WHAT MAKES AN EXPLANATION A GOOD EXPLANATION? ADULT LEARNERS' CRITERIA FOR ACCEPTANCE OF A GOOD EXPLANATION

ROSEMARY ROBERTS

What Makes An Explanation A Good Explanation?
Adult Learners' Criteria For Acceptance Of A Good Explanation

by

Rosemary Roberts

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Master's of Education

Department of Teaching and Learning
Faculty of Education
Memorial University of Newfoundland
1999

St. John's Newfoundland

Abstract

The problem that students have perceiving a need for proof is well-known to high school teachers and has been identified by researchers as a major problem in the teaching of proof. My research addresses the problem of teaching of proof, especially the role of proof as explanations for students. My study builds on Hoyles' (1997) and Reid's (1995a) studies to explore what qualities make an explanation a good explanation for the student.

Through a questionnaire, classroom observations, and interviews with students and their teachers I researched the kinds of explanations students prefer, what constitutes a good explanation for students and teachers, and whether or not students mirror teachers' explanations or if they have their own style of explaining.

Both quantitative and qualitative research methods were employed to collect and report the findings. A student questionnaire, set in two domains of mathematics, geometry and arithmetic/algebra, comprised the quantitative part of the study. To help determine the kind of explanation preferred, the questionnaire offered deductive, inductive and analogical explanations. The student questionnaire was administered to adult learners who were enrolled in the trades, technician, business, applied arts, and Adult Basic Education (ABE) programs at the College of the North Atlantic, Happy Valley-Goose Bay campus.

Interviews, participant observations and document analysis comprised the qualitative part of the study. Person-to-person, semi-structured interviews were conducted with eight adult learners enrolled in the ABE program at the same college. The two ABE mathematics instructors also participated in the person-to-person interviews. Both students and their

instructors were observed within their classroom setting. The interviews and observations helped to determine students' preference for a particular kind of explanation, what qualities make an explanation a good explanation for the student and for the teacher and whether or not students mirror teacher explanations. Document analysis involved an intense literature review of proof, proving, and the different purposes proving serves.

Students showed an overall preference for multiple example explanation and analogical explanation. It was the form of the explanation, namely its familiarity and accessibility, that students used as criteria for acceptance. The logical structure of an explanation was what the teacher used as criteria for acceptance. Conforming to teacher expectations was seen as a motivation for proving in the classes observed.

Acknowledgements

By writing this thesis, I have gained insight into my teaching methods and the way students learn. My thesis has broaden my perspective on what is proof, what is proving? Although it has been a learning experience, it has also been a difficult process. Many sacrifices have been made to further my education and learning. I am very grateful to my husband and son who have supported me through out the entire process. As well, I am very grateful to have friends like Cathy Jong who was willing to entertain my ideas and provide constructive feedback that helped to focus my thinking.

David Reid, as supervisor was always willing to listen to my views and to provide intellectual support that helped to broaden and focus my thinking. I must also acknowledge the students and teachers who took part in my study, whose names are hidden, but whose importance is obvious.

As I complete this thesis as partial fulfilment of the requirements for the degree of Master's of Education, I cannot help but think of my son, David who has just started kindergarten. I hope that he is successful in his pursuit of learning.

Dedication

This thesis is dedicated to the memory of my parents Edward Jerome White and Mary White (nee Dawson).

Table of Contents

List of Tables

List of Figures

Chapter I

STATEMENT OF THE PROBLEM

My research addresses the problem of the teaching of proof, especially the role of proof as explanations for students. More specifically, I researched what qualities make an explanation a good explanation in the eyes of students.

The problem that students have perceiving a need for proof is well-known to high school teachers and has been identified by researchers as a major problem in the teaching of proof. Wheeler (1990) comments on the fact that teaching proof will always be a difficult thing in a mathematics classroom because it was not put there for any clear reason other than to imitate the activities of mathematicians. De Villiers (1990) claims that the role and function of proof in the classroom has either been completely ignored, or it has been presented as a means of certainty. Hoyles (1997) claims that students' difficulty with proof may be attributed to proof's ambiguous meaning and the fact that proof requires the coordination of a range of competencies such as identifying assumptions and organizing logical arguments. Hoyles believes that many students have a limited awareness of what proof is about. "Students are unable to distinguish between empirical argument and deductive reasoning" (p. 7). She maintains that many students believe they have proved a conjecture if their examples verify the statement.

Deductive reasoning serves different purposes depending on the user. While in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation (Hanna, 1990). For Hanna, "a proof that explains" proves using evidence derived from the phenomenon itself while at the same time showing why a statement is true. Many other researchers (e.g., de Villiers 1990, Reid 1995a, Bell 1976) have investigated the need and function of proof in the search for an alternate way to teach proof.

For the mathematician, proof serves not only the need to verify, but also the need to understand the *why* of one's mathematical discoveries (Thurston, 1994). Ironically, the teaching of proof in schools continues to focus on proving to verify. The teacher may perceive proof as a means of establishing truth or validity while ignoring the *why*. These beliefs are then passed on to students, leaving students with the belief that proof establishes certainty, which may contribute to student difficulty in understanding proof (Hanna, 1990).

A. What is a proof? What is proving?

Researchers, mathematicians, math educators and students all differ in their perceptions of what a proof is. A logical or convincing argument using deductive reasoning is a consistent characteristic among these varying perceptions.

Among researchers, Hanna states that "a formal proof (the succession of statements according to rules of inference), mechanizable in nature, is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference and the last sentence is the one to be proved" (p.6). Barbeau (1990) describes a formal proof as "a succession of statements ordered according to rules of inference" (p.24). For Reid (1995a), proving is "investigating using deductive reasoning"; "reasoning that proceeds from agreed upon premises to conclusions, using logical arguments" (p.7). The deductive aspect of proof characterizes the way of investigating which Reid calls proving. I have used Reid's definition of proof as my working definition.

B. What is an explanation?

One of the many uses of proving is explanation. Bell (1976), de Villiers (1990), Thurston (1994), Hanna (1990), Reid (1995a) and Hoyles (1996), all suggest explaining as a need to prove. Hersh (1993) believes that the main function of proof is to explain. Both Manin (1981) and Bell (1976) believe that explanation is a criterion for a good proof. For Thurston (1994), " the measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics" (p. 163).

The Gage Canadian Dictionary (1983) defines *explain* as "to make clear or understandable; tell what something means or how something is done, organized, or used; give an acceptable reason for; excuse or justify." Reid (1995a, p.22) describes explaining as something which provides connections between what is known and what is being proven in a way that clarifies *why* a statement is true. An explanation for Hanna (1995) would be that which promotes understanding and makes us wiser. Hanna uses the term *explain* when a proof reveals and makes use of the mathematical ideas which motivate it. According to Hanna (1990), a proof that explains must provide a rationale based upon the mathematical properties that cause the asserted theorem or other mathematical statement to be true. The quest for an explanation is an attempt to find a rationale which may or may not be reduced to deductive proof (Sierpinska, 1994).

Based on the literature related to explanatory proofs, I proposed the following as my working definition of an explanation. An explanation uses mathematical properties to demonstrate why mathematical discoveries are true. The *why* may or may not be a deductive proof.

Chapter II

PURPOSE OF THE STUDY

My study builds on Hoyles' (1997) and Reid's (1995a) studies to explore explanations offered to students. Through investigation and observation, I addressed the following questions:

- What kinds of explanations do students prefer? (E.g., Deductive, inductive, or analogical)
- What constitutes a good explanation for teachers?
- What constitutes a good explanation for students?
- Do students mirror teachers' explanations or do they have their own style of explaining?

Using these four questions, I explored how the research of others fits with students' behaviour in Labrador, possibly adding to what others have discovered.

A. Kinds of Explanations

Many kinds of explanations have been identified by researchers in mathematics education. They fall into three main types: deductive, inductive, and analogical.

A. (1) Deductive Explanations

Reid (1995a) defines deductive reasoning as "reasoning that proceeds from agreed upon premises to conclusions, using logical arguments" (p.7). Deductive explanations can be further classified as: formulated or unformulated. Reid distinguishes between "formulated" and "unformulated" deductive reasoning with formulation based on the "provers' knowledge that they are proving" (p.25). The prover is unaware that he/she is proving when providing an unformulated explanation. Reid found that unformulated explanations as explanations for others were inadequate because of hidden assumptions and the lack of articulation. This lack of articulation and hidden assumptions accompanying unformulated proving makes it difficult for others to understand.

Deductive formulated and unformulated explanations may be presented in the form of a proof. Pre-formal, semi-formal, formal, and informal proofs are terms used to describe the degree of formality of a proof.

A pre-formal proof might appear in the working notes of a mathematician (Reid, 1995a). Such a proof might include references to inductive or analogical evidence and involve hidden assumptions, and the use of informal language.

A semi-formal proof consisting of deductive arguments using formal symbols and natural language is presented in a form suitable for publication. In a semi-formal proof, unusual assumptions are made explicit. If steps are omitted in a semi-formal proof, a note to the reader will show how these steps may be worked out.

A formal proof includes all steps and all assumptions are made explicit. The language of a formal proof is symbolic.

The prover is unaware that he/she is proving in an informal proof. Thus, an informal proof is considered to be unformulated but deductive.

Reid (1995a) describes an additional type of deductive proof called "formulaic" proof. According to Reid, a formulaic proof is not proving. A formulaic proof is constructed according to principles which are associated with the creation of the sorts of proofs teachers like. Formulaic proofs result from what Reid calls "teacher-games".

A. (2) Inductive Explanations

Inductive reasoning is a kind of reasoning in which the conclusion is based on several past observations. That is, an inductive explanation makes a generalization based on several specific cases. For example, making the generalization that the sum of two odd numbers is even from specific cases such as 3 + 5 = 8, 21 + 33 = 54, 67 + 45 = 112.

Inductive explanations can be further classified as: single example, multiple example or generic example. A single example explanation would involve the use of an example to show that the conjecture holds true. The example may be presented numerically, visually or symbolically.

Multiple example explanation would include more than one example as empirical evidence that a conjecture holds true. Multiple example explanation might be presented numerically, visually, or symbolically.

According to Reid (1998), a generic example is one where the example is not critical. Using 7 + 11, it can be shown through generic example that the sum of two odd numbers is even. Both eleven and seven can be written as the sum of two numbers, for example 11 = 10 + 1 and 7 = 6 + 1. Therefore 11 + 7 = 10 + 1 + 6 + 1. Since 6 + 10 is even, because it is the sum of two even numbers and 1 + 1 = 2 which is even implies that 6 + 10 + 2 is even. So, 11 + 7 = 18 (an even number). This example shows that for the two odd numbers 7 and 11 their sum is even and that this method can be seen to work for all cases.

A. (3) Analogical Explanations

An analogy involves making a conjecture based on similarities between two cases. For example, one might conjecture that the sum of two odd numbers is even from the principle which says that the product of two negative numbers is positive. In Reid's (1995a) study, some students used analogy for their explanations. Bill, a participant in Reid's study used the principle that the product of two negative numbers is positive to explain why the sum of two odd numbers is even. Using Bill's analogy, Reid explains how reasoning by analogy can be based on a case and lead to a case, or be based on a rule and lead to a rule. Bill's analogy shows how he reasoned from a rule to a rule. In Reid's study, students' analogies are described as strong or weak based on the level of understanding provided. Reid claims that a strong analogy may not only satisfy a need for explanation, but may also be preferred over a deductive explanation.

B. A Good Explanation

B. (1) What constitutes a good explanation for teachers?

De Villiers (1990) believes that proof is either completely ignored within the classroom or is presented as a means of certainty. Hoyles' (1997) discussions with teachers in the UK reveal that some teachers are comfortable with an informal explanation while others require a formally presented logical argument. Wheeler (1990) claims that teachers teach students algorithms as a recipe for understanding proof. Students in Hoyles study have a clear perception of what the teacher expects as an explanation. In Hoyles' (1997) study, she discovered that students chose general proofs as the type of proof which would be assigned the best mark; whereas, proofs that were both general and explanatory were chosen by students for understanding. Thus, the teacher must have a distinct method of explaining which is being relayed to students.

B. (2) What constitutes a good explanation for students?

It is clear from the research that explaining is important to students. Schoenfeld believes that students are like mathematicians in that students want to understand why a mathematical proposition is valid. De Villiers (1990) suggest that students develop an inner compulsion to understand why a conjecture is true if they have seen it to be true. Dreyfus and Hadas (1996) reports that students feel a need to prove in order to explain phenomena they could not explain otherwise or in order to convince themselves of counter-intuitive results.

"With reference to Mason's (1982) statement that when you prove, first you convince yourself, then convince a friend, and then convince an enemy, . . . It is my contention that in order for an argument to be considered a proof, the students have to not only convince themselves and others, but also explain" (Zack,1997, p.292). Hanna (1995) has asserted that in education proofs that explain should be favoured over those that merely prove; the children also seem to seek proofs which explain.

There seems to be little discussion in the literature of what features of an explanation are important for students' acceptance of them. Hoyles (1996) discovered that the form of an explanation was important for students' acceptance of them. In her study, students chose narrative explanations as their individual preference.

B. (3) Do students mirror teachers' explanations or do they have their own style of explaining?

Hoyles' (1997) study seeks to investigate the influence of the teacher on students' responses to proof. While some teachers were comfortable with informal explanations, Hoyles discovered that others would require a logical argument. In her study, students chose for themselves proofs that were general and explanatory. Yet, proofs that students believed would be assigned the best mark were evaluated as general but not explanatory. According to Hoyles, students in her study connect the requirement to prove with the prescribed format and language of presentation found in the investigations part of the UK curriculum. Hoyles found that students employed the same type of proof on every question regardless of whether it was appropriate. Hoyles' research demonstrates teachers' influence on students' understanding of proof. Her research highlights the role of the curriculum in shaping students' perceptions and approaches to proof.

Hoyles' (1997) study clearly shows that students' perceptions of what is acceptable as a proof and what is valued as a proof differs. Reid (1995a), Alibert (1988) and Schoenfeld (1983) have all recognized the importance of conforming to teacher expectations as a motivation for proving. Mok (1997) infers that the formulation of students' explanations may be a result of their learning experience in the course of instruction. Wong (1993) believes this to be true in Hong Kong where reception learning is the typically preferred model and students do not expect the opportunity to articulate their mathematical thoughts. De Villiers' (1992) study shows that students' strength of belief in, or attachment to a particular method is based on external rather than personal grounds.

Chapter III

SIGNIFICANCE AND LIMITATIONS OF THE STUDY

A. *Significance of the Study*

"Proof has been relegated to a less prominent role in the secondary mathematics curriculum in North America over the last thirty years or so." "This has come about in part because many mathematics educators have been influenced by certain developments in mathematics and mathematics education to believe that proof is no longer central to mathematical theory and practice and that its use in the classroom will not promote learning in any case" (Hanna, 1996). Hanna infers that mathematics educators have sought relief from the effort of teaching proof by avoiding it altogether. "The use of computer-assisted proofs, the growing recognition accorded mathematical experimentation, and the invention of new types of proof that do not fit the standard have led some to argue that mathematicians will come to accept such forms of mathematical validation in place of deductive proof" (Hanna, 1996). Such notions have caused great concern for researchers like Hanna, who believe that proof is a central feature of mathematics and a valuable tool for promoting mathematical understanding.

Do we need proof in mathematics education? Schoenfeld (1994) replies unequivocally: "Absolutely. Need I say More? Absolutely." (Schoenfeld quoted in Hanna, 1995, p.49). Proof and proving can promote mathematical understanding. Utilizing the many functions of proof within the mathematics classroom can help make proof a more meaningful activity. More specifically, the explanatory function of proof should be stressed in situations where conviction already exists to satisfy students' need for explanation.

My research addresses the problem of the teaching of proof, especially the role of proof as explanations for students. By investigating what makes an explanation a good explanation, my findings contributes to the teaching of proof as a meaningful activity. I expect my findings to be used to enhance the teaching of proof, in particular, its explanatory function. In addition, the study provides insight into ways teachers can formulate their explanations to enable better student understanding.

B. Limitations of the Study

The research reported in this thesis combines qualitative and quantitative techniques. The results from the quantitative part of the study (a questionnaire) are generalisable to the geographical location (Labrador) and student population (adult learners who were enrolled in the trades, technician, business, applied arts, and Adult Basic Education (ABE) programs at the College of the North Atlantic, Happy Valley-Goose Bay campus). The remainder of the study, in keeping with its qualitative character, seeks to understand and interpret how the various participants in a social setting construct the world around them. The study design focuses on in-depth interaction with eight research participants. The main research instrument is the researcher who interacts with the study's participants by observing them (in their mathematics classes) and by asking questions (in semi-structured interviews). This allows an element of subjectivity into the research as the researcher explores and interprets multiple possible realities. Because of the mixture of qualitative and quantitative techniques, the study as a whole is neither reproducible nor generalisable. Instead it offers qualitative insight into individual appreciations of mathematical explanations, which illuminate generalisations based on the quantitative results.

Chapter IV

LITERATURE REVIEW

The literature review looks at what researchers have to say about what a proof is, what purpose proof serves, the kinds of proof, the criteria for a good proof and the teaching of proof. Hoyles' discoveries of students' understanding of proof are reviewed as well.

A. What is a proof? What is proving?

In his doctoral thesis, Reid (1995a, p.6) asks "what is proving?". Responses provided include: "proving is making a proof" and "a proof is what results from proving". According to Reid, "proving is investigating using deductive reasoning" (p.7). "Deductive reasoning refers to reasoning that proceeds from agreed upon premises to conclusions, using logical arguments" (p.7). According to Hanna (1990), "a formal proof (the succession of statements according to rules of inference), mechanizable in nature is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference and the last sentence is the one to be proved" (p.6). A logical or convincing argument using deductive reasoning seems to be consistent characteristics in the many definitions of proof and proving found in the research. Yet, the many definitions of proof differ depending on what need proof fulfils.

B. Purposes of Proof

Proving satisfies many needs including: verification, explanation, exploration, systematization, communication, aesthetics, personal self-realization, developing logical thinking and teacher games.

B. (1) Verification

Verification (conviction or justification of the correctness of mathematical statements) has been the main focus and/or function of proof.

Bell (1976) has identified verifying as a need to prove. Verifying determines the truth or falsity of a statement whose value of truth is questioned. For Wilder (1944) a proof is " only a testing process and we apply to these suggestions of our intuitions" (p.318). Kline (1973) states that "a proof is only meaningful if it answers the student's doubts, when it proves what it is not obvious" (p.151). Alibert (1988) says that "the necessity, the functionality of proof can only surface in situations in which the student meets uncertainty about the truth of mathematical proposition" (p.31).

For Hanna (1989), " a proof is an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing" (p.20). Volminik (1990) in *Pythagoras* states that "we may regard proof as an argument sufficient to convince a reasonable skeptic" (p.10). Mason, Burton, and Stacey (1982) propose three stages in the putting up of a convincing argument, namely convincing oneself, convincing a friend and convincing an enemy. Movshovitz (1988) and Alibert (1988) have provided ways of presenting theorems and proofs, such as the stimulating response method, and the scientific debate method. In their presentations, proof is viewed as a valid argument.

De Villiers (1990) argues that conviction provides the motivation for a proof. Many teachers believe that a proof provides absolute authority in the establishment of the validity of a conjecture. For example, "a proof in mathematics consists of steps that show how one statement follows logically from other statements" (Jurgensen, Maier, & Donnelly, 1973). Alberta Education (1991) as cited in Reid (1995a, p.7) defines to prove as " to substantiate the validity of an operation, solution, formula or theorem in general and to provide logical arguments for each step in the process." In this sense proving is considered as logical, deductive and certain. Thus, proving is concerned with establishing validity. Students, themselves, view *proving* as making certain and believe that inductive evidence provided by examples is sufficient. Reid 's (1995a) research supports this claim.

B. (2) Explanation

For Balacheff, a proof is an explanation by virtue of it being a proof.

> We call an explanation the discourse of an individual who aims to establish for somebody else the validity of a statement. The validity of an explanation is initially related to the speaker who articulates it. We call proof an explanation which is accepted by a community at a given time. We call a mathematical proof a proof accepted by mathematicians. As a discourse, mathematical proofs have now-a-days a specific structure and follow well defined rules that have been formalized by logicians. (Balacheff, 1988 cited in Hanna, 1990, p.47).

For Balacheff, then, all proofs would seem to be explanations. Yet not all proofs have explanatory power (Hanna, 1990).

De Villiers (1990) recommends that the explanatory function of proof should be utilized to present proof as a meaningful activity to students rather than focusing on proof as a means of verification -- especially when a high level of conviction already exist.

According to de Villiers, it is not a question of making sure, but rather a question of explaining why. De Villiers believes that stressing the explanatory function of proof in situations where conviction already exists, may not only make proof potentially more meaningful to students, but it is in such cases probably more intellectually honest. Students do research like mathematicians. Both are easily convinced by authority and gain conviction by means of intuition and quasi-empirical testing. "Like mathematicians, students exhibit an independent need for explanation which seems to be satisfied by the production of some sort of logico-deductive argument" (de Villiers, 1990).

Within a social context, do students employ deductive reasoning to explain and to explore problem solving situations? Reid's (1995a) study examines this question. After analyzing student responses to problem prompts, Reid concluded that within a social context it was natural for students to explain but not necessarily to verify. Reid concludes that the purpose of proving in a classroom context, particularly, that of verification, should be replaced by explanation.

Reid found that analogy can be a powerful method of explaining in mathematical situations. Analogies were described as strong or weak based on the level of understanding. A strong analogy can satisfy a need for explanation. A strong analogy, as Reid discovered can be preferable to a deductive explanation. Explaining by analogy was more or less successful depending on the "strength of the analogy" (p.38) (see also Polya 1968).

Recognizing that a proof that proves and a proof that explains are both legitimate proofs, Hanna distinguishes between the two. She states that a proof that proves provides essential reasons to show that a theorem is true. Substantiation is its main concern: that is, with *Rationes cognoscendi*, - why-we-hold-it-to-be-so reasons. For Hanna (1995), a proof that explains also shows why the theorem is true; it provides a set of reasons that derive from the phenomenon itself. *Rationes essendi*, - why-it-is-so-reasons. Mathematical induction or syntactic considerations alone may suffice for a proof that proves. But a proof that explains must provide a rationale based upon the mathematical ideas involved, the mathematical properties that cause the asserted theorem to be true. When a proof reveals and makes use of the mathematical ideas which motivate it, it is a proof that explains. Like Steiner (1978), Hanna agrees that a proof explains when it shows what characteristic property entails the theorem it purports to prove.

Steiner (1978) is quoted by Hanna (1990), as saying that

> . . . an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the results depend on the property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response. (p.143)

Unlike Balacheff who believes that a proof is an explanation by virtue of it being a proof, Hanna prefers to use the term explain only when the proof reveals and makes use of the mathematical ideas which motivate it. In line with Steiner, Hanna says that a proof explains when it shows what characteristic property is included in the theorem that it purports to prove. According to Hanna (1989), the first step in promoting understanding through explanatory proof is to recognize that understanding is much more than confirming that all the links in a chain of deduction are correct, that in fact the completeness of detail in a formal deduction may obscure rather than enlighten, and that understanding requires some appeal to previous mathematical experience. For Manin (1981, p.1071) and Bell (1976, p.24), explanation is a criterion for a good proof when stating respectively that it is convincing " one which makes us wiser " and that it is expected to convey an insight into why the proposition is true.

Schoenfeld (1982) describes proving for the mathematician as . . .

> Proving is a means of coming to understand, and of coming to know what understanding is. In trying to prove something new, one is asking what makes it tick; in trying alternative proofs, rejecting them, modifying them, one is discovering things about its structure - and solidifying one's knowledge in the process. This is the deep reason for much of the emphasis on proof in mathematics. The mathematician comes to accept proving as a way (if not <u>the</u> way) of thinking, a way of demanding and insuring that he does indeed understand. (Schoenfeld, 1982, p.168, emphasis in original, cited in Reid, 1995, p.8)

B. (3) Exploration, Analysis, Discovery and Invention

For the mathematician, proof is a means of exploration, analysis, discovery and invention. Both de Villiers (1990) and Reid (1995a) believe exploring motivates proving. "Exploration extends the bounds of what is known" (Reid, 1995a, p.21).

B. (4) Systemazation and Communication

Proof is seen as a means of systematization of various known results into a deductive system of axioms, definitions and theorems. "It appears that proof is a form of discourse, a means of communication among people doing mathematics". (Volminik, 1990, p.8) as cited in de Villiers (1990, p.22). De Villiers (1990, p.22) quotes Davis and Hersh (1986, p.73): "In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is - namely, a human interchange based on shared meanings, not all of which are verbal or formulaic."

Developing logical thinking was seen as a purpose for teaching proof. Yet while research conducted by Sekiguchi, (1991, p.26) shows that there is little transference of proof skills learned in mathematics to other contexts, some teachers still believe that this is the primary function of proof.

B. (5) Teacher-games

In a teacher-game (Reid 1995a, p.23), students try to satisfy the implicit or explicit demands of the teacher. Attempting to achieve a high grade, or avoiding social discomfort are reasons why students play teacher-games. Playing a teacher-game as a motivation for proving has been recognized by Alibert (1988) and Schoenfeld (1983) who point to the importance of conforming to teacher expectations.

C. Kinds of Proof

As stated earlier, proving satisfies many needs including verification, explanation, exploration, systematization, communication, developing logical thinking and teacher games. Different kinds of proofs help to satisfy these needs. Reid (1995b) distinguishes between formulated and unformulated proving for the purpose of explanation.

According to Reid (1995a) formulated proving allows extended explanations beyond what analogy can provide. Mok (1997) cites Reid's (1995b) distinction of formulated and unformulated proving to explain. Formulation describes "the degree to which the proof is thought of and thought out and is related to the articulation and hidden assumptions while proving" (Reid, 1995b, p.137). Explaining can be done by proving, which can be more or less formulated. Formulated proving is not necessarily preferred over explaining by analogy. Appropriate social context is needed for formulated proving and for explaining. Either, a social context in which formulated proving is already occurring to address another need, or one in which there is a strong need to explain to others, or one in which a teacher (present or in the past) indicates that formulated proving should be used (Reid, 1995b).

Unformulated explanations are limited precisely because they are unformulated (Reid, 1995b). According to Reid, unformulated proving as an explanation is inadequate, because of its lack of articulation and hidden assumptions, which prevents other people from being able to understand it. Unformulated explanations in classroom situations are useless; but, as explanations for an individual, they may work, if the argument required is short.

Mok (1997) classifies students' formulation of an explanation into an hierarchy beginning with prestructural progressing through unistructural, to multistructural and extended abstract. Prestructural explanations involve those formulations of explanations in which students are not really engaged. Unistructural explanations are usually in the form of recalling familiar procedural rules. They are brief, suggesting quick closure and may be inconsistent. Unistructural/multistructural explanations tend to be short and straightforward although students do attempt to elaborate. Rational explanations are founded on relevant clues; i.e. the operations and variables. Extended abstract involves an attempt to prove. Students no longer rely on observed cases. At this level, students justify their formulated hypothesis through a chain of coherent arguments. Mok concluded from his study that if students thought an explanation was simply recalling facts/rules or carrying out routine manipulations, then they would not be likely to give high level explanations.

D. The Role of Proof in Mathematics Education

Hanna (1990) uses three different aspects of proof, formal proof, acceptable proof and the teaching of proof, to help distinguish among different perceptions of proof.

D. (1) Formal Proof

For Hanna, "a formal proof (the succession of statements according to rules of inference), mechanizable in nature is a finite sequence of sentences such that the first sentence is an axiom. each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference and the last sentence is the one to be proved" (p.6).

D. (2) Acceptance

Realizing that proofs may have different degrees of formal validity and still gain the same degree of acceptance, mathematicians and mathematics educators have come to reassess the role of formal proof. According to Hanna (1990) mathematicians freely admit that a proof may lack conviction when it is shown to be valid by virtue of its form alone, without regard to its content. The significance of what is proved rather than the correctness of a proof determines its acceptance. "The acceptance of a theorem by practising mathematicians is a social process which is more a function of understanding and significance than of rigorous proof" (p.8).

Hanna (1983) provides the following criteria for mathematicians' acceptance of proofs:

> Most mathematicians accept a new theorem when some combination of the following factors is present:
>
> 1. They understand the theorem, the concepts embodied in it, its logical antecedents, and its implications. There is nothing to suggest that it is not true;
> 2. The theorem is significant enough to have implications in one or more branches of mathematics (and thus important and useful enough to warrant detailed study and analysis);
> 3. The theorem is consistent with the body of accepted mathematical results;
> 4. The author has an unpeachable reputation as an expert in the subject matter of the theorem;
> 5. There is a convincing argument for it (rigorous or otherwise), of a type they have encountered before.
>
> If there is rank order of criteria for admissibility, then these five criteria all rank higher than rigorous proof. (p. 70)

Hanna offers an alternative approach to proof based on explanatory proofs, proofs that are acceptable from a mathematical point of view. For Hanna, an explanatory proof focuses on understanding rather than on syntax requirements and formal deductive methods.

D. (3) Teaching of proof

According to Bolzano as cited in Hanna (1990), making certain requires no more than a formal demonstration, while building a foundation demands an approach which also provides insight into the connections among ideas. The focus of an explanatory proof is not upon the deductive mechanism but upon understanding. The teacher's argument must indicate why the result is taught, whether for its beauty, usefulness, or critical importance in the development.

Wheeler (1990) claims that because it is so difficult to teach students proof, teachers teach them algorithms as a sort of recipe. The recipe has become what is commonly referred to as the *T-proof* or two column proof which Wheeler believes defeats the purpose of proofs. Hoyles (1997) argues that the ambiguity of proof makes proof difficult for students to master. Hoyles questions the existence of a hierarchy of proving competencies. Proof has the purpose of verification – confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. If proof simply follows conviction of truth rather than contributing to its construction and is only experienced as a demonstration of something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students (see De Villiers, 1990, Tall, 1992, Hanna and Jahnke, 1993 as cited in Hoyles 1997). Hoyles (1997) believes that school proofs should shed light upon the mathematical structures under study rather than seeking to verify correctness by providing insight as to *why* a statement is true. She suggests adding a social dimension to the explanatory process. A social dimension to explanatory proof exists where students explain their arguments to a peer or a teacher which helps to engage students and to enable them to claim ownership of the proving process.

Hanna (1990) implies that in the teaching of proof emphasis is placed on the "convincing argument". This is a result of educators recognizing proof as a means of communication and the social processes involved in the idea of proof. Volminik (1988) believes that if the curriculum were to place greater emphasis on the social criteria for the acceptance of a mathematical truth then mathematics education would benefit. Hanna (1995) believes that the role of proof in mathematics curriculum is to reflect mathematics itself, and furthermore that the main function of proof in the classroom reflects one of its key functions in mathematics itself - the promotion of understanding.

E. Hoyles' Discoveries of Students' Understandings of Proof

According to Hoyles (1996), there has been a huge outcry, among mathematicians, engineers and scientists in universities in the UK, complaining about the mathematical incompetence of entrants to their universities. The London Mathematical Society (1995) as cited by Hoyles (1996) states that the serious problems perceived by teachers in higher education result from:

- a serious lack of essential technical facility - the ability to undertake numerical and algebraic calculation with fluency and accuracy;
- a marked decline in analytical powers when faced with simple problems requiring more than one step;
- a changed perception of what mathematics is - in particular of the essential place within it of precision and proof (London Mathematical Society, 1995, p. 2).

Hoyles believes that many students prefer empirical argument over deductive reasoning. Many students judge that after having giving some examples which verify a conjecture they have proved it (Hoyles, 1996). Both Gonobolin and Chazan share similar beliefs to that of Hoyles. Gonobolin (1954) argues that students do not recognize the need for a logical proof of a geometric theorem when the theorem can be shown using empirical evidence. Chazan (1989) reports that high school students who are taught geometry and the method of deductive proof in a beginner's course seemed to hold two incorrect beliefs: the empirical evidence is proof for all cases, and that deductive proof is evidence for only one case (see also Fischbein and Kedem, 1992; Balacheff, 1988; Chazan, 1993; Finlow-Bates, 1994).

Hoyles completed a comprehensive study of students' views of proving and proof and the major influences on them. In her study, Hoyles examined students' perceptions of the nature of mathematical proof and its purposes. The identification and analysis of students' written responses to a range of questions concerning proof comprises the empirical core of her study. The meaning of what is required as a proof is not made explicit; neither is it clear what students have been taught, or what has been emphasized, or what forms of presentation have been deemed acceptable. Hoyles points out that proof is discussed either explicitly or implicitly in curricula. Where proof is discussed explicitly, definitions, logical deductions and acceptable forms of presentation of proofs are made apparent whereas, implicitly mathematical proof and its criteria are negotiated during the activity. Hoyles discovered that some teachers were comfortable with informal explanations while others would judge this to be inadequate and would require a logical argument. A logical argument would be representative of the ' two column proof '.

Hoyles' study compares the general proof to the explanatory proof to ascertain which type of proof is valued by the student and which proof the student perceives as acceptable. More specifically, Hoyles questions whether empirical examples help students explain their results. The 15 year old students, who were participants in Hoyles' study, chose for themselves proofs that are general and explanatory, while the proofs they thought would be assigned the best mark are general but not explanatory. Students in Hoyles' study chose formal presentation (correct or incorrect) as highly favoured for the best mark while the narrative proof was the favourite for individual choice.

Proof should be seen as a generative and not merely descriptive process. Hoyles believes that teachers must resist the temptation to assume that situations that engage students with proof must follow a linear sequence from induction to deduction. "The challenge remains to design situations that motivate students to use proof for all its functions and that help students to forge connections between these functions at every opportunity. Teachers need to engage students into a mathematical proving culture where students see a sense of purpose in proving and realize that proof is generative and not merely descriptive" (Hoyles, 1997).

Chapter V

METHODOLOGY

Methodology encompasses method, theory and epistemology (Guba, 1990). Using naturalistic or qualitative inquiry methods, I studied what makes a mathematical explanation a good explanation within a natural setting, without manipulating or controlling the setting or its members. Qualitative information consists of description and interpretations in narrative form collected and analyzed from interviews, participant and non-participant observations, interviews, documents and records.

A. Qualitative Research

Qualitative research is an approach to research that is evolutionary and emergent in nature, that takes place in the subjects' natural setting, and that uses sociological/anthropological methods (participant observation, document analysis, and interviews) as data collection techniques. The philosophical framework is phenomenology, and the belief system regarding reality is that it is pluralistic and socially constructed. Data are largely descriptive, and are reported using, as much as possible, the words and language of the subjects. The next three sections will address theory, epistemology and method.

A. (1) Theoretical Orientations

Guba (1990) says that the naturalistic approach, which is characteristic of qualitative research, uses a discovery oriented approach that minimizes constraints and places no prior constraints on what the outcome will be. Glaser and Strauss (1967) believe that hypotheses in qualitative research are grounded in the research and emerge with the collection and analysis of data. *Grounded theory* (Glaser and Strauss, 1967) is generated inductively through the discovery approach that occurs during research. Categories, properties, and hypotheses are three components of grounded theory. Through content analysis, conceptual categories are developed by looking for recurring regularities in the data. Categories should be analytical and sensitizing. Properties are concepts that describe a given category or attributes of categories. Hypotheses emerge during data collection and analysis for the qualitative researcher, who prefers the term working hypotheses, so as not to limit the scope and depth of the research.

The advantage of taking a qualitative approach is that I do not have a theory or hypotheses to verify (falsify) but rather an idea to explore and to interpret, allowing for a hypotheses to emerge. My hypotheses or theory were grounded in research and emerged through exploring and interpreting the context.

A. (2) Epistemology

Epistemology is concerned with what kinds of knowledge we have or can get by the various investigatory means at our disposal. The underlying framework of qualitative research, phenomenological inquiry focuses on understanding human beings in context specific settings. Phenomenological inquiry is inductive and has a holistic perspective. A phenomenological approach says something about our views as to what constitutes valuable knowledge, or epistemology and our perspective on the nature of reality, or ontology. Qualitative research is defined by the way the researcher sees the world.

A. (3) Qualitative Research Methods

Qualitative researchers use interviews, participant and non-participant observations, and document analysis as data collection methods.

A. 3(a) Interviewing

Interviewing is used to find out what others think and feel, what opinions they hold and to find out things that cannot be observed. Qualitative interviewing begins with the assumption that the perspective of others is meaningful, knowable, and can be made explicit. Interviewing is best used in conjunction with document analysis and observation.

Person-to-person interviews were conducted with eight adult learners enrolled in the Adult Basic Education (ABE) program to determine students' preference for a particular kind of explanation. Being able to ask why students preferred or rejected a particular explanation helped me discover what constitutes a good explanation for the student. In addition, the interviews allowed an opportunity to discover whether or not students mirror teacher explanations outside of the classroom setting. Person-to person interviews were also conducted with the two ABE mathematics instructors to determine what constitutes a good explanation for the teacher.

A. 3(b) Participant-observation

Participant-observation requires that the researcher be a genuine participant, such that his/her presence becomes accepted as part of the setting. Often in qualitative studies observation is limited to non-participatory observation because of time constraints. Observation is valuable in that the researcher is able to see with one's own eyes.

Participant-observation occurred on a daily basis over a five week period in which I observed ABE students within their classroom setting. Observing the kinds of explanations offered by students and teachers within their classroom setting provided data that helped to determine what kinds of explanations students prefer, and what constitutes a good explanation for the student and/or teacher. As well, observing the kinds of explanations offered by students that teachers deemed acceptable provided further insight into what constitutes a good explanation for the teacher.

A. 3(c) Document Analysis

For the purpose of qualitative research, a document is considered to be all written information about the entity under study. Documents are a rich source of data on participants' views of the situation under study. Data from documents can furnish descriptive information, verify emerging hypotheses, offer historical understanding and advance new themes and categories. Document analysis, like observation should be cross-checked with data collected from using other methodological approaches.

The document analysis involved an intense literature review of proof, proving, and the different purposes proving serves. The literature review assisted in defining the problem, selecting the methodology, and interpreting the findings. Surveying, interpreting, and synthesizing what has been studied and published about the teaching of proof provided the foundation and direction for the current research study.

A student questionnaire, set in two domains of mathematics arithmetic/algebra and geometry offering deductive, inductive and analogical explanations was administered to help determine the kind of explanation preferred. The different kinds of explanations offered on the student questionnaire served as categories for data analysis. The student questionnaire served as a rich source of data on students' views of what constitutes a good explanation.

A. (4) Methodological Issues

A. 4(a) Rigor of Qualitative Research

Lincoln and Guba (1986) stress that the criteria used to judge the rigor of scientific methods hold for naturalistic or qualitative inquiry. Trustworthiness and authenticity of naturalistic or qualitative inquiry are parallel terms for rigor of scientific or quantitative inquiry. The criteria of trustworthiness that parallel those of the conventional paradigm (truth, value, applicability, consistency and neutrality) are credibility, transferability, dependability, and confirmability. These criteria can be assessed using triangulation of data, member checks, persistent observation, prolonged engagement in the setting, external audits, and thick description. Triangulation of data refers to multiple data collection methods used as a means of establishing trustworthiness. Member checks allow the participant an opportunity to review data collected from interviews to ensure accuracy. Lincoln and Guba (1985) believe that enlisting an outsider to audit field notes, analysis and interpretations contributes to trustworthiness of findings. Thick description is a literal description of the entity being researched, the circumstances under which it is used and the characteristics of the people involved. By presenting balanced views of multiple realities, and being empathetic and understanding to all audiences, the researcher can achieve authenticity of findings. The qualitative or naturalistic researcher is concerned with credibility of findings rather than internal validity. Credibility is or will be established using cross-checking and triangulation to corroborate data. Lincoln and Guba refer to applicability as being enhanced with working hypotheses and with the use of thick description.

According to Sanders (1994), qualitative analysis involves an inductive, interactive, and iterative process whereby the researcher confirms and/or explains the purpose of the research and tests conclusions with relevant audiences. Sanders, in his article, *Analysis of Qualitative Information*, states that the researcher must assure accuracy of findings by seeking confirmatory evidence from more than one source and subjecting inferences to independent verification. Auditability of naturalistic inquiry will assure the evaluation's confirmability (using cross checking and member checks) which is analogous to an evaluator using quantitative inquiry guarantying the research's neutrality.

A. 4(b) Neutrality in Qualitative Research

Because the chief instrument of qualitative inquiry is the researcher, the issue of neutrality arises. Worthen and Sanders (1987) state, "because of their reliance on the human tendency to minimize the importance of instrumentation and group data, advocates of this approach have been criticized for loose and unsubstantiated findings". Such criticism of a participant-oriented approach is based in the scientific paradigm and the beliefs about objectivity. Researchers do not believe that qualitative research can be objective or neutral since it is emergent in design. The data collected are not specified in advance; there are no controls laid down; there are multiple realities capable of being explored to different depths by different researchers. Qualitative research is intensely subjective, but it does not make the findings less believable or biased.

A. (5) Establishing Rigor for the Current Research Study

Triangulation of observation, interviews, and questionnaire data corroborates my findings. Thick description and member checks were employed to establish rigor. Recognizing that data is not independent of its context whatever data was collected was not taken at a face value or unduly generalized.

Chapter VI

DESIGN OF STUDY

A. Instrumentation

In addition to employing qualitative methods of data collection (participant observation, interviewing and document analysis), a student questionnaire was administered to obtain empirical evidence. The questionnaire was divided into two sections, the first related to geometry and the second concerned with algebra. Both sections were presented in a multiple-choice format as illustrated in the sample questionnaire attached in Appendix A. The purpose of having a multiple choice question was to expose students to a range of possible ways of explaining -- namely, empirically, deductively and analogically. Student responses were analyzed to determine what the student perceives as a good explanation -- especially after having been exposed to a variety of types of explanations.

B. Study's Participants

Students attending the College of the North Atlantic, Happy Valley - Goose Bay Campus who were enrolled in the technician, Business Computer Studies (year I and II), Early Childhood Education and Adult Basic Education (ABE) programs were invited to participate in the study. Approximately 100 students were invited to participate. Of these students, 17 were enrolled in the Sheet Metal program, 12 in the Welding program, 12 in the Industrial Warehousing program, 16 in the Automotive Technician program, 12 in the Adult Basic Education, 6 in the Early Childhood certificate program and 28 in the Business Computer Studies (year I and II) program. Of the one hundred and three students invited to participate in the questionnaire, eighty-two actually did. Business Computer Studies, year II students were preparing for final exams and could not afford the time to participate in the study.

Students enrolled in these programs are required to complete Mathematics 1000 or Mathematics 1510 (with the exception of Early Childhood Education and Adult Basic Education students) (see Appendix C for course descriptions). Adult Basic Education students complete the equivalent mathematics program to that of the high school curriculum. Business Computer Studies, year I and year II students have to complete Mathematics 1510 in their first semester of their two year program. Early Childhood students are not required to complete a mathematics course. Although I teach at the same campus, I was not responsible for teaching any of the above students in the second semester of the 1997-1998 school year, which is when the study was conducted.

C. Instruments

C. (1) Questionnaire

C. 1(a) How was the questionnaire designed?

My questionnaire was based on that of Hoyles. Hoyles (1997) used two survey instruments – a student questionnaire and a school questionnaire. In order for the proofs to be accessible, familiar, and in tune with the UK curriculum, Hoyles chose mathematical content that was sufficiently straightforward. To ensure differentiation between student responses, the content was challenging. The student questionnaire, set in two domains of mathematics arithmetic/algebra and geometry was presented in a variety of forms -- exhaustive, visual, narrative, and symbolic (Hoyles, 1997). Different "proof types" were offered on the questionnaire to determine whether students were influenced by the form as well as the content of a proof. "Proof types included empirical, enactive, narrative, visual or formal, with two examples of formal proof, one correct and one incorrect" (Hoyles, 1997). Using an example to show that a mathematical statement holds true is characteristic of an empirical proof. For an enactive proof, the student discovers the mathematical statement to be true by doing. A visual proof uses visual representation to show why something is valid. A symbolic proof uses mathematical notation to verify a mathematical statement. A formal proof is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences and the last sentence is the one to be proved.

Many months were spent reviewing existing literature, discussing and brainstorming with the thesis supervisor, teachers, and students (who were not associated with the study) all of which aided in the design of the student questionnaire. Like Hoyles, I too wanted the mathematical content to be sufficiently straightforward for the explanations to be accessible, familiar and in keeping with the Newfoundland and Labrador high school curriculum and the mathematics curriculum of the college in which I work and where I conducted my study. The content was kept challenging enough so there would be differentiation amongst student responses. Keeping with what Hoyles did, the questionnaire presented explanations in a variety of forms -- exhaustive, visual, narrative and symbolic and set in two domains of mathematics -- algebra/arithmetic and geometry. A sample of the student questionnaire is attached in Appendix A. Students were given ample time to complete the questionnaire. The amount of time required to complete the student questionnaire was determined after the questionnaire was piloted. Student questionnaires were distributed in the second school semester. Over this same period, I observed student explanations within the classroom setting. Audio tape recordings were made, and formal interviews (voluntary and outside of normal class hours) were conducted. All information gathered in this study is strictly confidential and at no time were individuals identified. Consent was obtained from administration, teachers and students.

The questionnaire was divided into two sections, the first concerned with geometry and the second with algebra/arithmetic. Each of the three questions were presented in a multiple-choice format (see Appendix A). The purpose of having a multiple-choice question was to introduce students to a variety of possible meanings of 'to prove'.

The student questionnaire addressed three questions

- **Why does the sum of the interior angles of any triangle equal 180°?**
- **Why is the sum of two odd numbers even?**
- **Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?**

Inductive (single example, generic example, and multiple example), deductive (formulated and unformulated) and analogical explanations were offered. Students or participants could select their preferred explanation for each question.

Five different explanations were offered for the first question. Both the first question and the explanations offered were duplicated from Hoyles' (1997) study. The explanations included two (2) inductive/multiple example explanations, one (1) deductive, formulated, semi-formal explanation, one (1) deductive, formulated, pre-formal explanation and one (1) formulaic explanation.

Figure 1

Why does the sum of the interior angles of any triangle equal 180°?

Amanda's answer:

I tore the angles up and put them together. It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

Amanda's inductive explanation uses three different examples to explain why the sum of the interior angles of any triangle equals 180°. She tears up the angles of a triangle, puts them together and discovers that the angles form a straight line. Amanda tried the same thing for an equilateral triangle and an isosceles triangle only to discover that the same thing happens. She discovers that the sum of the interior angles of a triangle equals 180° by doing, which is indicative of an enactive proof. In an enactive proof, the prover discovers the conjecture to be true by doing. Amanda's inductive explanation uses a visual aid to show what she did.

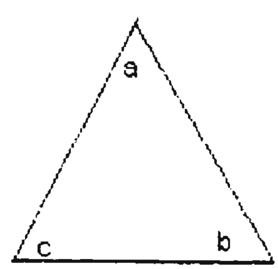
Figure 2

Barry's answer:

I drew an isosceles triangle, with c equal to 65°.

Statements	Reasons
a = 180° - 2c	Base angles in isosceles triangle equal
a = 50°	180° - 130°
b = 65°	180° - (a + c)
c = b	Base angles in isosceles triangle equal

therefore, a + b + c = 180°

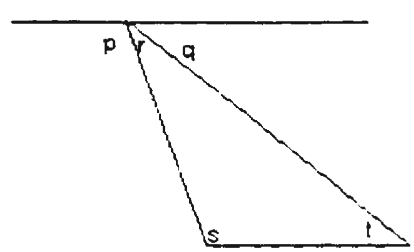


Barry's explanation uses the concept that the angles opposite the equal sides of an isosceles triangle are equal. His explanation is presented in the statements and reasons format. Barry's explanation would be considered a formulaic explanation, because it is constructed according to principles which are associated with the creation of the sorts of proofs teachers like. According to Barry's formula, the interior angles of an isosceles triangle equals 180°, because the base angles of an isosceles triangle are equivalent. A diagram accompanies Barry's formulaic explanation. Barry concludes his explanation with a formula.

Figure 3

Cynthia's answer:

I drew a line parallel to the base of the triangle



Statements	Reasons
p = s	Alternate angles between two parallel lines are equal
q = t	Alternate angles between two parallel lines are equal
p + q + r = 180°	Angles on a straight line
therefore s + t + r = 180°	

Cynthia's deductive, formulated, semi-formal explanation also uses statements and reasons format. Deductive reasoning "that proceeds from agreed upon premises to conclusions using logical arguments" (Reid, 1995a, p.7) is employed by Cynthia to explain why the sum of the interior angles of any triangle equals 180°. Cynthia uses parallel lines and congruent angles to explain why the sum of the interior angles of any triangle equals 180°. Since Cynthia makes reference to parallel lines and congruent angles, it is safe to say that she is aware that she is proving. Based on Reid's definition of formulation -- "prover's knowledge that he/she is proving" (1995a, p.25), I would describe Cynthia's deductive explanation as formulated. It is semi-formal in the fact that it is a deductive argument suitable for publication. Like Barry's formulaic explanation, Cynthia's deductive, formulated, semi-formal explanation is presented using statements and reasons format and concludes with a formula. However, the two explanations differ in that Barry is not proving.

Figure 4

Dylan's answer:

I measured the angles of all sorts of triangles accurately and made a table

a	b	c	total
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

They all added up to 180°

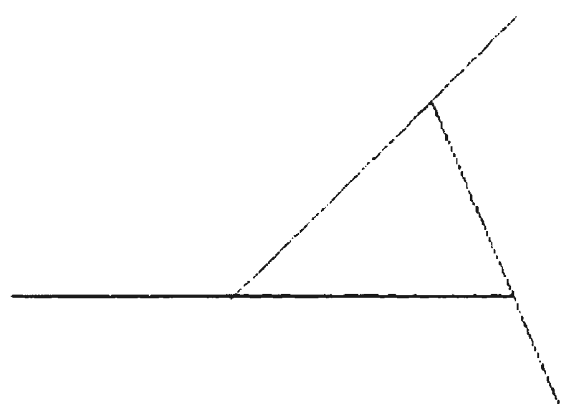
Dylan's inductive/multiple example explanation uses four numerical examples to show that the sum of the interior angles of any triangle equals 180°. Dylan does not use a diagram as part of his explanation. Of the five different explanations provided, Dylan's is the only one that does not include a diagram (see Figure 4).

Figure 5

Ewan's answer:

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°. You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line.

This makes a total of 540°. 540° - 360° = 180°.



Ewan's deductive, formulated, pre-formal explanation differs from Cynthia's deductive, formulated, semi-formal explanation in that Ewan's is more narrative and does not use the statements and reasons format. Ewan's explanation is deductive because he comes to the conclusion that the sum of the interior angles of any triangle equals 180° using the exterior angle theorem and the straight angle theorem. Because Ewan's argument uses informal language and hidden assumptions, it would be described as a pre-formal proof. A diagram accompanies Ewan's explanation (see Figure 5).

Why is the sum of two odd numbers even?

Four different explanations were provided for the second question. Since the intent of the study was to build on Hoyles' (1995) study and Reid's (1995a) study and since both used this question in their studies, I thought it pertinent to include it in my study. The explanations for this question were generated by the thesis supervisor and myself. They included one (1) deductive, formulated, semi-formal explanation, one (1) inductive/single example explanation, one (1) inductive/multiple example explanation and one (1) analogical explanation. The analogical explanation was a duplication of one offered by a participant in Reid's (1995a) study. A description of each of the explanations for this question follows.

Figure 6

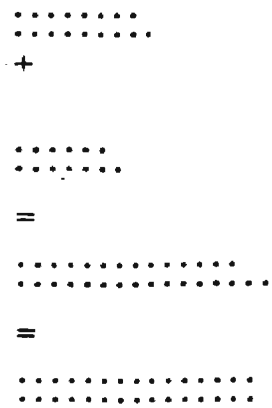
Andy's answer:

Let one odd number be $(2n - 1)$ and the another odd number $(2m - 1)$, then $(2n - 1) + (2m - 1) = 2(n + m) - 2$

Andy's deductive, formulated, semi-formal explanation is somewhat like a symbolic proof which uses mathematical notation to verify the mathematical statement that the sum of two odd numbers is even.

Figure 7

Bill's answer:



Because Bill uses a finite set of dots, his explanation would be considered to be a single example. His single example is strictly visual with the use of dots to represent numbers.

Figure 8

Cora's answer:

13 + 45 = 58
7 + 9 = 16
113 + 335 = 448
1077 + 517 = 1594

Although Cora's explanation is also inductive, she uses more than one example to show that the sum of two odd numbers is even. Unlike Bill's inductive/single example explanation, Cora's inductive/multiple example explanation is strictly numerical.

Figure 9

Drake's answer:

An odd number plus an odd number equals an even number because of the same principle which says a negative number times a negative number is a positive.

Drake uses an analogy to explain why the sum of two odd numbers is even. Drake's analogy shows how he reasoned from a rule to a rule. His analogy compares the sum of two odd numbers being even to a negative number times a negative number being positive.

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

The third question was suggested by the thesis supervisor. However the different explanations were generated by the researcher, supervisor and students (not associated with the research). Students enrolled in the Common First Year program were invited to formulate explanations for the trinomial square question. At the time of the invitation, these students were completing the topic, Review of Fundamental Algebra in their Mathematics 1100 course. Any formulated explanations were developed outside of regular classroom time and was on a voluntary basis. Five different explanations were provided to explain the third question. They included one (1) inductive/multiple generic example explanation, one (1) deductive, formulated, pre-formal explanation, one (1) inductive/multiple example explanation, one (1) deductive, formulated, semi-formal explanation and one (1) deductive, unformulated, informal explanation. The deductive, formulated, pre-formal explanation was offered by a student from the Common First Year program. The inductive/multiple generic example explanation, inductive/multiple example explanation, deductive, formulated, semi-formal explanation and the deductive, unformulated, informal explanations were generated by the thesis supervisor.

Figure 10

Lisa's answer:

If you take the number 144, then 144 is equal to $10^2 + 2(10)(2) + 2^2$
Likewise, $169 = 13^2$ is $10^2 + 2(10)(3) + 3^2$
Finally, $81 = 9^2$ is $8^2 + 2(8)(1) + 1^2$

Therefore, any perfect square number is equal to a binomial square which always multiplies out into the form $x^2 + 2bx + b^2$

The binomial is found by finding two numbers which add up to the number before it is squared. For example $9 = 8 + 1$ and $9^2 = 81$. Similarly, $13 = 10 + 3$ and $13^2 = 169$

Lisa's inductive/multiple generic example explanation uses three generic examples to explain "Why perfect trinomial squares have the form $x^2 + 2bx + b^2$?"

Figure 11

Julia's answer:

If you multiply two same binomials such as $(x + b)(x + b)$ using the FOIL method, then the first two terms of the two binomials will multiply to $x * x = x^2$; the two outside terms will be x times b = xb ; the two inside terms will be b times x = bx; and the two last terms of each of the two binomials multiplied together will be b^2. Combining like terms, the xb and bx will equal 2bx. Thus, $(x + b)(x + b)$ will always multiply into the form $x^2 + 2bx + b^2$.

Julia's explanation is deductive in that she uses the FOIL method (agreed upon premises) to come to the conclusion that perfect trinomial squares have the form $x^2 + 2bx + b^2$. With the use of the FOIL method to show that $(x + b)(x + b) = x^2 + 2bx + b^2$, Julia is aware that she is proving. Thus, her explanation is formulated. Yet, her explanation assumes the reader is familiar with the FOIL method. Because Julia's deductive, formulated explanation involves hidden assumptions and informal language, her explanation would be characteristic of a pre-formal proof. According to Reid (1995a, p.9), a pre-formal proof may involve hidden assumptions, and use of informal language and notation.

Figure 12

Jody's answer:

$(x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$
$(x + 3)(x + 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$
$(x - 5)(x - 5) = x^2 - 5x -5x + 25 = x^2 - 10x + 25$
$(3x + 4)(3x + 4) = 9x^2 + 12x + 12x + 16 = 9x^2 + 24x + 16$
$(2x - 3)(2x - 3) = 4x^2 - 6x -6x + 9 = 4x^2 - 12x + 9$

Therefore perfect trinomial squares always have the form $x^2 + 2bx + b^2$

Using multiple examples, Jody shows how a squared binomial multiplies into a perfect trinomial square.

Figure 13

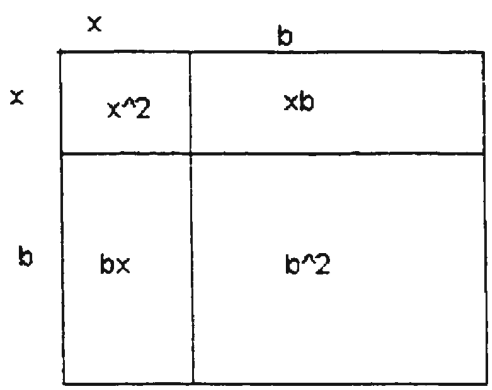
Dena' answer:
Using the distributive law:
$(x + b)(x + b)$
$(x + b)x = x^2 + bx$
$(x + b)b = xb + b^2$
$(x + b)(x + b) = x^2 + bx + xb + b^2$
The "2" comes because "xb" occurs in both distributions

Dena's explanation uses mathematical properties, namely the distributive law to show that $(x + b)(x + b) = x^2 + 2bx + b^2$. Although Dena's explanation explains why the "2" occurs – "The '2' comes because 'xb' occurs in both distributions." – she does not show a concluding step $x^2 + 2bx + b^2$. With the use of the distributive property, Dena is aware that she is proving. Thus, her explanation is formulated.

Figure 14

Cheryl's answer:

$(x + b)$ represents a line segment of length $(x + b)$



Cheryl's explanation for why do perfect trinomial squares have the form $x^2 + 2bx + b^2$ would be considered deductive but unformulated. Without a statement to show that she is multiplying two binomials or a concluding statement like $(x + b)(x + b) = x^2 + 2bx + b^2$, it is safe to say that Cheryl is unaware she is proving. That is, her deductive explanation is unformulated. Cheryl finds the area of a square with length $(x + b)$ to show that $(x + b)(x + b) = x^2 + 2bx + b^2$.

C. 1(b) *How was the questionnaire administered?*

The questionnaire was administered over a two week period (prior to the second semester break) by the researcher with the shop instructor or home room instructor present. Students were given as much time as needed to complete the questionnaire. The instructions for completing the questionnaire consisted of checking one of the students' explanations for each of the three questions presented. Participants were told that they could choose one explanation and were not asked to specify their reason for doing so.

C. (2) *Observations*

Adult Basic Education (ABE - level III) students were chosen for observation, because they were accessible and willing to participate in the classroom observations. At the end of the second school semester, most students had finished their program of studies or were on-the-job-training with the exception of the ABE students. ABE level III students and their mathematics instructors volunteered to be observed in their classroom setting.

Students' mathematical background varied given entrance requirements. For the most part a high school diploma is required as entrance into the Trades and Technology, Business and Applied Arts programs; but, often students are accepted through the mature student clause. The mature student clause states that applicants who do not meet the educational prerequisites for the program they wish to enter may be considered for admission on an individual basis provided that they are at least 19 years of age and have been out of school for at least one year. Students enrolled in the Level III, ABE program are representative of the school population since some are high school graduates seeking better grades or refresher courses as entrance requirements for university programs and others are mature students.

During the seven week intersession (April 27, 1998 - June 12, 1998), I was able to observe students and their teachers on a regular basis. Approximately one to two hours daily were spent observing students and teachers in their mathematics class. The classroom observations concluded at the end of the fifth week when all possible data had been collected.

Audio tapes were used to record data along with field notes. The intent of the classroom observations was to relate questionnaire responses to the classroom setting looking for commonalities among students' preferences. By observing students and teachers within the classroom setting, data was collected to determine the kinds of explanations students offer their classmates or teachers; kinds of explanations offered by students that teachers deemed acceptable; kinds of explanations students provided on the questionnaire and in classroom setting. These data collections helped to answer the four research questions.

C. (3) Interviews

C. 3(a) Student interviews

i. How were students selected?

Level III, ABE (both the academic and general mathematics) students who voluntarily participated in the classroom observations were invited to participate in the interviews. Eight of the twelve level III, ABE students were receptive to the invitation. Of the eight students who volunteered to be interviewed, six (6) students were from the academic mathematics stream and the other two (2) were from the general mathematics stream. The eight interviews were conducted over the last two weeks of the 1997-1998 school year. Most of the eight students had completed their required credits for graduation and had time to participate in the study.

ii. What questions were used?

The questionnaire questions were used for the interviews as a means of finding patterns in student responses to see if students' preference for a type of explanation was consistent among different settings. By using the questionnaire questions, the researcher was able to determine not only what kind of explanation students preferred but why they preferred it. This provided insight into what constitutes a good explanation for the student.

Other questions in addition to the questionnaire questions were used for the interviews. These additional questions came about from the classroom observations. The "Pick Up Charge" question and "Solving an Equation with Negative Numbers" question were asked to the same students who were observed solving them in their classroom settings. Both of these questions were chosen to collect data relating to whether or not students mirror teachers' explanations. Other questions, namely, "Why does .45/.99 reduce to 45/99?" and "Which of the following sequences is geometric and why?" were asked to the academic students who were interviewed. These questions were explained by their teacher in the classroom setting. By asking the same questions in the interviews, data could be collected to see if students parrot the teacher.

Pickup Charge

I asked student #5 and student #6 how they would calculate the pickup charge for 275 kg of a product if \$6.10 is charged for each 100 kg or fractional part. Both students could choose from the answers provided.

$\$6.10 * (275 \div 100) = \$16.775$ or $\$6.10 *(300 \div 100) = \$18.30$

Solving an equation with negative numbers

I asked student #7 to choose from the two different ways of solving the following equation.

$1020 = a \frac{(-255)}{-1}$	or	$1020 = a \frac{(-255)}{-1}$
$-1020 = a\,(-255)$	or	$1020 = a\,(255)$
$a = 4$	or	$a = 4$

Why does .45/.99 reduce to 45/99?

Which of the following sequence is geometric and why?

1 1/3 1/9 1/27

1 5 10 15 20

C. 3(b) Teacher interviews

The two ABE mathematics instructors were also receptive to being interviewed. The two teachers were busier than their students in the last two weeks of the school year when the student interviews were conducted. Thus, the teacher interviews were conducted during the summer vacation. The questionnaire questions were used for the teacher interviews to collect data that would help answer what constitutes a good explanation for the teacher.

Chapter VII

Results

The first part of the results section records the data collected from the questionnaire. The second part records the data from the interviews. The first section of part two of the results records the data collected from the student interviews. The second section compares the results from the questionnaire and the student interviews. The third section of part two of the results records the data collected from the teacher interviews. The fourth section records the data from student interviews using the questions developed from classroom observations.

A. Questionnaire Results

The student questionnaire addressed three questions set in two domains of mathematics, algebra/arithmetic and geometry.

- Why does the sum of the interior angles of any triangle equal $180°$?
- Why is the sum of two odd numbers even?
- Why do perfect trinomial squares have the form $x^2+2bx+b^2$?

Inductive (single example, generic example, and multiple example), deductive (formulated and unformulated) and analogical explanations were offered. Students or participants could choose the explanation of their choice for each of the three questions. The three questions and the explanations offered are explained in detail in the design chapter.

A. (1) Why does the sum of the interior angles of any triangle equal 180 °?

Five different explanations were offered for this question. They included two (2) inductive/multiple example explanations, one (1) deductive, formulated, semi-formal explanation, one (1) deductive, formulated, pre-formal and one (1) formulaic explanation.

Table 1

Why does the sum of the interior angles of any triangle equal 180°?

Students' Explanations	Kind of Explanation	Number from the questionnaire who chose a particular explanation n = 82
Amanda	Inductive/Multiple Example	16 (19.5%)
Barry	Formulaic	21 (25.6%)
Cynthia	Deductive, formulated, semi-formal	18 (22%)
Dylan	Inductive/Multiple Example	21 (25.6%)
Ewan	Deductive, formulated, pre-formal	6 (7.3%)

Of the eighty-two students who participated in the questionnaire, most chose either Barry's formulaic explanation (twenty-one students, 25.6%) or Dylan's inductive, multiple example explanation (also twenty-one students. 25.6%). Nearly as many preferred Cynthia's deductive, formulated, semi-formal explanation (eighteen students, 22%) or Amanda's inductive, multiple example explanation (sixteen students, 19.5%). Only six students (7.3%) preferred Ewan's deductive, formulated, pre-formal explanation.

*A. (2) **Why is the sum of two odd numbers even?***

Four different explanations were provided to explain why the sum of two odd numbers is even. They included one (1) deductive, formulated, semi-formal explanation, one (1) inductive/single example explanation, one (1) inductive/multiple example explanation and one (1) analogical explanation.

Table 2

Why is the sum of two odd numbers even?

Students' Explanations	Kinds of Explanations	Number from the questionnaire who chose a particular explanation n = 82
Andy	Deductive, formulated, semi-formal	11 (13.41%)
Bill	Inductive/Single Example/Visual	3 (3.67%)
Cora	Inductive/Multiple Example	35 (42.68%)
Drake	Analogy	33 (40.24%)

Of the eighty-two students who participated in the questionnaire, most chose either Cora's inductive/multiple example explanation (thirty-five students, 42.68%) or Drake's analogy (thirty-three students, 40.24%). Only eleven students (13.41%) preferred Andy's deductive, formulated, semi-formal explanation; fewer students preferred Bill's inductive, single example explanation (three students, 3.67%).

A. (3) Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Five different explanations were offered to explain why perfect trinomial squares have the form $x^2 + 2bx + b^2$. They included one (1) inductive/multiple generic example explanation, one (1) deductive, formulated, pre-formal explanation, one (1) deductive, formulated, semi-formal explanation, one (1) inductive/multiple example explanation, and one (1) deductive/unformulated/visual explanation.

Table 3

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Students' Explanations	Kinds of Explanations	Number from the questionnaire who chose a particular explanation n = 82
Lisa	Inductive/Generic Example	8 (9.76%)
Julia	Deductive, formulated, pre-formal	32 (39.02%)
Jody	Inductive/Multiple Example	17 (20.73%)
Dena	Deductive. formulated, semi-formal	16 (19.51%)
Cheryl	Deductive, unformulated/informal	9 (10.91 %)

Julia's deductive. formulated, pre-formal explanation was the most preferred with thirty-two students (39.02%) choosing it. Only half as many chose Jody's inductive, multiple explanation (seventeen students, 20.73%) with almost the same number of students choosing Dena's deductive, formulated, semi-formal explanation (sixteen students, 19.51%). The number of students who chose Lisa's inductive, generic example explanation (eight students. 9.76%) was about the same as Cheryl's deductive, unformulated, informal explanation (nine students, 10.91%).

B. Interviews

The student interviews included questions from the questionnaire and from classroom observations.

B. (1) Student interviews using questionnaire questions

B. 1(a)

Table 4 summarizes the data collected from the student interviews using the first questionnaire question.

Table 4

Why does the sum of the interior angles of any triangle equal 180°?

Students' Explanations	Kinds of Explanations	Number from the interview who chose a particular explanation n = 8
Amanda	Inductive/Multiple Example	4
Barry	Formulaic	1
Cynthia	Deductive, formulated, semi-formal	2
Dylan	Inductive/Multiple Example	1
Ewan	Deductive, formulated, pre-formal	0

Four of those students interviewed preferred Amanda's inductive/multiple example explanation. Cynthia's deductive, formulated, semi-formal explanation was the second most preferred. Of the eight students interviewed, one preferred Barry's formulaic explanation and another preferred Dylan's inductive/multiple example explanation. Of the eight students interviewed, no one chose Ewan's deductive, formulated, pre-formal explanation.

Table 5 specifically shows who from the interviews conducted preferred a particular explanation, the kind of explanation preferred and their reason for choosing the explanation for the questionnaire question, "Why does the sum of the interior angles of any triangle equal 180°?"

Table 5

Why does the sum of the interior angles of any triangle equal 180°?

Student interviewed	Choice of explanation	Kind of explanation	Reason for choosing explanation
Student #1	Amanda's	Inductive/Multiple Example	Familiar
Student #2	Amanda's	Inductive/Multiple Example	Easy
Student #3	Amanda's	Inductive/Multiple Example	Clear
Student #4	Amanda's	Inductive/Multiple Example	Obvious
Student #5	Barry's	Formulaic	Statements & Reasons
Student #6	Cynthia's	Deductive, formulated, semi-formal	Statements & Reasons
Student #7	Cynthia's	Deductive, formulated, semi-formal	Logical
Student #8	Dylan's	Inductive/Multiple Example	Straightforward

Because Amanda's inductive/multiple example explanation did not require any additional explanations, for example, why alternate angles are equal, it was student #3's preferred choice. Student #3 described Amanda's explanation as clean and clear and not warranting subsequent proofs. "I tell ya now. . . Amanda's answer hmm hmm because it's clear. It is very clean. It's not cluttered. It answers the question fully without going into long details" (see Appendix B).

Amanda's inductive/multiple example explanation was student #4's choice because of its obviousness that three bends would straighten into a straight line equalling 180°. ". . . three different bends. Then it's obvious. Three angles got to equal 180°" (see Appendix B).

Student #2 liked Amanda's inductive/multiple example explanation for its easiness. Student #2 felt that Amanda's explanation was easy because it did not involve statements and reasons. "It is easy. It doesn't involve statement and reasons" (see Appendix B).

Student #1 thought that Amanda's inductive/multiple example explanation was the most appropriate explanation, because he could relate what Amanda was saying to one-half of a circle which equals 180°. When I ran into student #1 a few weeks after the interview, I asked him how Amanda's inductive/multiple example explanation reminded him of a half of a circle. I finally understood that student #1 was relating Amanda's inductive/multiple example explanation to a half circle, because in Amanda's explanation she cuts up the angles which according to student #1 is like cutting a circle in half equalling 180°. "Think of circle -- 180° is 1/2 of a circle -- you add them all up together equal 180° -- so go with Amanda's response. She cut the angles and made a straight line. Like a circle is 360° but, ah, if you cut in half, then you get a straight line 180°" (see Appendix B). So, student #1 was explaining why he preferred Amanda's inductive/multiple example explanation using an analogy. For student #1, Amanda's inductive/multiple example explanation served as an analogy in which his reasoning by analogy was based on a case and lead to a case (see Appendix B).

Student #5 preferred Barry's formulaic explanation, because he uses statements and reasons providing a formula at the end. "Barry's because first he showed what he did – why in statements and reasons and then a formula at the end" (see Appendix B).

The format, namely statements and reasons, of Cynthia's deductive, formulated, semi-formal response was why student #6 preferred it as an explanation for -- Why does the sum of the interior angles of any triangle equal 180°°? "Cynthia's answer because she is using statements and reasons" (see Appendix B).

Student #7, on the other hand, preferred Cynthia's deductive, formulated, semi-formal response for its reasoning. According to student #7, Cynthia's explanation uses logical arguments with the use of the straight line and equivalent angles. "Cynthia's – she justifies using logical arguments and equivalent angles. Kind of liked Ewan's too because that one's using reasoning – all of them would be same triangle right – not going to change degrees" (see Appendix B).

Student #8 was convinced and did not require any further explanation why the sum of the interior angles of any triangle always equals 180° than what Dylan's inductive/multiple example explanation provided. Student #8 was convinced that when you add the interior angles of any triangle your answer will always be 180° by the simple fact that three different measurements added summed to 180°.

B. 1(b)

Table 6 summarizes the data collected from the student interviews using the second questionnaire question.

Table 6

Why is the sum of two odd numbers even?

Students' Explanations	Kinds of Explanations	Number from the interview who chose a particular explanation n = 8
Andy	Deductive, formulated, semi-formal	1
Bill	Inductive/Single Example/Visual	1
Cora	Inductive/Multiple Example	3
Drake	Analogy	3

Three of the eight students interviewed preferred Cora's explanation, and another three preferred Drake's analogy. One of the eight students interviewed preferred Andy's deductive, formulated, semi-formal explanation and another preferred Bill's inductive/single example explanation.

Table 7 specifically shows who from the interviews conducted preferred a particular explanation, the kind of explanation preferred and their reason for choosing the explanation for the questionnaire question -- Why is the sum of two odd numbers even?

Table 7

Why is the sum of two odd numbers even?

Student interviewed	Choice of explanation	Kind of explanation	Reason for choosing explanation
Student #5	Andy's	Deductive, formulated, semi-formal	Statements & Reasons (Formula)
Student #7	Bill's	Inductive/Single Example/Visual	Visual
Student #2	Cora's	Inductive/Multiple Example	Examples
Student #3	Cora's	Inductive/Multiple Example	Examples
Student #4	Cora's	Inductive/Multiple Example	Examples
Student #1	Drake's	Analogy	Familiar
Student #8	Drake's	Analogy	Familiar
Student #6	Drake's	Analogy	Written in words

Student #5 tended to prefer responses that were in the statements and reasons format. Because Andy's deductive, formulated, semi-formal explanation provided, in student #5's opinion, statements and reasons and then a formula at the end, it was her preferred explanation. "Andy's – he is saying what one number is and another in formula and then he went on to say why he did it – then the formula" (see Appendix B). Student #5 believed that Andy's deductive explanation provided statements and reasons; yet, he does not indicate why $2n + 1$ or $2m + 1$ equals an odd number and why $2(n + m) + 2$ is an even number.

Student #7 chose Bill's inductive/single example explanation because it provided a visual representation of why the sum of two odd numbers is even. Both Drake's analogy and

Cora's inductive/multiple examples explanations caught student #7's attention, but, she still preferred Bill's visual explanation. "I'm better with visuals sometimes. It depends on what I'm doing. If I can see things. Not that I would dispute that (Cora's explanation) or that" (Drake's explanation) (see Appendix B). When I asked student #7 what she thought about Andy's deductive, formulated, semi-formal explanation, she indicated that she could see the logic to it, but she could not think of the algebra at the time. "Andy's is alright, but like right now I can't think odd numbers. I'm trying to think of the algebra stuff -- the numbers -- the equations. It is logical to see where it worked out" (see Appendix B). Yet, logic was the reason student #7 had chosen Amanda's inductive/multiple example explanation to explain why the sum of the interior angles of any triangle equals 180°. However, for the sum of two odd numbers is even, student #7 chose the visual explanation over the 'logical' explanation (see Appendix B).

Cora's inductive/multiple example explanation showing the sum of two odd numbers is even through examples was enough to convince student #2. When asked if Cora's explanation would be proof that the sum of two odd numbers is always even, student #2 responded 'sure'. Student #2 indicated that Cora's inductive/multiple example explanation was a lot easier to see or understand. "I would say Cora's answer. This one is a lot easier to see. The numbers make it easier. You don't have to count the dots" (see Appendix B). Student #2 did not like Drake's analogy because it sounded too much like a word problem to her. "Drake's is like a word problem" (see Appendix B) Nor did she like Bill's inductive/single example explanation because she did not want to count the dots.

Both student #3 and student #4 also preferred Cora's inductive/multiple example explanation because Cora's explanation provided examples of why the sum of two odd numbers is even, and they would not have to figure out why a negative times a negative is positive. "Cora's answer on that one, because, not only, does she give more than one example and it's clear, again, it is the contents of it" (see Appendix B - student #3). "Given bunch of examples, right, which I think would be easier to do than just trying to explain something, okay, like a negative times together would give you a positive; whereas, if you were given an example, then I would say students would learn better, would understand better" (see Appendix B - student #4).

Student #1 decided on Drake's analogy. Student #1 found that he could relate an odd number to a negative number and an even number to a positive number. It was clear to student #1 that if a negative times a negative is positive , then an odd plus an odd is even. "Drake's answer because it goes along with a negative times a negative gives you a positive. So, an odd plus an odd is even. Ya, okay, an odd number is like a negative number and an even number is like a positive" (see Appendix B). For student #1, Drake's reasoning by analogy which is based on a rule and leads to a rule was strong. That is, Drake's analogy allowed student #1 to understand why the sum of two odd numbers is even.

Student #8 also preferred Drake's analogy because it related the sum of two odd numbers is even to a familiar principle that says the product of two negative numbers is positive; therefore, the sum of two odd numbers must be even. "Drake's because a negative times a negative number is positive; therefore, an odd number plus an odd number is an even number" (see Appendix B).

The fact that Drake's analogy uses words and not numbers, symbols or dots was reason enough for student #6 to choose his explanation. "Drake's because it's written out not using numbers" (see Appendix B).

B. 1(c)

Table 8 summarizes the data collected from the student interviews using the third questionnaire question.

Table 8

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Students' Explanations	Kinds of Explanations	Number from the interview who chose a particular explanation n = 5
Lisa	Inductive/Multiple Generic Example	1
Julia	Deductive, formulated, pre-formal	3
Jody	Inductive/Multiple Example	1
Dena	Deductive, formulated, semi-formal	0
Cheryl	Deductive, unformulated/informal	0

Julia's deductive, pre-formal explanation was the most preferred with three of the five students interviewed choosing it. Only one of the five students interviewed chose Lisa's inductive/multiple generic example explanation. Likewise, only one of the five chose Jody's inductive/multiple example explanation. No one from the interviews conducted selected Dena's deductive, formulated, semi-formal explanation or Cheryl's deductive, unformulated explanation.

Table 9 specifically shows who from the interviews conducted preferred a particular explanation, the kind of explanation preferred and their reason for choosing the explanation for the questionnaire question – Why do perfect trinomial squares have the form $x^2+2bx+b^2$?

Table 9

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Student interviewed	Choice of explanation	Kind of explanation	Reason for choosing explanation
Student #1	Lisa's	Inductive/Multiple Generic Example	Two different ways
Student #2	Julia's	Deductive, formulated, pre-formal	Familiar
Student #3	Julia's	Deductive, formulated, pre-formal	Straightforward
Student #4	Julia's	Deductive, formulated, pre-formal	Straightforward
Student #8	Jody's	Inductive/Multiple Example	Familiar

Student #1 seemed impressed that Lisa's inductive/multiple generic example explanation provided a different means to obtain the square of a number. He liked how 12^2 could be written as a binomial squared which multiplied out provided the same result as 12^2 "Lisa's answer because she shows two different ways" (see Appendix B).

Student #2 chose Julia's deductive, formulated, pre-formal explanation because of its familiarity. Julia's explanation uses the FOIL method of multiplication which was familiar to student #2. Student #2 did not know how her choice of explanations would have been different if the FOIL method had not been so familiar to her. "Julia's response because she is using the FOIL method – maybe because it's familiar" (see Appendix B).

Student #4, like student #2, chose Julia's deductive explanation, because she uses the FOIL method which student #4 thought was straightforward. "Ah, because she is using the FOIL method, right, and in my opinion, it is easier for students to understand and I mean – it is straightforward – First Outside Inside Last" (see Appendix B).

Student #3, too, chose Julia's deductive, formulated, pre-formal explanation, because she uses the FOIL method of multiplication which, according to student #3, is simple. In student #3's opinion, it is simple, because it uses straightforward instructions which you could follow like a recipe. "Say, Julia's answer for this one because it's -- she is explaining what she is doing. She is using the FOIL method of multiplication. Not only that, she'll go through every step of the FOIL method in each line. Again, it's simple -- simplicity, itself. This here is very simple, straightforward instructions. If you had different numbers and you were going to do this, you could almost follow like a recipe which she has here and learn and teach yourself how to do something like that." (see Appendix B).

Although student #8 preferred Jody's inductive/multiple example explanation, she did for the same reason as student #2, student #3 and student #4 chose Julia's deductive, formulated, pre-formal explanation. Student #8 chose Jody's explanation, because he uses the FOIL method. Student #8 seemed to focus more on the final product than the method, indicating that the perfect square trinomial works out evenly as a result of using the FOIL method.

B. 1(d) Summary of student interviews using questionnaire questions

Table 10 summarizes the student's choice of explanation, the kind of explanation for each of the three questions and why he/she preferred a particular explanation.

Table 10

Student interviewed	Choice of explanation for the sum of the interior angles in any triangle is 180° and why	Choice of explanation for the sum of two odd numbers is even and why	Choice of explanation for perfect trinomial squares and why
Student #1	Amanda's Inductive/Multiple Example **Familiar**	Drake's Analogy **Familiar**	Lisa's Inductive/Multiple Generic Example **Two different ways**
Student #2	Amanda's Inductive/Multiple Example **Easy**	Cora's Inductive/Multiple Example **Examples**	Julia's Deductive, formulated, semi-formal **Familiar**
Student #3	Amanda's Inductive/Multiple Example **Clear**	Cora's Inductive/Multiple Example **Examples**	Julia's Deductive, formulated, semi-formal **Straightforward**
Student #4	Amanda's Inductive/Multiple Example **Obvious**	Cora's Inductive/Multiple Example **Examples**	Julia's Deductive, formulated, semi-formal **Straightforward**
Student #8	Dylan's Inductive/Multiple Example **Straightforward**	Drake's Analogy **Familiar**	Julia's Deductive, formulated, semi-formal **Familiar**
Student #5	Barry's Formulaic **Statements & Reasons**	Andy's Deductive, formulated, semi-formal **Statements & Reasons (Formula)**	N/A
Student #6	Cynthia's Deductive, formulated, semi-formal **Statement & Reasons**	Drake's Analogy **Written out in words**	N/A
Student #7	Cynthia's Deductive, formulated, semi-formal **Logical**	Bill's Inductive/Single Example **Visual**	N/A

B. (2) Comparison of findings between questionnaire and student interviews

Tables 11, 12 and 13 compare the results from the questionnaire and the student interviews for the three questions

- Why does the sum of the interior angles of any triangle equal 180°?
- Why is the sum of two odd numbers even?
- Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

B. 2(a)

Table 11

Why does the sum of the interior angles of any triangle equal 180°?

Students' Explanations	Kinds of Explanations	Number of participants from the questionnaire who chose a particular explanation n = 82	Number of participants from the interviews who chose a particular explanation n = 8
Amanda	Inductive/Multiple Example	16 (19.5%)	4 (50%)
Barry	Formulaic	21 (25.6%)	1 (12.5%)
Cynthia	Deductive, formulated, semi-formal	18 (22%)	2 (25%)
Dylan	Inductive/Multiple Example	21 (25.6%)	1 (12.5%)
Ewan	Deductive, formulated, pre-formal	6 (7.3%)	0 (0%)

Although there are no obvious consistencies among the type of explanation preferred, students did seem to prefer the empirical explanations over the deductive explanations. It is interesting to note that only six of the eighty-two questionnaire participants (7.3%) and no one from the interviews preferred Ewan's deductive, formulated, pre-formal explanation.

B. 2(b)

Table 12

Why is the sum of two odd numbers even?

Students' Explanations	Kinds of Explanations	Number of participants from the questionnaire who chose a particular explanation n = 82	Number of participants from the interviews who chose a particular explanation n = 8
Andy	Deductive, formulated, semi-formal	11 (13.41%)	1 (12.5%)
Bill	Inductive/Single Example	3 (3.67%)	1 (12.5%)
Cora	Inductive/Multiple Example	35 (42.68%)	3 (37.5%)
Drake	Analogy	33 (40.24%)	3 (37.5%)

Although it was not obvious what kind of explanation students preferred for why the sum of the interior angles of any triangle equals 180°, it was obvious for why is the sum of two odd numbers even. Thirty-five of the eighty-two questionnaire participants (42.68%) and three of the eight students interviewed (37.5%) preferred Cora's inductive/multiple example explanation (see Table 12). An equal number of students interviewed (37.5%) (see

Table 2) and over forty percent (40.24%) (see Table 12) of those students who participated in the questionnaire preferred Drake's analogy.

B. 2(c)

Table 13

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Students' Explanations	Kinds of Explanations	Number of participants from the questionnaire who chose a particular explanation n = 82	Number of participants from the interview who chose a particular explanation n = 5
Lisa	Inductive/Multiple Generic Example	8 (9.76%)	1 (20%)
Julia	Deductive, formulated, pre-formal	32 (39.02%)	3 (60%)
Jody	Inductive/Multiple Example	17 (20.73%)	1 (20%)
Dena	Deductive, formulated, semi-formal	16 (19.51%)	0 (0%)
Cheryl	Deductive, unformulated/ informal	9 (10.91 %)	0 (0%)

Cora's inductive/multiple example explanation which was strictly numerical was favoured for why is the sum of two odd numbers even; while Julia's deductive, formulated, pre-formal explanation was favoured for why do perfect trinomial squares have the form

$x^2+2bx+b^2$. Close to forty percent (39.02%) of the students who participated in the questionnaire and sixty percent (60%) of those interviewed preferred Julia's deductive, formulated, pre-formal explanation (see Table 13).

B. 2(d) Summary of comparison

Tables 14, 15 and 16 show the chi-square test for each of the three questionnaire questions.

Table 14

Why does the sum of the interior angles in any triangle equal 180°?

Kind of Explanation	Number of students who preferred a particular explanation n = 82 (%)	Number of explanations offered n = 5 (%)	$\frac{(\text{observed - expected})^2}{\text{expected}}$
Inductive/Multiple Example Amanda's Dylan's	37 (45%)	2(40%)	0.54
Formulaic Barry's	21 (26%)	1 (20%)	1.29
Deductive Cynthia's Ewan's	24 (29%)	2 (40%)	2.36
			$\sum x^2 = 4.19$

The Chi-square statistic of 4.19, at 2 degrees of freedom indicates that there is about a 90% chance that students preferred inductive and formulaic explanations over deductive explanations for some other reason than chance.

Table 15

Why is the sum of two odd numbers even?

Kind of Explanation	Number of students who preferred a particular explanation n = 82 (%)	Number of explanations offered n = 4 (%)	$\frac{(\text{observed} - \text{expected})^2}{\text{expected}}$
Inductive/Single Example Bill's	3 (4%)	1 (25%)	14.94
Inductive/Multiple Example Cora's	35 (43%)	1 (25%)	10.26
Deductive Andy's	11 (13%)	1 (25%)	4.40
Analogical Drake's	33 (40%)	1 (25%)	7.62
			$\sum x^2 = 37.22$

The Chi-square statistic of 37.22, at 3 degrees of freedom indicates that there is more than a 99% chance that students preferred inductive and analogical explanations over deductive explanations for some other reason than chance.

Table 16

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Kind of Explanation	Number of students who preferred a particular explanation n = 82 (%)	Number of explanations offered n = 4 (%)	$\frac{(\text{observed - expected})^2}{\text{expected}}$
Inductive/Multiple Example Jody's Lisa's	25 (30%)	2 (40%)	1.85
Deductive Julia's Dena's Cheryl's	57 (69%)	3 (60%)	1.24
			$\sum x^2 = 3.09$

The Chi-square statistic of 3.09, at one degree of freedom indicates that there is about a 90% chance that students preferred deductive explanations over inductive explanations for some other reason than chance.

B. (3) Teacher Interviews

The results of the interviews with the two instructors showed that both preferred the deductive explanations, with one preferring formulated and the other not expressing that preference.

B. 3(a) Results from the teacher interviews using the first questionnaire question

Why does the sum of the interior angles of any triangle equal 180?

Both instructors preferred Cynthia's deductive, formulated, semi-formal explanation for the sum of the angles in a triangle question. According to instructor #1, Cynthia's deductive, formulated, semi-formal explanation uses postulates, theorems and proven statements which makes it the best choice. "Cynthia, she used postulates and theorems, proven statements." (see Appendix B). Instructor #2 liked Cynthia's deductive explanation, because "she shows an understanding of some aspects of math and application of them." (see Appendix B)

Dylan's inductive/multiple example explanation also interested instructor #1. Instructor #1 believed that Dylan's inductive/multiple example explanation was based on sound geometric principles. "Dylan's is trial and error where he measured angles. It's based on sound geometric principles." The fact that Cynthia's deductive, formulated, semi-formal explanation uses proven statements was reason enough for instructor # 1 to prefer it over Dylan's inductive/multiple example explanation (see Appendix B).

Instructor #1 did not like Amanda's inductive/multiple example explanation. The idea that Amanda's enactive proof tears up paper to show that the angles in a triangle form a straight line was not accurate enough for instructor #1. "Tearing up paper is not accurate, in my opinion." (see Appendix B). Instructor #2, like instructor #1, felt that Amanda's inductive/multiple example explanation was not accurate, because it did not provide proof for all cases, but rather for one particular case. "Amanda's is not proving for all cases: she shows that it is for one instance only -- no accuracy." (see Appendix B).

During the instructor interviews, I asked both instructors for their opinion of Ewan's deductive, formulated, pre-formal explanation. Both selected Cynthia's explanation which was also a deductive, formulated explanation and I expected that they would be equally comfortable with Ewan's. Instructor #1's first reaction to Ewan's deductive, formulated explanation was that she did not understand it. "I don't understand that at all." (see Appendix B). After a period of time studying Ewan's deductive explanation, instructor #1 said that his explanation "explains it in a round about manner" and that it was "no better than the others" (see Appendix B). In instructor #2's opinion, Ewan's deductive, formulated explanation does not apply mathematical concepts. "Ewan's uses no mathematical concepts." (see Appendix B)

B. 3(b) Results from the teacher interviews using the second questionnaire question

Why is the sum of two odd numbers even?

As was the case for the first question, both instructors preferred the deductive, formulated, semi-formal explanation for the second question. Both instructors preferred Andy's deductive, formulated, semi-formal explanation (see Appendix B). Instructor #1 liked the fact that Andy's deductive explanation "makes more sense because it's algebraically laid out" (see Appendix B). Instructor #2, noted that it "shows some thought given to mathematical aspects" (see Appendix B).

Instructor #1 noted the similarities between Cora's inductive/multiple example explanation and Dylan's inductive/multiple example explanation for the sum of the angles in a triangle question. "Cora's is the same as Dylan's, but it is trial and error and has only four, not enough to substantiate. It's only four, not a large sample size." Because Cora's inductive/multiple example explanation is based on trial and error and included a small sample size, it came second to Andy's deductive, formulated, semi-formal explanation (see Appendix B).

Both instructors disliked Bill's inductive/single example explanation saying they found the dots confusing. "I find the dots confusing, but a visual learner might like it." (see Appendix B - instructor #1). "Hated the dots." (see Appendix B - instructor #2).

According to instructor #1, Drake's analogy "makes no sense what so ever, because of two totally unrelated principles or cases are being compared" (see Appendix B). The fact that the product of two negative numbers is positive did not convince instructor #1 that thus the sum of two odd numbers is even

B. 3(c) Results from the teacher interviews using the third questionnaire question

Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

Again, instructor #1 chose the deductive, formulated explanation for the perfect trinomial square question. Instructor #1 preferred Julia's deductive, formulated, pre-formal explanation for why do perfect trinomial squares have the form $x^2 + 2bx + b^2$? However, instructor #2 preferred Cheryl's deductive, unformulated, informal explanation (see Appendix B).

Of the three deductive explanations, (Julia's, Dena's and Cheryl's) Julia's was instructor #1's preference. Instructor #1 thought that Julia gave a better explanation. "It explains step by step what you are doing." Although Julia's deductive, formulated, pre-formal explanation was instructor #1's favourite, she also liked to some extent Dena's deductive, formulated, semi-formal explanation. "Julia's and Dena's, they're both similar except Julia's gives an explanation" (see Appendix B). Like instructor #1, instructor #2 also had some positive comments regarding Dena's deductive, formulated, semi-formal explanation. "Dena's is not bad. She shows where the two (2) comes from, but Cheryl's provides a diagram" (see Appendix B).

Although instructor #1 preferred Julia's deductive, formulated, pre-formal explanation, she felt that Cheryl's deductive, unformulated, informal explanation would be more suitable for a visual learner (see Appendix B). Contrary to instructor #1, instructor #2 did prefer Cheryl's deductive, unformulated, informal explanation, because of its application with area to show why perfect trinomial squares have the form $x^2 + 2bx + b^2$. "Cheryl's shows through the use of application the process and should be easier to see for students as area" (see Appendix B).

Lisa's inductive/multiple generic example explanation was confusing and unclear to instructor #1. A small sample size was another thing about Lisa's inductive/multiple generic example explanation that instructor #1 disliked. "Lisa's is retarded. How would you know to break down 144. Not a whole lot of cases to support -- unclear" (see Appendix B).

Likewise, instructor #1 did not like Jody's inductive/multiple example explanation because of the small sample size which instructor #1 felt was inconclusive. "Jody's is trial and error. Five cases, small sample size is not enough to conclude" (see Appendix B). Instructor #2 supports instructor #1's claim that Jody's inductive/multiple example explanation is inconclusive. "Jody's doesn't prove anything. He only shows that it is for those particular cases, but not for all cases" (see Appendix B).

B. (4) Results from student interviews using questions from classroom observations

B. 4(a)

Pickup Charge

Two students were asked how they would calculate the pickup charge for 275 kg of a product if $6.10 is charged for each 100 kg or fractional part. Both students could choose from the answers provided.

$6.10 * (275 ÷ 100) = $16.775 or $6.10 *(300 ÷ 100) = $18.30

When deciding that the second response was the best, student #5 commented -- "The way we learned it is that 100 + 100 + 100 = 300; 275 is 75 more than 200 so you go to the next 100 kg up which brings you to 300 kg" (see Appendix B). For similar reasons, student #6 also chose the second response. Student #6 remarked that since 275 was 25 away from 300 kg you would have to use 300 kg (see Appendix B).

The manual which provided the question and answer showed $6.10 * (275 ÷100) = $16.775. Student #6 and student #5 were both familiar with the textbook answer. In class, their mathematics instructor told them that it was incorrect and should be changed to $6.10 * (300 ÷100) = $18.30. Without questioning, both student #5 and student #6 accepted that their instructor was right and the manual was wrong. This is evident in student #5's comment -- "The way we learned it is..." (see Appendix B).

B. 4 (b)

Solving an equation with negative numbers

I asked student #7 to choose from the two different ways of solving the following equation.

$1020 = a\,\frac{(-255)}{-1}$	or	$1020 = a\,\frac{(-255)}{-1}$
$-1020 = a\,(-255)$	or	$1020 = a\,(255)$
$a = 4$	or	$a = 4$

In class, student #7 solved the equation using the first method as was shown in the answer key. However, during the interview, she preferred to solve the equation using the second method, because she could eliminate the negative signs. "Because you are bringing the negative back over on this side. You still got the negative here. Just get rid of the negatives." (see Appendix B).

B. 4(c)

Why does .45/.99 reduce to 45/99?

Three of the five students interviewed using this question said that because .45 and .99 have the same number of decimal places, the decimal would cancel or be eliminated. Their explanations were much the same as what their instructor had provided.

"Because .45 and .99 have the same number of decimal places, it can reduce down to 45/99." (see Appendix B - Instructor #1). "Because there ah is two numbers after the decimal. You can just eliminate the decimal -- I guess. That's what I would do." (see

Appendix B - student #4). "Each has the same number of decimal places. Because decimal places are in the same spot; they are each the same amount of decimal places from -- so that it doesn't change it." (see Appendix B - student #3). "Well you know I don't have to use them because they're the same distance apart. That goes to that because where your decimal place is the same or both sets of numbers." (see appendix B - student #7). "Don't you have to multiply by 100?" (see Appendix B - student #2).

B. 4(d)

Which of the following sequence is geometric and why?

1	1/3	1/9	1/27	
1	5	10	15	20

Five of the six students questioned said that 1 1/3 1/9 1/27 was a geometric sequence. Student #1, student #4, student #7, and student #3 concluded that 1 1/3 1/9 1/27 was geometric after dividing the second number by the first and/or the third number by the second to determine the common ratio for each sequence. Since 5/1 does not equal 10/5, the second sequence was ruled out. Because 1/3/1 and 1/9/1/3 are equal, 1 1/3 1/9 1/27 was a geometric sequence. Student #3 also added that the sequence 1 1/3 1/9 1/27 was getting subsequently smaller by the same amount -- "each subsequent number is being multiplied by 1/3." Student #8 chose to work with the sequence 1 5 10 15 20 first. Because 15/10 = 1.5 and 10/5 = 2, student #8 concluded that this particular sequence was not geometric. Although student #8 appeared to know how to calculate the common ratio for a geometric sequence, she did not know how to calculate it for the fractional sequence. Like student #3, though, she knew the sequence was changing by a factor of 1/3 and that the next number in the sequence was 1/81 (see Appendix - B).

Student #2, on the other hand, decided on 1 5 10 15 20 as a geometric sequence. Student #2's calculations involved dividing five (5) by one (1) to get five (5) and then adding five (5) to each preceding number to obtain the next number in the sequence (see Appendix - B).

Chapter VIII

Analysis

The study investigated students' understanding of the role of proof as an explanation. Adult learners attending the College of the North Atlantic, Happy Valley-Goose Bay Campus enrolled in the technician, business, applied arts and Adult Basic Education programs participated in the study. Students' written responses to questions concerning proof made up the empirical core of the study. The meaning of what was required as a proof was not made explicit to the study's participants. Because each of the research questions required slightly different approaches to analysis, each question is considered separately.

A. *What kinds of explanations do students prefer (deductive, inductive, analogical)?*

Data collected from the eighty-two questionnaires was analyzed, looking for commonalities among responses. Any patterns found in student responses were then related to the student interviews and classroom setting to see if students' preference for a particular type of explanation was consistent. Students showed an overall preference for multiple example explanations and analogical explanations. Deductive argument was also preferred at times.

A. (1) Inductive/Multiple Example Explanations

The most popular kind of explanation among the study's participants was the multiple example explanation. Five different multiple example explanations were offered for the three questionnaire questions. Cora's multiple example explanation was the most preferred explanation for "Why is the sum of two odd numbers even?" Over forty percent (42.68%) of the questionnaire participants preferred Cora's multiple example explanation (see Table 2). Three of the five multiple example explanations (Amanda's explanation, Dylan's explanation, and Jody's explanation) were fairly favourable with about twenty percent or more of the questionnaire participants preferring either one of the three (Amanda's explanation - 19.5% see Table 1, Dylan's explanation - 25.6% see Table 1, Jody's explanation - 20.73% see Table 3). Less than ten percent (9.76%) of the questionnaire participants preferred Lisa's multiple generic example explanation. Students' preference for multiple example explanations supports the research which says that students prefer empirical evidence over deductive argument (see Fischbein and Kedem, 1982; Balacheff, 1988; Chazan, 1993; Finlow-Bates, 1994).

A. (2) Analogical Explanations

The only analogical explanation offered on the student questionnaire was Drake's analogy for "Why is the sum of two odd numbers even?" Over forty percent (40.24%) of the eighty-two students who participated in the questionnaire and thirty-seven percent (37.5%) of those interviewed indicated a preference for Drake's analogical explanation. Although analogical explanations only occurred once on the questionnaire, students' reaction to this kind of explanation was highly favorable. Students in my study, like Polya (1968) recognized the importance of analogical explanation. This supports Reid's (1995a) claim that a strong analogy can be preferable to a deductive explanation.

A. (3) Deductive Explanations

Six deductive explanations were offered on the student questionniare which included: Cynthia's, Ewan's, Andy's, Julia's, Dena's, and Cheryl's explanations. Five of the six deductive explanations were formulated with the other, Cheryl's explanation being unformulated. Students did not show an overall preference for deductive explanations, neither did they dismiss deductive argument. Although students did not indicate a preference for deductive explanations for the first two questionnaire questions, they did for the third question. Julia's deductive, formulated, pre-formal explanation was the most favored for "Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?" with close to forty percent (39.02%) of the questionnaire participants choosing it. Over twenty percent (22%) of the questionnaire participants preferred Cynthia's deductive, formulated, semi-formal explanation for the first question. About twenty percent (19.51%) of the questionnaire participants preferred Dena's deductive, formulated, semi-formal explanation for the third question. A little more than ten percent (13.41%) of the questionnaire participants preferred Andy's deductive, formulated, semi-formal explanation for the second question. Approximately ten percent (10.91%) of the questionnaire participants preferred Cheryl's deductive, unformulated, informal explanation for the third question. Less than ten percent (7.3%) of the questionnaire participants preferred Ewan's deductive, formulated, pre-formal explanation for the first question.

A. (4) Explanations that were not preferred

A. (4a) Inductive Explanations (Single Example and Generic Example)

Students and instructors indicated a dislike for Bill's inductive/single example/visual explanation. Only three of the eighty-two questionnaire participants preferred Bill's explanation. Lisa's multiple generic example explanation for the third questionnaire question was not well liked either. Only eight of the eighty-two questionnaire participants indicated a preference for it. Other explanations that were not well received were Ewan's and Cheryl's deductive explanations.

A. (5) What is it about these explanations that students rejected them? Why were the popular explanations preferred?

No kind of explanation was preferred by the majority on all three of the questionnaire questions. Deductive explanations were among the most common explanation but the least popular. On the first question the formulaic explanation, Barry's explanation was almost as popular as a correct deductive explanation, Cynthia's explanation and Dylan's inductive explanation. These results suggest that there is something other than the logical structure of the explanation that determines students' preference for a particular explanation.

B. What constitutes a good explanation for the student?

The data collected to answer this question came from analyzing the questionnaire, interviewing students, and observing students within their classroom setting. Students' preferences and dislikes for different explanations were analyzed to determine what constitutes a good explanation for the student. Students' preferences were also analysed using Hoyles (1997) "proof types", but no pattern of preference was found (see Chapter VI, section C. 1(a))

B. (1) Why were the popular explanations preferred?

The more popular explanations (Cora's multiple example explanation, Drake's analogy, and Julia's deductive, formulated, pre-formal explanation) were preferred for their familiarity, clarity, obviousness, easiness, and straightforwardness. In addition, Cora's use of examples was why students preferred it for the second questionnaire question.

Three of the eight students interviewed selected the same kind of explanation for each of the questionnaire questions: two multiple example explanations and a deductive explanation (see Table 10). In doing so, they described their preferred explanations as easy, familar, straightforward, and obvious.

Prior learning experience influenced these three students' choice of explanations. Their familiarity with the straight angle made Amanda's inductive/multiple example explanation seem easy, clear and obvious. Likewise their prior learning experience or familiarity with the FOIL method made Julia's deductive, formulated, pre-formal explanation straightforward. Cora's use of examples to show that the sum of two odd numbers is even provided clarity and made the conjecture easier to understand.

Student's familiarity with the mathematical principle that the product of two negative numbers is positive was the reason for at least two of the eight students interviewed preferring Drake's analogy.

B. (2) Why were the unpopular explanations rejected?

The presentation of Bill's inductive/single example explanation was what students and teachers disliked. That is, students and instructors rejected Bill's single example explanation because they did not want to count the dots.

Students' unfamiliarity with Ewan's deductive, formulated, pre-formal explanation, and Lisa's inductive/multiple generic example explanation was why they rejected these explanations. Ewan's walking around the triangle explanation is a type unfamiliar to the students and the instructors, and so was rejected. This is evident in the instructor's comments regarding Ewan's explanation for "Why the sum of the interior angles in a triangle equals 180°?" – "I don't understand that at all. It explains it in a round about manner." The same instructor refers to Lisa's inductive/multiple generic example explanation as "retarded" and says "How would you know to break down 144?" (see Appendix B).

So, what makes an explanation a good explanation for the student? Student #7 chose explanations based on their being logical and visual. Four of the five explanations for the first questionnaire question were visual, so logical may have been student #7's second criterion. For the other students, however, accessibility or familiarity seems to be more important than the kind of explanation offered (deductive, inductive, or analogical). Students in my study seemed to use the same kinds of criteria that Hanna (1983) says mathematicians use to determine acceptance of a proof (see Chapter IV, section D(2)). In particular, students seemed to use Hanna's criteria #1 and #5:

1. They understand the theorem, the concepts embodied in it, its logical

antecedents, and its implications. There is nothing to suggest that it is not true;

5. There is a convincing argument for it (rigorous or otherwise), of a type they have encountered before.

Students in my study, like the mathematicians Hanna refers to, accepted an explanation if there was nothing to suggest that it was false. More importantly, what seemed to matter to the students in my study was whether or not the explanation was of a type familiar to them. Thus, Hanna's criteria #5 for mathematicians' acceptance of proof was the same criterion students used to determine what constitutes as a good explanation.

C. *What constitutes a good explanation for the teacher?*

The data collected to answer this question came from interviewing and observing the two mathematics instructors that teach at the same college as the researcher. Explanations that teachers provided for the students were analyzed. Observing the kinds of explanations offered by students that teachers deemed acceptable helped to determine what constitutes a good explanation for the teacher.

The results of the interviews with the two mathematics instructors showed that both preferred the deductive explanations with one preferring formulated and the other not expressing that preference. For instructor # 1, a good explanation is one that is deductive and formulated, one that uses postulates, theorems and proven statements, and one that is clear - explaining step by step. In addition, trial and error or empirical explanations must include large sample sizes. For instructor # 2, a good explanation had to not only use logical arguments, but had to apply the question mathematically.

D. *Do students mirror teachers' explanations or do they have their own style of explaining?*

Do students respond differently in different situations; that is, do students respond differently in an informal setting than a classroom setting? Do students offer similar explanations on test questions as their teachers offered in class or do they tend to use their own style of explaining? Observing students and their teachers in the classroom, observing students in informal situations and interviewing students and teachers coupled with the empirical data collected from the questionnaire provided answers to this question.

The manual which provided the question and answer for the pick up charge question showed $6.10 * (275 ÷100) = $16.775. Both students interviewed (student # 5 and student # 6) were familiar with the textbook answer. In class, their mathematics instructor told them that it was incorrect and should be changed to $6.10 * (300 ÷100) = $18.30. Without questioning, both students accepted that their instructor was right and the manual was wrong. This is evident in one student's comment – "The way we learned it is..." This example shows the teacher's authority as a disincentive to explain. If things are the way they are solely because the teacher says so, then there is no reason for the student to explain. De Villiers' (1992) study shows that students' strength of belief in or attachment to a particular method is based on external rather than personal grounds. Both students showed an attachment to the way the teacher had explained the pickup charge question and had made sure they would answer any related test questions accordingly.

Solving mathematical equations according to the answer key is another example of how students are influenced by other authorities (see Results B. 4(b)). In this case, the answer key influenced student # 7's decision to carry the negative to other side rather than eliminating it from the beginning. Reliance on answer keys adversely affects students' confidence to do and to understand mathematics. How does the level of education or prior learning experience affect students' mathematical confidence? Are ABE students and other adult learners taught to rely on answer keys?

Perhaps, students were accustomed to formulating their explanations so as to conform to teacher expectations. This is especially evident with student # 5 and student #6 and the pickup charge question. The formulation of students' explanations was influenced by their learning experience in the course of instruction. Again, this is evident through students' parroting of the teachers' explanations (see Results B. 4. (c) and 4. (d)). There was only one student who seemed to use her own style of explaining. She either explained things differently, or she did not have a response for the question being asked in the interviews. This is interesting to note because prior to entering ABE she attended high school in Ontario; whereas, all of the others had attended high school in Labrador. Maybe her method of explaining was similar to what Ontario teachers used.

Chapter IX

Conclusion

No kind of explanation was preferred by the majority on all three questionnaire questions. Students showed an overall preference for multiple example explanation and analogical explanation. Deductive explanations were one of the most common kind of explanation offered but the least preferred. Students did not show an overall preference for deductive explanations; neither did they dismiss deductive argument. It was the form of the explanation, namely its familiarity and accessibility, that students used as criteria in determining its acceptance. Unlike the students, the instructors accepted or favored an explanation based on its logical structure. It was also found that students' reliance on answer keys adversely affects their confidence to do and understand mathematics. Students' parroting of teacher explanations shows how they are accustomed to formulating explanations so as to conform to teacher expectations as a motivation for proving (see Reid 1995a, Alibert 1988, Schoenfeld, 1983 and Wheeler, 1990). If things are the way they are solely because the teacher says so, then there is no reason for the student to explain.

Teachers need to be cognizant of the kinds of explanations offered to students and by students. Although the logical structure of an explanation seems to be important for the teacher, the student may perceive things differently. Teachers must be aware of students' criteria for acceptance of an explanation. That is, its accessibility and familiarity.

Students should be exposed to the different purposes proof serves, in particular, the explanatory function of proof. If proof serves a distinct purpose within the classroom it will be meaningful for students. The teaching of proof should shed light upon the mathematical structures under study by providing insight as to *why* a statement is true. Hanna (1995) claims that the main function of proof in the classroom is the promotion of understanding. Perhaps, math educators can promote understanding by placing greater emphasis on the social criteria for acceptance of an explanation.

Encouraging students to formulate their own explanations will enhance their confidence to do and understand mathematics. Exposing students to the different purposes proof serves will serve better as a motivation for proving than formulating explanations so as to conform to teacher expectations.

Finally, additional research will help to answer some of the questions my research has raised such as:

- Would students' preference for a kind of explanation differ if deductive explanations were made more accessible?
- What role could analogical explanations play in the teaching of proof?

REFERENCES

Alberta Education. (1991). *Math 30 Teacher Resource Manual*. Edmonton: Author.

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics, 8*(2), 31-35, 43.

Balacheff, N. (1988). Aspects of Proof in Pupils' Practice of School Mathematics. In D. Pimm (ed.), *Mathematics, Teachers and Children*, (pp. 216-235). London: Hodder and Stoughton.

Barbeau, E. J. (1990). Three faces of proof. *Interchange, 21*(1), 24-27.

Bell, A. (1976). A study of pupils' proof explanations in mathematical situations. *Educational Studies in Mathematics, 7*, 23-40.

Chazan, D. (1989). *Ways of knowing: High school students' conceptions of mathematical proof.* (Doctoral Dissertation, Havard University, 1989).

Chazan, D. (1993). High School Geometry Students' Justification for their Views of Empirical Evidence and Mathematical Proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Davis, P. J. & Hersh, J. (1986). *Descartes' dream*. New York: HBJ Publishers.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

de Villiers, M. (1991). Pupils' needs for conviction and explanation within the context of geometry. *Pythagoras, 24*, 17-24.

de Villiers, M. (1992). *Children's acceptance of theorems in geometry*. Poster presented at

The Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education. Durham NH.

Dreyfus, T. & Hadas, N. (1996). Proof as answer to the question why. *Zentralblatt fur Didaktik der Mathematik, 28*(1) 1-5.

Finlow-Bates, K. (1994). First Year Mathematics Students' Notions of the Role of Informal Proof and Examples. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 344-351). Libson, Portugal.

Fischbein, E., & Kedem, I. (1982). Proof and Certitude in the Development of Mathematics Thinking. In A. Vermandel (Ed.), *Proceedings of the Sixth Annual Conference of the International Group for the Psychology of Mathematics Education*, (pp. 128-131). Antwerp, Belgium.

Gage Canadian dictionary. (1983). Toronto, Ontario: Gage Publishing.

Glaser, B., and A. Strauss. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Gonobolin, F. (1954). Pupils' Comprehension of Geometric Proofs. In M. de Villiers (Ed). (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Guba, E. B., (Ed.). (1990). *The paradigm dialog*. Newbury Park, CA: Sage Publications.

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto: OISE Press.

Hanna, G. (1989). More than formal proof. *For the Learning of Mathematics, 9*(1), 20-23.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Hanna, G. & Jahnke, H. N. (1993). Proofs and applications. *Educational Studies in Mathematics*, *24*(4), 421-437.

Hanna, G. (1995). Challenges to importance of proof. *For the Learning of Mathematics*, *15*(3), 42-49.

Hanna, G. (1996). The ongoing value of proof. *The Twentieth Annual Conference of the Proceedings of the International Group for the Psychology of Mathematics Education*, (vol.1, pp. 21-34). Valencia, Spain.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies*, *24*(4), 389-399.

Hoyles, C. (1996). Beyond the classroom: The curriculum as a key factor in students' approaches to proof. In Yvonne Pothier, (Ed.), *Proceedings of the Annual Conference of the Canadian Mathematics Education Study Group*, (pp. 7-25).

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics*, *17*(1), 17-15.

Institute of Mathematics and its Applications, Royal Education Society (1995). *Tackling the mathematics problem*. London UK: London Mathematical Society.

Jurgensen, R., Maier, J. & Alfred, J. Donnelly. (1973). *Modern basic geometry*. Boston, Massachusetts: Houghton Mifflin.

Kline, M. (1973). *Why Johnny can't add: The failure of the new math*. New York: St. Martin's Press.

Lincoln, Y. S., and E. G. Guba. (1986). But is it rigorous? Trustworthiness and authencity in naturalistic evaluation. In D. D. Williams (Ed.), *Naturalistic evaluation. New directions for program evaluation* (pp. 73-84). San Francisco: Jossey-Bass.

Lincoln, Y. S., and E. G. Guba. (1985). *Naturalistic inquiry.* Beverly Hills, CA:Sage Publications.

Manin, YU. (1981). A digression on proof. *The Two-year College Mathematics Journal, 12*(2), 104-107.

Mason, J., Burton, L., and Stacey, K. (1982). *Thinking mathematically.* London: Addison-Wesley.

Mok, I. A. C. (1997). A hierarchy of students formulation of an explanation. In E. Pehkonen (Ed.), *Proceedings of the Twenty-first Annual Conference of the International Group for the Psychology of Mathematics Education,* (Vol.3, pp.248-255). Lahti, Finland.

Movshovitz-Hadar, N. (1988). Stimulating presentations of theorems followed by responsive proofs. *For the Learning of Mathematics, 8*(2), 12-19.

Polya, G. (1968). *Mathematics and plausible reasoning* (2nd ed.) Princton: Princton University Press.

Reid, D. A. (1995a). *The need to prove.* (Doctoral dissertation, University of Alberta, 1995).

Reid, D.A. (1995b). Proving to explain. In Meira, I., and Carraher, D. (Ed). *Proceedings of of the Nineteenth Annual Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 3, pp.137-143). Recife, Brazil.

Reid, D. A. (November 21, 1998). *Personal communication.*

Sanders, J. R. (1994). *The program evaluation standards*. Beverly Hills, CA: Sage Publications.

Schoenfeld, A. (1982). Psychological factors affecting students' performance on geometry problems. In S. Wagner (Ed.), *Proceedings of the Fourth Annual Conference of the International Group for the Psychology of Mathematics Education*, (pp. 168-174). Athens, GA.

Schoenfeld, A. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive Science*, 7, 329-363. (revised version of a paper presented at the 1982 meeting of AERA).

Schoenfeld, A. (1994). What do we know about curricula? *Journal of Mathematical Behavior*, *13*(1),55-80.

Sekiguchi, Y. (1991). *An investigation on proofs and refutations in the mathematics classroom*. (Doctoral dissertation, University of Georgia, 1991).

Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, *34*,135-152

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics*, (pp. 495-514). New York: Macmillan.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30*(2), 161-177.

Volminik, John. (1988). The role of proof in students' understanding of geometry. *Paper presented at the AERA annual meeting*, New Orleans.

Volminik, J. (1990). The nature and role of proof in mathematics education. *Pythagoras, 23*, 7-10.

Wheeler, D. (1990). Aspects of mathematical proof. *Interchange, 21*(1), 1-5.

Wilder, R. L. (1944). The nature of mathematical proof. *American Mathematical Monthly, 51*, 309-323.

Wong, N. Y. (1993). The psychology learning environment in the Hong Kong mathematics classroom. *Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 284-291). Tsukuba, Japan.

Worthen, B. R. & J. R. Sanders. (1987). *Educational evaluation*. Toronto: Copp Clark Pittman.

Zack, V. (1997). You have to prove us wrong: Proof at the elementary school level. In E. Pehkonen (Ed.), *Proceedings of the Twenty-First Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 299-306). Lahti, Finland.

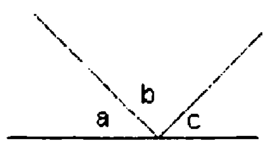
Appendix A

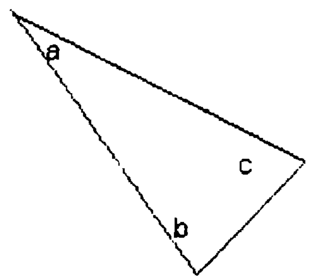
Sample of Student Questionnaire

Why does the sum of the interior angles of any triangle equal 180°?

Amanda's answer:

> I tore the angles up and put them together. It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.



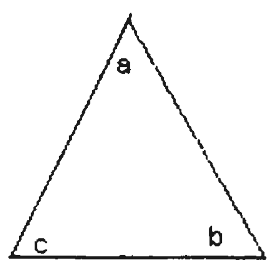


Barry's answer:

I drew an isosceles triangle, with c equal to 65°.

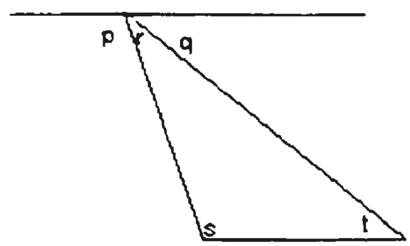
Statements	Reasons
$a = 180° - 2c$	Base angles in isosceles triangle equal
$a = 50°$	$180° - 130°$
$b = 65°$	$180° - (a + c)$
$c = b$	Base angles in isosceles triangle equal

therefore, $a + b + c = 180°$



Cynthia's answer:

I drew a line parallel to the base of the triangle



Statements	Reasons
$p = s$	Alternate angles between two parallel lines are equal
$q = t$	Alternate angles between two parallel lines are equal
$p + q + r = 180°$	Angles on a straight line

therefore $s + t + r = 180°$

Dylan's answer:

I measured the angles of all sorts of triangles accurately and made a table

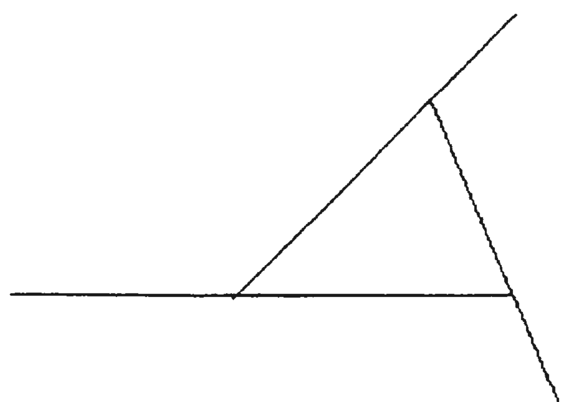
a	b	c	total
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

They all added up to 180°

Ewan's answer:

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°. You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line.

This makes a total of 540°. 540° - 360° = 180°.



WHY IS THE SUM OF TWO ODD NUMBERS EVEN?

Andy's answer:

Let one odd number be (2n + 1) and the another odd number (2m + 1),
then (2n + 1) + (2m + 1) = 2(n + m) + 2

Bill's answer:

• • • • • • • •

• • • • • • • • •

+

• • • • • •

• • • • • • •

=

• • • • • • • • • • • • • •

• • • • • • • • • • • • • • • •

=

• • • • • • • • • • • • • • •

• • • • • • • • • • • • • • •

Cora's answer:

13 + 45 = 58
7 + 9 = 16
113 + 335 = 448
1077 +517 = 1594

Drake's answer:

An odd number plus an odd number equals an even number because of the same principle which says a negative number times a negative number is a positive.

Why do perfect trinomial squares have the form $x^2+2bx+b^2$?

Lisa's answer:

If you take the number 144, then 144 is equal to $10^2 + 2(10)(2) + 2^2$
Likewise, $169 = 13^2$ is $10^2 + 2(10)(3) + 3^2$
Finally, $81 = 9^2$ is $8^2 + 2(8)(1) + 1^2$

Therefore, any perfect square number is equal to a binomial square which always multiplies out into the form $x^2 + 2bx + b^2$
The binomial is found by finding two numbers which add up to the number before it is squared. For example $9 = 8 + 1$ and $9^2 = 81$. Similarly, $13 = 10 + 3$ and $13^2 = 169$

Julia's answer:

If you multiply two same binomials such as $(x + b)(x + b)$ using the FOIL method, then the first two terms of the two binomials will multiply to $x * x = x^2$; the two outside terms will be x times b = xb ; the two inside terms will be b times x = bx; and the two last terms of each of the two binomials multiplied together will be b^2. Combining like terms, the xb and bx will equal 2bx. Thus, $(x + b)(x + b)$ will always multiply into the form $x^2 + 2bx + b^2$.

Jody's answer:

$(x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$
$(x + 3)(x +3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$
$(x - 5)(x - 5) = x^2 - 5x -5x + 25 = x^2 - 10x +25$
$(3x + 4)(3x + 4) = 9x^2 + 12x + 12x + 16 = 9x^2 + 24x + 16$
$(2x - 3)(2x - 3) = 4x^2 - 6x -6x + 9 = 4x^2 - 12x + 9$

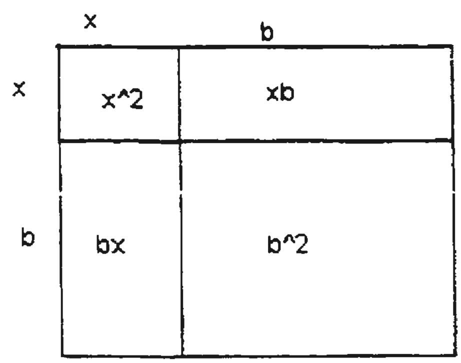
Therefore perfect trinomial squares always have the form $x^2 + 2bx +b^2$

Dena' answer:

Using the distributive law:
$(x + b)(x + b)$
$(x + b)x = x^2 + bx$
$(x + b)b = xb + b^2$
$(x + b)(x + b) = x^2 + bx + xb + b^2$
The "2" comes because "xb" occurs in both distributions

Cheryl's answer:

(x + b) represents a line segment of length (x + b)



Appendix B

1. Why does the sum of the interior angles of any triangle equal 180°?

1. (a) Students' responses to Amanda's inductive/multiple example explanation

Student #1

"Think of circle - 180° is 1/2 of a circle – you add them all up together equal 180° -- so go with Amanda's explanation. She cut the angles and made a straight line. Like a circle is 360° but, ah, if you cut it in half, then you get a straight line 180°."

Student #2

"Amanda's answer because she tore up the angles. It is easy. It doesn't involve statement and reasons."

Student #3

"I tell ya now... Amanda's answer hmm hmm because it's clear, it's not, it's very, ah. She is trying to solve the problem and she said well to add up to 180°. 'I took all the angles basically straighten them out and when I did so they all became a straight line:' so, therefore, that's why, right. It is very clean. It's not cluttered; whereas, I found some of the other answers – they're big equations, large equations and stuff like alternate angles between two parallel lines are equal. How do you know this? You don't. You have to prove that that subsequently. This says everything. It answers the question fully without going into great big long details."

Student #4

"...three different bends. Then it's obvious. Three angles got to equal 180°."

1.(b) Students' responses to Barry's formulaic explanation

Student #5

. "Barry's because first he showed what he did -- why in statement and reasons and then a formula at the end."

1. (c) Students' and instructors' responses to Cynthia's deductive, formulated, semi-formal explanation

Student #6

"Cynthia's answer because she is using statements and reasons."

Student #7

"Cynthia's -- she justifies using logical arguments -- straight line and equivalent angles."

"Kind of liked Ewan's too because that one's using reasoning -- all of them would be same triangle right -- not going to change degrees."

Instructor #1

"Dylan's is trial and error where he measured angles. It's based on sound geometric principles. Cynthia's is better. Cynthia, she used postulates and theorems, proven statements. Tearing up paper is not accurate, in my opinion (implying Amanda's explanation. I don't understand that at all (implying Ewan's explanation). It explains it in a round about manner. No better than the others."

Instructor #2

"Cynthia's, this answer shows an understanding of some aspects of math and application of them. Ewan's uses no mathematical concepts. Amanda's is not proving for all cases; she shows that it is for one instance only – no accuracy."

1. (d) Students' responses to Dylan's inductive/multiple example explanation

Student #8

"Dylan's – pretty straight forward – you take any angle and make the measure equal 180°."

1. (e) Instructors' responses to Ewan's deductive, formulated, pre-formal explanation

Instructor #1

"I don't understand that at all (implying Ewan's explanation). It explains it in a round about manner. No better than the others."

Instructor #2

"Ewan's uses no mathematical concepts."

2. *Why is the sum of two odd numbers even?*

2. (a) *Students' and instructors' responses to Andy's deductive, formulated, semi-formal explanation*

Student #5

"Andy's -- he is saying what one number is and another in formula and then he went on to say why he did it -- then the formula."

Instructor #1

"Andy's makes more sense because it's algebraically laid out. Cora's is the same as Dylan's, but it is trial and error and has only four, not enough to substantiate. It's only four, not a large sample size. I find the dots confusing, but a visual learner might like it (implying Bill's inductive/single example explanation). Drake's makes no sense what so ever, because of two totally unrelated principles or cases are being compared. I don't know how you can come to that conclusion. Do you?"

Instructor #2

"Andy's shows some thought to mathematical aspects and factoring. Hated the dots."

2. (b) *Student's response to Bill's inductive/single example/visual explanation*

Student #7

"I'm better with visuals sometimes. It depends on what I'm doing. If I can see things. Not that I would dispute that (Cora's explanation) or that (Drake's explanation). I know there is something better than that -- just the same. I don't know if it is in one of those tease testers -- some kind of book -- some explanation. You know like all kinds of game type of things. I know there was a whole bunch of stuff similar to that, but there's so many different things -- so many numbers. Andy's is alright, but like right now I can't think odd numbers. I'm trying to think of the algebra stuff -- the numbers -- the equations. It is logical to see where it worked out."

2. (c) Students' responses to Cora's inductive/multiple example explanation

Student #2

"Rose, this one here. I would say Cora's answer. This one here is a lot easier to see. The numbers makes it easier. You don't have to count the dots. Drake's is like a word problem.

Interviewer - "So, you would be convinced given a few examples that the sum of two odd numbers will always be even?"

Student #2 "Sure!"

Student #3

"Cora's answer on that one, because, not only, does she give more than one example and it's clear, again, it is the contents of it. If you read Drake's, he goes into the same principle which says a negative number times a negative number is positive. Well, if you didn't know that or if you weren't versed in algebra, you wouldn't know that; whereas this is basic addition. You can see that, well, they made the statement -- Why is the sum of two odd numbers even? Well, while Cora's answer doesn't explain why it is, she does show that it is. Okay. She does say -- you know what I mean -- okay. It may not be explained to a level of understanding, but it's taken as a given by the way she explains it. She says look at it -- no matter how many times you do it, it works out that way. Therefore, its got to be true."

Student #4

"Given bunch of examples, right, which I think would be easier to do than just trying to explain something, okay, like a negative times together would give you a positive; whereas, if you were given an example, then I would say students would learn better, would understand better."

2. (d) Students' responses to Drake's analogical explanation

Student #1

"Drake's answer because it goes along with a negative times a negative gives you a positive. So, an odd plus an odd is even. Ya, okay, an odd number is like a negative number and an even number is like a positive."

Student #8

"Drake's because a negative times a negative number is positive; therefore, an odd number plus an odd number is an even number."

Student #6

"Drake's because it's written out not using numbers."

3. Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

3. (a) Student's response to Lisa's inductive/multiple generic example explanation

Student #1

"Lisa's answer because she shows two different ways"

$12^2 = 144$

$12^2 = (10 + 2)^2 = 10^2 + 2(10)(2) + 2^2 = 100 + 40 + 4 = 144$

3. (b) Students' and instructors' responses to Julia's deductive, formulated, pre-formal explanation

Student #2

"Rose. Julia's response because she is using the FOIL method -- maybe because it's familiar."

Student #3

"Say, Julia's answer for this one because it's – she is explaining what she is doing. She is using the FOIL method of multiplication. Not only that, she'll go through every step of the FOIL method in each line. Again, it's simple – simplicity, itself. Stacks of numbers -- if you were just learning how to do this and I saw this – Jody's answer you wouldn't know what to make of it. It would be very difficult to follow because you be -- because there is such ah of information closely written together; whereas, this here is very simple, straight forward instructions. If you had different numbers and you were going to do this, you could almost follow like a recipe which she has here and learn and teach yourself how to do something like that. Because you are adding like terms, she goes on to explain what like terms are to a certain extend anyway -- combining like terms xb and bx leads to 2bx - so effective."

Student #4

"Ah, because she is using the FOIL method, right, and in my opinion, it is easier for students to understand and -- I mean -- it is straight forward -- First Outside Inside Last."

Instructor #1

Julia's and Dena's, they're both similar, except Julia's gives an explanation. In fact, Julia's is probably better. Perhaps, it is -- because I'm not sure what is really happening here. Dena's is a little unclear, because she starts with one equation, then splits in two and then makes it one or reverts back to one. I can see what she's done, but someone else might be confused by that. Julia explains step by step what you are doing. Dena does not explain that xb = bx. A visual learner would probably like that way sort of better (implying Cheryl's deductive, unformulated explanation). But it is not as good as the other misses (implying Julia's deductive, formulated, pre-formal explanation). Lisa's is retarded. How would you know to break down 144. Not a whole lot of cases to support -- unclear."

3. (c) Student's response to Jody's inductive/multiple example explanation

Student #8

"Jody's answer because she uses the FOIL method -- they work out evenly like

$(x + 2)(x + 2) = x^2 + 2x + 2x + 2^2 = x^2 + 4x + 4$."

3. (d) Instructors' responses to Dena's deductive, formulated, semi-formal explanation

Instructor #1

"Dena's is a little unclear, because she starts with one equation, then splits in two and then makes it one or reverts back to one. I can see what she's done, but someone else might be confused by that. Julia explains step by step what you are doing. Dena does not explain that xb = bx."

Instructor #2

"Dena's is not bad. She shows where the two (2) comes from, but Cheryl's provides a diagram."

3. (e) Instructor's response to Cheryl's deductive, unformulated explanation

Instructor #1

"A visual learner would probably like that way sort of better (implying Cheryl's deductive, unformulated, informal explanation)."

Instructor #2

"Cheryl's shows through the use of application the process and should be easier to see for students as area. Dena's is not bad. She shows where the two (2) comes from, but Cheryl's provides a diagram. Jody's doesn't prove anything. He only shows that it is for those particular cases, but not for all cases. Cheryl kind of explains it similar to Dena but with a diagram."

4. *Instructor and students' responses to "Why does .45/.99 reduce to 45/99?"*

Same Number of Decimal Places

Instructor #1

"Because .45 and .99 have the same number of decimal places, it can reduce down to 45/99. Okay."

Student #4

"Because there ah is two numbers after the decimal. You can just eliminate the decimal - I guess. That's what I would do."

Student #3

"Each has the same number of decimal places. Because decimal places are in the same spot; they are each the same amount of decimal places from -- so that it doesn't change it."

Student #7

"Well you know I don't have to use them because they're the same distance apart. That goes to that because where your decimal place is the same or both sets of numbers."

Multiply by 100

Student #2

"Don't you have to multiply by 100?"

Equivalent Fractions

Student #8

"I don't know why, but it works out on the calculator. You get the same answer for both (.45/.99 and 45/99) – so they're equivalent."

Summary

Three of the five students interviewed using this question said that because .45 and .99 have the same number of decimal places, the decimal would cancel or be eliminated. Their explanations were much the same as what their instructor had provided.

5. ***Instructor and students' responses to "Which of the following sequence is geometric and why?***

1	1/3	1/9	1/27	
1	5	10	15	20

Instructor #1

"Divide the second number in the sequence by the first and the third number by the second. It is nongeomtric if you get different answers."

Student #1

"Divide second number by first get 1/3 – 1/3 divided by 1 equals 1/3 and 1/9 divided by 1/3 equals 1/3. Five divded by one is five and 10 divided by five is two -- bottom one's not."

Student #3

"The first one because if you divide the second one by the first one or third by the second you are going to get a common ratio and if you times it together no matter what, you'll always get – if you times the common ratio by it right, hmm. I'm getting confused - right. Okay, basically, no matter how many numbers are there, right, you'll get a common ratio that you can multiply, right, to get the following one; I guess is what I'm trying to say, right."

Student #8

Lillian wrote 15/10 = 1.5 and 10/5 = 2 and said "so not geometric because different answers."

Student #7

"Same even number of digits and try to divide."

1/3 divided by 1 equals 1/3

1/9 divided by 1/3 equals 1/3

5/1 = 5 and 10/5 = 2 "Not geometric."

Student #3

"The first one (1 1/3 1/9 1/27), because it is getting progressively smaller by the same amount. It's, it's, ah, okay, because you're multiplying by 1/3, so each time each subsequent number is getting multiplied by 1/3 by 1/3."

Interviewer - "Student #3, how did you determine that?" (implying the 1/3 factor)?

Student #3 - "1 is greater than, less than or equaled; whereas, the r is less than, greater than 1. Ah, second divided by the first."

6. ***Student #7's response to "Solving an equation with negative numbers.'***

$$1020 = a\frac{(-255)}{-1} \quad \text{or} \quad 1020 = a\frac{(-255)}{-1}$$

$$-1020 = a(-255) \quad \text{or} \quad 1020 = a(255)$$

$$a = 4 \quad \text{or} \quad a = 4$$

Interviewer

"Which way do you prefer?"

Student #7

"This way." (Implying the second choice)

Interviewer

"Why didn't you like this way?" (Implying the first choice)

Student #7

"Because you are bringing the negative back over on this side. You still got the negative here. Just get rid of the negatives."

Interviewer

"When you were actually doing it you looked at the answer key and the answer key had it this way, carrying the negative over to the other side, so you left it according to the answer key. Why didn't you change it to the way you wanted?"

Student #7

"I don't know. Because I was probably going to ask about it sometime later, but never got around to it. But, that would be what I would go with, myself, right, get rid of all of the negative signs. I figure to get rid of all negatives, make it positive."

Appendix C - Course Descriptions

MATHEMATICS 1510

MATHEMATICS FOR COMPUTER STUDIES I

TIME ALLOCATION: Semester Length: 15 weeks @4 hours/week

TEXT: *Mathematics for Programming Computers*, F. J. Clarke.

CREDIT VALUE: FOUR (4)

PREREQUISITES: None

DESCRIPTION: This course involves the study of mathematical topics which are applicable to business computer studies.

COURSE OBJECTIVES:

1.0 Review related algebraic concepts

2.0 Use the decimal, binary, octal, and hexadecimal numeration systems.

3.0 Perform basic arithmetic operations in the four numeration systems.

4.0 Apply Boolean Algebra.

5.0 Use Determinants and Matrices.

MATHEMATICS 1000

ESSENTIAL MATHEMATICS

TYPES AND PURPOSE:	This is a course in basic mathematics designed to help alleviate specific weaknesses in students' mathematical skills. This course is a non-credit prerequisite for Mathematics 1100 for those students identified by the placement testing procedure.
CALENDAR ENTRY.	Operations with Whole Numbers and Fractions; Operations with Decimals and Percents; Operations with Integers and Exponents; Linear Equations, Operations with Algebraic Expressions; Operations with Fractional Expressions, Solving Formulas; Graphing, Systems of Linear Equations; Basic Geometry and Trigonometry
PREREQUISITES:	None
SCHEDULE:	Duration: 13 weeks Class Hours: 5 hours/week = 65 hours total
TEXT:	Zimmer, R. A., *Essential Mathematics* (Kendal/Hunt Publishing Co.)
COURSE AIMS:	1) To provide an opportunity for students to eliminate mathematical deficiencies as identified by the placement testing procedure. 2) To strengthen the student's mathematical skills in order to enhance the probability of success in his/her chosen technology program.
MAJOR TOPICS:	1.0 Operations with Whole Numbers and Fractions 2.0 Operations with Decimals and Percents 3.0 Operations with Integers and Exponents 4.0 Linear Equations, Operations with Algebraic Expressions 5.0 Operations Involving Fractional Expressions, Solving Formulas 6.0 Graphing, Systems of Linear Equations 7.0 Basic Geometry and Trigonometry