RELATIVE AND EQUIVARIANT COINCIDENCE THEORY

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by

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Abstract

In this thesis, we develop relative coincidence theory on the complement and equivariant coincidence theory. For two maps f and g from one pair of manifolds (X, A) to another (Y, B), a Nielsen number $N(f, g; X - A)$ is introduced which serves as a homotopy invariant lower bound for the number of coincidence points of f and g on $X - A$. We provide a method for computing the Nielsen numbers $N(f, g)$ and $N(f, g; X - A)$ when g_π is onto and $f_\pi(\pi_1(X)) \subset J(f)$. These results are also generalized to manifolds with boundary.

To estimate the number of coincidence points for equivariant maps, some Nielsen type invariants are developed. These invariants are introduced for the general cases first, and then explored further for the special case, when the fixed point set of the action is nonempty. A method is provided to compute these numbers and give an estimate of the number of coincidence points of a pair of equivariant maps. Finally, minimality is discussed for both relative and equivariant cases and we prove in some cases that these numbers are attainable within the appropriate homotopy classes.

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Contents

Introduction

Let X, Y be manifolds with the same dimension and $f, g : X \to Y$ maps. A point x in X is called a coincidence point of f and g if $f(x) = g(x)$. The goal of coincidence theory is to find a reasonable lower bound for the minimum number of coincidence points within the homotopy classes of a given pair of maps.

Nielsen fixed point theory can be thought of as a special case of coincidence theory though, of course, Nielsen fixed point theory came first. It is the case where $X = Y$ and g is identity. Nielsen theory was developed by Nielsen in the 1920's. In the past two decades, it has experienced a rapid new development. The famous Lefschetz fixed point theorem allows us to deduce that if the Lefschetz number of a map is non-zero, then the map has at least one fixed point. Nielsen fixed point theory goes further. It ensures not only the existence of fixed points but also gives a reasonable estimate of the number of fixed points within the homotopy class of the map. The Nielsen number $N(f)$ of a selfmap $f : X \to X$ of a compact connected ANR X, gives a lower bound for this number. However, for a long time, the Nielsen number could be computed only for two special cases, namely when X is simply connected, or when f is the identity. In 1962, Jiang in [JB1] gave for the first time a method to compute the Nielsen number in some nontrivial cases. A subgroup of the fundamental group of the space was introduced, called Jiang subgroup. It was proved that if the Jiang subgroup of the map f is equal to the fundamental group of the space X, then $N(f)$ is computable.

Though, under mild conditions, the number $N(f)$ is a sharp lower bound for the number of fixed points of f, this is not true in general. In particular it is known that for homeomorphisms with boundary the ordinary Nielsen number may be a poor lower bound. A relative Nielsen number for a selfmap of a pair of spaces (X, A) was introduced in [SH2]. This proved to be a better lower bound than the ordinary Nielsen number when one considered maps of pairs and in particular maps of manifolds with boundary. In [Z], Zhao considered the number of fixed points on the complement $X - A$ for a selfmap of a pair of spaces (X, A). Zhao's results provide the necessary background, for the introduction in [WP3], of invariants for an equivariant version of Nielsen theory.

Most concepts and results in fixed point theory can be generalized to coincidence theory. Unlike Nielsen fixed point theory, coincidence theory involves two spaces and two maps. Therefore the index of a coincidence point set is more difficult to define for arbitrary spaces. For this reason, most of the work in coincidence theory is on manifolds.

The index of an isolated coincidence point and the Nielsen number $N(f, g)$ of a pair of maps (f, g) was first introduced in [SH1]. If the dimension is greater than 2, the Nielsen number is a sharp lower bound of the minimal number of coincidence points within the homotopy classes of (f, g). The development of the theory was continued in [BR1]; a Reidemeister number for a pair of maps, which is relatively easier to compute in some cases was introduced, and the relationship between the Reidemeister number and the Nielsen number was established. Recently relative coincidence Nielsen numbers were introduced [JJ] and [JL]. Our work, which in may ways generalizes Zhao's work, bears the same relationship to relative coincidence theory as Zhao's does to relative fixed point theory.

Fixed point theory has been generalized in another direction namely to equivariant fixed point theory. The idea is to restrict attention to 'G-spaces', 'G-maps' and 'G-homotopies' for some fixed group G. As with any restricted Nielsen theory we are often able to detect more fixed points within the G-homotopy classes of a G-map. The Equivariant coincidence theory that we study here is a generalization of equivariant fixed point theory. A recent paper [FP] has made some progress in this direction. We will however consider general equivariant maps instead of the highly restrictive category of G-compactly coincident maps as in [FP]. We will discuss this further in Chapter 4.

The thesis is arranged as follows. In chapter 1, we present known results and techniques which we use later. In chapter 2, we introduce the Nielsen number on the complement in order to estimate the number of coincidence points on the complement. We also give a new method for computing the Nielsen number in some special cases. In chapter 3, we generalize a result of Brooks, which says that coincidence points can be coalesced, or removed by deforming only one of the maps involved. Our generalization is a relative version of Brooks' result. In addition, a local version of Brooks' theorem is proved. These results make the relative coincidence theory and the equivariant coincidence theory, which we develop, include the corresponding fixed point theories as a special cases. In chapter 4, we introduce equivariant coincidence theory. We first give several Nielsen type invariants, which are related to the isotropy subgroups of the action group. Finally we discuss the computation of the invariants and minimality.

Chapter 1

Preliminaries

In this chapter, we will introduce the basic concepts and results in coincidence theory, which can be found in [BR1], [JJ] and [SH1] etc.. In section 1, we define Reidemeister and coincidence classes for a pair of maps $(f, g) : X \to Y$. We use the universal covering space approach. In section 2, we describe Reidemeister classes using the fundamental group approach. We also prove that the two approaches are equivalent. In section 3, we introduce the concept of an index of a coincidence class when both the domain and the range of the maps are manifolds with the same dimension. In section 4, the computation of the coincidence Nielsen number is discussed. In section 5, relative coincidence theory is introduced. Finally, in section 6, we give the minimal theorem.

1.1 Reidemeister and coincidence classes

Let X and Y be connected topological spaces, and $f : X \to Y$ and $g : X \to Y$ be maps. We will use (f, g) to represent the pair of maps f and g. A coincidence point of (f, g) is a point $x \in X$ such that $f(x) = g(x)$. The set of all coincidence points of (f, g) is denoted by $\Gamma(f, g)$.

Let $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$ be the projections from the universal covering spaces of X and Y respectively. A lifting $\tilde{f}$ of a map $f : X \to Y$ is a map $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $f \circ p_X = p_Y \circ \tilde{f}$. A pair of maps $(\tilde{f}, \tilde{g})$ is called a lifting of (f, g) if $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a lifting of f and $\tilde{g} : \tilde{X} \to \tilde{Y}$ a lifting of g. Let $\Pi(X)$ denote the group of the covering transformations of $p_X : \tilde{X} \to X$, and $\Pi(Y)$ the group of covering transformations of $p_Y : \tilde{Y} \to Y$.

Definition 1.1.1 Two liftings $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ of (f, g) are said to be conjugate if there are elements $\tilde{\gamma}^X \in \Pi(X)$ and $\tilde{\gamma}^Y \in \Pi(Y)$ such that

$$(\tilde{f}', \tilde{g}') = (\tilde{\gamma}^Y \circ \tilde{f} \circ (\tilde{\gamma}^X)^{-1}, \tilde{\gamma}^Y \circ \tilde{g} \circ (\tilde{\gamma}^X)^{-1}).$$

For simplicity we denote $(\tilde{\gamma}^Y \circ \tilde{f} \circ (\tilde{\gamma}^X)^{-1}, \tilde{\gamma}^Y \circ \tilde{g} \circ (\tilde{\gamma}^X)^{-1})$ by $\tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1}$.

It is easy to see that conjugacy is an equivalent relation. A conjugacy class is called a Reidemeister class of (f, g). We denote the class containing $(\tilde{f}, \tilde{g})$ by $[(\tilde{f}, \tilde{g})]$. Note that $[(\tilde{f}, \tilde{g})] = \{\tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1} | \tilde{\gamma}^X \in \Pi(X), \tilde{\gamma}^Y \in \Pi(Y)\}$. The set of all Reidemeister classes is called the Reidemeister set of (f, g) and is denoted by $\mathcal{R}_{f,g}$. The number of Reidemeister classes is called the Reidemeister number of (f, g) and is denoted by $R(f, g)$.

Proposition 1.1.2 *Let (f, g) be a pair of maps from X to Y, we have*

(i) $\Gamma(f, g) = \bigcup_{(\tilde{f},\tilde{g})} p_X\Gamma(\tilde{f}, \tilde{g})$.

(ii) $p_X\Gamma(\tilde{f}, \tilde{g}) = p_X\Gamma(\tilde{f}', \tilde{g}')$, *if* $[(\tilde{f}, \tilde{g})] = [(\tilde{f}', \tilde{g}')]$.

(iii) $p_X\Gamma(\tilde{f}, \tilde{g}) \cap p_X\Gamma(\tilde{f}', \tilde{g}') = \emptyset$, *if* $[(\tilde{f}, \tilde{g})] \neq [(\tilde{f}', \tilde{g}')]$.

Proof: (i) Assume $x_0 \in \Gamma(f, g)$ and $y_0 = f(x_0) = g(x_0)$, choose $\tilde{x}_0 \in p_X^{-1}(x_0)$ and choose liftings $\tilde{f}$ of f and $\tilde{g}$ of g. Then we have $\tilde{f}(\tilde{x}_0), \tilde{g}(\tilde{x}_0) \in p_Y^{-1}(y_0)$ and there is an element $\alpha \in \pi(Y)$ such that $\tilde{f}(\tilde{x}_0) = \alpha \circ \tilde{g}(\tilde{x}_0)$. Hence $x_0 \in p_X\Gamma(\tilde{f}, \alpha \circ \tilde{g})$. Since $\alpha \circ \tilde{g}$ is also a lifting of g, $x_0 \in \bigcup_{(\tilde{f},\tilde{g})} p_X\Gamma(\tilde{f}, \tilde{g})$. This shows $\Gamma(f, g) \subset \bigcup_{(\tilde{f},\tilde{g})} p_X\Gamma(\tilde{f}, \tilde{g})$. $\bigcup_{(\tilde{f},\tilde{g})} p_X\Gamma(\tilde{f}, \tilde{g}) \subset \Gamma(f, g)$ is trivial.

(ii) Assume $(\tilde{f}', \tilde{g}') = \tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1}$, and $x_0 \in p_X\Gamma(\tilde{f}, \tilde{g})$, i.e. there is $\tilde{x}_0 \in (p_X)^{-1}(x_0)$ with $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. Then we have $\tilde{f}'(\tilde{\gamma}^X(\tilde{x}_0)) = \tilde{\gamma}^Y \circ \tilde{f} \circ (\tilde{\gamma}^X)^{-1}(\tilde{\gamma}^X(\tilde{x}_0)) = \tilde{\gamma}^Y \circ \tilde{f}(\tilde{x}_0) = \tilde{\gamma}^Y \circ \tilde{g}(\tilde{x}_0)) = \tilde{\gamma}^Y \circ \tilde{g} \circ (\tilde{\gamma}^X)^{-1}(\tilde{\gamma}^X(\tilde{x}_0)) = \tilde{g}'(\tilde{\gamma}^X(\tilde{x}_0))$, i.e. $x_0 = p_X(\gamma^X(\tilde{x}_0)) \in p_X\Gamma(\tilde{f}', \tilde{g}')$.

(iii) If $x_0 \in p_X\Gamma(\tilde{f}, \tilde{g}) \cap p_X\Gamma(\tilde{f}', \tilde{g}')$, there are $\tilde{x}_0, \tilde{x}_0' \in (p_X)^{-1}(x_0)$ such that $\tilde{x}_0 \in \Gamma(\tilde{f}, \tilde{g})$ and $\tilde{x}_0' \in \Gamma(\tilde{f}', \tilde{g}')$. Suppose $\tilde{x}_0' = \tilde{\gamma}^X(\tilde{x}_0)$. Since $p_Y(\tilde{f}(\tilde{x}_0)) = p_Y(\tilde{f}'(\tilde{x}_0'))$, There is $\tilde{\gamma}^Y \in \Pi(Y)$ such that $\tilde{f}'(\tilde{x}_0')) = \tilde{\gamma}^Y \tilde{f}(\tilde{x}_0))$. Thus $\tilde{g}'(\tilde{x}_0') = \tilde{\gamma}^Y \circ \tilde{g}(\tilde{x}_0')$ as well so that by Theorem 6.1 in [GH], which says that maps with equal projections that agree on a single point are identical, we have that $(\tilde{f}', \tilde{g}') = \tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1}$ as needed. □

Definition 1.1.3 The subset $p_X\Gamma(\tilde{f}, \tilde{g})$ of $\Gamma(f, g)$ is called the coincidence class of (f, g) determined by the Reidemeister class $[(\tilde{f}, \tilde{g})]$.

Proposition 1.1.4 $\Gamma(f, g)$ *splits into a disjoint union of coincidence classes.* □

The set of nonempty coincidence classes will be denoted by $\tilde{\Gamma}(f,g)$. We have an injective map $\rho_{\mathcal{R}_{f,g}}$ from $\tilde{\Gamma}(f,g)$ to $\mathcal{R}_{f,g}$, which sends a coincidence class S to the Reidemeister class $[(\tilde{f},\tilde{g})]$ if $S = p_X\Gamma(\tilde{f},\tilde{g})$.

Proposition 1.1.5 *Two coincidence points x_0, x_1 are in the same coincidence class if and only if there is a path α from x_0 to x_1 such that $g \circ \alpha$ and $f \circ \alpha$ are homotopic relative to endpoints, which will be denoted by $g \circ \alpha \sim f \circ \alpha$.*

Proof: $\Rightarrow$): Assume that x_0, x_1 are in the same coincidence class. Then there is a lifting $(\tilde{f},\tilde{g})$ of (f,g) such that $x_0, x_1 \in p_X\Gamma(\tilde{f},\tilde{g})$. Equivalently there are points $\tilde{x}_0 \in (p_X)^{-1}(x_0) \cap \Gamma(\tilde{f},\tilde{g})$, $\tilde{x}_1 \in (p_X)^{-1}(x_1) \cap \Gamma(\tilde{f},\tilde{g})$. Let $\tilde{\alpha}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$, then $\tilde{f} \circ \tilde{\alpha}$ and $\tilde{g} \circ \tilde{\alpha}$ have the same beginning and end points and hence their projections $p_Y \circ (\tilde{g} \circ \tilde{\alpha})$ and $p_Y \circ (\tilde{f} \circ \tilde{\alpha})$ homotopic. Since $p_Y \circ \tilde{g} = g \circ p_Y$ and $p_Y \circ \tilde{f} = f \circ p_Y$, this means that with $\alpha = p_X \circ \tilde{\alpha}$ then $f \circ \alpha$ and $g \circ \alpha$ are homotopic as needed.

$\Leftarrow$): Let α be a path from x_0 to x_1 with the property $g \circ \alpha \sim f \circ \alpha$. Let $\tilde{x}_0 \in (p_X)^{-1}(x_0)$ be a coincidence point of $(\tilde{f},\tilde{g})$. Let $\tilde{\alpha}$ be a lifting of α starting at $\tilde{x}_0$, $\tilde{\alpha}(1) \in (p_X)^{-1}(x_1)$, then $\tilde{f} \circ \tilde{\alpha}$ and $\tilde{g} \circ \tilde{\alpha}$ are liftings of $f \circ \alpha$ and $g \circ \alpha$ respectively. Since $g \circ \alpha \sim f \circ \alpha$, and $(\tilde{g} \circ \tilde{\alpha})(0) = \tilde{x}_0 = (\tilde{f} \circ \tilde{\alpha})(0)$, we have $\tilde{g}(\tilde{\alpha}(1)) = \tilde{f}(\tilde{\alpha}(1))$, but $p_X(\tilde{\alpha}(1)) = \alpha(1) = x_1$. So x_0 and x_1 are in the same class. $\square$

Note 1.1.6 Proposition 1.1.5 is actually Brooks' definition of coincidence class [BR1]. Thus our definition is equivalent to Brooks' when the class is nonempty.

Proposition 1.1.7 *If X is locally path-connected and Y is semilocally simply-connected, then each coincidence class is open in $\Gamma(f,g)$.*

Proof: Let x_0 be a coincidence point of (f, g). We want to find a neighbourhood U of x_0 such that any coincidence point $x_1 \in U$ is in the same class as x_0.

Let V be a neighbourhood of $f(x_0) = g(x_0)$ such that every loop in V at $f(x_0)$ is trivial in Y. Let U be a path-connected neighbourhood of x_0 such that $U \subset f^{-1}(V) \cap g^{-1}(V)$. Let $x_1 \in U$ be a coincidence point of (f, g), and let c be a path in U from x_0 to x_1. We have $f \circ c \sim g \circ c$ since $f \circ c$ and $g \circ c$ are both in V and have the same end points. Thus x_0 and x_1 are in the same class. □

1.2 An alternative description of Reidemeister classes

In this section, we will redefine the Reidemeister number using the fundamental group, which is used in [BR1], and then we prove that this definition is equivalent to the one defined in the last section.

Let $x_0 \in X$ and $y_0 \in Y$ be given, and ω_f and ω_g be paths from y_0 to $f(x_0)$ and $g(x_0)$ respectively. Define homomorphisms $f_\pi^{\omega_f} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_\pi^{\omega_g} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_\pi^{\omega_f}(\alpha) = \omega_f \cdot (f \circ \alpha) \cdot (\omega_f)^{-1}$ and $g_\pi^{\omega_g}(\alpha) = \omega_g \cdot (g \circ \alpha) \cdot (\omega_g)^{-1}$ respectively, by confusing a path and a class in fundamental groups. When $y_0 = f(x_0)$ and ω_f is a constant path at y_0, we use f_π to denote $f_\pi^{\omega_f}$.

Definition 1.2.1 Two elements $\alpha_1, \alpha_2 \in \pi_1(Y, y_0)$ are said to be $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruent if there is an element $\gamma \in \pi_1(X, x_0)$ such that $\alpha_2 = g_\pi^{\omega_g}(\gamma)\alpha_1(f_\pi^{\omega_f}(\gamma))^{-1}$. We denote this relation by $\alpha_1 \equiv \alpha_2 \bmod f_\pi^{\omega_f}, g_\pi^{\omega_g}$, or more briefly by, $\alpha_1 \equiv \alpha_2$.

Proposition 1.2.2 *$f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence is an equivalence relation.*

Proof: See Proposition 2 on p.28 in [BR1]. □

Definition 1.2.3 The set of all $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence classes is denoted by $\nabla(f,g;x_0,y_0,\omega_f,\omega_g)$. We will use $\nabla(f,g;x_0)$ to denote$\nabla(f,g;x_0,y_0,\omega_f,\omega_g)$ when $y_0 = f(x_0) = g(x_0)$, and ω_f and ω_g are constant paths at y_0. An element in $\nabla(f,g;x_0,y_0,\omega_f,\omega_g)$ containing $\alpha \in \pi_1(Y,y_0)$ will be denoted by $\overline{\alpha}$.

Definition 1.2.4 Let $(\tilde{f},\tilde{g})$ be a lifting of (f,g) and $\tilde{x}_0$ a point on $(p_X)^{-1}(x_0)$. Let $\tilde{\alpha}$ a path in $\tilde{Y}$ from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$. Define $\Theta_{f,g} : \mathcal{R}_{f,g} \to \nabla(f,g;x_0,y_0,\omega_f,\omega_g)$ by $\Theta_{f,g}([(\tilde{f},\tilde{g})]) = \overline{[\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}]}$.

Proposition 1.2.5 shows that the definition of $\Theta_{f,g}([(\tilde{f},\tilde{g})])$ does not depends on the choice of $(\tilde{f},\tilde{g})$, $\tilde{x}_0$ and $\tilde{\alpha}$.

Proposition 1.2.5 *$\Theta_{f,g}$ is well defined and is a bijection.*

Proof: It is obvious that $\Theta_{f,g}([(\tilde{f},\tilde{g})])$ does not depend on the choice of $\tilde{\alpha}$. We have to prove that it does not depend on the choices of $\tilde{x}_0$ and $(\tilde{f},\tilde{g})$.

Let $\tilde{x}_0' \in (p_X)^{-1}(x_0)$, and $\tilde{\eta}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0'$, then $[p_X \circ \tilde{\eta}] = [\eta]$ is an element in $\pi_1(X,x_0)$. Let $\tilde{\alpha}'$ be a path from $\tilde{g}(\tilde{x}_0')$ to $\tilde{f}(\tilde{x}_0')$, then $\tilde{\alpha} \sim (\tilde{g} \circ \tilde{\eta}) \cdot \tilde{\alpha}' \cdot (\tilde{f} \circ \tilde{\eta}^{-1})$. So we have $[\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}] = [\omega_g \cdot (p_Y \circ ((\tilde{g} \circ \tilde{\eta}) \cdot \tilde{\alpha}' \cdot (\tilde{f} \circ \tilde{\eta}^{-1}))) \cdot \omega_f^{-1}] = [\omega_g \cdot (p_Y \circ (\tilde{g} \circ \tilde{\eta})) \cdot \omega_g^{-1} \cdot \omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1} \cdot \omega_f \cdot (p_Y(\tilde{f} \circ \tilde{\eta}^{-1})) \cdot \omega_f^{-1}] = [\omega_g \cdot (p_Y \circ (\tilde{g} \circ \tilde{\eta})) \cdot \omega_g^{-1}][\omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1}][\omega_f \cdot (p_Y \circ (\tilde{f} \circ \tilde{\eta}^{-1})) \cdot \omega_f^{-1}]$ $= g_\pi^{\omega_g}(\eta)[\omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1}] f_\pi^{\omega_f}(\eta^{-1})$. This shows that $[\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}]$ and $[\omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1}]$ are in the same $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence class.

Suppose $(\tilde{f}', \tilde{g}')$ is another lifting of (f, g) in the class $[(\tilde{f}, \tilde{g})]$, i.e. there are $\gamma^X \in \Pi(X)$ and $\gamma^Y \in \Pi(Y)$ such that $(\tilde{f}', \tilde{g}') = \gamma^Y(\tilde{f}, \tilde{g})(\gamma^X)^{-1}$. Let $\tilde{x}_0' = \gamma^X(\tilde{x}_0)$, then $\gamma^Y \circ \tilde{\alpha}$ is a path from $\tilde{g}'(\tilde{x}_0')$ to $\tilde{f}'(\tilde{x}_0')$, and it is obvious that $\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1} \sim \omega_g \cdot (p_Y \circ \gamma^Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}$. So, $\Theta_{f,g}$ is well defined.

We show next that $\Theta_{f,g}$ is injective: Suppose $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ are liftings of (f, g) such that $\Theta_{f,g}([(\tilde{f}, \tilde{g})]) = \Theta_{f,g}([(\tilde{f}', \tilde{g}')])$. Let $\tilde{x}_0 \in p_X^{-1}(x_0), \tilde{\alpha}$ be a path from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$, and $\tilde{\alpha}'$ a path from $\tilde{g}'(\tilde{x}_0)$ to $\tilde{f}'(\tilde{x}_0)$. Then we have $\overline{[\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}]} = \overline{[\omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1}]}$, i.e. there is a $[\beta] \in \pi_1(X, x_0)$ such that $g_\pi^{\omega_g}([\beta])[\omega_g \cdot (p_Y \circ \tilde{\alpha}') \cdot \omega_f^{-1}] f_\pi^{\omega_f}([\beta]^{-1}) = [\omega_g \cdot (p_Y \circ \tilde{\alpha}) \cdot \omega_f^{-1}]$. This implies $(g \circ \beta) \cdot (p_Y \circ \tilde{\alpha}') \cdot (f \circ \beta^{-1}) \sim p_Y \circ \tilde{\alpha}$. Let $\tilde{\beta}$ be a lift of β ending at $\tilde{x}_0$. Since β is a loop, $\tilde{\beta}(0) \in p_X^{-1}(x_0)$ and $\tilde{g}' \circ \tilde{\beta}(0) \in p_Y^{-1}(g(x_0))$. So there is a $\gamma^Y \in \Pi(Y)$ such that $\gamma^Y \circ \tilde{g}' \circ \tilde{\beta}(0) = \tilde{g}(\tilde{x}_0)$. Now $\tilde{\alpha}'' = \gamma^Y \circ ((\tilde{g}' \circ \tilde{\beta}) \cdot \tilde{\alpha}' \cdot (\tilde{f}' \circ \tilde{\beta}^{-1})$ is a lifting of $(g \circ \beta) \cdot (p_Y \circ \tilde{\alpha}') \cdot (f \circ \beta^{-1})$, which is homotopic to $p_Y \circ \tilde{\alpha}$, and $\tilde{\alpha}''(0) = \gamma^Y \circ \tilde{g}' \circ \tilde{\beta}(0) = \tilde{g}(\tilde{x}_0) = \tilde{\alpha}(0)$. This implies $\tilde{\alpha}''(1) = \tilde{\alpha}(1) = \tilde{f}(\tilde{x}_0)$. However, $\tilde{\alpha}''(1) = \gamma^Y \circ \tilde{f}' \circ \tilde{\beta}^{-1}(1) = \gamma^Y \circ \tilde{f}' \circ \tilde{\beta}(0)$. So we have $\gamma^Y \circ \tilde{f}' \circ \tilde{\beta}(0) = \tilde{f}(\tilde{x}_0)$ as well. Let $\gamma^X \in \Pi(X)$ such that $\gamma^X(\tilde{\beta}(0)) = \tilde{x}_0$. Then we have $\gamma^Y \circ \tilde{g}' \circ (\gamma^X)^{-1}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. By the uniqueness of liftings, we have $\gamma^Y \circ \tilde{g}' \circ (\gamma^X)^{-1} = \tilde{g}$. Similarly, we have $\gamma^Y \circ \tilde{f}' \circ (\gamma^X)^{-1} = \tilde{f}$. So $[(\tilde{f}, \tilde{g})] = [(\tilde{f}', \tilde{g}')]$ and $\Theta_{f,g}$ is injective.

Finally, we show that $\Theta_{f,g}$ is surjective: Assume that $[\alpha]$ is an element in $\pi_1(Y, y_0)$, and let $\tilde{\alpha}$ be a lifting of α. Let $\tilde{\omega}_g$ be a lifting of ω_g starting at $\tilde{\alpha}(0)$, and $\tilde{\omega}_f$ be a lifting of ω_f starting at $\tilde{\alpha}(1)$. Then $\tilde{\omega}_g(1) \in p_Y^{-1}(g(x_0))$ and $\tilde{\omega}_f(1) \in p_Y^{-1}(f(x_0))$. By Theorem 6.1 in [GH], there are lifting $\tilde{g}$ and $\tilde{f}$ of g and f respectively, such that $\tilde{g}(\tilde{x}_0) = \tilde{\omega}_g(1)$ and $\tilde{f}(\tilde{x}_0) = \tilde{\omega}_f(1)$. It is easy to see that $\tilde{\omega}_g^{-1} \cdot \tilde{\alpha} \cdot \tilde{\omega}_f$ is a path from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$ and $\omega_g \cdot p_Y \circ (\tilde{\omega}_g^{-1} \cdot \tilde{\alpha} \cdot \tilde{\omega}_f) \cdot \omega_f^{-1} \sim \alpha$.

Therefore, we have $\Theta_{f,g}([(\tilde{f},\tilde{g})]) = \overline{[\alpha]}$, i.e. $\Theta_{f,g}$ is surjective. Hence, $\Theta_{f,g}$ is bijective. $\square$

Define $\rho_{\nabla_{f,g}} : \tilde{\Gamma}(f,g) \to \nabla(f,g;x_0,y_0,\omega_f,\omega_g)$ as follows. Let x be a coincidence point, and $C : I \to X$ be a path from x_0 to x, then $\omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}$ is a loop at y_0. Define $\rho_{\nabla_{f,g}}([x]) = \overline{[\omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}]}$. The definition of $\rho_{\nabla_{f,g}}$ is independent of the choices of C and x. Let C' be another path from x_0 to x and $\beta = C' \cdot C^{-1}$, then β is a loop at x_0 and we have $\omega_g \cdot g \circ C' \cdot (f \circ C')^{-1} \cdot \omega_f^{-1} \sim \omega_g \cdot g \circ C' \cdot g \circ C^{-1} \cdot g \circ C \cdot (f \circ C)^{-1} \cdot f \circ C \cdot (f \circ C')^{-1} \cdot \omega_f^{-1}$ $\sim \omega_g \cdot g \circ (C' \cdot C^{-1}) \cdot g \circ C \cdot (f \circ C)^{-1} \cdot f \circ (C \cdot (C')^{-1}) \cdot \omega_f^{-1} \sim \omega_g \cdot g \circ \beta \cdot g \circ C \cdot (f \circ C)^{-1} \cdot f \circ \beta^{-1} \cdot \omega_f^{-1}$ $\sim (\omega_g \cdot g \circ \beta \cdot \omega_g^{-1}) \cdot (\omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}) \cdot (\omega_f \cdot f \circ \beta^{-1} \cdot \omega_f^{-1})$. In other words, $[\omega_g \cdot g \circ C' \cdot (f \circ C')^{-1} \cdot \omega_f^{-1}] = g_\pi^{\omega_g}([\beta])[\omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}] f_\pi^{\omega_f}([\beta]^{-1})$, so $[\omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}]$ and $[\omega_g \cdot g \circ C' \cdot (f \circ C')^{-1} \cdot \omega_f^{-1}]$ are in the same class. Let x' be another coincidence point of (f,g) in the same class as x and and $\alpha : I \to X$ a path from x to x' such that $f \circ \alpha \sim g \circ \alpha$. Let $C_1 = C \cdot \alpha$, then C_1 is a path from x_0 to x' and we have $\omega_g \cdot g \circ C_1 \cdot (f \circ C_1)^{-1} \cdot \omega_f^{-1}$ $\sim \omega_g \cdot g \circ C \cdot g \circ \alpha \cdot (f \circ \alpha)^{-1} \cdot (f \circ C)^{-1} \cdot \omega_f^{-1} \sim \omega_g \cdot g \circ C \cdot (f \circ C)^{-1} \cdot \omega_f^{-1}$. The last $\sim$ holds because $g \circ \alpha \sim f \circ \alpha$. This means $\rho_{\nabla_{f,g}}([x]) = \rho_{\nabla_{f,g}}([x_1])$.

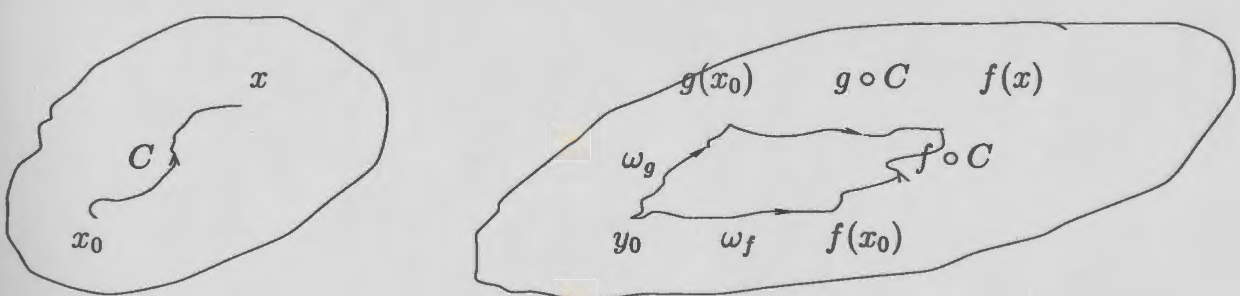

Figure 1.1:

Lemma 1.2.6 *The diagram*

$$\begin{tikzcd}
 & & \mathcal{R}_{f,g} \arrow[d, "\Theta_{f,g}"] \\
\tilde{\Gamma}(f,g) \arrow[urr, "\rho_{\mathcal{R}_{f,g}}"] \arrow[rr, "\rho_{\nabla_{f,g}}"] & & \nabla(f,g;x_0,y_0,\omega_f,\omega_g)
\end{tikzcd}$$

is commutative.

Proof: Let x be a coincidence point. Assume $(\tilde{f},\tilde{g})$ is a lifting of (f,g) such that $x \in p_X(\Gamma(\tilde{f},\tilde{g}))$. Then $\rho_{\mathcal{R}_{f,g}}([x]) = [(\tilde{f},\tilde{g})]$. Let $\tilde{x} \in p_X^{-1}(x)$ be a coincidence point of $(\tilde{f},\tilde{g})$ and $\tilde{x}_0 \in p_X^{-1}(x_0)$. Let $\tilde{\alpha}$ be a path from $\tilde{x}_0$ to $\tilde{x}$, then $\tilde{g}(\tilde{\alpha})\tilde{f}(\tilde{\alpha}^{-1})$ is a path from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$, so $\Theta_{f,g}([(\tilde{f},\tilde{g})]) = \overline{[\omega_g \cdot p_Y(\tilde{g}(\tilde{\alpha})) \cdot p_Y(\tilde{f}(\tilde{\alpha}^{-1})) \cdot \omega_f^{-1}]}$. Since $p_X(\tilde{\alpha})$ is a path from x_0 to x, $\rho_{\nabla_{f,g}}([x]) = [\omega_g \cdot g(p_X(\tilde{\alpha}))\cdot f(p_X(\tilde{\alpha}^{-1}))\cdot\omega_f^{-1}]$, which is equal to $[\omega_g \cdot p_Y(\tilde{g}(\tilde{\alpha}))\cdot p_Y(\tilde{f}(\tilde{\alpha}^{-1}))\cdot\omega_f^{-1}]=$ $\Theta_{f,g}([(\tilde{f},\tilde{g})])$. $\square$

1.3 The index of a coincidence class and the Nielsen number

In this section, we assume that X, Y are orientable closed manifolds with the same dimension n. Suppose that U is an open set of X, such that $U \cap \Gamma(f,g)$ is compact. Let V be an open set containing $U \cap \Gamma(f,g)$ such that $\overline{V} \subset U$. The inclusion $j : (U, U - V) \to (X, X - V)$ is an excision. Define $(f,g) : U \to Y \times Y$ by $(f,g)(x) = (f(x), g(x))$ and let $\Delta(Y)$ be the diagonal in $Y \times Y$. Consider the composition

$$H_n(X) \xrightarrow{i_*} H_n(X, X - V) \xrightarrow{j_*^{-1}} H_n(U, U - V) \xrightarrow{(f,g)_*} H_n(Y \times Y, Y \times Y - \Delta(Y)) \cong \mathbf{Z}$$

Definition 1.3.1 Let $\mu \in H_n(X)$ be the fundamental class of X. The index of f and g on U is defined to be $I(U; f, g) = < \xi_Y, (f, g)_* j_*^{-1} i_*(\mu) > \in \mathbf{Z}$, where $\xi_Y \in H^n(Y \times Y, Y \times Y - \Delta(Y))$ is the Thom class of Y, and $<, >$ is the Kronecker index. (cf. p.177 in [V])

Lemma 1.3.2 *The index is well defined.*

Proof: See p.177 in [V]. □

If $U = U_1 \cup U_2 \cup ... \cup U_k$ is disjoint union of open sets, and Γ_i denotes the compact set $\Gamma(f, g) \cap U_i$, and f_i and g_i the restrictions of f and g to U_i, then we have

Lemma 1.3.3 *The coincidence index is additive, that is* $I(U; f, g) = \sum_{i=1}^{k} I(U_i; f_i, g_i)$.

Proof: See Lemma 6.1 in [V]. □

Lemma 1.3.4 (Existence of coincidences) *If* $U \cap \Gamma(f, g) = \emptyset$, *then* $I(U; f, g) = 0$. *In other words, if* $I(U; f, g) \neq 0$, *then the pair* (f, g) *has at least one coincidence point in* U.

Proof: See Corollary 6.3 in [V]. □

Lemma 1.3.5 *The coincidence index is a homotopy invariant. In particular, if* f_t *and* $g_t : U \to Y$, $0 \leq t \leq 1$, *are homotopies and* $D = U \cap \bigcup_t \Gamma(f_t, g_t)$ *is compact, then*

$$I(U; f_0, g_0) = I(U; f_1, g_1).$$

Proof: See Lemma 6.4 of [V]. □

Definition 1.3.6 Let N be a compact and open set of $\Gamma(f,g)$. The index of (f,g) on N is defined by

$$I(N;f,g) = I(U;f,g),$$

where U is an open set of X such that $U \cap \Gamma(f,g) = N$.

Lemma 1.3.7 *$I(N;f,g)$ is independent to the choice of U.*

Proof: See Exercise 1 on p.177 in [V]. □

Notation: We often use $\mathrm{ind}(N;f,g)$ to denote $I(N;f,g)$. When f,g are clear, we use $\mathrm{ind}(N)$.

Now assume that $x \in X$ is an isolated coincidence point of (f,g). Let U be a neighborhood of x in X with $\Gamma(f,g) \cap \overline{U} = x$ and V a neighborhood of $y = f(x)$ in Y such that there are orientation-preserving homeomorphisms $h : (\overline{U}, x) \to (D^n, 0)$ and $k : (\overline{V}, y) \to (D^n, 0)$, where D^n is the unit ball of $\mathbf{R}^n$, and $f(\overline{U}), g(\overline{U}) \subset V$. Define $\phi : S^{n-1} \to S^{n-1}$ to be the composition

$$S^{n-1} \xrightarrow{h^{-1}} \partial\overline{U} \xrightarrow{(f,g)} \overline{V} \times \overline{V} - \Delta(\overline{V}) \xrightarrow{k\times k} D^n \times D^n - \Delta(D^n) \xrightarrow{F} D^n - 0 \xrightarrow{\pi} S^{n-1}$$

where $F(x,y) = \frac{1}{2}(y-x)$ and π denotes radial projection.

Lemma 1.3.8 *$I(x;f,g) =$ degree of ϕ.*

Proof: The proof is similar to the proof of Proposition 6.9 of [V]. □

This shows that the index defined here is the same as defined in [SH1] when the coincidence point is an isolated one.

Definition 1.3.9 A coincidence class N is said to be essential if $I(N; f, g) \neq 0$. It is said to be inessential if $I(N; f, g) = 0$.

The number of essential coincidence classes is called the Nielsen number of (f, g), and is denoted by $N(f, g)$.

By the definition of $N(f, g)$, we have that the Nielsen number of (f, g) is a lower bound of the number of coincidence points of (f, g).

Theorem 1.3.10 (Lower Bound) $\#\Gamma(f, g) \geq N(f, g)$. □

In fact, we have $\#\tilde{\Gamma}(f, g) \geq N(f, g)$. Since there is an injective map $\rho_{\mathcal{R}_{f,g}} : \tilde{\Gamma}(f, g) \to \mathcal{R}_{f,g}$, we have

Theorem 1.3.11 $R(f, g) \geq N(f, g)$. □

Definition 1.3.12 Suppose $F : X \times I \to Y$ is a homotopy and $C : I \to X$ is a path in X. Then $< F, C >$ is the path in Y defined by

$$< F, C > (t) = F(C(t), t), \forall t \in I.$$

Definition 1.3.13 Suppose $F, G : U \times I \to Y$ are homotopies of f and g respectively, and that $x_0 \in \Gamma(f, g), x_1 \in \Gamma(F(\cdot, 1), G(\cdot, 1))$. If there is path $\alpha : I \to X$ such that $\alpha(0) = x_0, \alpha(1) = x_1$ and if $< F, \alpha > \sim < G, \alpha >$ relative to $\{0, 1\}$, then x_0, x_1 are said to be F, G-related.

Lemma 1.3.14 *Suppose x_0, x_1 are F, G-related and x_i' are in the same equivalence class as x_i for $i = 0, 1$, then x_0' is F, G-related to x_1'.*

Proof: See Proposition 15 on p.19 in [BR1]. □

By Lemma 1.3.14, the relation F, G-related can be extended to coincidence classes.

Definition 1.3.15 Suppose $F, G : X \times I \to Y$ are homotopies. A coincidence class $\alpha_0 \in \tilde{\Gamma}(F(\cdot, 0), G(\cdot, 0))$ is F, G-related to a coincidence class $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$ if and only if some point $x_0 \in \alpha_0$ is F, G-related to some point $x_1 \in \alpha_1$.

Proposition 1.3.16 *Suppose F and G are homotopies. Each $\alpha_0 \in \tilde{\Gamma}(F(\cdot, 0), G(\cdot, 0))$ is F, G-related to at most one $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$. Each $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$ has at most one $\alpha_0 \in \tilde{\Gamma}(F(\cdot, 0), G(\cdot, 0))$ to which it is F, G-related.*

Proof: See Proposition 20 on p.24 in [BR1]. □

Proposition 1.3.17 *Let $F, G : X \times I \to Y$ be maps. Assume $\alpha_0 \in \tilde{\Gamma}(F(\cdot, 0), G(\cdot, 0))$ is F, G-related to $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$. Then the index of α_0 is equal to that of α_1. In particular, if there is no $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$ to which α_0 is F, G-related to, then the index of α_0 is zero.*

Proof: See Theorem 24 on p.81 in [BR1]. □

Corollary 1.3.18 *Let $F, G : X \times I \to Y$ be maps. If $\alpha_0 \in \tilde{\Gamma}(F(\cdot, 0), G(\cdot, 0))$ is essential, then there is $\alpha_1 \in \tilde{\Gamma}(F(\cdot, 1), G(\cdot, 1))$ such that α_0 is F, G-related to α_1.* □

Theorem 1.3.19 (Homotopy Invariance) *If $f' \sim f$ and $g' \sim g$, then $N(f', g') = N(f, g)$.*

Proof: This follows from definition of the Nielsen number, Proposition 1.3.16 and 1.3.17.

□

Let X, Y be closed connected oriented manifolds. We denote the Poincaré duality isomorphisms by $D_q(X) : H^{n-q}(X) \to H_q(X)$ and $D_q(Y) : H^{n-q}(Y) \to H_q(Y)$. Given two maps $f, g : X \to Y$, define $\theta_q(f, g) : H_q(X) \to H_q(Y)$ to be the composite

$$H_q(X) \xrightarrow{f_*} H_q(Y) \xrightarrow{D_q^{-1}(Y)} H^{n-q}(Y) \xrightarrow{g^*} H^{n-q}(X) \xrightarrow{D_q(X)} H_q(X).$$

The *Lefschetz coincidence number* $L(f, g)$ of f and g is defined by

$$L(f, g) = \sum_{q=0}^{n} (-1)^q tr\theta_q$$

where tr denotes the trace.

Theorem 1.3.20 (LEFSCHETZ COINCIDENCE THEOREM) *Assume X and Y are orientable closed manifolds with the same dimension. The coincidence index of the pair (f, g) on X is equal to the Lefschetz number of (f, g); that is*

$$I(f, g) = L(f, g),$$

and hence we have

$$L(f, g) = \sum_{N} I(N; f, g),$$

where the sum is taken over the collection N of coincidence classes.

Proof: See 6.13 in [V] for the first equality. The second one is from the definition of $I(N; f, g)$, Proposition 1.1.4 , Lemma 1.3.3 and the first equality of the theorem. □

1.4 Computation of the Nielsen number

Definition 1.4.1 Let $f : X \to Y$ be a map. A homotopy $F : X \times I \to Y$ is said to be a loop at f if $F(\cdot, 0) = f = F(\cdot, 1)$.

Definition 1.4.2 Let $f, g : X \to Y$ be maps, $x_0 \in \Gamma(f, g)$ and $y_0 = f(x_0)$. Suppose $F, G : X \times I \to Y$ are loops at f and g respectively. Then $< G, x_0 >< F, x_0 >^{-1}$ is a loop in Y at y_0, and therefore $[< G, x_0 >< F, x_0 >^{-1}] \in \pi_1(Y, y_0)$. The set of all such elements of $\pi_1(Y, y_0)$ for all such loops F and G is denoted by $T(f, g, x_0)$. The set of all $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence classes of $\pi_1(Y, y_0)$ that have representatives in $T(f, g, x_0)$ is denoted by $\tilde{T}(f, g, x_0)$. In other words, $\tilde{T}(f, g, x_0)$ is the image of $T(f, g, x_0)$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ under projection from $\pi_1(Y, y_0)$ to $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$.

Theorem 1.4.3 *If x_0 is in an essential coincidence class, then $N(f, g) \geq \#\tilde{T}(f, g, x_0)$.*

Proof: See Theorem 26 on p.51 in [BR1]. □

Brooks actually proves the following

Theorem 1.4.4 *If x_0 is in an essential coincidence class, then there are at least $\#\tilde{T}(f, g, x_0)$ essential coincidence classes of (f, g) each with index equal to the index of $[x_0]$.*

Proof: Suppose $[\alpha_1] \in \tilde{T}(f, g, x_0)$, where $\alpha_1 \in T(f, g, x_0)$. By Definition 1.4.2, there are loops $F : X \times I \to Y$ at f and $G : X \times I \to Y$ at g such that

$$[< G, x_0 >< F, x_0 >^{-1}] = \alpha_1.$$

Since $[x_0]$ is essential, there is a coincidence point x_1 of (f, g) such that $[x_0]$ is F^{-1}, G^{-1}-related to $[x_1]$ by Corollary 1.3.18. By Lemma 1.3.14, there is a path C from x_0 to x_1 in X such that

$$< F^{-1}, C > \sim < G^{-1}, C > .$$

Then we have

$$\begin{aligned}
\alpha_1 &= [< G, x_0 >< F, x_0 >^{-1}] \\
&= [< G, x_0 >< F^{-1}, C >< F, C^{-1} >< F, x_0 >^{-1}] \\
&= [< G, x_0 >< G^{-1}, C >< F, C^{-1} >< F, x_0 >^{-1}] \\
&= [< g, C >< F, C^{-1} >< F^{-1}, x_0 >] \\
&= [< g, C >< f, C^{-1} >] \\
&= [(g \circ C) \cdot (f \circ C)^{-1}].
\end{aligned}$$

This shows that $[\alpha_1]$ is the image of $[x_1]$ under $\rho_{\nabla_{f,g}}$ by definition. Since $[x_1]$ is F, G-related to x_0, the index of $[x_1]$ is equal to the one of $[x_0]$. Therefore, any element of $\tilde{T}(f, g, x_0)$ is an image of an element in $\tilde{\Gamma}(f, g)$ with the same index as $[x_0]$ under $\rho_{\nabla_{f,g}}$ and we have the result. □

A topological space X is a Jiang space if for any $x_0 \in X$, the set consisting of $[< F, x_0 >]$ is equal to the fundamental group $\pi_1(X, x_0)$, where F is a loop at id_X.

When Y is a Jiang space, $T(f, g, x_0)$ is equal to $\pi_1(Y, y_0)$ and therefore $\tilde{T}(f, g, x_0) = \nabla(f, g, x_0, y_0, \omega_f, \omega_g)$. In addition, $\pi_1(Y, y_0)$ is abelian, so $\pi_1(Y, y_0) \cong H_1(Y)$ and $\# \nabla (f, g, x_0, y_0, \omega_f, \omega_g) = \# Coker(g_* - f_*)$, where $f_*, g_* : H_1(X) \to H_1(Y)$ are the homomorphisms induced by f and g respectively on the first homology groups of X and Y.

Theorem 1.4.5 *Suppose Y is a Jiang space. If $L(f, g) \neq 0$, then $N(f, g) = \# Coker(f_* -$*

g_*); if $L(f,g) = 0$, then $N(f,g) = 0$.

Proof: See Corollary 37 on p.56 in [BR1]. □

1.5 Relative coincidence theory

The results in this section are those of [JJ] and [JL].

Let (X, A), (Y, B) be pairs of manifolds with dim X = dim Y, and dim A = dim B, and let $f, g : (X, A) \to (Y, B)$ be maps. Denote the restrictions of f and g on A by $f|_A$ and $g|_A$ respectively.

Lemma 1.5.1 *Let $\alpha \in \tilde{\Gamma}(f,g)$ and $\alpha_A \in \tilde{\Gamma}(f|_A, g|_A)$, then we have either*

(i) $\alpha_A \subset \alpha$, or

(ii) $\alpha_A \cap \alpha = \emptyset$. □

Definition 1.5.2 An essential coincidence class $\alpha \in \tilde{\Gamma}(f,g)$ is called a common essential coincidence class of (f,g) if there is an essential class $\alpha_A \in \tilde{\Gamma}(f|_A, g|_A)$ such that $\alpha_A \subset \alpha$. The number of common essential classes is denoted by $N(f, g; f|_A, g|_A)$.

Definition 1.5.3 The relative coincidence Nielsen number of a pair of maps (f,g) is defined to be

$$N(f, g; X, A) = N(f,g) + N(f|_A, g|_A) - N(f, g; f|_A, g|_A).$$

Definition 1.5.4 A pair of maps $(f', g') : (X, A) \to (Y, B)$ is homotopic to (f,g) if there are homotopies $F : (X, A) \times I \to (Y, B)$ and $G : (X, A) \times I \to (Y, B)$ such that $F(\cdot, 0) = f, F(\cdot, 1) = f'$ and $G(\cdot, 0) = g, G(\cdot, 1) = g'$.

Theorem 1.5.5 (Lower Bound) *Any pair of maps (f', g') homotopic to $(f, g) : (X, A) \to (Y, B)$ has at least $N(f, g; X, A)$ coincidence points.* □

1.6 Minimality

Lemma 1.6.1 *Assume $g : D^d \to R^n$ and $f : S^{d-1} \to R^n$ are maps with the properties:*

(1) $\#\Gamma(f, g|_{S^{d-1}})$ is finite.

(2) $d(f, g|_{S^{d-1}}) < \epsilon$.

Then there is an extension $\overline{f}$ of f to D^d such that

(1) if $\Gamma(f, g|_{S^{d-1}}) \neq \emptyset$ or $d < n$, then $\Gamma(f, g|_{S^{d-1}}) = \Gamma(\overline{f}, g)$.

(2) if $d = n$ and $\Gamma(f, g|_{S^{d-1}}) = \emptyset$, there is at most one coincidence point of $(\bar{f}, g)$ in $int(D^n)$.

(3) $d(\overline{f}, g) < \epsilon$.

Proof: A proof can be found in [FP]. However we give here a sketch of a different proof.

The proof can be found in [SH1] for the case when $\Gamma(f, g|_{S^{d-1}}) = \emptyset$. So we consider only the case that $\Gamma(f, g|_{S^{d-1}}) \neq \emptyset$.

Case 1: First, assume that $\#\Gamma(f, g|_{S^{d-1}}) = 1$ and that x_0 is the single coincidence point. Then any point x in D^d can be represented as

$$x = tx_0 + (1-t)x', \text{ for some } t \in I \text{ and } x' \in S^{d-1}.$$

Then the desired map is defined by $\overline{f}(x) = g(x) + (1-t)(f(x') - g(x'))$.

Case 2: assume $\#\Gamma(f, g|_{S^{d-1}}) = 2$, and x_0 and x_1 are the coincidence points. We can decompose D^d to be $D^d = D_0^d \cup D_1^d$ such that D_0^d and D_1^d are both d-disks, $D_2^{d-1} = D_0^d \cap D_1^d$ is an $d-1$-disk and $x_0 \in D^d - D_1^d$, $x_1 \in D^d - D_0^d$. Since there is no coincidence point of (f, g) on ∂D_2^{d-1}, there is an extension of f over D_2^{d-1} such that $\Gamma(f, g) \cap D_2^{d-1}$ is empty. Now on ∂D_0^d there is no coincidence point of (f, g), so by Case 1, f can be extended to D_0^d with no coincidence points in $\mathrm{int}(D_0^d)$. Similarly, we can extend f over D_1^d such that there is no coincidence point in $\mathrm{int}(D_1^d)$. Therefore, we have an extension of f such that $\Gamma(f, g) = \{x_0, x_1\}$. We proceed by induction on the number of points in $\Gamma(f, g|_{S^{d-1}})$. □

Lemma 1.6.2 *Let X, Y be closed manifolds of the same dimension, $A \subset X$, $B \subset Y$ submanifolds of the same dimension in X and Y, respectively, and $f, g : (X, A) \to (Y, B)$ maps with $\#\Gamma(f|_A, g|_A)$ finite. Then there is a map $f' : (X, A) \to (Y, B)$ such that $f' \sim f$ rel A and $\Gamma(f', g)$ is a finite set.*

Proof: The proof is similar to the one of Theorem 2 of [SH1], except we use Lemma 1.6.1 when the simplex we consider intersects with A. This lemma can also be thought of a special case of Theorem 4.6.5. □

Lemma 1.6.3 *Let $x_0, x_1 \in X$ be points in a manifold with dimension greater than 2, and α be a path from x_0 to x_1, then there exist an arc α' homotopic to α and a neighbourhood U of α', which is homeomorphic to $\mathbf{R}^n$ via a homeomorphism ϕ such that $\phi(\alpha')$ is a segment.*

Proof: See Lemma 7 in [SH1]. □

The ideas contained in the next lemma come from the proof of Theorem 2.4 in [JJ]. We need to state these ideas explicitly in order to show firstly that all the changes are local, that is that all changes are restricted within a neighborhood of an arc connecting the two given coincidence points. Secondly we need to see that the change to one of the maps is away from one end of the given arc.

Lemma 1.6.4 *Let X, Y be manifolds with dimensions greater then or equal to 3 and let $(f, g) : X \to Y$ be a pair of maps with a finite number of coincidence points. Let $x_0, x_1 \in \Gamma(f, g)$ and α be an arc from x_0 to x_1 such that $f \circ \alpha \sim g \circ \alpha$ and $\alpha((0, 1)) \cap \Gamma(f, g) = \emptyset$. Let U be a neighbourhood of $\alpha((0, 1])$ such that $\overline{U} \cong D^n$ and $x_0 \in \partial\overline{U}$. Then there are f_2 and g_1 such that $f_2 \sim f$ rel $X - U$ and $g_1 \sim g$ rel $X - U'$, and such that $\Gamma(f_2, g_1) = \Gamma(f, g) - \{x_1\}$, where $U' \subset \overline{U'} \subset U$.*

Proof: Let β be an arc in Y from $f(x_0)$ to $f(x_1)$ such that $\beta \sim f \circ \alpha$ rel $\{0, 1\}$ and V a neighbourhood of β homeomorphic to $\mathbf{R}^n$. Since $f(x_0) = g(x_0)$ and $f(x_1) = g(x_1) \in V$, there is an $\epsilon > 0$ such that both $f(\alpha([0, \epsilon]))$, $f(\alpha([1-\epsilon, 1]) \subset V$, and also $g(\alpha([0, \epsilon]))$, $g(\alpha([1-\epsilon, 1]) \subset V$. Let H_1 be a homotopy from $f \circ \alpha_\epsilon$ to a path in V and H_2 be a homotopy from $g \circ \alpha_\epsilon$ to another path in V, where α_ϵ is the path defined by $\alpha_\epsilon(t) = \alpha(t(1 - \epsilon) + (1 - t)\epsilon)$ (i.e. α_ϵ is the part of α from ϵ to $1 - \epsilon$). Since the dimension of Y is greater than or equal to 3, we may assume that H_1 and H_2 have no coincidence points. This is equivalent to saying that the image of the map $(H_1, H_2) : I \times I \to Y \times Y$ does not intersect with the diagonal ΔY of the product $Y \times Y$. Since the codimension of ΔY in $Y \times Y$ is at least 3 and the dimension of $I \times I$ is 2, we may deform (H_1, H_2) to a map whose image does not intersect with the diagonal ΔY.

By Lemma 1.6.3, we can assume that $\alpha([0, 1]) = \{(0, 0, ..., t_n) \in \mathbf{R}^n | -1 \leq t_n \leq 2\} =$

$S_{[-1,2]}$, that U is an open set containing $\alpha((0,1])$, and that f, g are maps from $\mathbf{R}^n$ to Y. We may assume further that $\alpha([0,\epsilon]) = \{(0,0,...,0,t_n) \in \mathbf{R}^n | -1 \le t_n \le 0\} = S_{[-1,0]}$, $\alpha([\epsilon, 1-\epsilon]) = \{(0,0,...,0,t_n) \in \mathbf{R}^n | 0 \le t_n \le 1\} = S_{[0,1]}$ and $\alpha([1-\epsilon,1]) = \{(0,0,...,0,t_n) \in \mathbf{R}^n | 1 \le t_n \le 2\} = S_{[1,2]}$. Since $S_{[0,1]} \subset U$, there is an $\eta > 0$ such that $U_\eta = \{t = (t_1, t_2, ..., t_n) \in \mathbf{R}^n | d(t, S_{[0,1]}) < \eta\}$ satisfies the property that $U_\eta \subset \overline{U_\eta} \subset U$. Since there are no coincidence points of (f,g) on $S_{[0,1]}$, we may also assume that there are no coincidence points of (f,g) on U_η.

Let $U_\eta = U'_\eta \cup U''_\eta \cup U'''_\eta$, where $U'_\eta = \{t = (t_1, t_2, ..., t_n) \in \mathbf{R}^n | \Sigma_{i=1}^n t_i^2 \le \eta, -\eta \le t_n \le 0\}$, $U''_\eta = \{t = (t_1, t_2, ..., t_n) \in \mathbf{R}^n | \Sigma_{i=1}^{n-1} t_i^2 \le \eta, 0 \le t_n \le 1\}$ and $U'''_\eta = \{t = (t_1, t_2, ..., t_n) \in \mathbf{R}^n | \Sigma_{i=1}^{n-1} t_i^2 + (t_n - 1)^2 \le \eta, 1 \le t_n \le 1+\eta\}$. Define $f' : \overline{U'_\eta} \to Y$, $f'' : \overline{U''_\eta} \to Y$ and $f''' : \overline{U'''_\eta} \to Y$ as follows:

$$f'(t) = \begin{cases} f((\frac{2|t|}{\eta} - 1)t) & \text{if } |t| \ge \frac{1}{2}\eta, \\ H_1(0, 1 - \frac{2|t|}{\eta}) & \text{if } |t| \le \frac{1}{2}\eta \end{cases}$$

$$f''(t) = \begin{cases} f((\frac{2|t'|}{\eta} - 1)t', t_n) & \text{if } t = (t', t_n), \text{ and } |t'| \ge \frac{1}{2}\eta, \\ H_1(t_n, 1 - \frac{2|t'|}{\eta}) & \text{if } t = (t', t_n), \text{ and } |t'| \le \frac{1}{2}\eta \end{cases}$$

$$f'''(t) = \begin{cases} f((\frac{2|t-e_n|}{\eta} - 1)(t - e_n) + e_n) & \text{if } |t - e_n| \ge \frac{1}{2}\eta, \\ H_1(1, 1 - \frac{2|t-e_n|}{\eta}) & \text{if } |t - e_n| \le \frac{1}{2}\eta \end{cases}$$

where $e_n = (0, ..., 0, 1) \in \mathbf{R}^n$.

Define $f_1 : U \to Y$ by

$$f_1(t) = \begin{cases} f(x) & \text{if } t \notin U_\eta, \\ f'(x) & \text{if } t \in U'_\eta, \\ f''(x) & \text{if } t \in U''_\eta, \\ f'''(x) & \text{if } t \in U'''_\eta. \end{cases}$$

We claim that $f \sim f_1$ rel $X - U_\eta$, but so as not to interrupt the flow of the proof we postpone the verification of the claim until the very end of the proof.

Next using the same procedure as above, but replacing H_1 by H_2, we obtain $g_1 \sim g$ rel $X - U_\eta$. It can be checked case by case that f_1 and g_1 have no coincidence points on U_η. For example, if $t \in U'_\eta$ and $|t| \geq \frac{1}{2}\eta$, then according the definitions of f_1 and g_1, $f_1(t) = f((\frac{2|t|}{\eta} - 1)t)$ and $g_1(t) = g((\frac{2|t|}{\eta} - 1)t)$. However, $(\frac{2|t|}{\eta} - 1)t$ is in U'_η and there are no coincidence points of (f, g) on U'_η, so we have $f_1(t) \neq g_1(t)$. The other cases are similar. Since f_1 and g_1 differ from f and g respectively only on U_η, we have $\Gamma(f, g) = \Gamma(f_1, g_1)$.

Now both f_1 and g_1 map $\alpha([0, 1])$ into V, so we can find a neighborhood $U_1 \subset U$ of $\alpha((0, 1])$ such that $\overline{U}_1 \cong D^n$, $f_1(\overline{U}_1), g_1(\overline{U}_1) \subset V$ and $x_0 = \alpha(0)$ is the only coincidence point of (f_1, g_1) on $\partial\overline{U}_1$. By Lemma 1.6.1, we have an extension $f'_1 : \overline{U}_1 \to V$ of $f_1|_{\partial\overline{U}_1}$ such that x_0 is the only coincidence point of $(f'_1, g_1|_{\overline{U}_1})$ on $\overline{U}_1$. Since both the images of $f_1|_{\overline{U}_1}$ and f'_1 are in V, which is homeomorphic to $\mathbf{R}^n$, we have $f_1|_{\overline{U}_1} \sim f'_1$ rel $\partial\overline{U}_1$. Define $f_2 : U \to Y$ by

$$f_2(t) = \begin{cases} f_1(x) & \text{if } t \notin U_1, \\ f'_1(x) & \text{if } t \in U_1, \end{cases}$$

We have $f_2 \sim f_1$ rel $X - U_1$ and $\Gamma(f_2, g_1) = \Gamma(f_1, g_1) - \{x_1\} = \Gamma(f, g) - \{x_1\}$. Note that $g \sim g_1$ rel $X - U_\eta$, so we have the result by setting $U' = U_\eta$.

Finally we prove the claim mentioned earlier. We define a homotopy f_s from f to f_1 as follows:

$$f_s(t) = \begin{cases} f(x) & \text{if } t \notin U_1, \\ f_s'(x) & \text{if } t \in U_\eta', \\ f_s''(x) & \text{if } t \in U_\eta'', \\ f_s'''(x) & \text{if } t \in U_\eta'''. \end{cases}$$

where f_s', f_s'', f_s''' are defined as follows:

$$f_s'(t) = \begin{cases} f((\frac{2|t|}{\eta} - s)t) & \text{if } |t| \geq \frac{1}{2}s\eta, \\ H_1(0, s - \frac{2|t|}{\eta}) & \text{if } |t| \leq \frac{1}{2}s\eta \end{cases}$$

$$f_s''(t) = \begin{cases} f((\frac{2|t'|}{\eta} - s)t', t_n) & \text{if } t = (t', t_n), \text{ and } |t'| \geq \frac{1}{2}s\eta, \\ H_1(t_n, s - \frac{2|t'|}{\eta}) & \text{if } t = (t', t_n), \text{ and } |t'| \leq \frac{1}{2}s\eta \end{cases}$$

$$f_s'''(t) = \begin{cases} f((\frac{2|t-e_n|}{\eta} - s)(t - e_n) + e_n) & \text{if } |t - e_n| \geq \frac{1}{2}s\eta, \\ H_1(1, s - \frac{2|t-e_n|}{\eta}) & \text{if } |t - e_n| \leq \frac{1}{2}s\eta \end{cases}$$

□

Lemma 1.6.5 *If x_0 is an isolated coincidence point of (f, g) with index zero and U is a neighbourhood of x_0, then there is an $f' \sim f$ rel $X - U$ such that $\Gamma(f', g) = \Gamma(f, g) - \{x_0\}$.*

Proof: See the proof of Lemma 2 in [SH1]. □

Theorem 1.6.6 *Let X, Y be manifolds with the same dimension and $\dim X \geq 3$. Then for any pair of maps $(f, g) : X \to Y$, there is a pair (f', g') with $(f', g') \sim (f, g)$, and such that $\#\Gamma(f', g') = N(f, g)$.*

Proof: By Lemma 1.6.2, we may assume that $\Gamma(f,g)$ is finite. Then applying Lemma 1.6.4, we can coalesce all the coincidence points in the same class to a single one and by Lemma 1.6.5, those inessential classes can be removed. $\square$

Definition 1.6.7 (Definition 5.1 of [SH2]) A subspace A of a topological space X can be bypassed if every path in X with end points in $X - A$ is homotopic to a path in $X - A$ keeping end points fixed.

Theorem 1.6.8 *If dim $A \geq 3$ and A can be bypassed in X, then for any pair of maps (f,g) : $(X,A) \to (Y,B)$, there is a pair (f',g') with $(f',g') \sim (f,g)$ and such that $\#\Gamma(f',g') = N(f,g;X,A)$.*

Proof: See Theorem 2.4 in [JJ]. $\square$

Chapter 2

Coincidence Points on the Complement

As stated in the introduction, relative Nielsen theory, which concerns a selfmap $f : (X, A) \to (X, A)$ of a pair of spaces, was developed in [SH2]. When such a map is considered, the ordinary Nielsen number may be a poor lower bound for the number of fixed points. The relative Nielsen number gives a better lower bound in this case. Zhao's work ([Z]) goes further, in that not only is the number of fixed points of a selfmap of a pair of spaces considered, but also the location of the fixed points. While for ordinary fixed point theory the location of fixed points does not affect the number of fixed points, in the equivariant case which is discussed in [WP3] there may be a difference.

Relative fixed point theory is generalized to relative coincidence theory in [JJ] and [JL]. In this chapter we generalize Zhao's work on the complement to coincidence theory. Some of the techniques and results developed here will be used in Chapter 4.

In this chapter then, we consider the coincidence points of a pair of maps $(f, g) : (X, A) \to (Y, B)$ that are located in the complement $X - A$. Here X and Y are manifolds with dimension n and A and B are submanifolds of X and Y respectively with dimension k. In the absolute case ($A = B = \emptyset$), it is not hard, using homotopies, to move a coincidence point to any given point in X. However, when we restrict to maps f, g of pairs manifolds, this is no longer true. More explicitly, if we consider maps $f, g : (X, A) \to (Y, B)$ of pairs, then it may not be possible to move all of the coincidence points in $X - A$ to A. We will discuss which coincidence points may be moved to A, and give a lower bound of the number of coincidence points on $X - A$ that cannot be so moved.

After this thesis was submitted, the paper [L] came to our attention. The paper sketches some of the results of section 2.1 and 2.3 (for example, Theorem 2.1.15, 2.1.16 and 2.3.2). However minimum theorem is not discussed there and the applications of our new Jiang type condition are absent. In addition, the result of Theorem 3.5 in [L] is incorrect when the subspace is not connected. There are no examples in [L].

This chapter is arranged as follows. In section 1, we introduce the concept of a coincidence Nielsen number of a pair of maps on the complement and give some basic properties of this number. In section 2, we develop a method to compute the Nielsen number when g_π is onto, and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$ (while Y is not necessarily a Jiang space). Then in section 3, we use the results of section 2 to give both some estimations of the Nielsen number on the complement, and also of the relative Nielsen number defined in [JJ] for manifolds. In section 4, we prove the minimum theorem and show how to create a coincidence point in the subspace A that is Nielsen equivalent to a coincidence point in $X - A$. We also show how to coalesce a coincidence point on $X - A$ to a coincidence point in A. These results are useful for equivariant coincidence theory which we will discuss in chapter 4.

2.1 Definitions and basic properties

Let $f, g : (X, A) \to (Y, B)$ be maps of pairs of spaces. Let $\hat{A} = \cup_{k=1}^{l} A_k$ be the disjoint union of all components A_k of A which are mapped by f and g into the same component B_s of B. We shall write $f_k, g_k : A_k \to B_s$ for the restrictions of f, g to A_k respectively.

For each A_k, we have the sets $\mathcal{R}_{f_k,g_k}$, $\tilde{\Gamma}(f_k, g_k)$, and also inclusions $\rho_{\mathcal{R}_{f_k,g_k}} : \tilde{\Gamma}(f_k, g_k) \to \mathcal{R}_{f_k,g_k}$. Suppose now that f_k, g_k map A_k into the same component B_s of B. We will define maps $i^{\tilde{\Gamma}}_{f_k,g_k} : \tilde{\Gamma}(f_k, g_k) \to \tilde{\Gamma}(f, g)$ and $i^{\mathcal{R}}_{f_k,g_k} : \mathcal{R}_{f_k,g_k} \to \mathcal{R}_{f,g}$ such that the following diagram is commutative.

$$\begin{array}{ccc}
\tilde{\Gamma}(f_k, g_k) & \xrightarrow{\rho_{\mathcal{R}_{f_k,g_k}}} & \mathcal{R}_{f_k,g_k} \\
\downarrow i^{\tilde{\Gamma}}_{f_k,g_k} & & \downarrow i^{\mathcal{R}}_{f_k,g_k} \\
\tilde{\Gamma}(f, g) & \xrightarrow{\rho_{\mathcal{R}_{f,g}}} & \mathcal{R}_{f,g}.
\end{array}$$

We start with the general case. By analogy with Proposition 1.5 in [JB2], we have the following lemma.

Lemma 2.1.1 *Let X_i and Y_i be connected, $i = 1, 2$, and let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$,*

$h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ be maps such that the following diagram is commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow h_X & & \downarrow h_Y \\ X_2 & \xrightarrow{f_2} & Y_2. \end{array}$$

Given a lifting $\tilde{f}_1$ of f_1, a lifting $\tilde{h}_X$ of h_X and a lifting $\tilde{h}_Y$ of h_Y, there is a unique lifting $\tilde{f}_2$ of f_2 such that the digram

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{f}_1} & \tilde{Y}_1 \\ \downarrow \tilde{h}_X & & \downarrow \tilde{h}_Y \\ \tilde{X}_2 & \xrightarrow{\tilde{f}_2} & \tilde{Y}_2. \end{array}$$

is commutative.

Proof: see the proof of Proposition 1.5 of [JB2]. □

Corollary 2.1.2 (cf. [JB2]) *Consider a commutative diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{id_{X_1}} & X_1 \\ \downarrow h & & \downarrow h \\ X_2 & \xrightarrow{id_{X_2}} & X_2. \end{array}$$

Then each lifting $\tilde{h}$ of h defines a map from $\Pi(X_1)$ to $\Pi(X_2)$. The map will be denoted by $\tilde{h}_\Pi : \Pi(X_1) \to \Pi(X_2)$.

Proof: This follows from Lemma 2.1.1. □

Definition 2.1.3 *Let $f_1, g_1 : X_1 \to Y_1$, $f_2, g_2 : X_2 \to Y_2$, $h_X : X_1 \to X_2$, $h_Y : Y_1 \to Y_2$ be maps such that the diagram*

$$\begin{array}{ccc} X_1 & \overset{f_1, g_1}{\rightarrow} & Y_1 \\ \downarrow h_X & & \downarrow h_Y \\ X_2 & \overset{f_2, g_2}{\rightarrow} & Y_2 \end{array}$$

is commutative. We define a map $j_{h_X, h_Y} : \mathcal{R}_{f_1, g_1} \to \mathcal{R}_{f_2, g_2}$ as follows. If we have a commutative diagram

$$\begin{array}{ccc} \tilde{X}_1 & \overset{\tilde{f}_1, \tilde{g}_1}{\rightarrow} & \tilde{Y}_1 \\ \downarrow \tilde{h}_X & & \downarrow \tilde{h}_Y \\ \tilde{X}_2 & \overset{\tilde{f}_2, \tilde{g}_2}{\rightarrow} & \tilde{Y}_2, \end{array}$$

where $\tilde{f}_i, \tilde{g}_i$ are liftings of f_i, g_i respectively for $i = 1, 2$. $\tilde{h}_X, \tilde{h}_Y$ are liftings of h_X, h_Y respectively, define $j_{h_X, h_Y}([(\tilde{f}_1, \tilde{g}_1)]) = [(\tilde{f}_2, \tilde{g}_2)])$.

Lemma 2.1.4 *j_{h_X, h_Y} is well defined.*

Proof: First j_{h_X, h_Y} is defined on all the elements in $\mathcal{R}_{f_1, g_1}$ by Lemma 2.1.1.

We first prove that the definition of j_{h_X, h_Y} is independent of the choice of $(\tilde{f}_1, \tilde{g}_1)$. Let $(\tilde{f}'_1, \tilde{g}'_1) = \tilde{\gamma}^{Y_1}(\tilde{f}_1, \tilde{g}_1)(\tilde{\gamma}^{X_1})^{-1}$ where $\tilde{f}'_2$ and $\tilde{g}'_2$ are liftings of f and g respectively such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{X}_1 & \overset{\tilde{f}'_1, \tilde{g}'_1}{\rightarrow} & \tilde{Y}_1 \\ \downarrow \tilde{h}_X & & \downarrow \tilde{h}_Y \\ \tilde{X}_2 & \overset{\tilde{f}'_2, \tilde{g}'_2}{\rightarrow} & \tilde{Y}_2. \end{array}$$

Let $\tilde{\gamma}^{X_2} = (\tilde{h}_X)_\Pi(\tilde{\gamma}^{X_1})$ and $(\tilde{\gamma}^{Y_2}) = (\tilde{h}_Y)_\Pi(\tilde{\gamma}^{Y_1})$, then the diagrams

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{\gamma}^{X_1}} & \tilde{X}_1 \\ \downarrow \tilde{h}_X & & \downarrow \tilde{h}_X \\ \tilde{X}_2 & \xrightarrow{\tilde{\gamma}^{X_2}} & \tilde{X}_2 \end{array} \qquad \begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\tilde{\gamma}^{Y_1}} & \tilde{Y}_1 \\ \downarrow \tilde{h}_Y & & \downarrow \tilde{h}_Y \\ \tilde{Y}_2 & \xrightarrow{\tilde{\gamma}^{Y_2}} & \tilde{Y}_2 \end{array}$$

are commutative. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "\tilde{f}'_1, \tilde{g}'_1"] \arrow[dr, "\tilde{\gamma}^{X_1}"] \arrow[dd, "\tilde{h}_X"'] & & \tilde{Y}_1 \arrow[dr, "\tilde{\gamma}^{Y_1}"] \arrow[dd, "\tilde{h}_Y"] & \\
& \tilde{X}_1 \arrow[rr, "\tilde{f}_1, \tilde{g}_1"] \arrow[dd, "\tilde{h}_X"'] & & \tilde{Y}_1 \arrow[dd, "\tilde{h}_Y"] \\
\tilde{X}_2 \arrow[rr, "\tilde{f}'_2, \tilde{g}'_2"] \arrow[dr, "\tilde{\gamma}^{X_2}"'] & & \tilde{Y}_2 \arrow[dr, "\tilde{\gamma}^{Y_2}"] & \\
& \tilde{X}_2 \arrow[rr, "\tilde{f}_2, \tilde{g}_2"'] & & \tilde{Y}_2
\end{tikzcd}$$

So $(\tilde{f}'_2, \tilde{g}'_2) = \tilde{\gamma}^{X_2}(\tilde{f}_2, \tilde{g}_2)(\tilde{\gamma}^{X_2})^{-1}$, and thus j_{h_X, h_Y} does not depend on the choice of $(\tilde{f}_1, \tilde{g}_1)$.

Next we show j_{h_X, h_Y} does not depend on the choices of $\tilde{h}_X$ and $\tilde{h}_Y$. Assume $\tilde{h}'_X = \tilde{\gamma}^{X_2}\tilde{h}_X$ and $\tilde{h}'_Y = \tilde{\gamma}^{Y_2}\tilde{h}_Y$. Let $\tilde{f}'_2$ and $\tilde{g}'_2$ be liftings of f_2 and g_2 respectively such that $\tilde{f}'_2\tilde{h}'_X = \tilde{h}'_Y\tilde{f}_1$ and $\tilde{g}'_2\tilde{h}'_X = \tilde{h}'_Y\tilde{g}_1$. This means that $(\tilde{\gamma}^{Y_2})^{-1}\tilde{f}'_2\tilde{\gamma}^{X_2}\tilde{h}_X = \tilde{h}_Y\tilde{f}_1$ and $(\tilde{\gamma}^{Y_2})^{-1}\tilde{g}'_2\tilde{\gamma}^{X_2}\tilde{h}_X = \tilde{h}_Y\tilde{g}_1$. By uniqueness, we have $\tilde{f}_2 = (\tilde{\gamma}^{Y_2})^{-1}\tilde{f}'_2\tilde{\gamma}^{X_2}$ and $\tilde{g}_2 = (\tilde{\gamma}^{Y_2})^{-1}\tilde{g}'_2\tilde{\gamma}^{X_2}$. This proves that j_{h_X, h_Y} does not depend on the choices of $\tilde{h}_X$ and $\tilde{h}_Y$. □

For connected subspaces $A \subset X$ and $B \subset Y$ there are inclusion maps $i_A : A \to X$ and

$i_B : B \to Y$, and the following diagram are commutative.

$$\begin{array}{ccc} A & \overset{f_A, g_A}{\to} & B \\ \downarrow i_A & & \downarrow i_B \\ X & \overset{f,g}{\to} & Y. \end{array}$$

Then we have the map $j_{i_A, i_B} : \mathcal{R}_{f_A, g_A} \to \mathcal{R}_{f,g}$ defined in Definition 2.1.3.

Notation: We will use the notation $i^{\mathcal{R}}_{f_A, g_A} : \mathcal{R}_{f_A, g_A} \to \mathcal{R}_{f,g}$ to denote j_{i_A, i_B}.

By the definition of $i^{\mathcal{R}}_{f_A, g_A}$, if an element $[(\tilde{f}, \tilde{g})] \in \mathcal{R}_{f,g}$ is in the image of $i^{\mathcal{R}}_{f_A, g_A}$, then there is a lifting $(\tilde{f}_A, \tilde{g}_A)$ of (f_A, g_A) such that the diagram

$$\begin{array}{ccc} \tilde{A} & \overset{\tilde{f}_A, \tilde{g}_A}{\to} & \tilde{B} \\ \downarrow \tilde{i}_A & & \downarrow \tilde{i}_B \\ \tilde{X} & \overset{\tilde{f}, \tilde{g}}{\to} & \tilde{Y} \end{array}$$

is commutative.

For any connected subspace A of X, we have a inclusion map $i^{\Gamma}_{f_A, g_A} : \Gamma(f_A, g_A) \to \Gamma(f, g)$, which induces a map $i^{\tilde{\Gamma}}_{f_A, g_A} : \tilde{\Gamma}(f_A, g_A) \to \tilde{\Gamma}(f, g)$.

Lemma 2.1.5 *The diagram*

$$\begin{array}{ccc} \tilde{\Gamma}(f_A, g_A) & \overset{\rho_{\mathcal{R}_{f_A, g_A}}}{\longrightarrow} & \mathcal{R}_{f_A, g_A} \\ \downarrow i^{\tilde{\Gamma}}_{f_A, g_A} & & \downarrow i^{\mathcal{R}}_{f_A, g_A} \\ \tilde{\Gamma}(f, g) & \overset{\rho_{\mathcal{R}_{f,g}}}{\longrightarrow} & \mathcal{R}_{f,g} \end{array}$$

is commutative.

Proof: Let x be a coincidence point on A. We use $[x]_{f_A, g_A}$ to denote the coincidence class of (f_A, g_A) containing x and use $[x]_{f,g}$ to denote the coincidence class of (f, g) containing

x. Then $i^{\tilde{\Gamma}}_{f_A,g_A}([x]_{f_A,g_A}) = [x]_{f,g}$. Assume that $\rho_{\mathcal{R}_{f,g}}([x]_{f,g}) = [(\tilde{f}, \tilde{g})]$, where $\tilde{f}$ and $\tilde{g}$ are liftings of f and g respectively such that for some $\tilde{x} \in p_X^{-1}(x)$, $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$. Let $\tilde{A}_1$ be the component of $p_X^{-1}(A)$ containing $\tilde{x}$ and let $\tilde{B}_1$ be the component of $p_X^{-1}(B)$ containing $\tilde{f}(\tilde{x})$. Let $\tilde{j}_{A\subset X} : \tilde{A} \to \tilde{A}_1$ ($\tilde{j}_{B\subset X} : \tilde{B} \to \tilde{B}_1$ respectively) be a lifting of the inclusion maps $i_A : A \hookrightarrow X$ ($i_B : B \hookrightarrow Y$ respectively) from the universal covering space $\tilde{A}$ ($\tilde{B}$ respectively) of A (B respectively) to $\tilde{A}_1$ ($\tilde{B}_1$ respectively). Note that $\tilde{j}_{A\subset X}$ and $\tilde{j}_{B\subset X}$ are actually covering projections. Let $(\tilde{f}_A, \tilde{g}_A)$ be a lifting of $(\tilde{f}|_{\tilde{A}_1}, \tilde{g}|_{\tilde{A}_1})$ with a coincidence point on $\tilde{j}^{-1}_{A\subset X}(\tilde{x})$. It is easy to check that $(\tilde{f}_A, \tilde{g}_A)$ is a lifting of (f_A, g_A), $\rho_{\mathcal{R}_{f_A,g_A}}([x]_{f_A,g_A}) = [(\tilde{f}_A, \tilde{g}_A)]$ and $i^{\mathcal{R}}_{f_A,g_A}([(\tilde{f}_A, \tilde{g}_A)] = [(\tilde{f}, \tilde{g})]$. So the diagram is commutative. □

Now we assume $A = \cup A_k$ and $B = \cup B_s$, where A_k and B_s are components of A and B respectively. We choose base points $x_0 \in X$ and $y_0 \in Y$, and for each component B_s of B we choose a base point $b_s \in B_s$. Similarly for each component A_k of A, we choose a base point a_k and a path u_k in X from x_0 to a_k. If A_k is mapped into B_s by both f_k and g_k, then we choose paths ω_{f_k} and ω_{g_k} in B_s from b_s to $f_k(a_k)$ and $g_k(a_k)$ respectively. (Note that there may be more than one component of A mapped into the same B_s, therefore there may be more than one ω_{f_k}'s for each B_s.)

Definition 2.1.6 A class $[x] \in \tilde{\Gamma}(f, g)$ is called a weakly common coincidence class if $\rho_{\mathcal{R}_{f,g}}([x])$ is in the image of $i_{\mathcal{R}_{f_k,g_k}}$ for some k. If $[x]$ is essential, it is called an essential weakly common coincidence class. The number of essential weakly common coincidence classes of f and g is denoted by $E(f, g; f_A, g_A)$.

From section 1.2 in Chapter 1, we have alternative description of the Reidemeister sets

$\mathcal{R}_{f_k,g_k}$ and $\mathcal{R}_{f,g}$ as $\nabla(f_k, g_k, a_k, b_s, \omega_{f_k}, \omega_{g_k})$ and $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$. This alternative description allows for an alternative description of a weakly common coincidence point. There is a map from $\nabla(f_k, g_k, a_k, b_s, \omega_{f_k}, \omega_{g_k})$ to $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$ defined as follows.

Definition 2.1.7 Define $v_{f_k,g_k} : \pi_1(B_s, b_s) \to \pi_1(Y, y_0)$ by

$$v_{f_k,g_k}(\alpha) = \omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot \alpha \cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}.$$

and define $\tilde{v}_{f_k,g_k}(\overline{\alpha}) = \overline{v_{f_k,g_k}(\alpha)}$. (where $\overline{\alpha}$ is the $f_\pi^{\omega_{f_k}}, g_\pi^{\omega_{g_k}}$-congruence class of α. See Definition 1.2.3.)

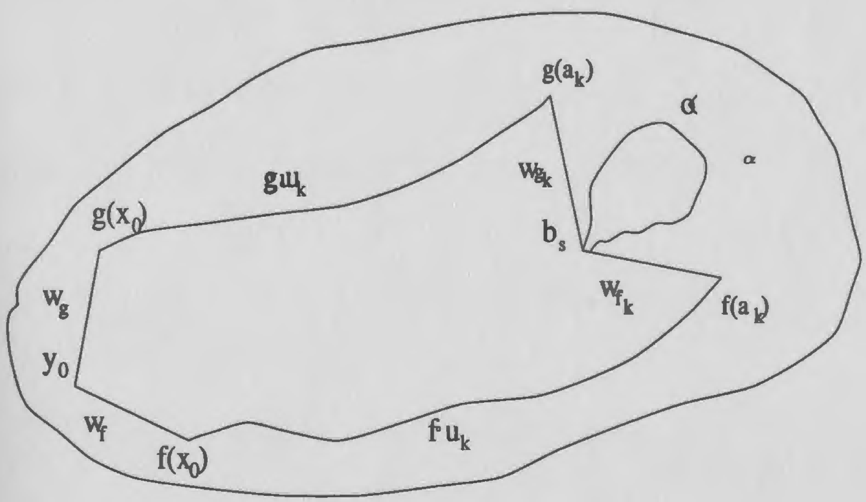

Figure 2.1:

Lemma 2.1.8 *The $\tilde{v}_{f_k,g_k}$ is well defined.*

Proof: Suppose that $\alpha, \beta \in \pi_1(B_s, b_s)$ are in the same Reidemeister class, i.e. there is an element $\gamma \in \pi_1(A_k, a_k)$ such that

$$\beta = (g_k)_\pi(\gamma) \cdot \alpha \cdot (f_k)_\pi(\gamma^{-1}).$$

Then $v_{f_k,g_k}([\beta]) =$

$$[\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot \beta \cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}\ ] =$$

$$[\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot (g_k)_\pi(\gamma) \cdot \alpha \cdot (f_k)_\pi(\gamma^{-1}) \cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] =$$

$$[\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot \omega_{g_k} \cdot (g_k \circ \gamma) \cdot \omega_{g_k}^{-1} \cdot \alpha \cdot \omega_{f_k} \cdot (f_k \circ \gamma^{-1}) \cdot \omega_{f_k}^{-1} \cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] =$$

$$[\omega_g \cdot (g \circ u_k) \cdot (g_k \circ \gamma) \cdot (g \circ u_k^{-1}) \cdot \omega_g^{-1} \cdot \omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot \alpha$$

$$\cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1} \cdot \omega_f \cdot (f \circ u_k) \cdot (f_k \circ \gamma^{-1}) \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] =$$

$$g_\pi(i_*(\gamma))v_{f_k,g_k}(\alpha)f_\pi(i_*(\gamma))^{-1}.$$

This proves that $v_{f_k,g_k}(\alpha)$ and $v_{f_k,g_k}(\beta)$ are in the same class of $\mathcal{R}_{f,g}$. □

Lemma 2.1.9 *Assume $A \subset X$ and $B \subset Y$. $f, g : (X, A) \to (Y, B)$ are maps, and $a \in A$ and $b \in B$ are basepoints of A and B respectively. Let μ be a path from x_0 to a. Then the diagram*

$$\begin{array}{ccc}
\mathcal{R}_{f_A,g_A} & \overset{\Theta_{f_A,g_A}}{\to} & \nabla(f_A, g_A; a, b, \omega_{f_A}, \omega_{g_A}) \\
\downarrow i^{\mathcal{R}}_{f_A,g_A} & & \downarrow \tilde{v}_{f_A,g_A} \\
\mathcal{R}_{f,g} & \overset{\Theta_{f,g}}{\to} & \nabla(f, g; x_0, y_0, \omega_f, \omega_g)
\end{array}$$

is commutative.

Proof: Assume that $[(\tilde{f}_A, \tilde{g}_A)] \in \mathcal{R}_{f_A,g_A}$. Let $[(\tilde{f}, \tilde{g})] = i^{\mathcal{R}}_{f_A,g_A}([(\tilde{f}_A, \tilde{g}_A)])$, then we have the commutative diagram

$$\begin{array}{ccc}
\tilde{A} & \overset{\tilde{f}_A,\tilde{g}_A}{\longrightarrow} & \tilde{B} \\
\downarrow \tilde{j}_{A\subset X} & & \downarrow \tilde{j}_{B\subset Y} \\
\tilde{X} & \overset{\tilde{f},\tilde{g}}{\longrightarrow} & \tilde{Y},
\end{array}$$

where $\tilde{j}_{A\subset X}$ and $\tilde{j}_{B\subset Y}$ are liftings of $i_A : A \to X$ and $i_B : B \to Y$ respectively. Let $\tilde{a} \in p_A^{-1}(a)$ and $\tilde{\alpha}_A$ be a path from $\tilde{g}_A(\tilde{a})$ to $\tilde{f}_A(\tilde{a})$. Denote $p_A \circ \tilde{\alpha}_A$ by α_A. Then $\Theta_{f_A,g_A}([(\tilde{f}_A, \tilde{g}_A)]) = \overline{[\omega_{g_A} \cdot \alpha_A \cdot \omega_{f_A}]}$. Let $\tilde{\mu}$ be a lifting of μ with the end at $\tilde{a}$ and denote $\tilde{\mu}(0)$ by $\tilde{x}_0$. It is obvious that $\tilde{x}_0 \in p_X^{-1}(x_0)$ and $(\tilde{g} \circ \tilde{\mu}) \cdot \tilde{\alpha}_A \cdot (\tilde{f} \circ \tilde{\mu}^{-1})$ is a path from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$. By the definition, we have

$$
\begin{aligned}
&\Theta_{f,g}([(\tilde{f}, \tilde{g})]) \\
&= \overline{[\omega_g \cdot p_X \circ ((\tilde{g} \circ \tilde{\mu}) \cdot \tilde{\alpha}_A \cdot (\tilde{f} \circ \tilde{\mu}^{-1})) \cdot \omega_f^{-1}]} \\
&= \overline{[\omega_g \cdot (g \circ \mu) \cdot \alpha_A \cdot (f \circ \mu^{-1}) \cdot \omega_f^{-1}]} \\
&= \overline{[\omega_g \cdot (g \circ \mu) \cdot \omega_{g_A}^{-1}][\omega_{g_A} \cdot \alpha_A \cdot \omega_{f_A}^{-1}][\omega_{f_A} \cdot (f \circ \mu^{-1}) \cdot \omega_f^{-1}]} \\
&= \tilde{v}_{f_A,g_A}([\omega_{g_A} \cdot \alpha_A \cdot \omega_{f_A}^{-1}]).
\end{aligned}
$$

□

Lemma 2.1.10 *The diagram*

$$
\begin{array}{ccc}
\tilde{\Gamma}(f_k, g_k) & \xrightarrow{\rho_{\nabla f_k,g_k}} & \nabla(f_k, g_k, a_k, b_s, \omega_{f_k}, \omega_{g_k}) \\
\downarrow i^{\tilde{\Gamma}}_{f_k,g_k} & & \downarrow \tilde{v}_{f_k,g_k} \\
\tilde{\Gamma}(f, g) & \xrightarrow{\rho_{\nabla f,g}} & \nabla(f, g, x_0, y_0, \omega_f, \omega_g)
\end{array}
$$

is commutative.

Proof: The proof follows from the commutativity of the diagrams in Lemma 2.1.5, 2.1.9 and 1.2.6. □

Corollary 2.1.11 *A class $[x] \in \tilde{\Gamma}(f, g)$ is a weakly common coincidence class if and only if $\rho_{\nabla_{f,g}}([x])$ is in the image of $\tilde{v}_{f_k,g_k}$ for some k.*

Proof: The result follows from Lemma 1.2.6, 2.1.5, 2.1.9 and 2.1.10. □

The next result generalizes Lemma 2.3 of [Z].

Lemma 2.1.12 *A coincidence point $x \in \Gamma(f, g)$ belongs to a weakly common coincidence class if and only if there is a path $\alpha : (I, 0, 1) \to (X, x, A)$ from x to A such that $f \circ \alpha \sim g \circ \alpha : (I, 0, 1) \to (Y, f(x), B)$.*

Proof: Assume that x is in a weakly common coincidence class. By Corollary 2.1.11, there is a path $C : I \to X$ from x_0 to x, such that $[\omega_g \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1}] \in \tilde{v}_{f_k, g_k}(\widetilde{[\beta]})$ for some element $[\beta] \in \pi_1(B_s, b_s)$ and k, where both f and g send A_k to B_s. In other words, there is an element $\gamma \in \pi_1(X, x_0)$ such that

$$g_\pi(\gamma)[\omega_g \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1}] f_\pi(\gamma^{-1}) = v_{f_k, g_k}([\beta]),$$

or

$$[\omega_g \cdot (g \circ \gamma) \cdot \omega_g^{-1} \cdot \omega_g \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1} \cdot \omega_f \cdot (f \circ \gamma^{-1}) \cdot \omega_f^{-1}] = [\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \cdot \beta \cdot \omega_{f_k} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}].$$

From above, we have

$$(g \circ u_k^{-1}) \cdot (g \circ \gamma) \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot (f \circ \gamma^{-1}) \cdot (f \circ u_k) \sim \omega_{g_k}^{-1} \cdot \beta \cdot \omega_{f_k},$$

or more briefly,

$$g \circ (u_k^{-1} \cdot \gamma \cdot C) \cdot f \circ (C^{-1} \cdot \gamma^{-1} \cdot u_k) \sim \omega_{g_k}^{-1} \cdot \beta \cdot \omega_{f_k}.$$

Note that the right hand side is contained in B_s, and if we set $\alpha = C^{-1} \cdot \gamma^{-1} \cdot u_k$, then we have $g \circ \alpha \sim f \circ \alpha : (I, 0, 1) \to (Y, f(x), B)$.

On the other hand, assume that there is a path $\alpha : (I, 0, 1) \to (X, x, A)$ from x to A such that $g \circ \alpha \sim f \circ \alpha : (I, 0, 1) \to (Y, f(x), B)$ and $\alpha(1) = a \in A_k$, a component of A, and such that both $f(A_k), g(A_k) \subset B_s$. Let $C : I \to X$ be a path from x_0 to x, we have to find an element $[\beta] \in \pi_1(B_s, b_s)$ such that $v_{f_k,g_k}([\beta]) = g_\pi(\gamma)[\omega_g \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1}]f_\pi(\gamma^{-1})$ for some $\gamma \in \pi_1(X, x_0)$.

Let C_a be a path from a_k to a. Let $l = F(1, \cdot)$, a path from $g \circ \alpha(1)$ to $f \circ \alpha(1)$ in B_s. Set $\gamma = u_k \cdot C_a \cdot \alpha^{-1} \cdot C^{-1}$ and $\beta = \omega_{g_k} \cdot (g_k \circ C_a) \cdot l \cdot (f_k \circ C_a^{-1}) \cdot \omega_{f_k}^{-1}$.

By assumption, we have $(g \circ \alpha) \cdot l \cdot (f \circ \alpha^{-1}) \sim 0$, or

$$l \sim (g \circ \alpha^{-1}) \cdot (f \circ \alpha).$$

$$\begin{aligned}
v_{f_k,g_k}([\beta]) &= [\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1} \beta \omega_{fk} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] \\
&= [(\omega_g \cdot (g \circ u_k) \cdot \omega_{g_k}^{-1})(\omega_{g_k} \cdot (g_k \cdot C_a) \cdot l \cdot (f_k \circ C_a^{-1}) \cdot \omega_{f_k}^{-1})(\omega_{fk} \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1})] \\
&= [\omega_g \cdot (g \circ u_k) \cdot (g_k \cdot C_a) \cdot l \cdot (f_k \circ C_a^{-1}) \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] \\
&= [\omega_g \cdot (g \circ u_k) \cdot (g_k \cdot C_a) \cdot (g \circ \alpha^{-1}) \cdot (f \circ \alpha) \cdot (f_k \circ C_a^{-1}) \cdot (f \circ u_k^{-1}) \cdot \omega_f^{-1}] \\
&= [\omega_g \cdot g \circ (u_k \cdot C_a \cdot \alpha^{-1}) \cdot f \circ (\alpha \cdot C_a^{-1} \cdot u_k^{-1}) \cdot \omega_f^{-1}] \\
&= [\omega_g \cdot g \circ (u_k \cdot C_a \cdot \alpha^{-1}) \cdot ((g \circ C^{-1}) \cdot \omega_g^{-1} \cdot \omega_g \cdot (g \circ C)) \cdot \\
&\quad ((f \circ C^{-1}) \cdot \omega_f^{-1} \cdot \omega_f \cdot (f \circ C)) \cdot f \circ (\alpha \cdot C_a^{-1} \cdot u_k^{-1}) \cdot \omega_f^{-1}] \\
&= [(\omega_g \cdot g \circ (u_k \cdot C_a \cdot \alpha^{-1}) \cdot (g \circ C^{-1}) \cdot \omega_g^{-1}) \cdot (\omega_g \cdot (g \circ C)) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1}) \cdot \\
&\quad (\omega_f \cdot (f \circ C) \cdot f \circ (\alpha \cdot C_a^{-1} \cdot u_k^{-1}) \cdot \omega_f^{-1})] \\
&= g_\pi(\gamma)[\omega_g \cdot (g \circ C) \cdot (f \circ C^{-1}) \cdot \omega_f^{-1}]f_\pi(\gamma^{-1}). \qquad \square
\end{aligned}$$

Corollary 2.1.13 *A coincidence class of (f, g) containing a coincidence point on A is a weakly common coincidence class.* □

Definition 2.1.14 The number of essential coincidence classes of $f, g : X \to Y$, which are not weakly common coincidence classes is called the Nielsen number of f, g on the complementary space $X - A$ and is denoted by $N(f, g; X - A)$. In other words, $N(f, g; X - A)$ is equal to $N(f, g) - E(f, g; f_A, g_A)$ (see Definition 2.1.6).

Theorem 2.1.15 *Any pair of maps $f, g : (X, A) \to (Y, B)$ has at least $N(f, g; X - A)$ coincidence points on $X - A$.*

Proof: Note that each essential coincidence class has at least one coincidence point. If this point is in A, by Corollary 2.1.13, the class is a weakly common coincidence class. Therefore, for each essential non weakly common coincidence class there is at least one coincidence point in $X - A$, and there are $N(f, g : X - A)$ such classes by the definition. □

Theorem 2.1.16 *$N(f, g; X - A)$ is a homotopy invariant.*

Proof: Assume $F : f \sim f' : (X, A) \to (Y, B)$ and $G : g \sim g' : (X, A) \to (Y, B)$ are homotopies. We know that $N(f, g) = N(f', g')$, so we only need to prove that if $x \in \Gamma(f, g)$ and $x' \in \Gamma(f', g')$ are F, G-related and x is in a weakly common coincidence class, then x' is too.

Assume that α is a path from x to x' such that $< F, \alpha > \sim < G, \alpha >$. Let $\bar{\alpha}$ be the path in $X \times I$ defined by $\bar{\alpha}(t) = (\alpha(t), t)$, then $F \circ \bar{\alpha} = < F, \alpha > \sim < G, \alpha > = G \circ \bar{\alpha}$. Let $\tilde{F}$ and $\tilde{G}$ be liftings of F and G respectively such that $\tilde{F}(\tilde{x}, 0) = \tilde{G}(\tilde{x}, 0)$ for some $\tilde{x} \in p_X^{-1}(x)$. Then $\tilde{F}(\tilde{x}', 1) = \tilde{G}(\tilde{x}', 1)$ for some $\tilde{x}' \in p_X^{-1}(x')$ since $\bar{\alpha}$ is a path from $(x, 0)$ to $(x', 1)$ and the lift of $F \circ \bar{\alpha}(G \circ \bar{\alpha}$ respectively) is $\tilde{F} \circ \tilde{\bar{\alpha}}(\tilde{G} \circ \tilde{\bar{\alpha}}$ respectively). So $\rho_{\mathcal{R}_{f,g}}([x]) = [(\tilde{F}(\cdot, 0), \tilde{G}(\cdot, 0))]$ and $\rho_{\mathcal{R}_{f',g'}}([x']) = [(\tilde{F}(\cdot, 1), \tilde{G}(\cdot, 1))]$.

Now let $F_k = F|_{A_k \times I}$ and $G_k = G|_{A_k \times I}$ and let $(\tilde{f}_k, \tilde{g}_k)$ be lifting of (f_k, g_k) such that $i^{\mathcal{R}}_{f_k, g_k}([(\tilde{f}_k, \tilde{g}_k)]) = [(\tilde{F}(\cdot, 0), \tilde{G}(\cdot, 0))]$, then the diagram

$$\begin{array}{ccc} \tilde{A}_k & \xrightarrow{\tilde{f}_k, \tilde{g}_k} & \tilde{B}_s \\ \downarrow \tilde{i}_k & & \downarrow \tilde{i}_s \\ \tilde{X} & \xrightarrow{\tilde{F}(\cdot,0), \tilde{G}(\cdot,0)} & \tilde{Y} \end{array}$$

is commutative. Let $\tilde{F}_k, \tilde{G}_k$ be liftings of F_k, G_k starting from $\tilde{f}_k, \tilde{g}_k$ respectively. By the unique lifting property of covering spaces, we have the following commutative diagram

$$\begin{array}{ccc} \tilde{A}_k \times I & \xrightarrow{\tilde{F}_k, \tilde{G}_k} & \tilde{B}_s \\ \downarrow \tilde{i}_k \times id & & \downarrow \tilde{i}_s \\ \tilde{X} \times I & \xrightarrow{\tilde{F}, \tilde{G}} & \tilde{Y}. \end{array}$$

This implies the diagram

$$\begin{array}{ccc} \tilde{A}_k & \xrightarrow{\tilde{F}_k(\cdot,1), \tilde{G}_k(\cdot,1)} & \tilde{B}_s \\ \downarrow \tilde{i}_k & & \downarrow \tilde{i}_s \\ \tilde{X} & \xrightarrow{\tilde{F}(\cdot,1), \tilde{G}(\cdot,1)} & \tilde{Y} \end{array}$$

is commutative, which means $i^{\mathcal{R}}_{f'_k, g'_k}([(\tilde{F}_k(\cdot, 1), \tilde{G}_k(\cdot, 1))]) = [(\tilde{F}(\cdot, 1), \tilde{G}(\cdot, 1))]$. □

2.2 The Reidemeister number and the Nielsen number

In this section, we give conditions under which the Reidemeister numbers and Nielsen numbers can be computed easily. Let $\theta_X : \pi_1(X, x_0) \to H_1(X)$ and $\theta_Y : \pi_1(Y, y_0) \to H_1(Y)$ be the abelianizing homomorphisms, and $\eta_Y : H_1(Y) \to Coker(g_* - f_*)$ the projection map, where f_* and g_* are the homomorphisms from $H_1(X)$ to $H_1(Y)$ induced by f and g respectively.

Lemma 2.2.1 *If $\alpha \sim \alpha'$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, then $\eta_Y\theta_Y(\alpha) = \eta_Y\theta_Y(\alpha')$. Therefore $\eta_Y \circ \theta_Y$ induces a map from $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ to $Coker(g_* - f_*)$. We will denote this map by*

$$h : \nabla(f, g; x_0, y_0, \omega_f, \omega_g) \to Coker(g_* - f_*).$$

Proof: We have to prove that when $\alpha \sim \alpha'$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, then $\theta_Y(\alpha') - \theta_Y(\alpha)$ is in the image of $g_* - f_*$. Assume that $\alpha \sim \alpha'$, i.e. there is an element $\gamma \in \pi_1(X, x_0)$ such that $\alpha' = g_\pi(\gamma)\alpha f_\pi(\gamma^{-1})$. Since θ_Y is a homomorphism and θ_Y is natural, we have by applying θ_Y to both sides that $\theta_Y(\alpha') = \theta_Y \circ g_\pi(\gamma) + \theta_Y(\alpha) - \theta_Y \circ f_\pi(\gamma) = g_* \circ \theta_X(\gamma) + \theta_Y(\alpha) - f_* \circ \theta_X(\gamma)$. So $\theta_Y(\alpha') - \theta_Y(\alpha) = g_* \circ \theta_X(\gamma) - f_* \circ \theta_X(\gamma) = (g_* - f_*)(\theta_X(\gamma))$, i.e $\theta_Y(\alpha') - \theta_Y(\alpha) \in$ Im $(g_* - f_*)$ as required. □

Theorem 2.2.2 *The following two conditions are equivalent.*

(i) $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence classes are independent of the choice of ω_f (respectively ω_g), or more precisely, if ω_f and ω_f' are paths from y_0 to $f(x_0)$, then the $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence class containing α and the $f_\pi^{\omega_f'}, g_\pi^{\omega_g}$-congruence class containing α are the same.

(ii) For any $\beta \in \pi_1(Y, y_0)$, if $\alpha \sim \alpha'$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, then $\alpha\beta \sim \alpha'\beta$ (respectively $\beta\alpha \sim \beta\alpha'$) in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$.

The following three conditions are equivalent.

(iii) For any $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, $\alpha\beta\gamma \sim \beta\alpha\gamma$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$.

(iv) For any $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, $\gamma\alpha\beta \sim \gamma\beta\alpha$ in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$.

(v) The composition $\eta_Y \circ \theta_Y : \pi_1(Y, y_0) \to Coker(g_* - f_*)$ of θ_Y and η_Y sends elements in different $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence classes to different elements of $Coker(g_* - f_*)$, and hence $\eta_Y \circ \theta_Y$

induces a one-to-one correspondence between $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ *and* $Coker(g_* - f_*)$.

In addition, (iii), (iv) and (v) together imply (i) and (ii), and when g_π *is onto, all formulations of the five statements are equivalent.*

Proof: (i)⇒(ii): Suppose $\alpha = [a]$,$\alpha' = [a']$ are elements in $\pi_1(Y, y_0)$ and $\alpha \sim \alpha'$, i.e. there is a $\gamma = [r] \in \pi_1(X, x_0)$ such that $\alpha' = g_\pi(\gamma)\alpha f_\pi(\gamma^{-1})$, or $a' \sim \omega_g \cdot (g \circ r) \cdot \omega_g^{-1} \cdot a' \cdot \omega_f \cdot (f \circ r)^{-1} \cdot \omega_f^{-1}$. Assume that $\beta = [b]$ is another element in $\pi_1(Y, y_0)$. Let $\omega_f' = \beta \cdot \omega_f$. As Reidemeister classes are not dependent on the choice of ω_f', there is an element $\gamma' = [r'] \in \pi_1(X, x_0)$, such that $\alpha' = g_\pi(\gamma')\alpha f_\pi^{\omega_f'}(\gamma')^{-1}$, or $a' \sim \omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot b \cdot \omega_f \cdot (f \circ r')^{-1} \cdot \omega_f^{-1} \cdot b^{-1}$, so we have $a' \cdot b \sim \omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot b \cdot \omega_f \cdot (f \circ r')^{-1} \cdot \omega_f^{-1}$, or $\alpha'\beta = g_\pi(\gamma')\alpha\beta f_\pi(\gamma')^{-1}$. Then $\alpha\beta \sim \alpha'\beta$. The proof is analogous for independence with respect to ω_g.

(ii)⇒(i): Suppose $\alpha = [a], \alpha' = [a'] \in \pi_1(Y, y_0)$ are in the same Reidemeister class, i.e. there is an element $\gamma \in \pi_1(X, x_0)$ such that $\alpha' = g_\pi(\gamma)\alpha f_\pi(\gamma^{-1})$. Let ω_f' be another path from y_0 to $f(x_0)$, and $\beta = [b] = [\omega_f' \cdot (\omega_f)^{-1})]$. By (ii), $\alpha\beta \sim \alpha'\beta$, therefore there is an element $\gamma' = [r'] \in \pi_1(X, x_0)$ such that $\alpha'\beta = g_\pi(\gamma')\alpha\beta f_\pi((\gamma')^{-1})$, or $a' \cdot b \sim \omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot b \cdot \omega_f \cdot (f \circ (r')^{-1}) \cdot \omega_f^{-1}$, then

$$a' \sim \omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot b \cdot \omega_f \cdot (f \circ (r')^{-1}) \cdot \omega_f^{-1} \cdot b^{-1} =$$

$\omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot (\omega_f' \cdot \omega_f^{-1}) \cdot \omega_f \cdot (f \circ (r')^{-1}) \cdot \omega_f^{-1} \cdot (\omega_f \cdot (\omega_f')^{-1}) =$

$\omega_g \cdot (g \circ r') \cdot \omega_g^{-1} \cdot a \cdot \omega_f' \cdot (f \circ (r')^{-1}) \cdot (\omega_f')^{-1}$, that is $\alpha' = g_\pi(\gamma')\alpha f_\pi^{\omega_f'}((\gamma')^{-1})$. So the Reidemeister classes are independent to the choice of ω_f'. The equality for $\beta\alpha$ and $\beta\alpha'$ implying independence of ω_g is analogous.

(iii)$\Rightarrow$(v): We have to prove that if $\eta_Y \circ \theta_Y(\alpha) = \eta_Y \circ \theta_Y(\alpha')$, then $\alpha \sim \alpha'$. We divide the proof into three steps.

(1). For any commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ and γ in $\pi_1(X, x_0)$, then by (iii),

$$[\alpha, \beta]\gamma = \alpha\beta(\alpha^{-1}\beta^{-1}\gamma) \sim \beta\alpha(\alpha^{-1}\beta^{-1}\gamma) = \gamma.$$

(2). If $\theta_Y(\gamma) = \theta_Y(\gamma')$, then $\gamma \sim \gamma'$.

In fact, in this case, $\gamma'\gamma^{-1} \in Ker\theta_Y$, the commutator subgroup. Therefore,

$$\gamma' = [\alpha_1, \beta_1] \ldots [\alpha_k, \beta_k]\gamma \qquad \text{for some } \alpha_i, \beta_i.$$

Repeating (1), we have $\gamma' \sim \gamma$.

(3) Now assume that $\eta_Y \circ \theta_Y(\alpha) = \eta_Y \circ \theta_Y(\alpha')$. By the definition of η_Y, there is an element $c \in H_1(X)$ such that $\theta_Y(\alpha') - \theta_Y(\alpha) = g_*(c) - f_*(c)$. Let $c = \theta_X(\gamma)$, where $\gamma \in \pi_1(X, x_0)$, then $g_*(c) = g_*(\theta_X(\gamma)) = \theta_Y(g_\pi(\gamma))$ and $f_*(c) = \theta_Y(f_\pi(\gamma))$. Hence, $\theta_Y(\alpha') = \theta_Y(\alpha) + \theta_Y(g_\pi(\gamma)) - \theta_Y(f_\pi(\gamma)) = \theta_Y(g_\pi(\gamma)\alpha f_\pi(\gamma^{-1}))$, and by (2), $\alpha' \sim g_\pi(\gamma)\alpha f_\pi(\gamma^{-1}) \sim \alpha$.

(v)$\Rightarrow$(iii): As $\theta_Y(\alpha\beta\gamma) = \theta_Y(\beta\alpha\gamma)$, we have $\eta_Y \circ \theta_Y(\alpha\beta\gamma) = \eta_Y \circ \theta_Y(\beta\alpha\gamma)$. By (v), we have $\alpha\beta\gamma \sim \beta\alpha\gamma$.

The proof that (iv)$\Leftrightarrow$(v) is similar to the proof of (iii)$\Leftrightarrow$(v).

(v)$\Rightarrow$(ii): Assume that $\alpha \sim \alpha'$, then $\eta_Y\theta_Y(\alpha) = \eta_Y\theta_Y(\alpha')$. So, $\eta_Y\theta_Y(\alpha\beta) = \eta_Y\theta_Y(\alpha) + \eta_Y\theta_Y(\beta) = \eta_Y\theta_Y(\alpha') + \eta_Y\theta_Y(\beta) = \eta_Y\theta_Y(\alpha'\beta)$. By (v), we have $\alpha\beta \sim \alpha'\beta$.

Assume that g_π is onto. (ii)$\Rightarrow$(iii): Assume $\alpha, \beta \in \pi_1(Y, y_0)$ and $g_\pi(a) = \alpha, g_\pi(b) = \beta$, where $a, b \in \pi_1(X, x_0)$. Then we have $\alpha \sim g_\pi(a^{-1})\alpha f_\pi(a) = f_\pi(a)$ and $\beta \sim g_\pi(b^{-1})\beta f_\pi(b) = f_\pi(b)$. $\alpha\beta \sim g_\pi((ab)^{-1})\alpha\beta f_\pi(ab) = f_\pi(ab) = f_\pi(a)f_\pi(b)$. By (ii), $f_\pi(a)f_\pi(b) \sim \alpha f_\pi(b)$

$\sim g_\pi(b)\alpha f_\pi(b)f_\pi(b^{-1}) = g_\pi(b)\alpha = \beta\alpha$. Therefore, we have $\alpha\beta \sim \beta\alpha$. By (ii) again, $\alpha\beta\gamma \sim \beta\alpha\gamma$. □

Corollary 2.2.3 *If conditions (iii), (iv) or (v) in 2.2.2 hold true, or g_π is onto and conditions (i) or (ii) in Theorem 2.2.2 hold true, we have*

$$R(f, g) = \#Coker(g_* - f_*).$$

In particular, if $\pi_1(Y, y_0)$ is abelian, then

$$R(f, g) = \#Coker(g_* - f_*).$$ □

Definition 2.2.4 A pair of maps $(f, g) : X \to Y$ is said to be R-commutative if

For any $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, $\alpha\beta\gamma \cong \beta\alpha\gamma$ mod $f_\pi^{\omega_f}, g_\pi^{\omega_g}$.

Lemma 2.2.5 *The property of being R-commutative is independent of the choices of ω_f, ω_g, x_0 and y_0.*

Proof: By 2.2.2 ($(iii) \Rightarrow (i)$ and (ii)), so R-commutativity does not depend on the choices of ω_f and ω_g. We have to prove that R-commutativity does not depend on the choices of x_0 and y_0.

Let x_1 be another point in X, and C be a path from x_0 to x_1. Let $\omega_f' = \omega_f \cdot f(C)$, $\omega_g' = \omega_g \cdot g(C)$. We only need to prove that if α, β are $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruent, then they are $f_\pi^{\omega_f'}, g_\pi^{\omega_g'}$-congruent.

Suppose that $\alpha, \beta \in \pi_1(Y, y_0)$ are $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruent, i.e. there is an element $[\gamma] \in \pi_1(X, x_0)$ such that $\beta = g_\pi^{\omega_g}([\gamma])\alpha f_\pi^{\omega_f}([\gamma]^{-1})$. However, $g_\pi^{\omega_g}([\gamma]) = [\omega_g \cdot (g \circ \gamma) \cdot \omega_g^{-1}] =$

$[\omega_g\cdot(g\circ C)\cdot(g\circ(C^{-1}\cdot\gamma\cdot C))\cdot(g\circ C^{-1})\cdot\omega_g^{-1}] = [\omega_g'\cdot(g\circ(C^{-1}\cdot\gamma\cdot C))\cdot(\omega_g')^{-1}] = g_\pi^{\omega_g'}(C_\#([\gamma]))$. Similarly, we have $f_\pi^{\omega_f}([\gamma]) = f_\pi^{\omega_f'}(C_\#([\gamma]))$. so $\beta = g_\pi^{\omega_g'}(C_\#([\gamma]))\alpha f_\pi^{\omega_f'}(C_\#([\gamma]^{-1}))$, and α and β are $f_\pi^{\omega_f'}, g_\pi^{\omega_g'}$-congruent as required.

The argument that R-commutativity does not depend on the choices of y_0 is similar. □

Note that $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$ is actually an partition of $\pi_1(Y, y_0)$, so we can compare an element α in $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$ with an element α' in $\nabla(f, g, x_0', y_0, \omega_f', \omega_g')$. If α and α' contain the same elements of $\pi_1(Y, y_0)$, we can say $\alpha = \alpha'$. If $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$ and $\nabla(f, g, x_0', y_0, \omega_f', \omega_g')$ represent the same partition of $\pi_1(Y, y_0)$, we say $\nabla(f, g, x_0, y_0, \omega_f, \omega_g) = \nabla(f, g, x_0', y_0, \omega_f', \omega_g')$.

Proposition 2.2.6 *If (f, g) is R-commutative, then $\nabla(f, g, x_0, y_0, \omega_f, \omega_g) = \nabla(f, g, x_0', y_0, \omega_f', \omega_g')$ and*

$$R(f, g) = \#Coker(g_* - f_*).$$

Proof: The first statement follows from the proof of Theorem 2.2.2 ((iii) $\Rightarrow$ (v)). The second one follows from Corollary 2.2.3. □

Definition 2.2.7 (cf. Definition 1.4.2) Let $x_0 \in X$ (not necessarily a coincidence point of (f, g)). Define $T(f, g; x_0, y_0, \omega_f, \omega_g) = \{\omega_g < G, x_0 > \omega_g^{-1}\cdot\omega_f < F, x_0 >^{-1} \omega_f^{-1} | F : f \sim f, G : g \sim g\} \subset \pi_1(Y, y_0)$, and $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g)$ to be the image of $T(f, g; x_0, y_0, \omega_f, \omega_g)$ in $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$ under projection from $\pi_1(Y, y_0)$ to $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$.

Definition 2.2.8 Let (f, g) be R-commutative, then the number of elements in the set $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g)$ is called the Nielsen orbit length of (f, g). It is denoted by $NL(f, g)$.

The following lemma shows that $NL(f,g)$ is well defined. From Proposition 2.2.6, we know that when (f,g) is R-commutative, $\nabla(f,g,x_0,y_0,\omega_f,\omega_g)=\nabla(f,g,x_0',y_0,\omega_f',\omega_g')$, so we can compare its two subsets $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)$ and $\tilde{T}(f,g;x_0',y_0,\omega_f',\omega_g')$.

Lemma 2.2.9 *Let $x_0, x_0' \in X$, ω_f and ω_f' be paths from y_0 to $f(x_0)$ and $f(x_0')$ respectively, and ω_g and ω_g' be paths from y_0 to $g(x_0)$ and $g(x_0')$ respectively. If (f,g) is R-commutative, then $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)=\tilde{T}(f,g;x_0',y_0,\omega_f',\omega_g')$.*

Proof: By Proposition 2.2.6, we know that $\nabla(f,g,x_0,y_0,\omega_f,\omega_g)=\nabla(f,g,x_0',y_0,\omega_f',\omega_g')$, so $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)$ and $\tilde{T}(f,g,x_0',y_0,\omega_f',\omega_g')$ are both subsets of $\nabla(f,g,x_0,y_0,\omega_f,\omega_g)$. We divide the proof into two steps.

(i) The set $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)$ is independent of the choices of ω_f and ω_g. Let ω_f' and ω_g' be paths from y_0 to $f(x_0)$ and $g(x_0)$ respectively. We want to show that if an element $\overline{\alpha}$ of $\nabla(f,g,x_0,y_0,\omega_f,\omega_g)$ is in $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)$, then it is in $\tilde{T}(f,g;x_0,y_0,\omega_f',\omega_g')$. Let F and G be loops at f and g respectively, such that $\omega_g < G,x_0 > \omega_g^{-1}\cdot\omega_f < F,x_0 > \omega_f^{-1}$ represents $\overline{\alpha}$. Now $\omega_g' < G,x_0 > (\omega_g')^{-1}\cdot\omega_f' < F,x_0 > (\omega_f')^{-1}$ represents an element in $\tilde{T}(f,g;x_0,y_0,\omega_f',\omega_g')$. However, $\omega_g' < G,x_0 > (\omega_g')^{-1}\cdot\omega_f' < F,x_0 > (\omega_f')^{-1} \sim \omega_g'\cdot(\omega_g^{-1}\cdot\omega_g)\cdot < G,x_0 > \cdot(\omega_g^{-1}\cdot\omega_g)\cdot(\omega_g')^{-1}\cdot\omega_f'\cdot(\omega_f^{-1}\cdot\omega_f)\cdot < F,x_0 > \cdot(\omega_f^{-1}\cdot\omega_f)\cdot(\omega_f')^{-1} \sim (\omega_g'\cdot\omega_g^{-1})\cdot(\omega_g\cdot < G,x_0 > \cdot\omega_g^{-1})\cdot(\omega_g\cdot(\omega_g')^{-1})\cdot(\omega_f'\cdot\omega_f^{-1})\cdot(\omega_f\cdot < F,x_0 > \cdot\omega_f^{-1})\cdot(\omega_f\cdot(\omega_f')^{-1})$. Since (f,g) is R-commutative, $[\omega_g' < G,x_0 > (\omega_g')^{-1}\cdot\omega_f' < F,x_0 > (\omega_f')^{-1}] = [\omega_g'\cdot\omega_g^{-1}][\omega_g\cdot < G,x_0 > \cdot\omega_g^{-1}][\omega_g\cdot(\omega_g')^{-1}][\omega_f'\cdot\omega_f^{-1}][\omega_f\cdot < F,x_0 > \cdot\omega_f^{-1}][\omega_f\cdot(\omega_f')^{-1}] \equiv [\omega_g'\cdot\omega_g^{-1}][\omega_g\cdot < G,x_0 > \cdot\omega_g^{-1}][\omega_g\cdot(\omega_g')^{-1}][\omega_f'\cdot\omega_f^{-1}][\omega_f\cdot(\omega_f')^{-1}][\omega_f\cdot < F,x_0 > \cdot\omega_f^{-1}] = [\omega_g'\cdot\omega_g^{-1}][\omega_g\cdot < G,x_0 > \cdot\omega_g^{-1}][\omega_g\cdot(\omega_g')^{-1}][\omega_f\cdot < F,x_0 > \cdot\omega_f^{-1}] \equiv [\omega_g'\cdot\omega_g^{-1}][\omega_g\cdot(\omega_g')^{-1}][\omega_g\cdot < G,x_0 > \cdot\omega_g^{-1}][\omega_f\cdot < F,x_0 > \cdot\omega_f^{-1}]$

$= [\omega_g \cdot < G, x_0 > \cdot \omega_g^{-1}][\omega_f \cdot < F, x_0 > \cdot \omega_f^{-1}]$. This shows that $[\omega_g' < G, x_0 > (\omega_g')^{-1} \cdot \omega_f' < F, x_0 > (\omega_f')^{-1}]$ represents $\overline{\alpha}$ too, and therefore $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g) \subseteq \tilde{T}(f, g, x_0, y_0, \omega_f', \omega_g')$. Similarly, we have $\tilde{T}(f, g, x_0, y_0, \omega_f', \omega_g') \subseteq \tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g)$.

(ii) Now let x_0' be another point in X, and ω_f' and ω_g' be paths from y_0 to $f(x_0')$ and $g(x_0')$ respectively. Let $C : I \to X$ be a path from x_0 to x_0', and $\omega_f'' = \omega_f \cdot f \circ C$ and $\omega_g'' = \omega_g \cdot g \circ C$. By (i), we only need to prove that $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g) = \tilde{T}(f, g, x_0', y_0, \omega_f'', \omega_g'')$. Let F and G be loops at f and g respectively. Since $< F, x_0 > \sim f \circ C < F, x_0' > f \circ C^{-1}$ rel $\{0, 1\}$ and $< G, x_0 > \sim g \circ C < G, x_0' > g \circ C^{-1}$ rel $\{0, 1\}$, we have $\omega_g'' < G, x_0' > (\omega_g'')^{-1} \cdot \omega_f'' < F, x_0' >^{-1} (\omega_f'')^{-1} = (\omega_g \cdot g \circ C) < G, x_0' > (g \circ C^{-1} \cdot \omega_g^{-1}) \cdot (\omega_f \cdot f \circ C) < F, x_0' >^{-1} (f \circ C^{-1} \cdot \omega_f^{-1}) \sim \omega_g < G, x_0 > \omega_g^{-1} \cdot \omega_f < F, x_0 > \omega_f^{-1}$. So we actually have $T(f, g; x_0, y_0, \omega_f, \omega_g) = T(f, g, x_0', y_0, \omega_f'', \omega_g'')$ and therefore their projections $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g)$ and $\tilde{T}(f, g, x_0', y_0, \omega_f'', \omega_g'')$ in $\nabla(f, g, x_0, y_0, \omega_f, \omega_g) = \nabla(f, g, x_0', y_0, \omega_f'', \omega_g'')$ are the same. □

So without ambiguity, when (f, g) is R-commutative, $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g)$ can be written as $\tilde{T}(f, g, y_0)$.

Lemma 2.2.10 *Let $x_0 \in X$, $y_0, y_1 \in Y$, ω_f and ω_g be paths from y_0 to $f(x_0)$ and $g(x_0)$ respectively, and ω_f' and ω_g' be paths from y_1 to $f(x_0)$ and $g(x_0)$ respectively. If (f, g) is R-commutative, there is a one to one correspondence between the pair*

$(\nabla(f, g; x_0, y_0, \omega_f, \omega_g), \tilde{T}(f, g, x_0, y_0, \omega_f, \omega_g))$ and $(\nabla(f, g; x_0, y_1, \omega_f', \omega_g'), \tilde{T}(f, g, x_0, y_1, \omega_f', \omega_g'))$.

Proof: Let $C : I \to Y$ be a path from y_1 to y_0. Let ω_f and ω_g be paths from y_0 to $f(x_0)$ and $g(x_0)$ respectively, and define $\omega_f' = C \cdot \omega_f$ and $\omega_g' = C \cdot \omega_g$. Then $C_\# : \pi_1(Y, y_0) \to$

$\pi_1(Y, y_1)$ induces a bijective map from $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ to $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, which sends $\tilde{T}(f, g, x_0, y_0, \omega_f, \omega_g)$ onto $\tilde{T}(f, g, x_0, y_1, \omega'_f, \omega'_g)$. □

Theorem 2.2.11 *If (f, g) is R-commutative and has at least one essential coincidence class, then there are at least $NL(f, g)$ essential coincidence classes with the same index and hence $N(f, g) \geq NL(f, g)$.*

Proof: Choose $x_0 \in \Gamma(f, g)$ and $y_0 = f(x_0)$, and choose ω_f and ω_g be constant loops, then we have the results by Theorem 1.4.4 and the invariance theorems 2.2.9 and 2.2.10 proved above. □

Corollary 2.2.12 *If $L(f, g) \neq 0$, then $N(f, g) \geq NL(f, g)$.* □

Corollary 2.2.13 *If $NL(f, g)$ is infinite, then $N(f, g) = 0$.*

Proof: Otherwise, $N(f, g)$ is finite and $N(f, g) \geq NL(f, g)$ by Theorem 2.2.11. □

Corollary 2.2.14 *If $NL(f, g) = R(f, g)$ or equivalently $\tilde{T}(f, g; x_0, y_0, \omega_f, \omega_g) = \nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, then*

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0 \\ R(f, g) & \text{if } L(f, g) \neq 0 \end{cases}$$

Proof: By Theorem 1.4.4, all the coincidence classes have the same index. So $L(f, g) = 0$ implies all the classes have index 0 whereas $L(f, g) \neq 0$ implies they are all essential. □

In practice $T(f, g; x_0, y_0, \omega_f, \omega_g)$ and $R(f, g)$ are difficult to compute. In what follows we give conditions under which $T(f, g; x_0, y_0, \omega_f, \omega_g) = R(f, g) = Coker(g_* - f_*)$. This last set is the homology cokernel.

Definition 2.2.15 Define $J(f, \omega_f, x_0, y_0) \subset T(f, g; x_0, y_0, \omega_f, \omega_g)$ to be the set

$\{[\omega_f < F, x_0 > \omega_f^{-1}] | F$ is a loop at $f\}$.

Definition 2.2.16 A pair of maps $(f, g) : X \to Y$ is said to have the weak Jiang property if

(i) (f, g) is R-commutative;

(ii) the restriction of $p : \pi_1(Y, y_0) \to \nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ on $J(f, \omega_f, x_0, y_0)$ is onto.

Theorem 2.2.17 *If (f, g) has the weak Jiang property, then*

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0 \\ \#Coker(g_* - f_*) & \text{if } L(f, g) \neq 0 \end{cases}$$

Proof: Since $\tilde{T}(f, g, x_0, y_0, \omega_f, \omega_g) \supset p(J(f, x_0, y_0, \omega_f)) = \nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, from Corollary 2.2.14, we have

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0 \\ R(f, g) & \text{if } L(f, g) \neq 0\,. \end{cases}$$

However, (f, g) is R-commutative, so we have $R(f, g) = \#Coker(g_* - f_*)$, by Proposition 2.2.6 and we have proved the theorem. □

Corollary 2.2.18 (Brooks [Corollary 37 of [BR1]]) *If Y is a Jiang space, then*

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0 \\ \#Coker(g_* - f_*) & \text{if } L(f, g) \neq 0 \end{cases}$$

Proof: When Y is Jiang space, $\pi_1(Y, y_0)$ is commutative and hence is R-commutative, and $J(f, \omega_f, x_0, y_0) = \pi_1(Y, y_0)$, so (f, g) has the weak Jiang property. □

Lemma 2.2.19 *If $g_\pi^{\omega_g}$ is onto, then every element in $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ has a representative in $f_\pi^{\omega_f}(\pi_1(X, x_0))$.*

Proof: We will prove that for any $\alpha \in \pi_1(Y, y_0)$, there is $\gamma \in \pi_1(X, x_0)$ such that $\alpha \sim f_\pi^{\omega_f}(\gamma)$. Since $g_\pi^{\omega_g}$ is onto, there is $\gamma \in \pi_1(X, x_0)$ such that $g_\pi^{\omega_g}(\gamma) = \alpha$. Now $\alpha \sim g_\pi^{\omega_g}(\gamma^{-1})\alpha f_\pi^{\omega_f}(\gamma) = f_\pi^{\omega_f}(\gamma)$. □

Lemma 2.2.20 *If g_π is onto and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$, then (f, g) has the weak Jiang property.*

Proof: From 2.2.19, we see that $p(J(f, \omega_f, x_0, y_0)) \supset p(f_\pi(\pi_1(X, x_0))) = \nabla(f, g; x_0, y_0, \omega_f, \omega_g)$. On the other hand, $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$ implies that $f_\pi(\pi_1(X, x_0))$ is abelian. We have to prove that for any $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, $\alpha\beta\gamma \sim \beta\alpha\gamma$, i.e. (f, g) is R-commutative. Since g_π is onto, we can choose $a, b, r \in \pi_1(X, x_0)$ such that $g_\pi(a) = \alpha, g_\pi(b) = \beta$ and $g_\pi(r) = \gamma$. Then we have $\alpha\beta\gamma \sim g_\pi((abr)^{-1})(\alpha\beta\gamma)f_\pi(abr) = f_\pi(abr) = f_\pi(a)f_\pi(b)f_\pi(r) = f_\pi(b)f_\pi(a)f_\pi(r) = f_\pi(bar) \sim g_\pi(bar)f_\pi(bar)f_\pi((bar)^{-1}) = g_\pi(bar) = g_\pi(b)g_\pi(a)g_\pi(r) = \beta\alpha\gamma$. Therefore, (f, g) has the weak Jiang property. □

Note 2.2.21 We note that the hypotheses in 2.2.20 is equivalent to saying that $\tilde{T}(f, g, x_0, y_0, \omega_f, \omega_g)$ is all of $\nabla(f, g, x_0, y_0, \omega_f, \omega_g)$. However this is not as easy to verify.

Corollary 2.2.22 *If g_π is onto and $f_\pi(\pi_1(X, y_0)) \subset J(f, \omega_f, x_0, y_0)$, then*

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0 \\ \#Coker(g_* - f_*) & \text{if } L(f, g) \neq 0 \end{cases}$$

Proof: By Lemma 2.2.20 and Theorem 2.2.17. □

Example 2.2.23 Let $T = S^1 \times S^1$, and let a and b be the circles $S^1 \times 0$ and $0 \times S^1$ respectively. The paths a and b are called the standard basis of T. Then let $T_2 = T\#T$ be the connected sum of two copies of T. It has a standard basis a_1, b_1 corresponding to one copy of T and a_2, b_2 corresponding to the other. More generally, let T_n be the connected sum of n copies of T with standard basis $a_1, b_1, \ldots, a_n, b_n$. Now let $X = T_4$, $Y = T_2$. Define $f : X \to Y$ as follows: First define $f_1 : X \to T$ by sending the first factor of T in X to T, and squeezing everything else to a single point.

Next define $f_2 : T \to S^1$ to be the projection to the first factor, finally define $f_3 : S^1 \to Y$ to be the map which sends S^1 to the inverse of a_1.

Let $f = f_3 \circ f_2 \circ f_1$. Then f projects the first factor of T of X to a_1^{-1} and sends all other points to a single point.

We define g as follows: The map g sends the first two factors of X to Y by the identity, and sends the other two factors to a single point.

Then g_π is onto, $f_\pi(\pi_1(X, x_0)) =< a_1 >$ and $J(f, \omega_f, x_0, y_0) \supset < a_1 >$, and $L(f, g) = 1 - (-1) + 0 = 2$, where x_0, y_0 and ω_f can be chosen arbitrarily. So by Corollary 2.2.22, we have $N(f,g) = R_{f,g} = \#Coker(g_* - f_*) = 2$.

Example 2.2.24 Let $X = Y = T_2$, and a_1, b_1, a_2, b_2 be the standard basis. Define $f : X \to Y$ as in the example above, and define g similar to f except that g sends b_1 to b_1. Then we have $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$ and $L(f, g) = 1$, $N(f, g) = 1$. However, the image of $g_* - f_* : H_1(X) \to H_1(Y)$ is the subgroup generated by a_1 and b_1 and therefore

$\#Coker(g_* - f_*)$ is infinite. Since $R(f,g) \geq \#Coker(g_* - f_*)$, $R(f,g)$ is infinite. This shows that the hypothesis that g_π is onto is necessary in Corollary 2.2.22.

2.3 The computation of the Reidemeister and Nielsen numbers over the complement

In this section, we generalize the results in section 4 of [Z].

For each component A_k of A such that f_k, g_k map A_k into the same component B_s of B, there is a commutative diagram

$$\begin{array}{ccccc}
\pi_1(B_s, b_s) & \xrightarrow{\theta_{B_s}} & H_1(B_s) & \xrightarrow{\eta_{B_s,k}} & Coker((g_k)_* - (f_k)_* : H_1(A_k) \to H_1(B_s)) \\
\downarrow i_s & & \downarrow i_s & & \downarrow j_k \\
\pi_1(Y, b_s) & \xrightarrow{\theta_Y} & H_1(Y) & \xrightarrow{\eta_Y} & Coker(g_* - f_* : H_1(X) \to H_1(Y)),
\end{array}$$

where i_s and j_k are induced by the inclusion maps. Define $\gamma_k = \omega_g \cdot g \circ u_k \cdot \omega_{g_k}^{-1} \omega_{f_k} \cdot f \circ u_k^{-1} \cdot \omega_f^{-1}$, and define $\mu_k : H_1(B_s) \to H_1(Y)$ by $\mu_k(c) = i_s(c) + \theta_Y(\gamma_k)$. Then μ_k induces a map $\bar{\mu}_k : Coker((g_k)_* - (f_k)_*) \to Coker(g_* - f_*)$, and we have

Lemma 2.3.1 *The following diagram*

$$\begin{array}{ccc}
\nabla(f_k, g_k, x_0, y_0, \omega_{f_k}, \omega_{g_k}) & \xrightarrow{\eta_{B_s,k} \circ \theta_{B_s}} & Coker((g_k)_* - (f_k)_*) \\
\downarrow \tilde{v}_{f_k,g_k} & & \downarrow \bar{\mu}_k \\
\nabla(f, g, x_0, y_0, \omega_f, \omega_g) & \xrightarrow{\eta_Y \circ \theta_Y} & Coker(g_* - f_*)
\end{array}$$

is commutative.

Proof: We only need to prove that the following diagram is commutative.

$$\begin{array}{ccc} \pi_1(B_s, b_s) & \xrightarrow{\theta_{B_s}} & H_1(B_s) \\ \downarrow v_{f_k, g_k} & & \downarrow \mu_k \\ \pi_1(Y, y_0) & \xrightarrow{\theta_Y} & H_1(Y) \end{array}$$

Let $\gamma_k' = \omega_g \cdot g \circ u_k \cdot \omega_{g_k}^{-1}$ and $\gamma_k'' = \omega_{f_k} \cdot f \circ u_k^{-1} \cdot \omega_f^{-1}$. Then for each $[\alpha] \in \pi_1(B_s, b_s)$, we have

$$\theta_Y \circ v_{f_k, g_k}([\alpha]) = \theta_Y([\gamma_k' \cdot \alpha \cdot \gamma_k'']) = \overline{\gamma_k' \cdot \alpha \cdot \gamma_k''} = \overline{\gamma_k' + \alpha + \gamma_k''}$$

$$= \overline{\alpha + \gamma_k' + \gamma_k''} = \overline{\alpha + \gamma_k} = \overline{\alpha} + \overline{\gamma_k} = \bar{\mu}_k \circ \theta_{B_s}([\alpha]). \qquad \square$$

Theorem 2.3.2 *Let $f, g : (X, A) \to (Y, B)$ be maps. Suppose (f, g) has the weak Jiang property. If $L(f, g) = 0$, then $N(f, g; X - A) = 0$; if $L(f, g) \neq 0$, then*

$$N(f, g; X - A) = \#Coker(g_* - f_*) - \#\{\cup_{i=1}^k \bar{\mu}_i Coker((g_i)_* - (f_i)_*)\}$$

Proof: If $L(f, g) = 0$, then by Theorem 2.2.17, all the coincidence classes have zero index. So $N(f, g; X - A) = 0$.

If $L(f, g) \neq 0$, all the coincidence classes will have nonzero index and $\eta_Y \circ \theta_Y$ induces an one-to-one correspondence between $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ and $Coker(g_* - f_*)$ by Theorem 2.2.2 since (f, g) has the weak Jiang property and hence is R-commutative. By Lemma 2.3.1, an element in $Coker(g_* - f_*)$ corresponds to a weakly common coincidence class if and only if it is in the image of $\bar{\mu}_i$ for some i and the result follows. $\square$

Corollary 2.3.3 *If either Y is a Jiang space or if g_π is onto and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$, then the formula in Theorem 2.3.2 holds.* $\square$

Corollary 2.3.4 *Suppose A is connected and either Y is a Jiang space, or g_π is onto and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$. If $L(f,g) = 0$, then $N(f,g; X - A) = 0$; if $L(f,g;) \neq 0$, then*

$$N(f,g; X - A) = \#Coker(g_* - f_*) - \#(i_B)_* Coker((g_A)_* - (f_A)_*)$$

Proof: Under the hypothesis, (f,g) has the weak Jiang property. If $L(f,g) = 0$, then $N(f,g; X - A) = 0$ by Theorem 2.2.17. If $L(f,g) \neq 0$, then all Nielsen classes are essential. Since the property of being R-commutative is independent of the choice of ω_f, ω_g, x_0 and y_0, we may assume $x_0 \in A$, $y_0 \in B$ and $\omega_f, \omega_g \subset B$, and we have that $\eta_Y \circ \theta_Y : \nabla(f, g, x_0, y_0, \omega_f, \omega_g) \to Coker(g_* - f_*)$ is bijective. Now we can choose $a = x_0$, $b = y_0$, $\omega_{f_A} = \omega_f$ and $\omega_{g_A} = \omega_g$ and the map $\tilde{\mu}$ is equal to i_*. □

Theorem 2.3.5 *Let $(f,g) : (X, A) \to (Y, B)$ be a pair of maps with $\hat{A} = \cup_{i=1}^k A_i$, such that (f,g) and (f_i, g_i) have the weak Jiang property for all components A_i of $\hat{A}$. If $L(f,g) \cdot \Pi_{i=1}^m L(f_i, g_i) \neq 0$ and $L(f_i, g_i) = 0$ for $m < i \leq k$, then*

$$\begin{aligned} N(f,g; X, A) &= N(f,g) + N(f_A, g_A) - N(f,g; f_A, g_A) \\ &= \#Coker(g_* - f_*) + \sum_{i=1}^m \#Coker((g_i)_* - (f_i)_*) \\ &\quad - \#\{\cup_{i=1}^m \tilde{\mu}_i Coker((g_i)_* - (f_i)_*)\}. \end{aligned}$$

Proof: By Theorem 2.2.17, we have $N(f,g) = \#Coker(g_* - f_*)$ and $N(f_A, g_A) = \sum_{i=1}^m N(f_i, g_i) = \sum_{i=1}^m \#Coker((g_i)_* - (f_i)_*)$. A coincidence class of (f,g) is a common essential coincidence class if and only if it contains a coincidence class of (f_i, g_i) for some

$1 \leq i \leq m$ since each coincidence class of (f, g) and (f_i, g_i) with $1 \leq i \leq m$ is essential, and none of (f_i, g_i) with $m < i \leq k$ is essential. By Lemma 2.1.10 and 2.3.1, such a class corresponds to an element of $\{\cup_{i=1}^{m} \bar{\mu}_i Coker((g_i)_* - (f_i)_*)\}$ and the result follows. □

Corollary 2.3.6 *Suppose A is connected and $L(f, g) \cdot L(f_A, g_A) \neq 0$, then if the one the following is valid,*

(i) Y and B are Jiang spaces.

(ii) Y is a Jiang space and $(g_A)_\pi$ is onto and $(f_A)_\pi(\pi_1(A, a)) \subset J(f_A, \omega_f, x_0, y_0)$.

(iii) g_π is onto and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$ and B is a Jiang space.

(iv) g_π and $(g_A)_\pi$ are onto, and $f_\pi(\pi_1(X, x_0)) \in J(f, \omega_f, x_0, y_0)$ and $(f_A)_\pi(\pi_1(A, x_0)) \subset J(f_A, \omega_f, x_0, y_0)$.

then we have

$$N(f, g; X, A) = \#Coker(g_* - f_*) + \#Coker((g_A)_* - (f_A)_*)$$

$$-\#i_* Coker((g_A)_* - (f_A)_*).$$

Proof: As in the proof of Corollary 2.3.4, we may assume that $x_0 \in A$, $y_0 \in B$ and $\omega_f, \omega_g \subset B$. The result then follows from Theorem 2.3.5. □

2.4 Manifolds with boundary

In this section, we will extend our results to the manifolds with boundary. Unlike the fixed point case the extension of the coincidence index to manifolds with boundary is not entirely straightforward. The method used here is based on [BS].

We will first give a brief description of the definition of the index of a coincidence point set in [BS] and give an extension of Theorem 5.8 in [BS], then we introduce a new invariant for a pair of maps from a manifold with boundary to another one preserving boundaries.

Let X and Y be oriented manifolds with boundaries ∂X and ∂Y respectively. Assume that $f : X \to Y$ and $g : (X, \partial X) \to (Y, \partial Y)$ are maps. Note that we do not require $f(\partial X) \subset \partial Y$ here. Let $(-X)$ be a copy of X with opposite orientation. For each point $x \in X$ we denote the corresponding element in $(-X)$ by $-x$. The double $2X$ of X is the oriented manifold without boundary obtained from $X \cup (-X)$ by identifying each $x \in \partial X$ with $-x \in (-\partial X)$. Let $i_{Y \subset 2Y} : Y \to 2Y$ be the inclusion of Y into its double. Let $r_X : 2X \to X$ be the retraction defined by $r(x) = r(-x) = x$. Define $\hat{f} = i_{Y \subset 2Y} f r_X : 2X \to 2Y$ and $2g : 2X \to 2Y$ by $2g(x) = g(x)$ and $2g(-x) = -g(x)$. Now $\hat{f}$ and $2g$ are maps from a manifold without boundary to another one and therefore the Nielsen number can be defined for $(\hat{f}, 2g)$. It is proved in [BS] that the coincidence classes of (f, g) are identical to those of $(\hat{f}, 2g)$. Hence the index of a class α of (f, g) is defined to be the index of the corresponding class of $(\hat{f}, 2g)$ and the Nielsen number $N(f, g)$ of (f, g) is defined to be $N(\hat{f}, 2g)$.

Let $\hat{D}_q(X) : H^{n-q}(X) \to H_q(X, \partial X)$ be the Lefschetz duality isomorphism. Let $\hat{\theta}^{n-q}(f, g) : H^{n-q}(X) \to H^{n-q}(X)$ be the composition

$$H^{n-q}(X) \xrightarrow{\hat{D}_q(X)} H_q(X, \partial X) \xrightarrow{g_*} H_q(Y, \partial Y) \xrightarrow{\hat{D}(Y)^{-1}} H^{n-q}(Y) \xrightarrow{f^*} H^{n-q}(X)$$

and the B-Lefschetz coincidence number $L_B(f, g)$ be

$$L_B(f, g) = \Sigma_{q=0}^{n} (-1)^{n-q} \mathrm{Tr}[\hat{\theta}^{n-q}(f, g)].$$

It is proven that $L_B(f, g) = L(\hat{f}, 2g)$ and $I(X; f, g) = L_B(f, g)$ (cf. [BS]).

Before we extend Theorem 5.8 in [BS], we first show that $\mathcal{R}_{f,g}$ can be considered as a subset of $\mathcal{R}_{\hat{f},2g}$, in other words, we have an injective map from $\mathcal{R}_{f,g}$ to $\mathcal{R}_{\hat{f},2g}$ induced by the inclusion map. Let $x_0 \in X$, $y_0 \in Y$ be the base points. The inclusion map $i : Y \to 2Y$ induces a homomorphism $(i_{Y\subset 2Y})_\pi$ from $\pi_1(Y, y_0)$ to $\pi_1(2Y, y_0)$. If we choose $\omega_{\hat{f}} = \omega_f$ and $\omega_{2g} = \omega_g$, we have a natural map induced by i_π from $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ to $\nabla(\hat{f}, 2g; x_0, y_0, \omega_{\hat{f}}, \omega_{2g})$, which we will denote by $(i_{Y\subset 2Y})_\nabla$.

Lemma 2.4.1 *$(i_{Y\subset 2Y})_\nabla : \nabla(f, g; x_0, y_0, \omega_f, \omega_g) \to \nabla(\hat{f}, 2g; x_0, y_0, \omega_{\hat{f}}, \omega_{2g})$ is injective.*

Proof: It is easy to see that $(i_{Y\subset 2Y})_\pi : \pi_1(Y, y_0) \to \pi_1(2Y, y_0)$ is one to one, since there is a retraction $r_Y : 2Y \to Y$ such that $r_Y \circ i_{Y\subset 2Y} = id_Y$.

Assume the elements $\alpha, \beta \in \pi_1(Y, y_0)$ are mapped into the same Reidemeister class of $(\hat{f}, 2g)$. We want to prove that α and β are in the same Reidemeister class of (f, g) too. By assumption, there is an element $\gamma \in \pi_1(2X, x_0)$ such that $\alpha = (2g)_\pi(\gamma) \cdot \beta \cdot \hat{f}_\pi(\gamma^{-1})$, or $\alpha^{-1} \cdot (2g)_\pi(\gamma) \cdot \beta \cdot \hat{f}_\pi(\gamma^{-1}) = 1$. Applying $(r_Y)_\pi$ to both sides, we have

$$(r_Y)_\pi(\alpha^{-1})(r_Y)_\pi((2g)_\pi(\gamma))(r_Y)_\pi(\beta)(r_Y)_\pi(\hat{f}_\pi(\gamma)) = 1.$$

Note $(r_Y)_\pi \circ \hat{f}_\pi = f_\pi \circ (r_X)_\pi$ and $(r_Y)_\pi \circ (2g)_\pi = g_\pi \circ (r_X)_\pi$, and $(r_Y)_\pi(\alpha) = \alpha$, $(r_Y)_\pi(\beta) = \beta$, since α, β are both in Y. Hence, we have

$$\alpha^{-1}(g)_\pi((r_X)_\pi(\gamma))\beta f_\pi(((r_X)_\pi(\gamma))^{-1}) = 1,$$

or

$$\alpha = (g)_\pi((r_X)_\pi(\gamma))\beta f_\pi(((r_X)_\pi(\gamma))^{-1}).$$

Since $(r_X)_\pi(\gamma) \in \pi_1(X, x_0)$, we proved the lemma. □

The following theorem is an extension of Theorem 5.8 in [BS].

Theorem 2.4.2 *Suppose $f : X \to Y$ and $g : (X, \partial X) \to (Y, \partial Y)$ are maps of manifolds with boundary. Suppose that g_π is onto, and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$. If $L_B(f, g) = 0$, then $N(f, g) = 0$; if $L_B(f, g) \neq 0$, then $N(f, g) = \#Coker(g_* - f_*)$.*

Proof: We will prove that there are only two cases, namely $L_B(f, g) = 0$ and $N(f, g) = 0$. and $L_B(f, g) \neq 0$ and $N(f, g) = \#Coker(g_* - f_*)$.

If there is no essential class, then $L_B(f, g) = 0$ and $N(f, g) = 0$.

Otherwise, assume that x_0 is in an essential class α of (f, g) and ω_f and ω_g are constant loops at $y_0 = f(x_0)$. It is obvious that $(i_{Y\subset 2Y})_\pi(J(f, \omega_f, x_0, y_0)) \subset J(\hat{f}, \omega_f, x_0, y_0)$. Then $\tilde{T}(\hat{f}, 2g, x_0) \supset p(J(\hat{f}, \omega_f, x_0, y_0)) \supset p((iY \subset 2Y)_\pi(J(f, \omega_f, x_0, y_0))) \supset (i_{Y\subset 2Y})_\nabla(\nabla(f, g, x_0, y_0, \omega_f, \omega_g))$, where p is the projection from $\pi_1(Y, y_0)$ to $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$. The last inclusion is due to Lemma 2.2.19. By Theorem 1.4.3, $N(\hat{f}, 2g) \geq \#\tilde{T}(\hat{f}, 2g, x_0) \geq \#(i_*(\nabla(f, g, x_0, y_0, \omega_f, \omega_g)))$. Since i_* is injective, we have $N(\hat{f}, 2g) \geq R(f, g) \geq Coker(g_* - f_*)$. By Lemma 5.1 of [BS], we have $N(f, g) \geq \#Coker(f_* - g_*)$. However, $N(f, g) \leq R(f, g)$ is always true and by Lemma 2.2.20 and 2.2.6, $R(f, g) = \#Coker(f_* - g_*)$. So we have $N(f, g) = \#Coker(f_* - g_*)$. Since each class has the same index, we have $L_B(f, g) \neq 0$. □

Note 2.4.3 For closed manifolds Corollary 2.2.22 is symmetric in f and g. However, as we show in the following example, Theorem 2.4.2 is not symmetric in f and g.

Example 2.4.4 Let $X = Y = S^1 \times S^1 - D^2$, and let a, b be the standard basis of $\pi_1(X)$. Let f be the identity. Define map g as follows:

Define $g_1 : X \to S^1 \times S^1 \vee I$ by squeezing a collar of the boundary to I; $g_2 : S^1 \times S^1 \vee I \to S^1 \vee I$ by projecting T to the second factor and sending I to I by identity. $g_3 : S^1 \vee I \to Y$ by sending S^1 to $-b$ and send I to an arc from ∂Y to b. Let $g = g_3 \circ g_2 \circ g_1$, then f_π is onto, $g_\pi(\pi_1(X)) \subset J(g)$, $L_B(f, g) = 1$ and $N(f, g) = 1$ (the set of coincidence points consists of the arc from ∂Y to b and another point in b with index 1. Those coincidence points on the arc can be removed by deforming f along with the arc). However. $R(f, g) \geq \#Coker(g_* - f_*) = 2$. So $N(f, g) \neq R(f, g)$.

As before, let X and Y be manifolds with boundary. Now however we assume that $f, g : (X, \partial X) \to (Y, \partial Y)$ are maps. i.e. both f and g are boundary preserving.

Lemma 2.4.5 $N(2f, \hat{g}; 2X, \partial X) = N(2g, \hat{f}; 2X, \partial X)$.

Proof: Without loss of generality, we may assume that f is transversal to the boundary ∂Y of Y, and that there are only finite number of coincidence points of (f, g). Let $S_1, \ldots, S_l, S_{l+1}, \ldots, S_p$ be the non-empty Nielsen classes of $(\partial f, \partial g)$, such that $S_1, \ldots, S_l$ are essential and $S_{l+1}, \ldots, S_p$ are inessential. Let $N_1, \ldots, N_s, N_{s+1}, \ldots N_t, N_{t+1}, \ldots, N_k$ be the non-empty Nielsen classes of (f, g), arranged in an order such that (i) each N_i in $N_1, \ldots, N_s$ does not contain any S_j, i.e. it is contained in int(X); and (ii) each N_i of $N_{s+1}, \ldots N_t$ contains at least one class of $S_1, \ldots, S_l$, i.e. contains an essential classes of $(\partial f, \partial g)$; and (iii) each of $N_{t+1}, \ldots N_k$ contains at least one class of $S_{l+1}, \ldots, S_k$, i.e. contains an inessential class of $(\partial f, \partial g)$ and does not contain any of $S_1, \ldots, S_l$. So the relative Nielsen number is the sum of l, the number of essential classes in $N_1, \ldots N_s$, and the number of essential classes in $N_{t+1}, \ldots N_k$.

Since for $1 \leq i \leq s$, N_i is contained in int(X), by the definition of index and Lemma 5.16 of [V], $\text{ind}(N_i; \hat{f}, 2g) = (-1)^n \text{ind}(N_i; \hat{g}, 2f)$. That is, (as far as the essentiality is concerned), the two ways defining the index of N_i are the same. Since each $N_{s+1}, \ldots N_t$ contains an essential class of $(\partial f, \partial g)$, these classes do not contribute anything to the relative Nielsen number. So we only need to prove the essentiality of each N_i in the list $N_{t+1}, \ldots, N_k$ is the same when we use $(\hat{f}, 2g)$, and when we use $(\hat{g}, 2f)$.

Now let $N_i = A \cup B$ be a Nielsen class with $t+1 \leq i \leq k$, where A is in int(X), and B is in ∂X. As above, $\text{ind}(A; \hat{f}, 2g) = (-1)^n \text{ind}(A; \hat{g}, 2f)$, and we will prove that $\text{ind}(B; \hat{f}, 2g) = (-1)^n \text{ind}(B; \hat{g}, 2f)$, so that $\text{ind}(N_i; \hat{f}, 2g) = (-1)^n \text{ind}(N_i; \hat{g}, 2f)$.

Assume that $B = \{x_1, \ldots, x_{u_i}\}$, and let $v_k = \text{ind}(x_k; \partial f, \partial g)$. By the assumption on N_i, $\sum_{j=1}^{u_i} v_j = \sum_{j=1}^{u_i} \text{ind}(x_j; \partial f, \partial g) = 0$. Consider an x_k with $1 \leq k \leq u_i$. For simplicity, we assume that x_k is the origin of $\mathbf{R}^n$, and $\hat{f}, \hat{g}, 2f, 2g$ are maps from $\mathbf{R}^n$ to $\mathbf{R}^n$ such that f and g map $\mathbf{R}^n_+ = \{(a_1, \ldots, a_n) \mid a_n \geq 0\}$ to $\mathbf{R}^n_+$. Let

$$h_1 = \frac{\hat{f} - 2g}{|\hat{f} - 2g|} \text{ and } h_2 = \frac{\hat{g} - 2f}{|\hat{g} - 2f|},$$

and note that $H_{n-1}(S^{n-1}, S^{n-2}) = H_{n-1}(D^{n-1}_+, S^{n-2}) \oplus H_{n-1}(D^{n-1}_-, S^{n-2})$. Then for $j = 1, 2$, we have commutative diagrams

$$\begin{array}{ccc}
H_{n-1}(S^{n-1}) \xrightarrow{i_*} & H_{n-1}(D^{n-1}_+, S^{n-2}) \oplus H_{n-1}(D^{n-1}_-, S^{n-2}) \xrightarrow{\partial} & H_{n-2}(S^{n-2}) \\
\downarrow (h_j)_* & \downarrow \overline{(h_j)}_* & \downarrow \partial(h_j)_* \\
H_{n-1}(S^{n-1}) \xrightarrow{i_*} & H_{n-1}(D^{n-1}_+, S^{n-2}) \oplus H_{n-1}(D^{n-1}_-, S^{n-2}) \xrightarrow{\partial} & H_{n-2}(S^{n-2}) .
\end{array}$$

In the diagrams, i_* sends the generator 1 to (1,1), and ∂ sends (λ, η) to $\lambda - \eta$. Note that $\mathrm{ind}(x_k; \hat{f}, 2g)$ is equal to the degree of h_1, $\mathrm{ind}\,(x_k; \hat{g}, 2f)$ is equal to the degree of h_2, $\mathrm{ind}(x_k; \partial f, \partial g)$ is equal to the degree of ∂h_1, and $\mathrm{ind}\,(x_k; \partial g, \partial f)$ is equal to the degree of ∂h_2. Because of the definition of $\hat{f}$ and $2g$ on D_-^{n-1} and the commutativity of the diagram, all the points in D_-^{n-1} are sent to D_+^{n-1} and hence $\overline{(h_1)}_*$ sends (0, 1) to $(-v_k, 0)$ by the commutativity of the right square of the diagram. Let $\overline{(h_1)}_*((1,0)) = (\eta_k, \lambda_k)$. Then by the commutativity of the right square of the diagram, we have that $\lambda_k = \eta_k - v_k$ The commutativity of the left square says that $i_*(h_1)_*(1) = (\deg(h_1), \deg(h_1))$ is equal to $\overline{(h_1)}_*(1,1) = (\eta_k, \lambda_k) + (-v_k, 0) = (\eta_k - v_k, \lambda_k)$, so $\deg(h_1) = \lambda_k$. Note that $\overline{(h_2)}_*$ sends $(0,1)$ to $(-v_i, 0)$, for the same reason that $\overline{(h_1)}_*$ does. Since on D_+^{n-1}. $h_1 = (-1)h_2$, $\overline{(h_2)}_*$ sends $(1,0)$ to $((-1)^n\lambda_k, (-1)^n\eta_k)$ (this is because that on $H_{n-1}(D_+^{n-1}, S^{n-2})$, $(h_2)_* = (\alpha)_* \circ (h_1)_*$, where $\alpha : S^{n-1} \to S^{n-1}$ is the antipodal map defined by $\alpha(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$, and $(\alpha)_*((a,b)) = ((-1)^n b, (-1)^n a)$ for any $(a,b) \in H_{n-1}(D_+^{n-1}, S^{n-2}) \oplus H_{n-1}(D_-^{n-1}, S^{n-2})$). Also, we have that $(-1)^n\eta_k = (-1)^n\lambda_k - v_k$ and the degree of h_2 is $(-1)^n\eta_k$. Now $\mathrm{ind}(B; \hat{f}, 2g) = \sum_{j=1}^{u_1} \mathrm{ind}(x_j; \hat{f}, 2g) = \sum_{j=1}^{u_1} \lambda_j$ and $\mathrm{ind}(B; \hat{g}, 2f) = \sum_{j=1}^{u_1} \mathrm{ind}(x_j; \hat{g}, 2f) = \sum_{j=1}^{u_1} (-1)^n \eta_j = \sum_{j=1}^{u_1} ((-1)^n\lambda_j - v_j) = \sum_{j=1}^{u_1} (-1)^n\lambda_j - \sum_{j=1}^{u_1} v_j = \sum_{j=1}^{u_1} (-1)^n\lambda_j = (-1)^n \mathrm{ind}(B; \hat{f}, 2g)$ since $\sum_{j=1}^{u_1} v_j = 0$. □

The theorem allows us to define $N(f, g; X, \partial X) = N(\hat{f}, 2g; 2X, \partial X)$. The definition of $N(f, g; X - \partial X)$ is the same as before. With these definitions, most of the results in Section 2.3 are still valid. As an example, we have the following.

Theorem 2.4.6 *Let X, Y be manifolds with boundary with the same dimension, and ∂X and ∂Y connected. Let $f, g : (X, \partial X) \to (Y, \partial Y)$ be maps. Suppose g_π is onto, and $f_\pi(\pi_1(X, x_0)) \subset J(f, \omega_f, x_0, y_0)$. If $L_B(f,g) = 0$, then $N(f, g; X - \partial X) = 0$; if $L_B(f,g) \neq 0$,* *then* $N(f,g;X-\partial X) = \#Coker(g_* - f_*) - \#(i_{\partial Y \subset Y})_* Coker((\partial g)_* - (\partial f)_*)$.

Proof: Since ∂X is connected, we may assume $x_0 \in \partial X$ and $y_0 \in \partial Y$, and that ω_f, ω_g are constant paths. If $L_B(f,g) = 0$, then $N(f,g) = 0$ by Theorem 2.4.2. Since $N(f,g;X-\partial X) \leq N(f,g)$, we have $N(f,g;X-\partial X) = 0$. If $L_B(f,g) \neq 0$, from the proof of Theorem 2.4.2, we can see that there are $\#Coker(g_* - f_*)$ essential Nielsen classes of (f,g) that correspond to elements in $(i_{Y\subset 2Y})_\nabla(\nabla(f,g,x_0,y_0,\omega_f,\omega_g))$. The weakly common essential classes correspond to elements in $(i_{\partial Y\subset 2Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g))$. So $N(f,g;X-\partial X) = \#(i_{Y\subset 2Y})_\nabla(\nabla(f,g,x_0,y_0,\omega_f,\omega_g)) - \#(i_{\partial Y\subset 2Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g))$. Since $(i_{\partial Y\subset 2Y})_\nabla$ is injective by Lemma 2.4.1, we have $\#(i_{Y\subset 2Y})_\nabla(\nabla(f,g,x_0,y_0,\omega_f,\omega_g)) = \# \nabla\ (f,g,x_0,y_0,\omega_f,\omega_g)$ and

$\#(i_{\partial Y\subset 2Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g)) = \#(i_{Y\subset 2Y})_\nabla \circ (i_{\partial Y\subset Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g)) = \#(i_{\partial Y\subset Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g))$. Under the assumptions of the theorem, we have $\# \nabla\ (f,g,x_0,y_0,\omega_f,\omega_g) = \#Coker(g_* - f_*)$, and $\#(i_{\partial Y\subset Y})_\nabla(\nabla(\partial f,\partial g,x_0,y_0,\omega_f,\omega_g)) = \#(i_{\partial Y\subset Y})_* Coker((\partial g)_* - (\partial f)_*)$, and the result follows. □

Similarly, we have

Theorem 2.4.7 *Let X, Y be manifolds with boundary, and ∂X and ∂Y connected. Let $f,g : (X,\partial X) \to (Y,\partial Y)$ be maps. Suppose that $L_B(f,g) \neq 0$, $L(\partial f,\partial g) = 0$ and $(\partial f,\partial g)$ has the weak Jiang property. If $J(f,\omega_f,x_0,y_0) = \pi_1(Y,y_0)$, then $N(f,g;X,\partial X) = R_{f,g}$. If (f,g) has the weak Jiang property, then $N(f,g;X,\partial X) = \#Coker(g_* - f_*)$.* □

Note: In 2.4.7, we can not change the condition $J(f,x_0,y_0,\omega_f) = \pi_1(Y,y_0)$ to $T(f,g,x_0,y_0,\omega_f,\omega_g) = \pi_1(Y,y_0)$. The following example illustrates this.

Example 2.4.8 Let $X = Y = S^1 \times S^1 - D^2$, and let a and b be the bases. Define $h : X \to S^1 \times S^1 \vee I$ by squeezing a collar of the boundary to I. Let $f_1 : S^1 \times S^1 \vee I \to S^1 \vee I$ be the map which projects $S^1 \times S^1$ to the first factor, and $g_1 : S^1 \times S^1 \vee I \to S^1 \vee I$ be the one which projects $S^1 \times S^1$ to the second factor. Let $f_2 : S^1 \vee I \to Y$ send S^1 to a and I to a path from the boundary to a, and let $g_2 : S^1 \vee I \to Y$ send S^1 to b and I to a path from the boundary to b. Now define $f = f_2 \circ f_1 \circ h$ and $g = g_2 \circ g_1 \circ h$.

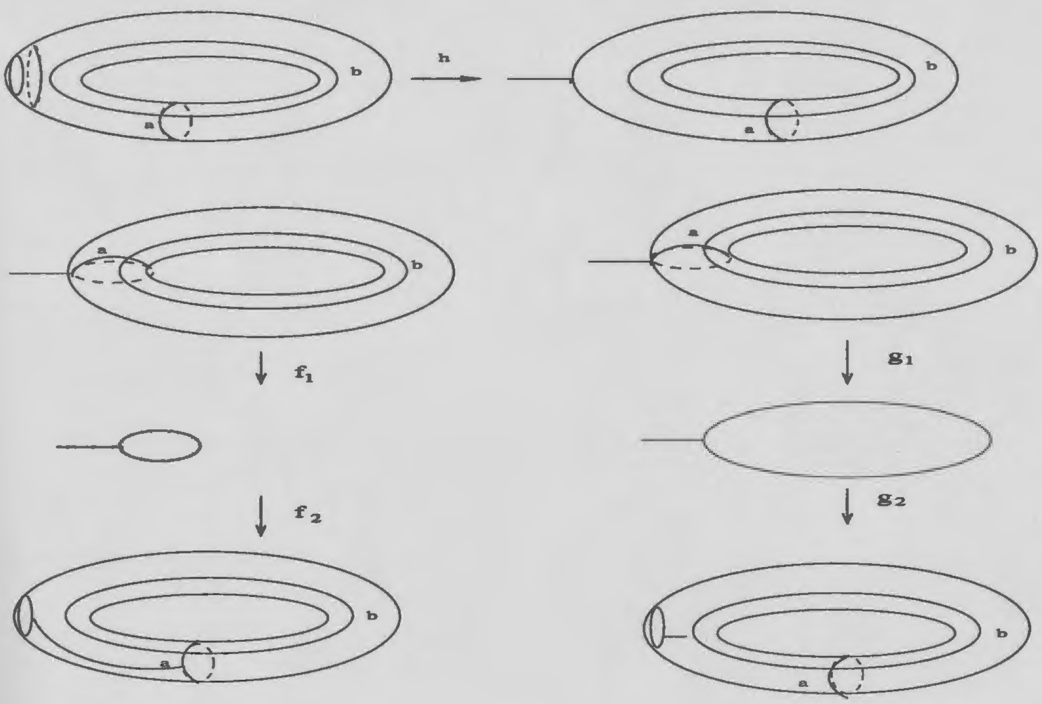

Figure 2.2:

It is easy to see that $N(f, g; X, \partial X) = 1$ (one coincidence point with index 1), and $N(\partial f, \partial g) = 0$. However, $R(f, g) \geq 2$. To see this, first note that $\pi_1(Y, y_0)$ is a free group of rank 2 of which a and b are the generators, and $f_\pi(a) = a$, $f_\pi(b) = 0$, $g_\pi(a) = 0$, and $g_\pi(b) = b$. We claim that ab and ba are in different f_π, g_π-congruence classes. Otherwise, we have $ba = g_\pi(l)abf_\pi(l^{-1})$=$b^s aba^t$ for some integers s and t. Then we have $b^{s-1}aba^{t-1} = 0$, which contradicts $\pi_1(Y, y_0)$ being a free group.

2.5 The minimum theorem

Lemma 2.5.1 *Let X, Y be manifolds, $A \subset X$, $B \subset Y$ submanifolds, and $f, g : (X, A) \to (Y, B)$ maps. Assume that A can be bypassed in X* (see Definition 1.6.7), *then a coincidence point $x \in \Gamma(f, g)$ belongs to a weakly common coincidence class if and only if there is a path $\alpha : (I, 0, I - \{1\}, 1) \to (X, x, X - A, A)$ from x to A such that $f \circ \alpha \sim g \circ \alpha : (I, 0, 1) \to (Y, f(x), B)$. Moreover, when f, g have only finite number of coincidence points, we may choose either that $\alpha(1) \notin \Gamma(f, g)$ or that the homotopy has the form $g \circ \alpha \sim f \circ \alpha : (I, 0, 1) \to (Y, f(x), f \circ \alpha(1))$.*

Proof: The proof is similar to the proof of Lemma 3.5 in [Z]. □

Lemma 2.5.2 *Suppose the dimension of A_k is greater than or equal to 2. For $x \in \Gamma(f, g)$, if there is a path $C : (I, 0, 1) \to (X, x, A_k)$ from x to A_k such that $g \circ C \overset{H}{\sim} f \circ C : (I, 0, 1) \to (Y, f(x), B)$, then for any point $a \in A_k - \Gamma(f, g)$, there exist maps f' and g' with $f' \sim f$ and $g' \sim g$ relative to $X - U(a)$, where $U(a)$ is a neighbourhood of a in X, and such that $\Gamma(f', g') = \Gamma(f, g) \cup \{a\}$, and x, a are in the same class.*

Proof: Let $l = H(1, \cdot)$, then l is a path from $g \circ C(1)$ to $f \circ C(1)$ in B. We may assume without loss of generality that $C(1) = a$. If $a \neq C(1)$, let $\alpha : I \to A_k$be a path from $C(1)$ to a. Then since $(g \circ C) \cdot (g \circ \alpha) \cdot (g \circ \alpha^{-1}) \cdot l \cdot (f \circ \alpha^{-1}) \cdot (f \circ \alpha) \cdot (f \circ C^{-1}) \sim (g \circ C) \cdot l \cdot (f \circ C^{-1}) \sim 0$ and $(g \circ \alpha^{-1}) \cdot l \cdot (f \circ \alpha) \subset B$, we can replace C by $C \cdot \alpha$. Let α_1, α_2 be paths in B such that

$\alpha_1(0) = g \circ C(1), \alpha_2(0) = f \circ C(1), \alpha_1(1) = \alpha_2(1)$ and $\alpha_1 \cdot \alpha_2^{-1} \sim l$ rel $\{0,1\}$, then it is easy to check that $(g \circ C) \cdot \alpha_1 \sim (f \circ C) \cdot \alpha_2$ rel $\{0,1\}$. Since dim $A_k \geq 2$, we may assume that for any $t \neq 1$, $\alpha_1(t) \neq \alpha_2(t)$. Let $U(a)$ be a neighbourhood of a in X such that there is a homeomorphism $\phi : (\overline{U(a)}, \overline{U(a)} \cap A_k) \to (D^n, D^m)$, where D^n, D^m are the closed unit balls in $\mathbf{R}^n$ and $\mathbf{R}^m \subset \mathbf{R}^n$ with n equal to the dimension of X and m equal to the dimension of A_k, and $U(a) \cap \Gamma(f,g) = \emptyset$. Then we can label each point $z \in U(a)$ by $z = (t,x)$, where $t \in I$ and $x \in \partial\overline{U(a)}$. Note that $z = (x,0)$ represents the center of the ball, for any x.

Define

$$f'(z) = \begin{cases} f(z) & \text{if } z \in X - U(a) \\ f((2t-1,x)) & \text{if } z = (t,x) \in U(a) \text{ and } t \geq 1/2 \\ \alpha_2(1-2t) & \text{if } z = (t,x) \in U(a) \text{ and } t \leq 1/2 \end{cases}$$

Define g' similarly by replacing f by g and α_2 by α_1. Obviously, $f' \sim f$. In fact, we can define the homotopy F as follows:

$$F(s,z) = \begin{cases} f(z) \text{ if } z \in X - U(a) \\ f((\frac{t-(s/2)}{1-(s/2)}, x)) \text{ if } z = (t,x) \text{ and } t \geq s/2 \\ \alpha_2(s-2t) \text{ if } z = (t,x) \text{ and } t \leq s/2. \end{cases}$$

And using this homotopy, we find that $f' \circ C \sim f \circ (C \cdot \alpha_2)$ rel $\{0,1\}$. To see this, let $C_0(t) = (C(t),0), C_1(t) = (C(t),1), C_x(t) = (x, 1-t), C_a(t) = (a,t)$ for paths in $X \times I$, then $C_1 \sim C_x \cdot C_0 \cdot C_a$, therefore $f' \circ C = F \circ C_1 \sim F \circ (C_x \cdot C_0 \cdot C_a) = e_{f(x)} \cdot (F \circ C_0) \cdot (F \circ C_a) \sim (F \circ C_0) \cdot (F \circ C_a) \sim (f \circ C) \cdot \alpha_2$, where $e_{f(x)}$ is the constant path at $f(x)$.

With the same argument, we have $g' \circ C \sim g \circ C \cdot \alpha_1$. Therefore, $f' \circ C \sim g' \circ C$, i.e. x and a are in the same coincidence class of f', g'. □

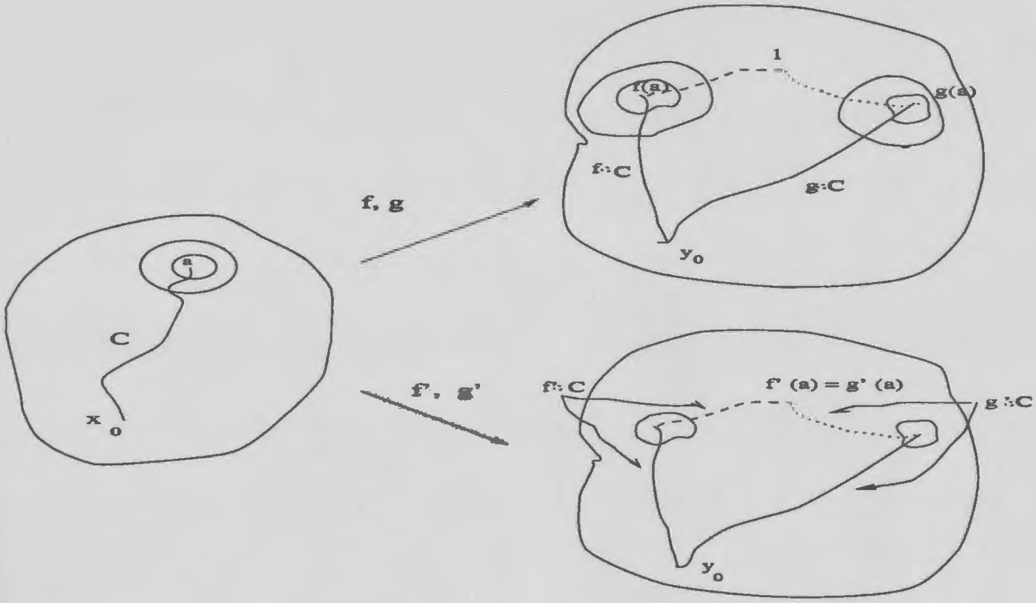

Figure 2.3:

Theorem 2.5.3 *Let $A \subset X$ be a submanifold such that A can be bypassed in X (cf. Definition 1.6.7), $B \subset Y$ be a submanifold, and $f, g : (X, A) \to (Y, B)$ be maps. If dim$(X) \geq 3$ and dim$(A) \geq 2$, then there are maps f' and g', from (X, A) to (Y, B), such that $f' \sim f$, $g' \sim g$ and (f', g') has $N(f, g; X - A)$ coincidence points on $X - A$.*

Proof: By Lemma 1.6.2, we can assume that there are only finite number of coincidence points on X. Since A can be bypassed in X, we can coalesce the coincidence points on $X - A$, which are in the same classes, by Lemma 1.6.4. We may therefore assume that each class contains at most one coincidence point on $X - A$, and each of them has non-zero index by Lemma 1.6.5. Now let $x_1 \in X - A$ be a coincident point, which is in a weekly common coincidence class. If there is a coincidence point $a \in A_k$, such that $x_1 \sim a$, let β be an arc from a to x_1, which demonstrates the equivalence, such that $\beta((0, 1]) \subset X - A$, and U be a neighborhood of $\beta((0, 1])$ such that $U \subset X - A$ and $\overline{U} \cap \Gamma(f, g) = a$. Since A is a submanifold of X such arc exists. By Lemma 1.6.4, we have $f' \sim f$ and $g' \sim g$ relative to $X - U$ such that $\Gamma(f', g') = \Gamma(f, g) - \{x_1\}$. If there is no such coincident point a, by 2.5.1, there is a path α from x_1 to A_k, a component of A, $\alpha : (I, 0, I - \{1\}, 1) \to (X, x_1, X - A, A_k)$ with $f \circ \alpha \sim g \circ \alpha$. Let $a_1 = \alpha(1)$, then by Lemma 2.5.2, there is an $f_1 \sim f$, such that $\Gamma(f_1, g) = \Gamma(f, g) \cup \{a_1\}$. Therefore, we have a coincidence point $a_1 \in A$ such that x_1 and a_1 are in the same class. Then we can find an arc β from a_1 to x_1 and a neighborhood U of $\beta((0, 1])$ as above, and by Lemma 1.6.4, we can get maps $f' \sim f_1$ and $g' \sim g$ such that $\Gamma(f', g') \cap (X - A) = \Gamma(f, g) \cap (X - A) - \{x_1\}$. □

Theorem 2.5.4 *If dim $A \geq 3$, and A can be bypassed in X, then there are maps $f' \sim f$ and $g' \sim g$, such that (f, g) has $N(f, g; X, A)$ coincidence points in X and $N(f, g; X - A)$ coincidence points on $X - A$.*

Proof: By Theorem 2.4 of [JJ], we can assume that (f, g) has $N(f, g; X, A)$ coincidence points. By Theorem 2.5.3, we can move any coincidence point $x \in X - A$ in a weakly common coincidence class to A. □

Chapter 3

A Local and Relative Version of a Brooks' Theorem

From the previous chapter, we see that in a sense, coincidence theory is more flexible than fixed point theory. In particular when we proved the minimum theorem, we were allowed to deform both f and g. Because of this, the minimum theorem in the fixed point theory is not in general a special case of coincidence theory. Brooks' theorem (see [BR2]) partially remedies this deficit. It says that if Y is a manifold and $f, g : X \to Y$ are maps, then for any $f', g' : X \to Y$ with $f' \sim f$ and $g' \sim g$, there is an $f'' \sim f$ such that $\Gamma(f'', g) = \Gamma(f', g')$. This makes the coincidence theory a "real" generalization of the fixed point theory when Y is a manifold.

In this chapter, we will give a local and relative version of the Brooks' theorem. This result will enable us to see that the results in the previous chapter, and results in the next chapter are a generalization of the corresponding results in fixed point theory.

We consider first the main idea in Brooks' paper [BR2]. A map from X to Y is equivalent to a graph in $X \times Y$. If we view $p_X : X \times Y \to X$ as a (trivial) bundle over X, this graph can be viewed as a section to this bundle. If f and g are maps from X to Y, then $\Gamma(f, g) = p_X(graph(f) \cap graph(g))$. A homotopy of f and g corresponds to a section of the bundle $p_{X\times I} : X \times I \times Y \to X \times I$. Now suppose that F, G are homotopies from f and g respectively. Let $F_1 : X \times I \to Y$ be a homotopy defined by $F_1(x,t) = F(x,1)$. It is obvious that $\Gamma(F(\cdot,1), G(\cdot,1)) = p_{X\times I}(graph(F_1) \cap graph(G_1)) \cap X \times \{t\}$ for any t if we identify $X \times \{t\}$ with X. If we can find an isomorphism (h, id), i.e. h is a fibre preserving homeomorphism over identity, from the bundle $p_{X\times I} : X \times I \times Y \to X \times I$ to itself such that $h(graph(G_1)) = graph(G)$, then $h(graph(F_1))$ is a section too. Let F' be a map from $X \times I \times Y$ to itself such that $graph(F') = h(graph(F_1))$. It is easy to see that $p_{X\times I}(graph(F') \cap graph(G)) = p_{X\times I}(graph(F_1) \cap graph(G_1))$ and therefore, $\Gamma(F'(\cdot,0), G(\cdot,0)) = p_{X\times I}(graph(F') \cap graph(G)) \cap X \times \{0\} = p_{X\times I}(graph(F_1) \cap graph(G_1)) \cap X \times \{0\} = \Gamma(F(\cdot,1), G(\cdot,1))$. It should be clear from this discussion why we need to discuss bundles. Since we will consider a homotopy from one pair of spaces to another, we introduce the concept of bundle triads.

This chapter is divided into three sections. In section 1, we introduce the concepts of bundle triads and give some basic properties. In section 2, we study a special case of bundle triads, namely the case where the total space is a product of the base space and the fiber space. In section 3, we prove the main theorem of this chapter.

3.1 Definition of bundle triads

Definition 3.1.1 A bundle triad over a pair of spaces (B, B_1) with $B_1 \subset B$ is a triple $(p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ of maps such that $E_0, E_1 \subset E$ and $p_0 : E_0 \to B$ and $p_1 : E_1 \to B_1$ are the restrictions of p on E_0 and E_1 respectively. We denote the bundle triad by ξ, and we use $E(\xi)$, $E_0(\xi)$ and $E_1(\xi)$ to denote E, E_0 and E_1 respectively.

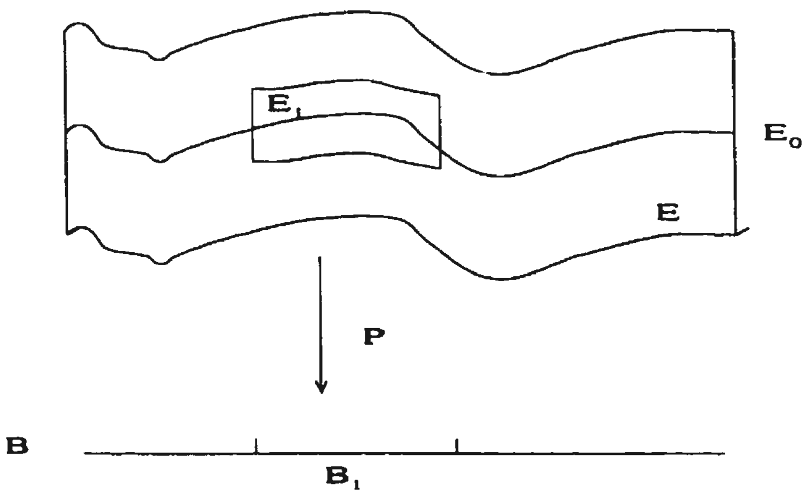

Figure 3.1:

Example 3.1.2 Let B and F be topological spaces, B_1 and F_1 subspaces of B and F respectively, and $f_0 \in F$ a point. Let $E = B \times F$, $E_0 = B \times \{f_0\}$ and $E_1 = B_1 \times F_1$, and let $p : E \to B$, $p_0 : E_0 \to B$ and $p_1 : E_1 \to B_1$ be the obvious projections. Then (p, p_0, p_1) is a bundle triad.

Example 3.1.3 Let M be a m-dimensional smooth manifold, $A \subset M$ be a n-dimensional submanifold. Let

$$E = TM, \text{the tangent space of } M;$$

$$E_0 = (TM)_0, \text{ the zero section of } TM;$$

$$E_1 = TA, \text{ the tangent space of } A;$$

And let $p : E \to M$ be the projection, and $p_0 = p|_{E_0}$, $p_1 = p|_{E_1}$. Then (p, p_0, p_1) is a bundle triad.

Definition 3.1.4 A bundle triad morphism from one bundle triad $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ to another $\xi' = (p' : E' \to B', p'_0 : E'_0 \to B', p'_1 : E'_1 \to B'_1)$ is a pair of maps: $f : (B, B_1) \to (B', B'_1)$ and $\tilde{f} : (E, E_0, E_1) \to (E', E'_0, E'_1)$ such that $p' \circ \tilde{f} = f \circ p$. The morphism is denoted by $(\tilde{f}, f)$. If $(B, B_1) = (B', B'_1)$ and $f = id$, then $(\tilde{f}, f)$ is called a (B, B_1)-morphism.

Two bundle triads ξ and ξ' over (B, B_1) are said to be $(B, B_1)-$isomorphic if there is a homeomorphism $h : E \to E'$ such that (h, id) is a (B, B_1)-morphism, and $h(E_0) = E'_0$ and $h(E_1) = E'_1$.

Example 3.1.5 Let $(B, B_1) = (D^2, S^1)$, $E = B \times D^3$, $E_0 = B \times \{(0,0,0)\}$ and $E_1 = B_1 \times D^2$. Let $p : E \to B$, $p_0 : E_0 \to B$ and $p_1 : E_1 \to B_1$ be the projections, then $\xi = (p, p_0, p_1)$ is a bundle triad. Using polar coordinates, a point in D^3 can be represented by a triple (r, θ, ϕ), and a point in D^2 can be represented by a pair (r, θ). Define $h : E \to E$ by

$$h((r_1, \theta_1), (r_2, \theta_2, \phi)) = ((r_1, \theta_1), (r_2, \theta_2 + r_1\pi, \phi)).$$

It is easy to check that h is a homeomorphism, $h(E_0) \subset E_0$, $h(E_1) \subset E_1$ and $ph = id_B p$. So (h, id_B) is a (B, B_1)-isomorphism.

Definition 3.1.6 Let (B, B_1) be a pair of spaces and $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ be a bundle triad over (B, B_1). Let $(A, A_1) \subset (B, B_1)$, with $A_1 = A \cap B_1$. The restriction bundle triad of ξ over (A, A_1), which will be denoted by $\xi|_{(A,A_1)}$, is defined to be the triple $(p' : E' \to A, p'_0 : E'_0 \to A, p'_1 : E'_1 \to A_1)$, where $E' = p^{-1}(A)$, $E'_0 = p_0^{-1}(A)$ and $E'_1 = p_1^{-1}(A_1)$, and p', p'_0 and p'_1 are the restrictions of p over E', E'_0 and E'_1 respectively.

Definition 3.1.7 Let $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ be a bundle triad over (B, B_1), and $f : (B', B'_1) \to (B, B_1)$ a map. The induced bundle triad of ξ under f, denoted by $f^*(\xi)$, is the bundle triad $(p' : E' \to B', p'_0 : E'_0 \to B', p'_1 : E'_1 \to B'_1)$, where $E' = \{(b', e) \in B' \times E \mid f(b') = p(e)\}$, $E'_0 = \{(b', e) \in B' \times E | f(b') = p(e) \textit{ and } e \in E_0\}$ and $E'_1 = \{(b', e) \in B' \times E | f(b') = p(e) \text{ and } b' \in B'_1, e \in E_1\}$, and p' is the restriction of the projection from $B' \times E$ to B'.

Let ξ be a bundle triad over (B, B_1) and $f : (B', B'_1) \to (B, B_1)$ a map. Define $f_\xi : E(f^*(\xi)) \to E(\xi)$ by $f_\xi(b, x) = x$. Then $(f_\xi, f) : f^*(\xi) \to \xi$ is a morphism which will call the canonical morphism from $f^*(\xi)$ to ξ.

The proof of the following proposition is similar to the analogous one for bundles (see Proposition 5.5 of [HD]).

Proposition 3.1.8 *If $(f_\xi, f) : f^*(\xi) \to \xi$ is the canonical morphism from $f^*(\xi)$ to ξ, where $f : (B', B'_1) \to (B, B_1)$ is a map, then for each $b' \in B'$ the restriction*

$$f_\xi : ((p')^{-1}(b'), (p'_0)^{-1}(b')) \to (p^{-1}(f(b')), p_0^{-1}(f(b')))$$

is a homeomorphism, and if $b' \in B'_1$, then the restriction

$$f_\xi : ((p')^{-1}(b'), (p'_0)^{-1}(b'), (p'_1)^{-1}(b')) \to (p^{-1}(f(b')), p_0^{-1}(f(b')), p_1^{-1}(f(b')))$$

is a homeomorphism. Moreover, if $(v, f) : \eta \to \xi$ is any bundle triad morphism, there exists a (B', B'_1)-morphism $w : \eta \to f^*(\xi)$ such that $f_\xi w = v$. With respect to this property, the morphism w is unique.

Proof: The fibre $(p')^{-1}(b') \subset \{b'\} \times E$ is the subspace of points $(b'. x) \in b' \times E$ such that $p(x) = f(b')$. or equivalently it is $\{b'\} \times p^{-1}(f(b'))$. Therefore. $f_\xi : \{b'\} \times p^{-1}(f(b')) \to p^{-1}(f(b'))$ is a homeomorphism. By the definition of $(p'_0)^{-1}(b')$ and $(p'_1)^{-1}(b')$. it is easy to see that $f_\xi((p'_0)^{-1}(b')) = (p_0)^{-1}(f(b'))$, and $f_\xi((p'_1)^{-1}(b')) = (p_1)^{-1}(f(b'))$ if $b' \in B'_1$.

To verify the second statement, define $w(y) = (p_\eta(y), v(y))$. Since (v, f) is a morphism, $f(p_\eta(y)) = p(v(y))$, and since for any $y \in E_0(\eta)$ respectively $E_1(\eta)$, $v(y) \in E_0$ respectively E_1, we have $w(y) \in E'_0$ respectively E'_1. Thus, w is a (B', B_1)-morphism. The property that $f_\xi w = v$ and uniqueness are easily checked. □

Proposition 3.1.9 *Let $g : (B'', B''_1) \to (B', B'_1)$ and $f : (B'. B'_1) \to (B, B_1)$ be maps, and let ξ be a bundle triad over (B, B_1), then $1^*(\xi)$ and ξ are (B, B_1)-isomorphic, and $g^*(f^*(\xi))$ and $(fg)^*(\xi)$ are (B'', B''_1)-isomorphic.*

Proof: Define $u : \xi \to 1^*(\xi)$ by the relation $u(x) = (p(x), x)$, then u is the inverse of the canonical morphism from $1^*(\xi)$ to ξ. Let $v : (fg)^*(\xi) \to g^*(f^*(\xi))$ be defined by $v(b'', x) = (b'', (g(b''), x))$. The composition of canonical morphisms from $g^*(f^*(\xi))$ to $f^*(\xi)$ and from $f^*(\xi)$ to ξ is a morphism from $g^*(f^*(\xi))$ to ξ. So by Proposition 3.1.8, there is a unique morphism h from $g^*(f^*(\xi))$ to $(fg)^*(\xi)$. It is easy to check that h is the inverse of v.

□

Note 3.1.10 Let (B, B_1) be a pair of spaces and ξ be a bundle triad over (B, B_1). Let $(A, A_1) \subset (B, B_1)$ and $i : (A, A_1) \to (B, B_1)$ be the inclusion map. Then $i^*(\xi)$ is isomorphic to $\xi|_{(A, A_1)}$. The proof is the same as Proposition 3.1.9.

(The following corollary will be used when we consider a bundle triad over $(B, B_1) \times I$.)

Corollary 3.1.11 *Let (B, B_1) and (B', B_1') be pairs of spaces, and let ξ be a bundle triad over (B, B_1). Assume $A \subset B$ and $A' \subset B'$, let $A_1 = A \cap B_1$, $A_1' = A' \cap B_1'$ and $f : (B', B_1', A') \to (B, B_1, A)$ be a map. Let $g = f|_{(A', A_1')} : (A', A_1') \to (A, A_1)$, then $g^*(\xi|_{(A, A_1)})$ is (A', A_1')-isomorphic to $f^*(\xi)|_{(A', A_1')}$.*

Proof: Let $i : (A, A_1) \to (B, B_1)$ and $i' : (A', A_1') \to (B', B_1')$ be inclusion maps, then $i \circ g = f \circ i'$. By the above note and Proposition 3.1.9, we have

$$f^*(\xi)|_{(A', A_1')} \cong (i')^* f^*(\xi) \cong (f \circ i')^*(\xi) \cong (i \circ g)^*(\xi) \cong g^* i^*(\xi) \cong g^*(\xi|_{(A', A_1')}). \qquad \square$$

Proposition 3.1.12 *Let $f : (B', B_1') \to (B, B_1)$ be a map, and η and ξ be bundle triads over (B, B_1). If η and ξ are $(B, B_1)-$isomorphic, then $f^*(\eta)$ and $f^*(\xi)$ are $(B', B_1')-$isomorphic.*

Proof: Let $h : E(\eta) \to E(\xi)$ be a homeomorphism such that $p_\xi \cdot h = p_\eta$. Then $id_{B'} \times h : B' \times E(\eta) \to B' \times E(\xi)$ is a homeomorphism. This restricts to $E(f^*(\eta))$ to give a homeomorphism from $E(f^*(\eta))$ to $E(f^*(\xi))$, which demostrates the required isomorphism.

□

Definition 3.1.13 *A bundle triad* $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ *is locally trivial if there is an open covering* $\{U_\alpha\}_{\alpha\in\Lambda}$ *of* B, *and for each* $\alpha \in \Lambda$ *there is topological space* F_α *and a subspace* $F_{0\alpha} \subset F_\alpha$, *and a homeomorphism* ϕ_α *such that the diagram*

$$\begin{array}{ccc} (p^{-1}(U_\alpha), p_0^{-1}(U_\alpha)) & \xrightarrow{\phi_\alpha} & (U_\alpha \times F_\alpha, U_\alpha \times F_{0\alpha}) \\ & p \searrow & \pi_\alpha \downarrow \\ & & (U_\alpha, U_\alpha) \end{array}$$

commutes, where $\pi_\alpha : U_\alpha \times F_\alpha \to U_\alpha$ *is the projection. If* $U_\alpha^1 = U_\alpha \cap B_1 \neq \emptyset$, *then* ϕ_α *can be regarded as a triad map and the commutative diagram becomes*

$$\begin{array}{ccc} (p^{-1}(U_\alpha), p_0^{-1}(U_\alpha), p_1^{-1}(U_\alpha^1)) & \xrightarrow{\phi_\alpha} & (U_\alpha \times F_\alpha, U_\alpha \times F_{0\alpha}, U_\alpha^1 \times F_{1\alpha}) \\ & p \searrow & \pi_\alpha \downarrow \\ & & (U_\alpha, U_\alpha, U_\alpha^1). \end{array}$$

The cover is called a trivializing cover and ϕ_α *is a trivialization over* U_α. *Note that* $p : E \to B$ *is a bundle.*

Example 3.1.14 The bundle triads in examples 3.1.2 and 3.1.3 are locally trivial.

Proof: We will discuss example 3.1.3 only, since it is obvious that the bundle triad in example 3.1.2 is locally trivial since it is globally trivial. For any point $x \in M$, there is a neighbourhood U, and a diffeomorphism $\psi : U \to \mathbf{R}^m$. If $x \in A$, ψ can be chosen such that $\psi(U \cap A) = \mathbf{R}^n \subset \mathbf{R}^m$. By Theorem 4.8 of [HS], $(\frac{\partial}{\partial(\psi_x)_1}, \frac{\partial}{\partial(\psi_x)_2}, \ldots, \frac{\partial}{\partial(\psi_x)_m})$ is a basis of

T_xM, and $(\frac{\partial}{\partial(\psi_x)_1}, \frac{\partial}{\partial(\psi_x)_2}, \ldots, \frac{\partial}{\partial(\psi_x)_n})$ is a basis of T_xA for any $x \in A$. Therefore, it is easy to contruct a trivialization over U. In this example, each of F_α, $F_{0\alpha}$, and $F_{1\alpha}$ is homeomorphic to (respectively) $\mathbf{R}^m$, a singleton, and $\mathbf{R}^n$. □

Definition 3.1.15 An open covering $\{U_\alpha | \alpha \in A\}$ of a space B is said to be numerable if it is locally finite and has a partition of unity subordinate to it. A bundle triad is numerable if its base has a numerable trivializing cover.

It is easy to see that every locally trivial bundle triad over a paracompact space is numerable.

Theorem 3.1.16 *Assume that* $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ *is a bundle triad, and* $f : (B', B'_1) \to (B, B_1)$ *is a map. If* ξ *is locally trivial (respectively numerable), then* $f^*(\xi)$ *is locally trivial (respectively numerable).*

Proof: Let $\xi' = f^*(\xi)$ and (f_ξ, f) be the canonical morphism from ξ' to ξ. Assume that $\xi|_{U_i}$ is trivial, where $\{U_i\}_{i \in A}$ is an open covering of B. Let $h_i : (p^{-1}(U_i), p_0^{-1}(U_i), p_1^{-1}(U_i \cap B_1)) \to (U_i \times F_i, U_i \times F_{0i}, U_i \cap B_1 \times F_{1i})$ be the trivialization. By Corollary 3.1.11, $f^*(\xi)|_{f^{-1}(U_i)}$ is isomorphic to $g^*(\xi|_{U_i})$, where $g = f|_{f^{-1}(U_i)}$. Therefore, by Proposition 3.1.12, we only need to prove that the induced bundle triad, of a trivial bundle triad, is trivial.

Since ξ is trivial, we have $E = B \times F$ for some F. Note that the projection $p_{(B',F)}$: $B' \times B \times F \to B' \times F$ and the map $i_f : B' \times F \to B' \times B \times F$ defined by $i_f(b', l) = (b', f(b'), l)$ are continuous, and that the restriction of $p_{(B',F)}$ on $E(f^*(\xi))$ is the inverse of i_f. Hence, $f^*(\xi)$ is trivial.

When $\{U_i\}_{i\in A}$ is locally finite, $\{f^{-1}(U_i)\}_{i\in A}$ is locally finite. So if ξ is numerable, then ξ' is numerable. □

Proposition 3.1.17 *Let ξ and ξ' be locally trivial bundle triads over (B, B_1), and $(\tilde{f}, id)$: $\xi' \to \xi$ a morphism. If $\tilde{f}|_{E_0(\xi')} : E_0(\xi') \to E_0(\xi)$ and $\tilde{f}|_{E_1(\xi')} : E_1(\xi') \to E_1(\xi)$ are onto, and for each $b \in B$, $\tilde{f}|_{(p')^{-1}(b)} : (p')^{-1}(b) \to p^{-1}(b)$ is a homeomorphism, and $(p')^{-1}(b)$ is locally connected and locally compact Hausdorff, then $(\tilde{f}, id)$ is a (B, B_1)-isomorphism.*

Proof: Obviously the map $\tilde{f} : E(\xi') \to E(\xi)$ is a one-to-one correspondence, and $\tilde{f}(E_0(\xi')) = E_0(\xi)$, and $\tilde{f}(E_1(\xi')) = E_1(\xi)$. Thus we only need to prove that $\tilde{f}$ is locally a homeomorphism. For each $b \in B$, there is a neighbourhood U_b such that

(a). both $\xi'|_{U_b}$ and $\xi|_{U_b}$ are trivial;

(b). there is a morphism $\tilde{f}|_{E(\xi'|_{U_b})} : U_b \times F_b \to U_b \times F_b$, which will be denoted by f_{U_b};

(c). for each $b' \in U_b$, the restriction $f_{U_b}|_{b'\times F_b} : b' \times F_b \to b' \times F_b$ is a homeomorphism.

By the proof of Lemma 2.2.1 of [PR], f_{U_b} is a homeomorphism. Since $E(\xi'|_{U_b})$ is open, $\tilde{f}$ is locally a homeomorphism. □

Proposition 3.1.18 *Let ξ and ξ' be locally trivial bundle triads over (B, B_1) and (B', B_1') respectively. Assume that for each point $b' \in B'$, $(p')^{-1}(b')$ is locally connected and locally compact Hausdorff. If there is a morphism $(\tilde{f}, f)$ from ξ' to ξ such that for each point $b' \in B'$, $\tilde{f}|_{(p')^{-1}(b')} : (p')^{-1}(b') \to p^{-1}(f(b'))$ is a homeomorphism with $\tilde{f}((p_0')^{-1}(b')) = p_0^{-1}(f(b'))$ and $\tilde{f}((p_1')^{-1}(b')) = p_1^{-1}(f(b'))$, then ξ' is $(B', B_1')-$isomorphic to the induced bundle triad of ξ under the map f, and the isomorphism is given by $h(x') = (p'(x'), \tilde{f}(x'))$.*

Proof: By Proposition 3.1.8, there is a morphism $(h, id) : \xi' \to f^*(\xi)$ such that $f_\xi \circ h = \tilde{f}$. Since on each fibre $(p')^{-1}(b')$ and $(p_{f_\xi})^{-1}(b')$, $\tilde{f}$ and f_ξ are homeomorphisms with $\tilde{f}((p_0')^{-1}(b')) = p_0^{-1}(f(b'))$, $\tilde{f}((p_1')^{-1}(b')) = p_1^{-1}(f(b'))$ and $f_\xi(((p_{f^*\xi})_0)^{-1}(b')) = p_0^{-1}(f(b'))$, $f_\xi(((p_{f^*\xi})_1)^{-1}(b')) = p_1^{-1}(f(b'))$ respectively, we have that $h|_{(p')^{-1}(b')}$ is a homeomorphism from $(p')^{-1}(b')$ to $(p_{f_\xi})^{-1}(b')$ with $h((p_0')^{-1}(b')) = ((p_{f^*\xi})_0)^{-1}(b')$ and $h((p_1')^{-1}(b')) = ((p_{f^*\xi})_1)^{-1}(b')$. By Proposition 3.1.17, (h, id) is a (B', B_1')-isomorphism. □

Lemma 3.1.19 *Let (A, A_1) be a pair of spaces, and $(B, B_1) = (A, A_1) \times [a, b]$ for some interval $[a, b]$, and let ξ be a bundle triad over (B, B_1). If for some $c \in [a, b]$, $\xi|_{(A\times[a,c], A_1\times[a,c])}$ and $\xi|_{(A\times[c,b], A_1\times[c,b])}$ are trivial, then ξ is trivial.*

Proof: Let $(B', B_1') = (A, A_1) \times [a, c]$ and $(B'', B_1'') = (A, A_1) \times [c, d]$. Assume that $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$, $\xi|_{(B', B_1')} = (p' : E' \to B', p_0' : E_0' \to B', p_1' : E_1' \to B_1')$, and $\xi|_{(B'', B_1'')} = (p'' : E'' \to B'', p_0'' : E_0'' \to B'', p_1'' : E_1'' \to B_1'')$. Let $u' : (B' \times F', B' \times F_0', B_1' \times F_1') \to (E', E_0', E_1')$ and $u'' : (B'' \times F'', B'' \times F_0'', B_1'' \times F_1'') \to (E'', E_0'', E_1'')$ be trivializations. Let $v' = u'|_{(A\times\{c\}\times F', A_1\times\{c\}\times F_1')}$ and $v'' = u''|_{(A\times\{c\}\times F'', A_1\times\{c\}\times F_1'')}$. Then $h = (v'')^{-1} \circ v'$ is an $(A \times \{c\}, A_1 \times \{c\})$−isomorphism. Assume that h has the form $h(a, c, x) = (a, c, h_a(x))$, where h_a is a homeomorphism from F' to F''. Define $u : B \times F' = A \times [a, d] \times F' \to E$ by

$$u(a, t, x) = \begin{cases} u'(a, t, x) & \text{if } a \le t \le c; \\ u''(a, t, h_a(x)) & \text{if } c \le t \le d; \end{cases}$$

Then u is a trivialization of ξ. □

Lemma 3.1.20 *Let ξ be a numerable bundle triad over $(B, B_1) \times I$, where B is a paracompact space. Then there is a numerable covering $\{U_j\}_{j\in S}$ of B, such that $\xi|_{(U_j, U_j\cap B_1)\times I}$ is trivial for each $j \in S$.*

Proof: For each $b \in B$ and $t \in I$, there are open neighbourhoods $U_b(t)$ of b in B, and $V_b(t)$ of t in I, such that $\xi|_{(U_b(t)\times V_b(t),(U_b(t)\cap B_1)\times V_b(t))}$ is trivial. Therefore, by the compactness of $[0,1]$, there exists a finite sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = 1$, and for each $0 \le i \le n$, there exists an open neighbourhood U_i of b in B such that $\xi|_{(U_i, U_i\cap B_1)\times[t_{i-1},t_i]}$ is trivial for $0 \le i \le n$. Let $U = \cap_{1\le i\le n} U_i$, then the bundle $\xi|_{(U,U\cap B_1)\times[0,1]}$ is trivial by $n-1$ applications of Lemma 3.1.19. Therefore, there is an open covering $\{U_j\}_{j\in S}$ of B such that $\xi|_{(U_j,U_j\cap B)\times I}$ is trivial. Since B is paracompact, we have the result. □

3.2 Properties of quasi-trivial bundle triads

In accordance with our purpose, we now restrict attention to the bundle triads $(p : E \to B, p : E_0 \to B, p_1 : E_1 \to B_1)$, where $E = B \times F$, $E_1 = B_1 \times F_1$, and where F is a locally connected and locally compact Hausdorff space and $F_1 \subset F$.

Definition 3.2.1 A bundle triad $\xi = (p : E \to B, p_0 : E_0 \to B, p_1 : E_1 \to B_1)$ is said to be quasi-trivial, if $E = B \times F, E_1 = B_1 \times F_1$, and if p is the projection to the first factor B.

Definition 3.2.2 An envelope of unity on the space X is a family $\{u_j\}_{j\in J}$ of non-negative maps $u_j : X \to \mathbf{R}$, such that for each $x \in X$, $\max_{j\in J}\{u_j(x)\} = 1$ and the set $\{j \in J \mid u_i(x) \neq 0\}$ is finite. If $\{U_j\}_{j\in J}$ is a point-finite open covering of X, i.e. for each $x \in X$, there are only a finite number of U_j's containing x, with the property that for each $j \in J$, $u_j^{-1}(0,1] \in U_j$, then $\{u_j\}_{j\in J}$ is said to be subordinate to $\{U_j\}_{j\in J}$.

If $\{U_j\}_{j\in J}$ is a point-finite open covering, and $\{\pi_j\}_{j\in J}$ is a partition of unity subordinate to $\{U_j\}_{j\in J}$, then the non-negative function $u : X \to \mathbf{R}$, given by

$$u(x) = \max_{j\in J}\{\pi_j(x)\}.$$

is continuous. It is easy to check that $\{u_j\}_{j\in J}$ is an envelope subordinate to $\{U_j\}_{j\in J}$, where

$$u_j = \frac{\pi_j(x)}{u(x)}$$

(see [JI] p.205).

Definition 3.2.3 Let $\xi = (p : E \to B, p_0 : E_0 \to B_0, p_1 : E_1 \to B_1)$ be a bundle triad over (B, B_1). The bundle triad $\xi \times I$ over $(B, B_1) \times I$ is defined to be the bundle triad $(p \times id : E \times I \to B \times I, p_0 \times id : E_0 \times I \to B \times I, p_1 \times id : E_1 \times I \to B_1 \times I)$.

We will identify $B \times I \times F$ with $B \times F \times I$ by the homeomorphism $T(b, x, t) = (b, t, x)$ for $b \in B, x \in F$, and $t \in I$. Let $r : (B, B_1) \times I \to (B, B_1) \times \{1\}$ be the projection map $r(b, t) = (b, 1)$.

Lemma 3.2.4 *If ξ is a locally trivial bundle triad over $(B, B_1) \times I$, and each fiber is locally connected and locally compact Hausdorff, then $\xi|_{(B,B_1)\times\{1\}} \times I$ is isomorphic to $r^*(\xi|_{(B,B_1)\times\{1\}})$ by the function which sends (l, t) to $((p(l), t), l)$. If ξ is quasi-trivial, the isomorphism sends $((b, 1, x), t)$ to $((b, t), (b, 1, x))$.*

Proof: Let $\tilde{r} : E(\xi|_{(B,B_1)\times\{1\}} \times I) \to E(\xi|_{(B,B_1)\times\{1\}})$ be defined by $\tilde{r}((l,t)) = l$. It is obvious that $(\tilde{r}, r)$ is a morphism, and on each fibre $\tilde{r}$ is a homeomorphism. The result follows from Proposition 3.1.18. □

Lemma 3.2.5 *If ξ is a locally trivial bundle triad over $(B, B_1) \times I$, and B is paracompact, then there is a morphism $(g, r) : \xi \to \xi|_{(B,B_1)\times\{1\}}$. Furthermore, if ξ is a quasi-trivial bundle triad, and if there is a closed set $C \subset B$ such that over $(B - C, B_1 - C) \times I$, $p_0^{-1}(B - C \times I) = p_0^{-1}((B - C) \times \{1\}) \times I$, then for any neighbourhood U of C, there is a morphism $(u, r) : \xi \to \xi|_{(B,B_1)\times\{1\}}$ with the property that u is a projection on $\xi|_{(B-U,B_1-U)\times I}$.*

Proof: By Lemma 3.1.20, there exists a locally finite open covering $\{U_s\}_{s\in S}$ of B such that $\xi|_{(U_s,U_s\cap B_1)\times I}$ is trivial. Let $\{\eta_s\}_{s\in S}$ be an envelope of unity subordinate to the open covering $\{U_s\}_{s\in S}$. Let $h_i : (U_i \times I \times F_i, U_i \times I \times F_{0i}, (U_i \cap B_1) \times I \times F_{1i}) \to (p^{-1}(U_i \times I), (p_0)^{-1}(U_i \times I), (p_1)^{-1}((U_i \cap B_1) \times I))$ be a trivialization.

Define $(u_s, r_s) : \xi \to \xi$ as follows. Firstly, $r_s(b, t) = (b, max(\eta_s(b), t))$ for each $(b, t) \in U_s \times I$. Secondly, u_s is the identity outside $p^{-1}(U_s \times I)$, and $u_s(h_s(b, t, x)) = h_s(b, max(\eta_s(b), t), x)$ for each $(b, t, x) \in U_s \times I \times F_s$. We well order the set S. For each $b \in B$, there is an open neighbourhood $U(b)$ of b such that $U_s \cap U(b)$ is nonempty only for $s \in S(b)$, where $S(b)$ is a finite subset of S. On $U(b) \times I$, we define $r = r_{s(n)} \circ \cdots \circ r_{s(1)}$, the composition of $r_{s(n)}, \cdots, r_{s(1)}$, and on $p^{-1}(U(b) \times I)$, we define $u = u_{s(n)} \circ \cdots \circ u_{s(1)}$, where $S(b) = \{s(1), ..., s(n)\}$ and $s(1) < s(2) < \cdots < s(n)$. Since for $s \notin s(b)$, r_s on $U(b) \times I$ and u_s on $p^{-1}(U(b) \times I)$ are identities, the maps r and u are infinite composition of maps where all but a finite number of terms are identities near a point. Since each u_s is a homeomorphism on each fibre, the composition u is a homeomorphism on each fibre.

Now we assume that ξ is quasi-trivial and on $(B-C, B_1-C)\times I$, $\xi|_{(B-C,B_1-C)\times I} = \xi|_{(B-C,B_1-C)\times\{1\}} \times I$. Then the open covering $\{U_s\}_{s\in S}$ can be chosen such that if $U_s \cap C \neq \emptyset$, then $U_s \subset U$, and for any $U_s \cap C = \emptyset$, we can choose the trivialization $h_s : (U_s \times I \times F_s, U_s \times I \times F_{0s}, (U_s \cap B_1) \times I \times F_{1s}) \to (p^{-1}(U_s \times I), (p_0)^{-1}(U_s \times I), (p_1)^{-1}((U_s \cap B_1) \times I))$ such that $h_s(u, t, x) = (u, t, k_{u,s}(x))$, where $k_{u,s}$ is a self-homeomorphism of F_s. So the map u_s can be defined by $u_s(b,t,x) = u_s(h_s(h_s^{-1}(b,t,x))) = u_s(h_s(b,t,k_{u,s}^{-1}(x))) = h_s(b, max(\eta_s(b), t), k_{u,s}^{-1}(x)) = (b, max(\eta_s(b), t), k_{u,s}(k_{u,s}^{-1}(x)) = (b, max(\eta_s(b), t), x)$. Now for any point $b \in B-U$, $u((b,t,x)) = (b, 1, x)$. $\square$

Lemma 3.2.6 *If ξ is a locally trivial bundle triad over $(B, B_1)\times I$, with B paracompact, and if the fibre F is a locally connected, locally compact Hausdorff space, then ξ is isomorphic to $(\xi|_{(B,B_1)\times\{1\}})\times I$. Furthermore, if ξ is quasi-trivial with the hypotheses of Lemma 3.2.5, then the isomorphism can be chosen to be a pair $(G, id) : \xi \to (\xi|_{(B,B_1)\times\{1\}})\times I$ with the restriction of G over $(B-U, B_1-U)\times I$ given by $G(b,t,x) = ((b,1,x),t)$.*

Proof: From Proposition 3.1.18, Lemma 3.2.4 and 3.2.5, we know that ξ is isomorphic to $(\xi|_{(B,B_1)\times\{1\}})\times I$. When ξ is quasi-trivial, and over $(B-C, B_1-C)\times I$, $p_0^{-1}((B-C)\times I) = p^{-1}((B-C)\times\{1\})\times I$, we have, by Lemma 3.2.4 and 3.2.5, that the isomorphism has the desired property. $\square$

3.3 Relative homotopy and quasi-trivial bundle triads

In this section, we will prove the following main theorem of this chapter, namely:

Theorem 3.3.1 *Suppose $f, g : (X, A) \to (Y, B)$ are maps of a pair of paracompact topological spaces (X, A) to a pair of manifolds (Y, B), and let f' and g' be homotopic to f and g respectively. Then there is a map f'' homotopic to f' (and therefore to f) such that $\Gamma(f'', g) = \Gamma(f', g')$. Furthermore, given any homotopy $\{g_t | t \in I\}$ from g to g', there is a homotopy $\{f_t | t \in I\}$ beginning at f' such that $\Gamma(f_{1-t}, g_t) = \Gamma(f', g')$ for all $t \in I$. Finally, if there is a closed subset $X_1 \subset X$, such that on $X - X_1$, $g_t(x) = g_0(x)$, then for any open set $U \supset X_1$, f_t, g_t can be chosen to be stationary outside of U, i.e. $f_t(x) = f_0(x), g_t(x) = g_0(x)$ on $(X - U)$.*

Let Y be a topological space, and $Y_1 \subset Y$ a subspace. Let $q : Y \times Y \to Y$ be the projection to the first factor, and $q_0 : D(Y) = \{(y, y) \in Y \times Y\} \to Y$, and $q_1 : Y_1 \times Y_1 \to Y_1$ the restrictions of q. Then (q, q_0, q_1) is a bundle triad. The following Proposition gives a condition on Y under which (q, q_0, q_1) is locally trivial.

Proposition 3.3.2 *If there exists an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of Y, such that for each α and for each pair $(x, y) \in U_\alpha \times U_\alpha$, there is a homeomorphism $\phi_{\alpha xy} : Y \to Y$ such that $\phi_{\alpha xy}(x) = y$, and if both x and y are in Y_1, then for any $z \in Y_1$, $\phi_{\alpha xy}(z) \in Y_1$, and considering $\phi_{\alpha xy}(z) \in Y$ as a function in x, y and z then $\phi_{\alpha xy}(z)$ is continuous with respect to (x, y, z). Then (q, q_0, q_1) is a locally trivial bundle triad.*

Proof: Suppose $\{U_\alpha\}_{\alpha\in\Lambda}$ is a covering of Y satisfying the conditions of the proposition. For each $\alpha \in \Lambda$, choose $u_\alpha \in U_\alpha$, such that if $U_\alpha \cap Y_1 \neq \emptyset$, then $u_\alpha \in U_\alpha \cap Y_1$. Let $F_\alpha = Y$, $F_{0\alpha} = \{u_\alpha\}$ and $F_{1\alpha} = Y_1$. Define $h_\alpha : q^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ by

$$h_\alpha(x,y) = (x, \phi_{\alpha x u_\alpha}(y)).$$

Then h_α is a homeomorphism. Note that if $x \in Y_1$, then $\phi_{\alpha x u_\alpha}(Y_1) = Y_1$, so h_α is actually a homeomorphism of pairs of spaces $(q^{-1}(U_\alpha), q_1^{-1}(U_\alpha \cap Y_1)) = (U_\alpha \times Y, U_\alpha \cap Y_1 \times Y_1)$ to $(U_\alpha \times F_\alpha, U_\alpha \cap Y_1 \times F_{1\alpha})$. Since $h_\alpha(x,x) = (x, \phi_{\alpha x u_\alpha}(x)) = (x, u_\alpha)$, then $h_\alpha(q_0^{-1}(U_\alpha)) = U_\alpha \times F_{0\alpha}$ and hence h_α is a trivialization. □

Proposition 3.3.3 *If Y is a smooth manifold of dimension n, and $Y_1 \subset Y$ is a submanifold of dimension k, then $(q : Y \times Y \to Y, q_0 : D(Y) \to Y, q_1 : Y_1 \times Y_1 \to Y_1)$ is a numerable bundle triad.*

Proof: Let $\{\psi_\alpha : U_\alpha \to \mathbf{R}^n | \alpha \in \Lambda\}$ be charts for Y with the property $\{U_\alpha\}$ is numerable and if $U_{1\alpha} = U_\alpha \cap Y_1 \neq \emptyset$, then $\psi_\alpha(U_{1\alpha}) = \mathbf{R}^k \subset \mathbf{R}^n$ (such charts exist as Y_1 is a submanifold of Y). Define

$$\phi_{\alpha x y}(z) = \begin{cases} z & \text{if } z \in Y - U_\alpha \\ \psi_\alpha^{-1}\left(\psi_\alpha(z) + \frac{\psi_\alpha(y)-\psi_\alpha(x)}{1+|\psi_\alpha(z)-\psi_\alpha(x)|}\right), & \text{if } z \in U_\alpha \end{cases}$$

then $\phi_{\alpha x y}$ satisfies the hypothesis of Proposition 3.3.2, and therefore we have the result. □

Lemma 3.3.4 *Suppose that $\{g_t : (X, A) \to (Y, B) | t \in I\}$ is a homotopy, where (Y, B) is a pair of manifolds. Then there is a isotopy $\{h_t : (X \times Y, A \times B) \to (X \times Y, A \times B) | t \in I\}$*

such that

$$h_t(x, g_t(x)) = (x, g_0(x))$$

for all $x \in X$ and $t \in I$. Furthermore, if there is a closed subset $X_1 \subset X$, such that on $X - X_1$, $g_t(x) = g_0(x)$, then for any open set $U \supset X_1$, h_t can be chosen to be stationary outside of U, that is $h_t(x, y) = (x, y)$ on $(X - U) \times Y$.

Proof: Define $G : (X \times I, A \times I) \to (Y, B)$ by $G(x, t) = g_t(x)$ for all $(x, t) \in X \times I$. Let $p : X \times I \times Y \to X \times I$ be the projection, and p_0 :Graph$(G) \to X \times I$ and $p_1 : A \times I \times B \to A \times I$ be the restriction of p on Graph(G) and $A \times I \times B$ respectively. Then $\xi = (p, p_0, p_1)$ is a pullback under G of $(q : Y \times Y \to Y, q_0 : D(Y) \to Y, q_1 : B \times B \to B)$. To see this define $\tilde{G} : (X \times I \times Y,$ Graph$(G), A \times I \times B) \to (Y \times Y, D(Y), B \times B)$ by $\tilde{G}(x, t, y) = (G(x, t), y)$. Then for any $(x_0, t_0) \in X \times I$, $\tilde{G}$ is a homeomorphism from $\{(x_0, t_0)\} \times Y$ to $\{G(x_0, t_0)\} \times Y$. Obviously $\tilde{G}$ sends Graph(G) to $D(Y)$. If $(x_0, t_0) \in A \times I$, then $\tilde{G}(\{(x_0, t_0)\} \times B) = \{G(x_0, t_0)\} \times B$. Thus $\tilde{G}$ is a morphism of bundle triads and $q \circ \tilde{G} = G \circ p$ and therefore by Proposition 3.1.18, ξ is a pullback under G. Now (q, q_0, q_1) is locally trivial by Proposition 3.3.3, thus (p, p_0, p_1) is locally trivial by Theorem 3.1.16. Since ξ is quasi-trivial and $X \times I \times Y$ is paracompact, by Lemma 3.2.6, we have the homeomorphism $H_1 : X \times I \times Y \to (X \times \{0\} \times Y) \times I$ such that $H_1(A \times I \times B) \subset (A \times \{0\} \times B) \times I$, and $H_1(\text{Graph}(G)) \subset Graph(g_0 \times id_I)$ and on $(X - U) \times I \times Y$, $H_1(x, t, y) = ((x, 0, y), t)$. Define $H_2 : (X \times \{0\} \times Y) \times I \to X \times I \times Y$ by $H_2((x, 0, y), t) = (x, t, y)$, and let H be the compositin of H_1 and H_2, then H is a homeomorphism, $H(A \times I \times B) \subset A \times I \times B$, and $H_1(\text{Graph}(G)) \subset Graph(g_0 \times id_I)$ and on $(X - U) \times I \times Y$, H is the identity. Let $h_t(x, y)$ be the projection of $H(x, t, y)$ in $X \times Y$, then h_t is the desired isotopy. □

We are now ready to prove the main theorem.

Proof of Theorem 3.3.1: Assume that $f', g' : (X, A) \to (Y, B)$ are homotopic to $f, g : (X, A) \to (Y, B)$ respectively. Let $\{g_t | t \in I\}$ be the homotopy from g to g'. We must find a homotopy $\{f_t\} : (X, A) \to (Y, B)$ beginning at f' such that $\Gamma(f_t, g_{1-t}) = \Gamma(f', g')$ for all $t \in I$.

Let $\{h_t : (X \times Y, A \times B) \to (X \times Y, A \times B)\}$ be the isotopy given in Lemma 3.3.4, such that on $(X - U) \times Y$, h_t is the identity. Let $\pi : (X \times Y, A \times B) \to (Y, B)$ be the projection. Then we define $f_t : (X, A) \to (Y, B)$ by

$$f_t(x) = \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x))$$

for every $x \in X$ and $t \in I$. If $x \in A$, then $f'(x) \in B$, so $(x, f'(x)) \in A \times B$. Now $h_1^{-1}(x, f'(x)) \in A \times B \Rightarrow h_{1-t} \circ h_1^{-1}(x, f'(x)) \in A \times B \Rightarrow \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x)) \in B$. So f_t is a homotopy from $f_0 : (X, A) \to (Y, B)$ to $f_1 : (X, A) \to (Y, B)$. If $x \in X - U$, then $f_t(x) = \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x)) = \pi \circ h_{1-t}(x, f'(x)) = \pi(x, f'(x)) = f'(x)$. This is because on $(X - U) \times Y$, h_t is identity. It remains to show that $\Gamma(f_t, g_{1-t}) = \Gamma(f_0, g_1) = \Gamma(f', g')$. The proof is exactly the same as in [BR2], but for completeness, we give the proof here. Let

$t \in I$, and suppose first that $x \in \Gamma(f_t, g_{1-t})$, so $f_t(x) = g_{1-t}(x)$. Then

$$\begin{aligned} f_0(x) &= \pi \circ h_1 \circ h_{1-t}^{-1} \circ h_{1-t} \circ h_1^{-1}(x, f_0(x)) \\ &= \pi \circ h_1 \circ h_{1-t}^{-1}(x, f_t(x)) \\ &= \pi \circ h_1 \circ h_{1-t}^{-1}(x, g_{1-t}(x)) \\ &= \pi \circ h_1(x, g_0(x)) \\ &= \pi(x, g_1(x)) \\ &= g_1(x), \end{aligned}$$

so $x \in \Gamma(f_0, g_1)$. Conversely, suppose $x \in \Gamma(f_0, g_1)$ so $f_0(x) = g_1(x)$. Then

$$\begin{aligned} f_t(x) &= \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x)) \\ &= \pi \circ h_{1-t} \circ h_1^{-1}(x, f_0(x)) \\ &= \pi \circ h_{1-t} \circ h_1^{-1}(x, g_1(x)) \\ &= \pi \circ h_{1-t}(x, g_0(x)) \\ &= \pi(x, g_{1-t}(x)) \\ &= g_{1-t}(x), \end{aligned}$$

so $x \in \Gamma(f_t, g_{1-t})$. □

Applying Theorem 3.3.1, Lemma 1.6.4 can be restated as following.

Lemma 3.3.5 *Let X, Y be manifolds with dimensions greater than or equal to 3 and let $(f, g) : X \to Y$ be a pair of maps with a finite number of coincidence points. Let $x_0, x_1 \in \Gamma(f, g)$ and α be an arc from x_0 to x_1 such that $f \circ \alpha \sim g \circ \alpha$ and $\alpha((0,1)) \cap \Gamma(f, g) = \emptyset$. Let U be a neighbourhood of $\alpha((0,1])$ such that $\overline{U} \cong D^n$ and $x_0 \in \partial\overline{U}$. Then there is f' such that $f' \sim f$ rel $X - U$, and with $\Gamma(f', g) = \Gamma(f, g) - \{x_1\}$.*

Proof: By Lemma 1.6.4, we have $f'' \sim f$ rel $X - U$ and $g' \sim g$ rel $X - U_1$ with $U_1 \subset \overline{U}_1 \subset U$ and $\Gamma(f'', g') = \Gamma(f, g) - \{x_1\}$. Let V be an open set such that $\overline{U}_1 \subset V \subset \overline{V} \subset U$. Since on $X - \overline{U}_1$, the homotopy from g to g' is constant to g, then by Theorem 3.3.1, we have an f'' with $f' \sim f''$ rel $X - V$ such that $\Gamma(f', g) = \Gamma(f'', g') = \Gamma(f, g) - \{x_1\}$. It is obvious that $f' \sim f$ rel $X - U$ since $V \subset U$. □

Chapter 4

Equivariant Coincidence Theory

Let W be a group, X and Y be W-spaces (see 4.1.4), and $f : X \to Y$ and $g : X \to Y$ be W-maps (see 4.1.6). Then we actually have a sequence of pairs of maps $\{(f^H, g^H)\}$, one for each isotropy group H on X. Here f^H and g^H are the restrictions of f, g respectively to the fixed point set X^H. If a point $x \in X$ is a coincidence point of (f, g), the orbit $Wx = \{wx \mid w \in W\}$ of x under the group action consists entirely of coincidence points since f, g are both W-maps. Because of this, it is natural to estimate the number of coincidence orbits instead of the number of coincidence points. Since the length of an orbit is dependent on the location of the orbit, we will employ the techniques of chapter 2 to locate those orbits. In order to do this, it is necessary to consider each isotropy group individually.

Two distinct categories are considered in equivariant fixed point theory. In [WP2], a class of W-maps, called W-compactly fixed maps, is discussed. A W-map $f : V \to X$ from an open invariant subset V of a W-manifold X to X is called a W-compactly fixed map if for any isotropy group H on V, the fixed point set Fix f_H is compact, where $f_H = f|_{V_H} : V_H \to X^H$ is the restriction of f on V_H (see section 4.1.1 for the definition of V_H). A k-tuple is defined for a W-compactly fixed map, which is a W-compactly homotopy invariant(as opposed to a W-homotopy invariant), where k is the number of the isotropy types of V. [FP] generalizes this work to coincidence theory. In [WP3], on the other hand, general W-maps are studied. The fixed points in X^H that can be moved to X^K for some $H \subset K$ are characterized, and the minimal number of fixed points on X_H estimated (This is not an issue in the previous category since coincidence points in X_H can not be moved to X_K with $H \subset K$ via a W-compactly fixed homotopy). Thus the theories in these two categories develop along very different lines. In addition, in [WP3] some methods of computation are given. These are absent from the corresponding theory in [WP2], and hence from the generalization of it in [FP]. In this chapter, we generalize the ideas in the latter category to equivariant coincidence theory. However, we use a different approach from [WP3] when we discuss computation. Throughout this chapter, except section 4.1, we assume that W is a finite group; $f, g : X \to Y$ are W-maps; X and Y are closed orientable smooth W-manifolds such that dim $X^H =$ dim Y^H for any subgroup H of W.

This chapter is arranged as follows. In section 1, we give the necessary preparation for the chapter. In particular, we introduce the concepts of a group action on a topological space, a complex and a manifold, and also the concept of equivariant maps. In addition, we discuss the homotopy properties of equivariant maps. In section 2, we introduce equivariant coincidence classes and equivariant Reidemeister classes through a covering space approach.

We then define an equivariant Nielsen number of a pair of equivariant maps $(f, g) : X \to Y$, which is a lower bound of the number of coincidence point orbits of a pair of equivariant maps. In addition, we describe the relationship between an equivariant coincidence class and an ordinary coincidence class and give some basic properties of the Nielsen number. In section 3, we discuss the computation of the equivariant Nielsen number introduced in section 2 and give an alternative description of the equivariant Reidemeister class using fundamental group approach. This allows us to compute the equivariant Reidemeister number in some special cases. In section 4, we introduce additional Nielsen type invariants for each isotropy group of the group. In section 5, we discuss the computation of the Nielsen type invariants defined in section 4. When the fixed point set of the group action is nonempty and some other conditions are satisfied, these invariants are computable in terms of homology groups. This approach is different from the one given in [WP3]. Finally, Minimality is discussed in section 6.

4.1 Group Actions

4.1.1 Definition of group action

Definition 4.1.1 Let W be a group and X be a set. By a W-action on X we mean a map

$$\phi : W \times X \to X$$

such that:

(1) $\phi(e, x) = x$ for all $x \in X$, where e is the identity of W;

(2) $\phi(w_2, \phi(w_1, x)) = \phi(w_2 w_1, x)$ for all $w_1, w_2 \in W$ and $x \in X$.

(X, ϕ) is called a W-set. We shall denote the W-set (X, ϕ) just by X.

For a $w \in W$, let

$$\phi_w : X \to X$$

be the map defined by $\phi_w(x) = \phi(w.x)$. ϕ_w is called the action of w. For simplity, we often use the notation $w \cdot x$ or wx for $\phi(w, x)$.

Notation and Basic Properties: The following definitions, notations and results can be found in [KK] and [tD].

(1) For a subset $A \subset X$, WA is the subset $\{wx \in X | x \in A, w \in W\}$. In particular, Wx is called the orbit of x, where x is an element of X. The number of elements in Wx is called the length of the orbit of x. If for any $x \in X$, $Wx = \{x\}$, we say that the W-action is trivial.

(2) For $x \in X$, the set $W_x = \{w \in W | wx = x\}$ is called the isotropy group at x.

(3) For a subgroup H of W, the set $X^H = \{x \in X | hx = x \text{ for all } h \in H\} = \{x \in X \mid W_x \supset H\}$ is called the H-fixed point set X. $X_H = \{x \in X | W_x = H\}$. It is obvious $X_H \subset X^H$.

(4) For subgroups H_1 and H_2 of W, recall that H_1 and H_2 are called conjugate in W if there exists $w \in W$ such that $H_2 = w^{-1} H_1 w$. We denote this equivalence relation by $\sim$ and write $H_1 \sim H_2$. The conjugacy class containing H is denoted by (H). If $H_1 \sim H_2$ and $H_2 \subset H$, we say that H_1 is subconjugate to H, denoted by $(H_1) \leq (H)$, or $(H_1) < (H)$ if $H_2 \neq H$. For a subgroup H of W, $X_{(H)} = \{x \in X | W_x \sim H\}$; $X^{(H)} = \{x \in X | (W_x) \geq (H)\}$. If H is an isotropy group at x for some $x \in X$, then (H) is called an isotropy type.

(5) If $(H_1), (H_2), ..., (H_n)$ are a finite number of isotropy types, then we can give them an ordering $\leq$ such that

$$(H_j) \leq (H_i) \Rightarrow i \leq j.$$

such an arrangement is called an admissible ordering on $\{(H_i)\}$.

(6) For a subgroup H of W, $X^{>H} = X^H - X_H = \bigcup_{H \subset K, K \neq H} X^K$. Also $X^{>(H)} = X^{(H)} - X_{(H)} = \bigcup_{(H)<(K)} X^{(K)}$.

Definition 4.1.2 A set W is called a topological group if W satisfies the following conditions.

(1) W is a Hausdorff space.

(2) W is a group.

(3) The composition map $\alpha : W \times W \to W$ and inverse map $\beta : W \to W$ defined by $\alpha(w_1, w_2) = w_1 w_2$ and $\beta(w) = w^{-1}$ respectively are continuous.

Example 4.1.3 A discrete group, for example a finite group, is a topological group with the discrete topology, i.e. every element subset is an open set.

Definition 4.1.4 Let W be a topological group and (X, ϕ) be a W-set. If X is a topological space and ϕ is continuous, (X, ϕ) is called a W-space. (Note that in this case, $\phi_w : X \to X$ is a homeomorphism for each $w \in W$. X is called a W-space.)

A W-action is said to be free if (the isotropy subgroup) W_x is trivial (i.e., $W_x = \{e\}$) for any $x \in X$. An action is said to be semifree if for each $x \in X$, $W_x = \{e\}$ or $W_x = W$. If W-action is not free and H is a isotropy group, W usually does not act on X^H because

X^H is not necessarily W-invariant. Let NH be the normalizer of H, i.e. $NH = \{w \in W \mid w^{-1}Hw = H\}$, we have

Lemma 4.1.5 *X^H is NH-invariant. Hence if H is a normal subgroup of W, then X^H is W-invariant.*

Proof: See Lemma 1.50 of [KK]. □

We denote the group NH/H by WH. It is called the Weyl group of H. Since H acts trivially on X^H, there is a natural action of WH on X^H. Note that WH acts freely on X_H.

Definition 4.1.6 Let X and Y be W-spaces, a map $f : X \to Y$ is called a W-map if for any $x \in X$ and $w \in W$, $f(wx) = wf(x)$. A homotopy $F : X \times I \to Y$ is called a W-Homotopy if it is a W-map, where W acts on I trivially. We note that, for each $t \in I$, $F(\cdot, t)$ is a W-map. If two W-maps f and g are homotopic via a W-homotopy, we say that f and g are W-homotopic and write $f \sim_W g$.

The proof of the next two results are trivial.

Proposition 4.1.7 *Let W be a group, X and Y be W-spaces and f and $g : X \to Y$ be W-maps. If $x \in X$ is a coincidence point, then so is each $y \in Wx$.* □

Proposition 4.1.8 *If $f : X \to Y$ is a W-map, then for any subgroup H of W, $f(X^H) \subset Y^H$. The restriction of f on X^H will be denoted by $f^H : X^H \to Y^H$.* □

Let X be a W-space, and $p_X : \tilde{X} \to X$ is a universal covering space of X and $\Pi(X)$ be the group of covering transformations. Define

$$\tilde{W}_X = \{\tilde{\gamma} \mid \tilde{\gamma} : \tilde{X} \to \tilde{X} \text{ is a homeomorphism and } p_X\tilde{\gamma} = \gamma p_X \text{ for some } \gamma \in W\},$$

Thus $\tilde{W}_X$ consists of all the liftings of $w : X \to X$ for all $w \in W$. It is easy to see $\Pi(X) \subset \tilde{W}_X$ and that the sequence

$$1 \to \Pi(X) \to \tilde{W}_X \to W \to 1$$

is exact.

Example 4.1.9 Let $X = S^1$, $W = \mathbf{Z}/2 =< \alpha >$ and an action of W on X be determined by

$$\alpha(x, y) = (x, -y).$$

$\tilde{W}_X$ consists of all the liftings of the identity map, which is the transformation group, and all the liftings of α. So

$$\begin{aligned} \tilde{W}_X &= \{\phi_n : \mathbf{R} \to \mathbf{R} \mid \phi_n(x) = x + n, n \in \mathbf{Z}\} \\ &\cup \{\psi_n : \mathbf{R} \to \mathbf{R} \mid \psi_n(x) = -x + n, n \in \mathbf{Z}\} \end{aligned}$$

4.1.2 Actions on complexes and manifolds

Definition 4.1.10 A simplicial complex K is called a W-complex, if there is a W-action on the set of the vertices of K, such that if $(x_0, x_1, ..., x_n)$ is a simplex of K, then $(wx_0, wx_1, ..., wx_n)$ is a simplex of K for any $w \in W$.

Suppose W acts on K. Let $w \in W$ and $\Delta = (x_0, x_1, ..., x_n)$ be a simplex of K, then w induces a map from Δ to $w\Delta$ as follows:

$$w(\sum a_i x_i) = \sum a_i(wx_i),$$

and hence w induces a continuous map from $|K|$ to itself, where $|K|$ denotes the polyhedron of K. By abuse of notation we also denote this map by w.

Consider the following property concerning a W-complex K.

Definition 4.1.11 (see [KK] p.229) Let K be a W-complex, if for any any $w \in W$ and any simplex s of K, w leaves $s \cap ws$ pointwise fixed, we say that K possesses property (P_1).

Suppose W acts on K, and K' denote the barycentric subdivision of K. Then the action of W on K induces an action of W on K' as follow: if a is the barycenter of a simplex $(x_0, \ldots, x_k)$ and $w \in W$, then $w(a)$ is defined to be the barycenter of the simplex $(w(x_0), \ldots, w(x_k))$.

Lemma 4.1.12 (p.229 in [KK]) *Let K be a W-complex having the property (P_1) and $s = (x_0, \ldots, x_k)$ be a simplex of K. If there exists a $w \in W$ with $wx_i = x_j$ for some i and j, then we have $i = j$.* □

Lemma 4.1.13 (p.230 in [KK]) *If K is a W-complex, then the induced W-action on the barycentric subdivision K' possesses Property (P_1).* □

Lemma 4.1.14 *Let K be a W-complex, and K_1 is an invariant subcomplex of K. Then the inclusion $i : |K_1| \to |K|$ has homotopy extension property for all W-maps $f : |K| \to Y$ and $\phi : |K_1| \times I \to Y$ with $\phi(a, 0) = f(a)$ for all $a \in |K_1|$. This means that given f and ϕ, there exists a W-map $\psi : |K| \times I \to Y$ such that $\psi|_{|K_1| \times I} = \phi$ and $\psi(x, 0) = f(x)$ for all $x \in |K|$.*

Proof: See exercise 3 on p.103 in [tD], or p.32 in [HP]. □

Let M be a smooth manifold, W a Lie group. If W acts on M and the action $\phi : W \times M \to M$ is smooth, the action is called a smooth W-action, and M is called a smooth W-manifold or simply a W-manifold.

Theorem 4.1.15 ([KK], Theorem 4.14 and Lemma 4.15) *Let W act smoothly on M, then for any subgroup H of W, the fixed point set M^H of H is a closed submanifold of M. In addition, M^H is a smooth WH-manifold, and M_H is an open set of M^H.* □

Definition 4.1.16 A W-triangulation (K, ϕ) of a W-manifold M consists of a W-complex K and a W-homeomorphism $\phi : |K| \to M$, where $|K|$ is given the W-action induced by the W-action on K.

Theorem 4.1.17 (see [IS]) *If W is a finite group and M is a W-manifold, then M has a W-triangulation.* □

Theorem 4.1.18 (p.305 in [BG]) *If W is a finite group and M is a W-manifold, then M has a W-invariant riemannian metric.* □

4.2 Equivariant coincidence classes

In this section, we will introduce the concepts of equivariant coincidence classes, the essentiality of such classes and also an equivariant Nielsen number of a pair of equivariant maps. As in the ordinary case, we use liftings of $(f, g) : X \to Y$ to define coincidence classes. However, since we consider coincidence point orbits, we use $\tilde{W}_X$ and $\tilde{W}_Y$ (see section 4.1.1) instead of $\Pi(X)$ and $\Pi(Y)$ to classify these liftings.

Definition 4.2.1 Let X and Y be orientable W-manifolds, and f and $g : X \to Y$ a pair of W-maps. Two liftings $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ of (f, g) are said to be W-conjugate if there are

$\tilde{\gamma}^X \in \tilde{W}_X$ and $\tilde{\gamma}^Y \in \tilde{W}_Y$ such that $(\tilde{f},\tilde{g})=\tilde{\gamma}^Y(\tilde{f}',\tilde{g}')(\tilde{\gamma}^X)^{-1}$. It is obvious that conjugacy is an equivalence relation. Denote the W-conjugacy class of $(\tilde{f},\tilde{g})$ by $[(\tilde{f},\tilde{g})]_W = \{(\tilde{f}',\tilde{g}')|(\tilde{f}',\tilde{g}') = \tilde{\gamma}^Y(\tilde{f},\tilde{g})(\tilde{\gamma}^X)^{-1}$, for some $\tilde{\gamma}^Y \in \tilde{W}_Y$, and $\tilde{\gamma}^X \in \tilde{W}_X\}$.

A W-conjugacy class is called a W-Reidemeister class of (f,g), and the set of all W-Reidemeister classes are called the W-Reidemeister set and denoted by $\mathcal{R}_{f,g}(W)$. The number of W-Reidemeister classes is called the W-Reidemeister number, and denoted by $R_W(f,g)$.

Note 4.2.2 Generally, $\tilde{\gamma}^Y(\tilde{f},\tilde{g})(\tilde{\gamma}^X)^{-1}$ is not necessarily a lifting of (f,g) for elements $\tilde{\gamma}^X \in \tilde{W}_X$ and $\tilde{\gamma}^Y \in \tilde{W}_Y$, since $\tilde{\gamma}^X$ and $\tilde{\gamma}^Y$ may be liftings of different elements $w_1, w_2 \in W$. So $[(\tilde{f},\tilde{g})]$ is not the set $\{\tilde{\gamma}^Y(\tilde{f},\tilde{g})(\tilde{\gamma}^X)^{-1}|\tilde{\gamma}^Y \in \tilde{W}_Y, \tilde{\gamma}^X \in \tilde{W}_X\}$. ($cf.$ Definition 1.1.1)

Proposition 4.2.3 *A W-Reidemeister class consists of a union of ordinary Reidemeister classes.*

Proof: Since $\Pi(X)$ is a subset of $\tilde{W}_X$ and $\Pi(Y)$ is a subset of $\tilde{W}_Y$, if two liftings $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ of (f,g) are conjugate then they are W-conjugate. So each ordinary Reidemeister class is entirely contained in a W-Reidemeister class; that is $\mathcal{R}_{f,g}(W)$ is a quotient of $\mathcal{R}_{f,g}$. □

The next proposition generalizes Proposition 2.2 of [WP3].

Proposition 4.2.4 *Let $(f,g) : X \to Y$ be a pair of W-maps and $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ be liftings of (f,g), then we have*

(1) If $[(\tilde{f},\tilde{g})]_W = [(\tilde{f}',\tilde{g}')]_W$, then $W(p_X(\Gamma(\tilde{f},\tilde{g}))) = W(p_X(\Gamma(\tilde{f}',\tilde{g}')))$.

(2) If $[(\tilde{f},\tilde{g})]_W \neq [(\tilde{f}',\tilde{g}')]_W$, then $W(p_X(\Gamma(\tilde{f},\tilde{g}))) \cap W(p_X(\Gamma(\tilde{f}',\tilde{g}'))) = \emptyset$.

Proof: (1) Suppose $(\tilde{f}', \tilde{g}') = \tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1}$ for some $\tilde{\gamma}^X \in \tilde{W}_X$ and $\tilde{\gamma}^Y \in \tilde{W}_Y$. We only need to prove $W(p_X\Gamma(\tilde{f}, \tilde{g})) \subseteq W(p_X\Gamma(\tilde{f}', \tilde{g}'))$, or $p_X\Gamma(\tilde{f}, \tilde{g}) \subseteq W(p_X\Gamma(\tilde{f}', \tilde{g}'))$ as the right hand side is closed under the action of W.

Let $x \in p_X\Gamma(\tilde{f}, \tilde{g}))$, and $\tilde{x} \in \Gamma(\tilde{f}, \tilde{g})$ be in the fiber over x. Then $\tilde{f}'(\tilde{\gamma}^X(\tilde{x})) = \tilde{\gamma}^Y \circ \tilde{f}(\tilde{\gamma}^X)^{-1}(\tilde{\gamma}^X(\tilde{x})) = \tilde{\gamma}^Y \circ \tilde{f}(\tilde{x}) = \tilde{\gamma}^Y \circ \tilde{g}(\tilde{x}) = \tilde{\gamma}^Y \circ \tilde{g}(\tilde{\gamma}^X)^{-1}(\tilde{\gamma}^X(\tilde{x})) = \tilde{g}'(\tilde{\gamma}^X(\tilde{x}))$, i.e. $\tilde{\gamma}^X(\tilde{x}) \in \Gamma(\tilde{f}', \tilde{g}')$. Note that $p_X(\tilde{\gamma}^X(\tilde{x})) = \gamma(p_X(\tilde{x})) = \gamma(x)$ for some $\gamma \in W$. Therefore, $\gamma(x) \in p_X\Gamma(\tilde{f}', \tilde{g}')$ and $x \in W(p_X\Gamma(\tilde{f}', \tilde{g}'))$.

(2) We prove that if $x \in W(p_X(\Gamma(\tilde{f}, \tilde{g}))) \cap W(p_X(\Gamma(\tilde{f}', \tilde{g}')))$ for some $x \in X$, then $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ are conjugate. Let $\gamma_1, \gamma_2 \in W$ such that $x_1 = \gamma_1(x) \in p_X\Gamma(\tilde{f}, \tilde{g})$ and $x_2 = \gamma_2(x) \in p_X\Gamma(\tilde{f}', \tilde{g}')$. Let $\tilde{x}_1$ be in the fiber over x_1 and in $\Gamma(\tilde{f}, \tilde{g})$, and $\tilde{x}_2$ in the fiber over x_2 and in $\Gamma(\tilde{f}', \tilde{g}')$. Let $\tilde{\gamma}_1^X$ and $\tilde{\gamma}_2^X$ be liftings of γ_1 and γ_2 respectively such that $\tilde{\gamma}_1^X \circ (\tilde{\gamma}_2^X)^{-1}(\tilde{x}_2) = \tilde{x}_1$. Let $\tilde{\gamma}_3^X = \tilde{\gamma}_1^X \circ (\tilde{\gamma}_2^X)^{-1}$. There is an element $\tilde{\gamma}^Y \in \tilde{W}^Y$ such that $\tilde{\gamma}^Y \tilde{f} \tilde{\gamma}_3^X = \tilde{f}'$. Then $\tilde{\gamma}^Y(\tilde{f}, \tilde{g})\tilde{\gamma}_3^X$ has $\tilde{x}_2$ as a coincident point. Since $\tilde{x}_2$ is also a coincidence point of $(\tilde{f}', \tilde{g}')$ and $\tilde{\gamma}^Y \tilde{f} \tilde{\gamma}_3^X = \tilde{f}'$, we have $\tilde{g}'(\tilde{x}_2) = \tilde{f}'(\tilde{x}_2) = \tilde{\gamma}^Y \tilde{f} \tilde{\gamma}_3^X(\tilde{x}_2) = \tilde{\gamma}^Y \tilde{g} \tilde{\gamma}_3^X(\tilde{x}_2)$. Since both $\tilde{g}'$ and $\tilde{\gamma}^Y \tilde{g} \tilde{\gamma}_3^X$ are liftings of g, $\tilde{g}' = \tilde{\gamma}^Y \tilde{g} \tilde{\gamma}_3^X$ by uniqueness. This shows $[(\tilde{f}, \tilde{g})]_W = [(\tilde{f}', \tilde{g}')]_W$. □

Proposition 4.2.4 allows us to define a W-coincidence class in the same way we define classes in ordinary coincidence theory.

Definition 4.2.5 Let (f, g) be a pair of W-maps from X to Y and $(\tilde{f}, \tilde{g})$ be a lifting of (f, g). The W-subset $W(p_X\Gamma(\tilde{f}, \tilde{g}))$ of $\Gamma(f, g)$ is called the W-coincidence class determined by the conjugacy class $[(\tilde{f}, \tilde{g})]_W$ (or briefly the W-class). The set of non-empty W-classes is denoted by $\tilde{\Gamma}_W(f,g)$. We have an injective map

$$\rho^W_{\mathcal{R}_{f,g}} : \tilde{\Gamma}_W(f,g) \to \mathcal{R}_{f,g}(W),$$

which sends a W-coincidence class S to the W-Reidemeister class $[(\tilde{f},\tilde{g})]_W$ if $W(p_X\Gamma(\tilde{f},\tilde{g})) = S$.

Proposition 4.2.6 *Two coincidence points $x_0, x_1 \in X$ are in the same W-class if and only if*

(1) $x_1 = wx_0$ for some $w \in W$, or

(2) there exists a path $\alpha : [0,1] \to X$ such that $\alpha(0) = x_0, \alpha(1) = wx_1$ for some $w \in W$ and $f \circ \alpha \sim g \circ \alpha$ rel $\{0,1\}$.

Hence, each W-class is the W-orbit of some ordinary coincidence classes and there are finitely many non-empty W-classes.

Proof: Suppose x_0 and x_1 are in the same W-class, then there is a lifting $(\tilde{f},\tilde{g})$ of (f,g) such that $x_0, x_1 \in W(p_X\Gamma(\tilde{f},\tilde{g}))$, and there are $w_1, w_2 \in W$ and $x_0', x_1' \in p_X\Gamma(\tilde{f},\tilde{g})$ such that $x_0 = w_0x_0'$, and $x_1 = w_1x_1'$. Now if $x_0' = x_1'$, then $x_1 = w_1x_1' = w_1x_0' = (w_1w_0^{-1})w_0x_0' = (w_1w_0^{-1})x_0$, so (1) is satisfied; if $x_0' \neq x_1'$, then there is a path α' from x_0' to x_1' such that $f\circ\alpha' \simeq g\circ\alpha'$. Let $\alpha = w_0\alpha'$, then α is a path from x_0 to $w_0x_1' = (w_0w_1^{-1})w_1x_1' = (w_0w_1^{-1})x_1$, and $f\circ\alpha \sim g\circ\alpha$ rel $\{0, 1\}$ since f and g are equivariant. So in this case (2) is satisfied.

For the converse, suppose that $x_0, x_1 \in \Gamma(f,g)$ and either (1) or (2) is satisfied. Suppose $x_0 \in p_X\Gamma(\tilde{f},\tilde{g})$ for some lifting $(\tilde{f},\tilde{g})$ of (f,g). If $x_1 = wx_0$ for some $w \in W$, then $x_1 \in w(p_X\Gamma(\tilde{f},\tilde{g})) \subset W(p_X(\Gamma(\tilde{f},\tilde{g}))$, i.e. x_0, x_1 are in the same W-class. If there is a path α from

x_0 to wx_1 such that $f \circ \alpha \simeq g \circ \alpha$ rel $\{0,1\}$, then x_0, and wx_1 are in the same Nielsen class, i.e. $wx_1 \in p_X\Gamma(\tilde{f},\tilde{g})$. Therefore, $x_1 \in w^{-1}(p_X\Gamma(\tilde{f},\tilde{g}))$, and so x_0 and x_1 are in the same W-class as required.

Finally from (2), it is easy to see that if a coincidence point is in a W-class, then the whole Nielsen class that contains it is in that same W-class. Since there is only a finite numbher of non-empty Nielsen classes, the number of non-empty W-classes is finite. □

Definition 4.2.7 Suppose that N^W is a W-class of (f,g). Choose an ordinary Nielsen class $N \subset N^W$. Then N^W is essential if and only if N is essential in the ordinary sense. This is well defined by Proposition 4.2.9.

Note 4.2.8 For convenience, we will identify the elements in $\tilde{\Gamma}_W(f,g)$ with some of the elements in $\mathcal{R}_{f,g}(W)$, by means of the injective map $\rho^W_{\mathcal{R}_{f,g}} : \tilde{\Gamma}_W(f,g) \to \mathcal{R}_{f,g}(W)$ (see Definition 4.2.5). We will call an element $\theta \in \mathcal{R}_{f,g}(W)$ essential if it is an image of an essential element of $\tilde{\Gamma}_W(f,g)$ under $\rho^W_{\mathcal{R}_{f,g}}$.

Proposition 4.2.9 *Suppose that two ordinary Nielsen classes N_1 and N_2 of (f,g) belong to a common W-class. Then $|ind(N_1)| = |ind(N_2)|$. Thus, the essentiality of a W-class is well defined.*

Proof: Since N_1 and N_2 belong to the same W class, then $N_1 = wN_2$ for some $w \in W$. Let U be an open set such that $U \cap \Gamma(f,g) = N_2$, and V open set such that $N_2 \subset V \subset \overline{V} \subset U$.

Let wU, wV be the images of U and V under w, the following diagram

$$\begin{array}{cccc}
H_n(X) \xrightarrow{i_*} & H_n(X, X-V) \xrightarrow{j_*^{-1}} & H_n(U, U-V) \xrightarrow{(f,g)_*} & H_n(Y\times Y, Y\times Y-\Delta(Y)) \\
w_* \downarrow & w_* \downarrow & w_* \downarrow & w_* \downarrow \\
H_n(X) \xrightarrow{i_*} & H_n(X, X-wV) \xrightarrow{j_*^{-1}} & H_n(wU, wU-wV) \xrightarrow{(f,g)_*} & H_n(Y\times Y, Y\times Y-\Delta(Y))
\end{array}$$

is commutative, where w_* is the isomorphism induced by w. In particular, the index $\mathrm{ind}(wN_2)$ of wN_2 is either $ind(N_2)$ or $(-1)\cdot ind(N_2)$. □

Definition 4.2.10 The number of essential W-classes is called the W-Nielsen number of (f, g), and is denoted by $N_W(f, g)$.

Proposition 4.2.11 *Let X, Y be W-manifolds with the same dimension, and $f, g : X \to Y$ W-maps. Then on X, there are at least $N_W(f, g)$ orbits of coincidence points of (f, g).*

Proof: Assume that N^W is an essential W-class and $N \subset N^W$ is an ordinary Nielsen class. By definition, N is essential and therefore contains at least one coincidence point. This implies that N^W contains at least one coincidence orbit. □

Note 4.2.12 Unfortunately since the number of points in the various orbits may vary we cannot always use $N_W(f, g)$ to give a good estimate of the number of coincidence points of (f, g).

Proposition 4.2.13 *If $f \sim_W f'$ and $g \sim_W g'$, then $N_W(f, g) = N_W(f', g')$.*

Proof: A pair of homotopies $(F, G) : X \times I \to Y$ from (f, g) to (f', g') induces a one to one correspondence between the Nielsen classes of (f, g) and those of (f', g'). Since F

and G are W-maps, the one to one correspondence induces a one to one correspondence between $\mathcal{R}_{f,g}(W)$ and $\mathcal{R}_{f',g'}(W)$. Let N^W be an essential W-class of (f,g), and $N \subset N^W$ is an ordinary Nielsen class contained in N^W. Suppose that N corresponds to N', an ordinary Nielsen class of (f',g'), then N' is essential. This implies the W-class containing N' is essential. Therefore each essential W-class of $\mathcal{R}_{f,g}(W)$ corresponds to an essential W-class of $\mathcal{R}_{f',g'}(W)$. □

The following simple example illustrates that $N_W(f,g)$ is not a homotopy invariant, although it is an equivariant homotopy invariant as shown in Proposition 4.2.13. Since we choose one of the maps be identity, this example shows that the corresponding number in fixed point theory is not a homotopy invariant either.

Example 4.2.14 Let X and W be the same as in Example 4.1.9, and $Y = X$.

Let $f : X \to Y$ be defined by

$$f(x,y) = (x,-y),$$

$f_1 : X \to Y$ be defined by

$$f_1(x,y) = (-x,y),$$

and $g = g_1 : X \to Y$ be the identity. We claim that $N_W(f,g) = 2$ and $N_W(f_1,g_1) = 1$ even though $f \sim f_1$ and $g = g_1$. However f and f_1 are not W-homotopic.

(1) $N_W(f,g) = 2$: there are two coincidence points, $(1,0)$ and $(-1,0)$, and both have non-zero indices. Since they are not ordinary Nielsen equivalent and they are fixed by the W, they are not in the same W-class. So $N_W(f,g) = 2$.

(2) $N_W(f_1, g_1) = 1$: there are two coincidence points, $(0,1)$ and $(0,-1)$, both have non-zero indices. Since they are in the same W-orbit, they are in the same W-class. So $N_W(f_1, g_1) = 1$.

The next propostion shows that just as in the ordinary case $N(f,g) \leq R(f,g)$, so the W-Reidemeister number is an upper bound for the W-Nielsen number.

Proposition 4.2.15 $N_W(f,g) \leq R_W(f,g)$.

Proof: The set of essential W-classes is a subset of $\tilde{\Gamma}_W(f,g)$, and there is an injection from $\tilde{\Gamma}_W(f,g)$ to $\mathcal{R}_{f,g}(W)$. So we have $N_W(f,g) \leq R_W(f,g)$. □

4.3 Computation of equivariant Nielsen numbers

The computation of the Nielsen numbers in any Nielsen theory is always difficult. The computation in the equivariant cases is worse since $\mathcal{R}_{f,g}(W)$ is much more complicated than $\mathcal{R}_{f,g}$, and $R_W(f,g)$ is not even a homotopy invariant as we saw in Example 4.2.14. The usual way to compute a Reidemeister number is to relate the Reidemeister set to the fundamental group. While this does not pose a problem in ordinary coincidence theory, it is more difficult in the equivariant case in general. When X^W is nonempty, however, it is possible to represent coincidence classes by means of the fundamental group of Y.

In this section, we will, first of all, relate $N_W(f,g)$ to $R_W(f,g)$, then we will describe the equivariant coincidence classes in terms of the fundamental group in cases where $X^W \neq \emptyset$, and both f and g send some component of X^W to the same component of Y^W. Finally, we establish conditions under which $N_W(f,g)$ and $R_W(f,g)$ may be computed easily.

The following theorem generalizes from the equivariant fixed point case, and improves upon the result of Theorem 4.10 in [WP3].

Theorem 4.3.1 *Let X and Y be W-manifolds, x_0 and y_0 be basepoints of X and Y respectively, and f, g be W-maps. If $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g) = \mathcal{R}_{f,g}$ (* see Definition 2.2.7 for the definition of $\tilde{T}(f,g;x_0,y_0,\omega_f,\omega_g)$*), then*

(1) $L(f,g) = 0 \Rightarrow N_W(f,g) = 0$, and

(2) $L(f,g) \neq 0 \Rightarrow N_W(f,g) = R_W(f,g)$.

Proof: By Corollary 2.2.14, the hypothesis guarantees that each ordinary coincidence class has the same index. If $L(f,g) = 0$, then every Nielsen class has index 0 by Theorem 1.3.20. By Proposition 4.2.6, each W-class is the union of several ordinary Nielsen classes and therefore its index is zero too. If $L(f,g) \neq 0$, then each ordinary Nielsen class has nonzero index. Thus each W-class must contain some ordinary Nielsen class with nonzero index, so it is essential by its definition. □

For the rest of this section, we will assume that $X^W \neq \emptyset$, $Y^W \neq \emptyset$, $x_0 \in X^W$, $y_0 \in Y^W$, and $f(x_0)$ and $g(x_0)$ are in the same component of Y^W as y_0. There is a W-action over $\pi_1(Y,y_0)$ defined by $w[\alpha] = [w\alpha]$ for each $w \in W$ and any loop $\alpha : (I,\{0,1\}) \to (Y,y_0)$ in Y, where $w\alpha$ is a path defined by $w\alpha(t) = w(\alpha(t))$.

Let ω_g, ω_f be two paths from y_0 to $g(x_0)$ and $f(x_0)$ respectively in Y^W, and let $g_\pi^{\omega_g}, f_\pi^{\omega_f}$ be the corresponding homomorphisms from $\pi_1(X,x_0)$ to $\pi_1(Y,y_0)$ defined using conjugation by ω_g, ω_f respectively. The fact that f and g are equivariant means that $g_\pi^{\omega_g}$ and $f_\pi^{\omega_f}$ are invariant with respect to this action on $\pi_1(Y, y_0)$ and a similar action on $\pi_1(X, x_0)$. A W-action on $\nabla(f, g; x_0, y_0, \omega_g, \omega_f)$ (see Definition 1.2.3) is defined by $w\overline{\beta} = \overline{w\beta}$ for any $w \in W$ and $\beta \in \pi_1(Y, y_0)$. The class of $\nabla(f, g; x_0, y_0, \omega_g, \omega_f)/W$ containing $\zeta \in \nabla(f, g; x_0, y_0, \omega_g, \omega_f)$ will be denoted as $W\zeta$.

Lemma 4.3.2 *The action defined above is well defined.*

Prove: Suppose $\overline{\beta} = \overline{\beta'}$, then by definition there is an element $\gamma \in \pi_1(X, x_0)$ such that $\beta' = g_\pi^{\omega_f}(\gamma)\beta f_\pi^{\omega_f}(\gamma^{-1})$. Then $w\beta' = w(g_\pi^{\omega_f}(\gamma)\beta f_\pi^{\omega_f}(\gamma^{-1})) = g_\pi^{\omega_g}(w\gamma)w\beta f_\pi^{\omega_f}(w\gamma^{-1})$, and therefore, $w\overline{\beta} = \overline{w\beta} = \overline{w\beta'} = w\overline{\beta'}$. □

Recall Definition 1.2.4 that there is a map $\Theta_{f,g}$ from $\mathcal{R}_{f,g}$ to $\nabla(f, g; x_0, y_0, \omega_g, \omega_f)$ and note that $\mathcal{R}_{f,g}(W)$ is actually a quotient set of $\mathcal{R}_{f,g}$. We define $\Theta_{f,g}^W : \mathcal{R}_{f,g}(W) \to \nabla(f, g; x_0, y_0, \omega_f, \omega_g)/W$ by

$$\Theta_{f,g}^W([(\tilde{f}, \tilde{g})]_W) = W\Theta_{f,g}([(\tilde{f}, \tilde{g})]).$$

Theorem 4.3.3 $\Theta_{f,g}^W$ *is well defined and is a bijection.*

Proof: We first prove that $\Theta_{f,g}^W$ is well defined. Assume that $[(\tilde{f}, \tilde{g})]_W = [(\tilde{f}', \tilde{g}')]_W$, that is there are $\tilde{\gamma}^X \in W_X$ and $\tilde{\gamma}^Y \in W_Y$, such that $(\tilde{f}', \tilde{g}') = \tilde{\gamma}^Y(\tilde{f}, \tilde{g})(\tilde{\gamma}^X)^{-1}$. We have to prove that $\Theta_{f,g}([(\tilde{f}', \tilde{g}')] \in W\Theta_{f,g}([(\tilde{f}, \tilde{g})])$.

Let $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$ be universal covering spaces of X and Y respectively, $\tilde{x}_0 \in p_X^{-1}(x_0)$ and $\tilde{y}_0 \in p_Y^{-1}(y_0)$. Let $\tilde{\theta} : I \to \tilde{Y}$ from $\tilde{g}(\tilde{x}_0)$ to $\tilde{f}(\tilde{x}_0)$, then $\Theta_{f,g}([(\tilde{f}, \tilde{g})]) = \overline{[\omega_g \cdot (p_Y \circ \tilde{\theta}) \cdot \omega_f^{-1}]}$. Let $\tilde{\eta} : I \to \tilde{X}$ is a path from $\tilde{x}_0$ to $(\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$. Since $p_X \circ (\tilde{\gamma}^X)^{-1}(\tilde{x}_0) =$

$(\gamma^X)^{-1} \circ p_X(\tilde{x}_0) = (\gamma^X)^{-1}(x_0) = x_0$ since $x_0 \in X^W$, we know that $p_X \circ \tilde{\eta}$ must be a loop at x_0. We denote it by η. Note that $(\tilde{g} \circ \tilde{\eta}^{-1}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\eta})$ is a path from $\tilde{g} \circ (\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$ to $\tilde{f} \circ (\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$ and $\tilde{\gamma}^Y((\tilde{g} \circ \tilde{\eta}^{-1}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\eta}))$ is a path from $\tilde{\gamma}^Y \tilde{g} \circ (\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$ to $\tilde{\gamma}^Y \tilde{f} \circ (\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$, i.e. a path from $\tilde{g}'(\tilde{x}_0)$ to $\tilde{f}'(\tilde{x}_0)$. So

$$
\begin{aligned}
&\Theta_{f,g}([(\tilde{f}', \tilde{g}')]) \\
&= \overline{[\omega_g \cdot (p_Y \circ \tilde{\gamma}^Y((\tilde{g} \circ \tilde{\eta}^{-1}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\eta}))) \cdot \omega_f^{-1}]} \\
&= \overline{[\omega_g \cdot (\gamma^Y \circ p_Y((\tilde{g} \circ \tilde{\eta}^{-1}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\eta}))) \cdot \omega_f^{-1}]} \\
&= \overline{[\gamma^Y(\omega_g \cdot (p_Y((\tilde{g} \circ \tilde{\eta}^{-1}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\eta}))) \cdot \omega_f^{-1})]} \\
&= \overline{[\gamma^Y(\omega_g \cdot (g \circ \eta^{-1}) \cdot (p_Y \circ \tilde{\theta}) \cdot (f \circ \eta)) \cdot \omega_f^{-1})]} \\
&= \gamma^Y \overline{[\omega_g \cdot (g \circ \eta^{-1}) \cdot \omega_g^{-1} \cdot \omega_g \cdot (p_Y \circ \tilde{\theta}) \cdot \omega_f^{-1} \cdot \omega_f(f \circ \eta)) \cdot \omega_f^{-1}]} \\
&= \gamma^Y \overline{g_\pi^{\omega_g}([\eta^{-1}])[\omega_g \cdot (p_Y \circ \tilde{\theta}) \cdot \omega_f^{-1}] f_\pi^{\omega_f}([\eta])}.
\end{aligned}
$$

This proves that $(\gamma^Y)^{-1}(\Theta_{f,g}([(\tilde{f}', \tilde{g}')])) = \Theta_{f,g}([(\tilde{f}, \tilde{g})])$, or equivalently, $\Theta_{f,g}([(\tilde{f}', \tilde{g}')]) \in W\Theta_{f,g}([(\tilde{f}, \tilde{g})])$ and hence $\Theta^W_{f,g}$ is well defined.

Since $\Theta_{f,g}$ is surjective, the surjectivity of $\Theta^W_{f,g}$ follows from the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{R}_{f,g} & \xrightarrow{\Theta_{f,g}} & \nabla(f, g; x_0, y_0, \omega_f, \omega_g) \\
\downarrow & & \downarrow \\
\mathcal{R}_{f,g}(W) & \xrightarrow{\Theta^W_{f,g}} & \nabla(f, g; x_0, y_0, \omega_f, \omega_g)/W,
\end{array}
$$

where the vertical maps are projections.

To prove that $\Theta^W_{f,g}$ is injective, let $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ be liftings of (f, g) such that $\Theta^W_{f,g}([(\tilde{f}, \tilde{g})]_W) = \Theta^W_{f,g}([(\tilde{f}', \tilde{g}')]_W)$. We want to prove that $[(\tilde{f}, \tilde{g})]_W = [(\tilde{f}', \tilde{g}')]_W$. Let $\tilde{\theta} : I \to \tilde{Y}$ be a path from $\tilde{g}(x_0)$ to $\tilde{f}(x_0)$, and $\tilde{\theta}' : I \to \tilde{Y}$ be a path from $\tilde{g}'(x_0)$ to $\tilde{f}'(x_0)$. Then

$W\overline{[\omega_g \cdot (p_Y \circ \tilde{\theta}) \cdot (\omega_f)^{-1}]} = \Theta^W_{f,g}([(\tilde{f},\tilde{g})]_W) = \Theta^W_{f,g}([(\tilde{f}',\tilde{g}')]_W) = W\overline{[\omega_g \cdot (p_Y \circ \tilde{\theta}') \cdot (\omega_f)^{-1}]}$, or equivalently, there is a $\gamma \in W$ and a $[\beta] \in \pi_1(X, x_0)$ such that

$$\omega_g \cdot (p_Y \circ \tilde{\theta}') \cdot (\omega_f)^{-1} \sim \gamma(g_\pi^{\omega_g}([\beta])[\omega_g \cdot (p_Y \circ \tilde{\theta}) \cdot (\omega_f)^{-1}]f_\pi^{\omega_f}([\beta]^{-1})).$$

Let $\tilde{\gamma}^X : \tilde{X} \to \tilde{X}$ be a lifting of $\gamma : X \to X$ such that there is a lifting $\tilde{\beta}$ of β starting with $(\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$ and ending with $\tilde{x}_0$. (Such $\tilde{\gamma}^X$ exists, it can be chosen as follows: since $\gamma(x_0) = x_0$ and there is a lifting $\tilde{\gamma}'$ of γ such that $\tilde{\gamma}'(\tilde{x}_0) = \tilde{x}_0$, let ϕ be a lifting of identity such that the lifting $\tilde{\beta}$ of β starting at $\phi^{-1}(\tilde{x}_0)$ ends at $\tilde{x}_0$, then the composition of ϕ and $\tilde{\gamma}'$ is the required $\tilde{\gamma}$.) By the same argument, we can find a lifting $\tilde{\gamma}^Y : \tilde{Y} \to \tilde{Y}$ of $\gamma : Y \to Y$ such that $\tilde{\gamma}^Y\tilde{g}(\tilde{\gamma}^X)^{-1}(\tilde{x}_0) = \tilde{g}'(\tilde{x}_0)$. So $\tilde{\gamma}^Y\tilde{g}(\tilde{\gamma}^X)^{-1} = \tilde{g}'$. Now $\tilde{\gamma}^Y((\tilde{g} \circ \tilde{\beta}) \cdot \tilde{\theta} \cdot (\tilde{f} \circ \tilde{\beta}^{-1}))$ is a path in $\tilde{Y}$ starting at $\tilde{g}'(\tilde{x}_0)$ and ending at $\tilde{\gamma}^Y\tilde{f}(\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$. Since $p_Y \circ \tilde{\theta}'$ is homotopic to $\gamma((g\circ\beta)\cdot(p_Y\circ\tilde{\theta})\cdot(f\circ\beta^{-1}))$, then $\tilde{\theta}'$ and $\tilde{\gamma}^Y((\tilde{g}\circ\tilde{\beta})\cdot\tilde{\theta}\cdot(\tilde{f}\circ\tilde{\beta}^{-1}))$ have the same end point, namely $\tilde{f}'(\tilde{x}_0) = \tilde{\gamma}^Y\tilde{f}(\tilde{\gamma}^X)^{-1}(\tilde{x}_0)$. This proves $\tilde{f}' = \tilde{\gamma}^Y\tilde{f}(\tilde{\gamma}^X)^{-1}$ and hence $[(\tilde{f},\tilde{g})]_W = [(\tilde{f}',\tilde{g}')]_W$. $\square$

In order to make the computation practical, we compare the Reidemeister classes with $Coker(g_* - f_*)$. This is because homology is much simpler than the fundamental group. Let X be a W-space, then there is a natural W-action on $H_1(X)$ defined as follows. Let $\sigma : I \to X$ be a simplex. For any $w \in W$, define $w \cdot \sigma : I \to X$ by $(w \cdot \sigma)(t) = w \cdot \sigma(t)$. For a chain $\Sigma_\sigma a_\sigma \sigma \in S_1(X)$, we define $w \cdot (\Sigma_\sigma a_\sigma \sigma) = \Sigma_\sigma a_\sigma (w \cdot \sigma)$. This gives a W-action on $S_1(X)$. Let $[z] \in H_1(X)$ and $w \in W$, we define a W-action on $H_1(X)$ by $w \cdot [z] = [w \cdot z]$. This is well defined since the action of w commutes with the boundary operator. This action induces an action on $Coker(g_* - f_*)$ defined by $w \cdot \{[z]\} = \{w \cdot [z]\}$, where $\{[z]\}$ be the element of $Coker(g_* - f_*)$ containing $[z]$. It is easy to check that this action is well defined since both f and g are W-maps. The class of $Coker(g_* - f_*)/W$ containing $\xi \in Coker(g_* - f_*)$ is denoted by $W\xi$. With this action on $Coker(g_* - f_*)$, we can define a map $h_W : \nabla(f, g; x_0, y_0, \omega_g, \omega_f)/W \to Coker(g_* - f_*)/W$ by $h_W(W\zeta) = Wh(\zeta)$, where $h : \nabla(f, g; x_0, y_0, \omega_g, \omega_f) \to Coker(g_* - f_*)$ is defined in Lemma 2.2.1.

Proposition 4.3.4 *The function h_W is well defined and h_W is bijective when h is.*

Proof: We first prove that h_W is well defined. To do this, we only need to show that h is a W-map. Let ζ be an element of $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$, which is represented by $\alpha \in \pi_1(Y, y_0)$. Then $h(\zeta) = h(\overline{\alpha}) = \{\theta_X(\alpha)\} = \{[\alpha]\}$, and for any $w \in W$, $h(w\zeta) = h(w\overline{\alpha}) = h(\overline{w\alpha}) = \{\theta_X(w\alpha)\} = \{[w\alpha]\} = \{w[\alpha]\} = w\{[\alpha]\} = wh(\zeta)$.

Since h_W is induced by h, then h_W is surjective when h is. Next we prove that h_W is injective when h is. Assume that ζ_1, ζ_2 are elements of $\nabla(f, g; x_0, y_0, \omega_f, \omega_g)$ such that $h_W(W\zeta_1) = h_W(W\zeta_2)$. This implies that $h(\zeta_1) \in Wh(\zeta_2)$, or explicitly, there is a $w \in W$ such that $h(\zeta_1) = wh(\zeta_2) = h(w\zeta_2)$. Since h is one to one, $\zeta_1 = w\zeta_2$. This shows that $W\zeta_1 = W\zeta_2$. □

Theorem 4.3.5 *Let X and Y be W-manifolds, and (f, g) a pair of W-maps from X to Y. Suppose that X^W is nonempty and f, g map some component of X^W into the same component of Y^W. Then if Y is a Jiang space or if (f, g) has the weak Jiang property* (see Definition 2.2.16), *then*

(1) $L(f, g) = 0 \Rightarrow N_W(f, g) = 0$, and

(2) $L(f, g) \neq 0 \Rightarrow N_W(f, g) = R_W(f, g) = \#(Coker(g_* - f_*)/W)$.

Proof: Under the assumptions of the theorem, we have either $N_W = 0$ or $N_W(f,g) = R_W(f,g)$ by Theorem 4.3.1. By Theorem 4.3.3 and Proposition 4.3.4, $R_W(f,g) = \#(Coker(g_* - f_*)/W)$ since h is bijective in this case from Proposition 2.2.6. □

When H is an isotropy subgroup of W, we have WH-manifolds X^H, Y^H by Theorem 4.1.15. In addition, f^H and g^H are WH-maps , and so the above theorem can be applied to $(f^H, g^H) : X^H \to Y^H$ to give the following corollary.

Corollary 4.3.6 *Let X and Y be W-manifolds, and (f,g) a pair of W-maps from X to Y. Suppose X^W is nonempty and f,g map some component of X^W into the same component of Y^W. Then if Y^H is a Jiang space or (f^H, g^H) has the weak Jiang property for an isotropy subgroup H of W, then*

(1) $L(f^H, g^H) = 0 \Rightarrow N_{WH}(f^H, g^H) = 0$, and

(2) $L(f^H, g^H) \neq 0 \Rightarrow N_{WH}(f^H, g^H) = R_{WH}(f^H, g^H) = \#(Coker(g_*^H - f_*^H)/WH)$. □

Example 4.3.7 Let $X = Y = S^1 \times S^1 \times S^2$, $W = Z/2 =< \alpha >$ be the cyclic group of order 2. We denote a point in S^2 by a cylindrical coordinate. Let the action of W on X be

$$\alpha \cdot (e^{i\theta_1}, e^{i\theta_2}, (r,\theta,z)) = (e^{i\theta_2}, e^{i\theta_1}, (r,\theta,-z)).$$

Then $X^W = \{(e^{i\theta}, e^{i\theta}, (r,\theta,0))\} \cong S^1 \times S^1$ is not empty.

Define $f : X \to X$ to be

$$f((e^{i\theta_1}, e^{i\theta_2}, (r,\theta,z))) = (e^{i2\theta_2}, e^{i2\theta_1}, (r,-\theta,-z)),$$

and define $g : X \to X$ to be

$$g((e^{i\theta_1}, e^{i\theta_2}, (r,\theta,z))) = (e^{i\theta_1}, e^{i\theta_2}, (r,3\theta,z)),$$

Then the set of coincidence points of f and g is $\Gamma(f,g) = \{(m, m^2, (1,\theta,0)) \mid m^3 = 1; \theta = 0, \pi/2, \pi, 3\pi/2\}$. Now $N(f,g) = 3$.

Note from the Kunneth formula for homology (see p.108 in [V]) that $H_1(X) = H_1(S^1 \times S^1 \times S^2) \cong H_1(S^1 \times S^1) \cong H_1(S^1) \times H_1(S^1)$. Let a_1 and a_2 be generators of the first and second factors respectively. So $H_1(X) =< a_1, a_2 >=< a_1 - 2a_2, a_1 - a_2 >$. The W-action on $H_1(X)$ is defined as follows: $\alpha \cdot a_1 = a_2$ and $\alpha \cdot a_2 = a_1$.

The homomorphism f_* induced by f is defined by $f_*(a_1) = 2a_2$ and $f_*(a_2) = 2a_1$, the homomorphism g_* induced by g is defined by $g_*(a_1) = a_1$ and $g_*(a_2) = a_2$. It is easy to see that Im $(g_* - f_*) =< a_1 - 2a_2, a_2 - 2a_1 >=< a_1 - 2a_2, 3(a_1 - a_2) >$. So $Coker(g_* - f_*) =< a_1 - 2a_2, a_1 - a_2 > / < a_1 - 2a_2, 3(a_1 - a_2) >= \{[0], [a_1 - a_2], [2(a_1 - a_2)]\}$. To find $Coker(g_* - f_*)/W$, we need to know the W-action on $Coker(g_* - f_*)$. We have $\alpha([a_1 - a_2]) = [\alpha(a_1 - a_2)] = [a_2 - a_1] = [a_2 - a_1 + 3(a_1 - a_2)] = [2(a_1 - a_2)]$ and $\alpha([0]) = [0]$. So $Coker(g_* - f_*)/W = \{[0]_W, [a_1 - a_2]_W\}$.

We find the Lefschetz number $L(f,g)$ as follows (we would like to thank Ross Geoghegan for helpful conversations about this calculation, many of the details of what follows may be found in chapter 5 sections 3 and 6 of [SE]). We write $X = X_1 \times X_2 = (S^1 \times S^1) \times S^2$, and $f = f_1 \times f_2$, $g = g_1 \times g_2$ in the obvious way. Note, again from the Kunneth formula for homology, that $H_0(X) \cong \mathbf{Z}$; that $H_1(X) \cong H_1(X_1) \cong \mathbf{Z} + \mathbf{Z}$; that $H_2(X) \cong H_2(X_1) + H_2(X_2) \cong \mathbf{Z}+\mathbf{Z}$; that $H_3(X) \cong H_1(X_1) \otimes H_2(X_2) \cong \mathbf{Z}+\mathbf{Z}$; and $H_4(X) \cong \mathbf{Z}$. Let U_1 and U_2 be the fundamental classes of X_1 and X_2 respectively, and $a_0 \in H_0(X_1)$, $a_{11}, a_{12} \in H_1(X_1)$ and $a_2 \in H_2(X_1)$ be generators. Let $a^0 \in H^0(X_1)$, $a^{11}, a^{12} \in H^1(X_1)$ and $a^2 \in H^2(X_1)$ be such that $D_0(X_1)(a^0) = U_1 \cap a^0 = a_2$, $D_1(X_1)(a^{11}) = U_1 \cap a^{11} = a_{11}$, $D_1(X_1)(a^{12}) = U_1 \cap a^{12} = a_{12}$, and $D_2(X_1)(a^2) = U_1 \cap a^2 = a_0$, where the D_i are the Poincaré duality isomorphisms, and $\cap$ is the cap product. Let $b_0 \in H_0(X_2)$ and $b_2 \in H_2(X_2)$ be generators, and $b^0 \in H^0(X_2)$ and $b^2 \in H^2(X_2)$ be such that $D_0(X_2)(b^0) = U_2 \cap b^0 = b_2$, and $D_2(X_2)(b^2) = U_2 \cap b^2 = b_0$. Then the (homology) cross product $a_0 \times b_0$ is a generator of $H_0(X)$, $a_{11} \times b_0$ and $a_{12} \times b_0$ are generators of $H_1(X)$, $a_2 \times b_0$ and $a_0 \times b_2$ are generators of $H_2(X)$, $a_{11} \times b_2$ and $a_{12} \times b_2$ are genrators of $H_3(X)$, and $a_2 \times b_2$ is a generator of $H_4(X)$.

Let $U_1 \times U_2$ be the (homology) cross product of the fundamental classes U_1 and U_2 given above. Then $U_1 \times U_2$ is defined in terms of tensor products (see Chapter 5.3 of [SE]). Since X_1 and X_2 are orientable manifolds, the Eilenberg-Zilber Theorem, and the Kunneth Formula allow us to deduce in the top dimensions of the both manifold factors that that (up to sign) the (homology) cross product of fundamental classes is a fundamental class. In this way up to sign $U_1 \times U_2$ in $H_4(X)$, can be regarded as the fundamental class. Now for any $0 \leq p, q \leq 2$, $D_{p+q}(X)(a^p \times b^q) = (-1)^s(U_1 \times U_2) \cap (a^p \times b^q) = (-1)^s(-1)^{p(2-q)}((U_1 \cap a^p) \times (U_2 \cap b^q))$ $= (-1)^s(-1)^{p(2-q)}(a_{2-p} \times b_{2-q}) = (-1)^{s+p(2-q)}(a_{2-p} \times b_{2-q})$, where $s = 0$ or 1. Let $k(s,p,q) = s + p(2-q)$, then we have $D_{p+q}(X)(a^p \times b^q) = (-1)^{k(s,p,q)}(a_{2-p} \times b_{2-q})$. Since $D_{p+q}(X)$ is an isomorphism, we have $D^{-1}_{p+q}(X)(a_{2-p} \times b_{2-q}) = (-1)^{k(s,p,q)}(a^p \times b^q)$. Applying this result to the calculation of $\theta_0(f,g)(a_0 \times b_0)$, we have

$$\begin{aligned}
\theta_0(f,g)(a_0 \times b_0) &= D_4(X)g^* D_4^{-1}(X)f_*(a_0 \times b_0) \\
&= D_4(X)g^* D_4^{-1}(X)((f_1)_* \times (f_2)_*)(a_0 \times b_0) \\
&= D_4(X)g^* D_4^{-1}(X)((f_1)_*(a_0) \times (f_2)_*(b_0)) \\
&= D_4(X)g^* D_4^{-1}(X)(a_0 \times b_0) \\
&= D_4(X)g^*((-1)^{k(s,0,0)}(a^2 \times b^2)) \\
&= (-1)^{k(s,0,0)} D_4(X)(g_1^* \times g_2^*)(a^2 \times b^2) \\
&= (-1)^{k(s,0,0)} D_4(X)(g_1^*(a^2) \times g_2^*(b^2)) \\
&= (-1)^{k(s,0,0)} D_4(X)(a^2 \times 3b^2) \\
&= (-1)^{k(s,0,0)} 3D_4(X)(a^2 \times b^2) \\
&= (-1)^{k(s,0,0)} 3(-1)^{k(s,0,0)}(a_0 \times b_0) \\
&= 3(a_0 \times b_0).
\end{aligned}$$

Similarly, we have

$\theta_1(f,g)(a_{11} \times b_0) = a_{12} \times b_0$; and $\theta_1(f,g)(a_{12} \times b_0) = a_{11} \times b_0$;

$\theta_2(f,g)(a_2 \times b_0) = (-12)(a_2 \times b_0)$; and $\theta_2(f,g)(a_0 \times b_2) = a_0 \times b_2$;

$\theta_3(f,g)(a_{11} \times b_2) = a_{12} \times b_2$; and $\theta_3(f,g)(a_{12} \times b_2) = a_{11} \times b_2$;

and $\theta_4(f,g)(a_2 \times b_2) = (-4)(a_2 \times b_2)$;

So the Lefschetz number $L(f,g) = \sum_{q=0}^{4}(-1)^q tr\theta_q(f,g) = 3 + 0 - 11 + 0 - 4 = -12 \neq 0$. Since X is a Jiang space and X^W is not empty, Theorem 4.3.5 can be applied, and $N_W(f,g) = \#(Coker(g_* - f_*)/W) = 2$. However at this point we can say very little about orbit length. In order to do this we need some other invariants.

4.4 More equivariant Nielsen type invariants

In order to determine the number of coincidence points of a pair of W-maps $(f, g) : X \to Y$, we need to know both the number of coincidence point orbits and the length of each orbit. However, the length of an orbit may vary dependent on the location of the orbit. If a coincidence point x of (f, g) is in X_H for some isotropy subgroup H, then the length of the orbit Wx is $[W : H]$, but if we deform f and g, this coincidence point may move to X^K with $K \supset H$, as a consequence, the length of this new orbit is $[W : K]$. So in order to find the minimum number of coincidence points, we have to distinguish those coincidence points from the others. Recall in chapter 2, in order to define the Nielsen number on the complement, we first defined maps $j_{f_k, g_k} : \mathcal{R}_{f_k, g_k} \to \mathcal{R}_{f,g}$, and then introduced the concept of weakly common coincidence class. These ideas will be used in this section. Each pair of W-maps $f, g : X \to Y$ induces a pair of maps (f^H, g^H) from X^H to Y^H for each isotropy subgroup H. If an isotropy subgroup H is a subgroup of another isotropy subgroup K, we have $X^K \subset X^H$, and there is a map from the WK-classes of (f^K, g^K) in X^K to the WH-classes of (f^H, g^H) in X^H (see Definition 4.4.5).

Note that if two isotropy subgroups H_1 and H_2 are in the same isotropy type (H), then there is a $w \in W$ such that the action of w induces a homeomorphism from X^{H_1} to X^{H_2}. This allows us to define a one to one correspondence between WH_1-classes and WH_2-classes, and then subsequently to define the W-orbit of a WH_1-class. To see if a coincidence point orbit Wx in $X^{(H)}$ can be moved to some $X^{(K)}$, we have to determine if the W-orbit of the WH-class that contains Wx, includes any W-orbit of a WK-class for some $K \supset H$. It may happen that two different W-orbits of WH-classes may contain the same W-orbit of a WK-class. This is different from the relative case where for a pair of maps $(f, g) : (X, A) \to (Y, B)$, a coincidence class of (f_A, g_A) on a subspace is contained in a unique coincidence class of (f, g).

We first define a map from the set of WH_1-classes to the set of WH_2-classes when H_1 and H_2 are conjugate. Let W be a finite group, and X and Y be compact W-manifolds. We will use the symbol $Iso(X)$ to denote the set of isotropy types of X.

For W-maps $f, g : X \to Y$, and each $H \in Iso(X)$, we have the restrictions $f^H, g^H : X^H \to Y^H$ of f and g respectively on X^H, which are WH-maps.

From now on, we will assume that X^H and Y^H are connected and orientable manifolds for each $(H) \in Iso(X)$. Suppose $f, g : X \to Y$ are W-maps and H is an isotropy subgroup of W. For each $w \in W$, let $H' = wHw^{-1}$ (H' could be equal to H). Then we have homeomorphisms $l^H_{w,X} : X^H \to X^{H'}$ and $l^H_{w,Y} : Y^H \to Y^{H'}$ defined by $l^H_{w,X}(x) = wx$ for any $x \in X^H$, and $l^H_{w,Y}(y) = wy$ for any $y \in Y^H$. Let $\tilde{l}^H_{w,X}$ and $\tilde{l}^H_{w,Y}$ be liftings of $l^H_{w,X}$ and $l^H_{w,Y}$ respectively. Then for each lift $(\tilde{f}^H, \tilde{g}^H)$ of (f^H, g^H), there is a (unique) lifting $(\tilde{f}^{H'}, \tilde{g}^{H'})$ of $(f^{H'}, g^{H'})$ such that the following diagram is commutative.

$$
\begin{tikzcd}
\tilde{X}^H \arrow[rr, "\tilde{f}^H{,}\tilde{g}^H"] \arrow[dr, "\tilde{l}^H_{w,X}"] \arrow[dd, "p_X^H"'] & & \tilde{Y}^H \arrow[dr, "\tilde{l}^H_{w,Y}"] \arrow[dd, "p_Y^H"] & \\
& \tilde{X}^{H'} \arrow[rr, "\tilde{f}^{H'}{,}\tilde{g}^{H'}"] \arrow[dd, "p_X^{H'}"] & & \tilde{Y}^{H'} \arrow[dd, "p_Y^{H'}"] \\
X^H \arrow[rr, "f^H{,}g^H"] \arrow[dr, "l^H_{w,X}"'] & & Y^H \arrow[dr, "l^H_{w,Y}"] & \\
& X^{H'} \arrow[rr, "f^{H'}{,}g^{H'}"'] & & Y^{H'}
\end{tikzcd}
$$

where the bottom square is commutative because f and g are W-maps.

Definition 4.4.1 Define $\phi_{H,H'} : \mathcal{R}_{f^H,g^H}(WH) \to \mathcal{R}_{f^{H'},g^{H'}}(WH')$ by $\phi_{H,H'}([(\tilde{f}^H,\tilde{g}^H)]_{WH}) = [(\tilde{f}^{H'},\tilde{g}^{H'})]_{WH}$, where $[(\tilde{f}^H,\tilde{g}^H)]_{WH}$ is the WH-class containing $(\tilde{f}^H,\tilde{g}^H)$.

Lemma 4.4.2 *$\phi_{H,H'}$ is well defined and is bijective. Furthermore, if $H' = w'Hw'^{-1}$ and $H'' = w''H'w''^{-1}$, then $\phi_{H,H''} = \phi_{H',H''} \circ \phi_{H,H'}$.*

Proof: We have to prove that $[(\tilde{f}^{H'},\tilde{g}^{H'})]$ is independent of the choice of $w \in W$ (and, hence, the choice of $\tilde{l}^H_{w,X}$ and $\tilde{l}^H_{w,Y}$).

Assume $w' \in W$ is another element such that $H' = w'Hw'^{-1}$, and $(\tilde{f'}^{H'},\tilde{g'}^{H'})$ the lifting of $(f^{H'},g^{H'})$ such that $(\tilde{f'}^{H'},\tilde{g'}^{H'}) = \tilde{l}^H_{w',Y}(\tilde{f}^H,\tilde{g}^H)(\tilde{l}^H_{w',X})^{-1}$. Then we have $(\tilde{f'}^{H'},\tilde{g'}^{H'}) = \tilde{l}^H_{w',Y}(\tilde{l}^H_{w,Y})^{-1}(\tilde{f}^{H'},\tilde{g}^{H'})\tilde{l}^H_{w,X}(\tilde{l}^H_{w',X})^{-1}$. Note that $H' = w'w^{-1}H'ww'^{-1}$, therefore, $w'w^{-1}$ represents an element in WH'. So $w'w^{-1}$ is an action of an element in WH' on $Y^{H'}$; $w'w^{-1}$ is an action of an element in WH' on $X^{H'}$; $\tilde{l}^H_{w',Y}(\tilde{l}^H_{w,Y})^{-1}$ is an action of an element in $\widetilde{WH'}_{\tilde{Y}^{H'}}$;

$\tilde{l}^H_{w',X}(\tilde{l}^H_{w,X})^{-1}$ is an action of an element in $W\tilde{H}'_{\tilde{X}^{H'}}$. By Definition 4.2.1, $[(\tilde{f}^{H'},\tilde{g}^{H'})]_{WH'}$ is equal to $[(\tilde{f}'^{H'},\tilde{g}'^{H'})]_{WH'}$.

$\phi_{H,H'}$ is bijective since it has an inverse defined using w^{-1}.

Now Assume that $H' = w'Hw'^{-1}$ and $H'' = w''H'w''^{-1}$. Then $H'' = w''w'Hw'^{-1}w''^{-1} = (w''w')H(w''w')^{-1}$. It is obvious that $l^{H'}_{w'',X} \circ l^H_{w',X} = l^H_{(w''w'),X}$, $l^{H'}_{w'',Y} \circ l^H_{w',Y} = l^H_{(w''w'),Y}$ and therefore $\tilde{l}^{H'}_{w'',X} \circ \tilde{l}^H_{w',X}$, $\tilde{l}^{H'}_{w'',Y} \circ \tilde{l}^H_{w',Y}$ are liftings of $l_{(w''w'),X}$ and $l^H_{(w''w'),Y}$ respectively. If $\phi_{H,H'}([(\tilde{f}^H,\tilde{g}^H)]_{WH}) = [(\tilde{f}^{H'},\tilde{g}^{H'})]_{WH'}$ and $\phi_{H',H''}([(\tilde{f}^{H'},\tilde{g}^{H'})]_{WH'}) = [(\tilde{f}^{H''},\tilde{g}^{H''})]_{WH''}$, then $(\tilde{f}^{H'},\tilde{g}^{H'}) = \tilde{l}^H_{w',Y}(\tilde{f}^H,\tilde{g}^H)(\tilde{l}^H_{w',X})^{-1}$ and $(\tilde{f}^{H''},\tilde{g}^{H''}) = \tilde{l}^{H'}_{w'',Y}(\tilde{f}^{H'},\tilde{g}^{H'})(\tilde{l}^{H'}_{w'',X})^{-1}$. Therefore, $(\tilde{f}^{H''},\tilde{g}^{H''}) = \tilde{l}^{H'}_{w'',Y}\tilde{l}^H_{w',Y}(\tilde{f}^H,\tilde{g}^H)(\tilde{l}^H_{w',X})^{-1}(\tilde{l}^{H'}_{w'',X})^{-1} = (\tilde{l}^{H'}_{w'',Y}\tilde{l}^H_{w',Y})(\tilde{f}^H,\tilde{g}^H)(\tilde{l}^{H'}_{w'',X}\tilde{l}^H_{w',X})^{-1}$. This shows $\phi_{H,H''}([(\tilde{f}^H,\tilde{g}^H)]_{WH}) = [(\tilde{f}^{H''},\tilde{g}^{H''})]_{WH''} = \phi_{H',H''} \circ \phi_{H,H'}([(\tilde{f}^H,\tilde{g}^H)]_{WH})$ and therefore $\phi_{H,H''} = \phi_{H',H''} \circ \phi_{H,H'}$. □

Definition 4.4.3 Let H be an isotropy subgroup of W, and θ a WH-class in $\mathcal{R}_{f^H,g^H}(WH)$. The W-orbit of θ is the set $\{\phi_{H,H'}(\theta)\}_{H'\in(H)}$, and is denoted by $W\theta$. If a lifting $(\tilde{f}^H,\tilde{g}^H)$ of (f^H,g^H) is in θ, we say $(\tilde{f}^H,\tilde{g}^H)$ represents $W\theta$, and denote $W\theta$ by $\{(\tilde{f}^H,\tilde{g}^H)\}$. The set of all W-orbits of WH-classes is denoted by $W\mathcal{R}_{f^H,g^H}(WH)$. Define $incl : \mathcal{R}_{f^H,g^H}(WH) \to W\mathcal{R}_{f^H,g^H}(WH)$ by $incl(\theta) = W\theta$.

Proposition 4.4.4 *$incl : \mathcal{R}_{f^H,g^H}(WH) \to W\mathcal{R}_{f^H,g^H}(WH)$ is bijective.*

Proof: It is sufficient to show that different WH-classes are in different W-orbits. By Lemma 4.4.2, we only need to show that if $\phi_{H,H}([(\tilde{f}^H,\tilde{g}^H)]_{WH}) = [(\tilde{f}^H_1,\tilde{g}^H_1)]_{WH}$ then $[(\tilde{f}^H,\tilde{g}^H)]_{WH} = [(\tilde{f}^H_1,\tilde{g}^H_1)]_{WH}$. Assume $(\tilde{f}^H,\tilde{g}^H)$ and $(\tilde{f}^H_1,\tilde{g}^H_1)$ are two liftings of (f^H,g^H) such that

$\phi_{H,H}([(\tilde{f}^H, \tilde{g}^H)]_{WH}) = [(\tilde{f}_1^H, \tilde{g}_1^H)]_{WH}$, then there is a $w \in W$ such that the diagram

$$\begin{array}{ccc} \tilde{X}^H & \xrightarrow{(\tilde{f}^H, \tilde{g}^H)} & \tilde{Y}^H \\ \downarrow l^H_{w,X} & & \downarrow l^H_{w,Y} \\ \tilde{X}^H & \xrightarrow{(\tilde{f}_1^H, \tilde{g}_1^H)} & \tilde{Y}^H \end{array}$$

is commutative. Since $H = wHw^{-1}$, $l^H_{w,Y}$ is in $\widetilde{WH}$, and $(\tilde{f}^H, \tilde{g}^H)$ and $(\tilde{f}_1^H, \tilde{g}_1^H)$ are in the same WH-class. $\square$

Assume that $H \subset K$ are isotropy subgroups of W. For any lifting $(\tilde{f}^{K'}, \tilde{g}^{K'})$ of $(f^{K'}, g^{K'})$, where $K' \in (K)$, there is $H' \in (H)$ and a lifting $(\tilde{f}^{H'}, \tilde{g}^{H'})$ of $(f^{H'}, g^{H'})$ such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{X}^{K'} & \xrightarrow{(\tilde{f}^{K'}, \tilde{g}^{K'})} & \tilde{Y}^{K'} \\ \downarrow \tilde{i}^X_{H' \subset K'} & & \downarrow \tilde{i}^Y_{H' \subset K'} \\ \tilde{X}^{H'} & \xrightarrow{(\tilde{f}^{H'}, \tilde{g}^{H'})} & \tilde{Y}^{H'}, \end{array}$$

where $\tilde{i}^X_{H' \subset K'}$ is a map from $\tilde{X}^{K'}$ to a component of $p^{-1}_{H'}(X^{H'})$ and $\tilde{i}^Y_{H' \subset K'}$ is a map from $\tilde{Y}^{K'}$ to a component of $p^{-1}_{H'}(Y^{H'})$.

Definition 4.4.5 Define $\tau_{(H)<(K)} : W\mathcal{R}_{f^K, g^K}(WK) \to W\mathcal{R}_{f^H, g^H}(WH)$ by $\tau_{(H)<(K)}(\{(\tilde{f}^{K'}, \tilde{g}^{K'})\})$ $= \{(\tilde{f}^{H'}, \tilde{g}^{H'})\}$.

The example which follows illustrates that $\tau_{(H)<(K)}$ can be a multivalued map when W is not commutative, and not necessarily singlevalued as stated in [WP3] (however, the results in [WP3] are not affected). So $\tau_{(H)<(K)}(\{(\tilde{f}^{K'}, \tilde{g}^{K'})\})$ is a set and we also use the notation $\{(\tilde{f}^{H'}, \tilde{g}^{H'})\} \in \tau_{(H)<(K)}(\{(\tilde{f}^{K'}, \tilde{g}^{K'})\})$ and denote $\bigcup_{W\theta \in W\mathcal{R}_{f^K, g^K}(WK)} \tau_{(H)\leq(K)}(W\theta)$ by Im $\tau_{(H)\leq(K)}$.

Example 4.4.6 Let $X = S^1 \times S^1 \times S^1 \times S^1$, $W = S_4$ the symmetric group of degree 4. The W-action on X is defined as follows: if (i_1, i_2) is a 2-cycle, then it acts on an element of X by exchanging the i_1-th and i_2-nd coordinates. For example, $(1,2)(x_1, x_2, x_3, x_4) = (x_2, x_1, x_3, x_4)$. Note that $X^W = \{(x, x, x, x) \mid x \in S^1\} = \Delta(S^1 \times S^1 \times S^1 \times S^1)$ is not empty.

Let $< (1,2) >$ be the subgroup generated by $(1,2)$, $< (3,4) >$ be the subgroup generated by $(3,4)$ and $< (1,2), (3,4) >$ be the subgroup generated by $(1,2)$ and $(3,4)$. Then

$$X^{<(1,2)>} = \{(x, x, x_3, x_4) \mid x, x_3, x_4 \in S^1\} \cong \Delta(S^1 \times S^1) \times S^1 \times S^1;$$

$$X^{<(3,4)>} = \{(x_1, x_2, x, x) \mid x, x_1, x_2 \in S^1\} \cong S^1 \times S^1 \times \Delta(S^1 \times S^1);$$

$$X^{<(1,2),(3,4)>} = \{(x, x, x', x') \mid x, x' \in S^1\} \cong \Delta(S^1 \times S^1) \times \Delta(S^1 \times S^1).$$

Note $X^{<(1,2),(3,4)>} \subset X^{<(1,2)>} \cap X^{<(3,4)>}$. Note also that $< (1,2) >$ and $< (3,4) >$ have the same normalizer, $< (1,2), (3,4) >$. So the Weyl-group $W < (1,2) >$ of $< (1,2) >$ is $< [(3,4)] >$ and the Weyl-group $W < (3,4) >$ of $< (3,4) >$ is $< [(1,2)] >$. The Weyl-group $W < (1,2), (3,4) >$ of $< (1,2), (3,4) >$ contains $[(2,3)(1,4)]$.

Now let $Y = X$ and $f = g = id$ and $x_0 = (1,1,1,1)$. Since X^W is non-empty, we can identify $\mathcal{R}_{f,g}(W)$ with $\nabla(f, g; x_0, x_0, \omega_f, \omega_g)/W$. We know that for this pair f and g each ordinary Reidemeister class contains one element of $\pi_1(Y, y_0)$. Let α be the loop represented by $\Delta(S^1 \times S^1) \times 1 \times 1$ and β the loop represented by $1 \times 1 \times \Delta(S^1 \times S^1)$. Let $[\alpha]_{W<(1,2)>}$ and $[\beta]_{W<(1,2)>} \in R_{f^{W<(1,2)>}, g^{W<(1,2)>}}(W < (1,2) >)$ be the $W < (1,2) >$-classes containing α and β respectively and let $[\alpha]_{W<(1,2),(3,4)>}$ and $[\beta]_{W<(1,2),(3,4)>} \in R_{f^{W<(1,2),(3,4)>}, g^{W<(1,2),(3,4)>}}(W < (1,2), (3,4) >)$ be the $W < (1,2), (3,4) >$-classes containing α and β respectively. Then

$[\alpha]_{W<(1,2)>} \neq [\beta]_{W<(1,2)>}$ since $W<(1,2)>=<[(3,4)]>$ and $(3,4)\alpha \neq \beta$. So the classes containing $[\alpha]_{W<(1,2)>}$ and $[\beta]_{W<(1,2)>}$ in $WR_{f^{W<(1,2)>}, g^{W<(1,2)>}}(W<(1,2)>)$ are not the same by Proposition 4.4.4. On the other hand, $[\alpha]_{W<(1,2),(3,4)>} = [\beta]_{W<(1,2),(3,4)>}$ since $W<(1,2),(3,4)>$ contains $[(2,3)(1,4)]$ and $(2,3)(1,4)\alpha = \beta$. Therefore the image of $[\alpha]_{W<(1,2),(3,4)>}$ under $\tau_{(<(1,2)>)<(<(1,2),(3,4)>)}$ has at least two elements containing $[\alpha]_{W<(1,2)>}$ and $[\beta]_{W<(1,2)>}$ respectively. This shows that τ is not singlevalued in general.

Definition 4.4.7 A finite set $\mathcal{G} \subset \bigcup_{(H)\leq(K)} W\mathcal{R}_{WK}(f^K, g^K)$ is said to be an essential basis of (f^H, g^H) over $X^{(H)}$, if for any essential WK'-class N' with $(H) \leq (K')$ there is a WK-class $N \in \mathcal{G}$ such that $WN' \in \tau_{(K')\leq(K)}(WN)$.

Definition 4.4.8 Let $f, g : X \to Y$ be W-maps. For each $H \in Iso(X)$ define

$NO_W(f_H, g_H) = \#\{$essential WH-classes N of $(f^H, g^H) | WN \notin \bigcup_{(H)<(K)} \text{Im } \tau_{(H)<(K)}\}$

$NO_W(f^H, g^H) = \min\{\#\mathcal{W} | \mathcal{W}$ is an essential basis of (f^H, g^H) over $X^{(H)}\}$

Proposition 4.4.9 (*W-Homotopy Invariance*) *$f \sim_W f'$, $g \sim_W g'$, then*

(1) $NO_W(f_H, g_H) = NO_W(f'_H, g'_H)$.

(2) $NO_W(f^H, g^H) = NO_W(f'^H, g'^H)$.

for every $(H) \in Iso(X)$.

Proof: (1) Note that $NO_W(f_H, g_H)$ is equal to $N_{WH}(f^H, g^H)$ minus the number of essential classes which are in Im $(\tau_{(H)<(K)})$ for some $K \supset H$. So to prove (1), we only need to prove that if a WH-class N^{WH} of (f^H, g^H) corresponds to a WH-class N'^{WH} of (f'^H, g'^H) under a pair of homotopies (F, G), then N^{WH} is in Im $(\tau_{(H)<(K)})$ if and only if N'^{WH} is in Im $(\tau_{(H)<(K)})$.

Assume that $(\tilde{f}^H, \tilde{g}^H)$ is a lifting of (f^H, g^H) and $N^{WH} = WH(p_X(\Gamma(\tilde{f}^H, \tilde{g}^H)))$ and $(\tilde{f}^K, \tilde{g}^K)$ is a lifting of (f^K, g^K) such that the diagram

$$\begin{array}{ccc} \tilde{X}^K & \stackrel{(\tilde{f}^K, \tilde{g}^K)}{\longrightarrow} & \tilde{Y}^K \\ \downarrow \tilde{i}^X_{H\subset K} & & \downarrow \tilde{i}^Y_{H\subset K} \\ \tilde{X}^H & \stackrel{(\tilde{f}^H, \tilde{g}^H)}{\longrightarrow} & \tilde{Y}^H \end{array}$$

is commutative.

Assume that (F, G) is a W-homotopy from (f, g) to (f', g'). Let $(\tilde{F}^H, \tilde{G}^H)$ be a lifting of (F^H, G^H) such that $(\tilde{F}^H, \tilde{G}^H)|_{X\times\{0\}} = (\tilde{f}^H, \tilde{g}^H)$, and let $(\tilde{F}^K, \tilde{G}^K)$ be a lifting of (F^K, G^K) such that $(\tilde{F}^K, \tilde{G}^K)|_{X\times\{0\}} = (\tilde{f}^K, \tilde{g}^K)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} \tilde{X}^K \times I & \stackrel{(\tilde{F}^K, \tilde{G}^K)}{\longrightarrow} & \tilde{Y}^K \\ \downarrow \tilde{i}^X_{H\subset K} \times id & & \downarrow \tilde{i}^Y_{H\subset K} \\ \tilde{X}^H \times I & \stackrel{(\tilde{F}^H, \tilde{G}^H)}{\longrightarrow} & \tilde{Y}^H \end{array}$$

Since N'^{WH} corresponds to N^{WH} under homotopies (F, G), we have

$$N'^{WH} = WH(p_X(\Gamma(\tilde{F}^H|_{X\times\{1\}}, \tilde{G}^H|_{X\times\{1\}}))).$$

This shows that N'^{WH} is in Im $\tau_{(H)<(K)}$. By running the homotopy backward we can show that if $N'^{WH} \in \mathrm{Im}(\tau_{(H)<(K)})$ then $N^{WH} \in \mathrm{Im}(\tau_{(H)<(K)})$.

(2) Assume $\mathcal{G}$ is an essential basis of (f^H, g^H) over $X^{(H)}$. Under the pair of homotopies (F, G), it corresponds to a set $\mathcal{G}' \subset \bigcup_{(H)\leq(K)} W\mathcal{R}_{WK}(f'^K, g'^K)$. We will prove that $\mathcal{G}'$ is an essential basis of (f'^H, g'^H) over $X^{(H)}$. Then we have $NO_W(f'^H, g'^H) \leq NO_W(f^H, g^H)$ by the definition of NO_W since $\mathcal{G}$ and $\mathcal{G}'$ have the same number of elements. With the same argument, we have $NO_W(f^H, g^H) \leq NO_W(f'^H, g'^H)$, and hence the equality.

Let N'^{WK} be an essential WK-class of (f'^K, g'^K), where $(H) \le (K)$, and N^{WK} be the essential WK-class of (f^K, g^K) corresponding to N'^{WK} under the homotopy (F, G). So N^{WK} is essential. By the choice of $\mathcal{G}$, there is an isotropy subgroup $K_1 \supset K$ and a WK_1-class $\theta_1 \in \mathcal{R}_{WK_1}(f^{K_1}, g^{K_1}) \cap \mathcal{G}$ such that $WN^{WK} \in \tau_{(K)<(K_1)}(W\theta_1)$. More explicitly there are liftings $(\tilde{f}^K, \tilde{g}^K)$ of (f^K, g^K), and $(\tilde{f}^{K_1}, \tilde{g}^{K_1})$ of (f^{K_1}, g^{K_1}), such that $N^{WK} = [(\tilde{f}^K, \tilde{g}^K)]$, and $\theta_1 = [(\tilde{f}^{K_1}, \tilde{g}^{K_1})]$. The diagram

$$\begin{array}{ccc}
\tilde{X}^{K_1} & \overset{(\tilde{f}^{K_1}, \tilde{g}^{K_1})}{\longrightarrow} & \tilde{Y}^{K_1} \\
\downarrow \tilde{i}^X_{K \subset K_1} & & \downarrow \tilde{i}^Y_{K \subset K_1} \\
\tilde{X}^K & \overset{(\tilde{f}^K, \tilde{g}^K)}{\longrightarrow} & \tilde{Y}^K
\end{array}$$

is commutative. As we saw in Theorem 2.1.16, this diagram leads to the commutative diagram

$$\begin{array}{ccc}
\tilde{X}^{K_1} & \overset{(\tilde{f'}^{K_1}, \tilde{g'}^{K_1})}{\longrightarrow} & \tilde{Y}^{K_1} \\
\downarrow \tilde{i}^X_{K \subset K_1} & & \downarrow \tilde{i}^Y_{K \subset K_1} \\
\tilde{X}^K & \overset{(\tilde{f'}^K, \tilde{g'}^K)}{\longrightarrow} & \tilde{Y}^K,
\end{array}$$

where $[(\tilde{f'}^K, \tilde{g'}^K)]_{WK}$ and $[(\tilde{f'}^{K_1}, \tilde{g'}^{K_1})]_{WK_1}$ correspond to $[(\tilde{f}^K, \tilde{g}^K)]_{WK} = N^{WK}$ and $[(\tilde{f}^{K_1}, \tilde{g}^{K_1})]_{WK_1} = \theta$ respectively. Hence $[(\tilde{f'}^K, \tilde{g'}^K)]_{WK} = N'^{WK}$, and $W[(\tilde{f'}^{K_1}, \tilde{g'}^{K_1})] \in \mathcal{G}'$. The diagram shows that $WN'^{WK} \in \tau_{(K)<(K_1)}(W[(\tilde{f'}^{K_1}, \tilde{g'}^{K_1})])$. □

Definition 4.4.10 Let $f, g : X \to Y$ be W-maps, define

$MO_W(f^{(H)}, g^{(H)})$=min{# coincidence W-orbits of (φ, ψ) on $X^{(H)} | \varphi \sim_W f, \psi \sim_W g$} ,

$M_W(f_{(H)}, g_{(H)})$ =min{# coincidences of (φ, ψ) on $X_{(H)} | \varphi \sim_W f, \psi \sim_W g$}.

$M_W(f^{(H)}, g^{(H)})$ =min{# coincidences of (φ, ψ) on $X^{(H)}|\varphi \sim_W f, \psi \sim_W g\}$.

Theorem 4.4.11 (*Lower Bound*) *Let X, Y be W-manifolds with the same dimension and $f, g : X \to Y$ W-maps. Assume for any $(H) \in Iso(X)$, dim X^H = dim Y^H, then we have*

(1). $MO_W(f^{(H)}, g^{(H)}) \geq NO_W(f^H, g^H)$.

(2). $M_W(f_{(H)}, g_{(H)}) \geq [W : H] \cdot NO_W(f_H, g_H)$.

(3). $M_W(f^{(H)}, g^{(H)}) \geq \sum_{(H)\leq(K)}[W : K] \cdot NO_W(f_K, g_K)$.

Proof: (1). Assume that $\{x_1, x_2, ..., x_s\} \in \Gamma(f^H, g^H)$ such that $\bigcup_{i=1}^s \{wx_i\}_{w\in W} = X^{(H)} \cap \Gamma(f, g)$ and $\{wx_i\}_{w\in W} \cap \{wx_j\}_{w\in W} = \emptyset$ for $i \neq j$, i.e. $\{wx_i\}$'s are the only orbits on $X^{(H)}$ and any two of them are distinct.

For each i let K_i be the largest subgroup such that $x_i \in X^{K_i}$, and let N_i be the element in $\mathcal{R}_{f^{K_i}, g^{K_i}}$ containing x_i. We will prove that $\{WN_i\}$ is an essential basis over $X^{(H)}$. Assume N' is an essential WK'-class with $H \subset K'$, then N' contains at least one coincidence point, say $x' \in X^{K'}$. Since $\bigcup_{i=1}^s \{wx_i\}_{w\in W} = X^{(H)} \cap \Gamma(f, g)$, $x' \in \{wx_i\}_{w\in W}$ for some i, or $x' = w_i x_i$ for some i and $w_i \in W$. Let $K'' = w_i^{-1}K'w_i$, then $x_i = w_i^{-1}x' \in X^{K''}$ and $x_i \in \Gamma(f^{K''}, g^{K''})$. Since K_i is the largest subgroup such that $x_i \in X^{K_i}$, we have $K'' \subset K_i$. It is not hard to see there are liftings $(\tilde{f}^{K_i}, \tilde{g}^{K_i})$ of (f^{K_i}, g^{K_i}) and $(\tilde{f}^{K''}, \tilde{g}^{K''})$ of $(f^{K''}, g^{K''})$ such that the diagram

$$\begin{array}{ccc} \tilde{X}^{K_i} & \xrightarrow{(\tilde{f}^{K_i}, \tilde{g}^{K_i})} & \tilde{Y}^{K_i} \\ \downarrow \tilde{i}^X_{K''\subset K_i} & & \downarrow \tilde{i}^X_{K''\subset K_i} \\ \tilde{X}^{K''} & \xrightarrow{(\tilde{f}^{K''}, \tilde{g}^{K''})} & \tilde{Y}^{K''} \end{array}$$

is commutative. Just choose lifts to both have coincidences which project down to a set containing x_i. This shows that the W-orbit of N' is in $\tau_{(K_i)\subset(K'')}(WN_i)$ and $\{WN_i\}$ is an essential basis over X^H. This proves (1).

(2). Assume that N is an essential WH-class, then N contains at least one coincidence point. If $WN \notin \bigcup_{(H)<(K)} \operatorname{Im} \tau_{(H)<(K)}$, then all the coincidence points in N lie in X_H. So there are at least $NO_W(f_H, g_H)$ coincidence points on X_H. By Proposition 4.4.4, any two classes in $NO_W(f_H, g_H)$ have coincidence points which are in different W-orbits. Note that the length of each W-orbit is $[W : H]$, so we have the inequality.

(3) follows from (2). □

Let $H =< 1 >$ in Theorem 4.4.11, we have

Corollary 4.4.12

$$\Sigma_{(K)}[W : K]NO_W(f_K, g_K)$$

is a lower bound for the number of coincidence points of (f', g'), for any pair of W-maps $f' \sim_W f$ and $g' \sim_W g$. □

4.5 Computations

In this section, we will discuss the computation of $NO_W(f_H, g_H)$ and $NO_W(f^H, g^H)$. As we pointed out in the last section, in order to estimate the number of coincidence points of a pair of W-maps (f, g), we have to find not only the number of coincidence point orbits, but also the location of each orbit so that we can know the length of that orbit. For each orbit in

X_H, its length is $[W : H]$ and hence the number $[W : H] \cdot NO_W(f_H, g_H)$ is a lower bound for the number of coincidence points of (f, g) on $X_{(H)}$ (Theorem 4.4.11). Though the technique used in Theorem 4.14 in [WP3] can be used here, we take a different approach. We make use of the results and discussion in Section 4.3, and the results in the relative case to compute $NO_W(f_H, g_H)$ under certain conditions. The relationship between the minimal number of coincidence points of (f, g) and $NO_W(f^H, g^H)$ is very complicated, and the computation of $NO_W(f^H, g^H)$ is generally more difficult than the that of $NO_W(f_H, g_H)$. However in some special cases, it can be reduced to the computation of $NO_W(f_H, g_H)$, which in turn can be reduced to the computation of $R_{WH}(f_H, g_H)$ or of $\#(Coker((g_H)_* - (f_H)_*)/W)$. Throughout this section and the next section, we assume that X^H and Y^H are connected and oriented manifolds for each $H \in Iso(X)$.

Theorem 4.5.1 *Let X, Y be W-manifolds, and (f, g) a pair of W-maps from X to Y. If Y^H is a Jiang space or (f^H, g^H) has the weak Jiang property for some isotropy subgroup H of W, then either*

(1) $L(f^H, g^H) = 0 \Rightarrow NO_W(f_H, g_H) = 0$, or

(2) $L(f^H, g^H) \neq 0 \Rightarrow NO_W(f_H, g_H) = \#W\mathcal{R}_{f^H, g^H}(WH) - \#(\bigcup_{(H)<(K)} Im\ \tau_{(H)<(K)})$

Proof: (1) is obvious since when $L(f^H, g^H) = 0$, there are no essential elements in $\mathcal{R}_{f^H, g^H}(WH)$.

To show (2), we only need to note that all WH-classes are essential, and, by Proposition 4.4.4, that $\mathcal{R}_{f^H, g^H}(WH)$ is in one to one correspondence with $W\mathcal{R}_{f^H, g^H}(WH)$. □

From Section 4.3, we know that if the assumptions of Theorem 4.5.1 are satisfied and if $X^{WH} \neq \emptyset$, then $\mathcal{R}_{WH}(f^H, g^H)$ is equal to $Coker(g^H_* - f^H_*)/WH$. Thus to compute $NO_W(f_H, g_H)$, we only need to know what $\#(\bigcup_{(H)<(K)} \text{ Im } \tau_{(H)<(K)})$ is. We take the next few pages to discuss the computation of $\#(\bigcup_{(H)<(K)} \text{ Im } \tau_{(H)<(K)})$, or more precisely, to identify the subset of $Coker(g_* - f_*)/WH$ corresponding to $(\bigcup_{(H)<(K)} \text{ Im } \tau_{(H)<(K)})$.

Throughout this section, we will assume that $X^W \neq \emptyset$, and for each isotropy subgroup H, X^H is connected. We choose $x_0 \in X^W$, $y_0 \in Y^W$ and $\omega_f, \omega_g \in Y^W$. Since for each isotropy subgroup H we have $X^W \subset X^H$, we can choose x_0 and y_0 as base points of X^H and Y^H respectively and set $\omega_{f^H} = \omega_f$ and $\omega_{g^H} = \omega_g$. We denote the inclusion map from X^K to X^H by $i_{H\subset K} : X^K \to X^H$ if $H \subset K$. The homomorphism induced by $i_{H\subset K}$ from $\pi_1(X^K, x_0)$ to $\pi_1(X^H, x_0)$ will be also denoted by $i_{H\subset K}$. Since X^K is connected and $X^W \subset X^K \subset X^H$, $i_{H\subset K}$ induces a map from $\nabla(f^K, g^K; x_0, y_0, \omega_f, \omega_g)$ to $\nabla(f^H, g^H; x_0, y_0, \omega_f, \omega_g)$, which is defined in Definition 2.1.7 (in this case, the basepoints x_0, y_0 and the paths from y_0 to $g(x_0)$ and $f(x_0)$ are all the same), and we will denote it by $\bar{i}_{H\subset K}$.

We will first identify the subset of $\mathcal{R}_{f^H, g^H}(WH)$ that corresponds to $(\bigcup_{(H)<(K)} \text{ Im } \tau_{(H)<(K)})$, and then identify the corresponding subset in $\nabla(f^H, g^H; x_0, y_0, \omega_f, \omega_g)$.

Define $E_{f^H, g^H}(WH) = \{WH\text{-class } \theta \mid W\theta \in \bigcup_{(H)<(K)} \text{ Im } \tau_{(H)<(K)}\} \subset \mathcal{R}_{f^H, g^H}(WH)$.

Lemma 4.5.2 *Let (X, A) and (Y, B) be pairs of manifolds with A and B connected, and $f, g : (X, A) \to (Y, B)$ be maps. Let $(\tilde{f}, \tilde{g})$ be a lifting of (f, g), then the following two statements are equivalent.*

(1) there is a lifting $(\tilde{f}_A, \tilde{g}_A)$ of (f_A, g_A) such that the diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{f}_A, \tilde{g}_A} & \tilde{B} \\ \downarrow \tilde{i}_{A\subset X} & & \downarrow \tilde{i}_{B\subset Y} \\ \tilde{X} & \xrightarrow{\tilde{f}, \tilde{g}} & \tilde{Y}. \end{array}$$

is commutative, where $\tilde{i}_{A\subset X}$ is a lifting of inclusion map $i_{A\subset X} : A \to X$;

(2) $\tilde{f}$ and $\tilde{g}$ map some component of $p_{\tilde{X}}^{-1}(A)$ to the same component of $p_{\tilde{Y}}^{-1}(B)$.

Proof: (1)$\Rightarrow$(2): Note that $\tilde{i}_{A\subset X}(\tilde{A})$ is a component of $p_{\tilde{X}}^{-1}(A)$. Since the diagram is commutative, it is sent to the same component $\tilde{i}_{B\subset Y}(\tilde{B})$ of $p_{\tilde{Y}}^{-1}(B)$.

(2)$\Rightarrow$(1): Assume $\tilde{A}_1$ is a component of $p_{\tilde{X}}^{-1}(A)$ which is sent to the same component $\tilde{B}_1$, of $p_{\tilde{X}}^{-1}(B)$, by $\tilde{f}$ and $\tilde{g}$. Let $\tilde{i}_{A\subset X}$ be the lifting of the inclusion map $i_{A\subset X} : A \to X$ which sends $\tilde{A}$ to $\tilde{A}_1$, and let $\tilde{i}_{B\subset Y}$ be the lifting of the inclusion map $i_{B\subset Y} : B \to Y$ which sends $\tilde{B}$ to $\tilde{B}_1$. It is easy to see that $\tilde{A}$ is a covering space of $\tilde{A}_1$, and that $\tilde{B}$ is a covering space of $\tilde{B}_1$. Let $\tilde{f}_A$ and $\tilde{g}_A$ be liftings of $\tilde{f}|_{\tilde{A}_1}$ and $\tilde{g}|_{\tilde{A}_1}$. It is easy to check that $\tilde{f}_A$ and $\tilde{g}_A$ are liftings of f_A and g_A respectively, and that the diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{f}_A, \tilde{g}_A} & \tilde{B} \\ \downarrow \tilde{i}_{A\subset X} & & \downarrow \tilde{i}_{B\subset Y} \\ \tilde{X} & \xrightarrow{\tilde{f}, \tilde{g}} & \tilde{Y}. \end{array}$$

is commutative. □

Lemma 4.5.3 *An element $\{(\tilde{f}^H, \tilde{g}^H)\} \in \mathcal{R}_{f^H, g^H}$ is in E_{f^H, g^H} if and only if there is an isotropy subgroup $K \supset H$, and a lifting $(\tilde{f}^K, \tilde{g}^K)$ of (f^K, g^K) such that the diagram*

$$\begin{array}{ccc} \tilde{X}^K & \xrightarrow{(\tilde{f}^K,\tilde{g}^K)} & \tilde{Y}^K \\ \downarrow \tilde{i}^X_{H\subset K} & & \downarrow \tilde{i}^Y_{H\subset K} \\ \tilde{X}^H & \xrightarrow{(\tilde{f}^H,\tilde{g}^H)} & \tilde{Y}^H, \end{array}$$

is commutative.

Proof: By Lemma 4.5.2, we only need to prove that if an element $\{(\tilde{f}^H,\tilde{g}^H)\} \in \mathcal{R}_{f^H,g^H}$ is in E_{f^H,g^H}, iff there is an isotropy subgroup $K \supset H$ and a component $\tilde{X}_1^K \subset (p_{X^H})^{-1}(X^K)$ such that $\tilde{f}^H$ and $\tilde{g}^H$ map $\tilde{X}_1^K$ to the same component $\tilde{Y}_1^K \subset (p_{Y^H})^{-1}(Y^K)$.

By the definitions of E_{f^H,g^H} and $\tau_{(H)<(K)}$, there are isotropy subgroups $H' \in (H)$ and $K' \supset H'$ and liftings $(\tilde{f}^{H'}, \tilde{g}^{H'})$ of $(f^{H'}, g^{H'})$, and $(\tilde{f}^{K'}, \tilde{f}^{K'})$ of $(f^{K'}, g^{K'})$ such that the diagrams

$$\begin{array}{ccc} \tilde{X}^{K'} & \xrightarrow{(\tilde{f}^{K'},\tilde{g}^{K'})} & \tilde{Y}^{K'} \\ \downarrow \tilde{i}^X_{H'\subset K'} & & \downarrow \tilde{i}^Y_{H'\subset K'} \\ \tilde{X}^{H'} & \xrightarrow{(\tilde{f}^{H'},\tilde{g}^{H'})} & \tilde{Y}^{H'}, \end{array}$$

and

$$\begin{array}{ccc} \tilde{X}^{H'} & \xrightarrow{(\tilde{f}^{H'},\tilde{g}^{H'})} & \tilde{Y}^{H'} \\ \downarrow \tilde{l}^{H'}_{w,X} & & \downarrow \tilde{l}^{H'}_{w,Y} \\ \tilde{X}^{H} & \xrightarrow{(\tilde{f}^{H},\tilde{g}^{H})} & \tilde{Y}^{H}, \end{array}$$

are commutative, where $w \in W$ and $H = wH'w^{-1}$. By Lemma 4.5.2, there is a component $\tilde{X}_1^{K'} \subset (p_{X^{H'}})^{-1}(X^{K'})$ such that $\tilde{f}^{H'}$ and $\tilde{g}^{H'}$ map $\tilde{X}_1^{K'}$ to the same component $\tilde{Y}_1^{K'} \subset (p_{Y^{H'}})^{-1}(Y^{K'})$. Let $K = wK'w^{-1}$, then $H \subset K$, $X^K \subset X^H$, and $Y^K \subset Y^H$. Note that w induces homeomorphisms from $X^{H'}$ to X^H, $X^{K'}$ to X^K, $Y^{H'}$ to Y^H and $Y^{K'}$ to

Y^K, so $(p_{X^{H'}})^{-1}(X^{K'})$ is mapped to $(p_{X^H})^{-1}(X^K)$ by $\bar{l}^{H'}_{w,X}$ and $(p_{Y^{H'}})^{-1}(Y^{K'})$ is mapped to $(p_{Y^H})^{-1}(Y^K)$ by $\bar{l}^{H'}_{w,Y}$. Assume that $\tilde{X}_1^{K'}$ is mapped to a component $\tilde{X}_1^K \in (p_{X^H})^{-1}(X^K)$ by $\bar{l}^H_{w,X}$, and $\tilde{Y}_1^{K'}$ is mapped to a component $\tilde{Y}_1^K \in (p_{Y^H})^{-1}(Y^K)$ by $\bar{l}^H_{w,Y}$. From the second commutative diagram, we have that $\tilde{X}_1^K$ is mapped to $\tilde{Y}_1^K$. □

Lemma 4.5.4 *The map incl defined in Definition 4.4.3 induces a one to one correspondence between $E_{f^H,g^H}(WH)$ and $\bigcup_{(H)<(K)}$ Im $\tau_{(H)<(K)}$.*

Proof: It is obvious from the definition of $E_{f^H,g^H}(WH)$ that $\bigcup_{(H)<(K)}$ Im $\tau_{(H)<(K)}$ is the image of $E_{f^H,g^H}(WH)$ under *incl*. By Proposition 4.4.4, *incl* is injective. So we have the result. □

By Lemma 4.5.4, we have $\#W\mathcal{R}_{f^H,g^H}(WH)-\#(\bigcup_{(H)<(K)}\text{ Im }\tau_{(H)\le(K)}) = \#(\mathcal{R}_{f^H,g^H}(WH)-E_{f^H,g^H}(WH))$. By Theorem 4.3.3, $\Theta^{WH}_{f^H,g^H}$ is a one to one correspondence between $\mathcal{R}_{f^H,g^H}(WH)$ and $\nabla(f^H,g^H;x_0,y_0,\omega_f,\omega_g)/WH$. So to calculate $\#W\mathcal{R}_{f^H,g^H}(WH)-\#(\bigcup_{(H)<(K)}\text{ Im }\tau_{(H)\le(K)})$, we need to know the image of $E_{f^H,g^H}(WH)$ in $\nabla(f^H,g^H;x_0,y_0,\omega_f,\omega_g)/WH$ under $\Theta^{WH}_{f^H,g^H}$.

Lemma 4.5.5 *An element $\theta \in \mathcal{R}_{f^H,g^H}(WH)$ is in $E_{f^H,g^H}(WH)$ if and only if for any $\alpha \in \Theta^{WH}_{f^H,g^H}(\theta)$, the $f_\pi^{\omega_f}, g_\pi^{\omega_g}$-congruence class containing α is in $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}$, where $\bar{i}_{H\subset K}$ is induced by the inclusion map $i_{H\subset K}: X^K \to X^H$ if $H \subset K$.*

Proof: Let $(\tilde{f}^H,\tilde{g}^H)$ be a lifting of (f^H,g^H), $\tilde{x}_0 \in (p_{X^H})^{-1}(x_0)$, and $\tilde{\alpha}$ be a path from $\tilde{g}^H(\tilde{x}_0)$ to $\tilde{f}^H(\tilde{x}_0)$, then $\Theta^{WH}_{f^H,g^H}([(\tilde{f}^H,\tilde{g}^H)]_{WH}) = WH\overline{[\omega_g\cdot(p_{X^{WH}}\circ\tilde{\alpha})\cdot\omega_f^{-1}]}$. By the proof of Proposition 1.2.5, we know that any $\alpha \in \nabla(f^H,g^H;x_0,y_0,\omega_g,\omega_f)/WH$ has this form. Now assume $\theta \in E_{f^H,g^H}(WH)$. By Lemma 4.5.3, there is a lifting $(\tilde{f}^H,\tilde{g}^H)$ of (f^H,g^H) in θ, and a lifting $(\bar{f}^K, \bar{g}^K)$ of (f^K, g^K) for some isotropy subgroup $K \supset H$ such that the diagram

$$\begin{array}{ccc}
\tilde{X}^K & \stackrel{(\bar{f}^K,\bar{g}^K)}{\longrightarrow} & \tilde{Y}^K \\
\downarrow \bar{i}^X_{H\subset K} & & \downarrow \bar{i}^Y_{H\subset K} \\
\tilde{X}^H & \stackrel{(\bar{f}^H,\bar{g}^H)}{\longrightarrow} & \tilde{Y}^H
\end{array}$$

is commutative. Let $\tilde{x}_0' \in (\bar{i}^X_{H\subset K})^{-1}(\tilde{x}_0) \subset \tilde{X}^K$ and $\tilde{\beta}$ be a path from $\bar{g}^K(\tilde{x}_0')$ to $\bar{f}^K(\tilde{x}_0')$. Then $\bar{i}^Y_{H\subset K}(\tilde{\beta})$ is a path from $\bar{g}^H(\tilde{x}_0)$ to $\bar{f}^H(\tilde{x}_0)$. So we have $\overline{[\omega_g \cdot (p_{X^{WH}} \circ \bar{i}^Y_{H\subset K}(\tilde{\beta})) \cdot \omega_f^{-1}]}$ is f_π, g_π-congruent to α. However, $\omega_g \cdot (p_{X^{WH}} \circ \bar{i}^Y_{H\subset K}(\tilde{\beta})) \cdot \omega_f^{-1}$ is a loop in X^K, so $\overline{\alpha}$ is in $\bigcup_{H\subset K} \mathrm{Im}\, \bar{i}_{H\subset K}$.

Now assume that $(\bar{f}^H, \bar{g}^H) \in \theta$, and $\overline{[\omega_g \cdot p_{X^H}(\alpha_{\bar{f},\bar{g}}) \cdot \omega_f]}$ is in $\mathrm{Im}\, \bar{i}_{H\subset K}$ for some isotropy subgroup $K \supset H$. By Proposition 1.2.5 and Lemma 2.1.9, $(\bar{f}^H, \bar{g}^H)$ is in $\mathrm{Im}\, \bar{i}^R_{f^K,g^K}$, so there is a commutative diagram

$$\begin{array}{ccc}
\tilde{X}^K & \stackrel{(\bar{f}^K,\bar{g}^K)}{\longrightarrow} & \tilde{Y}^K \\
\downarrow \bar{i}^X_{H\subset K} & & \downarrow \bar{i}^Y_{H\subset K} \\
\tilde{X}^H & \stackrel{(\bar{f}^H,\bar{g}^H)}{\longrightarrow} & \tilde{Y}^H.
\end{array}$$

This shows θ is in $E_{f^H,g^H}(WH)$ by Lemma 4.5.3. □

Lemma 4.5.6 *Assume $X^W \neq \emptyset$ and H is an isotropy subgroup of W. If an element $[\alpha]$ of $\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H})$ is in $\bigcup_{H\subset K} \mathrm{Im}\, \bar{i}_{H\subset K}$, then for any $w \in WH$, $w[\alpha]$ is in $\bigcup_{H\subset K} \mathrm{Im}\, \bar{i}_{H\subset K}$.*

Proof: Assume that K_1 is an isotropy subgroup of W such that $H \subset K_1$ and $[\alpha] \in \mathrm{Im}\, \bar{i}_{H\subset K_1}$. Then there is an element $[\alpha_1] \in \pi_1(Y^{K_1})$ such that $[\alpha] = \bar{i}_{H\subset K_1}([\alpha_1]) = [i_{H\subset K_1} \circ \alpha_1]$. Now for an element $w \in WH$, $wHw^{-1} = H$ and let $K_2 = wK_1w^{-1}$, then we have

$w[\alpha_1] = [w\alpha_1]$ is in $\pi_1(Y^{K_2})$ and $w[\alpha] = [w\alpha] = [w(i_{H\subset K_1} \circ \alpha_1)] = [i_{H\subset K_2} \circ (w\alpha_1)] = \bar{i}_{H\subset K_2}[w\alpha_1]$. □

Lemma 4.5.6 shows that $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}$ is closed under the action of WH on $\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H})$. Therefore, $\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H}) - \bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}$ is closed too. So $\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H})/WH$ can be split into $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}/WH$ and $(\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H})$ - $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K})/WH$.

Proposition 4.5.7 *Assume $X^W \neq \emptyset$. Then $\Theta^{WH}_{f^H, g^H}$ induces a one to one correspondence between $\mathcal{R}_{f^H, g^H}(WH) - E_{f^H, g^H}(WH)$ and $(\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H}) - \bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K})/WH$*

Proof: It is sufficient to observe that $\Theta^{WH}_{f^H, g^H}$ induces a one to one correspondence between $E_{f^H, g^H}(WH)$ and $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}/WH$. By Lemmas 4.5.5 and 4.5.6, an element θ is in $E_{f^H, g^H}(WH)$ if and only if $\Theta_{WH}(\theta)$ is in $\bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K}/WH$. □

Theorem 4.5.8 *Let X, Y be W-manifolds, (f, g) a pair of W-maps from X to Y. Suppose X^W is nonempty, and for every isotropy subgroup H, that X^H is connected. Then if Y^H is a Jiang space, or if (f^H, g^H) has the weak Jiang property for some isotropy subgroup H of W, then either*

(1) $L(f^H, g^H) = 0 \Rightarrow NO_W(f_H, g_H) = 0$, or

(2) $L(f^H, g^H) \neq 0 \Rightarrow NO_W(f_H, g_H) = \#((Coker(g^H_* - f^H_*) - \cup_{H\subset K}$ Im $\tilde{\mu}_{H\subset K})/WH)$.

Proof: (1) follows as in Theorem 4.5.1. So we assume that $L(f^H, g^H) \neq 0$. Since every WH-class is essential, by Lemma 4.5.4, 4.5.6, 4.3.4 and 2.3.1, we have that $NO_W(f_H, g_H) = \#(\mathcal{R}_{f^H, g^H}(WH) - E_{f^H, g^H}(WH)) = \#((\nabla(f^H, g^H; x_0, y_0, \omega_{g^H}, \omega_{f^H}) - \bigcup_{H\subset K}$ Im $\bar{i}_{H\subset K})/WH)$ $= \#((Coker(g^H_* - f^H_*) - \cup_{H\subset K}$ Im $\tilde{\mu}_{H\subset K})/WH)$. □

Note 4.5.9 Unlike Theorem 4.14 in [WP3], Theorem 4.5.8 does not require that W be abelian and that for any $\sigma \in \pi_1(X^H)$, $T_{WH}(\sigma) \subset \pi_1(X^H)$ (where $T_{WH}(\sigma) = \{\tilde{\gamma} \in \tilde{WH} \mid \tilde{\gamma}\sigma\phi_W(\tilde{\gamma})^{-1} = \sigma\}$, and $\tilde{WH}$ is the set of liftings of WH and ϕ_{WH} is a map from $\tilde{WH}$ to itself such that $\phi_{WH}(\tilde{\gamma})\bar{f} = \bar{f}\tilde{\gamma}$ for a given lifting $\bar{f}$ of $f : X \to X$ and $\tilde{\gamma} \in \tilde{WH}$. See [WP3] for details). However, it does require that X^W be nonempty (this is not required in Theorem 4.14 in [WP3].)

Theorem 4.5.10 *Let X, Y be W-manifolds, and (f, g) a pair of W-maps from X to Y. Suppose that X^W is nonempty and for every isotropy subgroup H, that X^H is connected. Then if Y^H is a Jiang space, or if (f^H, g^H) has the weak Jiang property for every isotropy subgroup H of W, and $\Pi_{H\in Iso(X)}L(f^H, g^H) \neq 0$, then*

$$NO_W(f^H, g^H) = \Sigma_{(H)\leq(K)}NO_W(f_K, g_K)$$

Proof: For each isotropy type $(K) \geq (H)$, choose an isotropy subgroup $K \supset H$ and the maximal set $\mathcal{G}_K$ of $NO_{WK}(f^K, g^K)$ WK-class such that for each $\theta_K \in \mathcal{G}_K$, $W\theta_K \notin \tau_{(K)<(K')}(W\theta_{K'})$ for any isotropy subgroup $K \subset K'$ and any element $\theta_{K'} \in \mathcal{R}_{f^{K'}, g^{K'}}$. Then $\cup_{(H)\leq(K)}\mathcal{G}_K$ is an essential basis over $X^{(H)}$. For any essential WK-class θ_K, if $W\theta_K \notin \tau_{(K)<(K')}(W\theta_{K'})$ for any isotropy subgroup $K \subset K'$, then $\theta_K \in \cup_{(H)\leq(K)}\mathcal{G}_K$. Let $W\theta_K \in \tau_{(K)<(K')}(W\theta_{K'})$ for some isotropy subgroup $K \subset K'$, and let K_1 be the smallest isotropy subgroup such that there is a WK_1-class θ_1 satisfying $W\theta_K \in \tau_{(K)<(K_1)}(W\theta_1)$. Then θ_1 is in $\cup_{(H)\leq(K)}\mathcal{G}_K$, otherwise, there is a contradiction to the assumption that K_1 is smallest such isotropy subgroup. From the definition of $\cup_{(H)\leq(K)}\mathcal{G}_K$, it is minimal. □

Example 4.5.11 Let us now compute $NO_W(f_H, g_H)$ and $NO_W(f^H, g^H)$ for the pair of maps (f, g) in Example 4.3.7.

There are only two subgroups of W, the trivial group $< 1 >$ and W itself. We already discussed the homology of X and the homomorphisms induced by f and g in Example 4.3.7. So we give the homology of X^W here and the homomorphisms induced by f^W and g^W. $X^W = \{(e^{i\theta}, e^{i\theta}, (x, y, 0))\} \cong S^1 \times S^1$, where the first factor S^1 is the diagonal of $S^1 \times S^1$, and the second factor S^1 is the equator of S^2. So $H_1(X^W) \cong H_1(S^1 \times S^1) \cong \mathbf{Z} \oplus \mathbf{Z}$. Let a and b be the generators of first and second factors respectively. Then the homomorphism i_* induced by inclusion map $i_{<1>\subset W} : X^W \to X$ is as follows: $i_*(a) = a_1 + a_2$ and $i_*(b) = 0$(see Example 4.3.7 for the definition of a_1 and a_2). The homomorphisms induced by f^W and g^W are as follows: $f_*^W(a) = 2a$, $f_*^W(b) = -b$ and $g_*^W(a) = a$, $g_*^W(b) = 3b$.

(1) $NO_W(f_W, g_W) = 4$. Since W is the action group, $NO_W(f_W, g_W) = N_W(f^W, g^W) = N(f^W, g^W)$. Since X^W is a Jiang space and $L(f^W, g^W) = 4$, $N(f^W, g^W) = Coker(g_*^W - f_*^W)$. Since Im $(g_* - f_*) =< a, 4b >$, we have $Coker(g_* - f_*) = \{[0], [b], [2b], [3b]\}$. So $NO_W(f^W, g^W) = N(f^W, g^W) = 4$.

(2) To compute $NO_W(f_{<1>}, g_{<1>})$, we need to find the image of $Coker(g_*^W - f_*^W) = \{[0], [b], [2b], [3b]\}$ under $\tilde{\mu}_{<1>\subset W}$ in $Coker(g_* - f_*)/W$. Since $i_*(a) = a_1 + a_2$ and $i_*(b) = 0$, and $[a_1 + a_2] = [a_1 + a_2 + (a_1 - 2a_2) + (a_2 - 2a_1)] = [0]$, we have Im $\tilde{\mu} = \{[0]\}$. Hence $NO_W(f_{<1>}, g_{<1>}) = \#(Coker(g_* - f_*) - \text{Im } \tilde{\mu})/W = \#\{[a_1]_W\} = 1$.

So we have $NO_W(f^{<1>}, g^{<1>}) = NO_W(f, g) = 1 + 4 = 5$. Since the length of an orbit in $X_{<1>}$ is 2, by Theorem 4.4.11, $2 \cdot NO_W(f_{<1>}, g_{<1>}) + NO_W(f_W, g_W) = 2 \cdot 1 + 4 = 6$ is a lower bound of the number of coincidence points of (f', g') for any W-maps $f' \sim_W f$ and $g' \sim_W g$. On the other hand the ordinary coincidence Nielsen number is 3.

The following example is presented in [WP3] by an alternative method. We use our method to calculate the equivariant Nielsen number, and point out some errors made in [WP3].

Example 4.5.12 Let $X = Y = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^2$ and $W = \mathbf{Z}_6 =<\alpha> \times <\beta>$ where $\mathbf{Z}_2 =<\alpha>$, $\mathbf{Z}_3 =<\beta>$. Let W act on X via

$\alpha \cdot (e^{i\theta_1}, ..., e^{i\theta_5}, (x, y, z)) = (e^{i\theta_2}, e^{i\theta_1}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, (x, y, -z))$,

$\beta \cdot (e^{i\theta_1}, ..., e^{i\theta_5}, (x, y, z)) = (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_5}, e^{i\theta_3}, e^{i\theta_4}, (x, y, z))$.

Then $X^{<\alpha>} = \{(e^{i\theta}, e^{i\theta}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, (x, y, 0))\} \approx T^5$, and

$X^{<\beta>} = \{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta}, e^{i\theta}, e^{i\theta}, (x, y, z))\} \approx T^3 \times S^2$,

$X^W = (e^{i\theta_1}, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_2}, e^{i\theta_2}, (x, y, 0))\} \approx T^3$.

Let $g : X \to Y$ be the identity, and $f : X \to Y$ the W-map defined by

$f(e^{i\theta_1}, ..., e^{i\theta_5}, (x, y, z)) = (e^{i2\theta_2}, e^{i2\theta_1}, e^{i2\theta_3}, e^{i2\theta_4}, e^{i2\theta_5}, (x, -y, -z))$.

All the conditions in Theorems 4.5.10 and 4.5.8 are satisfied, so we can use these theorems to compute $NO_W(f^H, g^H)$.

(1) $NO_W(f_{<1>}, g_{<1>}) = 0$.

Let $X_1 = S^1 \times S^1$, and $f_1 : X_1 \to X_1$ be defined by $f_1(e^{i\theta_1}, e^{i\theta_2}) = (e^{i2\theta_2}, e^{i2\theta_1})$.

Let $X_2 = S^1 \times S^1 \times S^1$, and $f_2 : X_2 \to X_2$ be defined by $f_2(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) = (e^{i2\theta_1}, e^{i2\theta_2}, e^{i2\theta_3})$.

Let $X_3 = S^2$, and $f_3 : X_3 \to X_3$ be defined by $f_3(x, y, z) = (x, -y, -z)$.

Then $X = X_1 \times X_2 \times X_3$ and $f = f_1 \times f_2 \times f_3$, and there is a one to one correspondence between $Coker(1 - f_*)$ and $Coker(1 - f_{1*}) \times Coker(1 - f_{2*}) \times Coker(1 - f_{3*})$.

Let $a_1, a_2 \in H_1(X_1)$ be generators represented by $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ respectively. Then $Coker(1 - f_{1*}) = \{[0], [a_1], [2a_1]\}$. It is easy to see that $Coker(1 - f_{2*}) = \{[0]\}$ and $Coker(1 - f_{3*}) = \{[0]\}$

Let $X_2' = (e^{i\theta}, e^{i\theta}, e^{i\theta}) \in X_2$. Then $X^{<\beta>} \approx X_1 \times X_2' \times X_3$ and the inclusion $i = id_{X_1} \times i_2 \times id_{X_3}$ from $X^{<\beta>}$ to X, where i_2 is the inclusion from X_2' to X_2. The image of $\tilde{\mu}_{<1>\subset<\beta>}$ is equal to Im $\tilde{\mu}_1 \times$ Im $\tilde{\mu}_2 \times$ Im $\tilde{\mu}_3$. Now it is easy to see that $\tilde{\mu}_{<1>\subset<\beta>}$ is onto since $\tilde{\mu}_i$ is onto for $i = 1, 2$, and 3. Therefore, we have $NO_W(f_{<1>}, g_{<1>}) = 0$. (Note that we do not need to consider the images of $\tilde{\mu}_{<1>\subset<\alpha>}$ and $\tilde{\mu}_{<1>\subset W}$).

(2) Similarly, we have $NO_W(f_{<\alpha>}, g_{<\alpha>}) = 0$.

(3) $NO_W(f_W, g_W) = N(f_W, g_W) = 2$.

(4) $NO_W(f_{<\beta>}, g_{<\beta>}) = 1$.

As in (1), we have $Coker(1 - f_*^{<\beta>}) = Coker(1 - f_{1*}^{<\beta>}) \times Coker(1 - f_{2*}^{<\beta>}) \times Coker(1 - f_{3*}^{<\beta>})$ and $Coker(1 - f_{1*}^{<\beta>}) = \{[0], [a_1], [2a_1]\}$, $Coker(1 - f_{2*}^{<\beta>}) = \{[0]\}$ and $Coker(1 - f_{3*}^{<\beta>}) = \{[0]\}$. So $Coker(1 - f_*^{<\beta>}) = \{([0], [0], [0]), ([a_1], [0], [0]), ([2a_1], [0], [0])\}$.

$X^W \approx X_1' \times X_2' \times X_3'$, where $X_1' = \{(e^{i\theta}, e^{i\theta}) \in X_1\}$, $X_2' = \{(e^{i\theta}, e^{i\theta}, e^{i\theta}) \in X_2\}$, and $X_3' = \{(x, y, 0) \in X_3\}$. The image of $\tilde{\mu}_{<\beta>\subset W}$ is $\{([0], [0], [0])\}$. The calculation is similar to Example 4.5.11. So $Coker(1 - f_*^{<\beta>}) -$ Im$\tilde{\mu}_{<\beta>\subset W}$ consists of two elements, namely $([a_1], [0], [0])$ and $([2a_1], [0], [0])$. To prove that $NO_W(f_{<\beta>}, g_{<\beta>}) = 1$, we only need to show that the action of the Weyl group of $< \beta >$, $W < \beta > \cong < \alpha >$, on $Coker(1 -$

$f_*^{<\beta>}) -$ Im $\tilde{\mu}_{<\beta>\subset W}$ is nontrivial. In fact, $\alpha\cdot([a_1],[0],[0]) = ([a_2],[0],[0])$, but $([a_2],[0],[0]) = ([2a_1],[0],[0])$ since $[a_2] = [a_2+(1-f^{<\beta>_{1*}})(-a_2)] = [2a_1]$. So we have $NO_W(f_{<\beta>}, g_{<\beta>}) = 1$.

Finally, we have that

$\Sigma_{<1>\subset(K)}[W : K]NO_W(f_K, g_K)$

$= 6\cdot NO_W(f_{<1>}, g_{<1>}) + 3\cdot NO_W(f_{<\alpha>}, g_{<\alpha>}) + 2\cdot NO_W(f_\beta, g_\beta) + 1\cdot NO_W(f_W, g_W)$

$= 6\cdot 0 + 3\cdot 0 + 2\cdot 1 + 1\cdot 2 = 4.$

So 4 is a lower bound of the number of coincidence points of (f', g') for any W-maps $f' \sim_W f$ and $g' \sim_W g$ by Theorem 4.4.11.

Note 4.5.13 There some errors in the computation of Example 3.9 of [WP3], where the spaces and maps are the same as in Example 4.5.12. In particular, $N_G(f_{<\beta>}) = 2$, not 4 as stated in [WP3]. In fact, the fixed points $(m, m^2, 1, 1, 1, (1, 0, 0))$ and $(m, m^2, 1, 1, 1, (-1, 0, 0))$ are in the same class. To see this let $a : I \to S^2$ be a path from $(1, 0, 0)$ to $(-1, 0, 0)$, and let C be the path in X defined by $C(t) = (m, m^2, 1, 1, 1, a(t))$, then C is a path between these two points and $f \circ C \sim C$. So by Theorem 3.7 in [WP3], $NO_G(f_{<\beta>}) = N_G(f_{<\beta>})/W < \beta >= 2/2 = 1$ instead of 2. This causes incorret results in the computation of $NO_G(f)$ and $m_G(f)$, which are the same as $NO_G(f^{<1>}, id^{<1>})$ and the minimal number of fixed points of maps in the G-homotopy class of f respectively, and the correct results should be $NO_G(f) = 3$ and $m_G(f) = 4$. As a consequence $\#\Gamma(f)$ is not the minimal number of fixed points of G-maps in the W-homotopic class of f, as claimed in [WP3] (see p.163 in [WP3] for detail).

4.6 Minimality

In this section, we will prove that in some cases the lower bounds $NO_W(f_H, g_H)$ and $NO_W(f^H, g^H)$ on the number of orbits can be attained. First, we will give an analog of Schirmer's method to obtain a pair of maps (f', g'), which is W-homotopic (f, g), and in which $\#\Gamma(f', g')$ is finite. Then we will coalesce the coincidence point orbits in the same W-class to a single orbit.

Lemma 4.6.1 *For any W-space X and $\epsilon > 0$, there is a $\delta > 0$ such that, if $f, g : X \to Y$ are equivariant maps and $d(f(x), g(x)) < \delta$ for all $x \in X$, then f and g are equivariantly ϵ-homotopic through a homotopy constant on the coincidence set of f and g.*

Proof: See Corollary 2.3 of [WD]. □

In [SH1], in order to prove the minimal theorem, a sequence of numbers $\{\epsilon'_i\}_{i=0}^n$ are introduced. We give the definition here. Let Y be a W-manifold of dimension n, $\epsilon > 0$ be arbitrary, with corresponding δ given by Lemma 4.6.1, and $\mathcal{U} = \{U_i\}_{i=1}^l$ be an open covering of Y such that for each $1 \le i \le l$ there is a homeomorphism $\phi_i : U_i \to B^n$, where B^n is a unit n-disk. Let $\delta' > 0$ be the Lebesque number of $\mathcal{U}$ and $\delta'' = \min\{\delta', \delta\}$.

1. $\epsilon'_{n+1} = \frac{1}{4}\delta''$.

2. Suppose ϵ'_{j+1} is defined. There is a δ'_{j+1} such that $0 < \delta'_{j+1} = \delta'_{j+1}(\epsilon'_{j+1}) < \frac{1}{2}$, and such that for each $y_1, y_2 \in U_i$, if $|\phi_i(y_1) - \phi_i(y_2)| < \delta'_{j+1}$, then $d(y_1, y_2) < \epsilon'_{j+1}$, $(i = 1, ..., l)$.

There is a γ'_{j+1} such that $0 < \gamma'_{j+1} = \gamma'_{j+1}(\delta'_{j+1}) < \frac{1}{2}$, and such that for each $y_1, y_2 \in U_i$, if $d(y_1, y_2) < \gamma'_{j+1}$, then $|\phi_i(y_1) - \phi_i(y_2)| < \delta'_{j+1}$, $(i = 1, ..., l)$, Set $\epsilon'_j = \text{Min}\ (\epsilon'_{j+1}, \gamma'_{j+1})$.

Then after $n+1$ steps, we have $0 < \epsilon'_0 \leq \epsilon'_1 \leq, \ldots, \leq \epsilon'_n \leq \epsilon'_{n+1} = \frac{1}{4}\delta''$.

The following lemma is an equivariant version of a generalization of Lemma 1 in [SH1].

Lemma 4.6.2 *Let K be a simplicial W-complex with property* (P_1) (see Definition 4.1.11), *$0 \leq d < n$ and K_1 be an invariant subcomplex of K such that $K^{d-1} \subset K_1$. Let $g : K \to Y$ be a W-map with the property:*

(1) $d_Y(g(a^m)) < \frac{1}{4}\epsilon'_0$ for all $a^m \in K$.

Let $f : K_1 \to Y$ be a W-map with properties:

(2) $\#\Gamma(f, g|_{K_1})$ is finite.

(3) $|f - g| < \epsilon'_i$ on K_1^i.

Then there is a W-extension $f' : K_1 \cup K^d \to Y$ of f such that:

(2)' $\Gamma(f', g|_{K_1 \cup K^d}) = \Gamma(f, g|_{K_1})$.

(3)' $|f' - g| < \epsilon'_i$ on $(K_1 \cup K^d)^i$.

Proof: We will prove the lemma by induction on the number l_d of the d-simplices in $K - K_1$.

If $l_d = 1$, let a^d the single d simplex. Since $d_Y(g(a^d)) < \frac{1}{4}\epsilon'_0$ by (1) and on ∂a^d, $d_Y(f, g) < \epsilon'_{d-1}$, $d_Y(g(a^d) \cup f(\partial a^d))$ is less than δ', the Lebesque number of the given open covering $\mathcal{U}$ of Y, and therefore, $g(a^d) \cup f(\partial a^d)$ is contained in some $U_i \in \mathcal{U}$. By the choice of ϵ'_{d-1}, we have $|\phi_i \circ f - \phi_i \circ g| < \delta'_d$. By Lemma 1.6.1, we have an extension $\phi_i \circ f'$ of $\phi_i \circ f$ such that $|\phi_i \circ f' - \phi_i \circ g| < \delta'_d$ and $\Gamma(\phi_i \circ f', \phi_i \circ g) = \Gamma(\phi_i \circ f|_{\partial a^d}, \phi_i \circ g|_{\partial a^d})$ since $d < n$. By composing ϕ_i^{-1} with $\phi_i \circ f'$, we get an extension f' of f over $K_1 \cup a^d$ with the properties that $d(f', g) < \epsilon'_d$ on a^d (by the choice of δ'_d), and $\Gamma(f', g|_{K_1 \cup a^d}) = \Gamma(f, g|_{K_1})$.

Assume the lemma is proven for $l_d \leq k$. Now assume $l_d = k+1$. Let a^d be a d-simplex in $K - K_1$. As above, we can extend f over $K_1 \cup a^d$. Since K has property (P_1), we can extend f over $K_1 \cup (\bigcup_w wa^d)$ by applying the W-action. Now let $K_2 = K_1 \cup \{wa^d\}$, then K_2 is an invariant subcomplex of K with the same properties as K_1, and the number of d-simplices in $K - K_2$ is less than or equal to l_d. By the induction hypothesis, f can be extended to $K_2 \cup K^d$ with the desired properties. □

The next lemma is for the case when $n = d$.

Lemma 4.6.3 *Let K be a simplicial W-complex with property (P_1) and K_1 an invariant subcomplex of K. Let $g : K \to Y$ be a W-map with the property that $d_Y(g(a^m)) < \frac{1}{4}\epsilon'_0$ for all $a^m \in K$. Let $f : K_1 \cup K^{n-1} \to Y$ be a W-map such that $|f - g| < \epsilon'_{n-1}$ and $\Gamma(f', g|_{K^{n-1} \cup K_1}) \subset K_1$, then there is a W-extension $f' : K^n \to Y$ of f such that $|f' - g| < \epsilon'_n$ and for any n-simplex a^n, if $\Gamma(f', g|_{K^{n-1} \cup K_1}) \cap \partial a^n \neq \emptyset$, then $\Gamma(f', g|_{K^{n-1} \cup K_1}) \cap (int\ a^n) = \emptyset$; if $\Gamma(f', g|_{K^{n-1} \cup K_1}) \cap \partial a^n = \emptyset$, then $\Gamma(f', g|_{K^{n-1} \cup K_1}) \cap (int\ a^n)$ is at most one point.*

Proof: Similar to the proof of Lemma 4.6.2. □

Corollary 4.6.4 *Let K be a W-complex of dimension n and K_1 be an invariant subcomplex of K, $g : |K| \to Y$ and $f : |K_1| \to Y$ be W-maps such that $d_Y(f, g) < \epsilon'_0$ on $|K_1|$, and $\Gamma(f, g|_{K_1})$ a finite set. Then there is a W-extension $f' : |K| \to Y$ such that $d_Y(f', g) < \epsilon'_n$ and $\#\Gamma(f', g)$ is finite.*

Proof: By Theorem 4.1.18, we may assume that the W-action is isometric. Subdividing K if necessary, we can assume that $d_Y(g(a^m)) < \epsilon'_0$. Applying Lemmas 4.6.2 and 4.6.3 several times, we have the corollary. □

Theorem 4.6.5 *For any pair of W-maps $(f,g) : X \to Y$, there is an $f' \sim_W f$ such that $\#\Gamma(f',g)$ is finite.*

Proof: By Theorem 4.1.18, we may assume that the W-action is isometric. Let $(H_1), (H_2), \ldots, (H_k) = \{1\}$ be an admissible ordering on $\{(H_i)\}$ with the property that $(H_i) \leq (H_j)$ implies $j \leq i$.

Consider the WH_i-space pair $(X^{H_i}, X^{>H_i})$, and assume that on $X^{(H_j)}$ the number of coincidence points of (f,g) is finite for any $j < i$. This implies that the number of coincidence points of (f,g) on $X^{>H_i}$ is finite. By Theorem 4.1.17, we may choose a triangulation $(K_{H_i}, K_{>H_i})$ for $(X^{H_i}, X^{>H_i})$ such that for any $a^d \in K_{H_i}$, $d_Y(f(a^d)) < \frac{1}{2}\epsilon'_0$ and $d_Y(g(a^d)) < \frac{1}{2}\epsilon'_0$. By Lemma 4.1.13, we may assume that $(K_{H_i}, K_{>H_i})$ has property (P1).

Let $A_i^+ = \{a^d \in K_{H_i} | \exists d_1 \geq d \text{ and } a^{d_1} \in K_{H_i}, \text{ such that } a^d \subset a^{d_1} \text{ and } |a^{d_1}| \cap \Gamma(f,g) \neq \emptyset\}$, and $A_i^- = \{a^d \in K_{H_i} | \, |a^d| \cap \Gamma(f,g) = \emptyset\}$. It is easy to check that both A_i^+ and A_i^- are invariant subcomplex of K_{H_i}, and $A_i^+ \cup A_i^- = K_{H_i}$. Let $A_{>i} = (A_i^+ \cap K_{>H_i}) \cup (A_i^+ \cap A_i^-)$, then $A_{>i}$ is an invariant subcomplex of A_i. In addition, $A_{>i}$ has the following properties:

(1) $d(f(x), g(x)) < \epsilon'_0$ for all $x \in |A_{>i}|$: in fact, for any $x \in A_{>i}$, there exist $a \in A_i^+$ and $x' \in a$, such that $x \in a$ and $f(x') = g(x')$. So we have $d(f(x), g(x)) \leq d(f(x), f(x')) + d(f(x'), g(x')) + d(g(x'), g(x)) \leq d_Y(f(a)) + 0 + d_Y(g(a)) < \frac{1}{2}\epsilon'_0 + \frac{1}{2}\epsilon'_0 = \epsilon'_i$.

(2) There are finite number of coincidence points of (f, g) on $A_{>i}$: there are finite number of coincidence points on $A_i^+ \cap K_{>H_i}$ by induction hypothesis, and there are no coincidence points on $A_i^+ \cap A_i^-$ by the definition of A_i^-.

By Lemmas 4.6.2 and 4.6.3, $f|_{A_{>i}}$ can be extended to a WH_i-map $f'|_{|A_i^+|}$ such that for any $x \in |A_i^+|$, $d(f'|_{|A_i^+|}(x), g(x)) < \delta$ and $\#\Gamma(f'|_{|A_i^+|}, g|_{|A_i^+|})$ is finite.

Define $f'|_{X^{H_i}} : X^{H_i} \to Y^{H_i}$ to be

$$f'|_{X^{H_i}}(x) = \begin{cases} f(x) & \text{if } x \notin |A_i^+| \\ f'|_{|A_i^+|}(x) & \text{if } x \in |A_i^+| \end{cases}$$

It is obvious that $f'|_{X^{H_i}}$ is continuous, and is a WH_i-map since both A_i^+ and A_i^- are invariant complex of K_{H_i} and on $A_i^+ \cap A_i^-$, $f(x) = f'|_{|A_i^+|}(x)$. Since $d(f|_{X^{H_i}}, f'|_{X^{H_i}}) < \delta$, we have $f|_{X^{H_i}} \sim_{WH_i} f'|_{X^{H_i}}$ by Lemma 4.6.1.

Applying the W-action, we get a W-map $f'|_{X^{(H_i)}} : X^{(H_i)} \to Y^{(H_i)}$, which is homotopic to $f|_{X^{(H_i)}}$ and has the property $\#\Gamma(f'|_{X^{(H_i)}}, g|_{X^{(H_i)}})$ is finite. By Lemma 4.1.14, $f'|_{X^{(H_i)}}$ can be extended to a W-map $f' : X \to Y$, such that $f' \sim_W f$ with the property that $\#\Gamma(f'|_{X^{(H_i)}}, g|_{X^{(H_i)}})$ is finite.

Applying this procedure to $(X^{H_1}, \emptyset)$ first, we get the result by induction. $\square$

Standard Hypotheses : Let W be a finite group and X a smooth compact W-manifold. For each $H \in Iso(X)$, we assume that X^H is connected, dim $X^H \geq 3$ and dim X^H - dim $(X^H - X_H) \geq 2$.

Note 4.6.6 : Under the standard hypotheses, for each $H \in Iso(X)$, $X^H - X_H$ can be bypassed in X^H.

Lemma 4.6.7 *Assume the standard hypotheses. Suppose that $f, g : X \to Y$ are W-maps; θ_0 and θ_1 are two distinct isolated WH-coincidence orbits belonging to the same WH-class of (f^H, g^H), for some $H \in Iso(X)$. Assume both θ_0 and θ_1 are in X_H. Then there exists a W-homotopy $\{f_t\}$ relative to $X^{>H}$ such that $f_0 = f$ and $\Gamma(f_1^H, g^H) = \Gamma(f^H, g^H) - \theta_1$.*

Proof: Let $x_0 \in \theta_0$, $x_1 \in \theta_1$ and $\sigma : [0,1] \to X^H$ be a path from x_0 to x_1 such that $f \circ \sigma \sim g \circ \sigma$ rel $\{0,1\}$. Since dim X^H - dim $(X^H - X_H) \geq 2$, we can assume that σ is in X_H. As in Lemma 5.4 in [WP2], we can find an arc α from x_0 to x_1 homotopic to σ and a neighborhood U of α such that $\overline{U} \cong D^n$ and for any $w \in WH$, $wU \cap U = \emptyset$. By Lemma 3.3.5, we have $f' \sim f$ rel $X - U$ such that $\Gamma((f')^H, g^H) = \Gamma(f^H, g^H) - \{x_1\}$. Applying the W-action, we have $f_1 \sim_W f$ with $\Gamma((f_1)^H, g^H) = \Gamma(f^H, g^H) - \{\theta_1\}$. □

Lemma 4.6.8 *Assume the standard hypotheses. Suppose that $f, g : X \to Y$ are W-maps; θ_0 and θ_1 are two distinct isolated WH coincidence orbits belonging to the same WH-class of (f^H, g^H), for some $H \in Iso(X)$. Furthermore, we assume that $\theta_0 \subset X_K$, $\theta_1 \subset X_H$ for some $K \in Iso(X)$ with $H \subset K$. Then there exists a W-homotopy $\{f_t\}$ relative to $X^{>H}$ such that $f_0 = f$ and $\Gamma(f_1^H, g^H) = \Gamma(f^H, g^H) - \theta_1$.*

Proof: Let $x_0 \in \theta_0$, $x_1 \in \theta_1$ and $\sigma : [0,1] \to X^H$ be a path from x_0 to x_1 such that $g \circ \sigma \sim f \circ \sigma$ rel $\{0,1\}$. As in Theorem 1.1 in [WP1], we can find an arc $\alpha \sim \sigma$ from x_0 to x_1 such that $\alpha([0,1)) \subset X_H$ and a neighborhood U of $\alpha([0,1))$ such that $\overline{U} \cong D^n$ and for $w \in WH$, $wU \cap U = \emptyset$. By Lemma 3.3.5, we have $f' \sim f$ rel $X - U$ such that $\Gamma((f')^H, g^H) = \Gamma(f^H, g^H) - \{x_1\}$. Applying the W-action, we have $f_1 \sim_W f$ with $\Gamma(f_1^H, g^H) = \Gamma(f^H, g^H) - \{\theta_1\}$. □

Lemma 4.6.9 *Let $\theta \subset X_H$ be an isolated WH-coincidence orbit of (f^H, g^H) for some $H \in Iso(X)$. Assume $x \in \theta$ and ind $(x) = 0$, then there exists a W-homotopy $\{f_t\}$ relative to $X^{>H}$ such that $f_0 = f$ and $\Gamma(f_1^H, g^H) = \Gamma(f^H, g^H) - \theta$.*

Proof: Let U be a neighborhood of x such that for any $w \in WH$ $wU \cap U = \emptyset$. By Lemma 1.6.5, we have $f' \sim f$ rel $X - U$ such that $\Gamma((f')^H, g^H) = \Gamma(f^H, g^H) - \{x_1\}$. Applying the W-action, we have $f_1 \sim_W f$ with $\Gamma(f_1^H, g^H) = \Gamma(f^H, g^H) - \{\theta_1\}$. □

Theorem 4.6.10 *Assume the Standard Hypotheses. For any pair of W-maps $(f, g) : X \to Y$, we have $M_W(f_{(H)}, g_{(H)}) = [W : H] \cdot NO_W(f_H, g_H)$ for all $H \in Iso(X)$.*

Proof: By Lemma 4.6.7, we may assume that each WH-class has at most one coincidence orbit. Assume $\{wx\}_{w \in WH}$ is an orbit of coincidence points. If $x \in X_H$ and $[x] \in$ Im $\tau_{(H)<(K)}$ for some isotropy subgroup K, we may create a coincidence orbit, which is in the same class with $[x]$, and then, by Lemma 4.6.8, coalesce $\{wx\}$ to it. Applying the W-action, we get the pair of maps with $NO_W(f_H, g_H)$ coincidence orbits in $X_{(H)}$ and the total number of coincidence points is $[W : H] \cdot NO_W(f_H, g_H)$. □

Theorem 4.6.11 *Assume the Standard Hypotheses. For any pair of W-maps $(f, g) : X \to Y$, we have $MO_W(f^{(H)}, g^{(H)}) = NO_W(f^H, g^H)$ for all $H \in Iso(X)$.*

Proof: Assume $\{\theta_i\}_{i=1}^k$ is an essential basis over $X^{(H)}$. We will prove that there are homotopies $f' \sim_W f$ and $g' \sim_W g$ such that $\Gamma(f', g') \cap X^{(H)} = \bigcup_{i=1}^k \{wx_i\}_{w \in W}$, where $\{wx_i\}_{w \in W}$ corresponds to θ_i. Applying this to an essential basis $\mathcal{G}$ with $NO_W(f^H, g^H)$ number of elements, we prove the theorem.

Let $(H_0), (H_1), ..., (H_k)$ be an admissible ordering on $\{(K) \mid (K) \geq (H)\}$. We will use induction on the number of the elements in $\{(K) \mid (K) \geq (H)\}$, which we will denote by I_H. If I_H is 1, we can assume that in each essential WH-class there is a unique coincidence point orbit and there are no other coincidence points since the dimension of X^H is greater than 2. Assume $\theta = [(\tilde{f}^H, \tilde{g}^H)]$ is an element of $\mathcal{G}$ and is inessential. Let $\alpha \in \pi_1(Y, y_0)$ be in $\Theta_{f^H, g^H}([(\tilde{f}^H, \tilde{g}^H)])$. Let $x \in X^H - \Gamma(f, g)$ be any point and let β be a path from x to x_0 in $X^H - \Gamma(f, g)$. Let l_f and l_g be arcs in Y^H such that $l_f(0) = f(x)$, $l_g(0) = g(x)$, $l_f(1) = l_g(1)$ and $l_g \cdot l_f^{-1} \sim (g \circ \beta) \cdot \omega_g^{-1} \cdot \alpha \cdot \omega_f \cdot (f \circ \beta^{-1})$. As in the proof of Lemma 2.5.2, we can change f^H and g^H in a small neighborhood of x. such that $\Gamma(f'^H, g'^H) = \Gamma(f^H, g^H) \cup \{x\}$. By using the WH-action on f', g', we can assume that f'^H, g'^H are WH-maps from X^H to Y^H. Repeating this procedure, we can get a pair of WH-maps (f'^H, g'^H) from X^H to Y^H homotopic to (f^H, g^H) such that for each element θ_i there is a unique WH-orbit $\{wx_i\}_{w \in WH}$ corresponding to it and there are no more coincidence points in X^H. By Theorem 4.1.17, X has a W-triangulation and by Lemma 4.1.14, f'^H and g'^H can be extended to W-maps f' and g'.

We proceed by induction on the cardinality of I_H. Suppose that for any H the statement is true for $I_H < k$. Let $I_H = k$. It is easy to see that if $\mathcal{G}$ is an essential basis of (f^H, g^H) over $X^{(H)}$, then for any $(H_i) > (H)$, $\mathcal{G} \cap \bigcup_{(H_i) \leq (K)} W\mathcal{R}_{WK}(f^K, g^K)$ is an essential basis of (f^{H_i}, g^{H_i}) over $X^{(H_i)}$. So we can assume that for any $\theta \in \mathcal{G} \cap \bigcup_{(H)<(K)} W\mathcal{R}_{WK}(f^K, g^K)$, there is a coincidence point orbit $\{wx\}$ in $\bigcup_{(H)<(H_i)} X^{(H_i)}$ that θ corresponds to, and that there are no other coincidence points on $\bigcup_{(H)<(H_i)} X^{(H_i)}$. Suppose that each WH-class contains at most one coincidence point orbit on $X_{(H)}$. Assume that $\{wx\}$ is a coincidence orbit, there are three cases:

(i) x is equivalent to some coincidence point x_1 in $\bigcup_{(H)<(H_i)} X^{(H_i)}$. As we did in Lemma 4.6.8, we can coalesce the two orbits into $\{wx_1\}$.

(ii) $[x]_{WH}$ is essential, but x is not equivalent to any coincidence point in $\bigcup_{(H)<(H_i)} X^{(H_i)}$. This must mean that $[x]_{WH}$ is in $\mathcal{G}$.

(iii) $[x]_{WH}$ is inessential. In this case $[x]_{WH}$ can be removed by Lemma 4.6.9 if it does not correspond to any $\theta \in \mathcal{G}$.

Finally, if there is a $\theta \in \mathcal{G}$ which corresponds no coincidence point orbit, we can create one in X_H as in the case $I_H = 1$ since $X^H - X_H$ can be bypassed in X^H (as indicated in Note 4.6.6).

This proves the statement and hence the theorem. □

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