following proof into two cases.
*Case 1*, $p \in (1, n/s)$ :
It follows from Maz'ya [49] that

$$cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(S) \approx \inf\left\{\|g\|^p_{L^p(\mathbb{R}^n)} : g \in L^p(\mathbb{R}^n), g \geq 0, S \subseteq \mathrm{Int}(\{x \in \mathbb{R}^n : I_s * g(x) \geq 1\})\right\}.$$

By this equivalent definition we can find $g_j(x) \in L^p(\mathbb{R}^n)$ such that

$$g_j \geq 0, I_s * g_j(x) \geq 1, \forall x \in O_j \text{ and } \|g_j\|^p_{L^p(\mathbb{R}^n)} \leq 2\ cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_j) \leq 4\ c^s_p(\mu; 2^j).$$

Given integers $i, k$ with $i < k$, define

$$g_{i,k} = \sup_{i \leq j \leq k} \left(\frac{2^j}{c^s_p(\mu; 2^j)}\right)^{\frac{1}{p-q}} g_j.$$

Since $L^p(\mathbb{R}^n)$ is a lattice, we can conclude that $g_{i,k} \in L^p(\mathbb{R}^n)$ and

$$\|g_{i,k}\|^p_{L^p(\mathbb{R}^n)} \leq \sum_{j=i}^{k} \left(\frac{2^j}{c^s_p(\mu; 2^j)}\right)^{\frac{p}{p-q}} \|g_j\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=i}^{k} \left(\frac{2^j}{c^s_p(\mu; 2^j)}\right)^{\frac{p}{p-q}} c^s_p(\mu; 2^j).$$

Note that for $i \leq j \leq k$,

$$x \in O_j \Longrightarrow I_s * g_{i,k}(x) \geq \left(\frac{2^j}{c^s_p(\mu; 2^j)}\right)^{\frac{1}{p-q}}.$$

It follows from Lemma 2.3.2 (d) that there exists a dimensional constant $\theta_2$ such that

$$(t, x) \in T(O_j) \Longrightarrow S_\beta(t^{2\beta})|I_s * g_{i,k}(x)|(x) \geq \left(\frac{2^j}{c^s_p(\mu; 2^j)}\right)^{\frac{1}{p-q}} \theta_2.$$

This gives

$$2^j < \mu\left(T(O_j)\right) \le \mu\left( E^{\beta,s}_{\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{1}{p-q}}\left(\frac{\theta_2}{2}\right)} \left(I_s * g_{i,k}(x)\right)\right).$$

Thus

$$\begin{aligned}
\left(J_{p,q}(\mu)\|g_{i,k}\|_{L^p(\mathbb{R}^n)}\right)^q &\gtrsim \int_{\mathbb{R}^{1+n}_+} |S_\beta(t^{2\beta})(I_s * g_{i,k}(x))|^q \mathrm{d}\mu(t,x)\\
&\approx \int_0^\infty \left(\inf\{\lambda : \mu\left(E^{\beta,s}_\lambda(I_s * g_{i,k}(x))\right) \le s\}\right)^q \mathrm{d}s\\
&\gtrsim \sum_{j=i}^{k}\left(\inf\{\lambda : \mu\left(E^{\beta,s}_\lambda(I_s * g_{i,k}(x))\right) \le 2^j\}\right)^q 2^j\\
&\gtrsim \sum_{j=i}^{k}\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} 2^j\\
&\gtrsim \left(\frac{\sum_{j=i}^k \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} 2^j}{\left(\sum_{j=i}^k \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} c_p^s(\mu;2^j)\right)^{\frac{q}{p}}}\right)\|g_{i,k}\|^q_{L^p(\mathbb{R}^n)}\\
&\approx \left(\sum_{j=i}^{k}\frac{2^{\frac{jp}{p-q}}}{\left(c_p^s(\mu;2^j)\right)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}}\|g_{i,k}\|^q_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This tells us

$$\sum_{j=i}^{k}\frac{2^{\frac{jp}{p-q}}}{\left(c_p^s(\mu;2^j)\right)^{\frac{q}{p-q}}} \lesssim \left(J_{p,q}(\mu)\right)^{\frac{pq}{p-q}}.$$

*Case 2*, $p = \frac{n}{s}$ : By the definition of $\mathrm{cap}_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_j)$, there is $f_j \in \dot{W}^{s,p}(\mathbb{R}^n)$ such that

$$f_j \ge 0, f_j(x) \ge 1, \forall x \in O_j \text{ and } \|f_j\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)} \le 2\ cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O_j) \le 4\ c_p^s(\mu;2^j).$$

Lemma 2.3.3 implies that for each $j$ there is $g_j(\cdot,\cdot) \in L^p(\mathbb{R}^{2n})$ such that

$$f_j(x) = I^{(2n)}_{2s} * g_j(x,0) = \mathcal{RE}f_j(x)$$

and

$$\|I^{(2n)}_{2s} * g_j\|_{\mathcal{L}^p_{2s}(\mathbb{R}^{2n})} = \|\mathcal{E}f_j\|_{\mathcal{L}^p_{2s}(\mathbb{R}^{2n})} \le \|f_j\|_{\dot{W}^{s,p}(\mathbb{R}^n)}. \tag{2.3.3}$$

Given integers $i,k$ with $i<k$, define

$$g_{i,k} = \sup_{i\le j\le k}\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{1}{p-q}} g_j.$$

Since $L^p(\mathbb{R}^{2n})$ is a lattice, we can conclude that $g_{i,k} \in L^p(\mathbb{R}^{2n})$ and $I_{2s}^{(2n)} * g_{i,k} \in \dot{\mathcal{L}}_{2s}^p(\mathbb{R}^{2n})$. Then (2.3.3) and Lemma 2.3.3 imply that

$$\begin{aligned}
&\|\mathcal{R}(I_{2s}^{(2n)} * g_{i,k})\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)} \\
\leq\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{p}{p-q}} \|\mathcal{R}(I_{2s}^{(2n)} * g_j)\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)} \\
\leq\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{p}{p-q}} \|I_{2s}^{(2n)} * g_j\|^p_{\dot{\mathcal{L}}_{2s}^p(\mathbb{R}^{2n})} \\
\lesssim\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{p}{p-q}} \|\mathcal{E}f_j\|^p_{\dot{\mathcal{L}}_{2s}^p(\mathbb{R}^{2n})} \\
\lesssim\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{p}{p-q}} \|f_j\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)} \\
\lesssim\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{p}{p-q}} c_p^s(\mu;2^j).
\end{aligned}$$

Note that for $i \leq j \leq k$,

$$x \in O_j \Longrightarrow \mathcal{R}(I_{2s}^{(2n)} * g_{i,k})(x) \geq \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{1}{p-q}}.$$

Then Lemma 2.3.2 (d) implies that

$$(t,x) \in T(O_j) \Longrightarrow S_\beta(t^{2\beta})|\mathcal{R}(I_s^{(2n)} * g_{i,k})|(x) \geq \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{1}{p-q}} \theta_2.$$

This gives

$$2^j < \mu\left(T(O_j)\right) \leq \mu\left(E^{\beta,s}_{\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{1}{p-q}}\left(\frac{\theta_2}{2}\right)}\left(\mathcal{R}(I_{2s}^{(2n)} * g_{i,k})(x)\right)\right).$$

Hence

$$
\begin{aligned}
&\left(J_{p,q}(\mu)\|\mathcal{R}(I_{2s}^{(2n)} * g_{i,k})\|_{\dot{W}^{s,p}(\mathbb{R}^n)}\right)^q \\
\gtrsim\ & \int_{\mathbb{R}^{1+n}_+} |S_\beta(t^{2\beta})\mathcal{R}(I_{2s} * g_{i,k})(x)|^q \mathrm{d}\mu(t,x) \\
\approx\ & \int_0^\infty \left(\inf\{\lambda : \mu\left(E_\lambda^{\beta,s}\mathcal{R}(I_{2s} * g_{i,k})(x)\right) \le r\}\right)^q \mathrm{d}r \\
\gtrsim\ & \sum_{j=i}^{k} \left(\inf\{\lambda : \mu\left(E_\lambda^{\beta,s}\mathcal{R}(I_{2s} * g_{i,k})(x)\right) \le 2^j\}\right)^q 2^j \\
\gtrsim\ & \sum_{j=i}^{k} \left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} 2^j \\
\gtrsim\ & \frac{\sum_{j=i}^{k}\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} 2^j}{\left(\sum_{j=i}^{k}\left(\frac{2^j}{c_p^s(\mu;2^j)}\right)^{\frac{q}{p-q}} c_p^s(\mu;2^j)\right)^{\frac{q}{p}}} \|\mathcal{R}(I_{2s}^{(2n)} * g_{i,k})\|^q_{\dot{W}^{s,p}(\mathbb{R}^n)} \\
\approx\ & \left(\sum_{j=i}^{k} \frac{2^{\frac{jp}{p-q}}}{\left(c_p^s(\mu;2^j)\right)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}} \|\mathcal{R}(I_{2s}^{(2n)} * g_{i,k})\|^q_{\dot{W}^{s,p}(\mathbb{R}^n)}.
\end{aligned}
$$

This tells us we obtain the same inequality as in the first case

$$
\sum_{j=i}^{k} \frac{2^{\frac{jp}{p-q}}}{\left(c_p^s(\mu;2^j)\right)^{\frac{q}{p-q}}} \lesssim (J_{p,q}(\mu))^{\frac{pq}{p-q}}.
$$

Note that the constant involved in the last inequality does not depend on $i$ and $k$. Letting $i \longrightarrow \infty$ and $k \longrightarrow \infty$, we have

$$
\int_0^\infty \left(\frac{t^{p/q}}{c_p^s(\mu;t)}\right)^{\frac{q}{p-q}} \frac{dt}{t} \lesssim \sum_{-\infty}^{\infty} \frac{2^{\frac{jp}{p-q}}}{\left(c_p^s(\mu;2^j)\right)^{\frac{q}{p-q}}} \lesssim (J_{p,q}(\mu))^{\frac{pq}{p-q}}.
$$

Therefore, (b) holds. □

### 2.3.2 Case: $0 < q < p = 1$

When $0 < q < p = 1$ we obtain necessary conditions for such embeddings.

**Theorem 2.3.5** *Let $s \in (0,n)$, $0 < q < p = 1$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) :*
*(a)*

$$
\|v(t^{2\beta},x)\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,1}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,1}(\mathbb{R}^n).
$$

(b)

$$\|v(t^{2\beta},x)\|_{L^{q,\infty}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,1}(\mathbb{R}^n)},\ \forall v_0 \in \dot{W}^{s,1}(\mathbb{R}^n).$$

(c)

$$\sup\left\{\frac{(\mu(T(O)))^{1/q}}{cap_{\dot{W}^{s,1}(\mathbb{R}^n)}(O)}:\ open\ \ O\subseteq\mathbb{R}^n\right\}<\infty.$$

(d)

$$\|v(t^{2\beta},x)\|_{L^{q,1}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,1}(\mathbb{R}^n)},\ \forall v_0 \in \dot{W}^{s,1}(\mathbb{R}^n).$$

**Proof**. Suppose $0<q<1$. Since the proof of $(a) \Longrightarrow (b) \Longrightarrow (c)$ is similar to that of $(b) \Longrightarrow (c) \Longrightarrow (e)$ of Theorem 2.3.6, we only need to verify $(c) \Longrightarrow (d)$. Let $(c)$ be true. Then Lemma 2.3.2 (a)–(c) imply

$$\begin{aligned}\mu\left(E_\lambda^{\beta,s}(v_0)\right) &\le \left(\mu\left(T(O_\lambda^{\beta,s}(v_0))\right)\right)\\ &\le (\mu(T(\{x\in\mathbb{R}^n:\theta_1\mathcal{M}v_0(x)>\lambda\})))\\ &\lesssim \left(cap_{\dot{W}^{s,1}(\mathbb{R}^n)}(\{x\in\mathbb{R}^n:\theta_1\mathcal{M}v_0(x)>\lambda\})\right)^q.\end{aligned}$$

This and Lemma 2.2.1 (b) imply that (d) holds. □

### 2.3.3 Case: $1\le p\le n/s$ and $p\le q<\infty$

If we change $1<p\le n/s$ and $0<q<p$ in Theorem 2.3.4 into $1\le p\le n/s$ and $p\le q<\infty$, then the conditions (a) and (b) of Theorem 2.3.4 can be replaced by a weak-type one and two simpler ones, respectively.

**Theorem 2.3.6** *Let $s\in(0,n)$, $1\le p\le n/s$, $p\le q<\infty$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then the following five conditions are equivalent:*

(a)

$$\|v(t^{2\beta},x)\|_{L^{q,p}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\ \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

(b)

$$\|v(t^{2\beta},x)\|_{L^{q}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\ \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

(c)

$$\|v(t^{2\beta},x)\|_{L^{q,\infty}(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\ \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

(d)

$$\sup_{t>0}\frac{t^{p/q}}{c_p^s(\mu;t)}<\infty.$$

(e)

$$\sup\left\{\frac{(\mu(T(O)))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O)}:\ bounded\ open\ \ O\subseteq\mathbb{R}^n\right\}<\infty.$$

**Proof**. Let $p < q$. The proof consists two parts.

*Part 1*: We prove $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (e) \Longrightarrow (a)$.

$(a) \Longrightarrow (b) \Longrightarrow (c)$. Since $\mu(E_\lambda(v_0))$ is nonincreasing in $\lambda$,

$$q\mu(E_\lambda^{\beta,s}(v_0))\lambda^{q-1} \le \frac{d}{d\lambda}\left(\int_0^\lambda \left(\mu(E_s^{\beta,s}(v_0))\right)^{p/q} ds^p\right)^{q/p}.$$

This gives, for $r > 0$

$$(r^q\mu(E_r^{\beta,s}(v_0)))^{\frac{p}{q}} \le \left(q\int_0^\infty \mu(E_\lambda^{\beta,s}(v_0))\lambda^{q-1}\mathrm{d}\lambda\right)^{\frac{p}{q}} \le \int_0^\infty \left(\mu(E_\lambda^{\beta,s}(v_0))\right)^{\frac{p}{q}} \mathrm{d}\lambda^p,$$

and establishes the desired implications.

If $(c)$ is true, then

$$K_{p,q}(\mu) = \sup_{v_0\in \dot{W}^{s,p}(\mathbb{R}^n), \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}>0} \frac{\sup_{\lambda>0}\lambda\left(\mu\left(\{(t,x)\in\mathbb{R}_+^{1+n} : |v(t^{2\beta},x)|>\lambda\}\right)\right)^{\frac{1}{q}}}{\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}} < \infty.$$

For a given $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$ and a bounded set $O \subseteq \mathrm{Int}\left(\{x\in\mathbb{R}^n : v_0(x) \ge 1\}\right)$, then Lemma 2.3.2 (d) implies

$$\left(\mu\left(T(O)\right)\right)^{1/q} \lesssim K_{p,q}(\mu)\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}$$

and hence (e) follows from the definition of $cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O)$. To prove $(e) \Longrightarrow (a)$, we assume $(e)$. Then

$$Q_{p,q}(\mu) = \sup\left\{\frac{\left(\mu\left(T(O)\right)\right)^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O)} : \text{ bounded open } O\subseteq\mathbb{R}^n\right\} < \infty.$$

If $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$ and $k = 1, 2, 3, \cdots$, then Lemmas 2.3.2 (a)-(c) and 2.2.1 (b) imply

$$\begin{aligned}
&\int_0^\infty \left(\mu\left(E_\lambda^{\beta,s}(v_0)\cap T(B(0,k))\right)\right)^{p/q}\mathrm{d}\lambda^p\\
\le\ & \int_0^\infty \left(\mu\left(T(O_\lambda^{\beta,s}(v_0)\cap B(0,k))\right)\right)^{p/q}\mathrm{d}\lambda^p\\
\le\ & \int_0^\infty \left(\mu_k\left(T(\{x\in\mathbb{R}^n : \theta_1\mathcal{M}v_0(x)>\lambda\}\cap B(0,k))\right)\right)^{p/q}\mathrm{d}\lambda^p\\
\le\ & \int_0^\infty \left(\mu\left(T(\{x\in\mathbb{R}^n : \theta_1\mathcal{M}v_0(x)>\lambda\}\cap B(0,k))\right)\right)^{p/q}\mathrm{d}\lambda^p\\
\lesssim\ & Q_{p,q}(\mu)\int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}\left(\{x\in\mathbb{R}^n : \theta_1\mathcal{M}v_0(x)>\lambda\}\cap B(0,k)\right)\mathrm{d}\lambda^p\\
\lesssim\ & Q_{p,q}(\mu)\int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}\left(\{x\in\mathbb{R}^n : \theta_1\mathcal{M}v_0(x)>\lambda\}\right)\mathrm{d}\lambda^p\\
\lesssim\ & Q_{p,q}(\mu)\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}^p.
\end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality we have

$$\int_0^\infty \left(\mu\left(E_\lambda^{\beta,s}(v_0)\right)\right)^{p/q} \mathrm{d}\lambda^p \lesssim Q_{p,q}(\mu)\|v_0\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

This derives (a).

*Part 2*: We verify $(c) \Longrightarrow (d) \Longrightarrow (a)$.

If (c) holds, then for any bounded open set $O \subseteq \mathbb{R}^n$, we have

$$\mu\left(T(O)\right)^{1/q} \lesssim K_{p,q}(\mu)\|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

Note that

$$t^{p/q} \lesssim (K_{p,q}(\mu))^p \, cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O) \text{ wthenever } 0 < t < \mu\left(T(O)\right).$$

Hence

$$t^{p/q} \lesssim (K_{p,q}(\mu))^p \, c_p^s(\mu; t).$$

Therefore (d) holds.

If $(d)$ holds, then Lemmas 2.3.2 (b)-(c) and 2.2.1 (b) imply that for each $k = 1, 2, 3, \cdots$,

$$\begin{aligned}
&\int_0^\infty \left(\mu\left(E_\lambda^{\beta,s}(v_0) \cap T(B(0,k))\right)\right)^{p/q} \mathrm{d}\lambda^p \\
\leq\ & \int_0^\infty \left(\frac{\left(\mu\left(E_\lambda^{\beta,s}(v_0) \cap T\left(B(0,k)\right)\right)\right)^{p/q}}{c_p^s\left(\mu; \mu\left(E_\lambda^{\beta,s}(v_0) \cap T\left(B(0,k)\right)\right)\right)}\right) cap_{\dot{W}^{s,p}(\mathbb{R}^n)}\left(O_\lambda(v_0) \cap B(0,k)\right) \mathrm{d}\lambda^p \\
\lesssim\ & \left(\sup_{t>0} \frac{t^{p/q}}{c_p^s(\mu;t)}\right) \int_0^\infty cap_{\dot{W}^{s,p}(\mathbb{R}^n)}\left(\{x \in \mathbb{R}^n : \theta_1 \mathcal{M} v_0(x) > \lambda\} \cap B(0,k)\right) \mathrm{d}\lambda^p \\
\lesssim\ & \left(\sup_{t>0} \frac{t^{p/q}}{c_p^s(\mu;t)}\right) \|v_0\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)}.
\end{aligned}$$

Letting $k \longrightarrow \infty$ in the previous inequality we have

$$\int_0^\infty \left(\mu\left(E_\lambda^{\beta,s}(v_0)\right)\right)^{p/q} \mathrm{d}\lambda^p \lesssim \left(\sup_{t>0} \frac{t^{p/q}}{c_p^s(\mu;t)}\right) \|v_0\|^p_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

This implies that (a) holds. □

**Corollary 2.3.7** *Let $s \in (0,n)$, $1 < p < \min\{q, n/s\}$ or $1 = p \leq q < \infty$, $\gamma \in (0,1]$, $\zeta > 0$ and $\zeta + n\gamma > n - ps$. If $d\mu_{\gamma,\zeta}(t,x) = t^{\zeta-1}|x|^{n(s-1)}dtdx$, then*

$$\left(\mu_{\gamma,\zeta}(T(O))\right)^{\frac{n-ps}{\zeta+n\gamma}} \lesssim cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O), \quad \textit{open } O \subseteq \mathbb{R}^n.$$

*Equivalently*

$$\|v(t^{2\beta}, x)\|_{L^{\frac{(\zeta+n\gamma)p}{n-ps}}(\mathbb{R}^{1+n}_+, \mu_{\gamma,\zeta})} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

**Proof.** This assertion follows from the case $q = p(\zeta + n\gamma)/(n - ps)$ and $\mu = \mu_{\gamma,\zeta}$ of Theorem 2.3.6. □

### 2.3.4 Case: $1 < p < \min\{q, n/s\}$ or $1 = p \leq q$

Furthermore, the family of all bounded open sets in the inequality (e) of Theorem 2.3.6 in some situation can be replaced by the family of all open balls. To see this, we need the following two lemmas.

**Lemma 2.3.8** *If $s \in (0,1]$, $s \in (0,n)$ and $(t,x) \in \mathbb{R}^{1+n}_+$, then*

$$\int_{\mathbb{R}^n} K^{\beta}_{t^{2\beta}}(y)|y-x|^{s-n}dy \lesssim (t^2+|x|^2)^{\frac{s-n}{2}}.$$

**Proof.** By Miao-Yuan-Zhang [53], Nishio-Shimomura-Suzuki [57] or Nishio-Yamada [58], we have the following point-wise estimate

$$|K^{\beta}_t(x)| \leq C\frac{t}{(t^{1/2\beta}+|x|)^{n+2\beta}}, \quad \forall (t,x) \in \mathbb{R}^{1+n}_+. \tag{2.3.4}$$

So, it suffices to verify

$$J(t,x) := \int_{\mathbb{R}^n} t^{2\beta}(t+|y|)^{-n-2\beta}|y-x|^{s-n}dy \lesssim (t^2+|x|^2)^{\frac{s-n}{2}}.$$

Changing variables: $x \longrightarrow tx$, $y \longrightarrow ty$, we see the previous estimate is equivalent to the following one:

$$J(1,x) \lesssim (1+|x|^2)^{\frac{s-n}{2}}.$$

Since $J(1,0) \lesssim 1$ we may assume that $|x| > 0$. Then

$$J(1,x) \lesssim \left(\int_{B(x,|x|/2)} + \int_{\mathbb{R}^n\setminus B(x,|x|/2)}\right)\frac{1}{(1+|y|)^{n+2\beta}|y-x|^{n-s}}dy = I_1(x)+I_2(x).$$

Since $|x-y| \leq |x|/2$ implies that $|y| \approx |x|$, we have

$$\begin{aligned} I_1(x) &= \int_{B(x,|x|/2)} \frac{1}{(1+|y|)^{n+2\beta}|y-x|^{n-s}}dy \\ &\lesssim (1+|x|)^{-n-2\beta}\int_{B(x,|x|/2)}\frac{1}{|y-x|^{n-s}}dy \\ &\lesssim (1+|x|)^{-n-2\beta}\int_0^{|x|/2} v^{s-1}dv \\ &\lesssim |x|^s(1+|x|)^{-n-2\beta} \\ &\lesssim (1+|x|)^{s-n}, \end{aligned}$$

with the last inequality using the fact $1 \leq (1+|x|)^{2\beta}$. If $|x-y| > |x|/2$, then

$$\begin{aligned} I_2(x) &= \int_{\mathbb{R}^n\setminus B(x,|x|/2)} \frac{1}{(1+|y|)^{n+2\beta}|x-y|^{n-s}}dy \\ &\lesssim |x|^{s-n}\int_{\mathbb{R}^n\setminus B(x,|x|/2)}\frac{1}{(1+|y|)^{n+2\beta}}dy \\ &\lesssim |x|^{s-n}, \end{aligned}$$

with the last inequalities using the fact $\frac{1}{(1+|y|)^{n+2\beta}} \in L^1(\mathbb{R}^n)$. Since $|x-y| > |x|/2$ implies $|y| < 3|x-y|$,

$$I_2(x) \lesssim \int_{\mathbb{R}^n \setminus B(x,|x|/2)} \frac{1}{(1+|y|)^{n+2\beta}|y|^{n-s}} dy \lesssim 1.$$

Thus $I_2 \lesssim (1+|x|)^{s-n}$ and $J(1,x) \lesssim (1+|x|^2)^{\frac{s-n}{2}}$. □

Using $f(x) = (-\triangle)^{-s/2}\left((-\triangle)^{s/2} f(x)\right)$ and the definition of Riesz potentials, we can easily derive an integral representation of homogeneous Sobolev functions.

**Lemma 2.3.9** *[6] Let $p \in (1, n/s)$, $0 < s < n$ and $f \in \dot{W}^{s,p}(\mathbb{R}^n)$. Then*

$$f(x) = \frac{1}{\gamma_s} \int_{\mathbb{R}^n} \frac{(-\triangle)^{s/2} f(y)}{|y-x|^{n-s}} dy,$$

*where $\gamma_s = \pi^{n/2} 2^s \Gamma(s/2)/\Gamma(\frac{n-s}{2})$.*

**Theorem 2.3.10** *Let $s \in (0,n)$ and $\mu$ a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. If $1 < p < \min\{q, n/s\}$ or $1 = p \le q < \infty$, then the following two conditions are equivalent:*

*(a)*

$$\|v(t^{2\beta}, x)\|_{L^q(\mathbb{R}^{1+n}_+, \mu)} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

*(b)*

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(T(B(x,r))))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r))} < \infty.$$

*But, this equivalence fails to hold when $1 < p = q < n/s$.*

**Proof.** *Part 1*: We prove $(a) \Longleftrightarrow (b)$.

It follows from Theorem 2.3.6 that it is enough to prove that (b) implies (c) or (e) in Theorem 2.3.6. We consider the following three cases.

*Case 1*, $1 = p \le q < \infty$ : If $(b)$ holds, then $\|u\|_{1,q} < \infty$. Suppose that $O \subseteq \mathbb{R}^n$ is a bounded open set and is covered by a sequence of dyadic cubes $\{I_j\}$ in $\mathbb{R}^n$ with $\sum_j |I_j|^{\frac{n-s}{n}} < \infty$. According to Dafni-Xiao [22, Lemma 4.1] there exists another sequence of dyadic cubes $\{J_j\}$ in $\mathbb{R}^n$ such that

$$\operatorname{Int}(J_j) \cap \operatorname{Int}(J_k) = \emptyset \quad \text{for} \quad j \ne k, \quad \bigcup_j J_j = \bigcup_k I_k,$$

$$\sum_j |J_j|^{\frac{n-s}{n}} \le \sum_k |I_k|^{\frac{n-s}{n}}, \quad T(O) \subseteq \bigcup_j T(\operatorname{Int}(5\sqrt{n} J_j)).$$

Then

$$\begin{aligned} \mu(T(O)) &\lesssim \|\mu\|_{1,q} \sum_j |5\sqrt{n} J_j|^{\frac{q(n-s)}{n}} \lesssim \|\mu\|_{1,q} \left(\sum_j |J_j|^{\frac{(n-s)}{n}}\right)^q \\ &\lesssim \|\mu\|_{1,q} \left(\sum_j |I_j|^{\frac{(n-s)}{n}}\right)^q. \end{aligned}$$

By Xiao [77] (see also Adams [4] or [5]) we have $cap_{\dot{W}^{s,1}(\mathbb{R}^n)}(\cdot) \approx H^{n-s}_{\infty}(\cdot)$, where the $H^d_{\infty}(\cdot)$ is the $d-$ dimensional Hausdorff capacity. Thus, these along with the definition of $H^{n-s}_{\infty}(O)$ imply

$$\mu(T(O)) \lesssim \|\mu\|_{1,q} \left(cap_{\dot{W}^{s,1}(\mathbb{R}^n)}(O)\right)^q;$$

that is, the inequality (e) in Theorem 2.3.6 holds.

*Case 2*: $1 < p < \min\{q, n/s\}$ : Let $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$ and $\mu_\lambda$ be the restriction of $\mu$ to $E^{\beta,s}_\lambda(\dot{W}^{s,p}(\mathbb{R}^n))$. If (b) holds, then

$$\|\mu\|_{p,q} := \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu\left(T(B(x,r))\right)}{r^{\frac{q(n-ps)}{p}}} < \infty.$$

It follows from Lemma 2.3.9 that

$$|f(x)| \lesssim \int_{\mathbb{R}^n} \frac{(-\Delta)^{s/2} f(y)}{|y-x|^{n-s}} \mathrm{d}y, \quad f \in \dot{W}^{s,p}(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

This inequality along with Lemma 2.3.8 and Fubini's theorem tell us

$$\begin{aligned}
\lambda\mu\left(E^{\beta,s}_\lambda(v_0)\right) &\lesssim \int_{E^{\beta,s}_\lambda(v_0)} |S_\beta(t^{2\beta})(v_0(x))| \mathrm{d}\mu(t,x) \\
&\lesssim \int_{E^{\beta,s}_\lambda(v_0)} \left| \int_{\mathbb{R}^n} K^\beta_{t^{2\beta}}(y)(v_0(x-y))\mathrm{d}y \right| \mathrm{d}\mu(t,x) \\
&\lesssim \int_{\mathbb{R}^{1+n}_+} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{K^\beta_{t^{2\beta}}(y)}{|z-(x-y)|^{n-s}} \mathrm{d}y \right) |(-\Delta)^{\frac{s}{2}} v_0(z)| \mathrm{d}z \right) \mathrm{d}\mu_\lambda(t,x) \\
&\lesssim \int_{\mathbb{R}^{1+n}_+} \left( \int_{\mathbb{R}^n} \left( (t^2+|z-x|^2)^{\frac{s-n}{2}} \right) |(-\Delta)^{s/2} v_0(z)| \mathrm{d}z \right) \mathrm{d}\mu_\lambda(t,x) \\
&\lesssim \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v_0(z)| \left( \int_{\mathbb{R}^{1+n}_+} \left( (t^2+|z-x|^2)^{\frac{s-n}{2}} \right) \mathrm{d}\mu_\lambda(t,x) \right) \mathrm{d}z \\
&\lesssim \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v_0(y)| \left( \int_0^\infty \mu_\lambda\left(T(B(y,r))r^{s-n-1}\right) \mathrm{d}r \right) \mathrm{d}y \\
&\lesssim (I_1(z) + I_2(z)),
\end{aligned}$$

where

$$I_1(z) = \int_0^z \left( \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v_0(y)| \mu_\lambda\left(T(B(y,r))\right) \mathrm{d}y \right) r^{s-n-1} \mathrm{d}r$$

and

$$I_2(z) = \int_z^\infty \left( \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v_0(y)| \mu_\lambda\left(T(B(y,r))\right) \mathrm{d}y \right) r^{s-n-1} \mathrm{d}r.$$

By the definition of $\|\mu\|_{p,q}$, we have

$$\mu_\lambda\left(T(B(y,r)\right) \leq \left(\mu_\lambda(T(B(y,r)))\right)^{1/p'} \|\mu\|^{1/p}_{p,q} r^{\frac{q(n-ps)}{p^2}}$$

for $\frac{1}{p} + \frac{1}{p'} = 1$. So, using Hölder's inequality and the estimate

$$\int_{\mathbb{R}^n} \mu_\lambda \left(T(B(x,r))\right) \mathrm{d}x \lesssim r^n \mu_\lambda \left(E_\lambda^{\beta,s}(v_0)\right),$$

we obtain

$$\begin{aligned}
I_1(z) &\lesssim \int_0^z \left(\int_{\mathbb{R}^n} |(-\triangle)^{s/2} v_0(y)| (\mu_\lambda(T(B(y,r))))^{1/p'} \|\mu\|_{p,q}^{\frac{1}{p}} r^{\frac{q(n-ps)}{p^2}} \mathrm{d}y\right) r^{s-n-1} \mathrm{d}r \\
&\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{1/p} \int_0^s \left(\int_{\mathbb{R}^n} \mu_\lambda \left(T(B(y,r))\right) \mathrm{d}y\right)^{1/p'} r^{\frac{q(n-ps)}{p^2}+s-n-1} \mathrm{d}r \\
&\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{1/p} \int_0^z \left(r^n \mu \left(E_\lambda^{\beta,s}(v_0)\right)\right)^{1/p'} r^{\frac{q(n-ps)}{p^2}+s-n-1} \mathrm{d}r \\
&\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{1/p} \left(\mu \left(E_\lambda^{\beta,s}(v_0)\right)\right)^{1/p'} z^{\frac{(q-p)(n-p\beta)}{p^2}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2(z) &\lesssim \int_z^\infty \left(\int_{\mathbb{R}^n} |(-\triangle)^{\frac{s}{2}} v_0(y)|^p \mu_\lambda(T(B(y,r)))\, \mathrm{d}y\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \mu_\lambda(T(B(y,r)))\, \mathrm{d}y\right)^{\frac{1}{p'}} r^{s-n-1} \mathrm{d}r \\
&\lesssim \int_z^\infty \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \left(\mu_\lambda \left(T(B(y,r))\right)\right)^{1/p} \left(\int_{\mathbb{R}^n} \mu_\lambda \left(T(B(y,r))\right) \mathrm{d}y\right)^{1/p'} r^{s-n-1} \mathrm{d}r \\
&\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \left(\mu \left(E_\lambda^{\beta,s}(v_0)\right)\right)^{1/p} \int_z^\infty r^{n/p'} \left(\mu(E_\lambda^{\beta,s}(v_0))\right)^{1/p'} r^{s-n-1} \mathrm{d}r \\
&\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \left(\mu \left(E_\lambda^{\beta,s}(v_0)\right)\right) z^{s-n/p}.
\end{aligned}$$

Combing the above estimates on $I_1(z)$ and $I_2(z)$ together, we have

$$\begin{aligned}
\lambda \mu(E_\lambda^{\beta,s}(v_0)) &\lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \mu_\lambda \left(E_\lambda^{\beta,s}(v_0)\right) \\
&\times \left(z^{s-n/p} + \left(\|\mu\|_{p,q} \left(\mu \left(E_\lambda^{\beta,s}(v_0)\right)\right)^{-1}\right)^{1/p} z^{\frac{(q-p)(n-ps)}{p^2}}\right).
\end{aligned}$$

Taking

$$z = \left(\|\mu\|_{p,q}^{-1} \left(\mu \left(E_\lambda^{\beta,s}(v_0)\right)\right)\right)^{\frac{p}{q(n-ps)}}$$

in the above inequality, we have

$$\lambda \left(\mu(E_\lambda^{\beta,s}(v_0))\right)^{1/q} \lesssim \|\mu\|_{p,q}^{1/q} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$

This implies the condition (c) of Theorem 2.3.6.

*Part 2*: We find a nonnegative Radon measure to show that if $1 < p = q < n/s$ then (b) does not imply (a) in general.

In fact, suppose $K \subseteq \mathbb{R}^n$ is a compact set with the $(n-p)-$dimensional Hausdorff measure $H^{(n-ps)}(K) > 0$, then by Maz'ya [49, p. 358, Proposition 3] we have $cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(K) = 0$, on the other hand by Adams-Hedberg [6, p. 132, Proposition 5.1.5 & p. 136, Theorem 5.1.12] we can find a nonnegative Radon measure $\nu$ on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n, r>o} \frac{\nu(B(x,r))}{r^{n-ps}} < \infty \text{ and } 0 < H_\infty^{n-ps}(K) \lesssim \nu(K).$$

Define $\mu(t,x) = \delta_1(t) \otimes \nu(x)$. Then (b) hold for this nonnegative Radon measure on $\mathbb{R}_+^{1+n}$. However, (a) is not true, otherwise, we would have $0 < \nu(K) \lesssim cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(K) = 0$. Contradiction. □

Working from $\mathbb{R}_+^{1+n}$ to $\mathbb{R}^n$, a trace inequality can be derived from $\dot{W}^{s,p}(\mathbb{R}^n)$.

**Theorem 2.3.11** *Let $s \in (0,n)$, $1 < p \leq q < \infty$, $p < n/s$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^n$. Then*

$$\|f\|_{L^q(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\ f \in \dot{W}^{s,p}(\mathbb{R}^n) \Leftrightarrow \sup_{open\ O\subseteq\mathbb{R}^n} \frac{(\mu(O))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O)} < \infty.$$

*If $1 = p \leq q < \infty$, or $1 < p < \min\{q, n/s\}$ then*

$$\|f\|_{L^q(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{W}^{s,1}(\mathbb{R}^n)},\ f \in \dot{W}^{s,p}(\mathbb{R}^n) \Leftrightarrow \sup_{x\in\mathbb{R}^n, r>0} \frac{(\mu(B(x,r)))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r))} < \infty.$$

Similarly to Theorem 2.3.6 & 2.3.10, we obtain the following result which covers Theorem 2.3.11.

**Theorem 2.3.12** *Let $s \in (0,n)$, $1 < p \leq q < \infty$, $p < n/s$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^n$. Then the following statements are equivalent:*

*(a)*

$$\|f\|_{L^{q,p}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\quad f \in \dot{W}^{s,p}(\mathbb{R}^n),$$

*(b)*

$$\|f\|_{L^q(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\quad f \in \dot{W}^{s,p}(\mathbb{R}^n),$$

*(c)*

$$\|f\|_{L^{q,\infty}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)},\quad f \in \dot{W}^{s,p}(\mathbb{R}^n),$$

*(d)*

$$(\mu(O))^{p/q} \lesssim cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(O),\quad open\ O \subseteq \mathbb{R}^n.$$

*If $1 = p \leq q < \infty$, or $1 < p < \min\{q, n/s\}$ then all of them are equivalent to*

*(e)*

$$\sup_{r>0, x\in\mathbb{R}^n} \frac{(\mu(B(x,r)))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r))} < \infty.$$

*If $0 < q < p = 1$, then (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a).*

**Remark 2.3.13** *The case $1 = p \le q < \infty$ of Theorems 2.3.12 was shown by Xiao in [77].*

**Corollary 2.3.14** *Let $\beta \in (0, 1]$, $s \in (0, n)$, $1 \le p < n/s$ and $\gamma \in (-1, \infty)$. Then the following two conditions hold:*
*(a)*

$$\left(\int_{\mathbb{R}^{1+n}_+} |v(t^{2\beta}, x)|^{\frac{p(1+n+\gamma)}{n-ps}} t^\gamma dtdx\right)^{\frac{n-ps}{p(1+n+\gamma)}} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

*(b)*

$$\sup_{t>0}\left(\int_{\mathbb{R}^n} |v(t^{2\beta}, x)|^{\frac{pn}{n-ps}} dx\right)^{\frac{n-ps}{pn}} \lesssim \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n).$$

**Proof.** In Theorem 2.3.10 we take

$$d\mu(t, x) = (1+\gamma)^{-1} t^\gamma dtdx, \; q = \frac{p(1+n+\gamma)}{n-ps}, \; \gamma > -1,$$

respective

$$d\mu(t, x) = \delta_{t_0}(t) \otimes dx, \; q = \frac{pn}{n-ps}, \; \gamma \longrightarrow -1,$$

where $\delta_{t_0}(t)$ is the Dirac measure at $t_0 > 0$, then an application of the capacitary estimate of ball (see Maz'ya [49, p. 356] for $p \in (1, n/\beta)$, Xiao [77, p. 833] for $p = 1$):

$$cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x, r)) \approx r^{n-ps}, x \in \mathbb{R}^n, r > 0,$$

we can finish the proof. □

According to Miao-Yuan-Zhang [53, Proposition 2.1], the condition (a) of Corollary 2.3.14 amounts to that $\dot{W}^{s,p}(\mathbb{R}^n)$ is embedded in the homogeneous Besov or Triebel-Lizorkin space (see Triebel [69] for more details about these spaces)

$$\dot{B}^{-\frac{\gamma+1}{q}}_{q,q}(\mathbb{R}^n) = \dot{F}^{-\frac{\gamma+1}{q}}_{q,q}(\mathbb{R}^n), \; q = \frac{p(1+n+\gamma)}{n-ps}.$$

At the same time, the condition (b) of Corollary 2.3.14 can be treated as extreme case of the condition (a) in Corollary 2.3.14.

Finally, according to Theorems 2.3.6 & 2.3.10, we can establish the following decay of the solutions of equation (2.0.2).

**Theorem 2.3.15** *If $v_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$ for $1 \le p < n/s$ and $s \in (0, n)$, then*

$$|v(t_0^{2\beta}, x_0)| \lesssim t_0^{ps-n} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall (t_0, x_0) \in \mathbb{R}^{1+n}_+.$$

*Equivalently*

$$|v(t_0, x_0)| \lesssim t_0^{\frac{ps-n}{2s}} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall (t_0, x_0) \in \mathbb{R}^{1+n}_+.$$

The special case $\beta = p = 1$ of Theorem 2.3.15 was proved by Xiao in [77].
**Proof.** From the proof of Theorems 2.3.6 & 2.3.10 for $1 \le p < n/s$ and $q > p$, we have

$$\begin{aligned} |||\mu|||_{p,q} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{(\mu(T(B(x,r))))^{\frac{p}{q}}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r))} < \infty \\ &\Rightarrow \left( \int_{\mathbb{R}^{1+n}_+} |v(t^{2\beta}, x)|^q \mathrm{d}\mu(t,x) \right)^{\frac{1}{q}} \lesssim |||\mu|||_{p,q} \|v_0\|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad \forall v_0 \in \dot{W}^{s,p}(\mathbb{R}^n). \end{aligned}$$

Given $(t_0, x_0) \in \mathbb{R}^{1+n}_+$. Let $q = \frac{np}{n-ps}$ and $\mu(t,x) = \delta_{(t_0,x_0)}$ be the Dirac measure at $(t_0, x_0)$. It suffices to prove $|||\delta_{(t_0,x_0)}|||_{p,q} \le t_0^{ps-n}$. In fact, if $(t_0, x_0)$ is not in $T(B(x,r))$, then $\delta_{(t_0,x_0)}(T(B(x,r))) = 0$. If $(t_0,x_0) \in T(B(x,r))$, then $B(x_0,t_0) \subseteq B(x,r)$ and $r^n \ge t_0^n$. This give the estimate

$$\delta_{(t_0,x_0)}(T(B(x,r))) \le \frac{r^n}{t_0^n} = t_0^{-n} r^{\frac{(n-ps)q}{p}}.$$

The above estimate and $cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r)) \approx r^{n-ps}$ verify

$$\frac{(\delta_{(t_0,x_0)}(T(B(x,r))))^{p/q}}{cap_{\dot{W}^{s,p}(\mathbb{R}^n)}(B(x,r))} \le t_0^{-\frac{np}{q}}.$$

Therefore, $|||\delta_{(t_0,x_0)}|||_{p,q} \le t_0^{ps-n}$. □

# Chapter 3

# Strichartz Type Estimates for Fractional Heat-Type Equations

This chapter studies Strichartz type estimates for the inhomogeneous initial problems associated with the fractional heat-type equations

$$\begin{cases} \partial_t v(t,x) + (-\triangle)^\beta v(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+^{1+n}, \\ v(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.0.1}$$

where $\beta \in (0,\infty)$ and $n \in \mathbb{N}$. The main goal is to determine pairs $(q,p)$ and $(q_1,p_1)$ ensuring

$$\|e^{-t(-\triangle)^\beta} f\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.0.2}$$

$$\left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} F(s,x)ds \right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \|F\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}, \tag{3.0.3}$$

where $I$ is either $[0,\infty)$ or $[0,T]$ for some $0 < T < \infty$, and $p_1' = \frac{p_1}{p_1-1}$ is the conjugate of a given number $p_1 \geq 1$.

## 3.1 Notations and Preliminaries

In the following, for a Banach space $X$, $L^p(X)$ (where $p \in [1,\infty)$) is used as the space of functions $f : X \longrightarrow \mathbb{R}$ with

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p dx\right)^{1/p} < \infty;$$

for a function space $F(\mathbb{R}^n)$ on $\mathbb{R}^n$, $L^q(I;F(\mathbb{R}^n))$ (where $q \in [1,\infty)$) represents the set of functions $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ for $I \subseteq \mathbb{R}$ with

$$\|f\|_{L^q(I;F(\mathbb{R}^n))} = \left(\int_I \|f(t,x)\|_{F(\mathbb{R}^n)}^q dt\right)^{1/q} < \infty.$$

Given an infinitely differential function $\eta$ with compact support in $\mathbb{R}^n$ satisfying

$$\eta(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2, \end{cases} \tag{3.1.1}$$

define the sequence $\{\psi_j\}_{j\in\mathbb{Z}}$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$\psi_j(\xi) = \eta\left(\frac{\xi}{2^j}\right) - \eta\left(\frac{\xi}{2^{j-1}}\right). \tag{3.1.2}$$

This sequence is applied to define the inhomogeneous and homogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $1 \le p, q \le \infty$ and $s \in \mathbb{R}^n$ :

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\} \text{ and } \dot{B}^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty\},$$

respectively. Here $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(\eta\mathcal{F}(f))\|_{L^p(\mathbb{R}^n)} + \begin{cases} \left(\sum\limits_{j=1}^{\infty} (2^{sj}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))\|_{L^p(\mathbb{R}^n)})^q\right)^{1/q} & \text{if } q < \infty, \\ \sup\limits_{j\ge 1} 2^{sj}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty \end{cases} \tag{3.1.3}$$

and

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum\limits_{j=-\infty}^{\infty} (2^{sj}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))\|_{L^p(\mathbb{R}^n)})^q\right)^{1/q} & \text{if } q < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{sj}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty. \end{cases} \tag{3.1.4}$$

On the other hand, Besov spaces can be defined by interpolation between the Lebesgue spaces and the Sobolev spaces of integer order (see Triebel [69]). It follows from Bergh and Löfström [8] that for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = [H^{s_1,p}(\mathbb{R}^n), H^{s_2,p}(\mathbb{R}^n)]_{\theta,q} \text{ and } \dot{B}^s_{p,q}(\mathbb{R}^n) = [\dot{H}^{s_1,p}(\mathbb{R}^n), \dot{H}^{s_2,p}(\mathbb{R}^n)]_{\theta,q},$$

where $s_1 \neq s_2$, $0 < \theta < 1$ and $s = (1-\theta)s_1 + \theta s_2$. Here $H^{s,p}(\mathbb{R}^n)$ and $\dot{H}^{s,p}(\mathbb{R}^n)$ are the inhomogeneous and homogeneous Sobolev spaces which are the completion of all infinitely differential functions $f$ with compact support in $\mathbb{R}^n$ with respect to the norms

$$\|f\|_{H^{s,p}(\mathbb{R}^n)} = \|(I-\triangle)^{s/2}f\|_{L^p(\mathbb{R}^n)}, \quad \text{and } \|f\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \|(-\triangle)^{s/2}f\|_{L^p(\mathbb{R}^n)}$$

respectively, where $(I-\triangle)^{s/2}f = \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}f(\xi))$.

$BMO(\mathbb{R}^n)$ is the set of equivalence classes of locally integrable functions $f$ modulo constants for which the following supremum is finite:

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_I l(I)^{-n}\int_I |f(x) - f_I|dx,$$

where $I$ is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $l(I)$ is the sidelength of $I$ and $f_I = (l(I))^{-n}\int_I f(x)dx$.

**Definition 3.1.1** *The triplet $(q,p,r)$ is called a $\sigma-$admissible triplet provided*

$$\frac{1}{q}=\sigma\left(\frac{1}{r}-\frac{1}{p}\right),$$

*where $1<r\leq p\leq\infty$ and $\sigma>0$.*

## 3.2 Strichartz Estimates Involving Norms in Lebesgue Spaces

We first establish the Strichartz estimates with norms in Lebesgue spaces for fractional heat equations. Before doing this, we need the following lemmas.

**Lemma 3.2.1** *For all $t>0$ and $\gamma,\beta>0$, we have*
*(a) $e^{-t(-\triangle)^\beta}(-\triangle)^\gamma=(-\triangle)^\gamma e^{-t(-\triangle)^\beta}$.*
*(b) $e^{-t(-\triangle)^\beta}(I-\triangle)^\gamma=(I-\triangle)^\gamma e^{-t(-\triangle)^\beta}$.*
*(c) $\langle e^{-t(-\triangle)^\beta}f,g\rangle=\langle f,e^{-t(-\triangle)^\beta}f\rangle,\quad\forall f,g\in L^2(\mathbb{R}^n)$.*

**Proof.** The proofs of (a) and (b) will follow from the definition of $e^{-t(-\triangle)^\beta}$, $(-\triangle)^\gamma$ and $(I-\triangle)^\gamma$. For (b), let $f,g\in L^2(\mathbb{R}^n)$. According to the Fourier transform and the Plancherel's identity we have

$$\begin{aligned}\langle e^{-t(-\triangle)^\beta}f,g\rangle &= \int(e^{-t(-\triangle)^\beta}f)\overline{g(x)}dx\\ &= \int\mathcal{F}^{-1}\left(e^{-t|\xi|^{2\beta}}\mathcal{F}f(\xi)\right)(x)\overline{g(x)}dx\\ &= \int e^{-t|\xi|^{2\beta}}\mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}d\xi\\ &= \int\mathcal{F}f(\xi)\overline{e^{-t|\xi|^{2\beta}}\mathcal{F}g(\xi)}d\xi\\ &= \int f(x)\overline{\mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}}\mathcal{F}g(\xi))(x)}dx\\ &= \langle f,e^{-t(-\triangle)^\beta}g\rangle.\end{aligned}$$

This finishes the proof of Lemma 3.2.1. □

Miao-Yuan-Zhang in [53] established the forthcoming two lemmas.

**Lemma 3.2.2** *[53] Let $1\leq r\leq p\leq\infty$ and $f\in L^r(\mathbb{R}^n)$. Then*

$$\|e^{-t(-\triangle)^\beta}f\|_{L^p_x(\mathbb{R}^n)}\lesssim t^{-\frac{n}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)}\|f\|_{L^r(\mathbb{R}^n)},$$

$$\|\nabla e^{-t(-\triangle)^\beta}f\|_{L^p_x(\mathbb{R}^n)}\lesssim t^{-\frac{1}{2\beta}-\frac{n}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)}\|f\|_{L^r(\mathbb{R}^n)}.$$

**Lemma 3.2.3** *[53] Let $(q,p,r)$ be any $\frac{n}{2\beta}-$admissible triplet satisfying*

$$p < \begin{cases} \frac{nr}{n-2\beta}, & n > 2\beta, \\ \infty, & n \leq 2\beta, \end{cases}$$

*and let $\varphi \in L^r(\mathbb{R}^n)$. Then $e^{-t(-\triangle)^\beta}\varphi \in L^q(I; L^p(\mathbb{R}^n))$ with the estimate*

$$\|e^{-t(-\triangle)^\beta}\varphi\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|\varphi\|_{L^r(\mathbb{R}^n)},$$

*for $I = [0,T)$, $0 < T \leq \infty$.*

Lemma 3.2.3 gives us the homogeneous Strichartz estimates of equation (3.0.1) except endpoint cases. To obtain the endpoint estimates we need the abstract Strichartz estimates of Keel-Tao [38].

**Lemma 3.2.4** *[38] Let $H$ be a Hilbert space and $X$ be a Banach space. Suppose that $U(t) : H \longrightarrow L^2(X)$ obeys the energy estimate:*

$$\|U(t)f\|_{L^2(X)} \lesssim \|f\|_H$$

*and the untruncated decay estimate, that is for some $\sigma > 0$,*

$$\|U(t)(U(s))^* f\|_{L^\infty} \lesssim |t-s|^{-\sigma}\|f\|_{L^1}, \ \forall s \neq t.$$

*Then the estimates*

$$\|U(t)f\|_{L^q_t L^p_x} \lesssim \|f\|_H,$$

$$\left\|\int (U(s))^* F(s)ds\right\|_H \lesssim \|F\|_{L^{q'}_t L^{p'}_x}$$

$$\left\|\int_{s<t} U(t)(U(s))^* F(s)ds\right\|_{L^q_t L^p_x} \lesssim \|F\|_{L^{q_1'}_t L^{p_1'}_x}$$

*hold for all $\sigma-$admissible triplets $(q,p,2)$ and $(q_1,p_1,2)$ with $q, q_1 \geq 2$, $(q,p,\sigma)$ and $(q_1,p_1,\sigma)$ are not $(2,\infty,1)$.*

**Proposition 3.2.5** *Let $(q,p,2)$ be $\frac{n}{2\beta}-$admissible. If $q \geq 2$ and $(q,p,\frac{n}{2\beta})$ is not $(2,\infty,1)$, then (3.0.2) holds.*

**Proof.** We only need to prove (3.0.2) for $I = [0,\infty)$ since the proofs for other cases are similar. Assume that $(q,p,2)$ is a $\frac{n}{2\beta}-$admissible triplet with $q \geq 2$ and $\left(q,p,\frac{n}{2\beta}\right)$ is not $(2,\infty,1)$. It follows from Lemma 3.2.2 that we have the energy estimate

$$\|e^{-t(-\triangle)^\beta} f\|_{L^2_x(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \forall t > 0, \tag{3.2.1}$$

and untruncated decay estimate

$$\|e^{-(t+s)(-\triangle)^\beta} f\|_{L^\infty(\mathbb{R}^n)} \lesssim |t+s|^{-\frac{n}{2\beta}}\|f\|_{L^1(\mathbb{R}^n)} \lesssim |t-s|^{-\frac{n}{2\beta}}\|f\|_{L^1(\mathbb{R}^n)}, \ \forall s \neq t, s,t \in (0,\infty). \tag{3.2.2}$$

By (3.2.1), (3.2.2) and Lemma 3.2.1, we can apply Lemma 3.2.4 with $U(t) = e^{-t(-\triangle)^\beta}$ for $t > 0$, $H = L^2(\mathbb{R}^n)$ and $X = \mathbb{R}^n$ to obtain (3.0.2). □

**Remark 3.2.6** *Proposition 3.2.5 extends Miao, Yuan and Zhang's [53, Lemma 3.2] to the cases:* $(q,p,r) = (2, \frac{2n}{n-2\beta}, 2)$ *when* $n > 2\beta$; $(q,p,r) = (\frac{4\beta}{n}, \infty, 2)$ *when* $n < 2\beta$.

It is well known that for the Schrödinger equations, there are pairs $(q,p)$ and $(q_1,p_1)$ such that $(q,p,2)$ and $(q_1,p_1,2)$ are not $n/2-$admissible but the inhomogeneous Strichartz estimates hold (see Cazenave-Weissler [17], Tao [65] and Vilela [70]). Similarly, we will prove that (3.0.3) holds for some pairs $(q,p)$ and $(q_1,p_1)$ satisfying the property

$$\left(\frac{1}{q_1'} - \frac{1}{q}\right) + \frac{n}{2\beta}\left(\frac{1}{p_1'} - \frac{1}{p}\right) = 1. \tag{3.2.3}$$

This property is weaker than the $\frac{n}{2\beta}-$admissibility of $(q,p,2)$ and $(q_1,p_1,2)$.

**Theorem 3.2.7** *Let* $1 \le p_1' < p \le \infty$ *and* $1 < q_1' < q < \infty$. *If* $(q,p)$ *and* $(q_1,p_1)$ *satisfy (3.2.3), then (3.0.3) holds.*

**Proof.** We only need to prove (3.0.3) for $I = [0.\infty)$, the proofs for other cases being similar. Assume that $(q,p,2)$ and $(q_1,p_1,2)$ satisfy $1 \le p_1' < p \le \infty$, $1 < q_1' < q < \infty$ and $\frac{1}{q_1'} + \frac{n}{2\beta}\left(\frac{1}{p_1'} - \frac{1}{p}\right) = 1 + \frac{1}{q}$. It follows from Lemma 3.2.2 that

$$\|e^{-(t-s)(-\triangle)^\beta}F(s,x)\|_{L^p_x(\mathbb{R}^n)} \lesssim |t-s|^{-\frac{n}{2\beta}\left(\frac{1}{p_1'}-\frac{1}{p}\right)}\|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)}, \quad \forall s < t.$$

Then the Hardy-Littlewood-Sobolev inequality implies that

$$\begin{aligned}
\left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} &\lesssim \left\|\int_0^t \|e^{-(t-s)(-\triangle)^\beta}F(s,x)\|_{L^p_x(\mathbb{R}^n)}ds\right\|_{L^q_t(I)} \\
&\lesssim \left\|\int_0^t |t-s|^{-\frac{n}{2\beta}\left(\frac{1}{p_1'}-\frac{1}{p}\right)}\|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)}ds\right\|_{L^q_t(I)} \\
&\lesssim \|F\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}.
\end{aligned}$$

This finishes the proof of (3.0.3). □

**Remark 3.2.8** *Since* $e^{-t(-\triangle)^\beta}$ *commutes with* $(-\triangle)^\gamma$ *and* $(I-\triangle)^\gamma$ *for* $\gamma > 0$, *if* $(q,p)$ *satisfies the assumption of Theorem 3.2.5 then (3.0.2) holds with* $\|\cdot\|_{L^p(\mathbb{R}^n)}$ *replaced by either* $\|\cdot\|_{\dot{H}^{\gamma,p}(\mathbb{R}^n)}$ *or* $\|\cdot\|_{H^{\gamma,p}(\mathbb{R}^n)}$. *Similarly, if* $(q,p)$ *and* $(q_1,p_1)$ *satisfy the assumption of Theorem 3.2.7, then (3.0.3) holds with the same replacement.*

**Corollary 3.2.9** *Let* $n \ge 2\beta$, $I = [0,T]$ *or* $[0,\infty)$. *Suppose* $V$ *is a real potential and*

$$V \in L^r_t(I;L^s_x(\mathbb{R}^n)),\ \frac{1}{r} + \frac{n}{2\beta s} = 1,$$

*for some fixed* $r \in (1,2) \cup (2,\infty)$ *and* $s \in (\frac{n}{2\beta}, \frac{n}{\beta}) \cup (\frac{n}{\beta}, \infty)$. *Let* $f \in L^2(\mathbb{R}^n)$ *and* $F \in L_t^{q_1'}(I; L_x^{p_1'}(\mathbb{R}^n))$ *for some* $\frac{n}{2\beta}$*−admissible triplet* $(q_1, p_1, 2)$ *with* $p_1' \in [1,2)$ *and* $q_1' \in (1,2)$. *Then equation (1.0.8) has a unique solution* $v(t,x)$ *satisfying*

$$\|v\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}, \tag{3.2.4}$$

*for all* $\frac{n}{2\beta}$*−admissible triplets* $(q,p,2)$ *with* $2 \le q < \infty$.

**Proof**. We shall prove this theorem for $n > 2\beta$. In the case $n = 2\beta$, we can replace in the sequel the space $L_t^2(J; L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))$ by any $L_t^q(J; L_x^p(\mathbb{R}^n))$ for 1−admissible $(q,p,2)$ with $p$ arbitrarily large. We consider the following two cases.

*Case 1,* $r \in (2,\infty)$: Let $(q,p,2)$ $(2 \le q < \infty)$ be $\frac{n}{2\beta}$−admissible. Let $J = [0,\varepsilon]$ where $\varepsilon > 0$ will be determined later and $(k,l,2)$ be $\frac{n}{2\beta}$−admissible with $q \le k < \infty$, and set

$$X = L_t^k(J; L_x^l(\mathbb{R}^n)) \cap L_t^2(J; L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))$$

with

$$\|v\|_X := \max\left\{ \|v\|_{L_t^k(J;L_x^l(\mathbb{R}^n))}, \|v\|_{L_t^2(J;L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))} \right\}.$$

By interpolation (see Triebel [69]), $X$ can be embedded into $L_t^{q_0}(J; L_x^{p_0}(\mathbb{R}^n))$ for each $\frac{n}{2\beta}$−admissible triplet $(q_0, p_0, 2)$ with $2 \le q_0 \le k$. Define $T(v)$ on $X$ by

$$T(v) = e^{-t(-\triangle)^\beta} f + \int_0^t e^{-(t-s)(-\triangle)^\beta} (F(s,x) - V(s,x)v(s,x))ds, \quad \forall v = v(t,x) \in X.$$

Applying Proposition 3.2.5 and Theorem 3.2.7, we have

$$\|T(v)\|_{L_t^{q_0}(J;L_x^{p_0}(\mathbb{R}^n))} \le C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|Vv\|_{L_t^{q_2'}(J;L_x^{p_2'}(\mathbb{R}^n))},$$

for all $\frac{n}{2\beta}$−admissible triplets $(q_0, p_0, 2), (q_1, p_1, 2)$, and $(q_2, p_2, 2)$ satisfying

$$2 \le q_0 \le k, \quad q_1' \in (1,2), \quad q_2' \in (1,2), \quad 1 \le p_1' < p_0 \le \infty, \quad 1 \le p_2' < p_0 \le \infty.$$

Here and later $C > 0$ is a constant. Clearly, Hölder's inequality implies

$$\|T(v)\|_{L_t^{q_0}(J;L_x^{p_0}(\mathbb{R}^n))} \le C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_{L_t^2(J;L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))}$$

provided

$$\frac{1}{q_2} = \frac{1}{2} - \frac{1}{r}, \frac{1}{p_2} = \frac{n+2\beta}{2n} - \frac{1}{s}.$$

This and the assumption on $r$ and $s$ imply that $q_2' \in (1,2)$, $p_2' \in [1,2)$ and

$$\frac{1}{q_2} + \frac{n}{2\beta}\frac{1}{p_2} = \frac{1}{2} + \frac{n}{2\beta}\frac{n+2\beta}{2n} - \left(\frac{n}{2\beta}\frac{1}{s} + \frac{1}{r}\right) = \frac{n}{4\beta}.$$

Again, by Hölder's inequality we have

$$\|T(v)\|_{L_t^{q_0}(J;L_x^{p_0}(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)}+C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}+C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_{L_t^\infty(J;L_x^2(\mathbb{R}^n))}.$$

Similarly, taking $(q_0, p_0, 2)$ be $(k, l, 2)$ and $(2, \frac{2n}{n-2\beta}, 2)$, we have

$$\|T(v)\|_X \leq C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_X.$$

The rest of the proof is similar to that of the first case. □

## 3.3 Strichartz Estimates Involving Norms in Other Spaces

In this section we obtain several Strichartz type estimates containing parabolic Strichartz estimates and other Strichartz type estimates with $L^p(\mathbb{R}^n)$ norms replaced by the norms in $BMO(\mathbb{R}^n)$, Sobolev and Besov spaces.

**Theorem 3.3.1** *Let $n = 2\beta$. Then*

$$\|e^{-t(-\triangle)^\beta} f\|_{L_t^2((0,\infty);BMO_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.3.1}$$

**Proof.** Let $n = 2\alpha$. Define $\varphi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq (1/2, 2)$, $\varphi(x) = 1$ for $x \in (3/4, 9/8)$ and $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}t) = 1$ for all $t > 0$. Let $\varphi_k(t) = \varphi(2^{-k}t)$. Define $P_k f = \mathcal{F}^{-1}(\mathcal{F}f(\cdot)\varphi_k(|\cdot|))$ be a Littlewood-Paley decomposition with respect to $\varphi_k$ (see [64]). Since $BMO(\mathbb{R}^n) = \dot{F}_\infty^{0,2}(\mathbb{R}^n)$ (see Frazier-Jawerth-Weiss [26]),

$$\|g\|_{BMO(\mathbb{R}^n)} \approx \left\| \left( \sum_{k\in\mathbb{Z}} |P_k g|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^n)}.$$

Let $M_k = B(0, 2^{k+1})\backslash B(0, 2^{k-1})$ and $\chi_{M_k}$ its characteristic function. Since $\varphi$ is supported in $(1/2, 2)$ and $n = 2\beta$, we have

$$\begin{aligned}
\|e^{-t(-\triangle)^\beta} P_k f\|^2_{L_t^2((0,\infty);L_x^\infty(\mathbb{R}^n))} &\leq \int_0^\infty \sup_x \left| \int_{\mathbb{R}^n} e^{-t|\xi|^n} e^{i\langle\xi,x\rangle} \mathcal{F}f(\xi)\varphi(2^{-k}|\xi|)d\xi \right|^2 dt \\
&\leq \int_0^\infty \int_{\mathbb{R}^n} \chi_{M_k}(\xi)d\xi \sup_x \int_{\mathbb{R}^n} \left| e^{-t|\xi|^n} e^{i\langle\xi,x\rangle} \mathcal{F}f(\xi)\varphi(2^{-k}|\xi|) \right|^2 d\xi dt \\
&\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty \int_{M_k} e^{-2t|\xi|^n} \left|\mathcal{F}f(\xi)\varphi(2^{-k}|\xi|)\right|^2 d\xi dt \\
&\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty e^{-t2^{(k-1)n+1}} dt \|f\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim (2^{2n-1} - 1/2)\|f\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Taking $(q_0, p_0, 2)$ be $(k, l, 2)$ and $(2, \frac{2n}{n-2\beta}, 2)$, we get

$$\|T(v)\|_X \le C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_X.$$

Hence $T(v) \in X$ and $T$ is a operator from $X$ to $X$. Since $r < \infty$, we may choose such an $\varepsilon > 0$ that

$$C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))} \le \frac{1}{2}. \tag{3.2.5}$$

This fact yields that

$$\|T(v_1) - T(v_2)\|_X \le \frac{1}{2}\|v_1 - v_2\|_X, \quad \forall v_1, v_2 \in X.$$

Thus $T$ is a contraction operator on $X$, and $T$ has a unique fixed point $v(t,x)$ which is the unique solution of equation (1.0.8) and $v$ satisfies

$$\|v\|_X \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}.$$

Since $X$ is embedded in $L_t^q(J; L_x^p(\mathbb{R}^n))$, one finds

$$\|v\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}.$$

Now, we can apply the previous arguments to any subinterval $J = [t_1, t_2]$ on which a condition like (3.2.5) holds, and obtain

$$\|v\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \lesssim \|v(t_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}. \tag{3.2.6}$$

Note that (3.2.6) implies

$$\|v(t_2)\|_{L^2(\mathbb{R}^n)} \lesssim \|v(t_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}. \tag{3.2.7}$$

If $I = [0, T]$ for $0 < T < \infty$, we can partition $I$ into a finite many of subintervals on which the condition (3.2.5) holds. If $I = [0, \infty)$, since $V \in L_t^r(I; L_x^s(\mathbb{R}^n))$ we can find $T_1 > 0$ such that $C\|V\|_{L_t^r((T_1,\infty);L_x^s(\mathbb{R}^n))} < \frac{1}{2}$ and partition $[0, T_1]$ similarly. Thus we can prove (3.2.4) by inductively applying (3.2.6) and (3.2.7).

*Case 2*, $r \in (1, 2)$. Since $(r, \frac{2s}{s+2})$ is the dual of $(r', \frac{2s}{s-2})$, our assumption on $r, s$ implies

$$\frac{1}{r'} + \frac{n}{2\beta}\frac{s-2}{2s} = \frac{n}{2\beta s} + \frac{n}{2\beta}\frac{s-2}{2s} = \frac{n}{4\beta}.$$

Thus $(r', \frac{2s}{s-2})$ is $\frac{n}{2\beta}$–admissible with $r \in (1, 2)$. In a fashion analogous to handling *Case 1*, we use Theorems 3.2.5 & 3.2.7, to obtain

$$\|T(v)\|_{L_x^{q_0}(J;L_x^{p_0}(\mathbb{R}^n))} \le C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|Vv\|_{L_t^r(J;L_x^{\frac{2s}{s+2}}(\mathbb{R}^n))}.$$

Again, by Hölder's inequality we have

$$\|T(v)\|_{L_t^{q_0}(J;L_x^{p_0}(\mathbb{R}^n))} \le C\|f\|_{L^2(\mathbb{R}^n)}+C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}+C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_{L_t^\infty(J;L_x^2(\mathbb{R}^n))}.$$

Similarly, taking $(q_0, p_0, 2)$ be $(k, l, 2)$ and $(2, \frac{2n}{n-2\beta}, 2)$, we have

$$\|T(v)\|_X \le C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_X.$$

The rest of the proof is similar to that of the first case. □

## 3.3 Strichartz Estimates Involving Norms in Other Spaces

In this section we obtain several Strichartz type estimates containing parabolic Strichartz estimates and other Strichartz type estimates with $L^p(\mathbb{R}^n)$ norms replaced by the norms in $BMO(\mathbb{R}^n)$, Sobolev and Besov spaces.

**Theorem 3.3.1** *Let $n = 2\beta$. Then*

$$\|e^{-t(-\triangle)^\beta} f\|_{L_t^2((0,\infty);BMO_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.3.1}$$

**Proof.** Let $n = 2\alpha$. Define $\varphi \in C^\infty(\mathbb{R})$ with $\text{supp}(\varphi) \subseteq (1/2, 2)$, $\varphi(x) = 1$ for $x \in (3/4, 9/8)$ and $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}t) = 1$ for all $t > 0$. Let $\varphi_k(t) = \varphi(2^{-k}t)$. Define $P_k f = \mathcal{F}^{-1}(\mathcal{F}f(\cdot)\varphi_k(|\cdot|))$ be a Littlewood-Paley decomposition with respect to $\varphi_k$ (see [64]). Since $BMO(\mathbb{R}^n) = \dot{F}_\infty^{0,2}(\mathbb{R}^n)$ (see Frazier-Jawerth-Weiss [26]),

$$\|g\|_{BMO(\mathbb{R}^n)} \approx \left\| \left( \sum_{k\in\mathbb{Z}} |P_k g|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^n)}.$$

Let $M_k = B(0, 2^{k+1})\backslash B(0, 2^{k-1})$ and $\chi_{M_k}$ its characteristic function. Since $\varphi$ is supported in $(1/2, 2)$ and $n = 2\beta$, we have

$$\begin{aligned}
\|e^{-t(-\triangle)^\beta} P_k f\|^2_{L_t^2((0,\infty);L_x^\infty(\mathbb{R}^n))} &\le \int_0^\infty \sup_x \left| \int_{\mathbb{R}^n} e^{-t|\xi|^n} e^{i\langle \xi, x\rangle} \mathcal{F}f(\xi)\varphi(2^{-k}|\xi|)d\xi \right|^2 dt \\
&\le \int_0^\infty \int_{\mathbb{R}^n} \chi_{M_k}(\xi)d\xi \sup_x \int_{\mathbb{R}^n} \left| e^{-t|\xi|^n} e^{i\langle\xi,x\rangle} \mathcal{F}f(\xi)\varphi(2^{-k}|\xi|)\right|^2 d\xi dt \\
&\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty \int_{M_k} e^{-2t|\xi|^n} \left|\mathcal{F}f(\xi)\varphi(2^{-k}|\xi|)\right|^2 d\xi dt \\
&\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty e^{-t2^{(k-1)n+1}} dt \|f\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim (2^{2n-1} - 1/2)\|f\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Take $\psi \in C^\infty(\mathbb{R})$ with $\text{supp}(\psi) \subseteq (1/4, 4)$ and $\psi(x)\varphi(x) = \varphi(x)$. Define

$$\widetilde{P}_k f = \mathcal{F}^{-1}((\mathcal{F}f)\psi_k).$$

Then we have

$$\begin{aligned}
\|e^{-t(-\triangle)^\beta} f\|^2_{L^2_t((0,\infty);BMO_x(\mathbb{R}^n))} &\lesssim \int_0^\infty \sup_x \left( \sum_k |e^{-t(-\triangle)^\beta} P_k f|^2 \right) dt \\
&\lesssim \sum_k \|e^{-t(-\triangle)^\beta} P_k \widetilde{P}_k f\|^2_{L^2_t((0,\infty);L^\infty_x(\mathbb{R}^n))} \\
&\lesssim \sum_k \|\widetilde{P}_k f\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

That is, (3.3.1) holds. □

**Theorem 3.3.2** *(a) Let $1 \leq r \leq p \leq \infty$ and $0 < T < \infty$. If $n < 2\beta$, then*

$$\int_0^T s^{-\frac{nr}{2p\beta}} \|e^{-s(-\triangle)^\beta} f\|^r_{L^p_x(\mathbb{R}^n)} ds \lesssim T^{1-\frac{n}{2\beta}} \|f\|^r_{L^r(\mathbb{R}^n)}. \tag{3.3.2}$$

*(b) Let $2 < p \leq \infty$. If $n = 2\beta$, then*

$$\int_0^\infty s^{-2/p} \|e^{-s(-\triangle)^\beta} f\|^2_{L^p_x(\mathbb{R}^n)} ds \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.3.3}$$

**Remark 3.3.3** *We can refer to (3.3.3) as a parabolic homogeneous Strichartz estimate. The special case $n = 2$ of (3.3.3) was proved by Tao in [68]. On the other hand, according to Miao, Yuan and Zhang's [53, Proposition 2.1], (3.3.3) amounts to that $L^2(\mathbb{R}^n)$ is embedded in the homogeneous Besov space*

$$\dot{B}^s_{p,2}(\mathbb{R}^n), \quad s = \frac{(2-p)n}{2p}, \quad 2 < p \leq \infty.$$

**Proof.** (a). Let $1 \leq r \leq p \leq \infty$ and $n < 2\beta$. It follows from Lemma 3.2.2 that

$$s^{-\frac{nr}{2p\beta}} \|e^{-s(-\triangle)^\beta} f\|^r_{L^p(\mathbb{R}^n)} \lesssim s^{-\frac{n}{2\beta}} \|f\|^r_{L^r(\mathbb{R}^n)}.$$

On the other hand, $n < 2\beta$ implies that

$$\int_0^T s^{-\frac{n}{2\beta}} ds = \frac{2\beta}{2\beta - n} T^{1-\frac{n}{2\beta}}.$$

Thus (3.3.2) holds.
(b). The following proof is essentially the same as the proof Tao's [68, Lemma 2.5]. For the

sake of completeness, it is provided here. We use the $TT^*$ method. Thus, by duality and the self-adjointness of $e^{-t(-\triangle)^\beta}$, it suffices to verify

$$\left\|\int_0^\infty s^{-1/p}e^{-s(-\triangle)^\beta}F(s,x)ds\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \int_0^\infty \|F(s,x)\|_{L_x^{p'}(\mathbb{R}^n)}^2 ds \tag{3.3.4}$$

for all test functions $F$. The left hand side of (3.3.4) can be written as

$$\int_0^\infty\int_0^\infty s_1^{-1/p}s^{-1/p}\left\langle e^{-\frac{s+s_1}{2}(-\triangle)^\beta}F(s,x), e^{-\frac{s+s_1}{2}(-\triangle)^\beta}F(s_1,x)\right\rangle_x dsds_1.$$

Let $g(s) = \|F(s,x)\|_{L_x^{p'}(\mathbb{R}^n)}$. According to Lemma 3.2.2, we have

$$\left|\left\langle e^{-\frac{s+s_1}{2}(-\triangle)^\beta}F(s,x), e^{-\frac{s+s_1}{2}(-\triangle)^\beta}F(s_1,x)\right\rangle_x\right| \lesssim (s+s_1)^{-2\left(\frac{1}{p'}-\frac{1}{2}\right)}g(s)g(s_1).$$

Hence, it suffices to prove that

$$\int_0^\infty\int_0^\infty \frac{g(s)g(s_1)dsds_1}{(s+s_1)^{1-2/p}s^{1/p}s_1^{1/p}} \lesssim \int_0^\infty g(s)^2ds. \tag{3.3.5}$$

On the other hand, by symmetry we can only consider the region $s_1 \le s$ which can be decomposed into the dyadic ranges $2^{-m}s \le s_1 \le 2^{-m+1}s$. Hence the left hand side of (3.3.5) can be bounded by

$$\begin{aligned}
&\lesssim \sum_{m=1}^\infty 2^{m/p}\int_0^\infty\int_{2^{-m}s\le s_1\le 2^{-m+1}s}\frac{g(s)g(s_1)}{s}ds_1ds\\
&\lesssim \sum_{m=1}^\infty 2^{m\left(\frac{1}{p}-\frac{1}{2}\right)}\int_0^\infty g(s)^2ds\\
&\lesssim \int_0^\infty g(s)^2ds
\end{aligned}$$

with the second inequality using the Schur's test of Tao [66]. □

Using the imbedding of $\dot{H}^{\beta,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)$ when $0 < 2\beta < n$, we prove the following result.

**Theorem 3.3.4** *Let $n > 2\beta > 0$, $p \in [1,2)$, $q \in (1,2)$. If $\frac{1}{q} + \frac{n}{2\beta}\left(\frac{1}{p}-\frac{1}{2}\right) = \frac{3}{2}$ then*

$$\left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^2(I;L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))} \lesssim \|F\|_{L_t^q(I;Z)} \tag{3.3.6}$$

*holds with $Z = \dot{H}_x^{\beta,p}(\mathbb{R}^n)$ or $H_x^{\beta,p}(\mathbb{R}^n)$.*

**Proof.** We only need to prove (3.3.6) for $Z = \dot{H}_x^{\beta,p}(\mathbb{R}^n)$. Suppose

$$n > 2\beta > 0,\quad p \in [1,2),\quad q\in(1,2)\ \text{ and }\ \frac{1}{q} + \frac{n}{2\beta}\left(\frac{1}{p}-\frac{1}{2}\right) = \frac{3}{2}.$$

Thus $\frac{n}{2\beta}\left(\frac{1}{p}-\frac{1}{2}\right)\in(0,1)$. According to the imbedding of $\dot{H}^{\beta,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)$, Lemmas 3.2.1 & 3.2.2 and the Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned}
\left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^2(I;L_x^{\frac{2n}{n-2\beta}}(\mathbb{R}^n))} &\lesssim \left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^2(I;\dot{H}_x^{\beta,2}(\mathbb{R}^n))}\\
&\lesssim \left\|(-\triangle)^{\beta/2}\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^2(I;L_x^2(\mathbb{R}^n))}\\
&\lesssim \left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}((-\triangle)^{\beta/2}F(s,x))ds\right\|_{L_t^2(I;L_x^2(\mathbb{R}^n))}\\
&\lesssim \left\|\int_0^t \|e^{-(t-s)(-\triangle)^\beta}((-\triangle)^{\beta/2}F(s,x))\|_{L_x^2(\mathbb{R}^n)}ds\right\|_{L_t^2(I)}\\
&\lesssim \left\|\int_0^t |t-s|^{-\frac{n}{2\beta}(\frac{1}{p}-\frac{1}{2})}\|(-\triangle)^{\beta/2}F(s,x)\|_{L_x^p(\mathbb{R}^n)}ds\right\|_{L_t^2(I)}\\
&\lesssim \|(-\triangle)^{\beta/2}F\|_{L_t^q(I;L_x^p(\mathbb{R}^n))}\\
&\lesssim \|F\|_{L_t^q(I;\dot{H}_x^{\beta,p}(\mathbb{R}^n))}.
\end{aligned}$$

This finishes the proof of (3.3.6). □

Using the Littlewood-Paley decomposition, we establish the following estimates in the Besov spaces.

**Theorem 3.3.5** *(a) Let $(q,p,2)$ be $\frac{n}{2\beta}$−admissible. If $q\geq 2$ and $(q,p,\frac{n}{2\beta})$ is not $(2,\infty,1)$, then*

$$\|e^{-t(-\triangle)^\beta}f\|_{L_t^q(I;X_1)}\lesssim\|f\|_{X_2}\tag{3.3.7}$$

*holds with $(X_1,X_2)=(B_{p,2}^s(\mathbb{R}^n),B_{2,2}^s(\mathbb{R}^n))$ or $(\dot{B}_{p,2}^s(\mathbb{R}^n),\dot{B}_{2,2}^s(\mathbb{R}^n))$.*
*(b) Let $1\leq p_1'<p\leq\infty$ and $1<q_1'<q<\infty$. If $(q,p)$ and $(q_1,p_1)$ satisfy (3.2.3) and $q_1\geq 2$, then*

$$\left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^q(I;Y_1)}\lesssim\|F\|_{L_t^{q_1'}(I;Y_2)}\tag{3.3.8}$$

*holds with $(Y_1,Y_2)=(B_{p,2}^s(\mathbb{R}^n),B_{p_1',2}^s(\mathbb{R}^n))$ or $(\dot{B}_{p,2}^s(\mathbb{R}^n),\dot{B}_{p_1',2}^s(\mathbb{R}^n))$.*

**Proof.** We only check (3.3.7) with $(X_1,X_2)=(\dot{B}_{p,2}^s(\mathbb{R}^n),\dot{B}_{2,2}^s(\mathbb{R}^n))$ and (3.3.8) with $(Y_1,Y_2)=(\dot{B}_{p,2}^s(\mathbb{R}^n),\dot{B}_{p_1',2}^s(\mathbb{R}^n))$ because the proofs of other cases are similar. We assume that $p<\infty$ since the case $p=\infty$ is similar. Let $\eta$ and $\psi_j$ satisfying (3.1.1) and (3.1.2).
*Part 1. Proof of (3.3.7).* We assume that $q<\infty$, note that the case $q=\infty$ is obvious. Define $u(t)=e^{-t(-\triangle)^\beta}f$. Then

$$\mathcal{F}^{-1}(\psi_j\mathcal{F}(u))=\mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}}\psi_j\mathcal{F}(f))=e^{-t(-\triangle)^\beta}(\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))).$$

Hence

$$\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} = \left(\int_I \left(\sum_j 2^{2sj}\|e^{-t(-\triangle)^\beta}(\mathcal{F}^{-1}(\psi_j\mathcal{F}(f)))\|^2_{L^p(\mathbb{R}^n)}\right)^{q/2} dt\right)^{2/q}.$$

Letting $A_j(t) = 2^{2sj}\|e^{-t(-\triangle)^\beta}(\mathcal{F}^{-1}(\psi_j\mathcal{F}(f)))\|^2_{L^p(\mathbb{R}^n)}$ and $k = q/2 \geq 1$, we have

$$\begin{aligned}
\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} &= \left(\int_I \left(\sum_j A_j(t)\right)^k dt\right)^{1/k} \\
&= \|\sum_j A_j(\cdot)\|_{L^k(I)} \\
&\leq \sum_j \|A_j(\cdot)\|_{L^k(I)} \\
&= \sum_j 2^{2sj}\|e^{-t(-\triangle)^\beta}(\mathcal{F}^{-1}(\psi_j\mathcal{F}(f)))\|^2_{L^q(I;L^p(\mathbb{R}^n))}.
\end{aligned}$$

Using Proposition 3.2.5, we deduce

$$\|u\|_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \left(\sum_j 2^{2sj}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}(f))\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} \lesssim \|f\|_{\dot{B}^s_{2,2}(\mathbb{R}^n)}.$$

Therefore, (3.3.7) holds.

*Part 2. Proof of (3.3.8).* Let $u(t) = \int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds$. Then

$$\begin{aligned}
2^{sj}\mathcal{F}^{-1}(\psi_j\mathcal{F}(u)) &= 2^{sj}\mathcal{F}^{-1}\int_0^t \psi_j\mathcal{F}(e^{-(t-s)(-\triangle)^\beta}F(s,x))ds \\
&= 2^{sj}\mathcal{F}^{-1}\int_0^t e^{-(t-s)|\xi|^{2\beta}}\psi_j\mathcal{F}(F(s,\xi))ds \\
&= 2^{sj}\int_0^t \mathcal{F}^{-1}\left(e^{-(t-s)|\xi|^{2\beta}}\psi_j\mathcal{F}(F(s,\xi))\right)ds \\
&= \int_0^t e^{-(t-s)(-\triangle)^\beta}\left(2^{sj}\mathcal{F}^{-1}\left(\psi_j\mathcal{F}(F(s,\xi))\right)\right)ds \\
&= \int_0^t e^{-(t-s)(-\triangle)^\beta}v_j(t)ds,
\end{aligned}$$

where $v_j(t) = 2^{sj}\mathcal{F}^{-1}\left(\psi_j\mathcal{F}(F(s,\xi))\right)$. Thus

$$\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \left(\int_I \left(\sum_j \left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}v_j(t)ds\right\|^2_{L^p(\mathbb{R}^n)}\right)^{q/2} dt\right)^{2/q}.$$

In a similar manner to verify (3.3.7), we have

$$\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \sum_j \left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} v_j(t)ds \right\|^2_{L^q_t(I;L^p(\mathbb{R}^n))}.$$

Applying Theorem 3.2.7, we get

$$\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \sum_j \|v_j\|^2_{L^{q_1'}_t(I;L^{p_1'}(\mathbb{R}^n))} \lesssim \sum_j \left(\int_I R_j(t)dt\right)^k,$$

where $R_j(t) = \|v_j(t)\|^{q_1'}_{L^{p_1'}(\mathbb{R}^n)}$ and $k = 2/q_1' \geq 1$. An application of the Minkowski inequality yields

$$\begin{aligned}
\|u\|^{2/k}_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} &\lesssim \left\| \int_I R_j(t)dt \right\|_{l^k(\mathbb{Z})} \\
&\lesssim \int_I \|R_j(t)\|_{l^k(\mathbb{Z})} dt \\
&\lesssim \int_I \left( \sum_j \|v_j(t)\|^2_{L^{p_1'}(\mathbb{R}^n)} \right)^{q_1'/2} dt \\
&\lesssim \|F\|^{q_1'}_{L^{q_1'}_t(I;\dot{B}^s_{p_1',2}(\mathbb{R}^n))}.
\end{aligned}$$

Thus (3.3.8) holds. □

We can obtain the following estimate by estimating $K^\beta_t(x)$ in mixed norm spaces.

**Theorem 3.3.6** *Let $\beta > 0$, $0 < T < \infty$, $1 \leq p_1' < p \leq \infty$, $1 \leq q_1' < q \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$ and $\frac{1}{h} = \frac{1}{q} + \frac{1}{q_1}$. If*

$$0 < \frac{nh}{2\beta}\left(1 - \frac{1}{r}\right) < 1,$$

*then*

$$\left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} F(s,x)ds \right\|_{L^q_t([0,T);X)} \lesssim T^{\frac{1}{h} - \frac{n}{2\beta}(1-\frac{1}{r})} \|F\|_{L^{q_1'}_t([0,T);Y)} \tag{3.3.9}$$

*holds with $(X,Y) = (L^p_x(\mathbb{R}^n), L^{p_1'}_x(\mathbb{R}^n))$, $(\dot{H}^{\gamma,p}_x(\mathbb{R}^n), \dot{H}^{\gamma,p_1'}_x(\mathbb{R}^n))$ or $(H^{\gamma,p}_x(\mathbb{R}^n), H^{\gamma,p_1'}_x(\mathbb{R}^n))$ for all $\gamma > 0$.*

**Proof**. We only prove the case $(X,Y) = (L^p_x(\mathbb{R}^n), L^{p_1'}_x(\mathbb{R}^n))$ since similar arguments apply to other cases. Assume that $T \in (0,\infty)$, $1 \leq p_1' < p \leq \infty$, $1 \leq q_1' < q \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$, $\frac{1}{h} = \frac{1}{q} + \frac{1}{q_1}$ and $\frac{nh}{2\beta}\left(1 - \frac{1}{r}\right) \in (0,1)$. Let $J = [0,T)$. According to the Young's inequality and the definition of $e^{-t(-\triangle)^\beta}$, we have

$$\begin{aligned}\left\|\int_0^t e^{-(t-s)(-\triangle)^\beta}F(s,x)ds\right\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} &\lesssim \left\|\int_0^t \|e^{-(t-s)(-\triangle)^\beta}F(s,x)\|_{L_x^p(\mathbb{R}^n)}ds\right\|_{L_t^q(J)}\\ &\lesssim \left\|\int_0^t \|K_{t-s}^\beta(x) *_x F(s,x)\|_{L_x^p(\mathbb{R}^n)}ds\right\|_{L_t^q(J)}\\ &\lesssim \left\|\int_0^t \|K_{t-s}^\beta(x)\|_{L_x^r(\mathbb{R}^n)}\|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)}ds\right\|_{L_t^q(J)}\\ &\lesssim \|K_t^\beta(x)\|_{L_t^h(J;L_x^r(\mathbb{R}^n))}\|F(s,x)\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}.\end{aligned}$$

Thus it suffices to prove $\|K_t^\beta(x)\|_{L_t^h(J;L_x^r(\mathbb{R}^n))} \lesssim T^{\frac{1}{h}-\frac{n}{2\beta}(1-\frac{1}{r})}$. In fact, it follows from Miao, Yuan and Zhang's [53, Lemma 2.1] that $K_1^\beta(x) \in L^k(\mathbb{R}^n)$ for all $1 \le k \le \infty$. Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$ and $p_1' < p$ imply that $r > 1$, $K_1^\beta(x) \in L^r(\mathbb{R}^n)$. Hence

$$\begin{aligned}\|K_t^\beta(x)\|_{L_t^h(J;L_x^r(\mathbb{R}^n))} &= \left(\int_0^T \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-t|\xi|^{2\beta}}d\xi\right)^r dx\right)^{\frac{h}{r}} dt\right)^{1/h}\\ &= \left(\int_0^T t^{-\frac{nh}{2\beta}(1-\frac{1}{r})}dt\right)^{1/h}\|K_1^\beta\|_{L^r(\mathbb{R}^n)}\\ &\lesssim T^{\frac{1}{h}-\frac{n}{2\beta}(1-\frac{1}{r})}.\end{aligned}$$

This finishes the proof of Theorem 3.3.6. □

# Chapter 4

# Some $Q-$Spaces of Several Real Variables

As mentioned in Chapter 1, in this chapter, we study the $Q_\alpha^\beta(\mathbb{R}^n)$ and its derivative space $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$.

**Definition 4.0.7** *For $\alpha \in (-\infty, \beta)$ and $\beta \in (1/2, 1)$, define $Q_\alpha^\beta(\mathbb{R}^n)$ as the set of all measurable complex-valued functions $f$ on $\mathbb{R}^n$ satisfying*

$$\|f\|_{Q_\alpha^\beta(\mathbb{R}^n)} = \sup_I \left( (l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dx dy \right)^{1/2} < \infty \tag{4.0.1}$$

*where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.*

In Section 4.1, we introduce some notation and some facts about homogeneous Besov spaces, Hausdorff capacity and Carleson measures. In Section 4.2, in order to establish Carleson measure characterization of $Q_\alpha^\beta(\mathbb{R}^n)$, we introduce a new type of tent spaces, their atomic decompositions and the predual space of $Q_\alpha^\beta(\mathbb{R}^n)$. The proofs of the main theorems in this section are similar to that of Dafni-Xiao [22]. For the completeness, we provide the details. In Section 4.3, via the Carleson measure characterization of $Q_\alpha^\beta(\mathbb{R}^n)$, we define $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$ as the derivative space of $Q_\alpha^\beta(\mathbb{R}^n)$ and investigate some properties for $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$. In Section 4.4, we establish the mean oscillation characterization of $Q_\alpha^\beta(\mathbb{R}^n)$. Using this characterization, in Section 4.5, we study John-Nirenberg type and Gagliardo-Nibenberg type inequalities in $Q_\alpha^\beta(\mathbb{R}^n)$.

## 4.1 Notations and Preliminaries

A ball in $\mathbb{R}^n$ with center $x$ and radius $r$ will be denoted by $B = B(x, r)$, its Lebesgue measure by $|B|$. A cube in $\mathbb{R}^n$ will always mean a cube in $\mathbb{R}^n$ with side parallel to the coordinate axes. The sidelength of a cube $I$ will be denoted by $l(I)$. Similarly, its volume will be denoted by $|I|$. The characteristic function of a set $A$ will be denoted by $1_A$.

Recall that $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the homogeneous Besov space (see Chapter 3). When $0 < s < 1$, we have the following equivalent characterization. If $1 \leq p, q < \infty$, then $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is equivalent to

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx \right)^{q/p} \frac{dy}{|y|^{n+qs}} < \infty; \quad (4.1.1)$$

if $0 < s < 1$ and $1 \leq p < q = \infty$, $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ amounts to

$$\sup_{y \in \mathbb{R}^n} |y|^{-s} \left( \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \right)^{1/p} < \infty. \quad (4.1.2)$$

The homogenous Besov spaces obey the following inclusion relations (see [8]).

**Theorem 4.1.1** *Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.*
*(i) If $1 \leq q_1 \leq q_2 \leq \infty$, then $\dot{B}^s_{p,q_1}(\mathbb{R}^n) \subseteq \dot{B}^s_{p,q_2}(\mathbb{R}^n)$;*
*(ii) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s_1 = s_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then $\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \subseteq \dot{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$.*

We recall the definition of fractional Carleson measures (see Essen-Janson-Peng-Xiao [25]) and their connection with Hausdorff capacity established by Dafni-Xiao in [22].

**Definition 4.1.2** *For $p > 0$, we say that a Borel measure $\mu$ on $\mathbb{R}^{1+n}_+$ is a $p-$Carleson measure provided that*

$$|||\mu|||_p = \sup \frac{\mu(S(I))}{(l(I))^{np}} < \infty \quad (4.1.3)$$

*where the supremum is taken over all Carleson boxes $S(I) = \{(t,x) : x \in I, t \in (0, l(I))\}$.*

Obviously, the $1-$Carleson measures are the usual Carleson measures. On the other hand, similar to the case $p = 1$, if we denote by

$$T(E) = \{(t,x) \in \mathbb{R}^{1+n}_+ : B(x,t) \subset E\}$$

the tent based on the set $E \subset \mathbb{R}^n$, then a Borel measure $\mu$ on $\mathbb{R}^{1+n}_+$ is a $p-$Carleson measure if and only if $|\mu|(T(B)) \leq C|B|^p$ holds for all balls $B \subset \mathbb{R}^n$. That is to say $p-$Carleson measures can be equivalently defined in terms of tents over balls.

We review some definitions and properties about Hausdorff capacity (see Adams [3], Dafni-Xiao [22] and Yang-Yuan [80]).

**Definition 4.1.3** *Let $d \in (0, n]$ and $E \subset \mathbb{R}^n$.*
*(i) The $d-$dimensional Hausdorff capacity of $E$ is defined by*

$$\Lambda^{(\infty)}_d(E); = \inf \left\{ \sum_j r_j^d : E \subset \cup_{j=1}^{\infty} B(x_j, r_j) \right\}, \quad (4.1.4)$$

*where the infimum is taken over all covers of $E$ by countable families of open (closed) balls with radii $r_j$.*
*(ii) The capacity $\widetilde{\Lambda}_d^{(\infty)}(E)$ in the sense of Choquet is defined (see Yang-Yuan [80]) by*

$$\widetilde{\Lambda}_d^{(\infty)}(E) := \inf\left\{\sum_j l(I_j)^d : E \subset (\cup_{j=1}^{\infty} I_j)^o\right\},$$

*where the infimum ranges only over covers of $E$ by dyadic cubes and $A^o$ means the interior part of $A$.*
*(iii) For a function $f : \mathbb{R}^n \longrightarrow [0,\infty]$, we define*

$$\int_{\mathbb{R}^n} f d\Lambda_d^{(\infty)} := \int_0^{\infty} \Lambda_d^{(\infty)}(\{x \in \mathbb{R}^n : f(x) > \lambda\})d\lambda.$$

**Remark 4.1.4** *(i) $\Lambda_d^{(\infty)}$ is not a capacity in the sense of Choquent. But, its dyadic counterpart $\widetilde{\Lambda}_d^{(\infty)}$ is a capacity since it is monotone, vanishes on the empty set, and satisfies the strong subadditivity condition*

$$\widetilde{\Lambda}_d^{(\infty)}(E_1 \cup E_2) + \widetilde{\Lambda}_d^{(\infty)}(E_1 \cap E_2) \leq \widetilde{\Lambda}_d^{(\infty)}(E_1) + \widetilde{\Lambda}_d^{(\infty)}(E_2),$$

*as well as the continuity conditions (see Adams [3] and Yang-Yuan [80]):*

$$\widetilde{\Lambda}_d^{(\infty)}(\cap_i K_i) = \lim_{i\longrightarrow\infty} \widetilde{\Lambda}_d^{(\infty)}(K_i), \quad \{K_i\} \text{ a decreasing sequence of compact sets,}$$

$$\widetilde{\Lambda}_d^{(\infty)}(\cup_i K_i) = \lim_{i\longrightarrow\infty} \widetilde{\Lambda}_d^{(\infty)}(K_i), \quad \{K_i\} \text{ an increasing sequence of sets.}$$

*(ii) There exist positive constants $C_1(n,d)$ and $C_2(n,d)$ such that*

$$C_1(n,d)\Lambda_d^{(\infty)}(E) \leq \widetilde{\Lambda}_d^{(\infty)}(E) \leq C_2(n,d)\Lambda_d^{(\infty)}(E) \quad \text{for all } E \subset \mathbb{R}^n. \tag{4.1.5}$$

*(iii) The integral with respect to $\widetilde{\Lambda}_d^{(\infty)}(E)$ satisfies Fatou's lemma*

$$\int_{\mathbb{R}^n} \liminf f_n d\widetilde{\Lambda}_d^{(\infty)} \leq \liminf f_n \int_{\mathbb{R}^n} d\widetilde{\Lambda}_d^{(\infty)}. \tag{4.1.6}$$

For $x \in \mathbb{R}^n$, let $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| < t\}$ be the cone at $x$. Define the nontangential maximal function $N(f)$ of a measurable function on $\mathbb{R}_+^{n+1}$ by

$$N(f)(x) := \sup_{(y,t)\in\Gamma(x)} |f(y,t)|.$$

In [22], Dafni-Xiao characterized the fractional Carleson measures as follows.

**Theorem 4.1.5** *([22, Theorem 4.2]) Let $d \in (0,n]$ and $\mu$ be a Borel measure on $\mathbb{R}_+^{1+n}$. Then $\mu$ is a $d/n-$Carleson measure if and only if the inequality*

$$\int_{\mathbb{R}_+^{1+n}} |f(t,y)|d|\mu| \leq A \int_{\mathbb{R}^n} N(f) d\Lambda_d^{(\infty)} \tag{4.1.7}$$

*holds for all Borel measurable functions $f$ on $\mathbb{R}_+^{1+n}$. If this is the case then in (4.1.7) the constant $A \approx |||\mu|||_{d/n}$ which is defined by (4.1.3).*

## 4.2 Carleson Measure Characterization of $Q_\alpha^\beta(\mathbb{R}^n)$

In this section, we establish the equivalent characterization (1.0.11). We first give some basic properties of $Q_\alpha^\beta(\mathbb{R}^n)$. Then inspired by Coifman-Meyer-Stein [19] and Dafni-Xiao [22], we introduce new tent spaces $T^1_{\alpha,\beta}$ and $T^\infty_{\alpha,\beta}$. Finally, we obtain the predual space of $Q_\alpha^\beta(\mathbb{R}^n)$.

### 4.2.1 Basic Properties of $Q_\alpha^\beta(\mathbb{R}^n)$

**Lemma 4.2.1** *Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if*

$$\sup_I (l(I))^{-n+2(\alpha+\beta-1)} \int_{|y|<l(I)} \int_I |f(x+y) - f(x)|^2 \frac{dxdy}{|y|^{n+2(\alpha-\beta+1)}} < \infty. \tag{4.2.1}$$

**Proof.** If the double integrals (4.0.1) and (4.2.1) are denoted by $U_1(I)$ and $U_2(I)$, respectively, then by the change of variable $y \longrightarrow x+y$ and simple geometry one obtains $U_1(I) \leq U_2(\sqrt{n}I)$ and $U_2(I) \leq U_1(3I)$. □

**Theorem 4.2.2** *Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then*
*(i) $Q_\alpha^\beta(\mathbb{R}^n)$ is decreasing in $\alpha$ for a fixed $\beta$, i.e.*

$$Q^\beta_{\alpha_1}(\mathbb{R}^n) \subseteq Q^\beta_{\alpha_2}(\mathbb{R}^n), \quad \text{if } \alpha_2 \leq \alpha_1;$$

*(ii) If $\alpha \in (-\infty, \beta - 1)$, then*

$$Q_\alpha^\beta(\mathbb{R}^n) = Q^\beta_{-\frac{n}{2}+\beta-1}(\mathbb{R}^n) := BMO^\beta(\mathbb{R}^n).$$

**Proof.** This theorem can be proved by a similar argument used in the proof of [25, Theorem 2.3]

In the following, we establish the connection between $Q_\alpha^\beta(\mathbb{R}^n)$ and homogeneous Besov spaces.

**Theorem 4.2.3** *Let $n \geq 2$ and $\max\{1/2, \alpha\} < \beta < 1$.*
*(i) If $1 \leq q \leq 2$ and $\alpha + \beta - 1 > 0$, then $\dot{B}^{\alpha-\beta+1}_{\frac{n}{\alpha+\beta-1},q}(\mathbb{R}^n) \subseteq Q_\alpha^\beta(\mathbb{R}^n)$.*
*(ii) Let $1 \leq q \leq \infty$, $\gamma_1 > (\alpha - \beta + 1)$ and $\gamma_2 > 0$. If $\gamma_1 - \gamma_2 = 2 - 2\beta$, then $\dot{B}^{\gamma_1}_{n/\gamma_2,q}(\mathbb{R}^n) \subseteq Q_\alpha^\beta(\mathbb{R}^n)$.*

**Proof** We can prove this theorem by using similar arguments applied for the special case $\beta = 1$ by Essen-Janson-Peng-Xiao in [25, Theorem 2.7].

### 4.2.2 New Tent Spaces

We introduce new tent spaces motivated by similar arguments in Dafni-Xiao [22].

**Definition 4.2.4** *For $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$, we define $T^{\infty}_{\alpha,\beta}$ be the class of all Lebesgue measurable functions $f$ on $\mathbb{R}^{1+n}_{+}$ with*

$$\|f\|_{T^{\infty}_{\alpha,\beta}} = \sup_{B\subset\mathbb{R}^n} \left( \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}} \int_{T(B)} |f(t,y)|^2 \frac{dtdy}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2} < \infty,$$

*where $B$ runs over all balls in $\mathbb{R}^n$.*

**Definition 4.2.5** *For $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$, a function $a$ on $\mathbb{R}^{1+n}_{+}$ is said to be a $T^1_{\alpha,\beta}-$atom provided there exists a ball $B \subset \mathbb{R}^n$ such that $a$ is supported in the tent $T(B)$ and satisfies*

$$\int_{T(B)} |a(t,y)|^2 \frac{dtdy}{t^{1-2(\alpha-\beta+1)}} \leq \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.$$

**Definition 4.2.6** *For $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$, the space $T^1_{\alpha,\beta}$ consists of all measurable functions $f$ on $\mathbb{R}^{1+n}_{+}$ with*

$$\|f\|_{T^1_{\alpha,\beta}} = \inf_{\omega} \left( \int_{\mathbb{R}^{1+n}_{+}} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} < \infty,$$

*where the infimum is taken over all nonnegative Borel measurable functions $\omega$ on $\mathbb{R}^{1+n}_{+}$ with*

$$\int_{\mathbb{R}^n} N\omega d\Lambda^{\infty}_{n-2(\alpha+\beta-1)} \leq 1$$

*and with the restriction that $\omega$ is allowed to vanish only where $f$ vanishes.*

**Lemma 4.2.7** *If $\sum_j \|g_j\|_{T^1_{\alpha,\beta}} < \infty$, then $g = \sum_j g_j \in T^1_{\alpha,\beta}$ with*

$$\|g\|_{T^1_{\alpha,\beta}} \leq \sqrt{C_1^{-1}(n,d)C_2(n,d)} \sum_j \|g_j\|_{T^1_{\alpha,\beta}},$$

*where $C_1(n,d)$ and $C_2(n,d)$ are the constants in (4.1.5).*

**Proof.** The proof of this lemma is similar to that of Dafni-Xiao [22, Lemma 5.3]. □

**Theorem 4.2.8** *Let $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$, then*
*(i) $f \in T^1_{\alpha,\beta}$ if and only if there is a sequence of $T^1_{\alpha,\beta}-$atoms $a_j$ and an $l^1-$sequence $\{\lambda_j\}$ such that $f = \sum_j \lambda_j a_j$. Moreover*

$$\|f\|_{T^1_{\alpha,\beta}} \approx \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}$$

*where the infimum is taken over all possible atomic decompositions of $f \in T^1_{\alpha,\beta}$. The right-hand side thus defines a norm on $T^1_{\alpha,\beta}$ which makes it into a Banach space.*
*(ii) The inequality*

$$\int_{\mathbb{R}^{1+n}_+} |f(t,y)g(t,y)|\frac{dtdy}{t} \le C\|f\|_{T^1_{\alpha,\beta}}\|g\|_{T^\infty_{\alpha,\beta}} \qquad (4.2.2)$$

*holds for all $f \in T^1_{\alpha,\beta}$ and $g \in T^\infty_{\alpha,\beta}$.*
*(iii) The Banach space dual of $T^1_{\alpha,\beta}$ can be identified with $T^\infty_{\alpha,\beta}$ under the following pairing*

$$\langle f, g\rangle = \int_{\mathbb{R}^{1+n}_+} f(t,y)g(t,y)\frac{dtdy}{t}.$$

**Proof.** (i) Let $a$ be a $T^1_{\alpha,\beta}$ atom. Then we can find a ball $B = B(x_B, r) \subset \mathbb{R}^n$ such that $\operatorname{supp}(a) \subset T(B)$ and

$$\int_{T(B)} |a(t,y)|^2\frac{dtdy}{t^{1-2(\alpha-\beta+1)}} \le \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.$$

Fix $\varepsilon > 0$ and define

$$\omega(t,x) = \kappa r^{-n+2(\alpha+\beta-1)}\min\left\{1, (\frac{r}{\sqrt{|x-x_B|^2+t^2}})^{n-2(\alpha+\beta-1)+\varepsilon}\right\},$$

where $\sqrt{|x-x_B|^2+t^2}$ is the distance between $(t,x)$ and $(0,x_B)$. For $x \in \mathbb{R}^n$, the distance in $\mathbb{R}^{1+n}_+$ from the cone $\Gamma(x)$ to $(0,x_B)$ is $\frac{|x-x_B|}{\sqrt{2}}$. So

$$\begin{aligned} N\omega(x) &= \sup_{(t,y)\in\Gamma(x)}\left|\kappa r^{-n+2(\alpha+\beta-1)}\min\left\{1, \left(\frac{r}{\sqrt{|x-x_B|^2+t^2}}\right)^{n-2(\alpha+\beta-1)+\varepsilon}\right\}\right| \\ &\le \kappa r^{-n+2(\alpha+\beta-1)}\min\left\{1, \left(\frac{\sqrt{2}r}{|x-x_B|}\right)^{n-2(\alpha+\beta-1)+\varepsilon}\right\}. \end{aligned}$$

Thus

$$\kappa^{-1}\int_{\mathbb{R}^n} N\omega d\Lambda^\infty_{n-2(\alpha+\beta-1)} \le \int_0^\infty \Lambda^\infty_{n-2(\alpha+\beta-1)}(\{x : N\omega(x) > \lambda\})d\lambda.$$

If $\lambda < N\omega(x)$, then $|x-x_B| \le \sqrt{2}\left(\frac{r^\varepsilon}{\lambda}\right)^{\frac{1}{n-2(\alpha+\beta-1)+\varepsilon}}$. Meanwhile, $\lambda < N\omega(x) \le \kappa r^{-n+2(\alpha+\beta-1)}$, so we obtain

$$\kappa^{-1}\int_{\mathbb{R}^n} N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)} \le \int_0^{r^{-n+2(\alpha+\beta-1)}}\left(\frac{r^\varepsilon}{\lambda}\right)^{\frac{n-2(\alpha+\beta-1)}{n-2(\alpha+\beta-1)+\varepsilon}} d\lambda \lesssim 1.$$

Moreover, on $T(B)$ we have $\omega^{-1}(t,x) = r^{n-2(\alpha+\beta-1)}$. By the definition of $T^1_{\alpha,\beta}$–atom, we get

$$\int_{T(B)} |a(t,y)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \lesssim 1.$$

Thus $a \in T^1_{\alpha,\beta}$ with $\|a\|_{T^1_{\alpha,\beta}} \lesssim 1$. For any sum $\sum_j \lambda_j a_j$ with $\|\{\lambda_j\}\|_{l_1} = \sum |\lambda_j| < \infty$ and $T^1_{\alpha,\beta}$−atoms $a_j$, Lemma 4.2.7 implies that the sum converges in the quasi-norm to $f \in T^1_{\alpha,\beta}$ with $\|f\|_{T^1_{\alpha,\beta}} \lesssim \sum_j |\lambda_j|$.

Conversely, suppose that $f \in T^1_{\alpha,\beta}$. There exists a Borel measurable function $\omega \geq 0$ on $\mathbb{R}^{1+n}_+$ such that

$$\int_{\mathbb{R}^{1+n}_+} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \leq 2\|f\|^2_{T^1_{\alpha,\beta}}.$$

For each $k \in \mathbb{Z}$, let $E_k = \{x \in \mathbb{R}^n : N\omega(x) > 2^k\}$. According to Dafni-Xiao [22, Lemma 4.1], there exists a sequence of dyadic cubes $\{I_{j,k}\}$ with disjoint interiors such that

$$\sum_j l(I_{j,k})^{n-2(\alpha+\beta-1)} \leq 2\widetilde{\Lambda}^{(\infty)}_{n-2(\alpha+\beta-1)}(E_k) \quad \text{and} \quad T(E_k) \subset \cup_j S^*(I_{j,k}).$$

Here we have used a Carleson box: $S^*(I_{j,k}) = \{(t,y) \in \mathbb{R}^{1+n}_+ : y \in I_{j,k}, t < 2\mathrm{diam}(I_{j,k})\}$ to replace the tent $T(I^*_{j,k})$ over the dilated cube $I^*_{j,k} = 5\sqrt{n} I_{j,k}$. Consequently, if we define $T_{j,k} = S^*(I_{j,k}) \setminus \cup_{m>k} \cup_l S^*(I_{l,m})$, these will have disjoint interiors for different values of $j$ or $k$. Now

$$\cup^K_{k=-K} \cup_j T_{j,k} = \cup_j S^*(I_{j,-K}) \setminus \cup_{m>K} \cup_l S^*(I_{l.m}) \supseteq T(E_{-K}) \setminus \cup_{m>K} \cup_l S^*(I_{l.m}).$$

Similar to the discussion in the proof of Dafni-Xiao [22, Theorem 5.4], we have

$$\cup_k \cup_j T_{j,k} \supseteq \cup_k T(E_k) \setminus \cap_k \cup_{m>k} \cup_l S^*(I_{l,m}) = \{(t,x) \in \mathbb{R}^{1+n}_+ : \omega(t,x) > 0\} \setminus T_\infty$$

with $\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}(T_\infty) = |T_\infty| = 0$. Since $\omega$ is allowed to vanish only where $f$ vanishes, $f = \sum f 1_{T_{j,k}}$ a.e. on $\mathbb{R}^{1+n}_+$. Defining $a_{j,k} = f 1_{T_{j,k}} (\lambda_{j,k})^{-1}$ and

$$\lambda_{j,k} = \left( (l(I^*_{j,k}))^{n-2(\alpha+\beta-1)} \int_{T_{j,k}} |f(t,x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2},$$

we get $f = \sum_{j,k} \lambda_{j,k} a_{j,k}$ almost everywhere. Since $S^*(I_{j,k}) \subset T(B_{j,k})$ where $B_{j,k}$ is the ball with the same center as $I_{j,k}$ and radius $l(I^*_{j,k})/2$. $a_{j,k}$ is supported in $T(B_{j,k})$ and

$$\begin{aligned}
&\int_{T(B_{j,k})} |a_{j,k}(t,y)|^2 \frac{dtdy}{t^{1-2(\alpha-\beta+1)}} \\
\leq\ & (l(I^*_{j,k}))^{-n+2(\alpha+\beta-1)} \left( \int_{T_{j,k}} |f(t,x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{-1} \left( \int_{T(B_{j,k})} |f(t,x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right) \\
\leq\ & (l(I^*_{j,k}))^{-n+2(\alpha+\beta-1)} \leq |B_{j,k}|^{-1+2(\alpha+\beta-1)/n}.
\end{aligned}$$

Thus each $a_{j,k}$ is a $T^1_{\alpha,\beta}$−atom.

Next, we prove that $\{\lambda_{j,k}\}$ is $l^1$−summable. Noting that $\omega \leq 2^{k+1}$ on

$$T_{j,k} \subset (\cup_l S^*(I_{l,k+1}))^c \subset (T(E_{k+1}))^c$$

and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\sum_{j,k}|\lambda_{j,k}| &\le \sum_{j,k}(l(I^*_{j,k}))^{\frac{n}{2}-(\alpha+\beta-1)}\left(\int_{T_{j,k}}|f(t,x)|^2\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\\
&\le \sum_{j,k}\sup_{T_{j,k}}\omega^{1/2}(l(I^*_{j,k}))^{\frac{n}{2}-(\alpha+\beta-1)}\left(\int_{T_{j,k}}|f(t,x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\\
&\le \left(\sum_{j,k}2^{(k+1)}(l(I^*_{j,k}))^{n-2(\alpha+\beta-1)}\right)^{1/2}\left(\sum_{j,k}\int_{T_{j,k}}|f(t,x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\\
&\lesssim \|f\|_{T^1_{\alpha,\beta}}\left(\sum_k 2^k\sum_j(l(I_{j,k}))^{n-2(\alpha+\beta-1)}\right)^{1/2}\\
&\lesssim \|f\|_{T^1_{\alpha,\beta}}\left(\sum_k 2^k\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}(E_k)\right)^{1/2}\\
&\lesssim \|f\|_{T^1_{\alpha,\beta}}\left(\int_{\mathbb{R}^n}N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}\right)^{1/2}\lesssim \|f\|_{T^1_{\alpha,\beta}}.
\end{aligned}$$

Thus $T^1_{\alpha,\beta}$ is a Banach space since it is complete in the quasi-norm (Lemma 4.2.7) and

$$\|f\|_{T^1_{\alpha,\beta}} \approx |||f|||_{T^1_{\alpha,\beta}} = \inf\left\{\sum_j|\lambda_j| : f=\sum_j\lambda_j a_j\right\}$$

where the infimum is taken over all possible atomic decompositions of $f\in T^1_{\alpha,\beta}$ and $|||\cdot|||_{T^1_{\alpha,\beta}}$ is a norm.
(ii) Let $\omega$ be a nonnegative Borel measurable function on $\mathbb{R}^{1+n}_+$ satisfying $\int_{\mathbb{R}^n}N\omega d\Lambda^{(\infty)}_{\alpha,\beta}\le 1$. For $g\in T^\infty_{\alpha,\beta}$,

$$\left(\int_{T(B)}|g(t,y)|^2\frac{dtdy}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2}\lesssim |B|^{1-2(\alpha+\beta-1)/n}.$$

Thus, $d\mu_{g,n-2(\alpha+\beta-1)}(t,x)=|g(t,x)|^2t^{-1-2(\alpha-\beta+1)}dtdx$ is a $1-2(\alpha+\beta-1)/n$−Carleson measure. Then (4.1.7) tells us, with $A\approx |||\mu_{g,n-2(\alpha+\beta-1)}|||_{n-2(\alpha+\beta-1)/n}\approx\|g\|^2_{T^\infty_{\alpha,\beta}}$,

$$\int_{\mathbb{R}^{1+n}_+}\omega(t,x)|g(t,x)|^2\frac{dtdx}{t^{1+2(\alpha-\beta+1)}}\lesssim\|g\|^2_{T^\infty_{\alpha,\beta}}\int_{\mathbb{R}^n}N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}\lesssim\|g\|^2_{T^\infty_{\alpha,\beta}}.$$

Thus if $f\in T^1_{\alpha,\beta}$, then

$$\int_{\mathbb{R}^{1+n}_+}|f(t,x)g(t,x)|\frac{dtdx}{t}\le\left(\int_{\mathbb{R}^{1+n}_+}|f(t,x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\|g\|_{T^\infty_{\alpha,\beta}}.$$

Hence we finish the proof of (ii) by taking the infimum on the right over all admissible $\omega$.
(iii) Form (ii), we know that for every $g \in T^{\infty}_{\alpha,\beta}$, the pairing

$$\langle f, g\rangle = \int_{\mathbb{R}^{1+n}_+} f(t,y)g(t,y)\frac{dtdy}{t}$$

defines a bounded linear functional on $T^1_{\alpha,\beta}$. Now we prove the converse. Let $L$ be a bounded linear functional on $T^1_{\alpha,\beta}$. Fix a ball $B = B(x_B, r) \subset \mathbb{R}^n$. If $f$ is supported on $T(B)$ with $f \in L^2(T(B), t^{-1}dtdx)$ then

$$\begin{aligned}
\int_{T(B)} |f(t,x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} &\le r^{2(\alpha-\beta+1)} \int_{T(B)} |f(t,x)|^2 \frac{dtdx}{t} \\
&\lesssim \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}} r^{n-2(\alpha+\beta-1)+2(\alpha-\beta+1)} \int_{T(B)} |f(t,x)|^2 \frac{dtdx}{t} \\
&\lesssim \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}} r^{n-4\beta+4} \|f\|^2_{L^2(T(B),t^{-1}dtdx)}.
\end{aligned}$$

This tells us that $f(t,x)$ is a multiple of a $T^1_{\alpha,\beta}$–atom and $L$ is a bounded linear functional on $L^2(T(B), t^{-1}dtdx)$ which can be represented by the inner-product with some function $g_B \in L^2(T(B), t^{-1}dtdx)$. Taking $B_j = B(0,j)$, $j \in \mathbb{N}$, then $g_{B_j} = g_{B_{j+1}}$ on $T(B_j)$. So we get a single function $g$ on $\mathbb{R}^{1+n}_+$ that is locally in $L^2(t^{-1}dtdx)$ such that

$$L(f) = \int_{\mathbb{R}^{1+n}_+} f(t,x)g(t,x)\frac{dtdx}{t}$$

whenever $f \in T^1_{\alpha,\beta}$ is supported in some tent $T(B)$. By the atomic decomposition, the subset of such $f$ is dense in $T^1_{\alpha,\beta}$. We only need to prove $g \in T^{\infty}_{\alpha,\beta}$ with $\|g\|_{T^{\infty}_{\alpha,\beta}} \lesssim \|L\|$.

For a ball $B \subset \mathbb{R}^n$ and every $\varepsilon > 0$, we set

$$f_\varepsilon(t,x) = t^{-2(\alpha-\beta+1)}\overline{g(t,x)}1_{T^\varepsilon(B)}(t,x)$$

where $T^\varepsilon(B)$ is the truncated tent $T(B)\bigcap\{(t,x) : t > \varepsilon\}$. Since $g \in L^2(T(B))$, we have

$$\int_{T(B)} |f_\varepsilon(t,x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} = \int_{T^\varepsilon(B)} |g(t,x)|^2 \frac{dtdx}{t^{1+2(\alpha-\beta+1)}} \lesssim \infty.$$

Hence we can obtain that $f_\varepsilon$ is a multiple of a $T^1_{\alpha,\beta}$–atom with

$$\|f_\varepsilon\|^2_{T^1_{\alpha,\beta}} \lesssim r^{n-2(\alpha+\beta-1)} \int_{T^\varepsilon(B)} |g(t,x)|^2 \frac{dtdx}{t^{1+2(\alpha-\beta+1)}}.$$

According to the representation above, we also get

$$\int_{T^\varepsilon(B)} |g(t,x)|^2 \frac{dtdx}{t^{1+2(\alpha-\beta+1)}} \lesssim \|L\| \left(r^{n-2(\alpha+\beta-1)} \int_{T^\varepsilon(B)} |g(t,x)|^2 \frac{dtdx}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2}.$$

This gives us

$$\left(r^{-n+2(\alpha+\beta-1)}\int_{T^{\varepsilon}(B)}|g(t,x)|^2\frac{dtdx}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2}\lesssim\|L\|,$$

that is, $g\in T^{\infty}_{\alpha,\beta}$ with $\|g\|_{T^{\infty}_{\alpha,\beta}}\lesssim\|L\|$. This completes the proof of Theorem 4.2.8. □

### 4.2.3 The Preduality of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$

In this subsection, we introduce a new space which can be viewed as the predual space of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. Then, we give an atomic decomposition for this space. For this purpose we need the following lemma which is Lemma 1.1 in [26].

**Lemma 4.2.9** *Fix $N\in\mathbb{N}$. Then there exists a function $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that*
*(1) supp $(\phi)\subset\{x\in\mathbb{R}^n:|x|\le 1\}$;*
*(2) $\phi$ is radial;*
*(3) $\phi\in C^{\infty}(\mathbb{R}^n)$;*
*(4)$\int_{\mathbb{R}^n}x^{\gamma}\phi(x)dx=0$ if $\gamma\in\mathbb{N}^n$, $x^{\gamma}=x_1^{\gamma^1}x_2^{\gamma^2}\cdots x_n^{\gamma^n}$, $|\gamma|=\gamma_1+\gamma_2+\cdots+\gamma_n$;*
*(5) $\int_0^{\infty}(\mathcal{F}\phi(t\xi))^2\frac{dt}{t}=1$ if $\xi\in\mathbb{R}^n\backslash\{0\}$.*

For $\phi$ satisfying the conditions of Lemma 4.2.9 and any $f\in\mathcal{S}'(\mathbb{R}^n)$, we have the well known Calderón reproducing formula

$$f=\int_0^{\infty}f*\phi_t*\phi_t\frac{dt}{t}=\lim_{\varepsilon\longrightarrow 0,N\longrightarrow\infty}\int_{\varepsilon}^{N}f*\phi_t*\phi_t\frac{dt}{t}. \tag{4.2.3}$$

We introduce the notation of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ in the sense of distributions.

**Definition 4.2.10** *For $\phi$ as in above lemma, $\alpha>0$, $\alpha+\beta\ge 1$ and $\max\{1/2,\alpha\}<\beta<1$, we define the Hardy-Hausdorff space $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ to be the class of all distributions $f\in\dot{L}^2_{-\frac{n}{2}+2(\beta-1)}(\mathbb{R}^n)$ with*

$$\|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}:=\|f*\phi_t(\cdot)\|_{T^1_{\alpha,\beta}}<\infty.$$

**Theorem 4.2.11** $\|\cdot\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}$ *is a quasi-norm. Furthermore, $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ is complete under this quasi-norm.*

**Proof.** Obviously, $\|\cdot\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}$ is a quasi-norm according to the linearity of $\rho_{\phi}(t,x)=f*\phi_t(x)$ and the corresponding property of $\|\cdot\|_{T^1_{\alpha,\beta}}$. Suppose that $\{f_j\}$ is a Cauchy sequence. By the Calderón reproducing formula and Theorem 4.2.3, we get $\dot{H}^{\frac{n}{2}-2(\beta-1),2}(\mathbb{R}^n)\hookrightarrow Q^{\beta}_{\alpha}(\mathbb{R}^n)$ and for every $\psi\in\mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}|\langle f_j-f_k,\psi\rangle| &\lesssim \|\rho_{\phi}(f_j-f_k)\|_{T^1_{\alpha,\beta}}\|\phi_t*\psi\|_{T^{\infty}_{\alpha,\beta}}\\ &\lesssim \|\rho_{\phi}(f_j-f_k)\|_{T^1_{\alpha,\beta}}\|\psi\|_{Q^{\beta}_{\alpha}(\mathbb{R}^n)}\\ &\lesssim \|\rho_{\phi}(f_j-f_k)\|_{T^1_{\alpha,\beta}}\|\psi\|_{\dot{H}^{\frac{n}{2}-2(\beta-1),2}(\mathbb{R}^n)}.\end{aligned}$$

This deduces that $\{f_j\}$ is a Cauchy sequence in $\dot{H}^{-\frac{n}{2}+2(\beta-1),2}(\mathbb{R}^n)$. By completeness, $f = \lim f_n$ exists in $\dot{H}^{-\frac{n}{2}+2(\beta-1),2}(\mathbb{R}^n)$. Thus there exists a subsequence such that $f = f_1 + \sum_{j\geq 1}(f_{j+1} - f_j)$ in $\mathcal{S}'(\mathbb{R}^n)$ with $\sum \|f_{j+1} - f_j\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)} < \infty$. Then we have

$$\|\rho_\phi(f)\|_{T^1_{\alpha,\beta}} \lesssim (\|\rho_\phi(f_1)\|_{T^1_{\alpha,\beta}} + \sum \|\rho_\phi(f_{j+1} - f_j)\|_{T^1_{\alpha,\beta}}) < \infty$$

and so $f \in HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. Similarly we can prove $f_j \longrightarrow f$ in $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. □

**Definition 4.2.12** *Let $\alpha > 0$, $\alpha+\beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$. A tempered distribution $a$ is called an $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ atom if $a$ is supported in a cube $I$ and satisfies the following two conditions:*
*(i) a local Sobolev–$(\alpha - \beta + 1)$ condition: for all $\psi \in \mathcal{S}$*

$$|\langle a, \psi\rangle| \leq diam(I)^{-\frac{n}{2}+\alpha+\beta-1}\left(\int_I\int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{2(\alpha-\beta+1)}}dxdy\right)^{1/2};$$

*(ii) a cancelation condition: $\langle a, \psi\rangle = 0$ for any $\psi \in \mathcal{S}$ which coincides with a polynomial of degree $\leq \frac{n}{2} + 1$ in a neighborhood of $I$.*

In [22], Dafni-Xiao established the following factional Poincaré inequality which will help us to understand the previous definition.

**Lemma 4.2.13** *Let $\psi \in C^\infty(\mathbb{R}^n)$ and $I$ be a cube. Denote by $\psi(I)$ the average of $\psi$ over $I$. If $0 \leq \alpha_1, \alpha_2 < \beta$ for a fixed $\beta \in (1/2, 1)$, then*

$$\begin{aligned}
\|\psi - \psi(I)\|_{L^2(\mathbb{R}^n)} &\leq n^{n/4} diam(I)^{\alpha_1-\beta+1}\left(\int_I\int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2(\alpha_1-\beta+1)}}dxdy\right)^{1/2} \\
&\leq n^{n/4} diam(I)^{\alpha_2-\beta+1}\left(\int_I\int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2(\alpha_2-\beta+1)}}dxdy\right)^{1/2} \\
&\leq C diam(I)\|\nabla\psi\|_{L^2(I)}
\end{aligned}$$

*with $C$ depending only on the dimension and $\alpha_2$. If in addition $\int_I \frac{\partial\psi}{\partial x_k}dx = 0$ for all $k = 1, \cdots, n$, then the quantities above are also bounded by*

$$C diam(I)\|\nabla\psi - (\nabla\psi)_I\|_{L^2(I)} \leq Cn^{n/4} diam(I)^{\alpha_1-\beta+2}\left(\int_I\int_I \frac{|\nabla\psi(x) - \nabla\psi(y)|^2}{|x-y|^{n+2(\alpha_1-\beta+1)}}dxdy\right)^{1/2}.$$

*Here $(\nabla\psi)_I$ denotes the vector whose coordinates are the means $(\frac{\partial\psi}{\partial x_k})(I)$, $k = 1, \cdots, n$.*

**Remark 4.2.14** *Similar to Remark (2) after Lemma 6.2 of Dafni-Xiao [22], we can prove that an $HH^1_{-\alpha,\beta}$–atom $a$ belongs to the homogeneous Sobolev spaces $\dot{H}^{-s,2}(\mathbb{R}^n)$ with $\alpha + \beta - 1 \leq s \leq \frac{n}{2} + 1$. Particularly, we have*

$$|\langle a, \psi\rangle| \lesssim (diam(I))^{-\frac{n}{2}+\alpha+\beta-1}\|\psi\|_{\dot{H}^{\alpha-\beta+1,2}(\mathbb{R}^n)}.$$

*This deduces* $\|a\|_{\dot{H}^{-(\alpha-\beta+1),2}(\mathbb{R}^n)} \lesssim (diam(I))^{-\frac{n}{2}+\alpha+\beta-1}$. *Meanwhile,*

$$|\langle a,\psi\rangle| \lesssim (diam(I))\|\psi\|_{\dot{H}^{\frac{n}{2}-2\beta+3,2}(\mathbb{R}^n)}$$

*and so* $\|a\|_{\dot{H}^{-(\frac{n}{2}-2\beta+3),2}(\mathbb{R}^n)} \lesssim diam(I)$.

We can obtain the atomic decomposition of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ as follows.

**Theorem 4.2.15** *Let* $\alpha > 0$, $\alpha+\beta \geq 1$ *and* $\max\{1/2,\alpha\} < \beta < 1$. *A tempered distribution* $f$ *on* $\mathbb{R}^n$ *belongs to* $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ *if and only if there exist* $HH^1_{-\alpha,\beta}(\mathbb{R}^n)-$*atoms* $\{a_j\}$ *and an* $l^1-$*summable sequence* $\{\lambda_j\}$ *such that* $f = \sum_j \lambda_j a_j$ *in the sense of distributions. Moreover,*

$$\|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)} \approx \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\}.$$

**Proof.** *Part 1.* "$\Longleftarrow$" By the completeness of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$, we only need to prove that if $a$ is an $HH^1_{-\alpha,\beta}-$atom then $a$ is in $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ with the quasinorm bounded by a constant. Since $a$ is an $HH^1_{-\alpha,\beta}(\mathbb{R}^n)-$atom and $\alpha+\beta-1 \leq \frac{n}{2}-2(\beta-1) \leq \frac{n}{2}+1$, Remark 4.2.14 implies that $a \in \dot{H}^{-\frac{n}{2}+2(\beta-1),2}(\mathbb{R}^n)$ with norm bounded by a constant. On the other hand, assume that $I$ is the support of $a$ and $x_I$ represents its center. For $\varepsilon \in (0,2)$, let

$$\omega(t,x) = \kappa(l(I))^{-n+2(\alpha+\beta-1)} \min\left\{1, \left(\frac{l(I)}{\sqrt{(x-x_I)^2+t^2}}\right)^{n-2(\alpha+\beta-1)+\varepsilon}\right\}$$

where $\kappa$ is a constant to be chosen later. Similar to the proof of Theorem 4.2.8, we have

$$N\omega(x) \leq \kappa(l(I))^{-n+2(\alpha+\beta-1)} \min\left\{1, \left(\frac{\sqrt{2}l(I)}{|x-x_I|}\right)^{n-2(\alpha+\beta-1)+\varepsilon}\right\}$$

and so $\int_{\mathbb{R}^n} N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)} \lesssim \kappa \leq 1$ by choosing $\kappa$ small enough.

Now, let $B_I = B(x_I, \mathrm{diam}(I))$, $E_I = (0,\mathrm{diam}(I)) \times B_I$ and $E_I^c = \mathbb{R}^{1+n}_+ \backslash E_I$. Suppose $S_a$ is the support of $a * \phi_t(x)$ in $\mathbb{R}^{1+n}_+$. We have

$$\int_{\mathbb{R}^{1+n}_+} |a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}} = \left(\int_{E_I} + \int_{E_I^c\cap S_a}\right)|a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}.$$

By the definition of the cylinder $E_I$ in $\mathbb{R}^{1+n}_+$, we can find a half-ball centered at $(0,x_I)$ to cover $E_I$. Thus we have $\omega^{-1} \lesssim (l(I))^{n-2(\alpha+\beta-1)}$ on $E_I$. This fact implies that

$$\begin{aligned}
&\int_{E_I} |a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \\
\leq\ & (l(I))^{n-2(\alpha+\beta-1)} \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{F}a(\xi)|^2|\mathcal{F}\phi(t\xi)|^2 d\xi \frac{dt}{t^{1-2(\alpha-\beta+1)}} \\
\leq\ & (l(I))^{n-2(\alpha+\beta-1)}\int_{\mathbb{R}^n} |\mathcal{F}a(\xi)|^2|\xi|^{-2(\alpha-\beta+1)}d\xi\int_0^\infty |\mathcal{F}\phi(t)|^2\frac{dt}{t^{1-2(\alpha-\beta+1)}} \\
\leq\ & (l(I))^{n-2(\alpha+\beta-1)}\|a\|_{\dot{H}^{-(\alpha-\beta+1),2}(\mathbb{R}^n)} \lesssim 1.
\end{aligned}$$

For the integral on $E_I^c \bigcap S_a$. If $z \in I$, $x \notin B_I$ and $t \leq |x - x_I|/2$, then

$$|x - z| \geq |x - x_I| - \text{diam(I)}/2 \geq |x - x_I|/2 \geq t,$$

and $a * \phi_t(x) = \int a(z)\phi_t(x - z)dz = 0$. Otherwise, we have

$$\begin{aligned} |a * \phi_t(x)| &\leq \|a\|_{\dot{H}^{-\frac{n}{2}+2\beta-3,2}(\mathbb{R}^n)} \|\phi_t^x\|_{\dot{H}^{\frac{n}{2}-2\beta+3,2}(\mathbb{R}^n)} \\ &\leq \text{diam}(I) t^{-(n-2\beta+3)} \left( \int_{\mathbb{R}^n} |\mathcal{F}\phi(\xi)|^2 |\xi|^{n-4\beta+6} d\xi \right)^{1/2} \\ &\leq \text{diam}(I) t^{-(n-2\beta+3)}. \end{aligned}$$

It is easy to check $t \approx \sqrt{|x - x_I|^2 + t^2} := r(t,x) >$diam$I$. This implies that

$$\omega^{-1}(t,x) \approx \kappa^{-1}(l(I))^{n-2(\alpha+\beta-1)} \frac{t^{n-2(\alpha+\beta-1)+\varepsilon}}{(l(I))^{n-2(\alpha+\beta)+\varepsilon}} \lesssim (l(I))^{-\varepsilon} t^{n-2(\alpha+\beta-1)+\varepsilon}.$$

Then we can get

$$\begin{aligned} &\int_{E_I^c \bigcap S_a} |a * \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \lesssim (l(I))^{2-\varepsilon} \int_{E_I^c \bigcap S_a} t^{\varepsilon-n-3} dtdx \\ \lesssim\ & (l(I))^{2-\varepsilon} \int_{r(t,x) \geq \text{diam}(I)} r(t,x)^{\varepsilon-n-3} dt \lesssim (l(I))^{2-\varepsilon+\varepsilon-2} \lesssim 1. \end{aligned}$$

*Part 2.* "$\Longrightarrow$" Suppose $f \in HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. Note that the Calderón reproducing formula (4.2.3) holds in the sense of distributions. Since the support of $\phi$ is the unit ball, we can denote

$$f^{\varepsilon,N}(x) = \int_{S^{\varepsilon,N}} F(t,y)\phi_t(x-y) \frac{dtdy}{t}$$

where $F(t,y) = f * \phi_t(y)$ and $S^{\varepsilon,N}$ is the strip $\{(t,x) \in \mathbb{R}^{1+n}_+ : \varepsilon \leq t \leq N\}$. Similar to the proof of Theorem 4.2.8, there exists an $\omega \geq 0$ on $\mathbb{R}^{1+n}_+$ such that $\int_{\mathbb{R}^n} N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)} \leq 1$ and

$$\int_{\mathbb{R}^{1+n}_+} |F(t,x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \leq 2\|F\|_{T^1_{\alpha,\beta}}.$$

Let $T_{j,k}$ be the corresponding structures over the set $E_k = \{N\omega > 2^k\}$ as those in Theorem 4.2.8 (i). Noting that $T_{j,k}$ have mutually disjoint interiors and $F = \sum F\chi_{T_{j,k}}$ a.e. on $\mathbb{R}^{1+n}_+$, we let

$$g^{\varepsilon,N}_{j,k}(x) = \int_{S^{\varepsilon,N} \bigcap T_{j,k}} F(t,y)\phi_t(x-y) \frac{dtdy}{t}.$$

Since $T_{j,k} \subset T(I^*_{j,k})$, these smooth functions in $x$ is supported in $\{x : \Gamma(x) \cap T_{j,k} \neq \emptyset\} \subset I^*_{j,k}$ and have the same number moments as $\phi$. We want to verify that there are distributions $g_{j,k}$ such that $g^{\varepsilon,N}_{j,k} \longrightarrow g_{j,k}$ as $\varepsilon \longrightarrow 0$ and $N \longrightarrow \infty$ with $f = \sum_{j,k} g_{j,k}$ in $\mathcal{S}'(\mathbb{R}^n)$. To see this, noting that $\omega \leq 2^{k+1}$ on $T_{j,k}$, we have

$$
\begin{aligned}
|\langle g_{j,k}^{\varepsilon,N}, \psi\rangle| &= \left|\int_{\mathbb{R}^n}\left(\int_{S^{\varepsilon,N}\bigcap T_{j,k}} F(t,y)\phi_t(x-y)\frac{dtdy}{t}\right)\psi(x)dx\right| \\
&\leq 2^{(k+1)/2}\left(\int_{S^{\varepsilon,N}\bigcap T_{j,k}} |F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \\
&\quad\times\left(\int_{S^{\varepsilon,N}\bigcap T_{j,k}} |\psi*\phi_t(y)|^2\frac{dtdy}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2} \\
&\leq 2^{(k+1)/2}\left(\int_{S^{\varepsilon,N}\bigcap T_{j,k}} |F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \\
&\quad\times\left(\int_{3I_{j,k}^*}\int_{3I_{j,k}^*}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}dtdy\right)^{1/2}.
\end{aligned}
$$

Similarly, we obtain that for $\varepsilon_1 < \varepsilon_2$ and $N_1 > N_2$,

$$
\begin{aligned}
&|\langle g_{j,k}^{\varepsilon_1,N_1} - g_{j,k}^{\varepsilon_2,N_2}, \psi\rangle| \\
&\leq C_k\left(\int_{(S^{\varepsilon_1,N_1}\setminus S^{\varepsilon_2,N_2})\bigcap T_{j,k}} |F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\|\psi\|_{\dot{H}^{\alpha-\beta+1,2}(\mathbb{R}^n)}.
\end{aligned}
$$

This gives us that $\|g_{j,k}^{\varepsilon_1,N_1} - g_{j,k}^{\varepsilon_2,N_2}\|_{\dot{H}^{-(\alpha-\beta+1),2}(\mathbb{R}^n)} \longrightarrow 0.$ as $\varepsilon_1, \varepsilon_2 \longrightarrow 0$ and $N_1, N_2 \longrightarrow \infty$. Thus, $g_{j,k}^{\varepsilon,N} \longrightarrow g_{j,k} \in \dot{H}^{-(\alpha-\beta+1),2}(\mathbb{R}^n)$ in the sense of distributions and $g_{j,k}$ is supported in $I_{j,k}^*$ with

$$
\|g_{j,k}\|_{\dot{H}^{-(\alpha-\beta+1),2}(3I_{j,k}^*)} \lesssim 2^{(k+1)/2}\left(\int_{T_{j,k}} |F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}.
$$

Let

$$
a_{j,k} = g_{j,k}\|g_{j,k}\|^{-1}_{\dot{H}^{-(\alpha-\beta+1),2}(3I_{j,k}^*)}(l(3I_{j,k}^*))^{(\alpha+\beta-1)-\frac{n}{2}}
$$

and

$$
\lambda_{j,k} = \|g_{j,k}\|_{\dot{H}^{-(\alpha-\beta+1),2}(3I_{j,k}^*)}(l(3I_{j,k}^*))^{\frac{n}{2}-(\alpha+\beta-1)}.
$$

Then

$$
\begin{aligned}
&|\langle a_{j,k}, \psi\rangle| \\
&\lesssim \frac{1}{\|g_{j,k}\|_{\dot{L}^2_{-(\alpha-\beta+1)}(3I_{j,k}^*)}}(l(3I_{j,k}^*))^{(\alpha+\beta-1)-\frac{n}{2}} \\
&\times\left(\int_{S^{\varepsilon,N}\bigcap T_{j,k}} |F(t,y)|^2 w(t,y)^{-1}\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{\frac{1}{2}}\left(\int_{3I_{j,k}^*}\int_{3I_{j,k}^*}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}dxdy\right)^{\frac{1}{2}}.
\end{aligned}
$$

This means that $a_{j,k}$ are $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$–atoms. On the other hand, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&\sum_{j,k}|\lambda_{j,k}| \\
&\lesssim \left(\sum_{j,k}2^{k+1}(l(3I^*_{j,k}))^{n-2(\alpha+\beta-1)}\right)^{1/2}\left(\sum_{j,k}\int_{T_{j,k}}|F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \\
&\lesssim \left(\sum_{j,k}2^{k+1}\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}(3I^*_{j,k})\right)^{1/2}\left(\sum_{j,k}\int_{\mathbb{R}^{1+n}_+}|F(t,y)|^2\omega^{-1}(t,y)\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \\
&\lesssim \left(\sum_{k}\int_{E_K}2^{k+1}d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}(E_k)\right)^{1/2}\|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)} \\
&\lesssim \left(\sum_{k}\int_{E_k}N\omega(x)d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}\right)^{1/2}\|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}\lesssim\|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}.
\end{aligned}$$

The above estimates tell us that $\sum g_{j,k}=\sum\lambda_{j,k}a_{j,k}$ converges to a distribution $g$ in $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. We need to verify that $g=f$. Since for a fix $\psi\in\mathcal{S}(\mathbb{R}^n)$, every $0<\varepsilon<N$,

$$\begin{aligned}
&|\langle g^{\varepsilon,N}_{j,k},\psi\rangle| \\
&\lesssim 2^{(k+1)/2}\left(\int_{T_{j,k}}|F(t,y)|^2w(t,y)^{-1}\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \\
&\quad\times\left(\int_{3I^*_{j,k}}\int_{3I^*_{j,k}}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}dtdy\right)^{1/2} \\
&\lesssim 2^{(k+1)/2}(l(3I^*_{j,k}))^{\frac{n}{2}-(\alpha+\beta-1)}\left(\int_{T_{j,k}}|F(t,y)|^2w(t,y)^{-1}\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\|\psi\|_{Q_{\alpha,\beta}(\mathbb{R}^n)} \\
&\lesssim \|f\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)}\|\psi\|_{Q_{\alpha,\beta}(\mathbb{R}^n)}.
\end{aligned}$$

Then, $\lim_{\varepsilon\longrightarrow0,N\longrightarrow\infty}\sum_{j,k}g^{\varepsilon,N}=\sum_{j,k}g_{j,k}=g$. Meanwhile, we can also obtain that

$$\sum_{j,k}\int_{\mathbb{R}^{1+n}_+}1_{S^{\varepsilon,N}\bigcap T_{j,k}}(t,y)F(t,y)\phi_t*\psi(y)\frac{dtdy}{t}=\int_{S^{\varepsilon,N}}F(t,y)\phi_t*\psi(y)\frac{dtdy}{t}=\langle f^{\varepsilon,N},\psi\rangle.$$

This tells us $\sum_{j,k}g^{\varepsilon,N}_{j,k}=f^{\varepsilon,N}\longrightarrow f$ in $\mathcal{S}'(\mathbb{R}^n)$. Therefore $f=g$ in $\mathcal{S}'(\mathbb{R}^n)$. □

**Lemma 4.2.16** *(i) If $a$ is an $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$–atom, then there exists a nonnegative function $\omega$ on $\mathbb{R}^{1+n}_+$ with $\int_{\mathbb{R}^n}N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)}\le1$ and*

$$\sigma_\delta(a,\omega)=\sup_{|y|\le\delta}\left(\int_{\mathbb{R}^{1+n}_+}|a*\psi_t(x-y)-a*\psi_t(x)|^2\omega(t,x)^{-1}\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\longrightarrow0.$$

*(ii)* $HH^1_{-\alpha,\beta}(\mathbb{R}^n)\cap C_0^\infty(\mathbb{R}^n)$ *is dense in* $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$.

**Proof.** (i) For a fixed $\varepsilon\in(0,2)$, the same $\omega$ defined in the proof of Theorem 4.2.15, $y\in B(0,\delta)$ and $x\in\mathbb{R}^n$, we have $a*\phi_t(x-y)-a*\phi_t(x)=\langle a,\phi_t^{x-y}-\phi_t^x\rangle$ and

$$|(\mathcal{F}\phi_t^{x-y}-\mathcal{F}\phi_t^x)(\xi)|=|1-e^{2\pi iy\cdot\xi}||\mathcal{F}\phi_t(\xi)|\le C\min\{2,\delta|\xi|\}|\mathcal{F}\phi_t(\xi)|. \qquad (4.2.4)$$

Note that

$$\begin{aligned}&\sup_{|y|<\delta}\left(\int_{\mathbb{R}^{1+n}_+}|a*\phi_t(x-y)-a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\\ \lesssim&\left((\sup_{|y|<\delta}\int_{E_I}+\sup_{|y|<\delta}\int_{E_I^c\bigcap S_{a,\delta}})|a*\phi_t(x-y)-a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2},\end{aligned}$$

where $B_I$ is the ball $B(x_I,2\mathrm{diam}(I))$, and $E_I=(0,2\mathrm{diam}(I))\times B_I$. By Fourier transforms, we can estimate the first term as

$$\begin{aligned}&\sup_{|y|<\delta}\int_{E_I}|a*\phi_t(x-y)-a*\phi_t(x)|^2\omega^{-1}(t,x)\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\\ \lesssim&(l(I))^{n-2(\alpha+\beta-1)}\sup_{|y|<\delta}\int_0^\infty\int_{\mathbb{R}^n}|a*(\phi_t^y-\phi_t)(x)|^2\frac{dtdx}{t^{1-2(\alpha-\beta+1)}}\\ \lesssim&(l(I))^{n-2(\alpha+\beta-1)}\sup_{|y|<\delta}\int_{\mathbb{R}^n}|\widehat{a}(\xi)|^2\min\{2,\delta|\xi|\}^2\int_0^\infty|\mathcal{F}\phi(t|\xi|)|^2\frac{dt}{t^{1-2(\alpha-\beta+1)}}d\xi\\ \lesssim&(l(I))^{n-2(\alpha+\beta-1)}\sup_{|y|<\delta}\int_{\mathbb{R}^n}|\mathcal{F}a(\xi)|^2\delta^2|\xi|^2|\xi|^{-2(\alpha-\beta+1)}\int_0^\infty|\mathcal{F}\psi(t)|^2\frac{dt}{t^{1-2(\alpha-\beta+1)}}d\xi\to0\end{aligned}$$

as $\delta\longrightarrow0$ according to the dominated convergence theorem.

For the second term. Since $\mathrm{supp}(a)=I$, when $x\notin B_I$ and $t\le|x-x_I|/4$, we obtain $|y|<\mathrm{diam}(I)<\frac{1}{2}|x-x_I|$ for $y\in B(0,\delta)$ with $\delta<\mathrm{diam}(I)$. Therefore

$$|x-y-z|>|x-x_I|-|z-x_I|-|y|\ge\frac{3}{4}|x-x_I|\ge t.$$

On the other hand $|x-z|\ge\frac{3}{4}|x-x_I|>t$. These estimates imply that $a*[\phi_t(x-y)-\phi_t(x)]=0$. Otherwise, we have

$$\begin{aligned}|a*\phi_t(x-y)-a*\phi_t(x)|&\lesssim\|a\|_{\dot H^{-(\frac{n}{2}-2\beta+3),2}(\mathbb{R}^n)}\|\phi_t^{x-y}-\phi_t^x\|_{\dot H^{\frac{n}{2}-2\beta+3,2}(\mathbb{R}^n)}\\ &\lesssim\mathrm{diam}(I)\left(\int_{\mathbb{R}^n}|\mathcal{F}\phi_t^{x-y}(\xi)-\mathcal{F}\phi_t^x(\xi)|^2|\xi|^{n-4\beta+6}d\xi\right)^{1/2}\\ &\lesssim\mathrm{diam}(I)\left(\int_{\mathbb{R}^n}\min\{2,\delta|\xi|\}^2|\mathcal{F}\phi_t(\xi)|^2|\xi|^{n-4\beta+6}d\xi\right)^{1/2}\\ &\lesssim\mathrm{diam}(I)\delta\left(\int_{\mathbb{R}^n}|\mathcal{F}\phi_t(\xi)|^2|\xi|^{n-4\beta+8}d\xi\right)^{1/2}\\ &\lesssim\mathrm{diam}(I)\delta t^{2\beta-4-n}.\end{aligned}$$

Using the above estimates and the fact $\omega^{-1} \lesssim t^{n-2(\alpha+\beta-1)+\varepsilon}$, we have

$$\begin{aligned}
&\int_{E_I^c \bigcap S_{a,\delta}} |a*\phi_t(x-y) - a*\phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \\
\lesssim\ & \delta^2 (l(I))^{2-\varepsilon} \int_{E_I^c \bigcap S_{a,\delta}} t^{-n-5+\varepsilon} dtdx \\
\lesssim\ & \delta^2 (l(I))^{2-\varepsilon} \int_{l(I)}^{\infty} \lambda^{\varepsilon-5} d\lambda \longrightarrow 0
\end{aligned}$$

as $\delta \longrightarrow 0$. Thus $\sigma_\delta(a,\omega) \longrightarrow 0$ as $\delta \longrightarrow 0$.

(ii) For an $HH^1_{\alpha,\beta}(\mathbb{R}^n)$–atom $a$, take $\eta \in C^\infty(\mathbb{R}^n)$ with support in $B(0,1)$ and $\int \eta = 1$. Then $a*\eta_j \in C_0^\infty(\mathbb{R}^n)$ and $\eta_j = j^n \eta(jx)$ form an approximate identity, $a*\eta_j \longrightarrow a$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \longrightarrow \infty$. For any nonnegative function $\omega$ on $\mathbb{R}^{1+n}_+$ with $\int_{\mathbb{R}^n} N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)} \leq 1$, we have

$$\begin{aligned}
&\left( \int_{\mathbb{R}^{1+n}_+} |a*\eta_j*\phi_t(x) - a*\phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \\
\lesssim\ & \int_{\mathbb{R}^n} |\eta_j(y)| \left( \int_{\mathbb{R}^{1+n}_+} |a*\phi_t(x-y) - a*\phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \\
\lesssim\ & \sigma_{\frac{1}{j}}(a,\omega).
\end{aligned}$$

From (i), we know that for every $\varepsilon > 0$ there exists an $\omega$ such that $\sigma_{\frac{1}{j}}(a,\omega) < \varepsilon$ with $j$ large enough. Taking the infimum over all $\omega$ induces

$$\|a*\eta_j - a\|_{HH^1_{-\alpha,\beta}(\mathbb{R}^n)} < \varepsilon \text{ for large } j,$$

that is, $a*\eta_j \longrightarrow a$ in $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$. Hence, we can get the desired density from the fact that every $f \in HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ can be approximated by finite sums of atoms. □

**Lemma 4.2.17** *For $\alpha > 0$, $\alpha+\beta \geq 1$, $\max\{\alpha, 1/2\} < \beta < 1$, $f \in L^2_{loc}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 0$, let*

$$d\mu_{f,\phi,\alpha,\beta}(t,x) = |(f*\phi_t)(y)|^2 t^{-1-2(\alpha-\beta+1)} dtdy.$$

*Then there is a constant $C$ such that for any cubes $I$ and $J$ in $\mathbb{R}^n$ with center $x_0$ and $l(J) \geq 3l(I)$,*
*(i)*

$$\mu_{f,\phi,\alpha,\beta}(S(I)) \leq \int_J\int_J \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dxdy + [l(I)]^{n-2(\alpha-\beta)} \left( \int_{\mathbb{R}^n \setminus \frac{2}{3}J} \frac{|f(x)-f(y)|}{|x-x_0|^{n+1}} dx \right)^2.$$

*(ii) If in addition supp$(\phi) \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ then*

$$\mu_{f,\phi,\alpha,\beta}(S(I)) \leq C \int_J\int_J \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dxdy.$$

**Proof.** This lemma is a special case of Dafini-Xiao [22, Lemma 3.2]. □

**Theorem 4.2.18** *Let $\phi$ be a function as in Lemma 4.2.9, $0 < \alpha < \beta$, $\alpha + \beta \geq 1$ and $1/2 < \beta < 1$. If $f \in Q_\alpha^\beta(\mathbb{R}^n)$ then $d\mu_{f,\phi,\alpha,\beta}(t,x) = |(f * \phi_t)(x)|^2 t^{-1-2(\alpha-\beta+1)}dtdx$ is a $1-2(\alpha+\beta-1)/n-$Carleson measure.*

**Proof.** The proof follows from (ii) Lemma 4.2.17 by taking $J = 3I$. □

To establish the equivalent (1.0.11) we need another theorem which contains the converse of Theorem 4.2.18.

**Theorem 4.2.19** *Consider the operator $\pi_\phi$ defined by*

$$\pi_\phi(F) = \int_0^\infty F(t,\cdot) * \phi_t \frac{dt}{t}. \tag{4.2.5}$$

*(i) The operator $\pi_\phi$ is a bounded and surjective operator form $T^\infty_{\alpha,\beta}$ to $Q_\alpha^\beta(\mathbb{R}^n)$. More precisely, if $F \in T^\infty_{\alpha,\beta}$ then the right-hand side of the above integral converges to a function $f \in Q_\alpha^\beta$ and*

$$\|f\|_{Q_\alpha^\beta(\mathbb{R}^n)} \lesssim \|F\|_{T^\infty_{\alpha,\beta}}$$

*and any $f \in Q_{\alpha,\beta}(\mathbb{R}^n)$ can be thus represented.*
*(ii) The operator $\pi_\psi$ initially defined on $F \in T^1_{\alpha,\beta}$ with compact support in $\mathbb{R}^{1+n}_+$ extends to a bounded and surjective operator form $T^1_{\alpha,\beta}$ to $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$.*

**Proof.** (i) Taking $f = \pi_\phi(F)$, we only need to prove $\sup_I D_{f,\alpha,\beta}(I) < \infty$ where

$$D_{f,\alpha,\beta}(I) = [l(I)]^{2\alpha-n+2\beta-2} \int_{|y|<l(I)} \int_I |f(x+y) - f(y)|^2 \frac{dxdy}{|y|^{n+2(\alpha-\beta+1)}}.$$

Denote the function $x \to f(x+y)$ by $f_y$ and note that the integral in (4.2.5) is valid in $\mathcal{S}'(R^n)$ modulo constants, that is, when it acts on test functions of integration zero, we obtain

$$f_y - f = \int_0^\infty [(F(t,\cdot) * \phi_t)_y - (F(t,\cdot) * \phi_t)] \frac{dt}{t} \text{ in } \mathcal{S}'(R^n).$$

Fix a cube $I$ and $y \in B(0, l(I))$. For any $g \in C_0^\infty(I)$, we write

$$\begin{aligned}
|\langle f_y - f, g\rangle| &\leq \int_0^{|y|} \int_{\mathbb{R}^n} |F(t,x)||\phi_t * (g_{-y} - g)(x)| \frac{dtdx}{t} \\
&\quad + \int_{|y|}^{l(I)} \int_{\mathbb{R}^n} |(F(t,\cdot) * \phi_t)(x+y) - (F(t,\cdot) * \phi_t)(x)||g(x)| \frac{dtdx}{t} \\
&\quad + \int_{l(I)}^\infty \int_{\mathbb{R}^n} |F(t,x)||\phi_t * (g_{-y} - g)(x)| \frac{dtdx}{t} \\
&:= A_1(g,y) + A_2(g,y) + A_3(g,y).
\end{aligned}$$

For $A_1(g,y)$, $|y| < l(I)$ verifies that $g_{-y} - g$ is supported in the dilated cube $3I$. Also if $t \leq |y|$ we have that $\phi_t * (g_y - g)$ is supported in the large cube $J = 5I$. Then we can get

$$\begin{aligned} A_1(g,y) &\leq \int_0^{|y|} \left( \int_J |F(t,x)|^2 dx \right)^{1/2} \|\phi_t * (g_{-y} - g)\|_{L^2} \frac{dt}{t} \\ &\lesssim \|\phi\|_{L^1(\mathbb{R}^n)} \|g\|_{L^2(I)} \int_0^{|y|} \left( \int_J |F(t,x)|^2 dx \right)^{1/2} \frac{dt}{t}. \end{aligned}$$

For $A_2$, if $|y| < t$, by changing variable $z - y = z$, we get

$$\begin{aligned} & |(F(t,\cdot) * \phi_t)(x+y) - (F(t,\cdot) * \phi_t)(x)| \\ \lesssim & \int_{\mathbb{R}^n} |\phi(t^{-1}y + z) - \phi(z)||F(t, x - tz)| dz \\ \lesssim & \; t^{-1}|y| \sup_{|\xi|\leq 1} |\nabla\phi(\xi)| \int_{|z|\leq 2} |F(t, x - tz)| dz \\ \lesssim & \; C_\phi t^{-1}|y| \int_{|z|\leq 2} |F(t, x - tz)| dz \end{aligned}$$

with $C_\phi = \sup |\nabla\phi| < \infty$. Fubini's theorem and the fact that $g$ is supported in $I$ imply that

$$\begin{aligned} A_2(g,y) &\leq C_\phi |y| \int_{|y|}^{l(I)} \int_{|z|\leq 2} \int_I |F(t, x - tz)||g(x)| dx dz \frac{dt}{t^2} \\ &\lesssim CC_\phi \|g\|_{L^2(\mathbb{R}^n)} |y| \int_{|y|}^{l(I)} \int_{|z|\leq 2} \left( \int_I |F(t, x - tz_t)|^2 dx \right)^{1/2} dz \frac{dt}{t^2} \end{aligned}$$

where $|z_t| \leq 2$ and $C = Vol(B(0,2))$.

For $A_3$, let $G_y(t,x) = \phi_t * (g_{-y} - g)(x) 1_{\{(t,x):t\geq|y|\}}$. Then the inequality (4.2.2) implies that

$$A_3 = \int_{\mathbb{R}^{1+n}_+} |F(t,x) G_y(t,x)| \frac{dtdx}{t} \lesssim \|F\|_{T^\infty_{\alpha,\beta}} \|G_y\|_{T^1_{\alpha,\beta}}$$

if we claim that $G_y \in T^1_{\alpha,\beta}$. To prove $G_y \in T^1_{\alpha,\beta}$, we follow the proof of Lemma 4.2.16 (i) and choose $\omega$ be the same function as that in Theorem 4.2.15 with $0 < 2(\alpha + \beta - 1) < \varepsilon < 2 - 4 + 4\beta$. Note that if $S_y :=$ supp $(G_y)$, then we obtain $\omega^{-1}(x) \simeq l(I)^{-\varepsilon} t^{n-2(\alpha+\beta-1)+\varepsilon}$.

Hence

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} |G_y(t,x)|^2\omega^{-1}(t,x)\frac{dxdt}{t^{1-2(\alpha-\beta+1)}}\\
\le\;& l(I)^{-\varepsilon}\int_{l(I)}^{\infty}\int_{\mathbb{R}^n}|\phi_t*(g_{-y}-g)(x)|^2dxt^{n-2(\alpha+\beta-1)+\varepsilon}\frac{dt}{t^{1-2(\alpha-\beta+1)}}\\
\le\;& l(I)^{-\varepsilon}\|g\|^2_{L^1(\mathbb{R}^n)}\int_{l(I)}^{\infty}\|\phi_t^y-\phi_t\|^2_{L^2}t^{n-4\beta+3+\varepsilon}dt\\
\le\;& l(I)^{n-\varepsilon}\|g\|^2_{L^2(\mathbb{R}^n)}\int_{l(I)}^{\infty}\int_{\mathbb{R}^n}|(\mathcal{F}\phi_t^y-\mathcal{F}\phi_t)(\xi)|^2d\xi t^{n-4\beta+3+\varepsilon}dt\\
\le\;& l(I)^{n-\varepsilon}\|g\|^2_{L^2(\mathbb{R}^n)}\int_{l(I)}^{\infty}\int_{\mathbb{R}^n}|1-e^{2\pi i y\cdot\xi}|^2|\mathcal{F}\phi(t|\xi|)|^2d\xi t^{n-4\beta+4+\varepsilon}\frac{dt}{t}\\
\le\;& l(I)^{n-\varepsilon}\|g\|^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|1-e^{2\pi i y\cdot\xi}|^2}{|\xi|^{n-4\beta+4+\varepsilon}}d\xi\int_0^{\infty}|\mathcal{F}\phi(t)|^2t^{n-4\beta+4+\varepsilon}\frac{dt}{t}\\
\le\;& C_\phi l(I)^{n-\varepsilon}\|g\|^2_{L^2(\mathbb{R}^n)}|y|^{-4\beta+4+\varepsilon}.
\end{aligned}
$$

In the last inequality we have used the fact:

$$
\int_{\mathbb{R}^n}\frac{|1-e^{2\pi i y\xi}|^2}{|\xi|^{n-4\beta+4+\varepsilon}}d\xi\lesssim |y|^{\varepsilon-4\beta+4}.
$$

In fact, we can write

$$
\begin{aligned}
\int_{\mathbb{R}^n}\frac{|1-e^{2\pi i y\xi}|^2}{|\xi|^{n-4\beta+4+\varepsilon}}d\xi &\lesssim \int_{R^n}\frac{|1-e^{2\pi i y\xi}|^2}{|y\xi|^{n-4\beta+4+\varepsilon}}|y|^{n-4\beta+4+\varepsilon}\frac{d(y\xi)}{|y|^n}\\
&\lesssim |y|^{\varepsilon-4\beta+4}\left(\int_{|z|\le 1}+\int_{|z>1|}\right)\frac{|1-e^{2\pi i z}|^2}{|z|^{n-4\beta+4+\varepsilon}}dz\\
&:= |y|^{\varepsilon-4\beta+4}(I_1+I_2).
\end{aligned}
$$

It is easy to see that

$$
I_2=\int_{|z|\ge 1}\frac{|1-e^{2\pi i z}|^2}{|z|^{n+\varepsilon-4\beta+4}}dz\lesssim\int_{|z|\ge 1}\frac{|z|^{n-1}}{|z|^{n+\varepsilon-4\beta+4}}d|z|\lesssim 1,
$$

$$
\begin{aligned}
I_1 &= \int_{|z|<1}\frac{|1-e^{2\pi i z}|^2}{|z|^{n+\varepsilon-4\beta+4}}dz\lesssim\int_{|z|<1}\left|\sum_{k=1}^{\infty}\frac{(2\pi i z)^{k-1}}{k!}\right|^2\frac{|z|^2}{|z|^{n+\varepsilon-4\beta+4}}dz\\
&\lesssim \int_{|z|<1}|z|^{1-\varepsilon+4\beta-4}d|z|\lesssim 1.
\end{aligned}
$$

Then $\|G_y\|_{T^1_{\alpha,\beta}} \le \|g\|_{L^2(\mathbb{R}^n)}\sqrt{l(I)^{n-\varepsilon}|y|^{\varepsilon-4\beta+4}}$. Thus we get

$$\begin{aligned}
\|f_y - f\|_{L^2(I)} &\le \sup_{g\in C_0^\infty(I),\|g\|_2\le 1} |\langle f_y - f, g\rangle| \\
&\lesssim \int_0^{|y|}\left(\int_J |F(t,x)|^2 dx\right)^{1/2}\frac{dt}{t} + |y|\int_{|y|}^{l(I)}\left(\int_I |F(t,x-tz_t)|^2 dx\right)^{1/2}\frac{dt}{t^2} \\
&\quad + \|F\|_{T^\infty_{\alpha,\beta}} l(I)^{(n-\varepsilon)/2}|y|^{\varepsilon/2-2\beta+2}.
\end{aligned}$$

Then, by Hardy's inequality(see Stein [64]), we have

$$\begin{aligned}
&\int_{|y|<l(I)}\int_I |f(x+y)-f(x)|^2\frac{dxdy}{|y|^{n+2(\alpha-\beta+1)}} \\
&\lesssim \int_0^{l(I)}\left(\int_0^s\left(\int_J |F(t,x)|^2 dx\right)^{1/2}\frac{dt}{t}\right)\frac{ds}{s^{1+2(\alpha-\beta+1)}} \\
&\quad + \int_0^{l(I)}\left(\int_s^{l(I)}\left(\int_I |F(t,x-tz_t)|^2 dx\right)^{1/2}\frac{dt}{t^2}\right)^2\frac{ds}{s^{2(\alpha-\beta+1)-1}} \\
&\quad + \|F\|^2_{T^\infty_{\alpha,\beta}} l(I)^{n-\varepsilon}\int_0^{l(I)}\frac{s^{n-1}s^{\varepsilon-4\beta+4}}{s^{n+2(\alpha-\beta+1)}}ds \\
&\lesssim \int_0^{l(I)}\int_J |F(t,x)|^2 t^{-1-2(\alpha-\beta+1)}dxdt \\
&\quad + \int_0^{l(I)}\int_I |F(t,x-tz_t)|^2 t^{-1-2(\alpha-\beta+1)}dtdx \\
&\quad + \|F\|^2_{T^\infty_{\alpha,\beta}} l(I)^{n-\varepsilon} l(I)^{\varepsilon-2(\alpha+\beta-1)} \\
&\lesssim l(I)^{n-2(\alpha+\beta-1)}\|F\|^2_{T^\infty_{\alpha,\beta}}
\end{aligned}$$

since for each $t \le l(I)$, $|z_t| \le 2$ implies $I - tz_t \subset J = 5I$. Then we get $\sup_I D_{f,\alpha,\beta}(I) \lesssim \|F\|^2_{T^\infty_{\alpha,\beta}} < \infty$, that is $f \in Q^\beta_\alpha(\mathbb{R}^n)$ and $\|f\|_{Q^\beta_\alpha(\mathbb{R}^n)} \lesssim \|F\|_{T^\infty_{\alpha,\beta}}$.

(ii) Firstly, we verify that for a $T^1_{\alpha,\beta}-$ atom $a$, the integral in (4.2.5) converges in $\dot{H}^{-n/2-2+2\beta,2}$ to a distribution which is a multiple of an $HH^1_{-\alpha,\beta}(\mathbb{R}^n)-$atom. Assume $a(x,t)$ is supported in $T(B)$ for some $B$. For $\varepsilon > 0$, let

$$\pi^\varepsilon_\phi(a) = \int_\varepsilon^\infty a(t,\cdot) * \phi_t(x)\frac{dxdt}{t}$$

and $T^\varepsilon(B)$ be the truncated tent $T(B)\cap\{(t,x) : t > \varepsilon\}$. The Cauchy-Schwarz inequality and (ii) of Lemma 4.2.17 imply that

$$\begin{aligned}
&|\langle \pi^\varepsilon_\phi(a), \psi\rangle| \\
&\le \left(\int_{T^\varepsilon(B)} |a(t,x)|^2\frac{dxdt}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2}\left(\int_{T^\varepsilon(B)} |\psi * \phi_t(x)|^2\frac{dxdt}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2} \\
&\lesssim \left(l(\widetilde{B})^{2\alpha-n+2\beta-2}\int_{\widetilde{B}}\int_{\widetilde{B}}\frac{|\psi(x)-\psi(y)|^2}{t^{n+2(\alpha-\beta+1)}}dxdy\right)^{1/2}
\end{aligned}$$

hold for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, where $\widetilde{B}$ is some fixed dilate of the ball $B$. Since the right-hand side is dominated by $\|\psi\|_{Q_\alpha^\beta(\mathbb{R}^n)} \le \|\psi\|_{\dot{H}^{n/2+2-2\beta,2}(\mathbb{R}^n)}$, the same argument also gives, for $0 < \varepsilon_1 < \varepsilon_2$,

$$|\langle \pi_\phi^{\varepsilon_1}(a) - \pi_\phi^{\varepsilon_2}(a), \psi\rangle| \le \left(\int_{T^{\varepsilon_1}(B)\setminus T^{\varepsilon_2}(B)} |a(t,x)|^2 \frac{dxdt}{t^{1-2(\alpha-\beta+1)}}\right)^{1/2} \|\psi\|_{\dot{H}^{n/2+2-2\beta,2}(\mathbb{R}^n)}.$$

Thus $\pi_\phi(a) = \lim_{\varepsilon \longrightarrow 0} \pi_\phi^\varepsilon(a)$ exists in $\dot{H}^{-n/2-2+2\beta,2}(\mathbb{R}^n)$. This distribution is supported in $\widetilde{B}$ and satisfies condition (i) of Definition 4.2.12 since $\phi$ satisfies the same condition. Therefore $\pi_\phi(a)$ is a multiple of an $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$–atom. For a function $F = \sum_j \lambda_j a_j$ in $T^1_{\alpha,\beta}$ and a test function $\psi \in \mathcal{S}(\mathbb{R}^n)$, by Theorem 4.2.8, we have

$$\int_{\mathbb{R}^{n+1}_+} (F(t,\cdot) * \phi_t)(x)\psi(x)\frac{dxdt}{t} = \sum_j \lambda_j \langle \pi_\phi a_j, \psi\rangle = \left\langle \sum_j \lambda_j \pi_\phi a_j, \psi \right\rangle,$$

since $\rho_\phi(\psi)(t,x) = (\phi_t * \psi)(x)$ is a function in $T^\infty_{\alpha,\beta}$. So $\pi_\phi(F) = \sum_j \lambda_j \pi_\phi a_j \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\|\pi_\phi(F)\|_{HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n)} \le \inf \sum_j |\lambda_j| \approx \|F\|_{T^1_{\alpha,\beta}}$$

the infimum being taken over all possible atomic decompositions of $F$ in $T^1_{\alpha,\beta}$. This finishes the proof of Theorem 4.2.19. □

By Theorem 4.2.18, Lemma 4.2.16 and Theorem 4.2.19, using a similar argument of Dafni-Xiao [22, Theorem 7.1], we can prove the following duality theorem.

**Theorem 4.2.20** *The duality of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ is $Q_\alpha^\beta(\mathbb{R}^n)$ in the following sense: if $g \in Q_\alpha^\beta$ then the linear functional*

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)dx,$$

*defined initially for $f \in HH^1_{-\alpha,\beta}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$, has a bounded extension to all elements of $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ with $\|L\| \le C\|g\|_{Q_\alpha^\beta(\mathbb{R}^n)}$. Conversely, if $L$ is a bounded linear functional on $HH^1_{-\alpha,\beta}(\mathbb{R}^n)$ then there is a function $g \in Q_\alpha^\beta(\mathbb{R}^n)$ so that $\|g\|_{Q_\alpha^\beta(\mathbb{R}^n)} \le C\|L\|$ and $L$ can be written in the above form for every $f \in HH^1_{-\alpha,\beta}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$.*

## 4.3 Some Properties of $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$

**Definition 4.3.1** *For $0 < \alpha < \beta$, $\alpha + \beta \ge 1$ and $1/2 < \beta < 1$, we say that a tempered distribution $f \in Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$ if and only if*

$$\sup_{x\in\mathbb{R}^n, r\in(0,\infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |K_t^\beta * f(y)|^2 t^{-\frac{\alpha}{\beta}} dydt < \infty.$$

**Remark 4.3.2** *In Definition 4.3.1, if we take $\beta = 1$, the space $Q^{1,-1}_{\alpha,\infty}(\mathbb{R}^n)$ becomes the space $Q^{-1}_{\alpha,\infty}(\mathbb{R}^n)$ introduced by Xiao in [78].*

In the next theorem, we prove an useful characterization of $Q^{\beta,-1}_{\alpha,\infty}$. For this purpose, we need the following lemma.

**Lemma 4.3.3** *For $\alpha \in (0,1)$, $\alpha+\beta \geq 1$ and $\max\{\alpha, \frac{1}{2}\} < \beta < 1$, let $f_{j,k} = \partial_j\partial_k(-\triangle)^{-1}f (j,k = 1,2,\cdots,n)$. If $f \in Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n)$, then $f_{j,k} \in Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n)$.*

**Proof**. Take $\phi \in C_0^\infty(\mathbb{R}^n)$ with supp $(\phi) \subset B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Write $\phi_r(x) = r^{-n}\phi(\frac{x}{r})$ and define $g_r(t,x) = \phi_r * \partial_j\partial_k(-\triangle)^{-1}e^{-t(-\triangle)^\beta}f(x)$. Then

$$e^{-t(-\triangle)^\beta}f_{j,k}(x) = \partial_j\partial_k(-\triangle)^{-1}e^{-t(-\triangle)^\beta}f(x) = f_r(t,x) + g_r(t,x).$$

Since $\dot{B}^{2\beta-1}_{1,1}(\mathbb{R}^n)$ is the predual of the homogeneous Besov space $\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$ and $Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$ (see Remark 4.3.5 and Theorem 4.3.6 below), we have

$$\|g_r(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \|\phi\|_{\dot{B}^{2\beta-1}_{1,1}(\mathbb{R}^n)} \left\|\partial_j\partial_k(-\triangle)^{-1}e^{-t(-\triangle)^\beta}f\right\|_{\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)} \lesssim Cr^{1-2\beta}\|f\|_{\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)}.$$

Therefore

$$\int_0^{r^{2\beta}}\int_{|y-x|<r} |g_r(t,y)|^2t^{-\alpha/\beta}dydt \lesssim r^{n-2\alpha-2\beta+2}\|f\|^2_{\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)} \lesssim r^{n-2\alpha-2\beta+2}\|f\|^2_{Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n)}.$$

To estimate $f_r$ we take $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B(0,10) = \{x \in \mathbb{R}^n : |x| < 10\}$ and define $\varphi_{r,x} = \varphi(\frac{y-x}{r})$. Then $f_r = F_{r,x} + G_{r,x}$ with

$$G_{r,x} = \partial_j\partial_k(-\triangle)^{-1}\varphi_{r,x}e^{-t(-\triangle)^\beta}f - \phi_r * \partial_j\partial_k(-\triangle)^{-1}\varphi_{r,x}e^{-t(-\triangle)^\beta}f.$$

Using Plancherel's identity, we have

$$\begin{aligned}
&\int_0^{r^{2\beta}} \|\partial_j\partial_k(-\triangle)^{-1}\varphi_{r,x}e^{-t(-\triangle)^\beta}f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}}\\
\lesssim\ &\int_0^{r^{2\beta}} \left(\int_{\mathbb{R}^n} \left|\xi_j\xi_k|\xi|^{-2}\mathcal{F}(\varphi_{r,x}e^{-t(-\triangle)^\beta}f)(\xi)\right|^2 d\xi\right)\frac{dt}{t^{\alpha/\beta}}\\
\lesssim\ &\int_0^{r^{2\beta}} \|\mathcal{F}(\varphi_{r,x}e^{-t(-\triangle)^\beta}f)\ \|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha\beta}}\\
\lesssim\ &\int_0^{r^{2\beta}} \|\varphi_{r,x}e^{-t(-\triangle)^\beta}f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}}.
\end{aligned}$$

Similarly we can prove

$$\int_0^{r^{2\beta}} \|\phi_r * \partial_j\partial_k(-\triangle)^{-1}\varphi_{r,x}e^{-t(-\triangle)^\beta}\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^{r^{2\beta}} \|\varphi_{r,x}e^{-t(-\triangle)^\beta}f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}}.$$

Thus, we obtain

$$\int_0^{r^{2\beta}} \|G_{r,\cdot}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^{r^{2\beta}} \|\varphi_{r,x} e^{-t(-\triangle)^\beta} f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}}.$$

To bound $F_{r,x}$, noting that

$$\int_{|y-x|<r} |F_{r,x}(t,y)|^2 dy \lesssim r^{n+1} \int_{|w-x|\geq r} |e^{-t(-\triangle)^\beta} f(w)|^2 |x-w|^{-(n+1)} dw,$$

we establish

$$\begin{aligned}
&\int_0^{r^{2\beta}} \left( \int_{|y-x|<r} |F_{r,x}(t,y)|^2 dy \right) \frac{dt}{t^{\alpha/\beta}} \\
\lesssim\ & r^{n+1} \int_{|w-x|\geq r} |x-w|^{-(n+1)} \left( \int_0^{r^{2\beta}} |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \\
\lesssim\ & \sum_{k=1}^{\infty} 2^{-k(n+1)} \int_{|w-x|\leq 2^{k+1} r} \left( \int_0^{r^{2\beta}} |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \\
\lesssim\ & \sum_{k=1}^{\infty} 2^{-k(n+1)} \left( \int_0^{(2^{k+1}r)^{2\beta}} \int_{|w-x|\leq 2^{k+1} r} |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \\
\lesssim\ & r^{n-2\alpha-2\beta+2} \|f\|_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)} \sum_{k=1}^{\infty} 2^{-k(2\alpha+2\beta-1)} \\
\lesssim\ & r^{n-2\alpha-2\beta+2} \|f\|_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Now we have proved that

$$\int_0^{r^{2\beta}} \int_{|y-x|<r} |f_r(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim r^{n-2\alpha-2\beta+2} \|f\|^2_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)},$$

that is, $f_{j,k} \in Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$. □

Using Lemma 4.2, we can prove the following theorem. By this theorem, we can regard $Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n)$ as derivatives of $Q^\beta_\alpha(\mathbb{R}^n)$.

**Theorem 4.3.4** *$Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla\cdot(Q^\beta_\alpha(\mathbb{R}^n))^n$, where a tempered distribution $f \in \mathbb{R}^n$ belongs to $\nabla\cdot(Q^\beta_\alpha(\mathbb{R}^n))^n$ if and only if there are $f_j \in Q^\beta_\alpha(\mathbb{R}^n)$ such that $f = \sum_{j=1}^n \partial_j f_j$.*

**Proof**. This can be proved similarly to that of [78, (iii), Theorem 1.2].

**Remark 4.3.5** *$Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ is critical for equations (1.0.12) for $h=0$ since $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ is the derivative space of $Q^\beta_\alpha(\mathbb{R}^n)$ and $Q^\beta_\alpha(\mathbb{R}^n)$ is invariant under the scaling $f(x) \longrightarrow \lambda^{2\beta-2} f(\lambda x)$.*

In the following theorem we apply the arguments in the proof of the "minimality of $\dot{B}^0_{1,1}(\mathbb{R}^n)$" used by Frazier-Jaweth-Weiss in [26] to prove that $\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$ contains all critical spaces for equations (1.0.12) for $h=0$. The special case $\beta=1$ of this theorem was proved by Cannone in [14].

**Theorem 4.3.6** *If a translation invariant Banach space of tempered distributions $X$ is a critical space of the generalized Navier-Stokes equations (1.0.12) for $h = 0$. Then $X$ is continuously embedded in the Besov space $\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$.*

**Proof.** It follows from the assumption that $X \hookrightarrow \mathcal{S}'$ and for any $f \in X$

$$\|f(\cdot)\|_X = \|\lambda^{2\beta-1} f(\lambda \cdot -x_0)\|_X, \quad \lambda > 0, x_0 \in \mathbb{R}^n. \tag{4.3.1}$$

$X \hookrightarrow \mathcal{S}'$ implies that there exists a constant $C$ such that

$$|\langle K_1^{2\beta}, f\rangle| \le C\|f\|_X, \forall f \in X.$$

According to the transformation invariant of $X$, we have

$$\|e^{-(-\triangle)^\beta} f\|_{L^\infty(\mathbb{R}^n)} = \|K_1^{2\beta} * f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_X \quad \text{for } \forall f \in X.$$

Using the fact $\mathcal{F}f(\lambda x)(\xi) = \lambda^{-n}\mathcal{F}f(\xi/\lambda)$, the definition of $e^{-(-\triangle)^\beta} f(x)$ and the scaling property (4.3.1), we obtain that

$$\lambda^{2\beta-1}\|e^{-\lambda^{2\beta}(-\triangle)^\beta} f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_X.$$

It follows from Miao-Yuan-Zhang [53, Prorposition 2.1] that for $s < 0$, $f \in \dot{B}^s_{\infty,\infty}(\mathbb{R}^n)$ if and only if

$$\sup_{r>0} r^{-s}\|e^{-r^{2\beta}(-\triangle)^\beta} f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Thus $X \hookrightarrow \dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$. □

**Theorem 4.3.7** *Let $\alpha > 0$, $\alpha + \beta \ge 1$ and $\max\{\alpha, \frac{1}{2}\} < \beta < 1$. If $1 \le q \le \infty$, $2 < p < \infty$ and $\alpha + \beta < 1 + \frac{n}{p} < 2\beta$, then $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n)$ and $\dot{B}^{1+\frac{n}{2}-2\beta}_{2,q}(\mathbb{R}^n)$ are continuously embedded in $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$.*

**Proof.** We first prove $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n) \hookrightarrow Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. Since $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n) \subset \dot{B}^{1+\frac{n}{p}-2\beta}_{p,\infty}(\mathbb{R}^n)$. Assume that $q = \infty$, it follows form $1 + \frac{n}{p} - 2\beta < 0$ and Proposition 2.1 of [53] that for any $f \in \dot{B}^{1+\frac{n}{p}-2\beta}_{p,\infty}(\mathbb{R}^n)$,

$$\sup_{r>0} r^{-(1+\frac{n}{p}-2\beta)/2\beta}\|e^{-r(-\triangle)^\beta} f\|_{L^p(\mathbb{R}^n)} < \infty.$$

Then we have

$$\begin{aligned}
&\int_0^{r^{2\beta}}\int_{|y-x|<r} |e^{-t(-\triangle)^\beta} f(y)|^2 t^{-\alpha/\beta} dy dt \\
\lesssim\ & r^{n(p-2)/p}\int_0^{r^{2\beta}} \|e^{-t(-\triangle)^\beta} f\|^2_{L^p(\mathbb{R}^n)} t^{-\alpha/\beta} dt \\
\lesssim\ & r^{n(p-2)/p}\int_0^{r^{2\beta}} \left(\sup_{t>0} t^{-(1+\frac{n}{p}-2\beta)/2\beta}\|e^{-t(-\triangle)^\beta} f\|_{L^p(\mathbb{R}^n)}\right)^2 t^{(1+\frac{n}{p}-2\beta)/\beta} t^{-\alpha/\beta} dt \\
\lesssim\ & r^{n(p-2)/p}\int_0^{r^{2\beta}} t^{(1+\frac{n}{p}-2\beta)/\beta} t^{-\alpha/\beta} dt \\
\lesssim\ & r^{n-2(\alpha+\beta-1)}.
\end{aligned}$$

Thus $f \in Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. Now we prove $\dot{B}^{1+\frac{n}{2}-2\beta}_{2,q}(\mathbb{R}^n) \hookrightarrow Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. Since $0 < \alpha < \beta$ and $1/2 < \beta < 1$, we can find $p \in (2,\infty)$ large enough such that $\alpha + \beta < 1 + \frac{n}{p} < 2\beta$ and $1 + \frac{n}{2} - 2\beta = 1 + \frac{n}{p} - 2\beta + n\left(\frac{1}{2} - \frac{1}{p}\right)$. Then (ii) of Theorem 4.1.1 implies

$$\dot{B}^{1+\frac{n}{2}-2\beta}_{2,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n) \hookrightarrow Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n).$$

This finishes the proof. □

## 4.4 Mean Oscillation Characterization of $Q^\beta_\alpha(\mathbb{R}^n)$

The main goal of this section is to establish the following theorem.

**Theorem 4.4.1** *Let $-\infty < \alpha < \beta$ and $\beta \in (1/2, 1]$. Then $Q^\beta_\alpha(\mathbb{R}^n)$ equals the space of all measurable functions $f$ on $\mathbb{R}^n$ such that $\sup_I \Psi_{f,\alpha,\beta}(I)$ is finite, where $I$ ranges over all cubes in $\mathbb{R}^n$. Moreover, the square root of this supremum is a norm on $Q^\beta_\alpha(\mathbb{R}^n)$, equivalent to $\|f\|_{Q^\beta_\alpha(\mathbb{R}^n)}$ as defined above.*

We need recall the definition of $\Psi_{f,\alpha,\beta}(I)$ and some facts about square mean oscillation over cubes, see Essen-Janson-Peng-Xiao [25].

For any cube $I$ and an integrable function $f$ on $I$, we define

$$f(I) = \frac{1}{|I|}\int_I f(x)dx \tag{4.4.1}$$

the mean of $f$ on $I$, and

$$\Phi^q_f(I) = \frac{1}{|I|}\int_I |f(x) - f(I)|^q dx \tag{4.4.2}$$

the square mean oscillation of $f$ on $I$. Obviously, $\Phi_f(I) := \Phi^2_f(I) < \infty \Longleftrightarrow f \in L^2(I)$. Note the well-known identities

$$\frac{1}{|I|}\int_I |f(x) - a|^2 dx = \Phi_f(I) + |f(I) - a|^2 \tag{4.4.3}$$

for any complex number $a$, and

$$\frac{1}{|I|^2}\int_I\int_I |f(x) - f(y)|^2 = 2\Phi_f(I). \tag{4.4.4}$$

Moreover, if $I \subset J$, then we have

$$\Phi_f(I) \le \frac{|J|}{|I|}\Phi_f(I) \tag{4.4.5}$$

and

$$|f(I) - f(J)|^2 \le \frac{|J|}{|I|}\Phi_f(I). \tag{4.4.6}$$

Let $\mathcal{D}_0 = \mathcal{D}_0(\mathbb{R}^n)$ be the set of unit cubes whose vertices have integer coordinates, and let, for any integer $k \in \mathbb{Z}$, $\mathcal{D}_k = \mathcal{D}_k(\mathbb{R}^n) = \{2^{-k}I : I \in \mathcal{D}_0\}$, then the cubes in $\mathcal{D} = \cup_{-\infty}^{\infty}\mathcal{D}_k$ are called dyadic. Furthermore, if $I$ is any cube, we set $\mathcal{D}_k(I)$, $k \geq 0$, denote the set of the $2^{kn}$ subcubes of edge length $2^{-k}l(I)$ obtained by $k$ successive bipartitions of each edge of $I$. Moreover, put $\mathcal{D}(I) = \cup_0^{\infty}\mathcal{D}_k(I)$. For any cube $I$ and a measurable function $f$ on $I$, we define

$$\begin{aligned}\Psi_{f,\alpha,\beta}(I) &= l(I)^{4\beta-4}\sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{(2(\alpha-\beta+1)-n)k}\Phi_f(J)\\ &= (l(I))^{4\beta-4}\sum_{J\in\mathcal{D}(I)}\left(\frac{l(J)}{l(I)}\right)^{n-2(\alpha-\beta+1)}\Phi_f(J). \qquad (4.4.7)\end{aligned}$$

We first establish the following lemmas.

**Lemma 4.4.2** *Let $\alpha \in (-\infty,\beta)$ and $\beta \in (1/2,1]$. For any cube $I$ and $f \in L^2(I)$, with $J$ ranging over the $2^n$ subcubes in $\mathcal{D}_1(I)$,*

$$\Phi_f(I) = 2^{-n}\sum_{J\in\mathcal{D}_1(I)}\Phi_f(J) + 2^{-n}\sum_{J\in\mathcal{D}_1(I)}|f(J)-f(I)|^2 \qquad (4.4.8)$$

*and*

$$\Psi_{f,\alpha,\beta}(I) \approx \sum_{J\in\mathcal{D}_1(I)}\Psi_{f,\alpha,\beta}(J) + (l(I))^{4\beta-4}\sum_{J\in\mathcal{D}_1(I)}|f(J)-f(I)|^2. \qquad (4.4.9)$$

**Proof**. (4.4.8) is a consequence of (4.4.1). For (4.4.9), since $\mathcal{D}_k(I) = \cup_{J\in\mathcal{D}_1(I)}\mathcal{D}_{k-1}(J)$ for $k \geq 1$, (4.4.8) implies that

$$\begin{aligned}\Psi_{f,\alpha,\beta}(I) &= (l(I))^{4\beta-4}\Phi_f(I) + \sum_{k=1}^{\infty}\sum_{J\in\mathcal{D}_1(I)}\sum_{K\in\mathcal{D}_{k-1}(J)}(l(I))^{4\beta-4}2^{(2(\alpha-\beta+1)-n)k}\Phi_f(K)\\ &= (l(I))^{4\beta-4}\Phi_f(I) + \sum_{J\in\mathcal{D}_1(I)}2^{(2(\alpha-\beta+1)-n)}\Psi_{f,\alpha,\beta}(J)\\ &\approx \sum_{J\in\mathcal{D}_1(I)}(\Psi_{f,\alpha,\beta}(J) + (l(I))^{4\beta-4}\Phi_f(J) + (l(I))^{4\beta-4}|f(J)-f(I)|^2),\end{aligned}$$

which gives (4.4.9), since $\Psi_{f,\alpha,\beta}(J) + (l(I))^{4\beta-4}\Phi_f(J) \approx \Psi_{f,\alpha,\beta}(J)$. □

**Lemma 4.4.3** *If $\alpha < \beta - 1$ and $\beta \in (1/2,1]$ then $\Psi_{f,\alpha,\beta}(I) \approx (l(I))^{4\beta-4}\Phi_f(I)$.*

**Proof**. By Lemma 4.4.2 and induction, we have $\sum_{J\in\mathcal{D}_k(I)}2^{-nk}\Phi_f(J) \leq \Phi_f(I)$, and hence

$$(l(I))^{4\beta-4}\Phi_f(I) \leq \Psi_{f,\alpha,\beta}(I) \leq \sum_{k=0}^{\infty}2^{2(\alpha-\beta+1)k}(l(I))^{4\beta-4}\Phi_f(I).$$

**Lemma 4.4.4** *Let $\alpha \in (-\infty, \beta)$ and $\beta \in (1/2, 1]$. Then, for any cube $I$ and $f \in L^2(I)$,*

$$\Psi_{f,\alpha,\beta}(I) \le C(l(I))^{2(\alpha+\beta-2)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{2(\alpha-\beta+1)+n}} dxdy. \tag{4.4.10}$$

**Proof**. According to (4.4.5) and (4.4.2),

$$\begin{aligned}\Psi_{f,\alpha,\beta}(I) &= (l(I))^{4\beta-4} \sum_{k=0}^{\infty} \sum_{J \in \mathcal{D}_k(I)} 2^{(2(\alpha-\beta+1)-n)k} \frac{1}{2} (2^{-nk}|I|)^{-2} \int_J \int_J |f(x)-f(y)|^2 dxdy \\ &= (l(I))^{4\beta-4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_I(x,y) |f(x)-f(y)|^2 dxdy,\end{aligned}$$

where

$$g_I(x,y) = (l(I))^{4\beta-4} \frac{1}{2} \sum_{k=0}^{\infty} \sum_{J \in \mathcal{D}_k(I)} 2^{(2(\alpha-\beta+1)+n)k} |I|^{-2} 1_J(x) 1_J(y). \tag{4.4.11}$$

We divide the following proof into two cases.
*Case 1*: $\alpha > \beta - 1 - \frac{n}{2}$. Since $x, y \in J \in \mathcal{D}_k(I)$, we have $|x-y|_\infty \le l(J) = 2^{-k} l(I)$. Thus $2^k \le \frac{l(I)}{|x-y|_\infty} \le C \frac{l(I)}{|x-y|}$. Then we have

$$\begin{aligned} g_I(x,y) &\le (l(I))^{4\beta-4} \sum_{2^k \le l(I)/|x-y|_\infty} 2^{(2(\alpha-\beta+1)+n)k} |I|^{-2} \\ &\le C \left( \frac{l(I)}{|x-y|} \right)^{2(\alpha-\beta+1)+n} |I|^{-2} (l(I))^{4\beta-4} \\ &\le C (l(I))^{2(\alpha+\beta-1)-n} |x-y|^{-2(\alpha-\beta+1)-n}, \end{aligned}$$

furthermore $g_I(x,y) = 0$ unless $x, y \in I$. Thus, the desired inequality holds.
*Case 2*: $\alpha \le \beta - 1 - \frac{n}{2}$. If $x, y \in I$, then the set $\{z \in I : \min(|x-z|, |y-z|) > \frac{1}{8} l(I)\}$ has measure at least $\frac{1}{2}|I|$ and thus for $-2\alpha - n + 2\beta - 2 > 0$,

$$\int_I \min\left\{|x-z|^{-2\alpha-n+2\beta-2}, |y-z|^{-2\alpha-n+2\beta-2}\right\} dz \ge C[l(I)]^{-2\alpha-n+2\beta-2}[l(I)]^n \ge C[l(I)]^{-2\alpha+2\beta-2}.$$

Hence we can get

$$\begin{aligned} & [l(I)]^{-2n+4\beta-4} \int_I \int_I |f(x)-f(y)|^2 dxdy \\ \le\ & [l(I)]^{2\alpha-2n+2\beta-2} \int_I \int_I \int_I |f(x)-f(y)|^2 \min(|x-z|^{-2\alpha-n+2\beta-2}, |y-z|^{-2\alpha-n+2\beta-2}) dxdydz \\ \lesssim\ & (l(I))^{-n+2(\alpha+\beta-1)} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dxdy. \end{aligned}$$

This tells us

$$(l(I))^{4\beta-4} \Phi_f(I) \le C(l(I))^{2(\alpha+\beta-2)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{2(\alpha-\beta+1)+n}} dxdy.$$

Combining this fact with Lemma 4.4.3, we can finish the proof of (4.4.10). □

**Lemma 4.4.5** *Let $\alpha \in (-\infty, \beta)$ and $\beta \in (1/2, 1]$. For any cube $I$ and $f \in L^2_{loc}(\mathbb{R}^n)$,*

$$
\begin{aligned}
& (l(I))^{2(\alpha+\beta-2)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{2(\alpha-\beta+1)+n}} dxdy \\
\leq\ & \frac{C}{|I|} \int_{|t|_\infty < l(I)} \Psi_{f,\alpha,\beta}(I+t)dt + C\Psi_{f,\alpha,\beta}(I) \\
\leq\ & C \sup_{|t|_\infty < l(I)} \Psi_{f,\alpha,\beta}(I+t).
\end{aligned}
$$

**Proof**. According to the proof of Lemma 4.4.4 and Fubini's theorem,

$$
\frac{1}{|I|} \int_{|t|_\infty < l(I)} \Psi_{f,\alpha,\beta}(I+t)dt = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|I|} \int_{|t|_\infty < l(I)} g_{I+t}(x,y)dt|f(x)-f(y)|^2 dxdt.
$$

Thus it suffices to prove

$$
\frac{1}{|I|} \int_{|t|_\infty < l(I)} g_{I+t}(x,y)dt + g_I(x,y) \geq Cl(l(I))^{2(\alpha+\beta-1)-n}|x-y|^{-2(\alpha-\beta+1)-n}, \ x, y \in I. \tag{4.4.12}
$$

Suppose that $x, y \in I$ with $|x-y|_\infty \leq \frac{1}{2}l(I)$ and take $j \geq 0$ such that

$$
2^{-j-2}l(I) < |x-y|_\infty \leq 2^{-j-1}l(I).
$$

If $|t|_\infty > l(I)$, then $x \notin I+t$ and so $g_{I+t}(x,y) = 0$. Thus

$$
\begin{aligned}
& \frac{1}{|I|} \int_{|t|_\infty < l(I)} g_{I+t}(x,y)dt \\
\geq\ & \frac{1}{|I|} \int_{\mathbb{R}^n} (l(I))^{4\beta-4} \frac{1}{2} \sum_{J \in \mathcal{D}_j(I+t)} 2^{(2(\alpha-\beta+1)+n)j} |I|^{-2} 1_J(x) 1_J(y) \\
=\ & (l(I))^{4\beta-4} \frac{2^{(2(\alpha-\beta+1)+n)j}}{2|I|^3} \sum_{J \in \mathcal{D}_j(I)} \int_{\mathbb{R}^n} 1_{J+t}(x) 1_{J+t}(y)dt \\
\geq\ & C(l(I))^{2(\alpha+\beta-1)-2n}|x-y|^{-2(\alpha-\beta+1)-n} \sum_{J \in \mathcal{D}_j(I)} \int_{\mathbb{R}^n} 1_{J+t}(x) 1_{J+t}(y)dt.
\end{aligned}
$$

Note that $1_{J+t}(x)1_{J+t}(y) = 1_{J-x}(-t)1_{J-y}(-t)$. Thus $\int_{\mathbb{R}^n} 1_{J+t}(x)1_{J+t}(y)dt = |(J-x) \cap (J-y)|$, which for each $J$ is a rectangular box with edges at least $l(I) - |x-y|_\infty \geq \frac{1}{2}l(I)$, and thus volume at least $2^{-n}|J|$. Consequently, the sum over $J$ is at least $2^{-n}|I|$. Thus (4.4.12) holds for $|x-y| \leq \frac{1}{2}l(I)$. Finally, if $x, y \in I$ with $|x-y| > \frac{1}{2}l(I)$, then taking $k = 0$ in (4.4.11), we have

$$
g_I(x,y) \geq \frac{1}{2}|I|^{-2}(l(I))^{4\beta-4} \geq C(l(I))^{2(\alpha+\beta-1)-n}|x-y|^{-2(\alpha-\beta+1)-n}
$$

and so (4.4.12) holds in this case. □

**Remark 4.4.6** *The case $\beta = 1$ for Lemmas 4.4.2-4.4.5 and Theorem 4.4.1 were established by Essen-Janson-Peng-Xiao in [25]. Here we follow their ideas to prove these lemmas. Theorem 4.4.1 can be deduced from Lemmas 4.4.4 and 4.4.5.*

## 4.5 John-Nirenberg and Gagliado-Nirenberg Type Inequalities in $Q_\alpha^\beta(\mathbb{R}^n)$

This section studies John-Nirenberg type inequalities in $Q_\alpha^\beta(\mathbb{R}^n)$ by the mean oscillation characterization of $Q_\alpha^\beta(\mathbb{R}^n)$. Then, we obtain Gagliado-Nirenberg type inequalities in $Q_\alpha^\beta(\mathbb{R}^n)$ by a special John-Nirenberg type inequality in $Q_\alpha^\beta(\mathbb{R}^n)$.

### 4.5.1 John-Nirenberg Type Inequalities in $Q_\alpha^\beta(\mathbb{R}^n)$

Using the mean oscillation characterization of $Q_\alpha^\beta(\mathbb{R}^n)$ and the following two lemmas, we can obtain John-Nirenberg type inequalities.

**Lemma 4.5.1** *Assume that $\alpha < \beta - \frac{1}{2}$ and $\beta \in (1/2, 1]$. Let $I^1, \cdots, I^j$ be $j$ cubes of the same size, that is, $|I^1| = \cdots = |I^j| = V$, for some $V > 0$. If a cube $I \subset I^1 \cup \cdots \cup I^j$, with $V \leq |I| < 2^n V$, then,*

$$\Phi_f(I) \leq \sum_{i=1}^{j} \Phi_f(I^i) + \frac{2(j-1)}{j^2} \sum_{1 \leq m,n \leq j} |f(I^m) - f(I^n)|^2, \tag{4.5.1}$$

*and*

$$\Psi_{f,\alpha,\beta}(I) \leq C \left[ \sum_{i=1}^{j} \Psi_f(I^i) + (l(I))^{4\beta-4} \sum_{1 \leq m,n \leq j} |f(I^m) - f(I^n)|^2 \right]. \tag{4.5.2}$$

**Proof**. Inequality (4.5.1) was proved by Yue-Dafni in [81]. The proof of (4.5.2) follows from [23, Lemma 2.6] used a similar argument in Essen-Janson-Peng-Xiao [25, Lemma 5.6]. □

We need the Caderón-Zygmund decomposition [11].

**Lemma 4.5.2** *Assume that $f$ is a nonnegative function in $L^1(\mathbb{R}^n)$ and $\xi$ is a positive constant. There is a decomposition $\mathbb{R}^n = P \cup \Omega$, $P \cap \Omega = \emptyset$, such that*
*(a) $\Omega = \cup_{k=1}^{\infty} I_k$, where $I_k$ is a collection of cubes whose interiors are disjoint;*
*(b) $f(x) \leq \xi$ for a.e. $x \in P$;*
*(c) $\xi < \frac{1}{|I|}\int_I f(x)dx \leq 2^n \xi$, for all $I$ in the collection $\{I_k\}$.*
*(d) $\xi|\triangle| \leq \int_\triangle f(x)dx \leq 2^n \xi |\triangle|$, if $\triangle$ is any union of cubes $I$ from $\{I_k\}$.*

**Theorem 4.5.3** *Let $-\infty < \alpha < \beta$, $\beta \in (1/2, 1]$ and $0 \leq p < 2$. If there exist positive constants $B, C$ and $c$, such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,*

$$(l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{m_I(t)}{|J|} \leq B \max\left\{1, \left(\frac{C}{t}\right)^p\right\} \exp(-ct), \tag{4.5.3}$$

*then $f$ is a function in $Q_\alpha^\beta(\mathbb{R}^n)$.*

**Proof**. We apply similar arguments used by Yue-Dafni in the proof of [81, Theorem 1]. According to Theorem 4.4.1, it suffices to prove that $\Psi_{f,\alpha,\beta}(I)$ is bounded independent of $f$ or $I$. More specially, we will prove the for any $q < p$, we have

$$\Psi^q_{f,\alpha,\beta}(I) := (l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \Phi^q_f(J) \le qBK_{C,c,q,p}, \tag{4.5.4}$$

where $B, C, c$ are the constants appearing in (4.5.3), and $K_{C,c,q,p}$ is a constant depending only on $C, c, p$, and $q$. When $q = 2$, $\Psi^q_{f,\alpha,\beta}(I) = \Psi_{f,\alpha,\beta}(I)$, so this implies the theorem.

For a fixed cube $I$, and any $J \in \mathcal{D}_k(I)$, let $\int_J |f(x) - f(J)|^q dx = q\int_0^\infty t^{q-1} m_J(t)dt$. Using the Monotone Convergence Theorem and the inequality (4.5.3), we have

$$\begin{aligned}
\Psi^q_{f,\alpha,\beta}(I) &= (l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{q}{|J|}\int_0^\infty t^{q-1} m_J(t)dt \\
&= q\int_0^\infty t^{q-1}\left((l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(I)}{|J|}\right)dt \\
&\le q\int_0^\infty t^{q-1}B(1+\left(\frac{C}{t}\right)^p)e^{-ct}dt \\
&= qB\left(c^{-q}\int_0^\infty u^{-u}du + C^p c^{-(q-p)}\int_0^\infty u^{q-p-1}e^{-u}du\right) \\
&= qB(c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p))
\end{aligned}$$

where $\Gamma(y) = \int_0^\infty u^{y-1}e^{-u}du$. Since $0 \le p < q$, $\Gamma(q)$ and $\Gamma(q-p)$ are finite. Thus, we can get the desired inequality by taking $K_{C,c,p,q} = c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p)$. □

**Theorem 4.5.4** *Let $-\infty < \alpha < \beta$, $\beta \in (1/2, 1]$ and $f \in Q^\beta_\alpha(\mathbb{R}^n)$. Then there exist positive constants $B$ and $b$, such that*

$$(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|} \le B\max\left\{1, \left(\frac{\|f\|_{Q^\beta_\alpha}}{t}\right)^2\right\}\exp\left(\frac{-bt}{\|f\|_{Q^\beta_\alpha}}\right) \tag{4.5.5}$$

*holds for $t \le \|f\|_{Q^\beta_\alpha(\mathbb{R}^n)}$ and any cubes $I \subset \mathbb{R}^n$, or for $t > \|f\|_{Q^\beta_\alpha(\mathbb{R}^n)}$ and cubes $I \subset \mathbb{R}^n$ with $(l(I))^{2\beta-2} \ge 1$. Moreover, there holds*

$$(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|} \le B \tag{4.5.6}$$

*for $t > \|f\|_{Q^\beta_\alpha(\mathbb{R}^n)}$ and cubes $I \subset \mathbb{R}^n$ with $(l(I))^{2\beta-2} < 1$.*

**Proof**. Assume that $f$ is a nontrivial element of $Q^\beta_\alpha(\mathbb{R}^n)$. Then $\gamma = \sup_I(\Psi_{f,\alpha,\beta}(I))^{1/2} < \infty$. For all cubes $I$ we have

$$(l(I))^{2\beta-2}\frac{1}{|I|}\int_I |f(x) - f(I)|dx \le ((l(I))^{4\beta-4}\Phi^2_f(I))^{1/2} \le (\Psi_{f,\alpha,\beta}(I))^{1/2} \le \gamma. \tag{4.5.7}$$

For a cube $I$ and each $J \in \mathcal{D}_k(I)$, we have by the Chebyshev inequality, for $t > 0$,

$$m_J(t) \le t^{-2} \int_J |f(x) - f(J)|^2 dx.$$

Thus we get

$$(l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|} \le t^{-2}\Psi_{f,\alpha,\beta}(I) \le t^{-2}\gamma^2. \tag{4.5.8}$$

Thus, if $t \le \gamma$, then (4.5.5) holds with $B = e$ and $b = 1$.

To consider the case of $t > \gamma$, we will apply Lemma 4.5.2. In the following we fix a cube $I$. For $\xi = t(l(I))^{2-2\beta}$ with any $t > 0$, we apply the Calderón-Zygmund decomposition to $|f(x) - f(J)|$ on a subcube $J \in \mathcal{D}_k(I)$. Set $\Omega = \Omega_J(t)$, $P = J\backslash\Omega_J(t)$.

From Cauchy-Schwarz inequality and (d) of Lemma 4.5.2, we get

$$(t(l(I))^{2-2\beta})^2|\triangle| \le \int_\triangle |f(x) - f(J)|^2 dx \tag{4.5.9}$$

for any union $\triangle$ of the cubes $K$ in the decomposition of $\Omega_J(t)$. Inequality (4.5.9) with $\triangle = \Omega_J(t)$ gives us a variant of inequality (4.5.8):

$$(l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{|\Omega_J(t)|}{|J|} \le \frac{\Psi_{f,\alpha,\beta}(I)}{(t(l(I))^{2-2\beta})^2} \le \left(\frac{\gamma}{(t(l(I))^{2-2\beta})}\right)^2 \tag{4.5.10}$$

for all $t > 0$.

When $t \ge \gamma$, we can strengthen the estimate (c) in Lemma 4.5.2 as follows:

$$t(l(I))^{2-2\beta} < \frac{1}{|K|}\int_K |f(x) - f(J)|dx \le (2^n\gamma + t)(l(I))^{2-2\beta} \tag{4.5.11}$$

for all cubes $K$ in the decomposition of $\Omega_J(t)$. In fact, note that $K$ is such a cube, then $K \ne J$. Otherwise, (4.5.7) implies

$$\frac{1}{|J|}\int_J |f(x) - f(J)|dx \le \gamma(l(I))^{2-2\beta} \le t(l(I))^{2-2\beta}.$$

This contradicts (c). It follows from the proof of the Calderón-Zygmund decomposition (see, Stein [64] ) that $K$ must have a "parent" cube $K^* \subset J$ satisfying $K \in \mathcal{D}_1(K^*)$, $l(K^*) = 2l(K)$ and

$$|f(K^*) - f(J)| \le |K^*|^{-1}\int_{K^*} |f(x) - f(J)|dx \le t(l(I))^{2-2\beta}.$$

Then (4.5.7) implies

$$\begin{aligned} t(l(I))^{2-2\beta} < \frac{1}{|K|}\int_K |f(x) - f(J)|dx &\le \frac{1}{|K|}\int_K |f(x) - f(K^*)|dx + |f(K^*) - f(J)| \\ &\le \frac{2^n}{|K^*|}\int_{K^*} |f(x) - f(K^*)|dx + t(l(I))^{2-2\beta} \\ &\le (2^n\gamma + t)(l(I))^{2-2\beta}. \end{aligned}$$

There holds $\Omega_J(t') \subset \Omega_J(t)$ for $0 < t < t'$. In fact, for any cube $K \in \Omega_J(t')\backslash\Omega_J(t)$, we get $K \subset J\backslash\Omega_J(t)$. So, property (b) tells us

$$t(l(I))^{2-2\beta} \geq \frac{1}{|K|}\int_K |f(x) - f(J)|dx > t'(l(I))^{2-2\beta}.$$

This is a contradiction.

Letting $t' = t + 2^{n+1}\gamma$ for $t \geq \gamma$, we claim that

$$|\Omega_J(t')| \leq 2^{-n}|\Omega_J(t)|. \tag{4.5.12}$$

To prove this, take a cube $K$ in the decomposition for $\Omega_J(t)$. Then (4.5.11) implies that

$$\frac{1}{|K|}\int_K |f(x) - f(J)|dx \leq (2^n\gamma + t)(l(I))^{2-2\beta} < t'(l(I))^{2-2\beta}.$$

Thus, $K$ is not a cube in the decomposition of $\Omega_J(t')$, and was further subdivided. Set $\triangle' = K \cap \Omega_J(t')$. If $\triangle' \neq \emptyset$, it must be a union of cubes from the decomposition of $\Omega_J(t')$. Thus, according to (d) of Lemma 4.5.2, (4.5.7) and (4.5.11),

$$\begin{aligned}
t'(l(I))^{2-2\beta} &\leq |\triangle'|^{-1}\int_{\triangle'} |f(x) - f(J)|dx \\
&\leq |\triangle'|^{-1}\int_{\triangle'} |f(x) - f(K)|dx + |f(K) - f(J)| \\
&\leq |\triangle'|^{-1}|K|\frac{1}{|K|}\int_{\triangle'} |f(x) - f(K)|dx + \frac{1}{|K|}\int_K |f(x) - f(J)|dx \\
&\leq |\triangle'|^{-1}|K|\gamma(l(K))^{2-2\beta} + (2^n\gamma + t)(l(I))^{2-2\beta} \\
&\leq |\triangle'|^{-1}|K|\gamma(l(I))^{2-2\beta} + (2^n\gamma + t)(l(I))^{2-2\beta}
\end{aligned}$$

since $2 - 2\beta > 0$ and $K \subset I$. Replacing $t'$ by $t + 2^{n+1}\gamma$, dividing by $(l(I))^{2-2\beta}$, subtracting $t$ and dividing by $\gamma$, we have

$$(2^{n+1} - 2^n) \leq |\triangle'|^{-1}|K| \quad \text{and} \quad |K \cap \Omega_J(t')| = |\triangle'| \leq 2^{-n}|K|$$

for any cube $K$ in the decomposition of $\Omega_J(t)$. Summing over all such $K$, and noting that $\Omega_J(t') = \Omega_J(t) \cap \Omega_J(t')$, we prove (4.5.12).

For each $J \in \mathcal{D}_k(I)$, property (b) of the decomposition for $|f - f(J)|$ implies that

$$m_J(t(l(I))^{2-2\beta}) = |\{x \in J : |f(x) - f(J)| > t(l(I))^{2-2\beta}\}| \leq |\Omega_J(t)|. \tag{4.5.13}$$

For $t > \gamma$, let $j$ be the integer part of $\frac{t-\gamma}{2^{n+1}\gamma}$ and $s = (1 + j2^{n+1})\gamma$. Then $\gamma \le s \le t$. Thus one obtains from (4.5.13) that

$$\begin{aligned}
& (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \\
= \; & (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J((l(I))^{2-2\beta}t(l(I))^{2\beta-2})}{|J|} \\
\le \; & (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J((l(I))^{2-2\beta}s(l(I))^{2\beta-2})}{|J|} \\
\le \; & (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J((1+j2^{n+1})\gamma(l(I))^{2\beta-2})|}{|J|} \\
\le \; & (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2}+j2^{n+1}\gamma)|}{|J|} \\
\le \; & 2^{-n}(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2}+(j-1)2^{n+1}\gamma)|}{|J|}
\end{aligned}$$

if $(l(I))^{2\beta-2} \ge 1$, by using (4.5.12) for

$$t = ((l(I))^{2\beta-2} + (j-1)2^{n+1})\gamma \quad \text{and} \quad t' = ((l(I))^{2\beta-2} + j2^{n+1})\gamma.$$

Iterating the previous estimate $j$ times and using (4.5.10) with $t = \gamma(l(I))^{2\beta-2}$, one has

$$\begin{aligned}
& (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \\
\le \; & 2^{-nj}(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2})|}{|J|} \\
\le \; & 2^{-nj}\gamma^2\gamma^{-2} \\
\le \; & 2^{-n\left(\frac{t-\gamma}{2^{n+1}\gamma}-1\right)} \\
= \; & 2^{-\frac{n}{2^{n+1}}(t/\gamma)}2^{\frac{n}{2^{n+1}}+n}.
\end{aligned}$$

Taking $B = 2^{n/2^{n+1}+n}$ and $b = \frac{n}{2^{n+1}}\ln 2$, we get (4.5.5) when $(l(I))^{2\beta-2} \ge 1$.

If $(l(I))^{2\beta-2} < 1$, using (4.5.13) and (4.5.9), one has

$$\begin{aligned}
& (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \\
\le \; & (l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(t(l(I))^{2\beta-2})|}{|J|} \\
\le \; & \gamma^2 t^{-2} \le 1
\end{aligned}$$

which yields (4.5.6). □

### 4.5.2 Gagliardo-Nibenberg Type Inequalities in $Q_\alpha(\mathbb{R}^n)$

When $k = 0$ and $\alpha = -\frac{n}{2} + \beta - 1$, (4.5.5) implies a special JN type inequality, that is, for $f \in L^2(\mathbb{R}^n) \cap BMO^\beta(\mathbb{R}^n)$ and $t \le \|f\|_{BMO^\beta(\mathbb{R}^n)}$,

$$|\{x \in \mathbb{R}^n : |f| > t\}| \le \frac{B\|f\|^2_{L^2(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right). \tag{4.5.14}$$

When $t > \|f\|_{BMO^\beta(\mathbb{R}^n)}$, we get a weaker form of (4.5.14) from which we establish the GN type inequalities in $Q_\alpha(\mathbb{R}^n)$.

In the following, $C_{*,\cdots,*}$ denotes a constant which depends only on the quantities appearing in the subscript indexes.

**Proposition 4.5.5** *Let $\beta \in (1/2, 1]$. If $f \in BMO^\beta(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then*
*(i) (4.5.14) holds for all $t \le \|f\|_{BMO^\beta(\mathbb{R}^n)}$;*
*(ii)*

$$|\{x \in \mathbb{R}^n : f(x) > t\}| \le \frac{B\|f\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}} \tag{4.5.15}$$

*holds for all $t > \|f\|_{BMO^\beta(\mathbb{R}^n)}$.*

**Proof.** Taking $k = 0$ and $\alpha = -\frac{n}{2} + \beta - 1$ in (4.5.5), we get that

$$(l(I))^{4\beta-4}\frac{m_I(t)}{|I|} \le B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)$$

holds for $t \le \|f\|_{BMO^\beta(\mathbb{R}^n)}$ and any cube $I$. Thus for $t \le \|f\|_{BMO^\beta(\mathbb{R}^n)}$ and any cube $I$, we have

$$\begin{aligned}
&(l(I))^{4\beta-4}\frac{m_I(t)}{|I|}\int_I |f(x) - f(I)|^2 dx \\
\le\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_I |f(x) - f(I)|^2 dx \\
\le\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_I |f(x)|^2 dx \\
\le\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n} |f(x)|^2 dx.
\end{aligned}$$

This tells us

$$\begin{aligned}
&m_I(t)\frac{(l(I))^{4\beta-4}}{|I|}\int_I |f(x) - f(I)|^2 dx \\
\le\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n} |f(x)|^2 dx.
\end{aligned} \tag{4.5.16}$$

According to the definition of $BMO^{\beta}(\mathbb{R}^n)$, see Theorem 4.2.2, we have

$$f \in BMO^{\beta}(\mathbb{R}^n) \Longleftrightarrow \|f\|^2_{BMO^{\beta}(\mathbb{R}^n)} = \sup_I \frac{(l(I))^{4\beta-4}}{|I|} \int_I |f(x) - f(I)|^2 dx < \infty.$$

Thus, we get

$$m_I(t)\|f\|^2_{BMO^{\beta}(\mathbb{R}^n)} \leq B\frac{\|f\|^2_{BMO^{\beta}(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^{\beta}(\mathbb{R}^n)}}\right) \int_{\mathbb{R}^n} |f(x)|^2 dx, \tag{4.5.17}$$

for $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$. Then, taking an increasing sequence of cubes covering $\mathbb{R}^n$, we obtain

$$|\{x \in \mathbb{R}^n : f(x) > t\}| \leq \frac{B}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^{\beta}(\mathbb{R}^n)}}\right) \int_{\mathbb{R}^n} |f(x)|^2 dx \tag{4.5.18}$$

for $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$, since $f(I) \longrightarrow 0$ as $l(I) \longrightarrow \infty$. Finally, we get (4.5.14). Similarly, we can prove (4.5.15) since $\exp\left(\frac{-bt}{\|f\|_{BMO^{\beta}(\mathbb{R}^n)}}\right) \leq 1$ for $t > \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$. □

We can prove the following Gagliardo-Nibenberg type inequalities in $Q_\alpha(\mathbb{R}^n)$ from (4.5.15) or [18, Theorem 2] and [25, Theorem 2.3].

**Theorem 4.5.6** *Let $-\infty < \alpha < 1$, and $1 \leq r \leq p < \infty$. Then, one has*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_n p \|f\|^{r/p}_{L^r(\mathbb{R}^n)} \|f\|^{1-r/p}_{Q_\alpha(\mathbb{R}^n)}, \tag{4.5.19}$$

*for $f \in L^r(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n)$.*

As an application of Theorem 4.5.6, we establish the Trudinger-Moser type inequality which implies a generalized John-Nirenberg type inequality.

**Corollary 4.5.7** *(i) There exists a positive constant $\gamma_n$ such that for every $0 < \zeta < \gamma_n$*

$$\int_{\mathbb{R}^n} \Phi_p\left(\zeta\left(\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)\right) dx \leq C_{n,\zeta}\left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^p \tag{4.5.20}$$

*holds for all*

$$f \in L^p(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \text{with} \quad 1 < p < \infty \quad \text{and} \quad -\infty < \alpha < 1.$$

*Here $\Phi_p$ is the function defined by*

$$\Phi_p(t) = e^t - \sum_{j<p, j\in\mathbb{N}\cup\{0\}} \frac{t^j}{j!}, t \in \mathbb{R}.$$

*(ii) There exists a positive constant $\gamma_n$ such that*

$$|\{x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}} \frac{1}{\left(\exp\left(\frac{t\gamma}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) - 1 - \frac{\gamma t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)} \tag{4.5.21}$$

*holds for all* $t > 0$ *and*

$$f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{with} \quad -\infty < \alpha < 1,$$

*In particular, we have*

$$|\{x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}} \exp\left(-\frac{t\gamma}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) \tag{4.5.22}$$

*holds for all* $t > \|f\|_{Q_\alpha(\mathbb{R}^n)}$ *and*

$$f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{with} \quad -\infty < \alpha < 1.$$

**Proof**. (i) According to Theorem 4.5.6, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi_{p,r}\left(\zeta \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) dx &= \int_{\mathbb{R}^n} \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \left(\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^j dx \\
&\leq \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \frac{\|f\|^j_{L^j(\mathbb{R}^n)}}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \\
&\leq \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \frac{\left(Cj\|f\|^{p/j}_{L^p(\mathbb{R}^n)}\|f\|^{1-p/j}_{Q_\alpha(\mathbb{R}^n)}\right)^j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \\
&\leq \sum_{j \geq p, j \in \mathbb{N}} a_j (\zeta C)^j \left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^p
\end{aligned}$$

with $a_j = \frac{j^j}{j!}$. Since $\lim_{j \longrightarrow \infty} \frac{a_j}{a_{j+1}} = e^{-1}$, the power series of the above right hand side converges provided $\zeta C < e^{-1}$ i.e. $\zeta < \gamma := (Ce)^{-1}$.
(ii) According to (i) with $p = 2$, we have

$$\int_{\mathbb{R}^n} \left(\exp\left(\gamma \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) - 1 - \gamma \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) dx \leq C \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}}.$$

On the other hand, since the distribution function $m(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$ is non-increasing, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\exp\left(\gamma\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)-1-\gamma\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)dx\\
=&\sum_{j=2}^{\infty}\frac{\gamma^j}{j!}\frac{\|f\|^j_{L^j(\mathbb{R}^n)}}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}}\\
=&\sum_{j=2}^{\infty}\frac{\gamma^j}{j!}\frac{j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}}\int_0^\infty m(s)s^{j-1}ds\\
\geq&\ m(t)\sum_{j=2}^{\infty}\frac{\gamma^j}{j!}\frac{j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}}\int_0^t s^{j-1}ds\\
=&\ m(t)\sum_{j=2}^{\infty}\frac{1}{j!}\left(\frac{\gamma t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^j\\
=&\ m(t)\left(\exp\left(\frac{\gamma t}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}}\right)-1-\frac{\gamma t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)
\end{aligned}$$

for all $t > 0$. Thus, we have

$$m(t)\leq C\frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}}\frac{1}{\left(\exp\left(\frac{\gamma t}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}}\right)-1-\frac{\gamma t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)}.$$

□

We can also deduce from Theorem 4.5.6 that the following Brezis-Gallouet-Wainger type inequalities hold.

**Theorem 4.5.8** *For every* $1 < q < \infty$ *and* $n/q < s < \infty$, *we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)}\leq C_{n,p,q,s}\left(1+(\|f\|_{L^p(\mathbb{R}^n)}+\|f\|_{Q_\alpha(\mathbb{R}^n)})\log(e+\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)})\right)\quad(4.5.23)$$

*holds for all* $(-\triangle)^{s/2}f\in L^q(\mathbb{R}^n)$ *satisfying*

$$f\in L^p(\mathbb{R}^n)\cap Q_\alpha(\mathbb{R}^n)\quad when\quad 1\leq p<\infty\quad and\quad -\infty<\alpha<1.$$

**Proof.** For any $g(x)$ in the Schwartz class of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$, define

$$v_g(t,x)=e^{-(\triangle)^{s/2}}g(x).$$

be the solution of fractional heat equation

$$\partial_t v(t,x)+(-\triangle)^{s/2}v(t,x)=0$$

with initial data $g$. Fix $f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n)$ with $(-\triangle)^{s/2} f \in L^q$. Then

$$\begin{aligned}\int_0^t \langle -(-\triangle)^{s/2} f(x), v(s,x)\rangle ds &= \int_0^t \langle f(x), -(-\triangle)^{s/2} v(s,x)\rangle ds \\ &= \int_0^t \langle f(x), \partial_s v(s,x)\rangle dt \\ &= \langle f(x), v(t,x)\rangle - \langle f(x), g(x)\rangle.\end{aligned}$$

Thus

$$|\langle f, g\rangle| \le |\langle f(x), v(t,x)\rangle| + \int_0^t |\langle (-\triangle)^{s/2} f(x), v(s,x)\rangle| ds = I_1 + I_2$$

for all $t > 0$. Here $\langle\cdot,\cdot\rangle$ denote the inner-product in $L^2$. Thus Hölder inequality, Lemma 3.2.2 and Theorem 4.5.6 imply that

$$\begin{aligned}I_1 &\le \|f\|_{L^{q_1}(\mathbb{R}^n)} \|v(t,\cdot)\|_{L^{q_1'}(\mathbb{R}^n)} = \|f\|_{L^{q_1}(\mathbb{R}^n)} \|e^{-t(-\triangle)^{s/2}} g\|_{L^{q_1'}(\mathbb{R}^n)} \\ &\le C q_1 t^{-\frac{n}{sq_1}} (\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha(\mathbb{R}^n)}) \|g\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

for all $t > 0$ and $p \le q_1 < \infty$. Similarly, we have

$$\begin{aligned}I_2 &\le \int_0^t \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|v(s,\cdot)\|_{L^{q'}(\mathbb{R}^n)} ds \\ &= \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \int_0^t \|e^{-t(-\triangle)^{s/2}} g\|_{L^{q'}(\mathbb{R}^n)} ds \\ &\le C \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \int_0^t s^{-\frac{n}{sq}} ds \\ &\le C t^{1-\frac{n}{sq}} \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

for all $t > 0$. Combing the duality argument and these two estimates about $I_1$ and $I_2$, we have

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^n)} &= \sup_{\|g\|_{L^1(\mathbb{R}^n)} \le 1, g \in \mathcal{S}} |\langle f, g\rangle| \\ &\le C \left( q_1 t^{-\frac{n}{sq_1}} (\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha(\mathbb{R}^n)}) + t^{1-\frac{n}{sq}} \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \right)\end{aligned}$$

for all $t > 0$ and $p \le q_1 < \infty$. Take

$$q_1 = \log(1/t),\ t = \left( e^p + \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)}^{(1-\frac{n}{sq})^{-1}} \right)^{-1}.$$

Then $t^{-n/(sq_1)} = (t^{1/\log t})^{n/s} = e^{n/s}$ and

$$t^{1-\frac{n}{sq}} \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} = \left( e^p + \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)}^{(1-\frac{n}{sq})^{-1}} \right)^{-(1-\frac{n}{sq})} \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \le 1.$$

Note that we can find constant $C$ such that $q_1 \le C \log\left(e + \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)}\right)$. Therefore, (4.5.23) holds. □

# Chapter 5

# Well-Posedness and Regularity of Fractional Navier-Stokes Equations

In this chapter, we study the well-posedness and regularity of fractional Navier-Stokes equations in some Lebsgue spaces or critical $Q-$spaces.

## 5.1 Well-Posedness and Regularity of Fractional Navier-Stokes Equations in Some Lebesgue Spaces

In this section, we use the notation $L^p$ indiscriminately for scalar and vector valued functions.

**Proposition 5.1.1** *Let $\beta > 1/2$ and $T > 0$. Assume that $u, v \in L^q([0,T]; L^p(\mathbb{R}^n))$ with $p, q$ satisfying*

$$\max\left\{\frac{n}{2\beta - 1}, 2\right\} < p < \infty, \quad 2\beta - 1 = \frac{2\beta}{q} + \frac{n}{p}.$$

*Then the operator*

$$B(u, v) = \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (u \otimes v) ds$$

*is bounded from $L^q([0,T]; L^p(\mathbb{R}^n)) \times L^q([0,T]; L^p(\mathbb{R}^n))$ to $L^q([0,T]; L^p(\mathbb{R}^n))$ with*

$$\|B(u, v)\|_{L^q([0,T];L^p(\mathbb{R}^n))} \lesssim \|u\|_{L^q([0,T];L^p(\mathbb{R}^n))} \|v\|_{L^q([0,T];L^p(\mathbb{R}^n))}.$$

**Proof.** By Lemma 3.2.2 and $L^p$–boundness of Riesz transform, we have

$$\begin{aligned}\|B(u,v)\|_{L^p(\mathbb{R}^n)} &\lesssim \int_0^t \|\nabla e^{-(t-s)(-\triangle)^\beta} P(u(s,\cdot)\otimes v(s,\cdot))\|_{L^p(\mathbb{R}^n)}ds \\ &\lesssim \int_0^t \frac{1}{|t-s|^{\frac{1}{2\beta}+\frac{n}{2\beta}\left(\frac{2}{p}-\frac{1}{p}\right)}}\|(u(s,\cdot)\otimes v(s,\cdot))\|_{L^{p/2}(\mathbb{R}^n)}ds \\ &\lesssim \int_0^t \frac{1}{|t-s|^{\frac{1}{2\beta}+\frac{n}{2p\beta}}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\|v(s,\cdot)\|_{L^p(\mathbb{R}^n)}ds.\end{aligned}$$

Since $\beta > \frac{1}{2}$ and $p > \frac{n}{2\beta-1}$,

$$0 < \frac{1}{2\beta} + \frac{n}{2p\beta} < 1.$$

It follows from $2\beta - 1 = \frac{2\beta}{q} + \frac{n}{p}$ and the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}\|B(u,v)\|_{L^q([0,T];L^p(\mathbb{R}^n))} &\lesssim \left\|(\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\|v(s,\cdot)\|_{L^p(\mathbb{R}^n)})\right\|_{L^{q/2}([0,T])} \\ &\lesssim \|u\|_{L^q([0,T];L^p(\mathbb{R}^n))}\|v\|_{L^q([0,T];L^p(\mathbb{R}^n))}.\end{aligned}$$

We can obtain the following estimate from Lemma 3.2.2.

**Lemma 5.1.2** *Let* $1/2 < \beta \le 1, T > 0$, *and* $p, q$ *satisfy*

$$p > \frac{n}{2\beta - 1}, \quad 2\beta - 1 = \frac{2\beta}{q} + \frac{n}{p}.$$

*Assume that* $f \in L^r(\mathbb{R}^n)$ *with* $\frac{n}{2\beta-1} < r \le p$. *Then we have*

$$\|e^{-t(-\triangle)^\beta} f\|_{L^q([0,T];L^p(\mathbb{R}^n))} \lesssim T^{1-\frac{n}{2\beta}\left(\frac{1}{n}+\frac{1}{r}\right)}\|f\|_{L^r(\mathbb{R}^n)}.$$

Applying Theorems 3.2.7 & 3.3.6, Proposition 5.1.1 and Lemma 5.1.2, we obtain the global existence and uniqueness of solutions for system (1.0.12).

**Theorem 5.1.3** *Let* $\beta \in (1/2, 1]$, $0 < T < \infty$, $p > \frac{n}{2\beta-1}$ *and* $\frac{n}{p} + \frac{2\beta}{q} = 2\beta - 1$.
*(a) Assume that* $\frac{n}{2\beta-1} < r \le p$, $1 \le p_1' < p < \infty$, $1 \le q_1' < q \le \infty$ *and*

$$0 < \frac{n}{2\beta}\left(\frac{1}{q} + \frac{1}{q_1}\right)\left(1 - \frac{1}{p} - \frac{1}{p_1}\right) < 1.$$

*If there exists a constant* $C > 0$ *such that*

$$T^{1-\frac{n}{2\beta}\left(\frac{1}{n}+\frac{1}{r}\right)}\|g\|_{L^r(\mathbb{R}^n)} + T^{\frac{1}{q}+\frac{1}{q_1}-\frac{n}{2\beta}\left(\frac{1}{p_1'}-\frac{1}{p}\right)}\|h\|_{L_t^{q_1'}([0,T];L_x^{p_1'}(\mathbb{R}^n))} \le C \tag{5.1.1}$$

*holds for all* $g \in L^r(\mathbb{R}^n)$ *with* $\nabla \cdot g = 0$ *and* $h \in L_t^{q_1'}([0,T]; L_x^{p_1'}(\mathbb{R}^n))$, *then (1.0.12) has a unique strong solution* $v \in L_t^q([0,T]; L_x^p(\mathbb{R}^n))$ *in the sense of*

$$v = e^{-t(-\triangle)^\beta} g(x) + \int_0^t e^{-(t-s)(-\triangle)^\beta} P[h(s,x) - \nabla \cdot (v \otimes v)(s,x)]ds,$$

*(b) Assume that* $g \in L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)$ *and* $h \in L_t^{q_1'}([0,\infty); L_x^{p_1'}(\mathbb{R}^n))$ *with* $q_1'$ *and* $p_1'$ *satisfying* $1 < q_1' < q < \infty$,

$$1 \le p_1' < p < \begin{cases} \frac{n^2}{(n-2\beta)(2\beta-1)}, & 2\beta < n, \\ \infty, & 2\beta \ge n, \end{cases} \quad \text{and } \frac{n}{p_1} + \frac{2\beta}{q_1} = 2\beta - 1.$$

*If* $\|g\|_{L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)} + \|h\|_{L_t^{q_1'}([0,\infty); L_x^{p_1'}(\mathbb{R}^n))}$ *is small enough and* $\nabla \cdot g = 0$, *then (1.0.12) has a unique strong solution* $v \in L_t^q([0,\infty); L_x^p(\mathbb{R}^n))$.

**Proof.** (a) Under the assumption of (a), let $X = L^q([0,\varepsilon]; L^p(\mathbb{R}^n))$ for some positive $\varepsilon$ which is to be determined later. Define

$$Tv = e^{-t(-\triangle)^\beta} g + \int_0^t e^{-(t-s)(-\triangle)^\beta} P(h - \nabla \cdot (v \otimes v)(s,x)ds. \tag{5.1.2}$$

We will prove that if $\varepsilon$ is small enough then $T$ is a contraction operator on the ball $B_R$ in $X$ with radius $R$ which is to be determined later. For any $v^1$, $v^2 \in B_R$, we have

$$\begin{aligned} \|T(v_1) - T(v_2)\|_X &= \left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (v_1 \otimes v_1)ds - \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (v_2 \otimes v_2)ds \right\|_X \\ &= \|B(v_1 - v_2, v_1) - B(v_2, v_1 - v_2)\|_X \\ &\le \|B(v_1 - v_2, v_1)\|_X + \|B(v_2, v_1 - v_2)\|_X, \end{aligned}$$

where

$$B(u,v) = \int_0^t (e^{-(t-s)(-\triangle)^\beta}) P\nabla \cdot (u \otimes v)(s)ds.$$

It follows from Proposition 5.1.1 that $B$ is bounded on $X$. Thus

$$\|T(v_1) - T(v_2)\|_X \le C\|v_1 - v_2\|_X \|v_1\|_X + C\|v_2\|_X \|v_1 - v_2\|_X,$$

where $C > 0$ is only dependent on $\beta, p$ and $q$. Thus

$$\|T(v_1) - T(v_2)\|_X \le C(\|v_1\|_X + \|v_2\|_X)\|v_1 - v_2\|_X \le CR\|v_1 - v_2\|_X.$$

To estimate $\|Tv\|_X$ for $v \in B_R$, we use

$$T(0) = e^{-t(-\triangle)^\beta} g + \int_0^t e^{-(t-s)(-\triangle)^\beta} Ph(s,x)ds$$

to obtain

$$\|T(0)\|_X \le Ca := C\varepsilon^{1-\frac{n}{2\beta}\left(\frac{1}{n}+\frac{1}{r}\right)}\|g\|_{L^r(\mathbb{R}^n)} + C\varepsilon^{\frac{1}{q}+\frac{1}{q_1}-\frac{n}{2\beta}\left(\frac{1}{p_1'}-\frac{1}{p}\right)}\|h\|_{L^{q_1'}([0,\varepsilon];L^{p_1'}(\mathbb{R}^n))}$$

according to Theorem 3.3.6 and Lemma5.1.2. Consequently,

$$\|T(v)\|_X = \|T(v) - T(0) + T(0)\|_X \le \|T(v-0)\|_X + \|T(0)\|_X \le CR\|v\|_X + Ca.$$

Letting $R = 2Ca$ and letting $\varepsilon$ be small enough with

$$CR \le \frac{1}{2}, \tag{5.1.3}$$

we have

$$\|T(v^1) - T(v^2)\|_X \le \frac{1}{2}\|v^1 - v^2\|_X \text{ and } \|T(v)\|_X \le R.$$

It follows from the Banach contraction mapping principle that there exists a unique $v \in X = L_t^q([0,\varepsilon]; L_x^p(\mathbb{R}^n))$. According to assumption (5.1.1), we can apply the above arguments on any interval $[t_1, t_2]$ on which (5.1.3) holds and prove (a) inductively.
(b) Note that $\frac{n}{p} + \frac{2\beta}{q} = 2\beta - 1$ implies that $(q, p, \frac{n}{2\beta-1})$ is $\frac{n}{2\beta}$–admissible. By Lemma 3.2.3, we get

$$\|e^{-t(-\triangle)^\beta} g\|_{L_t^q([0,\infty);L_x^p(\mathbb{R}^n))} \lesssim \|g\|_{L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)}.$$

On the other hand, Theorem 3.2.7 implies

$$\left\| \int_0^t e^{-(t-s)(-\triangle)^\beta} h(s,x) ds \right\|_{L_t^q([0,\infty);L_x^p(\mathbb{R}^n))} \lesssim \|h\|_{L_t^{q_1'}([0,\infty);L_x^{p_1'}(\mathbb{R}^n))}.$$

Applying Proposition 5.1.1 for $T = \infty$ and fixed point arguments, we can prove (b). □

We show that the solution established in Theorem 5.1.3 is smooth in spatial variables. For a non-negative multi-index $k = (k_1, \cdots, k_n)$ we define

$$D^k = \left(\frac{\partial}{\partial_{x_1}}\right)^{k_1} \cdots \left(\frac{\partial}{\partial_{x_n}}\right)^{k_n}$$

and $|k| = k_1 + \cdots + k_n$.

**Proposition 5.1.4** *Under the hypothesis of Theorem 5.1.3 we assume further that for a non-negative multi-index k*

$$D^k g \in L^r(\mathbb{R}^n) \text{ and } D^k h \in L^{q_1'}([0,T]; L^{p_1'}(\mathbb{R}^n)).$$

*Then the solution v established in Theorem 5.1.3 satisfies for any non-negative multi-index j with* $|j| \le |k|$,

$$D^j v \in L^q([0,T]; L^p(\mathbb{R}^n)). \tag{5.1.4}$$

**Proof**. The proof is similar to that of Theorem 5.1.3. We only demonstrate the case $|j| = 1$, since similar arguments apply to the cases $|j| = 2, 3, \cdots, |k|$. Define

$$\overline{T}(Dv) = e^{-t(-\triangle)^{\beta}}(Dg) + \int_0^t e^{-(t-s)(-\triangle)^{\beta}} P(Dh) - B(Dv, v) - B(v, Dv). \tag{5.1.5}$$

Consider the integral equation $Dv = \overline{T}(Dv)$. Then $\overline{T}$ is a mapping of the space $X$ of function $v$ with

$$v \in L^q([0,T]; L^p(\mathbb{R}^n)) \text{ and } Dv \in L^q([0,T]; L^p(\mathbb{R}^n)).$$

The norm in $X$ is defined by

$$\|v\|_X = \|v\|_{L^q([0,T];L^p(\mathbb{R}^n))} + \|Dv\|_{L^q([0,T];L^p(\mathbb{R}^n))}.$$

The assumption on $Dg$ and $Dh$ implies that the first two terms in the right hand side of (5.1.5) are bounded in $X$. The boundness of the other terms follows from Proposition 5.1.1. Thus, like $T$ in the proof of Theorem 5.1.3, $\overline{T}$ is a contraction mapping of $X$ into itself. Thus it has a unique fixed point in $X$. Therefore, the solution $v$ established in Theorem 5.1.3 satisfies $Dv \in L^q([0,T]; L^p(\mathbb{R}^n))$. □

## 5.2 Well-Posedness of Fractional Navier-Stokes Equations in $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$

In $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla\cdot(Q^{\beta}_{\alpha})^n(\mathbb{R}^n)$, we study the well-posedness of the generalized Navier-Stokes equations on the half-space $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$, $n \geq 2$ :

$$\begin{cases} \partial_t u + (-\triangle)^{\beta} u + (u\cdot\nabla)u - \nabla p = 0, & (t,x)\in\mathbb{R}^{1+n}_+; \\ \nabla\cdot u = 0, & (t,x)\in\mathbb{R}^{1+n}_+; \\ u|_{t=0} = a, & x\in\mathbb{R}^n \end{cases} \tag{5.2.1}$$

with $\beta \in (1/2, 1)$.

### 5.2.1 Several Technical Lemmas

We prove several technical lemmas used in the proof of our well-posedness result.

**Lemma 5.2.1** *Given $\alpha \in (0,1)$. For a fixed $T \in (0,\infty]$ and a function $f(\cdot,\cdot)$ on $\mathbb{R}^{1+n}_+$, let $A(t) = \int_0^t e^{-(t-s)(-\triangle)^{\beta}}(-\triangle)^{\beta} f(s,x)ds$. Then*

$$\int_0^T \|A(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^T \|f(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}}. \tag{5.2.2}$$

**Proof.** According to the definition of $e^{-(t-s)(-\triangle)^\beta}$, by Fubini's and Plancherel's theorem, we have

$$\begin{aligned}
I_A &= \int_0^\infty \|A(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&= \int_0^\infty \| \int_0^t |\xi|^{2\beta} e^{-(t-s)|\xi|^{2\beta}} \mathcal{F}f(s,\xi) d\xi \|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \int_0^t \frac{|\xi|^{2\beta}}{\exp{(t-s)|\xi|^{2\beta}}} |\mathcal{F}f(s,\xi)| ds \right)^2 d\xi \right) \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^\infty 1_{\{0\le s\le t\}} \frac{|\xi|^{2\beta}}{\exp{(t-s)|\xi|^{2\beta}}} |\mathcal{F}f(s,\xi)| ds \right)^2 \frac{dt}{t^{\alpha/\beta}} \right) d\xi \\
&\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^\infty 1_{\{0\le s\le t\}} \frac{|\xi|^{2\beta}}{e^{(t-s)|\xi|^{2\beta}}} ds \right) \left( \int_0^t \frac{|\xi|^{2\beta}}{e^{(t-s)|\xi|^{2\beta}}} |\mathcal{F}f(s,\xi)|^2 ds \right) ds \frac{dt}{t^{\alpha/\beta}} \right) d\xi.
\end{aligned}$$

Since $\int_0^t |\xi|^{2\beta} e^{-(t-s)|\xi|^{2\beta}} ds \le e^{-t|\xi|^{2\beta}} (e^{t|\xi|^{2\beta}} - 1) \le 1$, we have

$$\begin{aligned}
I_A &\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^t \frac{|\xi|^{2\beta}}{\exp{(t-s)|\xi|^{2\beta}}} |\mathcal{F}f(s,\xi)|^2 ds \right) ds \frac{dt}{t^{\alpha/\beta}} \right) d\xi \\
&\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty |\mathcal{F}f(s,\xi)|^2 e^{s|\xi|^{2\beta}} \left( \int_s^\infty \frac{|\xi|^{2\beta}}{\exp t|\xi|^{2\beta}} \right) dt \frac{ds}{s^{\alpha/\beta}} \right) d\xi \\
&\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty |\mathcal{F}f(s,\xi)|^2 e^{s|\xi|^{2\beta}} (-e^{-t|\xi|^{2\beta}}|_s^\infty) \frac{ds}{s^{\alpha/\beta}} \right) d\xi \\
&\lesssim \int_0^\infty ||f(t,\cdot)||^2_{L^2(\mathbb{R}^n)} \frac{dt}{s^{\alpha/\beta}}.
\end{aligned}$$

This finishes the proof. □

**Lemma 5.2.2** *For $\beta \in (1/2,1)$ and $N(t,x)$ defined on $(0,1)\times\mathbb{R}^n$, let $A(N)$ be the quantity*

$$A(\alpha,\beta,N) = \sup_{x\in\mathbb{R}^n, r\in(0,1)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |f(t,x)| \frac{dxdt}{t^{\alpha/\beta}}.$$

*Then for each $k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ there exists a constant $b(k)$ such that the following inequality holds:*

$$\int_0^1 \left\| t^{\frac{k}{2}} (-\triangle)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\triangle)^\beta} \int_0^t N(s,\cdot) ds \right\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \le b(k) A(\alpha,\beta,N) \int_0^1 \int_{\mathbb{R}^n} |N(s,x)| \frac{dxds}{s^{\alpha/\beta}}. \tag{5.2.3}$$

**Proof**. Using the inner-product $\langle\cdot,\cdot\rangle$ in $L^2$ with respect to the spatial variable $x\in\mathbb{R}^n$, we obtain

$$I = \int_0^1 \left\| t^{\frac{k}{2}}(-\triangle)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\triangle)^\beta} \int_0^t N(s,\cdot)ds \right\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}}$$

$$= \int_0^1 \langle \int_0^t t^{\frac{k}{2}}(-\triangle)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\triangle)^\beta} N(s,\cdot)ds, \int_0^t t^{\frac{k}{2}}(-\triangle)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\triangle)^\beta} N(h,\cdot)dh\rangle_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}}$$

$$= 2\mathcal{R}e\left( \int\int_{0<h<s<1} \langle N(s,\cdot), \int_s^1 t^k(-\triangle)^{k\beta+1} e^{-t(-\triangle)^\beta} N(h,\cdot)dt\rangle_{L^2(\mathbb{R}^n)} \frac{dhds}{s^{\alpha/\beta}}\right)$$

$$= 2\mathcal{R}e\left( \int\int_{0<h<s<1} \langle N(s,\cdot), (-\triangle)^{1-\beta} \int_s^1 (t(-\triangle)^\beta)^k e^{-t(-\triangle)^\beta} N(h,\cdot)d(t(-\triangle^\beta))\rangle_{L^2(\mathbb{R}^n)} \frac{dhds}{s^{\alpha/\beta}}\right)$$

$$\le \int_0^1 \left| \langle N(s,\cdot), (-\triangle)^{1-\beta} \int_0^s (L_k(1) - L_k(s))N(h,\cdot)dh\rangle_{L^2(\mathbb{R}^n)}\right| \frac{ds}{s^{\alpha/\beta}}$$

where $L_k(t) = \sum_{m=0}^k b_m(k)t^m(-\triangle)^{m\beta}e^{-t(-\triangle)^\beta}$.

We consider the $\nu-$th derivative of the kernel $K_t^\beta(x)$ and let

$$(K_1^\beta)^\nu(x) = (-\triangle)^{\nu/2}K_1^\beta(x) \quad \text{and} \quad (K_t^\beta)^\nu(x) = (-\triangle)^{\nu/2}K_t^\beta(x).$$

Using the estimates

$$(K_1^\beta)^\nu(x) \lesssim \frac{1}{(1+|x|)^{n+\nu}} \quad \text{and} \quad (K_t^\beta)^\nu(x) = t^{-\frac{\nu}{2\beta}}t^{-\frac{n}{2\beta}}(K_1^\beta)^\nu\left(\frac{x}{t^{1/2\beta}}\right)$$

(see Miao-Yuan-Zhang[53, Lemma 2.2 and Remark 2.1]), we get the kernel of the above operator satisfies the estimate:

$$\begin{aligned}(-\triangle)^{1-\beta}L_k(t)(x,y) &\lesssim \sum_{m=0}^k t^{m-\frac{2m\beta+n+2-2\beta}{2\beta}} \frac{b_m(k)}{(1+t^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}} \\ &\lesssim t^{-\frac{n+2-2\beta}{2\beta}} \sum_{m=0}^k \frac{b_m(k)}{(1+t^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}},\end{aligned}$$

we have

$$\begin{aligned}&\left|\int_0^s (-\triangle)^{1-\beta}L_k(s)N(h,x)dh\right| \\ \lesssim\ & s^{-\frac{n+2-2\beta}{2\beta}} \int_0^s\int_{\mathbb{R}^n} \sum_{m=0}^k b_m(k) \frac{|N(h,y)|dydh}{(1+s^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}} \\ \lesssim\ & s^{-\frac{n+2-2\beta}{2\beta}} \sum_{m=0}^k b_m(k) \sum_{k\in\mathbb{Z}^n} \int_0^s \int_{\frac{x-y}{t^{1/2\beta}}\in k+[0,1]^n} \frac{|N(h,y)|dydh}{(1+s^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}} \\ \lesssim\ & b(k) \sup_{x\in R^n}\sup_{0<t<1} t^{-\frac{n+2-2\beta}{2\beta}} \int_0^t \int_{|x-y|<t^{1/2\beta}} |N(h,y)|dydh \\ \lesssim\ & b(k) \sup_{x\in R^n}\sup_{0<\rho<1} \rho^{2\alpha-n+2\beta-2} \int_0^{\rho^{2\beta}} \int_{|x-y|<\rho} |N(h,y)|\frac{dydh}{h^{\alpha/\beta}}.\end{aligned}$$

Hence we can get

$$I \lesssim b(k)\left(\int_0^1\int_{\mathbb{R}^n}|N(s,x)|\frac{dsdx}{s^{\alpha/\beta}}\right)A(\alpha,\beta,N).$$

This completes the proof. □

**Remark 5.2.3** *Similarly when $k=0$, we can prove the following inequality:*

$$\int_0^1\left\|(-\triangle)^{\frac{1}{2}}e^{-t(-\triangle)^\beta}\int_0^t N(s,\cdot)ds\right\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}}\lesssim A(\alpha,\beta,N)\int_0^1\int_{\mathbb{R}^n}|N(s,x)|\frac{dxds}{s^{\alpha/\beta}}. \quad (5.2.4)$$

**Lemma 5.2.4** *For $1\le j,k\le n$ and $t>0$, the operator $Q^\beta_{j,k,t}=\frac{1}{\triangle}\partial_j\partial_k e^{-t(-\triangle)^\beta}$ is a convolution operator with the kernel $K^\beta_{j,k,t}(x)=\frac{1}{t^{n/2\beta}}K^\beta_{j,k}(\frac{x}{t^{1/2\beta}})$ for a smooth function $K^\beta_{j,k}$ such that for all $\alpha\in\mathbb{N}^n$*

$$(1+|x|)^{n+|\alpha|}\partial^\alpha K^\beta_{j,k}\in L^\infty(\mathbb{R}^n).$$

**Proof.** Since $\mathcal{F}K^\beta_{j,k}(\xi)=\frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^{2\beta}}$, we have $\mathcal{F}\partial^\alpha K^\beta_{j,k}(\xi)\lesssim|\xi|^{|\alpha|}\frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^{2\beta}}$ and

$$\int\mathcal{F}\partial^\alpha K^\beta_{j,k}(\xi)d\xi<\infty.$$

Thus $\partial^\alpha K^\beta_{j,k}(x)\in L^\infty(\mathbb{R}^n)$.

For $|x|\le 1$, we have

$$|(1+|x|)^{n+|\alpha|}\partial^\alpha K^\beta_{j,k}(x)|\lesssim|\partial^\alpha K_{j,k}(x)|\lesssim 1.$$

For $|x|>1$, we write $K^\beta_{j,k}=(I-S_0)K^\beta_{j,k}+\sum_{l<0}\Delta_l K^\beta_{j,k}$ where $(I-S_0)K^\beta_{j,k}\in\mathcal{S}$ and $\Delta_l K^\beta_{j,k}=2^{ln}\omega^\beta_{j,k,l}(2^l x)$ with $\mathcal{F}\omega^\beta_{j,k,l}=\psi(\xi)\frac{\xi_j\xi_k}{|\xi|^2}e^{-|2^l\xi|^{2\beta}}\in L^1$. Then the set $\{\omega^\beta_{j,k,l}:l<0\}$ is bounded in $\mathcal{S}$ and there exists an uniform constant $C_N$ such that

$$(1+2^l|x|)^N 2^{l(n+|\alpha|)}|\partial^\alpha\Delta_l K^\beta_{j,k}(x)|\le C_N.$$

Thus

$$|\partial^\alpha S_0 K_{j,k}(x)|\lesssim\sum_{2^l|x|\le 1}2^{l(n+|\alpha|)}+\sum_{2^l|x|>1}2^{l(n+|\alpha|-N)}|x|^{-N}\lesssim|x|^{-n-|\alpha|}.$$

This finishes the proof. □

### 5.2.2 Well-Posedness

In this subsection, we establish the well-posedness result for the solutions to the equations (5.2.1). Throughout this subsection, we always assume $\beta\in(\frac{1}{2},1)$. Our results can be regarded as a generalization of the result of Koch-Tataru [41] when $\alpha=0,\beta=1$ and that of Xiao [78] when $\alpha\in(0,1),\beta=1$.

**Definition 5.2.5** *Let $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{1/2, \alpha\} < \beta < 1$.*
*(i) A tempered distribution $f$ on $\mathbb{R}^n$ belongs to $Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)$ provided*

$$\|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r^{2\beta}\in(0,T)} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |K^\beta_t * f(y)|^2 t^{-\frac{\alpha}{\beta}} dydt \right)^{1/2} < \infty;$$

*(ii) A tempered distribution $f$ on $\mathbb{R}^n$ belongs to $\overline{VQ^{\beta,-1}_\alpha}(\mathbb{R}^n)$ provided $\lim_{T\longrightarrow 0} \|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)} = 0$;*
*(iii) A function $g$ on $\mathbb{R}^{1+n}_+$ belongs to the space $X^\beta_{\alpha;T}(\mathbb{R}^n)$ provided*

$$\begin{aligned}\|g\|_{X^\beta_{\alpha;T}(\mathbb{R}^n)} &= \sup_{t\in(0,T)} t^{1-\frac{1}{2\beta}} \|g(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{x\in\mathbb{R}^n, r^{2\beta}\in(0,T)} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |g(t,y)|^2 t^{-\alpha/\beta} dydt \right)^{1/2} < \infty.\end{aligned}$$

**Theorem 5.2.6** *Let $n \geq 2$, $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then*
*(i) The fractional Navier-Stokes system (5.2.1) has a unique small global mild solution in $(X^\beta_{\alpha;\infty}(\mathbb{R}^n))^n$ for all initial data $a$ with $\nabla \cdot a = 0$ and $\|a\|_{(Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n))^n}$ being small.*
*(ii) For any $T \in (0, \infty)$ there is an $\varepsilon > 0$ such that the fractional Navier-Stokes system (5.2.1) has a unique small mild solution in $(X^\beta_{\alpha,T}(\mathbb{R}^n))^n$ on $(0,T) \times \mathbb{R}^n$ when the initial data $a$ satisfies $\nabla \cdot a = 0$ and $\|a\|_{(Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n))^n} \leq \varepsilon$. In particular for all $a \in (\overline{VQ^{\beta,-1}_\alpha}(\mathbb{R}^n))^n$ with $\nabla \cdot a = 0$ there exists a unique small local mild solution in $(X^\beta_{\alpha;T}(\mathbb{R}^n))^n$ on $(0,T) \times \mathbb{R}^n$.*

**Proof.** By Picard's contraction principle, it sufficient to verify the bilinear operator

$$B(u,v) = \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (u \otimes v) ds$$

is bounded from $(X^\beta_{\alpha;T}(\mathbb{R}^n))^n \times (X^\beta_{\alpha;T}(\mathbb{R}^n))^n$ to $(X^\beta_{\alpha;T}(\mathbb{R}^n))^n$.
*Part 1. $L^2-$bound.* We want to establish that if $x \in \mathbb{R}^n$ and $r^{2\beta} \in (0,T)$ then

$$r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |B(u,v)|^2 dy \frac{ds}{s^{\alpha/\beta}} \lesssim \|u\|^2_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|^2_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n}. \tag{5.2.5}$$

To this aim, define $1_{r,x}(y) = 1_{|y-x|<10r}(y)$, i.e., the indicate function on the ball $\{y \in \mathbb{R}^n : |y-x| < 10r\}$. We divide $B(u,v)$ into three parts: $B(u,v) = B_1(u,v) + B_2(u,v) + B_3(u,v)$, where

$$\begin{aligned}
B_1(u,v) &= \int_0^s e^{-(s-h)(-\triangle)^\beta} P\nabla \cdot ((1-1_{r,x})u \otimes v) dh, \\
B_2(u,v) &= (-\triangle)^{-1/2} P\nabla \cdot \int_0^s e^{-(s-h)(-\triangle)^\beta} (-\triangle)((-\triangle)^{-1/2}(I - e^{-h(-\triangle)^\beta})(1_{r,x})u \otimes v) dh \\
B_3(u,v) &= (-\triangle)^{-1/2} P\nabla \cdot (-\triangle)^{1/2} e^{-s(-\triangle)^\beta} \int_0^s ((1_{r,x})u \otimes v) dh.
\end{aligned}$$

At first, we estimate $B_2(u,v)$ as

$$\begin{aligned}
I &= \int_0^{r^{2\beta}} \|B_2(u,v)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} \| \int_0^s e^{-(s-h)(-\triangle)^\beta}(-\triangle)((-\triangle)^{-1/2}(I-e^{-h(-\triangle)^\beta})(1_{r,x})u\otimes v)dh\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} \| \int_0^s e^{-(s-h)(-\triangle)^\beta}(-\triangle)^\beta((-\triangle)^{1/2-\beta}(I-e^{-h(-\triangle)^\beta})(1_{r,x})u\otimes v)dh\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} \|(-\triangle)^{1/2-\beta}(I-e^{-h(-\triangle)^\beta})(1_{r,x})u\otimes v dh\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}}.
\end{aligned}$$

Since $\sup_{s\in(0,\infty)} s^{1-2\beta}(1-e^{-s^{2\beta}})<\infty$ for $\frac{1}{2}<\beta<1$, we can obtain that $(-\triangle)^{1/2-\beta}(I-e^{-s(-\triangle)^\beta})$ is bounded on $L^2(\mathbb{R}^n)$ with operator norm $\lesssim s^{1-\frac{1}{2\beta}}$. Write $(1_{r,x})u(s,x)\otimes v(s,x) = M(s,x)$. Thus, using the Cahchy-Schwarz inequality, we have

$$\begin{aligned}
I &\lesssim \int_0^{r^{2\beta}} s^{2-\frac{1}{\beta}} \|M(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{ds}{s^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} s^{2-\frac{1}{\beta}} \int_{|y-x|<r} |u(s,y)v(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}} \\
&\lesssim \left(\sup_{s\in(0,T)} s^{1-\frac{1}{2\beta}}\|u(s,y)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\sup_{s\in(0,T)} s^{1-\frac{1}{2\beta}}\|v(s,y)\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\quad \times \left(\int_0^{r^{2\beta}}\int_{|y-x|<r} |u(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}}\right)\left(\int_0^{r^{2\beta}}\int_{|y-x|<r} |v(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}}\right) \\
&\lesssim r^{n-2\alpha-2(\beta-1)} \|u\|^2_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|^2_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n}.
\end{aligned}$$

Now by Lemma 5.2.2 with $k=0$, we estimate the term $B_3$ as follows.

$$\begin{aligned}
&\int_0^{r^{2\beta}} \|B_3(u,v)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} \left\|(-\triangle)^{1/2} e^{-t(-\triangle)^\beta}(\int_0^t M(s,\cdot)dh)\right\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim r^{n-2\alpha+6\beta-2} \int_0^1 \left\|(-\triangle)^{1/2} e^{-\tau(-\triangle)^\beta}\left(\int_0^\tau M(r^{2\beta}\theta, r\cdot)d\theta\right)\right\|^2_{L^2(\mathbb{R}^n)} \frac{d\tau}{\tau^{\alpha/\beta}} \\
&\lesssim r^{n-2\alpha+6\beta-2} \int_0^1 \|M(r^{2\beta}s, r\cdot)\|_{L^1(\mathbb{R}^n)} \frac{ds}{s^{\alpha/\beta}} C(\alpha,\beta;f) \\
&= r^{n-2\alpha+6\beta-2} \times II \times A(\alpha,\beta; M(r^{2\beta}s, ry)).
\end{aligned}$$

For $II$, we have

$$\begin{aligned} II &= r^{2\alpha-n-2\beta}\int_0^{r^{2\beta}}\int_{|z-x|<r}|M(t,z)|\frac{dzdt}{t^{\alpha/\beta}} \\ &\lesssim r^{2-4\beta}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}. \end{aligned}$$

For $C(\alpha,\beta;M(r^{2\beta}s,ry))$, we have

$$\begin{aligned} C(\alpha,\beta;M(r^{2\beta}s,ry)) &\lesssim \rho^{2\alpha-n+2(\beta-1)}\int_0^{\rho^{2\beta}}\int_{|y-x|<\rho}|M(r^{2\beta}s,ry)|\frac{dyds}{s^{\alpha/\beta}} \\ &\lesssim \rho^{2\alpha-n+2(\beta-1)}r^{2\alpha-n-2\beta}\int_0^{(r\rho)^{2\beta}}\int_{|z-x|<r\rho}|M(t,z)|\frac{dzdt}{t^{\alpha/\beta}} \\ &\lesssim r^{2-4\beta}(r\rho)^{2\alpha-n+2(\beta-1)}\int_0^{(r\rho)^{2\beta}}\int_{|z-x|<r\rho}|M(t,z)|\frac{dzdt}{t^{\alpha/\beta}} \\ &\lesssim r^{2-4\beta}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}. \end{aligned}$$

Therefore we get

$$\begin{aligned} &\int_0^{r^{2\beta}}\|B_3(u,v)\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}} \lesssim r^{n-2\alpha+6\beta-2}r^{2-4\beta}r^{2-4\beta}\|u\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n} \\ \lesssim\; & r^{n-2\alpha-2\beta+2}\|u\|^2_{(X^{\beta}_{\alpha,T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}, \end{aligned}$$

that is,

$$r^{2\alpha-n+2(\beta-2)}\int_0^{r^{2\beta}}\|B_3(u,v)\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t^{\alpha/\beta}} \lesssim \|u\|^2_{(X^{\beta}_{\alpha,T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.$$

For the estimate of $B_1$. According to Lemma 5.2.4, we have

$$e^{-t(-\triangle)^{\beta}}P\nabla\cdot f(x) = \int\nabla K^{\beta}_{j,k,t}(x-y)f(y)dy$$

and $$\nabla K^{\beta}_{j,k,t}(x-y) \lesssim \frac{1}{t^{\frac{n}{2\beta}+\frac{1}{2\beta}}}\frac{1}{\left(1+t^{-1/2\beta}|x-y|\right)^{n+1}} \lesssim \frac{1}{(t^{1/2\beta}+|x-y|)^{n+1}}.$$

Thus

$$\begin{aligned} |B_1(u,v)| &\leq \left|\int_0^s e^{-(s-h)(-\triangle)^{\beta}}P\nabla\cdot((1-1_{r,x})u\otimes v)dh\right| \\ &\lesssim \int_0^s\int_{|z-x|\geq 10r}\frac{|u(h,z)||v)(h,z)|}{((s-h)^{1/2\beta}+|z-y|)^{n+1}}dzdh. \end{aligned}$$

When $|z-x|\geq 10r$, $0<s<r^{2\beta}$ and $|y-x|<r$, we have $|y-z|\gtrsim|z-x|-|y-x|\geq 9r > 9|y-x|$. Thus $|x-z|\leq|x-y|+|y-z|\leq\frac{1}{9}|y-z|+|y-z|=\frac{10}{9}|y-z|$. This gives us

$$|B_1(u,v)| \lesssim \int_0^{r^{2\beta}}\int_{|z-x|\geq 10r}\frac{|u(h,z)||v(h,z)|}{|x-z|^{n+1}}dzdh = I_1\times I_2.$$

where

$$\begin{aligned}
I_1 &= \left(\int_0^{r^{2\beta}}\int_{|z-x|\geq 10r}\frac{|u(h,z)|^2}{|x-z|^{n+1}}dzdh\right)^{1/2}\\
&\lesssim \left(\sum_{j=3}^{\infty}\int_0^{r^{2\beta}}\int_{2^jr\leq|z-x|\leq 2^{j+1}r}\frac{|u(h,z)|^2}{(2^jr)^{n+1}}dzdh\right)^{1/2}\\
&\lesssim \left(\sum_{j=3}^{\infty}\frac{1}{(2^jr)^{n+1}}(r^{2\beta})^{\alpha/\beta}(2^jr)^{2\beta-2}(2^jr)^{2-2\beta}\int_0^{r^{2\beta}}\int_{2^jr\leq|z-x|\leq 2^{j+1}r}|u(h,z)|^2\frac{dzdh}{h^{\alpha/\beta}}\right)^{1/2}\\
&\lesssim \left(\sum_{j=3}^{\infty}(2^jr)^{2\alpha-n}(2^jr)^{-1}(2^jr)^{2\beta-2}(2^jr)^{2-2\beta}\int_0^{r^{2\beta}}\int_{|z-x|\leq 2^{j+1}r}|u(h,z)|^2\frac{dzdh}{h^{\alpha/\beta}}\right)^{1/2}\\
&\lesssim \left(\frac{1}{r^{2\beta-1}}\right)^{1/2}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.
\end{aligned}$$

Similarly, we obtain $I_2 \lesssim \left(\frac{1}{r^{2\beta-1}}\right)^{1/2}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}$. Thus

$$|B_1(u,v)| \lesssim \frac{1}{r^{2\beta-1}}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.$$

When $0<\alpha<\beta$, we have

$$\begin{aligned}
\int_0^{r^{2\beta}}\int_{|y-x|<r}|B_1(u,v)|^2\frac{dydt}{t^{\alpha/\beta}} &\lesssim \frac{1}{r^{4\beta-2}}r^n\int_0^{r^{2\beta}}\frac{dt}{t^{\alpha/\beta}}\|u\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\\
&\lesssim r^{n-2\alpha-2\beta+2}\|u\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.
\end{aligned}$$

This implies that

$$r^{2\alpha-n+2(\beta-1)}\int_0^{r^{2\beta}}\int_{|y-x|<r}|B_1(u,v)|^2\frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|^2_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.$$

*Part 2. $L^\infty-$bound.* The aim of this part is to prove

$$\|B(u,v)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{1}{2\beta}-1}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}, \quad \forall t\in(0,T).$$

If $\frac{t}{2}\leq s<t$ then

$$\begin{aligned}
&\|e^{-(t-s)(-\triangle)^\beta}P\nabla\cdot(u\otimes v)\|_{L^\infty} \lesssim \frac{\|u\|_{L^\infty(\mathbb{R}^n)}\|v\|_{L^\infty(\mathbb{R}^n)}}{(t-s)^{\frac{1}{2\beta}}}\\
&\lesssim (t-s)^{-\frac{1}{2\beta}}s^{\frac{1}{\beta}-2}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.
\end{aligned}$$

If $0 < s < \frac{t}{2}$ then $t - s \approx t$ and so

$$\begin{aligned}
& |e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (u \otimes v)| \\
\lesssim & \int_{\mathbb{R}^n} \frac{|u(s,y)||v(s,y)|}{\left((t-s)^{\frac{1}{2\beta}} + |x-y|\right)^{n+1}} dy \\
\lesssim & \int_{\mathbb{R}^n} \frac{|u(s,y)||v(s,y)|}{\left(t^{\frac{1}{2\beta}} + |x-y|\right)^{n+1}} dy \\
\lesssim & \sum_{k\in\mathbb{Z}^n} \int_{x-y\in t^{\frac{1}{2\beta}}(k+[0,1]^n)} \frac{|u(s,y)||v(s,y)|}{(t^{\frac{1}{2\beta}}(1+|k|))^{(n+1)}} dyds.
\end{aligned}$$

This gives us

$$\begin{aligned}
|B(u,v)| \lesssim & \int_0^{t/2} |e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (u \otimes v)| ds + \int_{t/2}^t |e^{-(t-s)(-\triangle)^\beta} P\nabla \cdot (u \otimes v)| ds \\
\lesssim & \sum_{k\in\mathbb{Z}^n} (t^{\frac{1}{2\beta}}(1+|k|))^{-(n+1)} \int_0^{t/2} \int_{x-y\in t^{\frac{1}{2\beta}}(k+[0,1]^n)} |u(s,y)||v(s,y)| dy \\
& + \int_{t/2}^t (t-s)^{-\frac{1}{2\beta}} s^{\frac{1}{\beta}-2} ds \|u\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \\
:= & I_3 + I_4.
\end{aligned}$$

Here,

$$\begin{aligned}
I_4 \lesssim & \int_{t/2}^t (t-s)^{-\frac{1}{2\beta}} s^{\frac{1}{\beta}-2} ds \|u\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \\
\lesssim & t^{\frac{1}{\beta}-2} t^{1-\frac{1}{2\beta}} \|u\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \\
\lesssim & t^{\frac{1}{2\beta}-1} \|u\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n} \|v\|_{(X^\beta_{\alpha;T}(\mathbb{R}^n))^n}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
I_3 \lesssim & \sum_{k\in\mathbb{Z}^n} (t^{\frac{1}{2\beta}}(1+|k|))^{-(n+1)} \left( \int_0^{t/2} \int_{|x-y|\lesssim t^{\frac{1}{2\beta}}} |u(s,y)|^2 dyds \right)^{1/2} \\
& \times \left( \int_0^{t/2} \int_{|x-y|\lesssim t^{\frac{1}{2\beta}}} |v(s,y)|^2 dyds \right)^{1/2} \\
:= & \sum_{k\in\mathbb{Z}^n} (t^{\frac{1}{2\beta}}(1+|k|))^{-(n+1)} I_{3,1} \times I_{3,2}.
\end{aligned}$$

Here,

$$\begin{aligned} I_{3,1} &= \left(\int_0^{t/2}\int_{|x-y|\lesssim t^{\frac{1}{2\beta}}} |u(s,y)|^2 dyds\right)^{1/2} \\ &= \left(t^{\frac{1}{2\beta}(n-2\beta+2)} t^{\frac{1}{2\beta}(2\alpha-n+2\beta-2)} \int_0^{t/2}\int_{|x-y|\lesssim t^{\frac{1}{2\beta}}} |u(s,y)|^2 \frac{dyds}{s^{\alpha/\beta}}\right)^{1/2} \\ &\lesssim t^{\frac{1}{4\beta}(n-2\beta+2)} \|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}. \end{aligned}$$

Similarly, we get $I_{3,2} \lesssim t^{\frac{1}{4\beta}(n-2\beta+2)}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}$. These estimates about $I_{3,1}$ and $I_{3,2}$ imply that

$$I_3 \lesssim t^{-\frac{1}{2\beta}(n+1)} t^{\frac{1}{2\beta}(n-2\beta+2)} \|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n} \lesssim t^{\frac{1}{2\beta}-1}\|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.$$

Thus

$$t^{1-\frac{1}{2\beta}}\|B(u,v)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}\|v\|_{(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n}.$$

Therefore, we establish the boundedness of $B(u,v)$ and finish the proof of (i) and (ii) by taking $T=\infty$ and $T\in(0,\infty)$, respectively. □

## 5.3 Regularity of Fractional Navier-Stokes Equations in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$

In this section, we study the regularity of the solutions to the equations (5.2.1) with $\beta\in(1/2,1]$. For $\beta=1$, that is, the classical Naiver-Stokes equations, the regularity has been studied by several authors. In [28], Germain-Pavlović-Staffilani analyzed the regularity properties of the solutions constructed by Koch-Tataru. More precisely, they showed that under certain smallness condition of initial data in $BMO^{-1}(\mathbb{R}^n)$, the solution $u$ to the classical Naiver-Stokes equations constructed in [41] satisfies the following regularity property:

$$t^{\frac{k}{2}}\nabla^k u \in X^0(\mathbb{R}^n), \text{for all } k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}$$

where $X^0(\mathbb{R}^n)$ denotes the space where the solution constructed by Koch and Tataru belongs.

In this section, we establish a similar result for the solutions of the equations (5.2.1) evolving initial data in $Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)$ with $\beta\in(1/2,1]$. In fact we get the solution $u$ to the equations (5.2.1) satisfies:

$$t^{\frac{k}{2\beta}}\nabla^k u \in X^{\beta,0}_{\alpha}(\mathbb{R}^n) \quad \text{for all } k$$

where $X^{\beta,0}_{\alpha}(\mathbb{R}^n)$ is the space $X^{\beta}_{\alpha;\infty}(\mathbb{R}^n)$ constructed in (iii) of Definition 5.2.5 for $\beta\in(1/2,1)$ and $X^{1}_{\alpha;\infty}(\mathbb{R}^n)$ in Xiao [78] for $\beta=1$. For convenience of the study, we introduce a class of spaces $X^{\beta,k}_{\alpha}(\mathbb{R}^n)$ as follows.

**Definition 5.3.1** *For a nonnegative integer* $k$, $\alpha > 0$, $\alpha+\beta \geq 1$ *and* $\max\{\alpha, 1/2\} < \beta \leq 1$, *we introduce the space* $X_\alpha^{\beta,k}(\mathbb{R}^n)$ *which is equipped with the following norm:*

$$\|u\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)} = \|u\|_{N_{\alpha,\infty}^{\beta,k}(\mathbb{R}^n)} + \|u\|_{N_{\alpha,C}^{\beta,k}(\mathbb{R}^n)}$$

*where*

$$\|u\|_{N_{\alpha,\infty}^{\beta,k}(\mathbb{R}^n)} = \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_t t^{\frac{2\beta-1+k}{2\beta}} \|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)},$$

$$\|u\|_{N_{\alpha,C}^{\beta,k}(\mathbb{R}^n)} = \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_{x_0,r} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |t^{\frac{k}{2\beta}} \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2}.$$

In the following, we will denote $\nabla^k u = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u$ with $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $k = \alpha_1 + \cdots + \alpha_n$.

### 5.3.1 Several Technical Lemmas

Before stating the main result of this section, we prove several preliminary lemmas associated with the fractional heat semigroup $e^{-t(-\triangle)^\beta}$. Recall that $e^{-t(-\triangle)^\beta} f(x) = K_t^\beta * f(x)$ where $K_t^\beta$ is defined by $\mathcal{F}K_t^\beta(\xi) = e^{-t|\xi|^{2\beta}}$ and $P$ is the Helmboltz-Weyl projection.

**Lemma 5.3.2** *Let* $\beta \in (1/2, 1)$. *There exists a constant* $C > 0$ *depending only on* $n$ *such that*

$$|\partial_x^k P\nabla K_t^\beta(x)| \leq C^k k^{k/2\beta} t^{-k/2\beta} (k^{-\frac{1}{2\beta}} t^{\frac{1}{2\beta}} + |x|)^{-n-1}$$

*for all* $t > 0, x \in \mathbb{R}^n$ *and* $k \in \mathbb{N}$.

**Proof**. By a dilation argument, we have

$$\partial_x^k P\nabla K_t^\beta(x) = t^{\frac{-k-1}{2\beta}} t^{-\frac{n}{2\beta}} \partial_x^k P\nabla K_1^\beta(x/t^{\frac{1}{2\beta}}).$$

If we could prove $|\partial_x^k P\nabla K_1^\beta(x)| \leq C^k k^{k/2\beta}(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}$, then we have

$$\begin{aligned} |\partial_x^k P\nabla K_t^\beta(x)| &\leq t^{-\frac{k+1}{2\beta}} t^{-\frac{n}{2\beta}} C^k k^{k/2\beta} \left(k^{-\frac{1}{2\beta}} + \left|\frac{x}{t^{1/2\beta}}\right|\right)^{-n-1} \\ &\leq c^k k^{k/2\beta} t^{-\frac{k}{2\beta}} (t^{\frac{1}{2\beta}} k^{-\frac{1}{\beta}} + |x|)^{-n-1}. \end{aligned}$$

Hence we obtain the desired.

By the semigroup property, it is easy to see that $\partial_x^k P\nabla K_1^\beta = P\nabla K_{1/2}^\beta * \partial_x^k K_{1/2}^\beta$. So we need to prove the following two estimates:

$$|P\nabla K_{1/2}^\beta(x)| \leq C(1+|x|)^{-n-1} \tag{5.3.1}$$

$$|\partial_x^k K_{1/2}^\beta(x)| \leq C^{k-1} k^{\frac{k-1}{2\beta}} (k^{-\frac{1}{2\beta}} + |x|)^{-n-1}. \tag{5.3.2}$$

For (5.3.1). Taking $\alpha = 1$ in the Lemma 5.2.4, we have

$$(1+|x|)^{n+1}|P\nabla K^{\beta}_{1/2}(x)| \le C,$$

that is, (5.3.1) is obvious.

For (5.3.2), we claim that $|\partial_i K^{\beta}_{1/2}(x)| \le C(1+|x|)^{-n-1}$. In fact when $|x|<1$,

$$(1+|x|)^{n+1}|\partial_i K^{\beta}_{1/2}(x)| \le 2^{n+1}\int_{\mathbb{R}^n} |i\xi_i| e^{-|\xi|^{2\beta}/2} d\xi \lesssim C.$$

When $|x|>1$, we define the operator

$$L(x,D) = \frac{x\cdot\nabla_\xi}{i|x|^2}, \text{ that is } L(x,D)e^{ix\cdot\xi} = e^{ix\cdot\xi}$$

and choose a $C_0^\infty(\mathbb{R}^n)$−function $\rho(x)$ satisfying:

$$\rho(\xi) = \begin{cases} 1, & |\xi|\le 1, \\ 0, & |\xi|>2, \end{cases}$$

we have

$$\begin{aligned} |\partial_i K^{\beta}_{1/2}(x)| &\le \left|\int_{\mathbb{R}^n} \rho\left(\frac{\xi}{\delta}\right) i\xi_i e^{-|\xi|^{2\beta}/2} e^{ix\cdot\xi} d\xi\right| \\ &+ \left|\int_{\mathbb{R}^n} [1-\rho\left(\frac{\xi}{\delta}\right)] i\xi_i e^{-|\xi|^{2\beta}/2} e^{ix\cdot\xi} d\xi\right| \\ &:= I_3 + I_4. \end{aligned}$$

For $I_3$, we have

$$I_3 \lesssim \int_{\mathbb{R}^n} \rho\left(\frac{\xi}{\delta}\right)|\xi| e^{-|\xi|^{2\beta}/2} d\xi \lesssim \int_{|\xi|<2\delta} \delta d\xi \lesssim \delta^{n+1}.$$

For $I_4$, using the integration by parts and $L^* = -\frac{x\cdot\nabla_\xi}{i|x|^2}$, we have

$$\begin{aligned} I_4 &= \left|\int_{\mathbb{R}^n} (L^*)^N \left([1-\rho\left(\frac{\xi}{\delta}\right)] i\xi_i e^{-|\xi|^{2\beta}/2}\right) e^{ix\cdot\xi} d\xi\right| \\ &\lesssim C_N|x|^{-N}\int_{\mathbb{R}^n}\left|\sum_{k=0}^{N} C_N^k \nabla_\xi^k\left[1-\rho\left(\frac{\xi}{\delta}\right)\right]\nabla_\xi^{N-k}(i\xi e^{-|x i|^{2\beta}/2})\right| d\xi \\ &\lesssim C_N|x|^{-N}\int_{|\xi|>\delta}\sum_{k=1}^{N}|\xi|^{2\beta k-N+1}e^{-|\xi|^{2\beta}/2}d\xi \\ &+ C_N|x|^{-N}\int_{\delta\le|\xi|\le 2\delta}\sum_{k=1}^{N} C_N^k\delta^{-k}\sum_{l=0}^{N-k} C_{N-k}^l |\xi|^{2\beta l-N+k+1}e^{-|\xi|^{2\beta}/2}d\xi \\ &\lesssim C_N|x|^{-N}\int_{|\xi|>\delta}|\xi|^{1-N}d\xi + C_N|x|^{-N}\int_{\delta<|\xi|<2\delta}\delta^{-k}|\xi|^{1-N+k}d\xi \\ &\lesssim C_N|x|^{-N}\delta^{n+1-N}. \end{aligned}$$

So we get, taking $\delta = |x|^{-1}$,

$$|\partial_i K^\beta_{1/2}(x)| \lesssim \delta^{n+1} + C_N|x|^{-N}\delta^{n+1-N} \lesssim C_N|x|^{-(n+1)} \lesssim C_N(1+|x|)^{-(n+1)}.$$

Then we have

$$|\partial_i K^\beta_{\frac{1}{2k}}(x)| \lesssim k^{\frac{1}{2\beta}} k^{\frac{n}{2\beta}} |\partial_i K^\beta_{1/2}(k^{1/2\beta}x)| \lesssim C(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}.$$

Because the following integral inequality(see [54]):

$$\int_{\mathbb{R}^n} (a+|x-y|)^{-n-1}(b+|y|)^{-n-1}dy \le ca^{-1}(a+|x|)^{-n-1} \text{ for } 0<a\le b,$$

we have

$$\begin{aligned} |\partial^2_{i,j} K^\beta_{\frac{1}{k}}(x)| &= \left|\int_{\mathbb{R}^n} \partial_i K^\beta_{\frac{1}{2k}}(x-y)\partial_j K^\beta_{\frac{1}{2k}}(y)dy\right| \\ &\lesssim \int_{\mathbb{R}^n} (k^{-\frac{1}{2\beta}} + |x-y|)^{-n-1}(k^{-\frac{1}{2\beta}} + |y|)^{-n-1}dy \\ &\lesssim k^{\frac{1}{2\beta}}(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}. \end{aligned}$$

Operating the process above $k-1$ times, we get $|\partial^k_x K^\beta_{1/2}(x)| \lesssim k^{\frac{k-1}{2\beta}}(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}$ and

$$\begin{aligned} |\partial^k_x P\nabla K^\beta_1(x)| &= \left|\int_{\mathbb{R}^n} P\nabla K_{1/2}(x-y)\partial^k_x K^\beta_{1/2}(y)dy\right| \\ &\lesssim \int_{\mathbb{R}^n} c'(1+|x-y|)^{-n-1}c^{k-1}k^{\frac{k-1}{2\beta}}(k^{-\frac{1}{2\beta}} + |y|)^{-n-1}dy \\ &\lesssim C^k k^{\frac{1}{2\beta}}(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}k^{\frac{k-1}{2\beta}} \\ &\lesssim C^k k^{\frac{k}{2\beta}}(k^{-\frac{1}{2\beta}} + |x|)^{-n-1}. \end{aligned}$$

This completes the proof of this lemma. □

The following lemma can be regarded as a generalization of Proposition 3.2 of [28].

**Lemma 5.3.3** *If $r$ is a natural number, $\alpha \in (0,1)$, $\alpha+\beta \ge 1$ and $\max\{\alpha, \frac{1}{2}\} < \beta \le 1$, the operator*

$$P^\beta_r f(t,x) = \int_0^t e^{-(t-s)(-\triangle)^\beta}(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^r \nabla^{r+2\beta} f(s,x)ds$$

*is bounded on $L^2([0,T], L^2(\mathbb{R}^n, dx), \frac{dt}{t^{\alpha/\beta}})$ for any $T \in [0,\infty]$ with constants $p(r)$ and $q(r)$.*

**Proof**. By Plancherel's theorem and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \|P_r^\beta f(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
\lesssim\ & \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^t e^{-(t-s)|\xi|^{2\beta}} (t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^r |\xi|^{r+2\beta} \mathcal{F}f(s,\xi) ds \right)^2 \frac{dt}{t^{\alpha/\beta}} d\xi \\
\lesssim\ & \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^t e^{-(t-s)|\xi|^{2\beta}} (t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^r |\xi|^{r+2\beta} ds \right) \\
& \times \left( \int_0^t e^{-(t-s)|\xi|^{2\beta}} (t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^r |\xi|^{r+2\beta} |\mathcal{F}f(s,\xi)|^2 ds \right) \frac{dt}{t^{\alpha/\beta}} d\xi.
\end{aligned}$$

Because $t^{1/2\beta} - s^{1/2\beta} \le (t-s)^{1/2\beta}$ for $2\beta > 1$ and $0 < s < t$, it is easy to see that

$$\begin{aligned}
\int_0^t e^{-(t-s)|\xi|^{2\beta}} (t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^r |\xi|^{r+2\beta} ds &\le \int_0^t e^{-(t-s)|\xi|^{2\beta}} (t-s)^{\frac{r}{2\beta}} |\xi|^{r+2\beta} ds \\
&\lesssim \int_0^\infty e^{-v} v^{r/2\beta} dv \lesssim 1.
\end{aligned}$$

Then we have, by $t^{1/2\beta} - s^{1/2\beta} \le (t-s)^{1/2\beta}$ for $2\beta > 1$ and $0 < s < t$ again,

$$\begin{aligned}
&\int_0^\infty \|P_r^\beta f(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t^{\alpha/\beta}} \\
\lesssim\ & \int_{\mathbb{R}^n} \int_0^\infty |\mathcal{F}f(s,\xi)|^2 \left( \int_s^\infty e^{-(t-s)|\xi|^{2\beta}} (t-s)^{\frac{r}{2\beta}} |\xi|^{r+2\beta} dt \right) \frac{ds d\xi}{s^{\alpha/\beta}} \\
\lesssim\ & \int_{\mathbb{R}^n} \int_0^\infty |\mathcal{F}f(s,\xi)|^2 \left( \int_0^\infty e^{-u|\xi|^{2\beta}} u^{\frac{r}{2\beta}} |\xi|^{r+2\beta} du \right) \frac{ds d\xi}{s^{\alpha/\beta}} \\
\lesssim\ & \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{F}f(s,\xi)|^2 \frac{ds}{s^{\alpha/\beta}} d\xi.
\end{aligned}$$

This completes the proof of this lemma. □

**Lemma 5.3.4** *For any $k \ge 0$ and $\beta \in (1/2, 1]$, there exists a constant $C(k)$ such that*

$$\|e^{-t(-\triangle)^\beta} u\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)} \le C(k) \|u\|_{Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)}.$$

**Proof**. Because $\|u\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)} = \|u\|_{N_{\alpha,\infty}^{\beta,k}(\mathbb{R}^n)} + \|u\|_{N_{\alpha,C}^{\beta,k}(\mathbb{R}^n)}$, we split the proof into two parts.

For the $L^\infty(\mathbb{R}^n)$ part of the norm. Because $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^{1-2\beta}(\mathbb{R}^n)$ and

$$\nabla^k : \dot{B}_{\infty,\infty}^{1-2\beta}(\mathbb{R}^n) \longrightarrow \dot{B}_{\infty,\infty}^{1-2\beta-k}(\mathbb{R}^n),$$

we have

$$\begin{aligned}
&\|\nabla^k e^{-t(-\triangle)^\beta} u\|_{L^\infty(\mathbb{R}^n)} \le t^{\frac{1-2\beta-k}{2\beta}} \|\nabla^k u\|_{\dot{B}_{\infty,\infty}^{1-2\beta-k}(\mathbb{R}^n)} \\
\le\ & t^{\frac{1-2\beta-k}{2\beta}} \|u\|_{\dot{B}_{\infty,\infty}^{1-2\beta}(\mathbb{R}^n)} \le t^{\frac{1-2\beta-k}{2\beta}} \|u\|_{Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)}.
\end{aligned}$$

Then we can get $t^{\frac{2\beta-1+k}{2\beta}}\|\nabla^k e^{-t(-\triangle)^\beta}u\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)}$.

For the Carleson part. Because $u \in Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla\cdot(Q^\beta_\alpha)^n(\mathbb{R}^n)$, there exists a sequence $\{f_j\} \subset Q^\beta_\alpha(\mathbb{R}^n)$ such that $u = \sum_j \partial_j f_j$. We only need to prove

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n, r>0} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x|<r}\left|t^{\frac{k}{2\beta}}\nabla^k e^{-t(-\triangle)^\beta}\partial_j f_j(x)\right|^2\frac{dydt}{t^{\alpha/\beta}}\\ \lesssim\;& C(k)\|\partial_j f_j\|^2_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)} = C(k)\|f_j\|^2_{Q^\beta_\alpha(\mathbb{R}^n)}.\end{aligned} \tag{5.3.3}$$

Taking

$$\psi(x) = \nabla^k\partial_j e^{-(-\triangle)^\beta}(x) = \int_{\mathbb{R}^n}(i\xi)^k i\xi_j e^{-|\xi|^{2\beta}}e^{2\pi i x\cdot\xi}d\xi,$$

we can justify the function $\psi(x)$ with $\mathcal{F}\psi_t(\xi) = (it\xi)^k(it\xi_j)e^{-t^{2\beta}|\xi|^{2\beta}}$ satisfying the conditions in (1.0.10):

$$|\psi(x)| \lesssim (1+|x|)^{-n-1}, \quad \psi(x)\in L^1 \quad \text{and} \quad \int_{\mathbb{R}^n}\psi(x)dx = 0.$$

By the equivalent characterization of $Q^\beta_\alpha(\mathbb{R}^n)$ (see(1.0.11)), we have

$$\sup_{x,r} r^{2\alpha-n+2\beta-2}\int_0^r\int_{|y-x|<r}\left|s^k\nabla^k e^{-s^{2\beta}(-\triangle)^\beta}(s\partial_j)f_j(x)\right|^2\frac{dyds}{s^{1+2\alpha-2(\beta-1)}} \lesssim C(k)\|f_j\|^2_{Q^\beta_\alpha(\mathbb{R}^n)}.$$

By a change of variable: $t = s^{2\beta}$, we get the desired result (5.3.3). □

### 5.3.2 Regularity

Now we state our regularity result.

**Theorem 5.3.5** *Let $\alpha > 0$, $\alpha+\beta \ge 1$ and $\max\{\alpha, 1/2\} < \beta < 1$. There exists an $\varepsilon = \varepsilon(n)$ such that if $\|u_0\|_{Q^{\beta;-1}_{\alpha;\infty}(\mathbb{R}^n)} < \varepsilon$, the solution $u$ to equations (5.2.1) verifies:*

$$t^{\frac{k}{2\beta}}\nabla^k u \in X^{\beta,0}_\alpha(\mathbb{R}^n) \quad \textit{for any } k \ge 0.$$

**Proof**. We can see that the solution to the equations (5.2.1) can be represented as

$$u(t,x) = e^{-t(-\triangle)^\beta}u(0,x) - B(u,u)(t,x),$$

where

$$B(u,v)(t,x) = \int_0^t e^{-(t-s)(-\triangle)^\beta}P\nabla\cdot(u(s,x)\otimes v(s,x))ds.$$

Here $u\otimes v$ denotes the tensor product of $u$ and $v$. For the linear term $e^{-t(-\triangle)^\beta}u(0,x) := e^{-t(-\triangle)^\beta}u_0$, by Proposition 5.3.4, we have $\|e^{-t(-\triangle)^\beta}u_0\|_{X^{\beta,k}_\alpha(\mathbb{R}^n)} \le C(k)\|u_0\|_{Q^{\beta,-1}_\alpha(\mathbb{R}^n)}$. Now we estimate the nonlinear term. We write $\widetilde{X}^{\beta,k}_\alpha(\mathbb{R}^n) = \cap^k_{l=0}X^{\beta,k}_\alpha(\mathbb{R}^n)$ equipped with the norm $\sum^k_{l=0}\|\cdot\|_{X^{\beta,l}_\alpha(\mathbb{R}^n)}$. We shall prove that the bilinear operator maps

$$B(u,v) : \widetilde{X}^{\beta,k}_\alpha(\mathbb{R}^n)\times\widetilde{X}^{\beta,k}_\alpha(\mathbb{R}^n) \longrightarrow \widetilde{X}^{\beta,k}_\alpha(\mathbb{R}^n).$$

*Part 1* $N^{\beta,k}_{\alpha,\infty}(\mathbb{R}^n)$ norm. Here we shall prove that

$$\begin{aligned}\|B(u,v)\|_{N^{\beta,k}_{\alpha,\infty}(\mathbb{R}^n)} &\lesssim C_0(k)\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)} + C(k)\sum_{l=1}^{k-1}\|u\|_{N^{\beta,l}_{\alpha,\infty}(\mathbb{R}^n)}\|v\|_{N^{\beta,k-l}_{\alpha,\infty}(\mathbb{R}^n)}\\ &+ C_1\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,k}_\alpha(\mathbb{R}^n)} + C_1\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,k}_\alpha(\mathbb{R}^n)}.\end{aligned}$$

If $0 < s < t(1-\frac{1}{m})$, $\frac{t}{m} < t-s < t$. By Lemma 5.3.2, we have

$$\begin{aligned}I &= \int_0^{t(1-\frac{1}{m})}\left|\nabla^k e^{-(t-s)(-\triangle)^\beta}P\nabla\cdot(u(s,x)\otimes v(s,x))\right|ds\\ &= C^k k^{\frac{k}{2\beta}}\int_0^{t(1-\frac{1}{m})}\int_{\mathbb{R}^n}\frac{|u(s,y)||v(s,y)|}{(t-s)^{\frac{k}{2\beta}}(t-s)^{\frac{n+1}{2\beta}}[k^{-\frac{1}{2\beta}}+\frac{|x-y|}{(t-s)^{1/2\beta}}]^{n+1}}dyds\\ &\le C^k k^{\frac{k}{2\beta}}\left(\frac{m}{t}\right)^{(n+k+1)/2\beta}\sum_{q\in\mathbb{Z}^n}\int_0^{t(1-\frac{1}{m})}\int_{\frac{x-y}{t^{1/2\beta}}\in q+[0,1]^n}\frac{|u(s,y)||v(s,y)|}{[k^{-\frac{1}{2\beta}}+|q|]^{n+1}}dyds.\end{aligned}$$

Because $\sum_{q\in\mathbb{Z}^n}\frac{1}{[k^{-\frac{1}{2\beta}}+|q|]^{n+1}} \approx k^{1/2\beta}$, we have

$$\begin{aligned}I &\le C^k k^{k/2\beta}\left(\frac{m}{t}\right)^{(n+k+1)/2\beta}k^{\frac{1}{2\beta}}\int_0^{t(1-\frac{1}{m})}\int_{x-y\in t^{\frac{1}{2\beta}}(q+[0,1]^n)}|u(s,y)||v(s,y)|dyds\\ &\le C^k k^{k/2\beta}\left(\frac{m}{t}\right)^{(n+k+1)/2\beta}k^{\frac{1}{2\beta}}t^{\alpha/\beta}\int_0^t\int_{|x-y|<t^{1/2\beta}}|u(s,y)||v(s,y)|\frac{dyds}{s^{\alpha/\beta}}\\ &\le C^k k^{\frac{k+1}{2\beta}}m^{\frac{n+k+1}{2\beta}}t^{\frac{-k-2\beta+1}{2\beta}}\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\\ &:= C_0(k)t^{\frac{-k-2\beta+1}{2\beta}}\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\end{aligned}$$

where $C_0(k) = C^k k^{\frac{k+1}{2\beta}}m^{\frac{n+k+1}{2\beta}}$.

If $t(1-\frac{1}{m}) \le s \le t$, by Young's inequality, we have

$$\begin{aligned}&|\nabla^k e^{-(t-s)(-\triangle)^\beta}P\nabla\cdot(u(s,x)\otimes v(s,x))|\\ =\;& |P\nabla e^{-(t-s)(-\triangle)^\beta}\nabla^k(u(s,x)\otimes v(s,x))|\\ \le\;& \|P\nabla e^{-(t-s)(-\triangle)^\beta}(x)\|_{L^1(\mathbb{R}^n)}\|\nabla^k(u(s,x)\otimes v(s,x))\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

By the estimate for the generalized Oseen kernel:

$$|P\nabla^{l+1}e^{-(-\triangle)^\beta}(x)| \lesssim \frac{1}{(1+|x|)^{n+1+l}}$$

and

$$P\nabla^{l+1}e^{-u(-\triangle)^\beta} = u^{-\frac{n+1+l}{2\beta}}P\nabla^{l+1}e^{-(-\triangle)^\beta}(\frac{x}{u^{1/2\beta}}),$$

we have

$$|P\nabla^{l+1}e^{-u(-\triangle)^\beta}| \lesssim u^{-\frac{n+l+1}{2\beta}}\frac{1}{\left(1+\frac{|x|}{u^{1/2\beta}}\right)^{l+n+1}} \lesssim \frac{1}{\left(u^{1/2\beta}+|x|\right)^{l+n+1}}.$$

Then we take $l=0$ and have

$$\|P\nabla e^{-u(-\triangle)^\beta}\|_{L^1(\mathbb{R}^n)} \lesssim \int_0^\infty \frac{|x|^{n-1}}{(u^{\frac{1}{2\beta}}+|x|)^{n+1}}d|x| \lesssim \frac{1}{u^{1/2\beta}}.$$

Hence we can get

$$\begin{aligned} & |\nabla^k e^{-(t-s)(-\triangle)^\beta} P\nabla(u(s,x)\otimes v(s,x))| \\ \lesssim & \frac{1}{(t-s)^{1/2\beta}}\sum_{l=0}^{k}\binom{k}{l}\|\nabla^l u(s,\cdot)\|_{L^\infty(\mathbb{R}^n)}\|\nabla^{k-l}v(s,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\ \lesssim & \frac{1}{(t-s)^{1/2\beta}}\sum_{l=0}^{k}\binom{k}{l}\frac{\|u\|_{N^{\beta,l}_{\alpha,\infty}(\mathbb{R}^n)}\|v\|_{N^{\beta,k-l}_{\alpha,\infty}(\mathbb{R}^n)}}{s^{(2\beta-1+l)/2\beta}s^{(2\beta-1+k-l)/2\beta}}. \end{aligned}$$

So we have

$$\begin{aligned} & \left|\int_{t(1-\frac{1}{m})}^{t}\nabla e^{-(t-s)(-\triangle)^\beta}P\nabla^{k+1}(u(s,x)\otimes v(s,x))ds\right| \\ & \lesssim \sum_{l=0}^{k}\binom{k}{l}\|u\|_{N^{\beta,l}_{\alpha,\infty}(\mathbb{R}^n)}\|v\|_{N^{\beta,k-l}_{\alpha,\infty}(\mathbb{R}^n)}\int_{t(1-\frac{1}{m})}^{t}\frac{1}{(t-s)^{1/2\beta}}\frac{1}{s^{(4\beta-2+k)/2\beta}}ds. \end{aligned}$$

For the integral in the last inequality, we make the change of variable: $s=zt$. Because $t\left(1-\frac{1}{m}\right)<s<t$ implies $\left(1-\frac{1}{m}\right)<z<1$, we have

$$\begin{aligned} II &= \int_{t\left(1-\frac{1}{m}\right)}^{t}\frac{1}{(t-s)^{1/2\beta}}\frac{1}{s^{\frac{4\beta-2+k}{2\beta}}}ds \\ &\lesssim t^{\frac{1-2\beta-k}{2\beta}}\left(1-\frac{1}{m}\right)^{-\frac{4\beta-2+k}{2\beta}}\int_{1-\frac{1}{m}}^{1}(1-z)^{-\frac{1}{2\beta}}dz \\ &= t^{\frac{1-2\beta-k}{2\beta}}\left(1-\frac{1}{m}\right)^{-\frac{4\beta-2+k}{2\beta}}\left(\frac{1}{m}\right)^{1-\frac{1}{2\beta}}. \end{aligned}$$

Denote $g(m)=\left(1-\frac{1}{m}\right)^{-\frac{4\beta-2+k}{2\beta}}\left(\frac{1}{m}\right)^{1-\frac{1}{2\beta}}$ and take $m=m(k)=k^{\frac{k-3}{n+k+1}}$. We can prove that $g(m)\longrightarrow 0$ as $k\longrightarrow\infty$. Then we have $II\lesssim Ct^{\frac{1-2\beta-k}{2\beta}}$ for $k\geq 1$.

Therefore we have

$$\begin{aligned} & \left|\int_{t\left(1-\frac{1}{m}\right)}^{t}\nabla e^{-(t-s)(-\triangle)^\beta}P\nabla^{k+1}(u(s,x)\times v(s,x))ds\right| \\ & \lesssim Ct^{\frac{1-2\beta-k}{2\beta}}\left[\sum_{l=1}^{k}\binom{k}{l}\|u\|_{N^{\beta,l}_{\alpha,\infty}}\|v\|_{N^{\beta,k-l}_{\alpha,\infty}}+\|u\|_{N^{\beta,0}_{\alpha,\infty}}\|v\|_{N^{\beta,k}_{\alpha,\infty}}+\|u\|_{N^{\beta,k}_{\alpha,\infty}}\|v\|_{N^{\beta,0}_{\alpha,\infty}}\right]. \end{aligned}$$

*Part 2* $N^{\beta,k}_{\alpha,C}$ norm. We split $B(u,v)$ as follows: $B(u,v) = B_1(u,v) + B_2(u,v)$ with

$$B_1(u,v)(t,x) = \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla \left[1 - \phi\left(\frac{x-x_0}{R^{1/2\beta}}\right)\right] u(s,x)\otimes v(s,x)ds$$
$$B_2(u,v)(t,x) = \int_0^t e^{-(t-s)(-\triangle)^\beta} P\nabla\phi\left(\frac{x-x_0}{R^{1/2\beta}}\right) u(s,x)\otimes v(s,x)ds$$

where $\phi_{R^{\frac{1}{2\beta}},x_0} = \phi((x-x_0)/R^{\frac{1}{2\beta}})$ for a smooth function $\phi$ supported in $B(0,15)$ and equals to 1 on $B(0,10)$.

For the estimate for $B_1$. Because $|P\nabla^{k+1}e^{-(t-s)(-\triangle)^\beta}(x)| \lesssim \frac{K(k)}{\left[(t-s)^{1/2\beta}+|x-y|\right]^{n+k+1}}$ and $0<t<R$, we have

$$\begin{aligned}
& |t^{\frac{k}{2\beta}}\nabla^k B_1(u,v)(t,x)| \\
\lesssim\ & t^{\frac{k}{2\beta}}\left|\nabla^k\int_0^t\int_{|y-x_0|\geq 10R^{1/2\beta}} e^{-(t-s)(-\triangle)^\beta}P\nabla(x-y)u(s,y)v(s,y)dyds\right| \\
\lesssim\ & K(k)t^{\frac{k}{2\beta}}\int_0^t\int_{|y-x_0|\geq 10R^{1/2\beta}} \frac{|u(s,y)||v(s,y)|dyds}{\left[(t-s)^{1/2\beta}+|x-y|\right]^{n+k+1}} \\
\lesssim\ & K(k)R^{\frac{k}{2\beta}}\int_0^R\int_{|y-x_0|\geq 10R^{1/2\beta}} \frac{|u(s,y)||v(s,y)|}{R^{\frac{n+k+1}{2\beta}}\left[(\frac{t-s}{R})^{\frac{1}{2\beta}}+\frac{|x-y|}{R^{1/2\beta}}\right]^{n+k+1}}dyds \\
\lesssim\ & K(k)R^{\frac{k}{2\beta}-\frac{n+k+1}{2\beta}}\sum_{q\in\mathbb{Z}^n}\frac{1}{|q|^{n+k+1}}R^{\alpha/\beta}\int_0^R\int_{|y-x_0|<R^{1/2\beta}}|u(s,y)||v(s,y)|\frac{dyds}{s^{\alpha/\beta}} \\
\lesssim\ & K(k)D(k)R^{\frac{1-2\beta}{2\beta}}\|u\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|_{X^{\beta,0}_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

Then we have, taking $R = r^{2\beta}$,

$$\begin{aligned}
& r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x|<r}\left|t^{\frac{k}{2\beta}}\nabla^k B_1(u,v)(t,y)\right|^2\frac{dydt}{t^{\alpha/\beta}} \\
& \lesssim (K(k)D(k))^2 r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x|<r} r^{2-4\beta}\|u\|^2_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|^2_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\frac{dydt}{t^{\alpha/\beta}} \\
& \lesssim (K(k)D(k))^2\|u\|^2_{X^{\beta,0}_\alpha(\mathbb{R}^n)}\|v\|^2_{X^{\beta,0}_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

For the estimate for $B_2$. We further split $B_2$ as $B_2 = B_2^1 + B_2^2$ with

$$B_2^1 = \frac{1}{\sqrt{-\triangle}}P\nabla\int_0^t e^{-(t-s)(-\triangle)^\beta}\frac{\triangle}{\sqrt{-\triangle}}\left(I - e^{-s(-\triangle)}\right)\left(\phi_{R^{\frac{1}{2\beta}},x_0}u(s,x)\otimes v(s,x)\right)ds,$$

$$B_2^2 = \frac{1}{\sqrt{-\triangle}}P\nabla e^{-t(-\triangle)^\beta}\int_0^t \phi_{R^{\frac{1}{2\beta}},x_0}u(s,x)\otimes v(s,x)ds.$$

At first we estimate the term $t^{\frac{k}{2\beta}}\nabla^k B_2^1$. Without loss of the generalization, we assume $k$ is odd. The proof of the case that $k$ is even is similar. If $k$ is odd, we have $k = 2K+1$ for $K \in \mathbb{Z}_+$. Because $\frac{1}{2} < \beta < 1$, we have

$$\begin{aligned} t^{\frac{k}{2\beta}} &= \left(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}} + s^{\frac{1}{2\beta}}\right)^{2K}\left(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}} + s^{\frac{1}{2\beta}}\right) \\ &= \sum_{l=0}^{2K}\begin{pmatrix} 2K \\ l \end{pmatrix}\left(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}}\right)^{2K-l} s^{\frac{l}{2\beta}}\left(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}} + s^{\frac{1}{2\beta}}\right) \\ &= \sum_{l=0}^{2K}\begin{pmatrix} 2K \\ l \end{pmatrix}\left(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}}\right)^{2K-l+1} s^{\frac{l}{2\beta}} + \sum_{l=0}^{2K}\begin{pmatrix} 2K \\ l \end{pmatrix}(t^{\frac{1}{2\beta}} - s^{\frac{1}{2\beta}})^{2K-l} s^{\frac{l+1}{2\beta}}. \end{aligned}$$

Then we have, setting $M(s,x) = \phi_{R^{\frac{1}{2\beta}},x_0}(x)u(s,x)\otimes v(s,x)$,

$$\begin{aligned} t^{\frac{l}{2\beta}}\nabla^k B_2^1 &= \sum_{l=0}^{2K-1}\begin{pmatrix} 2K \\ l \end{pmatrix}\frac{P\nabla}{\sqrt{-\triangle}}P^\beta_{2K-l+1}\left((-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{l}{2\beta}}\nabla^l M(s,x)\right) \\ &+\frac{P\nabla}{\sqrt{-\triangle}}P^\beta_1\left((-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{2K}{2\beta}}\nabla^{2K} M(s,x)\right) \\ &+\sum_{l=0}^{2K-1}\begin{pmatrix} 2K \\ l \end{pmatrix}\frac{P\nabla}{\sqrt{-\triangle}}P^\beta_{2K-l}\left((-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{l+1}{2\beta}}\nabla^{l+1} M(s,x)\right) \\ &+\frac{P\nabla}{\sqrt{-\triangle}}P^\beta_0\left((-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{2K+1}{2\beta}}\nabla^{2K+1} M(s,x)\right). \end{aligned}$$

Since $\sup_{s\in(0,\infty)} s^{1-2\beta}(1-e^{-s^{2\beta}}) < \infty$ for $\frac{1}{2} < \beta < 1$, we can obtain that $(-\triangle)^{1/2-\beta}(I - e^{-s(-\triangle)^\beta})$ is bounded on $L^2$ with operator norm $\lesssim s^{1-\frac{1}{2\beta}}$. By Lemma 5.3.3 and the $L^2-$boundedness of Riesz transform, we have

$$\begin{aligned} & r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-x_0|<r}\left|\frac{P\nabla}{\sqrt{-\triangle}}P^\beta_{2K-l}(-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{l+1}{2\beta}}\nabla^{l+1}M(s,x)\right|^2\frac{dxds}{s^{\alpha/\beta}} \\ &\le p(2K-l)r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{\mathbb{R}^n}\left|(-\triangle)^{\frac{1}{2}-\beta}(I-e^{-s(-\triangle)^\beta})s^{\frac{l+1}{2\beta}}\nabla^{l+1}M(s,x)\right|^2\frac{dxds}{s^{\alpha/\beta}} \\ &\le p(2K-l)r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-x_0|<r}\left|s^{1-\frac{1}{2\beta}}s^{\frac{l+1}{2\beta}}\nabla^{l+1}M(s,x)\right|^2\frac{dxds}{s^{\alpha/\beta}}. \end{aligned}$$

Because $0 < s < r^{2\beta}$ and

$$\begin{aligned} & s^{\frac{l}{2\beta}+1}\nabla^{l+1}M(s,x) = s^{\frac{l}{2\beta}+1}\nabla^{l+1}(\phi_{R^{\frac{1}{2\beta}},x_0}u(s,x)\otimes v(s,x)) \\ = & \sum_{m+\eta\le l+1}\left[s^{\frac{2\beta-1+m}{2\beta}}\nabla^m u(s,x)\right]\left[s^{\frac{\eta}{2\beta}}\nabla^\eta v(s,x)\right]\left[s^{\frac{l}{2\beta}+1-\frac{2\beta-1+m}{2\beta}-\frac{\eta}{2\beta}}\nabla^{l+1-m-\eta}\phi_{R^{\frac{1}{2\beta}},x_0}\right], \end{aligned}$$

then we can get, taking $R = r^{2\beta}$,

$$\begin{aligned} & r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-x_0|<r} \left| s^{1-\frac{1}{2\beta}} s^{\frac{l+1}{2\beta}} \nabla^{l+1} M(s,x) \right|^2 \frac{dxds}{s^{\alpha/\beta}} \\ \leq & \sum_{m+\eta\leq l+1} \|u\|^2_{N^{\beta,m}_{\alpha,\infty}(\mathbb{R}^n)} \|v\|^2_{N^{\beta,\eta}_{\alpha,C}(\mathbb{R}^n)}. \end{aligned}$$

In a similar way, we have

$$\begin{aligned} & r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-x_0|<r} \left| \frac{P\nabla}{\sqrt{-\triangle}} P_0^\beta \left( (-\triangle)^{\frac{1}{2}-\beta} (I - e^{-s(-\triangle)^\beta}) s^{\frac{2K+1}{2\beta}} \nabla^{2K+1} M(s,x) \right) \right|^2 \frac{dxds}{s^{\alpha/\beta}} \\ & \leq p(0) r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{\mathbb{R}^n} \left| s^{1-\frac{1}{2\beta}} s^{\frac{2K+1}{2\beta}} \nabla^{2K+1} M(s,x) \right|^2 \frac{dxds}{s^{\alpha/\beta}} \\ & \leq p(0) r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-x_0|<r} \left| s^{\frac{2K}{2\beta}+1} \sum_{m+\eta\leq 2K+1} \nabla^m u \nabla^\eta v \nabla^{2K+1-m-\eta} \phi_{R^{\frac{1}{2\beta}},x_0} \right|^2 \frac{dxds}{s^{\alpha/\beta}} \\ & \leq p(0) \left( \|u\|^2_{N^{\beta,0}_{\alpha,\infty}} \|v\|^2_{N^{\beta,2K+1}_{\alpha,C}} + \|v\|^2_{N^{\beta,0}_{\alpha,\infty}} \|u\|^2_{N^{\beta,2K+1}_{\alpha,C}} \right) + r(K) \|u\|^2_{\tilde{X}^{\beta,2K}_\alpha} \|v\|^2_{\tilde{X}^{\beta,2K}_\alpha}. \end{aligned}$$

Similarly we can estimate the terms associated with $P_1^\beta$ and $P^\beta_{2K-l+1}$. Combining all the estimates together, we can prove

$$\begin{aligned} & \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| t^{\frac{k}{2\beta}} \nabla^k B_2^1(u,v)(t,x) \right|^2 \frac{dxdt}{t^{\alpha/\beta}} \right)^{1/2} \\ & \leq C_1 \|u\|_{X^{\beta,0}_\alpha} \|v\|_{\tilde{X}^{\beta,k}_\alpha} + C_1 \|v\|_{X^{\beta,0}_\alpha} \|u\|_{\tilde{X}^{\beta,k}_\alpha} + C(k) \|u\|_{\tilde{X}^{\beta,k-1}_\alpha} \|u\|_{\tilde{X}^{\beta,k-1}_\alpha}. \end{aligned}$$

Now we estimate the term $B_2^2$. Taking the change of variables: $s = r^{2\beta}\theta$, $x = rz$ and $t = r^{2\beta}\tau$, we have

$$\begin{aligned} I &= r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| t^{\frac{k}{2\beta}} \nabla^k B_2^2(u,v)(t,x) \right|^2 \frac{dxdt}{t^{\alpha/\beta}} \\ &= r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| t^{\frac{k}{2\beta}} \nabla^k \frac{P\nabla}{\sqrt{-\triangle}} \sqrt{-\triangle} e^{-t(-\triangle)^\beta} \int_0^t M(s,x) ds \right|^2 \frac{dxdt}{t^{\alpha/\beta}} \\ &\leq r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{\mathbb{R}^n} \left| t^{\frac{k}{2\beta}} \nabla^{k+1} e^{-t(-\triangle)^\beta} \int_0^t M(s,x) ds \right|^2 \frac{dxdt}{t^{\alpha/\beta}} \\ &\leq r^{2\alpha-n+2\beta-2} \int_0^1 \int_{\mathbb{R}^n} \left| (\tau r^{2\beta})^{\frac{k}{2\beta}} \frac{1}{r^{k+1}} \nabla_z^{k+1} e^{-\tau(-\triangle_z)^\beta} \int_0^\tau M(r^{2\beta}\theta, rz) r^{2\beta} d\theta \right|^2 \frac{r^{n+2\beta} dz d\tau}{r^{2\alpha} \tau^{\alpha/\beta}} \\ &\leq r^{8\beta-4} \int_0^1 \int_{\mathbb{R}^n} \left| \tau^{k/2\beta} \nabla_z^{k+1} e^{-\tau(-\triangle_z)^\beta} \int_0^\tau M(r^{2\beta}\theta, rz) \right|^2 \frac{dz d\tau}{\tau^{\alpha/\beta}}. \end{aligned}$$

Denote by $\nabla_z^\nu e^{-\tau(-\triangle_z)^{\beta/2}}(x,y)$ the kernel of the operator $\nabla_z^\nu e^{-\tau(-\triangle_z)^\beta/2}, \nu>0$. Because $\frac{1}{2}<\beta\le 1$, we have

$$\left|\tau^{\frac{k(1-\beta)}{2\beta}}\nabla_z^{k(1-\beta)}e^{-\tau(-\triangle_z)^\beta/2}(x,y)\right| \lesssim \tau^{\frac{k(1-\beta)}{2\beta}}\frac{1}{(\tau/2)^{\frac{k(1-\beta)+n}{2\beta}}}\frac{1}{(1+\frac{|x-y|}{\tau^{1/2\beta}})^{n+k(1-\beta)}} \in L^1(\mathbb{R}^n)$$

uniformly in $\tau$. By Young's inequality and Lemma 5.2.2, we have

$$\begin{aligned} I &= r^{8\beta-4}\int_0^1\int_{\mathbb{R}^n}\left|\tau^{\frac{k(1-\beta)}{2\beta}}\nabla_z^{k(1-\beta)}e^{-\tau(-\triangle_z)^\beta/2}\tau^{\frac{k}{2}}\nabla_z^{k\beta+1}e^{-\tau(-\triangle_z)^\beta/2}\int_0^\tau M(r^{2\beta}\theta,rz)\right|^2\frac{dzd\tau}{\tau^{\alpha/\beta}}\\ &\le r^{8\beta-4}\int_0^1\int_{\mathbb{R}^n}\left|\tau^{\frac{k}{2}}\nabla_z^{k\beta+1}e^{-\tau(-\triangle_z)^\beta/2}\int_0^\tau M(r^{2\beta}\theta,rz)\right|^2\frac{dzd\tau}{\tau^{\alpha/\beta}}\\ &\le r^{8\beta-4}b(k)A(\alpha,\beta,M)\int_0^1\int_{\mathbb{R}^n}|M(r^{2\beta}\theta,rz)|\frac{dzd\theta}{\theta^{\alpha/\beta}}\\ &:= r^{8\beta-4}b(k)A(\alpha,\beta,M)I_M. \end{aligned}$$

For $A(\alpha,\beta,M)$, we have

$$\begin{aligned} A(\alpha,\beta,M) &= \rho^{2\alpha-n+2\beta-2}\int_0^{\rho^{2\beta}}\int_{|y-x|<\rho}|M(r^{2\beta}s,ry)|\frac{dsdy}{s^{\alpha/\beta}}\\ &\le r^{2-4\beta}(r\rho)^{2\alpha-n+2\beta-2}\int_0^{(r\rho)^{2\beta}}\int_{|z-rx|<r\rho}|M(t,z)|\frac{dzdt}{t^{\alpha/\beta}}\\ &\le r^{2-4\beta}\|u\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}\|v\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}. \end{aligned}$$

For $I_M$, we have

$$\begin{aligned} &\int_0^1\int_{\mathbb{R}^n}|M(r^{2\beta}\theta,rz)|\frac{dzd\theta}{\theta^{\alpha/\beta}}\\ \le &\int_0^{r^{2\beta}}\int_{\mathbb{R}^n}|M(t,z)|\frac{r^{-2\beta-n}dtdz}{r^{-2\alpha}t^{\alpha/\beta}} \le r^{2-4\beta}\|u\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}\|v\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}. \end{aligned}$$

Then we get

$$r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}\left|t^{\frac{k}{2\beta}}\nabla^k B_2^2(u,v)(t,x)\right|^2\frac{dxdt}{t^{\alpha/\beta}} \le b(k)\|u\|^2_{X_\alpha^{\beta,0}(\mathbb{R}^n)}\|v\|^2_{X_\alpha^{\beta,0}(\mathbb{R}^n)}.$$

Now we have proved that

$$\begin{aligned} \|B(u,v)\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)} &\le C_0(k)\|u\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}\|v\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)} + C(k)\|u\|_{\tilde{X}_\alpha^{\beta,k-1}(\mathbb{R}^n)}\|v\|_{\tilde{X}_\alpha^{\beta,k-1}(\mathbb{R}^n)}\\ &\quad +C_1\|u\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}\|v\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)} + C_1\|u\|_{X_\alpha^{\beta,k}(\mathbb{R}^n)}\|v\|_{X_\alpha^{\beta,0}(\mathbb{R}^n)}. \end{aligned}$$

Similar to the method applied in Lemma 4.3 of [28], if we construct the approximating sequence $u^j$ by

$$u^{-1}=0,\ u^0=e^{-t(-\triangle)^\beta}u_0,\ u^{j+1}=u^0+B(u^j,u^j),$$

we can get the following lemma and hence complete the proof of Theorem 5.3.5. □

**Lemma 5.3.6** *Let $\alpha > 0$, $\alpha + \beta \geq 1$ and $\max\{\frac{1}{2}, \alpha\} < \beta < 1$. Suppose $u_0$ be small enough in $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. Then for any $k \geq 0$, there exist constants $D_k$ and $E_k$ such that*

$$\|u^j\|_{\widetilde{X}^{\beta,k}_{\alpha}(\mathbb{R}^n)} \lesssim D_k \quad \text{and} \quad \|u^{j+1} - u^j\|_{\widetilde{X}^{\beta,k}_{\alpha}(\mathbb{R}^n)} \lesssim E_k \left(\frac{2}{3}\right)^j.$$

*In particular, for any $k \geq 0$, $u^j$ converges in $\widetilde{X}^{\beta,k}_{\alpha}(\mathbb{R}^n)$.*

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