SYNTHETIC ACTIVITY AND THE CONTINUUM
A CRITICISM OF RUSSELL'S ACCOUNT OF EXTENSIVE MAGNITUDE

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Synthetic Activity and the Continuum

A Criticism of Russell's Account of Extensive Magnitude

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Abstract

This essay deals with the concept of continuity, as it has been developed both in philosophy and in mathematics during the 19^{th} and 20^{th} Centuries. In particular, the problem of focus is the relation which the continuum has to number. The debate on whether the continuum can be given as a class of atomic individuals is the principle item of consideration in this work. The negation of this claim is argued for.

The two sides of this debate are presented in terms of the philosophical characterizations of extensive magnitude found in the writings of Bertrand Russell (representing the claim that such a reduction is possible) and of Immanuel Kant (representing the negation of this claim). In particular, the epistemological distinction between the two figures is connected with their relative positions on this debate, discussed mainly in connexion with the issue of synthetic *a priori* judgments.

The principal claim argued for in this paper is that the classical analysis of the geometric continuum, and hence Russell's logical reduction of space and time, tacitly presupposes an original undifferentiated continuum among its initial principles. This point is intended to lend support to the more general view of the continuum holding the undifferentiated whole to be utterly prior over its parts. In addition to Kant, one should attach to this view the names of Peirce and Brouwer. In particular, I shall attempt to establish an understanding of the 'spatial point' as an entity which can be individuated only as the result of a synthetic act. Finally, an examination of the relation of intuitionist choice sequences with the classical set of real numbers is presented, concluding with the conjecture that no law-like system can exhaust all possible positions on the line.

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Symbolic Definitions

$\forall$:= 'for all ...'
$\exists$:='there exists ...'
$\exists!$:= 'there exists an unique ...'
$\Rightarrow$:= 'implies'
$\in$:= membership
$\wedge$:= 'and'
$\vee$:= 'or'

A single italic Latin or Greek letter denotes a variable, unless otherwise specified.

The lowercase Greek letter π is used here with its standard meaning, as the ratio of the length of the circumference of a circle with the length of its diameter.

Introduction

The irreconcilable duality of the formal conditions of continuity and discretion form one of the oldest conceptions of the nature of the geometric line. Brouwer in his early Intuitionism[1] gave great credence to this division, though it seems to be of lesser concern in the later phase of his program.[2] Apart from the Intuitionists (and a few isolated figures such as Peirce), this dualism has largely been abandoned over the last two centuries in favor of an atomic understanding of the continuum based on Cantor's theory of non-denumerable sets. The reduction of the continuum to the 'discrete'[3] by means of atomic analysis is not unique to Russell and Logicism, but rather has been the dominant interpretation of the geometric line for much of the last two centuries. The view that the *continuum* is an infinite class of irreducible individuals, and that the formal condition of *continuity* is a feature of serial order shall be called into question in the Chapters that follow. This atomic conception of the continuum is given the most coherent philosophical characterization in Russell's work, towards which the criticism presented in this discussion will be primarily directed.

A thoroughly comprehensive consideration of the philosophical literature that has been written on this subject would be far too extensive a task to be undertaken in this essay.

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[1]For a full development of this conception of the continuum, refer to Brouwer's Ph. D. thesis, in his *Collected Works, Book I* pp. 11-101.

[2]For an interesting discussion on this issue, consult M. S. P. R. van Atten. *Phenomenology of Choice Sequences*. Quaestiones Infinitae: Publications of the Zeno Institute of Philosophy vol. XXXI. Netherlands: CS Utrecht, 1999. pp. 29-32.

[3]The term 'discrete' is intended, not in the strict ordinal sense, but rather in terms of the structure of a collection of individuals treated as basic, or non-composite.

Rather, in this work, I hope to cast a new light on a very old problem. I have presented the continuum debate in terms of the philosophy of two primary figures, namely Bertrand Russell and Immanuel Kant (who is associated with the claim that the continuum cannot be given as a collection of atomic parts). These figures were selected as representatives of the respective positions on this issue because they indicate the urgent relevance which it has to western epistemology as it has been developed over the twentieth century. In particular, I attempt to show the relevance that this issue bears to the debate on synthetic *a priori* judgements, moulding the distinction between the epistemology of Kant and the followers of Russell. The following essay should not be viewed as a comprehensive defense of the Kantian epistemology. On the contrary, as will be illustrated in Chapter 5, the full acceptance of the anti-atomic account of continuity is contrary to other principles which Kant insisted on, in particular those associated with the treatment of the relation of causation in the first *Critique*. What will be defended here is the general conception of the continuum as irreducible to a collection of atomic constituents. In particular, the departure which Russell takes from this view, as manifest in both the *Principles of Mathematics* and in *Principia Mathematica*, shall be the focal target of the criticism of this essay. In other words, the main task of the following essay is to elucidate the failure of Russell's complete reduction of the continuum to a set of atomic constituents capable of being given prior to the whole of which they form the divisions.

This essay consists of five chapters together with an opening statement on the nature of activity, the significance of which will emerge throughout the essay, particularly in

Chapters 4 and 5. Chapters 1 through 3 are largely explicatory in character, whereas Chapters 4 and 5 will offer a battery of objections to, and finally a rejection of, Russell's position. Chapter 1 will be an exegesis of the existential commitments of mathematical realism, giving consideration in particular to the school of mathematical Logicism (of which Russell is taken as the representative proponent). In Chapter 2, I introduce the ideal concept of continuity in terms of its connexion with the extensive structures of space and time. I then move to a description of the work of Cantor and Dedekind, in order to ground Russell's position in its mathematical origin. In Chapter 3, I will first discuss the relevance which this question has to Russell's epistemology in connexion with his dissent from the Kantian doctrine of synthetic *a priori* judgements. I then explain the details of the method by which Russell reduces the continuum to a set of points. In Chapter 4, I challenge the adequacy of the conception of a spatial point as identifiable with an infinite decimal expansion. I then present the central argument of this thesis, namely that an original undifferentiated continuum must be tacitly presupposed in Russell's reduction of the geometric line to a class of points. Finally, in Chapter 5, I examine the conception of a spatial point as a limit of a Cauchy sequence, and argue that such a definition of geometrical position presupposes conditions which would require the admission of points generated by intuitionist choice sequences, thereby undermining any attempt such as Russell's to capture the full set of divisions of the continuous line in terms of a set of laws. Chapter 5 concludes with an illustration of how such objects as choice sequences are incompatible more generally with Kant's epistemology as well.

As a technical note, the meaning of the term number is largely invariant throughout this essay, and refers to an object that is finitely traceable in its derivation to the inductive or natural numbers with the capacity to hold a (1) combinable (algebraic) and (2) comparable (ordinal) relation with any arbitrary other instance of its kind. For instance, ratios of whole numbers meet this criterion, as do the negative integers. Because of the lexicon of classical mathematics, this notion generalized to some extent in the context of a real number, particularly with regards to the condition of finite traceability to the inductive numbers.

A Preliminary Statement on Activity

Throughout the course of this essay it is presupposed that an instance of intuition should be understood as an *act* carried out by an *agent.* This interpretation of the nature of intuition will be of central importance in the argument to be presented, and hence shall be addressed at the very outset, so as to best clarify the steps taken in the discussion that follows.

Essentially, an *act* is both (1) non-simple and (2) irreducible. An act is non-simple in that it is relational. However, it is irreducible in that any constituent element cannot be given save as a feature comprehensible only in terms of the whole.

Activity is purely relational, in that *act* and *agent* cannot be characterized in isolation. The unity of an act is anterior to the completeness of each pole respectively. The objects to be *related* are in every sense inseparable from each other as well as from the completed act which *relates*. In particular, we shall not be interested in any kind of act other than that which is *synthetic*, or which produces unity among difference.

Of concern to us here is the nature of the agent or subject.[4] A synthetic agent must be understood as the 'first' condition of any act, since every *act* presupposes an *actor*. This agent or subject must abide throughout the entirety of an act - this is the crucial point. An act is irreducible, hence the subject is ineliminable in any instance of activity. The subject in

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[4]It is an open question whether this subject abides continuously, as would be held by Kant and Brouwer, or serially as in the case of Whitehead and Cassirer. The former conception is assumed in this essay on the grounds that it provides the simplest possible account. A full discussion of the philosophical implications of this conjecture would diverge greatly from the current topic, and moreover does not influence the conclusions arrived at. In either case, the subject is understood as a necessary pre-condition for any instance of an act. This is all that is required for present purposes.

all its basic characteristics (including that of freedom) must therefore be taken into consideration in the context of any object which is traceable to an act.

There are two kinds of act which shall be of concern in the discussion that follows. Firstly, there are acts of *intuition*, for which the issue of freedom does not arise. Intuition is a mode of knowledge. In particular, it has the formal property of being unmediated knowledge. The relation of the knowing subject to the object or matter of intuition is immediate, and shall be understood as an act. That which is *given* is by definition inseparable from that which *receives*, and conversely there can be no reception unless there is a matter to be received. In the case of intuitive knowledge, subject and object mutually condition each other. It is the point of view of the author that the appeal to intuition as a type of knowledge, described in this way, although being compatible with Kant, does not necessitate a commitment to his epistemology in the general sense.[5]

Secondly, we shall be concerned with acts of *algebraic extension* which involve the positing of a new element to an algebraic system, thereby changing the system. In this case, we shall be concerned with a system in which (1) it is possible to formulate a non-contradictory denoting complex for which (2) there is no element of the system that satisfies the description. In other words, we shall consider cases of a system S in which there can be assembled a description p such that:

i. the satisfaction of this description (the claim $\exists x : px$) entails no contradiction;

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[5]For instance, Husserl makes use of such a notion of 'intuition,' yet cannot be regarded as properly standing in the Kantian tradition.

ii. there is no constituent *s* in *S* such that the *ps* holds.

It is in this context that the notion of freedom becomes of interest. This relation will be discussed at length in Chapters 4 and 5. However, something must be said at the outset. Freedom is here to be understood in the most basic sense as self-conditioning. An act is said to be free when it is not exhaustively traceable to anything besides the agent by whom it is undertaken. An act of intuition shall not be understood here as a characterizing instance of a free act, because it is not certain that the subject is entirely unconditioned by the *matter* that is intuited. For instance, it would seem that such is the case when I arrive at the knowledge *that this apple before me is red.*[6]

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[6]A position contrary to that indicated here is not impossible, and in fact has been defended by others. For our purposes however, in particular the conclusions established in Chapters 4 and 5, the role which freedom plays in the attachment of sensible properties to experienced physical objects is irrelevant, and shall not be further discussed. For our purposes, the meaning of a free act is not best exemplified in such cases as 'colour predication.' (eg. an apple)

Chapter 1: Mathematical and Logical Realism

1.1 Mathematical Existence

The nature of mathematical existence is perhaps the most fundamental question in the philosophy of mathematics. One major position on this question is mathematical realism, or Platonism, the explication of which is the purpose of the current Chapter.

In the most basic sense, mathematical realism can be characterized as a view which holds the existence of mathematical entities to be independent of any act of cognition. A distinction is made between the intelligibility of mathematical objects and the notion of their having an existential dependence on mind. The structures and entities of mathematics are in their essence intelligible; that is to say, they are thoroughly *for mind*. Yet this is not to imply that the existence of mathematical objects is in some essential way supported by or derivative from the activity of thought.

The realist denies any claim to a generative or productive relation which thought may have to mathematical objects. Cognition, in this picture, has no causal role regarding the existence of structures or configurations of mathematics. The actual instances of mathematical cognition for the realist are in every sense posterior to the independent being of the entity which is cognized. Even if there were no thinking subjects in the universe at all, these mathematical entities would nevertheless be real. For the realist, mathematical entities have objective being and are to be known through discovery rather than cognitive construction, so that in this picture, our knowledge of mathematics consists of a sort of 'intelligible survey'. The realist position maintains a strong distinction between the existence of a mathematical object as a mind-independent entity, and the determinations by which we know and judge propositions about it.

The principles of realism eliminate the possibility of some condition having existential priority over a mathematical object on the grounds that mathematical existence is of a fixed, universal nature, and as such cannot have any antecedent determining conditions. For the realist, the absolute persistence of the determinations of a mathematical object indicates the impossibility of its 'introduction.' In particular, the objects of mathematics could never be in a state of becoming, and hence are not produced, generated, nor brought about by any other prior existent, cognitive or otherwise. In virtue of this, the determinations of mathematical entities are held to be universally anterior to everything else in existence.

The realist will grant that there can be uncertainty concerning propositions regarding mathematical objects and their features. However, if there is some defect which prevents us from knowing the truth status of an undecided mathematical proposition, it must occur on the epistemic side. That is, this defect must be purely our own. The essential feature of realism is the view that mathematical objects are fixed and hence utterly determinate (that is, in principle, decidable) in all of their features. If this is the nature of all mathematical entities then it follows that any sensible proposition which qualifies a mathematical object must be unambiguous; that is, the truth status of any mathematical proposition which makes 'sense' must in principle be either true or false. That is to say, the *Law of the Excluded Middle* (hereafter LEM) holds with absolute generality over all predicates extending over mathematical objects and their collections.

Consider the familiar ratio of circumference of a perfect circle to its diameter, π. Any attempt to subsume this number under the notation of decimal fractions results in a non-terminating, non-repetitive process of approximation. There is no finite decimal representation of the number π, nor is there any law-like[7] pattern which can be identified in its decimal expansion. As far as this number has been calculated to date, the string 123456789 has never occurred in its decimal expansion.[8] For the realist, the existence of the number π is utterly independent from the method by which we calculate it. Rather, the realist would view the method of calculation as conforming to the determinations of a fixed anterior entity. Furthermore, the infinite series of base ten decimal fractions generated in this approximation are both as individuals and as an infinite series completely realized, in a sense significantly independent from the approximative method. That is, the sequence

$$3.14, 3.141, 3.1415, \ldots$$

has a reality which is fully independent of the denoting complex 'ratio of the circumference of an arbitrary circle to its diameter.' The terms, the set that is their totality, together with their serial order, form an eternal, and hence fully realized structure, the determinations as such having priority over their relation to the number π, over any algorithmic method by which they are calculated, and over the mathematical subject cognizing their connexion. For the realist, the question 'Does the sequence 123456789 occur in the base ten decimal

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[7]That is, one that can be described with a finite arrangement of symbols.

[8]An example closely linked to this one can be found in Heyting *Intuitionism: An Introduction.* p. 16.

expansion of π?'[9] must in principle be decidable. On this analysis, there is a real fact in the world, an objectively valid proposition which determines φ as true or false. On this point the distinction between Platonism and one of its prevalent counter positions, mathematical constructivism, is made clear. In the case of verifying the presence of this string in the decimal expansion of π (answering φ affirmatively), the constructivist would insist on the provision of some effectively calculable method which not only showed that the string was included in the expansion, but would do so by indicating the exact place in the series of digits in which it occurs.[10]

Implicit in the realist position is the admission of a fully realized totality of mathematical entities. Since this eternal, mind-independent nature is common to all mathematical objects, they must mutually 'co-exist.' If all mathematical things in their nature stand utterly fixed in eternal independence from minds, then their totality must be equally fixed and independent as well. That is to say, the totality of mathematical objects is itself a well-defined and complete unity, and hence should in principle be able to participate as an argument for some propositional function. We should note here that the acceptance of a real infinity is an ineliminable aspect of the Platonist view. Clearly, there can be no finite number of mathematical objects, since each natural number is itself an individual. Moreover, if these objects are to be eternal and fixed, then they form a real collection which is infinite. For the

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[9]Hereafter referred to as φ.

[10]To be 'effectively calculable' in this sense is exclusively to involve algorithmic (that is finitely executable) procedures. For instance, multiplication and addition are such processes. Therefore, the only 'constructive proof' which answers φ is one that actually calculates some initial segment of the decimal expansion of π, in which 123456789 can be shown to occur.

realist, every mathematical individual is fully determined in all its features, and therefore the same is true for the whole that is the totality of all such individuals. If the realist ontology of mathematical objectivity holds, then there must exist a unique totality of exhaustively realized mathematical entities, in which all *possible* structures exist in utter *completeness*. If this static, eternal nature is the universal characteristic of all mathematical objects, then to admit the reality of any one individual mathematical possibility necessitates the admission of the reality of all others.

1.2 Logicism

The school of mathematical Logicism is fundamentally committed to a realist conception of the existence of mathematical entities. Logicism is primarily associated with the work of Frege, Russell and Whitehead.[11] Kant's first Critique had an enormous influence on the intellectual progress of Europe, and his interpretation of mathematical truth as the canonical instance of synthetic *a priori* judgement quickly became the dominant epistemological picture of mathematical propositions. However, in the late nineteenth century, the role of mathematical truth which for Kant formed the corner stone of his doctrine of synthetic necessity had degenerated into a simple psychologism. That is to say, the necessity of mathematical judgement was considered to be merely a feature of our

[11] Although Whitehead co-authored *Principia Mathematica* (Hereafter PM) with Russell, it is evident from his later work that Whitehead shifted his view of the propositional function in favor of a more constructive conception. Nevertheless his name is included here, as PM forms the most comprehensive formalization of the Logicist project. For a superb account of this transition in Whitehead's philosophy, see Bradley, J. "The Speculative Generalization of the Function: A Key to Whitehead" *Tijdschrift voor Filosofie* 64/2002, pp. 253-71.

psychology, or rather to depend entirely on the arbitrary idiosyncratic determinations of our cognition, thereby stripping the truths of mathematics of any genuine universality which we might claim they may have. Psychologism negates any claim to the ontological significance of mathematical propositions and their necessity. Frege regarded this approach as untenable[12], and initiated the chain of thought which was to lead to the Logicist project forming a sort of 'rebuttal' to the psychologism popular in his time.

Unlike mathematics, logic is thoroughly indispensable to cognition. Any employment of even the simplest act of cognition necessitates the validity of the law of non-contradiction. In analyzing the relation of consciousness to objects in terms of the saturation of a map, Frege seated the nature of cognition in the transcendental function, thereby granting the most absolutely basic ontological status to functional logic.

The relation of concepts to objects was a central feature of both Frege's and Kant's philosophy. Kant's epistemology is principally concerned with *judgement*; that is, the conditions under which we affirm or deny the subsumption of an object under a concept. Frege's approach was similar but he emphasized the linguistic entity which represents the connexion of concept and object with judgement, namely the *proposition*. In his analysis, propositions take the focal place of inquiry and hence posit the elements of functional logic at the most basic level of cognition.

A realism of the basic elements of logic is tacitly implied in Frege's conception of the function as transcendental. For Frege, the function is the highest possible principle in the

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[12] See Frege *Foundations of Arithmetic* on psychologism.

description of reality. Nothing lies beyond it, and the same is true for its correlative lattice, namely logic. The structure of logic is equally fundamental in Frege's analysis of consciousness as is the function. The totality of logical possibilities is complete and therefore fully realized, forming the possibility conditions for consciousness. No act of cognition could produce these elements, since their coherent totality is a necessary condition for thought in the first place. In Frege's picture, consciousness can only take place within the matrix of logical possibility. In other words, Frege held logic to be prior to thought, hence eliminating the possibility of some generative role for cognition in the existence of logical principles. As we shall see, Frege's logical realism passes immediately into mathematical realism as a result of his conception of the relation of logic to mathematical existence.

The transcendental status of the function in Frege's system makes logic absolutely universal over, and therefore prior to, subjectivity. Furthermore, it is difficult directly to criticize this position, as logic must in some sense be presupposed in any counter-argument. Logic is *given*; in contrast, the truths of mathematics are at least not obviously so. Whereas the necessity of mathematics gives rise to a serious epistemological problem, the truths of logic are held to be canonical features of thought, and hence do not stand in the same obvious need of justification.

Frege set out to establish the derivation of mathematical entities from logic. In this way, the necessity of mathematics is seen as an implicit feature of logical necessity: the structures and objects of mathematics therefore have the same transcendental status as do the elements of logic. Hence the objects of mathematics constitute the possibility conditions of thought, and have priority over any act or instance of cognition. In Frege's analysis, the entities of mathematics are elements of logic as are the familiar connectives and quantifiers. Mathematical objects differ from the canonical operators of logic in that the latter form the basis of logic, and hence occupy a position of epistemological firstness whereas the former are derivative, standing in need of elucidation. Apart from this, there is no difference between mathematics and logic for Frege, in that he considered all mathematical objects to be complexes of purely logical constituents. In Frege's epistemology, no extra-logical elements are required in the explanation of mathematical necessity.[13]

A crucial step in establishing the relation of derivation between mathematics and logic was to define number purely in terms of logical principles. Before Frege, there was no satisfactory philosophical definition of *number* that did not itself refer to some principle of magnitude, quantity or other numbers. Frege found a method to avoid such circularity in Cantor's theory of set cardinality, thereby establishing a definition of natural number in terms of a second order predicate.[14] In this way, number is defined strictly in terms of logical principles. For instance, a class Σ has 'two elements' when the following proposition obtains:

$$(\exists\, x \in \Sigma) \wedge (\exists\, y \in \Sigma) : (x \neq y) \wedge [\forall\, z \in \Sigma\ (z = x) \vee (z = y)]$$

[13]The hostility held towards the Kantian principle of synthetic necessity, characteristic of analytic philosophy, gains its first footing in the apparently marginal yet critical deviation from Kantian epistemology which Frege takes here. This will be discussed in detail in Chapter 3.

[14]First order propositional functions take simples or objects as their arguments (e.g, 'x is red'). Second order propositions, on the other hand receive first order functions as their arguments. For instance 'there exists x such that Γx', where 'Γ' is the variable to be saturated with a first order predicate (for instance, Γ = 'red'). This definition can be easily generalized to any predicate type of order greater than or equal to two, in that a number of objects of type *n* is a predicate of order *n* + 1.

Where the '=' symbol is not intended in the ordinary arithmetical sense, in terms of the comparison of magnitudes, but rather refers to the simple relation of identity which is admissible as a logical constant. 'Number' here is a feature of collections or sets, and is in this sense defined without circularity, as no element of magnitude is required in Frege's definition. The number 'three' can be defined similarly:

$$(\exists\, w \in \Sigma) \wedge (\exists\, x \in \Sigma) \wedge (\exists\, y \in \Sigma) : (w \neq x) \wedge (w \neq y) \wedge (x \neq y) \wedge [\forall\, z \in \Sigma \;\; (z = w) \vee (z = x) \vee (z = y)]$$

Clearly, all of the natural numbers can be defined in this way. Taking over this conception of number, Russell joined Frege in pioneering an attempt to display the full subsumption of mathematics under logical principles. Frege and Russell's version of mathematical realism is best understood as an intersection of two claims: (1) that of *logical realism*, in which the objects and structures of logic are held to be fixed, real entities; and (2) that of *logical reductionism* with respect to the nature of mathematical entities, namely that all mathematical entities are *essentially* configurations of logical elements. Mathematical realism is a necessary outcome of the conjunction of (1) and (2), though not all types of mathematical realism necessarily imply such a logical reductionism.

1.3 Problems of Logicism

The theory of mathematical foundations developed by Frege and Russell was the first of its kind, and has been criticized in the philosophical literature more than any other school. The principal target of these criticisms was the tendency which any Platonist program has to the formulation of set theoretical paradoxes. There are several such important paradoxes of set theory, all of which call into question the adequacy of a realist ontology of mathematical objects. Although, any satisfactory treatment of the paradoxes discovered at the end of the 19th and beginning of the 20th centuries would be impossible to complete in this discussion, it would be advantageous to include an example of such a paradox, in particular that discovered in Frege's system by Russell himself, in virtue of its immediate clarity, and relevance to the subject at hand.

Russell's Paradox is as follows: if y is a class let $y \in x \Leftrightarrow y \notin y$. Then, $x \notin x \Leftrightarrow x \in x$. In prose, one might say 'Let x be the class of all classes which do not contain themselves as members.' In this case, each of the two propositions 'x is a member of itself' and 'x is not a member of itself' implies the other. We should note here that if the law of excluded middle is accepted, then any such paradox (a statement p for which $p \Leftrightarrow -p$ holds) gives way immediately to a contradiction ($p \wedge -p$). In virtue of his Platonistic commitments, Russell accepts this law and consequently never uses the term paradox, but rather refers to such constructions as contradictions.

Russell developed a theory of types in order to eliminate propositions with unbounded quantification. He regarded these set theoretical paradoxes as stemming from an error in the construction of sets, one which breaches the conceptual order that individuals have over the collections which include them. No collection should involve a member which can only be defined in terms of the whole. Russell's theory of types is essentially a hierarchical division of classes into categories, each of which can be associated with a natural number. A class C composed of individuals of type n is itself an individual, and is considered to be of type $n+1$.

Genuine sets, for Russell, can only be quantified over a single type. Hence the question of whether a set belongs to itself or not is eliminated at the start.

The inadequacy of Russell's response to the problem of paradox, in the discovery of which he ironically played a pivotal role, is that it prevents such paradoxical formulations by way of censoring the language of set construction. That is, Russell's theory of types is essentially methodological in its prescription, whereas the problem is ontological in its nature. If true Platonism is to be upheld, one must grant the unqualified co-presence of all mathematical objects with each other. For instance, one natural number cannot be considered in any *active* sense the ancestor of another (as would be held to be the case by Whitehead and perhaps Wittgenstein), as the lattice that they form is a fixed and completely realized structure. For the Platonist, the relation of ancestry among the naturals is to be explained strictly in terms of asymmetrical transitive relations (namely serial order). In the conception of class and membership adopted by Frege and Russell, namely that of Cantor's set theory, such order relations presuppose the completeness of both domain and co-domain as a condition of their possibility.[15]

In particular, the Platonist position holds a set as co-present with its members in the ontological sense, despite the fact that such a proposition can be formally eliminated with conventions of symbolic nomenclature. Russell addresses the problem in terms of such formal restrictions with his theory of types. Moreover, there must be a set of all sets which is itself a completely realized collection, in which there exists a well defined sub-collection of all sets

[15]See further Chapter 2 on relations and cross products in Cantor's set theory.

which do not admit the peculiar property of self-membership. Regardless of the artificial technical bounds with which Russell attempts to confine and eliminate the problem, if mathematical realism is to be upheld, then the collection of all sets as well as its sub-collection of all sets which do not admit themselves as members must in principle be complete and conceptually valid entities.

This interpretation of the nature of mathematical Platonism, namely that it is doomed to admit contradictory entities, if not semantically then at least *ontologically,* has led many to seek a theory of foundations which makes no commitment to such realism as in the case of Russell and Frege. Much of the criticism of the Logicist school has been motivated by this problem. Nevertheless, the hope of resolving such paradoxes with technical modifications such as the theory of types has inspired many to maintain their Platonist view.

In this essay, I shall criticize Russell's Logicism on different grounds, namely on his analysis of the geometric continuum, thereby showing a second and hence decisive failure of this school to explain the nature of mathematics. Therein, we shall find an error which is not specific to Russell and the analytic philosophers, but rather has its roots in the mathematical work of the nineteenth century, especially in figures such as Cantor and Dedekind. In addition to the problem of paradox, I hope that the criticism of Russell's characterization of the geometric continuum which I shall present in the following four Chapters will establish new, and more intuitively immediate grounds for holding the fundamental tenets of the Logicist project in question.

Chapter 2: The Continuum

2.1 The Continuity of Space

For our purposes, we might think of the 'continuum' as being an abstraction from a simple line that connects two points. The science of geometry is fundamentally grounded upon the notion of continuity. The ideal entity which mathematicians refer to as the *continuum* is wholly traceable to this geometric heritage. That is to say, the continuum is an abstraction or idealization of the formal nature of space. This property of continuity is however not peculiar to space alone but rather is an essential common ground between space and time.[16] One might think of the continuum as a conceptual *general*, an abstraction under which both space and time fall as instances. In this sense, one could claim that the ordinary translational motion of physical objects could only subsist given the *formal homogeneity*[17] of

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[16]We should note here that this common nature is presupposed in the use of Cartesian coordinate systems in physics.

[17] Both are homogeneous, in that both must be part-wise (in particular, point-wise) continuous, in a 'topological sense,' i.e., in that there is such a likeness of form between them that a complete, or 'form preserving' map from one to the other is possible. Such a map (Ψ) would be one in which any formal property (such as a relation between constituents of the domain) would be preserved under mapping. In more concrete terms, we could consider, for an arbitrary formal relation $\Diamond$ among constituents of the domain that (i) there is a corresponding relation, $\square$, in the co-domain, such that (ii) whenever $x\Diamond y$, $\Psi x \square . \Psi y$. One example of such a form preserving map would be that of an algebraic homomorphism. For our purposes, the idea of the formal homogeneity among space and time could be summarized with this notion of a form preserving map in conjunction with the idea of a *neighbourhood* such as that in the theory of Hausdorff spaces (about which see Alexandrof, p. 9) For our purposes let us refer to a neighborhood as a part of space which can (1) *contain* multiple points (a property denoted *functionally*, or by juxtaposition, with the neighborhood 'multiplying' on the left), and (2) can *include* other neighbourhoods, both *totally* (in which no points of the included neighbourhood are not contained in the including neighbourhood) and *partially* (the case in which two neighbourhoods both share also points, but also at least one of them contains points which are not contained in the respective other). In the sense of Hausdorff, to each point in space x there corresponds at least one neighbourhood Π such that Πx. If both Πx and ξx hold, then there corresponds a neighbourhood Δ which is totally included in each of Π and ξ, such that Δx. If Πx holds, and a point y is contained in Π then there corresponds a neighbourhood Λ included in Π such that Λy. Finally, if x and y are two distinct points, then each is contained in one of two disjoint

space with time, insofar as they are both continuous.[18]

In virtue of this primary association with extensive magnitude, I shall begin this chapter with the remarks of Immanuel Kant on the continuum. The position which shall be defended in this thesis shares a crucial similarity with Kant, on the question of whether the continuous line can be exhaustively reduced to a set of points. The position defended here therefore, stands in strong contrast to Russell. However, as we shall see in Chapter 5, from the constructivist point of view, Kant's own position is contradictory on this very point.[19] Given that Kant lived and wrote in a time well before both the 'classical' analysis of Dedekind and Cantor, as well as that of the intuitionist mathematics of the 20[th] century had been developed, we shall not make this inconsistency in Kant a focal point of discussion. However, this example serves to elucidate the selectivity with which the author wishes the audience to view his adherence to the Kantian epistemology. The main focus of criticism here is the

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neighbourhoods: i.e., there correspond Π and ξ, such that Πx, ξy, and $\forall z$ -($\Pi z \wedge \xi z$). It is important to note that this notion of homogeneity should be understood as defined in full absence of any reference to the metric of space: what is claimed here is not that space is essentially Euclidean, nor any other result which would make reference to a theory of distance. Hence, this definition of 'neighbourhood' does not imply that we ought to think of space as Euclidean, as it is formed in such a way that neither length nor angle are determined. In topology, a function $f: A \rightarrow B$ is said to be continuous when for every $x \in A$ and every neighbourhood N of fx in B, $f^{1}N$ is a neighbourhood of x. That is, the set of points in A, the image of which lie in N in B, form a neighbourhood of x in A. See *Basic Topology*. p. 13.

[18]One interpretation of this relation is that the continuity of space is grasped by consciousness through time- it is by *following* the line (imaginatively or aesthetically) that we are shown its continuity. In the imagination of the geometer, space is held up and checked against time, its yard stick. Time is given, and our conception of its formal nature (i.e., its continuity) cannot be explained without reference to intuition or unmediated knowledge. The continuity of space on the other hand is something which is understood in its comparison with time. It is in 'following' the geometric line that we apprehend its continuity.

[19]See Brouwer *Collected Works*. In particular, Brouwer's Ph. D. thesis, vol. 1.

reduction of the continuity of space as endorsed by Russell. In Russell's own writings, he clarifies his position by explicitly indicating its opposition to Kant. The position which is argued for here is, in the main, negative, in that it holds such a decomposition as that allegedly undertaken by Russell in *Principia Mathematica* as fundamentally impossible. In the sense that Russell viewed his own work on the subject as 'diametrically opposed to that of Kant[20],' returning to the philosophical remarks on space in the first *Critique* will provide both valuable insight on the so-called 'anti-atomist side,' as well as illuminating the nature of Russell's view through direct contrast. In the present chapter, we shall begin with a discussion of Kant and move on to the mathematical work of figures such as Cantor and Dedekind, which as will be shown in the following chapter form the basis of Russell's analysis.

Let us recall the role which continuity played in Kant's analysis of extensive magnitude in the *Transcendental Aesthetic*:

> "The property of magnitudes by which no part of them is the smallest possible, that is by which no part is simple, is called their continuity. Space and time are
>
> *quanta continua*, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space or a time."[21]

The experiential modalities of space and time are here included under the single concept of extensive magnitude. Kant's transcendental analysis is in search of possibility conditions; in this case, as space and time are essentially formal in nature, what is sought after is the

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[20] *Principles of Mathematics*. p. 456.

[21] CPR: A169/B211

relational conditions upon which these modalites operate. From the above passage, we can see that Kant is categorizing space and time insofar as they are extensive quantities as essentially continuous. Continuity is characterized by the property of having no smallest part. There are at least two possible interpretations of what is intended here by the use of the term 'smallest.' The first involves an appeal to the metric of space and time respectively. As will be shown, this approach is the standard interpretation of continuity in classical mathematics, best exemplified in the work of Cantor and Dedekind. The second interpretation makes no reference to anything outside the abstract relation of whole and part.

The term 'smallest' in the above passage indicates that the relation of 'more and less' is of pivotal consideration in the deduction of the formal nature of extensive quantity. In the ordinary sense, the terms 'more' and 'less' are always employed within a pre-established lattice of quantity, in the context of which the relative terms obtain their meaning. In the case of the *Transcendental Aesthetic* however, such a narrow interpretation of this relation is insufficient. As we are discussing the possibility conditions of extensive magnitude in general, any appeal to the metric of space and time must be omitted. The formal condition of extension must be conceptually prior to extensive magnitude. The appeal to geometric intuition, in which 'smallness' is understood as defined within the metric of space (or time) is an insufficient method of characterizing the property of continuity. Reflecting the same sort of objection which logicians raise to self-referential constructions such as the set of all sets, this appeal to a geometric interpretation of 'smallness' similarly breaches the order of conceptual priority. It is a universal feature of all definitions that the *definiendum* cannot participate as a constituent term in its own definition. The appeal to geometric imagination in the definition of continuity commits such a breach of conceptual order, by insufficiently differentiating *definiendum* from *definiens*. To say that there is no smallest part of space with no reference to anything other than spatial size results in a vicious circle, for any appeal to the metric of space presupposes its coherent unity, thereby violating the conceptual priority which the formal condition of continuity has over any such unity. Such an interpretation of Kant's definition is therefore untenable - we must seek an alternative.

As mentioned above, the relative terms 'more' and 'less' ordinarily occur within the context of a coherent lattice of quantity. We are forbidden any use of such terms in the analysis of extensive magnitude in that we are investigating the conditions which make such quantity possible. 'More' and 'less' form an order relation, in that they are both *asymmetric* (in that we may distinguish one as 'first' over the other by means of the category 'greater') and *transitive* (if x and y are a 'more' and a 'less' respectively, and further if y and z are a 'more' and a 'less' respectively then it is always true that x and z form a respective pair of a 'more' and a 'less').[22]

The notion of continuity which Kant raises is clearly posterior to the more general relation of whole and part. If we consider the canonical whole-part relation of subsumption (alternatively, inclusion or 'containment') we find a relation which is both asymmetric and transitive. That is to say, subsumption is also an order relation, and is therefore formally homogeneous with the relation of 'more and less.' A better interpretation of Kant's intention

[22]See Appendix.

when he uses the term 'smallest' in the above citation is that the continuum is held to contain no parts which do not contain another part.

Let us define the following terms:

C	:=	continuum (total unity)
$x, y, z \ldots$	:=	parts of the continuum ('intervals')
$\lhd$	:=	the relation of subsumption: if x is subsumed in y we write $x \lhd y$

Note, it holds in every case that $x \lhd C$

Kant's claim can be translated verbatim as:

$$-\exists\, x \lhd C : \forall\, y \lhd C :-[y \lhd x]$$

Which when expressed positively states:

$$\forall\, x \lhd C\, \exists\, y \lhd \mathrm{C} : y \lhd x$$

Upon the second interpretation of the meaning of 'smallest' in the above passage, we might say therefore that the continuum, insofar as it is a whole-part structure, is distinctly non-atomic in that one of its essential features is to have no ultimate or irreducible parts. Further, this interpretation seems to reflect Kant's intentions which are tacitly evident from the use of the terms 'part' and 'simplicity.' The continuum has no part which is simple; therefore the continuum has no atoms or irreducible parts.

2.2 Cantor: The Continuum as a Class

The current mathematical conception of continuity is largely inherited from Cantor's theory of classes. Much of the set-theoretical language applied in the modern analysis of the continuum is traceable to his work. Underlying Cantor's development of logic and set theory is an ontological commitment regarding the nature of the whole-part relation. Namely, his methods suggest that he held the set-element relation as the canonical instance of whole and part. For Cantor, the relation of whole to part, insofar as it has logical import, is to be exhaustively explained with the apparatus of class and membership. Wholes relate to their parts in terms of some structure. Every part of a whole must constitute an individual. If one takes, as Cantor does, a realist view of the logical categories of whole and part, namely that the parts comprising a whole must be real and therefore fully *determined*, then the whole must in principle be comprehensible as a fixed totality of completed parts. The whole-part relation is thereby reduced to that of collection and membership - for Cantor, these relations are one and the same. The implications of this conjecture will come into greater relevance in Chapter 4; for now we shall refer to it as the Cantorian Dogma.

Cantor was the first to make the explicit connexion between relation and number. In Cantor's analysis, all mathematical objects are reducible to sets; in particular, his notion of the function[23] was to be explained in these terms, grounding Russell's extensional analysis of the propositional function.[24] We shall now briefly explain Cantor's development of relation, as it is of fundamental importance in the epistemology of Russell and Frege.

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[23]The idea of 'Set Product,' from which the set-theoretical definitions of both relation and function follow, is not derived in Cantor's theory, but is rather held as basic. In this sense, Cantor's theory can offer no explanation of the 'dimension of space.'

[24]For Russell the function is exhaustively explained in terms of its domain and co-domain, deliberately omitting any reference to any active principle which would explain the relation which maps argument to value. In using the term 'extensional,' we mean to indicate this view.

The cross product of two sets A and B (denoted by $A \times B$)[25] is a third set, the elements of which are ordered pairs. The cross product of A and B is the set of all possible pairs in which the first coordinate is an element of A and the second an element of B. For instance, if $A = \{1, 2, 3\}$ and $B = \{5, 7, 8\}$ then $A \times B = \{(1,5), (1,7), (1,8), (2,5), (2,7), (2,8), (3,5), (3,7), (3,8)\}$. In this context, Cantor defined a relation from a class A to a class B as any non-empty subset of $A \times B$. In this example, the set $R = \{(1,5), (1,7), (2,8), (3,8)\}$ would be such a subset of $A \times B$ and therefore constitutes a relation from A to B. The concept of relation is therefore for Cantor reducible to the concept of class.

Cantor defined the function as a relation F in which

$$\forall a \in A \; \exists! \, b \in B : (a, b) \in F$$

In other words, a relation satisfies Cantor's definition of the function when every element of A occurs in exactly one pair in F. That is, every element of A (1) is 'mapped' to an element of B, and (2) is mapped unambiguously; if (a, x) and (a, y) are both elements of F, then $x = y$. In this way the notion of relation as well as of the function collapse into that of set and subset. Furthermore, Cantor defined a bijective function as one that satisfies the following two properties:

(1) $Fx = Fy \Leftrightarrow x = y$[26] (If this is true then we call F an *injective* function)[27]

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[25]For a thorough discussion of the cross product and other related set theoretical operations, see Lin. [1985] pp. 33-88.

[26]Note, the left direction of this bi-conditional is already guaranteed in the definition of the function, so the above could equivalently read $Fx = Fy \rightarrow x = y$.

[27]For the purpose of expediency, we shall adopt the standard notation in which $Fx = y$ has the same meaning as $(x, y) \in F$.

(2) $\forall\, b \in B\; \exists\, a \in A : Fa = b$ (If this is true then we call F a *surjective* function)

Property (1) ensures that no two elements of the domain map to the same element of the co-domain. Property (2) ensures that no element of the co-domain lies outside the range of the function; that is, no element of B is not mapped to by some element of A. It is easy to deduce that in the finite case no two sets could be bijectively mapped to one another unless they are equal in number. If A is larger than B then there aren't enough elements in B to satisfy property (1). Similarly, if B is larger than A, then there aren't enough elements in A to exhaust the co-domain (since the definition of the function prevents any argument from mapping to more than one element), thereby breaching condition (2).

Let us consider the sets A and B from our earlier example, and define a map F from A to B, in which $F1 = 5$, $F2 = 7$ and $F3 = 8$. Clearly, this is a bijection; notice that both A and B have three elements. If B had been $\{5, 7\}$, then no such map could obtain; since every argument must be mapped to something in B, there would be at least one element of B to which two elements of A would be mapped. Further, if $B = \{5, 7, 8, 9\}$ then there would be at least one element of B which remained unmapped to (since each argument can map to at most one value). From this example it is clear in the finite case that a bijection can only subsist between two sets of equal number. Reversing the interpretation of this relation of functional possibility to magnitude, Cantor considered this the condition of number itself. Two sets are said to be equipotent if there exists a bijective map between them. Each set is associated with a determinate cardinal condition such that two sets between which there subsists a bijective function are said to have the same cardinality. In this analysis, the natural numbers are the names of the possible cardinal types of finite sets. For instance, if we consider our original example, the cardinal number of A is three, as is that of B. Hence it is possible to construct a bijective map. In this sense, number is used for the first time to denote a relation 'type' or capacity rather than simple magnitude. Two sets have the same cardinal number if and only if they can be related bijectively to one another. Number therefore takes on the role of a 'class of classes,' a notion which Russell explicitly takes up in PM.[28]

Cantor extended this intuition to the nature of infinite sets, attempting to give coherence to the notion that there is more than one logically apprehensible condition of infinitude, and further that each of these types can be characterized as determinate mathematical entities. He opened the subject of transfinite mathematics with the question "What is the cardinal status of the set of all natural numbers?" Clearly, this set is infinite, hence the answer to this question cannot be surmised by simple survey, as in the case of our example above. The naturals are infinite, yet they are ordered discretely. That is, for each natural number n there exists a unique successor $n + 1$ such that (relative to the standard ordinal relation of 'less than') it can be said of no natural number m that $n < m < n + 1$. The integers can be bijectively mapped to the naturals as follows:

$$\begin{array}{ccccccccccc} 1 & , & 2 & , & 3 & , & 4 & , & 5 & , & \ldots \\ | & & | & & | & & | & & | & & \\ 0 & , & -1 & , & 1 & , & -2 & , & 2 & , & \ldots \end{array}$$

Any set which might be listed in this fashion, such that there is a definite first element, and each element has a unique successor, can be bijectively mapped to the natural numbers by the

[28]PM: vol. II, part III, p. 4. (summary of section A)

method suggested in the above diagram. A set with this property is said to be countably or denumerably infinite.

The rational numbers on the other hand exhibit quite a different behavior than this under the same numerical relation of 'less than' - for in the case of whole number fractions, any two, no matter how 'close' contain an infinity of rational numbers between them. This can be easily verified in the context of iterating the process of calculating an arithmetical average: if x and y are rational, then there exist integers a, b, c and d (both b and d non-zero) such that $x = a/b$ and $y = c/d$. Their arithmetical average

$$z = \frac{ad + bc}{2bd} \quad \text{(where } ad + bc \text{ and } 2bd \text{ are whole numbers, } 2bd \neq 0\text{)}$$

is strictly between x and y. A second arithmetical average between x and z obtains a further rational number which is strictly between x and y. This process clearly will not terminate at any finite stage, thereby establishing an infinity of rationals between any two. It would seem as though if there were more than one type of infinite magnitude, then the rationals might be significantly differentiated from the natural numbers, between any two of which, no matter how distant there is always merely a finite number of intermediate elements. This characteristic of the rationals is referred to as their 'compactness.'[29] However, this lack of discretion in the series of rational numbers is strictly a feature of the way in which they are ordered. The rationals do not differ from the naturals in their magnitude, but rather are equipotent. They are in a sense one and the same 'logical mass' organized by two differing order relations.

[29]Russell, B. *Principles of Mathematics* p193.

Cantor proved this using the following array

$$
\begin{array}{cccccccc}
1/1 & \rightarrow & 1/2 & & 1/3 & \rightarrow & 1/4 & \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
2/1 & & 2/2 & & 2/3 & & 2/4 & \ldots \\
\downarrow & \nearrow & & \swarrow & & & & \\
3/1 & & 3/2 & & 3/3 & & 3/4 & \ldots \\
 & \swarrow & & & & & & \\
4/1 & & 4/2 & & 4/3 & & 4/4 & \ldots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Clearly, this array lists all possible positive fractions (in fact, there is a great deal of repetition, such as in the case of 2/4, 3/6, *et cetera*.) Following the order which is prescribed by the arrows above, skipping repetitions, we obtain an arrangement of the positive rational numbers such that they can be listed denumerably:

1/1, 1/2, 2/1, 3/1, 2/2, 1/3, 1/4, 2/3, 3/2, 4/1, 5/1, 3/3, ...

thereby showing the rationals to have equal cardinal status as that of the natural numbers.

The same could not be said of an interval of real numbers, however. Consider the set of all possible magnitudes between the numbers 0 and 1. Insofar as these entities may be subject to scriptural representation, Cantor understood that each real number can be characterized by an infinite determinate sequence of digits. Clearly, there are infinitely many magnitudes between 0 and 1, since we have established above there are infinitely many rational values between 0 and 1. The cardinal number of the reals therefore must be at least equal to that of the naturals.[30] Cantor offers the following argument:[31] suppose the interval

[30]Let x be the cardinal of a set X, and y of a set Y. We say that $x \leq y$ when there exists a subset Z of Y such that X can be bijectively mapped onto Z. We say $x < y$ (strict inequality) if the above holds, and

(0,1) could be bijectively mapped onto the natural numbers. If this were the case, then these numbers can in principle be written as a non-terminating discrete series of infinite decimal expansion with an initial term. Suppose the list is as follows:

1: $0.a(1,1)a(1,2)a(1,3)....$
2: $0.a(2,1)a(2,2)a(2,3)...$
3: $0.a(3,1)a(3,2)a(3,3)...$
4: $0.a(4,1)a(4,2)a(4,3)...$

and so on, in which $a(n, m)$ is the m^{th} decimal place in the n^{th} term in the series. If we define the number b as $0.b1b2b3b4b5...$ in which the n^{th} decimal of b, namely $bn = 2$ if $a(n, n) = 1$ or $bn = 1$ if $a(n, n) \neq 1$, then we obtain a number which clearly must lie between the numerical values of 0 and 1 and yet differs from each term in our list by at least one term in its decimal expansion. Hence the supposition that the interval (0,1) is a denumerable set is contradictory. We should note here that it is crucial for Cantor's proof that every real number corresponds uniquely to a denumerably infinite and determinate string of symbols selected from a finite list. The cardinal number of the reals is therefore strictly larger than that of the natural numbers. Such an infinite set is said to be 'uncountable' or 'non-denumerable.'

The nature of the continuum, here associated with the set of points on the real line, or of geometrically possible magnitudes, is characterized in Cantor's set theory as a transfinite cardinal number. Further, if one admits that all real numbers correspond to an unique decimal

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the range of such a map is necessarily a proper subset of Y.

[31]Lin: p127.

expansion, then the real numbers in fact are equipotent with the power set of the naturals.[32] Cantor's work was of pivotal importance in the development of the theory of classes, and this conception of continuity as a cardinal class of infinite sets has dominated the mathematical interpretation of this subject since his time. The Cantorian Dogma obtains its fullest expression here in the explanation of the whole-part structure of the continuum in terms of the relation of classes to their members. Continuity is therefore reduced to merely one form of cardinal determination, the second in the non-terminating series of transfinite cardinal numbers, purely derivative of the notions of sets, membership and relations.

2.3 Dedekind: Cutting the Line

Upon the advent of the differential calculus in light of its application to Newton's mechanics and therefore to natural science in general, the subject of continuity dominated the attention of many mathematicians in the 19th Century. As did Cantor, Richard Dedekind emerged as one of the most important theorists regarding this subject at the time. In 1858 Dedekind lectured on differential calculus at the Polytechnic School in Zürich, concluding at the end of the course that the concept of geometric continuity which the differential calculus relied on so fundamentally stood in need of scientific definition. The mathematical conception

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[32] In Cantor's cardinal arithmetic, the number obtained from the exponentiation of one cardinal number over another (i.e., x^y where x is the cardinality of a set X and y of Y) is the cardinality of the set of all possible maps in which Y is the domain and X is the co-domain. If we denote the cardinal number of the naturals with the symbol Γ, and consider the exponentiation of 2^Γ, we have obtained the cardinality of the set of all maps from the natural numbers to the set $\{1,0\}$. Further, if we let '1' denote the property of inclusion and '0' that of exclusion, then the cardinality of 2^Γ is equal to the set of all possible subsets of the naturals (referred to as its 'power set'). As our base ten notation can be converted into binary script, these infinite decimal expansions are in essence maps from the natural numbers to the set $\{0,1\}$.

of continuity as a cardinal type needed to be conjoined with some reference to sensible relations such as 'closeness' in order to apply this structure to instances of spatial modeling. Dedekind saw this appeal to geometric intuition as an "exceedingly useful"[33] technique for the instruction of students taking a first course in the calculus, and that it afforded great expediency in introducing the notion of continuity on which the calculus was grounded.[34] This appeal to geometric intuition in explaining the foundations of the calculus, however, was in his opinion insufficiently scientific given the context of its application. Upon this realization, he formulated a rigorous definition of the "essence of continuity" that is reflected in the standard axiomatic representation of the reals that was to follow.[35]

The negative numbers along with whole number fractions are definable solely in terms of the natural numbers taken along with the fundamental operations of arithmetic. Dedekind thought that a rigorous definition of the real numbers must involve no elements extraneous to the operations and ideas of arithmetic alone. He considered the appeal to geometric intuition to be such an extraneous element in the explanation of the calculus, and hence was to be discarded.

However, the applicability of the differential calculus to physical science relied on the formal homogeneity of the structure of the real numbers with that of the geometric line. The

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[33]Dedekind: p.1

[34]For instance, teachers of the elementary calculus often introduce the notion of a continuous function to their students as one in which the graph can be sketched with a single pen stroke.

[35]It is a standard practice question in most elementary texts on the subject of real analysis to prove the validity of Dedekind's concept of the continuum. A proof of this is included in the Appendix.

validity of Newton's application of the calculus to his physics required such perfect correspondence. The homogeneity of the real numbers with the geometric line hence had to be preserved; Dedekind sought not to eliminate their connexion, but rather to dissolve any relation of dependence or derivation which spatial intuition may have over the structure of real numbers, insisting that their formal nature be explained exhaustively in terms of arithmetical notions. This correspondence was obtained by an appeal to the order relation of 'less' and 'greater' on the reals, since this relation is understood as isomorphic in its behavior with that of the spatial relation of 'left' and 'right.' In particular, every point on the geometric line divides the line into two distinct segments, of which it can be said that one lies strictly to the left of the other. Further, the rational numbers, which are immediately derivative from the canons of arithmetic, standing in no such need of arithmetical cleansing, obey the same property; every rational number divides the set of rational numbers into two mutually exclusive, all inclusive subsets, in which each element of one set is 'less than' every element of the other. Dedekind referred to this kind of set division as a 'cut' [*Schnitt*], in virtue of its natural spatial analogy.

There are such cuts on the series of rational numbers, which cannot be defined by a ratio of whole numbers representing the 'place' of the cut. A standard example of this would be the root of a prime number. Such quantities obtain in instances of geometrical measurement such as the comparison of the diagonal of a square with its side. The incommensurability of the side of a square with its diagonal was known in classical geometry, violating the Pythagorean claim that all quantitative relations instantiate a ratio of whole numbers. Natural considerations of relations in space do not necessarily conform to such a ratio; furthermore, transcendental numbers such as π cannot be associated with any finite algebraic expression of which they are a solution (such as $\sqrt{2}$, in the case of the diagonal of the unit square.)[36] Space and time are quantitative entities, and all quantity in principle should be comparable to quantities of the same type. For instance, part of what is essential to any spatial length is that it is comparable to any other spatial length. Given any two distinguishable spatial lengths, they must as a pair admit of a more and of a less. A simple application of the law of identity would insist that given any spatially fixed denoting complex X (the object of which is not in flux), a unique quantity may be associated with X. For the purpose of expediency, let us call this quantity the 'length of X,' symbolized with $l(X)$. Surely, no one would deny in our example that for the fixed circle, both the length of its diameter $l(D)$ and of its circumference $l(C)$ remain themselves fixed. As all pairs of distinct homogeneous quantities admit of a more and a less, so must each pair admit of a 'fixed difference' and therefore the notion of their proportion

$$l(C) : l(D)$$

is unambiguous.[37] In the sphere of denoting complexes, nothing more needs be said here,

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[36]The division of the irrational numbers into two classes, namely the algebraic and the transcendental is of far greater significance in an anti-realist philosophy of mathematics than in the classical view. The former are posited solutions to finite objects (algebraic equations of one variable) whereas the latter usually obtain in finitude only in the context of some act of intuition which connects them with an extra-logical denoting complex. For instance, π can be associated with the finite expression 'ratio of the circumference of a circle with its diameter.' However, this magnitude cannot be identified analytically (i.e., in strictly algebraic terms) without the appeal to some infinite structure (such as a convergent infinite series).

[37] Two fixed homogeneous quantities admit of a definite comparison - this is essential to the concept of quantity. For instance, the symbol '½' is no more than the name of the rational (proportional)

since the phrase p : 'the ratio of the circumference of a circle with its diameter' obtains unique extension. This is ensured by the formal nature of spatial magnitude and the static nature of the intuitions presupposed. Hence, if we restrict our attention to that of prose descriptions, there is no problem with the quantity π. It is the introduction of this quantity into 'numberhood' which posits a difficulty, since it is not only essential to numbers that they be comparable but also that they are *combinable* (in at least the context of the fundamental operations of arithmetic). As the comparison of quantities is possible only if certain requirements of modal homogeneity are satisfied[38], the combination of quantities too requires a kind of conceptual homogeneity among the combinative constituents. The standard approach to this problem is to associate new numbers with definitions, the constituents of which are exhaustively expressible in terms of their antecedents. For instance, -2 is associated with the equation $x + 2 = 0$, ½ with $2x = 1$, and similarly $\sqrt{2}$ with $x^2 - 2 = 0$. Each of -2, ½ and $\sqrt{2}$ are names of entities which are exhaustively traceable as solutions to their associated equations. These equations *define* -2, ½ and $\sqrt{2}$ respectively, and involve no terms extraneous to the natural numbers (inclusive here of 0) along with the fundamental operations of arithmetic. The reconciliation of the unambiguous extension of denoting complexes referring to spatial and temporal intuitions such as (1) the ratio of the side of a square with its diagonal and (2) the ratio of the circumference of the circle with its diameter would be expediently

relation of the whole numbers 1 and 2. The extension of this name is unambiguous because 1 and 2 are fixed homogeneous quantities.

[38]For instance, it is senseless to say that an 'hour' is longer or shorter than a 'metre.'

addressed in the case where every such denoting phrase could be identified with a finite arithmetical expression. In the case of (1), a simple application of the Pythagorean theorem obtains such an expression: if the side of this square is of unit length, then the length (d) of the diagonal must satisfy the following equation:

$$1^2 + 1^2 = d^2 = 2 \Rightarrow d = \sqrt{2}$$

There is no such equation however which may be associated with (2). There are many quantities such as π that obtain from unambiguous denoting complexes that cannot be identified in terms of a finite algebraic object. The most desirable solution to subsumption of such quantities under the concept of 'number' is to form a definition of the real numbers which provides definite numerical status to the arbitrary quantity, thus solving the problem not only for π, but for any such quantity given in the most general sense. Dedekind was aware of this, and offered a definition of continuity which accommodated this principle:

> "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, thus severing of the straight line into two portions."[39]

This definition obtains the desired result by making number conceptually posterior to the division of the straight line, reminiscent of Kant's fundamental association of continuity with the formal character of the aesthetic. Each real number is the name of a determinate cut on the line, thus ensuring that a numerically coherent value obtains in each instance of a denoting complex which refers to a fixed extensive magnitude. Addition can be identified here

[39]Dedekind: 11

with extensive concatenation; multiplication, subtraction and division likewise follow. In particular, what this definition ensures is that given any (arbitrary) denoting complex which refers unambiguously to an unique extensive quantity, there must exist a unique real number (point) on the line which marks the associated 'cut.' In other words, we are ensured from the beginning that any *possible* extensive quantity is *actual* in that there exists a determinate (unique) concept corresponding to the quantity in question. Moreover, due to the identification of number here with the notion of 'point,' this determinate concept is thoroughly homogeneous, and therefore combinable with all other numbers, natural or otherwise.

Underlying this analysis of the real number system is a conception of the continuum as a series of points, in which the individual real numbers are the ultimate constituents. This theory is thoroughly compatible with Cantor's dogma; we might only surmise from Dedekind's definition that the analysis which was to follow the work of these men confined the mathematical conception of the continuum as an atomic whole-part structure. In other words, the analysis of the continuum in terms of whole and part on Dedekind's conception would terminate at the level of the individual real numbers (points), which alone form the ultimate irreducible constituents of the line.

What has been developed by Cantor, Dedekind, and most mathematicians since them is an atomic theory of the continuum. If the points on the line are the sole and ultimate constituents of the geometric line, then what results is a classification of the continuum which is fully contrary to that which is offered by Kant, cited at the beginning of this Chapter. No theory of the continuum can uphold Cantor's dogma without resulting in such an atomism.

In this context, we can clearly see that there is a major divergence in western thought concerning the nature of the continuum and its divisions. The first position presented, namely that of Immanuel Kant has been taken up by a number of other figures, C. S. Peirce being perhaps the most explicitly outspoken about the non-atomic nature of continuity.[40] For the sake of expediency we shall refer to this position as anti-atomism. Although this view has been introduced in terms of Kant's analysis, it cannot be said to have its origins here. Traces of it can be found among the philosophy of the classical period. In the *Philebus,* Plato writes:

> "...for whenever *the terms 'exceedingly' and 'gently'* occur they do not allow of the existence of quantity - they are always *introducing degrees into actions*, instituting a comparison of a more or a less excessive or a more or a less gentle, and at each creation of more and less, quantity disappears. For, as I was just now saying, if quantity and measure did not disappear, but were allowed to intrude on the sphere of more and less and the other comparatives, these last would be driven out of their own domain. When definite quantity is once admitted, there can be no longer a "hotter" or a "colder" (for these are always progressing and are never in one stay); but definite quantity is at rest, and has ceased to progress. Which proves that comparatives, such as the hotter and colder, are to be ranked in the class of the infinite."[41] (My italics)

Here, Socrates is pursuing his four-fold division of reality (the limitless, the limited, their mixture and the cause of their mixture) in the context of comparative relations, such as that of 'more and less.' Admittedly, the relevance of this passage to our current topic is not immediately obvious. However, if one considers the remarks of Kant immediately following the passage which was cited in the beginning of this chapter, one can see a correlation

[40]See C. S. Peirce *Collected Papers* 6:164-76. We should note, contrary to what is being claimed in this essay, that Peirce regarded both Aristotle's and Kant's conceptions of continuity as insufficient and likened his conception more to that of Cantor. This interpretation differs from the one that is presented here, and hence requires mention.

[41]*Philebus* [24]: The first italic selection replaces the single term 'they' in the translation consulted.

between their respective conjectures:

> "Points and instants are only limits, that is, mere positions which limit space and time. But positions always *presuppose the intuitions which they limit or are intended to limit*; and out of mere positions, viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed"[42] (My italics)

The connexion of these two passages displays an essential characteristic of the kind of anti-atomism which we associate with Kant, namely that points in space as well as instants in time are derivative rather than basic, and moreover that they are derived indexically. That is, an instant in time is inseparable from the concept of the present, and the present indexical 'now' is essentially and dependently related to some object of intuition. In the cases of both Plato and Kant, an *act of intuition* is held as the necessary condition for the existence of such an individuated element of time or space. Furthermore, as Kant indicates, the unities of space and time cannot be constructed or ultimately constituted by some multiplicity of atomic individuals, but rather these instants and points are themselves derivative from their respective wholes. These atoms are cognized through acts of intuition. If a stone is thrown up into in the air, it will rise and fall forming a parabolic path, at the end of which it will come to rest upon colliding with the ground. In the discussion of such events, we might speak of both the moment and the point at which the pebble first makes contact with the earth.[43] For the atomic Realist, there exists a fixed point which abides eternally, providing a pre-established

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[42]CPR [A169-70/B211]

[43]For instance, Isaac Newton might wonder *exactly* how fast is the pebble moving when it is at this point? We should not forget that there are pragmatic motivations for assembling such an atomic account of the continuum as in the classical case.

discrete receptacle in which the event can occur. In other words, the point abides, being exactly itself and nothing more regardless of whether such an event occurs. For the anti-atomist, space includes no such determinate point prior to the intuition in question, but rather the point is essentially correlative to the intuition itself.

Cantor's atomism forms the basic model of Russell's understanding of the continuity of space and time. Having in this Chapter contrasted Kant's anti-atomism with the theories of the continuum which have been established by Cantor and Dedekind, we can now move to a discussion of how this difference has been taken up by Russell and his followers, both in terms of the epistemological status of the real numbers as well as in the explanation of time. In the next Chapter, we shall examine the role which the continuum plays in the larger epistemological differences between Kant and Russell.[44]

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[44]I.e., Russell's basic viewpoint on the continuum at the time of writing both *The Principles of Mathematics*, and *Principia Mathematica.*

Chapter 3: Logical Reductionism and the Dissolution of Kant's Theory of Synthetic *A Priori* Judgements

3.1 Russell on *A Priori Judgements*

In the previous Chapter, two contrary accounts of the continuum were presented. The first, that of anti-atomism, has been associated here primarily with Kant. The second, namely atomism, has its origins in the mathematics of Cantor and Dedekind. As indicated earlier, Russell subscribed to the latter, drawing heavily on Cantor's set theoretical principles in his own analysis. This distinction is not arbitrary - we should not suspect that Russell accepted Cantor's atomic account on the grounds that it was the standard understanding of the continuum in classical mathematics. Rather, Russell's adherence to an atomic account of continuity is rooted in a much larger philosophical conviction, one that is best understood in contrast with its opposite - namely Kant's doctrine of synthetic necessity.

Russell's attitude regarding our epistemological conduct in the analysis of mathematical foundations is most concisely summarized in what was dubbed the 'Supreme Maxim of Scientific Philosophizing'[45] namely, that in any analysis or explicative theory concerning an object or state of affairs that is to be scientifically described, wherever possible we should substitute logical constructions for inferred entities. It is clear that Russell appeals to this principle in his analysis of continuity. We should note, however, that the maxim indicates that such a logical reduction is not always possible. However, Russell speaks of continuity as a 'purely ordinal notion,' and therefore assumes that such a reduction is possible

[45]Russell, B. *Mysticism and Logic*

in the case of extensive magnitude (insofar as it is continuous):

> "... mathematics will deal explicitly with the class of relations possessing the formal properties of temporal priority - properties which will be summed up the notion of continuity"[46]

By *priority*, we understand Russell as speaking of *order*. Later, he states:

> "*Continuity* applies to series (and only to series) ..."[47]

Series, and order are the gates through which Russell draws the concept of continuity into logical discourse. An order relation is one that is asymmetric and transitive,[48] and a series is a total order, in which for any two objects under the domain of relation, one must necessarily stand before the other (we should imagine a 'chain' as opposed to a 'tree-like' structure, in which not every two objects are comparable.) Order is a logically admissible concept, in that it requires nothing more than variables, symbolically definable relations and the logical connectives to define rigorously. As he suggests that continuity is to be understood *strictly in terms of order*, it is clear that Russell believes that a reduction in accordance with his maxim is indeed possible concerning the continuity of space. As temporal and spatial extension form, in large part, the basic concepts of physics[49], it is

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[46]*Principles of Mathematics*, p.11.

[47]*Principles of Mathematics*, p.193.

[48]See appendix.

[49]Consider our fundamental ideas of physics, both mass and energy. After Einstein, we might consider the former as subsumed under the latter, which is in turn definable strictly in terms of 'metres' and 'seconds.' Further, let us consider the case of Newton's idea of mass. Insofar as we can quantify our experience, Newton found that (in ordinary, or 'room temperature' conditions) the quantum specifying the degree to which a body 'resists' a change in its movement (be it either still, or in rectilinear motion) is perfectly proportional to that which can be abstracted from its gravitational potential (i.e., the degree to which it attracts and is attracted to other extended bodies.) The quantum of 'mass' need not be (and in

assumed here that Russell likely saw the reduction of space and time to logical constructions as providing a means to disassemble most if not all physically inferred entities into similar constructions. Whether Russell was correct in viewing physical science this way, is another matter which we have not the space to address here. I merely point out that such an approach to physics is greatly facilitated by an adequate reduction of its basic concepts.

Like Frege, Russell understood the function to be the most elementary description of the adjunction of concepts to objects. The concepts of *complexity* and *dynamism,* insofar as they are to have scientific import, are understood by Russell as *conditions of plurality,* and therefore must be accounted for in terms of a function and its extension. Multiplicity is to be subsumed under the notion of class and member, a relation which Russell took to be fundamentally identifiable with the propositional function. We should note here that Russell opposed the extreme kind of extensionalism to be found in other Platonist view points[50], favoring a predicative understanding of number which would explain definite plurality in terms of the logic of the function. That is, Russell did not accept the existence of arbitrary

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light of the developments since Einstein, should not be) understood as some ultimate quantity of 'physical substance,' but rather a notational expediency, combining the sense of two different quantities into one (as experience, in the form of measurable experiment would attest). In this sense, a body can be thoroughly described through the ideas of volume, spatial occupancy, and its spatio-temporal relations with itself, and to other bodies in the physical universe. It is likely that such a conception of physics is what Russell may have hoped for, however this may differ from the ideas of current physics.

[50]Before the influence of Wittgenstein, Ramsey held all possible (i.e., non-contradictory) functions to exist in pure extension. That is, given any two non-empty sets, every possible function between them is held to exist regardless of whether the specification of each ordered pair belonging to the function is determined by a law, or predicate. Upon this interpretation, the function $f: A \rightarrow B$ is nothing more than a set of ordered pairs, and hence, the question of its *existence* is fully separated from that of its *definability*, in terms of some specification of the set by a predicate p which can assume arguments from A such that $\{(x, y) : x \in A \wedge y \in B \wedge y = px\}$. Such a view would be understood as 'strong extensionalism.'

infinite associations, but rather only those that were generated by the extension of some determining function F, in which a pair of associates (x, y) obtains derivatively from the satisfaction of an antecedent predicate (in our case, $y = Fx$). We might say of Ramsey's arbitrary maps that they are essentially *combinational* in their nature, and rely on no principle of orderly configuration from which associations can be derived such as a function. Rather, such objects would understood to exist as implicitly actualized structures. For Russell, on the other hand, the function provides the sole grounds on which multiplicity and order are to be understood. Most importantly, classes are, for Russell, to be grounded in the extension of 'definable[51]' *functions*.

What Russell intended to eliminate with the application of this maxim is the appeal to concepts which cannot be exhaustively broken down into atomic constituents by means of logical representation in an instance of explanation. Logic alone provides the best material from which rigorous explanations can be woven. In other words, Russell's basic view was that the nature of scientific explanation itself is exhaustive or thorough logical representation. For example, let us consider the current periodic table. Provided that the investigation in question is limited to ordinary chemical (i.e., non-nuclear) events, we can consider protons, neutron and electrons as the elementary components to be configured. In an analysis such as Russell's, each element can be thought of as a triplet of natural numbers, (p, e, n), where p

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[51] *Definable* in the sense that fx holds if and only if x satisfies a description, or *legitimately formed* denoting complex. In particular, such a denoting complex must not involve reference to the object which is to be defined. Such a definition is called 'predicative.' This notion will have greater relevance in subsequent chapters.

= the number of protons, e = the number of electrons, and n = the number of neutrons. Each piece of determining data relating to the 'atom' can be given logical expression in terms of either a finitely constructed *order*[52] relation (as would be the case in terms of electron valency), or a real-valued[53] quantity (for instance, mass, velocity, etc.)

In this sense, as the periodic table suggests, the elements of matter can be catalogued in terms of some order relation (for instance, ordered by the value of p.) The idea of an 'element of matter' is reducible to an arithmetical construction together with certain real-valued descriptions. Upon the translation of sense data into this arithmetical scheme (which for Russell, we should recall, is no more than a type of logical construction) scientific inquiry can be best conducted. Here, the basic components of matter are to be understood *combinatorially;* they are no more than finite structures of basic parts (protons, neutrons and electrons), which are given logically admissible definitions (charge, mass, and other properties, catalogued in terms of either an integer, or real-valued variable.) In this sense, the chemist's idea of the 'atom[54]' is reducible to a logical construction. Russell's maxim should be interpreted as advocating the most comprehensive possible application of this method in science.

The imperative set forth by Russell to eliminate from philosophy all inferred entities

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[52]See appendix.

[53]For more on this, see Chapter 4.

[54]We should not confuse this term with the ordinary sense in which we have used it in this essay. Here 'atom' is used strictly in the sense of chemistry, and should not be understood as being ultimate, irreducible, nor free of parts.

which are not given in terms of logical constructions was taken up by the tradition of analytic philosophy that followed him. This largely contributed to the general hostility of the analytic school towards Kant's doctrine of synthetic judgements. It is difficult to generalize over such a heterogeneous and 'atraditional' group as the analytic philosophers. However, one characteristic which seems prevalent in this school is their unanimous rejection of Kant's theory of synthetic *a priori* knowledge. The analytic philosophers generally deny the existence of necessary synthetic judgments, often insisting that (1) all worldly, or *informative* (synthetic) judgements are *a posteriori* along with the corresponding claim that (2) all necessity is traceable to analyticity.[55]

In this sense, Russell's objection to Kant's notion of the synthetic *a priori* can be likened to that of Hume's objection to claims about necessity in the sensible world, to which Kant's doctrine of synthetic judgement may be considered a counter position. We must qualify this claim, in that the intention here is not to draw a full analogy between the respective epistemologies of Russell and Hume. In particular, Russell and Hume differed on the point of inductive reasoning; for Hume, all such arguments are to be considered invalid, whereas for Russell, such were accepted on the grounds that they were ineliminable aspect of knowledge. The analogy is limited to their understanding of propositions, concerning what types of propositions there are. Unlike Quine, Russell did not challenge the analytic/synthetic

[55]See 'The *a priori*' in Putnam and Benacerraf (ed.) *Philosophy of Mathematics*. This is not the view, however, of Quine, who would rejects Kant's epistemology on the grounds of rejecting the coherence of the analytic/synthetic distinction in the first place. See Quine 1953. 'Two Dogmas of Empiricism' in *From a Logical Point of View*. Cambridge, Massachusetts: Harvard University Press. This is one example of the heterogeneity of the so-called analytic philosophers.

distinction directly, but rather argued for the 'vacancy' or *eliminability* of Kant's mysterious 'third propositional type,' namely, that of synthetic *a priori* judgments. In this sense, what results is very much like the 'dualistic' epistemology for Hume (the same cannot be said of Quine and others).

Since the publication of the *Critique*, a multitude of positions differing from that of Kant have emerged on the relation of sense experience to knowledge. Quine's view of the analytic/synthetic distinction as ill-founded is one of many such positions. However, we shall here focus on Russell's, which for our purposes should be understood as significantly coinciding with the Humean epistemology, in that he argues for a wider understanding of the range of analytic propositions, to subdue those principal examples of synthetic necessity (for example, arithmetical sums) which are of crucial importance in the Kantian argument. In this way, Russell can be thought of as partially restoring the epistemology of Hume, in that he argues , not for the incoherence of the analytic/synthetic distinction, but rather for the vacancy of Kant's 'realm' of synthetic *a priori* of propositions. If successful, Russell would thereby restore Hume's original epistemic dichotomy. Recall, Hume's epistemology divides judgements into two categories,[56] those which relate to matters of fact (i.e. of sense experience), and those which express relations of ideas. Hume claimed that necessity can only subsist in judgements of the latter sort, and that all factual truths, that is truths about the world, are *a posteriori.* Hume held all necessary judgements to be uniformly analytic in

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[56]See Hume, D. *An Enquiry Concerning Human Understanding.* Prometheus Books: Amherst, 1988. p. 21.

nature. All informative propositions, that is propositions regarding objects and states of affairs in the world, are to be established solely on the grounds of sense data. In particular, Hume objected to empirical claims of a causal nature, insisting that any such proposition involves a confusion of the two categories of judgement. Hume differentiated causal claims from those that involve simple regular succession by claiming that the former contains the latter together with the additional clause of *necessary connexion*. Spatio-temporal causal claims involve a proposition which (1) qualifies a *sensible* object as falling under a particular concept, and (2) attaches this concept to its object with necessity. In this sense, all claims of physical causality breach the division with which Hume's epistemology begins.

Kant opens the first *Critique*[57] with an alternative to Hume's epistemic dualism. He classified four separate determinations of cognition, consisting of two pairs of opposite conditions. The first pair, the *a priori* and the *a posteriori* refer to the presence or absence of necessity in a judgement, whereas the second pair, namely that of the analytic/synthetic distinction refers to a relation among concepts. The latter two express the distinction between two instances of the adjunction of a concept to an object. Analytic propositions are those in which the concept is included in the subject term, i.e., the concept to be attached to the object is a constituent of its definition. Synthetic propositions are defined negatively, as those in which such an inclusion of predicate in subject term does not occur. What results from this separation is a 2×2 matrix of epistemic possibilities in which one of the four categories is omitted on the grounds of impossibility (for clearly there can be no *a posteriori*

[57]CPR: A6-10/B10-14

truths that are analytically valid.) In contrast with Hume's dualism, Kant's epistemology is constituted by a triad of judgmental regions, none of which are devoid of instances. In this conception of cognition, claims of necessity can be either analytic or synthetic in nature. In the introduction of the first *Critique* Kant suggests that the truths of arithmetic are synthetic, since the concept of the sum need not be included in the concepts of the respective summands. In virtue of this canonical instance, Kant developed an epistemology in which the account of synthetic necessity occupies a central role, later addressing Hume's objection to causal claims.[58]

Intuition plays a key role in Kant's doctrine of synthetic judgements, and is defined in the following:

> "In whatever manner and by whatever means a mode of knowledge may relate to objects, *intuition* is that through which it is in immediate relation to them, and to which all thought as a means is directed. But intuition takes place only insofar as the object is given to us."[59]

Knowledge is a relation of cognition to objects and relations can either be mediated or immediate. Intuition, insofar as it is a relation, is of the latter sort, and only through it could necessary synthetic judgements be possible. Kant further divided intuitions into two types, namely those that are pure and those that are impure:

> "I term all representations *pure* (in the transcendental sense) in which there is nothing that belongs to sensation. The pure form of sensible intuitions in which all the manifold is intuited in certain relations, must be found in the mind *a priori*. This pure form of sensibility may also itself be called *pure*

[58]CPR: [A189-211/B232-56]

[59]CPR: [A19/B32]

intuition."[60]

Empirical intuitions are only possible within the *a priori* structures of space and time. As the formal conditions of the manifold of appearances, space and time are both necessary for, and prior to, the objects of sensible experience. Because of this priority, space and time must fall under the category of pure intuition, as any reference to empirical sensibility presupposes their coherence. In order to explain the nature of sensory experience, Kant begins with a science of the *a priori* conditions of sensibility, namely the *Transcendental Aesthetic*. The science of geometry is concerned solely with these *a priori* conditions; geometric propositions are both necessary and aesthetic in nature. As space and time are the antecedent conditions for sensible experience, the science of geometry is likewise antecedent to that of physics.

In light of this, we can better understand Kant's conception of the continuum. Space and time taken by themselves are subjects of pure intuition. Moreover, they are *a priori* structures, and therefore admit of necessary qualification. Among the determinations of these formal conditions that Kant deduces in the *Transcendental Aesthetic*, continuity is one such aspect of space and time. Unlike empirical objects, the qualification of which can be verified only in the *a posteriori* sense, we know the nature of the continuity of space and time intuitively (i.e., without mediation). The *Transcendental Analytic*, however, is not to be uniquely associated with the anti-atomist position. It is merely one method of arriving at such

[60]CPR: [A20/B34]

a conclusion. We mention it here for the purposes of elucidating the connexion with Russell, who wanted to explain space and time in a way which is entirely independent of the activity of the cognitive *subject*. For Kant, as for Brouwer, it is the subject who divides the continuum, which is given in intuition as undifferentiated.

It is exactly this subject-dependent notion of intuition that Russell meant to eliminate from scientific epistemology through the application of his maxim. The only formal restriction on an instance of intuitive knowledge is that its matter be *given*. There is no common 'substance' to which an arbitrary instance of an intuited entity can be reduced. It is possible to have two intuitions, the matter of which in each case is incomparable to the other. There is no necessary mediation between two arbitrary matters of intuition apart from the cognizing subject, who in an act of synthesis, draws them together in unity.

Russell's logical reduction introduces such a mediating element, which we might view as making this reference to subjective activity totally superfluous, and hence eliminable from scientific explanation. The objects of scientific discourse would be reducible, and hence comparable to one another in terms of a common set of logical terms. We might say that Russell sought a principle of 'commensurability' which would serve to unite scientific discourse. He saw the unrestricted appeal to extraneous 'inferred entities' which any epistemology involving intuition as a central feature would seem to necessitate. Scientific 'explanations,' for Russell, in principle obtain their highest rigor when conducted among homogeneous comparables. The possibility of this homogeneity is at least jeopardized, if not omitted in Kant's epistemology, as the entities of intuition cannot universally be brought under a single operating principle besides the subject who cognizes them. The appeal to intuition therefore results in a failure to pursue explanation to its fullest extent, which in Russell's view, sacrifices scientific rigor. Objecting to this epistemological limitation, Russell (along with others, often for different reasons than those presented here) treated the Kantian epistemology with open hostility. An exegesis of all arguments which have been presented against the Kantian doctrine of synthetic necessity would be far too much of a digression from the current topic, and hence we shall focus solely on Russell's dissent from the Kantian position. It should be noted, however, that Russell's approach to this issue had extensive influence on his students and followers. One sees here the influence which Russell's maxim had on the semantic tradition, which many understand as directly opposed to Kant, with the doctrine of synthetic necessity as one of its primary targets.[61]

The standard Anglo-empiricist reading[62] of Kant held the doctrine of synthetic necessity as a counter-position intended to address Hume's skepticism regarding sensible judgements. It is helpful to understand Russell's work as in certain essential ways connected with the empiricist tradition. Unlike Hume, Russell accepted the epistemic legitimacy of inductive reasoning, on the grounds that it is unavoidable. Despite this crucial difference, Russell shared the Humean dualistic account of propositions. The problematic 'intersection zone' between judgments which are both synthetic, and *a priori,* was not accounted for in Hume's epistemology. In the wake of the powerful Kantian tradition, the status of

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[61]See Coffa, Chapter 1.

[62]An excellent example of this would be Sir Peter Strawson's *Bounds of Sense,* pp. 15-42.

mathematical propositions presented the greatest difficulty for the strategy of eliminating the appeal to intuition in philosophy. Russell set out to re-establish Hume's dualistic epistemology. As the truths of logic are necessary, canonical and universally anterior to all instances of cognition or judgement, they must also be analytic as well.[63] Grounding mathematics as essentially a derivative feature of logic obtains a decisive blow to Kantian epistemology; recall, Kant opened the question of synthetic necessity with the canonical instance of an arithmetical sum. If the judgement '7 + 5 = 12' were shown to be an analytic truth in disguise, then the faith in the subsistence of necessary synthetic judgements would be irrevocably usurped, thereby toppling the entire Kantian system. Kant mentions the status of this sum in the introduction to the first *Critique* quite deliberately; it is the initial evidence necessitating a doctrine of synthetic *a priori* judgements.[64] In its absence, the appeal of Kant's epistemology is drastically reduced.

3.2 Russell on Continuous Structures

The nature of inductive number and its canonical concepts such as the arithmetical sum and product, the algebraic distribution of the former over the latter (a(b+c) = ab + ac) along with the principle of mathematical induction were crucial in the establishment of Russell's logicism. Russell consequently addressed the status of natural number with greater

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[63]As logic delimits the space of potential thoughts, it must be the case that each logical entity includes all of its own concepts. Any instance of a synthetic proposition presupposes the logical calculus, in which all logical canons must be conceptually complete entities.

[64]CPR: B15

thoroughness than most other areas of mathematical investigation. In contrast with the natural numbers, the continuum in particular was not an item of such focal debate, often being raised only in the context of categorizing determinations of infinitude. That is, Cantor's conception of the continuum was dominant at the time, and apart from the remarks of certain isolated figures such as Peirce, it received minimal criticism on the grounds of its atomism. It is on this issue however that I shall criticize Russell's logical reduction of mathematics in the upcoming Chapters. Before we move to this, however, we must take a closer look at the way Russell understood this structure. As has been indicated, the continuum is essentially an abstraction from the nature of space and time. To shed light on the technical development of the continuum offered in the *Principles of Mathematics*, let us consider Russell's analysis of time.

The standard Cambridge analysis of time, to which Russell emphatically subscribed, is best understood negatively in terms of the understanding of time to which it is opposed, namely that of tense theory. Tense theory holds the categories of past, present and future as irreducible, and as being essentially ontological as opposed to purely subjective categories. An aspect of this theory of interest in the present discussion is on the nature of the 'instant' in time. Recall, from the first passage which was referred to above, that for Kant:

> "... positions always presuppose the intuitions which they limit or are intended to limit; and out of mere positions, viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed."[65]

If one admits the category of the present as an essential feature of time, then a theory

[65]CPR: [A169-70/B211]

of indexicals is sufficient to account for the nature of temporal instant which our experience seems to imply. For instance, in the case of a falling body, we refer to the 'moment' the object comes into contact with the ground. In correspondence with Kant's understanding of *quanta continua* as indicated in the earlier citation, tense theory is capable of holding the nature of a temporal instant as essentially derivative from an indexical act. That is, the instant is inseparable from an act of (sensible) intuition, which in turn holds the coherence of space and time as antecedent conditions. Time, in tense theory, is not built up from some collection of instants. Alternatively, instants result derivatively from the integration of subjective activity with the *a priori* structure of time, which must be a well formed whole if such an act of intuition is to take place. Subjectivity and the notions of act and presence are at the heart of tense theory. Moreover, tense theory can account for genuine freedom, as the future can be held to be an essentially unrealized domain within time. In this picture, real change is possible, as well as free acts on the part of the subject.

In opposition to this theory which has obvious Kantian commitments, Russell developed what has come to be called the date theory of time. This theory eliminates the essentiality of the tenses to time, understood as being features of our psychology. The tenses are at most held to be the operative mechanism by which we survey reality insofar as it is temporally extended. We may 'follow' the line of time in a way to which the three tenses are essential, but this is left as a mere feature of the subjective apparatus through which we represent reality. Real time, as it were, is a lattice of atomic individuals universally connected to one another by way of a transitive asymmetrical relation (namely, before and after). In opposition to tense theory, date theory holds time to be essentially characterizable in terms of atomic constituents, in which the term 'now' has no genuine ontological meaning. The status of indexicals is reduced to that of mere psychology, whereas the subsistence of date relations is held to be actual. A Platonist spirit clearly underlies this theory, which holds the entirety of the matrix of time as an exhaustively realized whole. We should note here that any Platonist analysis of the temporal, under pain of contradiction, holds time to be infinite in both directions of its extension. Date theory removes the activity of the cognitive subjective from the nature of time altogether, and therefore can admit neither of genuine change nor of freedom. This picture can be best contrasted with tense theory in that it makes the instant ultimate, explaining the whole of time in terms of a configuration of atoms connected to one another by way of asymmetrical relations.

Unlike Kant, there is for Russell no undivided continuum given as an *a priori* whole as such. In Kant's theory, continuity can be understood as the defining property of the *continuum*, and is rightly understood as the qualification of a coherent whole. For Russell, continuity is a determination of order relations, as indicated in the title of the first section of Chapter XXXVI of the *Principles of Mathematics*: "Continuity is a purely ordinal notion." For Russell, the term '*continuum*' referred to the collection of real numbers:

> "we shall have to reserve the word continuous for the sense which Cantor has given it."[66]

For Russell as for Cantor the continuum is identical with the set of real numbers. A real number, Russell explicitly states

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[66] *Principles of Mathematics*: p271

"... is nothing but a certain class of rational numbers"[67]

The picture of the continuum which Russell held is very similar to that of Cantor, and more still as we shall see to Dedekind and his use of cuts. In particular, a real number is a bounded class of rational numbers. Recall, for Dedekind a cut is a division of the rational numbers into two classes A and B, such that every member of A is less than any arbitrary selection of B. In Chapter 2 we discussed this picture in context with the completed real line. Let us consider this notion of a *cut* in relation to the rational numbers: a fraction a/b is said to be less than c/d when $ad < bc$.[68] Note, this is a relation of magnitude which stands in no need of geometric grounding; both ad and bc are integers, hence their ordering requires no reference to space or time. A cut on the rationals can be of two sorts, the first, and simpler being the case in which the rationals are divided in terms of a unique selection among them. Every individual rational number q defines a cut on the set of all rationals, in which the first set is composed of all fractions less than q and the second of those larger than q. For instance, the following cut is produced by the ratio ½

$$A = \{q \in Q: q < ½\} \text{ and } B = \{q \in Q: q > ½\}$$

In such an instance, it is said that ½ produces the cut. Alternatively, if we consider the following criterion:

$$A = \{q: q^2 < 2\} \quad \text{and} \quad B = \{q: q^2 > 2\} \ *$$

then A and B also constitute a cut, but unlike the first case every rational number falls into

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[67]*Principles of Mathematics*: p270

[68]Consider: $a/b = ad/bd$, and $c/d = bc/bd$. In the instance of two fractions with the same denominator, they are ordinally comparable solely in terms of their numerators.

one of the two segments. There is no individual rational number which can be said to *produce* this cut.[69]

Let us refer to the first kind of cut as a rational cut, and the second as irrational. We must note here that in an instance of an irrational cut, the rational numbers inexhaustibly produce values of arbitrary closeness to the 'place of the cut.' That is, due to their compactness[70], there can never be a 'last' fraction closer to the location of the cut than any other. In the instance of a rational cut, produced by the fraction q, we might say that the difference $a < q$ for some $a \in A$ in the smaller half is never absolute, and that there is an infinite chain of fractions closer to q than any such selection of A. In the case of an irrational cut, however, there is no 'q' by which we can hold the respective elements against as was the case above. Reference must be made to the denoting complex which is implemented in the definition of the cut. If we consider *, we can define this cut in terms of an algebraic criterion[71]. Recall that for any magnitude $x > 1$, it is also true that $x^2 > x$. Clearly then if $a^2 < 2$ and $2 < b^2$, and a and b are both positive numbers then it is certain that $a < b$. Criterion * does therefore produce a cut since every element falling under the first class is strictly less than every element in the second.[72] There is no 'q' as in the case of a rational cut mediating

[69]Recall, the Pythagoreans proved the irrationality of $\sqrt{2}$.

[70]I.e., there exists a third rational number between any two, regardless of the magnitude of their difference. For a fuller development of this notion, see *Basic Topology* pp. 43-64.

[71]I.e., a finite construction of executable or computable rules.

[72]Strictly speaking this equation cuts only the non-negative rational numbers. This awkwardness is trivial however- clearly we could simply adjoin the negative rationals to the lesser set, thereby satisfying the strict definition of a cut.

the division, but rather the sheer absence of such a number. An irrational cut divides the series into two segments in the absence of a constituent term mediating the division.

The rational numbers are incapable of providing a comprehensive description of extensive magnitude, as exemplified in the case of the diagonal line joining opposite corners of a square. Grounding magnitudes like $\sqrt{2}$ in a class of rational numbers in spite of the fact that such a value cannot be identified with a unique ratio of integers, Russell may be thought of as appealing to a sort of neo-Pythagoreanism, in which the identity of extensive magnitude with number was to be fully restored. Russell's definition of real number appeals exclusively to the set of all rational cuts. A real number is said to be rational when it corresponds to a rational cut, and conversely, irrational if it corresponds to an irrational cut. That is, the names of the real numbers are equivalent to those of the constituent elements the set of divisions on the rationals. Whenever there is an irrational cut, we simply 'insert' an additional element (such as the number $\sqrt{2}$).

With this approach, Russell has not only classified those spatial lengths which correspond to a whole number ratio of some given length[73], but has also given a method to analyse irrational spatial lengths under one and the same system.[74]

The rational numbers are defined strictly in terms of integral numbers by means of equations. For instance, 4/5 is identified with the solution of the equation $5x = 4$. Russell

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[73]I.e., the unit length.

[74]Russell emphasized the difference between a rational number and the real number with which it is associated. The latter is what comes into application in terms of extensive magnitudes, whereas the former is to be understood strictly in terms of inductive (natural) number and equations therein defined. See *Principles of Mathematics* p.270.

claims that the set of rational cuts exhausts all possible divisions of the continuum, in that all discontinuities on the rationals are satisfied with the insertion of a point at every instance of this problem. This insertion is mechanized in a law-like way,[75] given in terms of the general case, such that any arbitrary discontinuity is 'automatically' filled in. Embedding this method axiomatically, all possible discontinuities are in principal 'bridged,' thus eliminating all ordinal gaps in the system.

Russell's reduction of the continuum can be summarized as follows: given a continuous, or geometric line, we select an arbitrary origin, and 'lay' the lattice of rational numbers against the line. Some spatial positions can be identified with a rational coordinate: for instance the length of the diagonal of a rectangle with sides of length 3 and 4 respectively is such a value (namely 5).[76] In other cases, such as the diagonal of the square, no rational value could solve the intended equation. Let us consider this mesh of logical atoms which was 'laid against' the line; it is certainly the case that every determinate spatial position would divide the set of all rational numbers[77] into two disjoint sets such that each point in one set exceeds the length in question, whereas each point in the second set falls short. This follows from the formal homomorphism of the spatial relations 'right and left' with the general form of 'greater and less.' There is no position in space, regardless of whether it can be identified

[75]See Axiom 14, appendix.

[76]Recall Pythagoras' equation: in this case, the rational values 3, 4 and 5 solve the equation $x^2 + y^2 = z^2$.

[77]Barring of course the case in which the magnitude of the position coincides perfectly with a ratio of integers, in which the element which produces the cut constitutes a third, singleton class.

with a value analytically derived from the inductive numbers, that does not divide the set of all fractions in this way. Russell attempts to establish the reverse relation; namely that the class of all spatial points can be exhausted by the class of such divisions on the rationals. He turns his attention briefly away from the geometric line, and considers the set of rational numbers taken as a purely analytical instrument. It is here that the philosophical principles of mathematical realism come to have great significance. If one assumes the mind-independent existence of a realized infinity of the inductive numbers, then it is a marginal step to hold the set of all proportions among natural numbers to be a realized totality as well. A further step, equally admissible under Logicist principles, would be to consider the power set of all rational numbers, of which we might consider an interesting subclass, namely that of all cuts. By the tenets of mathematical realism, such a set must be considered as a well-defined and complete infinite structure, in which all possible divisions are co-extensively real.

At this point, Russell needs only to concern himself with the nature of this class of cuts, for as discussed above, every determinate spatial position must divide the set of rational numbers in this way. By obtaining a complete analysis of the structure of cuts on the rational numbers, it would seem that Russell accounts for every possible spatial position.

We may therefore characterize Russell's interpretation of the continuum as one that is distinctly atomic, as is reflected in both his understanding of time as a lattice of dates, as well as in the case of the set of real numbers. Continuity is therefore reduced to (1) discreteness together with (2) an ordinal determination. This atomism coincides perfectly with the theme of mathematical logicism, in that the nature of continuity is here reduced to the rational, and hence the natural numbers. As indicated in Section 1.2 above, the natural numbers are for Russell purely logical in their nature. Hence, the reduction of the continuum to relations of natural numbers gives way to a reduction to logical principles. In this case, Russell has obeyed his maxim thoroughly, replacing the intentional content of the continuum with the structure of the rational numbers, taken with respect to its possible divisions. Rational numbers are themselves ratios of natural numbers, and hence any set of relations on the set of all fractions can be translated into a configuration of relations among the naturals.

Chapter 4: The Unit Length

4.1 An Artificial Extension of the Concept of Number

The essential problem in Russell's approach to the continuum is his reversal of the appropriate priority relations between the continuum and its parts. Russell assembled an aggregate[78] of which the constituent features, although in their algebra and order are defined only in terms of the system, nevertheless stand atomically against one another. A set is literally nothing taken in isolation from its constituents.[79] Consider:

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)^{80}$$

Usually the meaning of the identity symbol is not stipulated- the relation of identity has a universal and reserved sense. In this case however, it is not the definition of '=' which is to be discussed, but rather the meaning of a set itself. Identity is well defined, as is the subset relation; the above is a qualification of the nature of set. It stipulates that a set is nothing more than its members, for two constructions differing in sense result in one and the same object provided they have identical elements. For instance, the class of 'all multiples of 10' is identical with the intersection of the class of 'all multiples of 5' with the class of 'all multiples of 2.' This says more about the nature of set than it does about the relation '='.

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[78]I would here direct the reader's attention to sections 419 and 422 in the *Principles of Mathematics*. In the latter Russell explicitly defends the claim that 'space is an aggregate of points, rather than a unity.' pp. 440-5.

[79]The sense of the empty-class is obtained only in conjunction with the universal negation of any claim that an individual is held as a member. In this way, the empty-class is given its sense by means of 'hypothetical constituents.' Since this determination has logical import, the empty set cannot be construed as purely 'nothing.'

[80]See Lin: p. 37.

The existence of a set is utterly dependent on the existence and well-formed nature of its members. This view is reflected in Russell's Theory of Types, in which the objects constituting a class hold logical priority over the class, in such a way that a collection cannot be a constituent of itself. The set-theoretical model is capable therefore only of giving an *aggregative* and hence *exhaustive* account of the continuum. By the former, I mean an account which holds the parts as prior over the whole. This naturally leads to an analysis which fits the latter, in that any interpretation of an instance of whole and part which holds such a relation of priority among them must also be *exhaustive* in that the existence of the parts is both a necessary and antecedent condition of the unity of the whole, and hence there can be no 'extra' parts not given in this primary sense. Any set theoretical account of the continuum is both atomic and aggregative, and hence must be accompanied with the claim of total exhaustion concerning its possible divisions. In consideration of the basic principles of mathematical realism as presented in Chapter 1, this conception of the continuum is a necessary outcome of Russell's basic understanding of mathematical existence. For a Platonist such as Russell, if the continuum is to be a genuine mathematical (i.e., logical) object, then its determinations must be utterly fixed and realized in the fullest possible sense. The pursuit of an analysis which obtains the total exhaustion of all such divisions would be for Russell the only admissible approach.

An opposed view holds the continuum, insofar as it is a whole, to be prior to its parts. This is the Kant's view, as indicated in the citation from the First Critique in Chapter 2. For Kant, the entirety of space is presupposed in any act of sensible intuition; similarly, any part of the continuum is intelligible only in terms of the whole. This starting point lends itself to the *inexhaustive* conception of the continuum, in which there can be no Platonic assumptions such as that of a fully realized set of irreducible constituent points. Rather, this view holds the opposite to be the case; namely that the continuum divides without bound, and there is no real sense to the denoting complex 'all divisions of the continuous line.' Such a classification of the continuum in this non-atomic way, runs contrary to Cantor's dogma. Hence, any analysis committed to this description must rely on basic objects different from sets.[81]

Considered as a unity first, no construction nor act of division on the continuous line will produce discontinuous parts.[82] Understood this way, the parts of the continuum essentially presuppose the coherent unity of the whole. Moreover, there is a relation of homogeneity among all such divisions and the whole itself. Namely, the parts are one and all homogeneous with the whole, in that any true part of the continuum is itself continuous. That is, there are no 'points' on the line as such, but rather only intervals. Points correlate to these intervals as limits or endpoints, and hence can be regarded as derivatively obtained. They merely 'name' or mark the divisions of the continuum. These divisions are determinate and correspond to relations of magnitude, hence we mark them with a punctual 'dot' in the instance of a geometrical diagram. Nevertheless, there is no reason to include a point as a

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[81]For an example of how such an analysis can be constructed, consult J. Bell *A Primer of Infinitesimal Analysis.*

[82]This is hardly surprising, given that the any and all such constructions must be of countable size.

genuine part of the line with which it is associated; the interval is a part, whereas the point specifies some feature of the part (i.e., where it terminates). There is no information, therefore, which can be said about the point which is not traceable to the determinations of the interval which it delimits.[83] Points are thoroughly correlative with these parts, and hence need not be regarded as having any relation to the whole other than that of labeling its divisions. In this sense, points taken as geometrical entities are not constituent of the solid line as such, but are rather always 'outside', in that they do nothing other than delimit, or 'mark,' that which is continuous. Points represent a uniquely characterizing feature of an interval (given any two distinguishable points, a unique interval is defined), and are in this sense names. Proper names, as such (unlike that which they might name), are always atomic. In this case, the parts which they name however are *quanta continua*. This distinction is crucial. The use of a name indicates the stability of what is named, in this case, that the division to be named is determinate. This must be distinguished from the suggestion that the thing which is named is itself irreducible, insofar as it is a part of a whole. Points are not parts of the line as such, but rather the names of certain divisions, in particular those simple divisions where the whole is breached exactly once (Dedekind cuts). We should note that these 'cuts' do not name a single part, but rather a determinate decomposition of the solid line into two parts. Instants and points are inseparable from the times or spaces they delimit, and their derivation presupposes the coherence of the whole of which they are parts. This

[83]This is especially the case in the analysis of irrational numbers, which can only enter into true computation insofar as we choose an approximate point from some interval they delimit.

view, contrary to that of Russell and Cantor, has been defended not only by Kant, but characterizes a long standing philosophical tradition to which one should attach the names of Aristotle, Leibniz, Peirce, Weyl and Brouwer.[84]

There are no grounds to assume that what constitutes an irrational cut on the system of rational numbers (i.e., a cut on the rationals which is produced by no mediating whole number ratio) is categorically reconcilable with the class of all fractions. The 'cut' is understood to produce a 'point' because the fractions around it neighbor with limitless proximity. That is, one cannot say that the gap has any width, since any interval which contains the gap, no matter how 'thin,' must contain an infinity of rational approximations which show the interval to be 'too large.' Nevertheless, there are no grounds to call this 'gap' a 'point,' since upon any attempt to associate it with a single number results in nothing but sheer difference and further implies the existence of other numbers which are even better approximations. The fundamental problem with this method of building the reals from the rational numbers is that there is nothing corresponding to a 'point' in the instance of an irrational cut, in the sense that each rational number defines a point. In the latter case, a place corresponds to an exact value, and hence is 'positively punctual.' By this, I mean that the place is uniquely associated with a fixed analytical object (such as a pair of integers), and from this association inherits its 'fixedness.' All there is to be found in an 'irrational cut' is

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[84]Aristotle. *Physics*. Cambridge, Massachusetts: Harvard University Press, 1980. vol. II, bk. 6, chp. 1.; Leibniz: quoted in Rescher, N. *The Philosophy of Leibniz* New Jersey: prentice Hall, 1967. p. 109; Peirce: *Philosophical Writings of Peirce* p. 354; Weyl: *The Continuum: A Critical Examination of The Foundation of Analysis* Translated by S. Pollard and T. Bole. Philadelphia: Thomas Jefferson University Press, 1987. p. 94; Brouwer: 'Intuitionism and Formalism' in Benacerraf and Putnam *Philosophy of Mathematics*. p. 80.

a mere separation of such points. Each division on the continuum is determinate - hence the name of any particular division can be regarded as a logical simple. An erroneous leap is made by Russell and the classical mathematicians, in that they include both rational and irrational cuts under the same concept, conducting an axiomatic construction among names with stipulated ordinal and algebraic properties. Each such stipulation is fully traceable to the algebraic definitions of 1 and 0 together with the ordering on the natural numbers. However the 'individuals' corresponding to determinate irrational cuts should not be called numbers at all, in that they stubbornly maintain the simple *negation* of any object which we would otherwise call a number. From the finite perspective, all that is ever positively 'produced' in the method of cuts is an everlastingly shrinking series of intervals, in which no single point is ever carved out. It is an unwarranted presupposition to hold that it is admissible to give numerical status to the moment of a cut; particularly when the number to be associated can only enter real computations insofar as it is *approximated* - that is, abandoned in favor of that which is 'close enough'. All that abides is an infinitude of indeterminate intervals between each successive approximation and its intended value.

Dedekind's view of the continuum can be given algebraic expression in terms of the axiomatic system of the real numbers (**R**[85]). For our purposes, it will be convenient to speak about Dedekind's analysis through its equivalent, **R**. Recall, each rational number q constitutes a cut $\{r \in Q : r \leq q\}$. It can be easily verified by the reader that axioms 1 through 13 in the appendix are satisfied by the set of rational numbers. In the axiomatic picture, the

[85]See appendix.

reals can be thought of as a kind of algebraic *extension* of the rationals; that is, the real numbers contain the rationals, in a way such that all algebraic and ordinal properties are preserved, but new components added, which in turn are given algebraic and ordinal properties so as to preserve those of the rational components. For example, the numbers π and $1/\pi$ are both irrational; however, their product is 1, a rational number. Hence, it must be the case that $\pi(1/\pi) < 2$. The extension of an algebraic structure is very different from that of the union of sets, in that the extensions must be 'woven' into the lattice of the original structure, which results in the implicit addition of new elements which are both distinct from but also generated by the object which was to be added. For instance, if we add π to the rational numbers, in order to consider the new structure as algebraically *closed*[86], we must acknowledge that in addition to the extension of the rationals to π, we have implicitly included all values of the form:

$$(qn)\pi^n + q(n-1)\pi^{n-1} + \ldots + q1\pi + q0 + p1/\pi + \ldots + pm/\pi^m$$

Where *m* and *n* are arbitrary natural numbers, and *qi* and *pi* range over all possible rational numbers. Hence, algebraic extension is not as simple a matter as is that of set union, because in doing so, we produce values which are neither traceable to the structure which is to be extended, nor to the object to which the structure is extended, but rather from the synthesis of the two.

The set of the irrational reals (i.e., those corresponding to cuts which are not

[86] A structure X (in which the operation is denoted by juxtaposition) is closed when $\forall x \in X, \forall y \in X$: $xy \in X$.

produced by an individual rational number) are not algebraically *closed* as the example of $\pi(1/\pi)$ would indicate. The structure which emerges can be characterized by the preservation of the first 13 axioms[87] and their action on the set of rational values. Hence, the meaning of π (and hence $\pi(1/\pi)$) must conform to the pre-established rules governing the order and combination of the rational numbers.

The axioms of **R,** which represent the 'natural[88]' algebraic and ordinal properties of the set of cuts on the rational numbers, form a coherent and consistent system, in which an uncountably infinite number of constituent terms are definable such that each real number is both comparable (in terms of their serial order) and combinable (through their respective field operations, multiplication and addition) with an arbitrary other real number. The entire system of the reals can be constructed from 0 and 1 together with the axioms of their algebra and order. In particular, what differentiates **R** from any other ordered field is the provision of the completeness axiom, otherwise known as the *supremum principle,* which states:

$$\forall X \subseteq \mathbf{R}\ (X \neq \varnothing)\ [(\exists\ b \in \mathbf{R} : X \Subset b^{89}) \Rightarrow (\exists!\ c \in \mathbf{R} : X \Subset c \wedge (\forall\ b \in \mathbf{R} : X \Subset b \Rightarrow c \leq b))]$$

In other words, this axiom states that for any subset X of the real numbers, if it is bounded above (i.e., if it can be said that their exists a real number b that exceeds every member of X), then there is a unique least upper bound. For instance, consider the set of reals, the square

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[87]See Appendix.

[88]The intended meaning of the word 'natural' can be illustrated in the following example: if $\alpha = (A, B)$ and $\beta = (C, D)$ are both irrational cuts, such that there is a rational number in B which is also a member of C, then α is said to be less than β..

[89]Here, $X \Subset b$ indicates that X is a non-empty and b is a real number such that $\forall\ x \in X\ \ x \leq b$.

of which is less than the value of 2. Such a set is bounded above, for if one squares 10, the result is 100. The number 10 therefore exceeds the value of any member of our set. The irrational number $\sqrt{2}$ is the least such upper bound. Dedekind's conception of the reals in terms of cuts is entirely consistent with the system **R**; the proof[90] of this fact follows easily and is usually presented as a practice problem for a student taking a first course in real analysis.

What is crucial about this axiom is the fact that it defines a *unique* ordered field. That is to say, there is no algebraic structure that possesses this form of (ordinal) continuity which cannot be mapped isomorphically onto the reals. Up to isomorphism, there is only one such field. In the absence of any plurality among the analytical instruments we may implement in our description of space, mathematicians have regarded **R** as not merely a model which represents the geometric line, but rather as capturing its true formal character. This formal character is unique and intrinsic to the nature of extensive magnitude and cannot be modified or improved upon significantly.[91] With the axiomatic construction of **R**, the atomic analysis of the continuum would seem complete.

Unlike Euclid, who made provision for synthetic postulates such as that of parallel lines in his axiomatic system, one might say that Russell stands in the Pythagorean tradition, seeking an exhaustive reduction of all (in particular, geometric) relations to natural number

[90]For a proof of this claim, see Appendix.

[91]Insofar as we consider space as represented by an ordered algebraic system. It is desirable to consider space this way, in that spatial lengths can be concatenated, and therefore combined to produce new lengths.

and ratio or proportion. That is, in Russell's picture all possible determinate positions, or rather all constituent elements of **R** are analytically traceable to the numbers 1 and 0. Moreover, the positive reals can all be defined strictly in terms of the number 1. Standing alone, **R** is a perfectly consistent logical structure and would seem to be implicitly suggested in the natural numbers in a way similar to the integers and rationals.

But we may ask, how does this construction relate to the original problem of describing the nature of space and time? In the purely hypothetical sense, that is, apart from any reference to sensibility, **R** stands as a consistent and rich logical structure among which the constituent terms need not be defined in any aspect other than that which refers to their combinable properties (i.e., sums and products with other constituents) and their relations of order to one another insofar as they form a series. It is an obvious presupposition in Russell's reduction of extensive continuity to an ordinal, and hence, 'logical construction,' that this structure is meant to grasp the formal character of real[92] space and time, providing the best possible way to represent our experience quantitatively.[93] What is obtained in the

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[92]The term 'real' is taken here with a special meaning. The author does not intend to indicate that either Kant or Russell hold a true realism of space. However, insofar as space is a modal condition by which objects of sense-data can be given to us, it is subject to *formal determination*. That is, second order propositions, containing 1st order variables referring spatio-temporal 'locations' or *parts*, and 2nd order variables referring to first order propositions, which qualify relations between locations (and propositions about relations) conform to objective of truth conditions. To make the claim that 'things in space relate to one another in *such and such a way*' need not presuppose a realism of space, as constituting a mind independent object which can be intelligibly separated from acts of sense-perception.

[93]That the purpose of real analysis and its associated ordinal conception of continuity is to reflect necessary formal truths about the nature of space (and time) is taken as our point of departure here; its is clearly presupposed by Russell, as well as the author. Hence, the argument here defends a hypothetical judgment of the form 'if **R** is intended to grasp the formal characteristics of spatio-temporal extension, then Φ.' We should here note that in standard analysis, and more still in classical point-set topology, the idea of a four-dimensional manifold (which has been used in more contemporary theories of space and

construction of **R** is the precise identification of every possible spatial magnitude in terms of some pre-established standard, or unit length. The naturals, the integers and the rationals all proceed through finite operations and equations reducible to the numbers 1 and 0. Similarly, grounding **R** in the rational numbers is an attempt to bridge the 'Pythagorean gap,' thereby reducing every instance of the comparison of two spatial magnitudes to an instance of 'proportion[94]'. Any two different magnitudes stand in a determinate relation of quantitative difference. That is, among them there is a more and a less, and the degree to which they differ is itself a fixed third object. The Pythagoreans believed that every such relation was really an instance of a ratio of natural numbers. This approach differs from Russell's on the grounds of complexity more than intent. This is explicitly indicated in the opening of Section 164 in the *Principles of Mathematics*:

> "It is one of the assumptions of educated common-sense that two magnitudes of the same kind must be numerically comparable"[95]

The construction of **R** offers an exhaustive account of the general commensurateness of bodies in the manifold. In principle, any two fixed distances differ determinately, and hence either one can be given in terms of a 'fraction' of the other. In other words, each should mutually account for the length of the other strictly in terms of its own conditions.

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time) is formulated as an iterated 'direct product (see chapter 2)' of $\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, or $\mathbf{R}^4$. The number of dimensions in consideration is superfluous to our discussion.

[94]Intended here in a loose sense, proportion referring to the identification of a pair of magnitudes with a relation defined solely in terms of natural numbers.

[95]*Principles of Mathematics*: 176.

If this notion is carried out to its completion, what obtains is a method of rigorously classifying arbitrary distances in terms of some fixed unit of metric. If one attempts to construct an analytical tool by which all distances can be related in terms of a common metric using nothing other than structures derivative from the natural numbers alone, then **R** is the inevitable result. All that is achieved in this construction is the ability to systematically characterize all lengths in terms of a single given length. The rational numbers are well understood, and derive simply from the naturals. It is the status of the irrational real numbers in which we find a flaw in Russell's picture. Let us consider the Platonistic conception of a real number. Insofar as it may be rendered in script, we often speak of the infinite decimal expansion of an irrational number. Any terminating string of digits placed after the decimal place is obviously a rational number. Consider the symbol 0.918654: it is really 918654/1,000,000. Any attempt to give an irrational number in terms of a decimal expansion therefore results in a non-terminating, non-repetitive process. For the realist, there is a well defined, totally realized infinite series of digits which corresponds uniquely with (i.e., identifies) a determinate spatial magnitude. This identification of each real with an infinite string of symbols selected from a finite set is essential to Cantor's understanding of the reals, as indicated in his diagonal proof.[96] Committing Russell to a Pythagorean view is his (and Cantor's) presupposition that an infinite decimal expansion can uniquely identify, and conversely be identified with, a spatial point. The two notions are equated from the start. Is this an oversight on the part of Cantor, when he proves the uncountability of the real

[96]See Chapter 2.

numbers? Is this merely an artificial extension of the algorithm of numerical nomenclature with which we write computable approximations into the infinite? There is no reason to assume that the correspondence of an infinite string of figures selected from a finite set of terms with the totality of possible geometric points on a line segment is a bijection. Cantor's uncountability theorem, although demonstrating the non-denumerability of the reals, proves this condition to be true of a set of 'ideal'[97] or 'unwritable figures.' Cantor's proof tacitly indicates that he may have subscribed to a kind of Pythagorean presupposition similar to Russell's.

There is a contradiction inherent in this conception of **R**. Consider the role of the number 1: it is primordial[98], given and defines the grounds on which all other numbers (apart from 0) proceed. It is sheer numerical unity, and is therefore an irreconcilably different concept from that to be associated with the ideal figure 0.999... in which the symbol '9' is understood to repeat indefinitely. If we are to name the constituents of **R** systematically then our method of nomenclature must apply without redundancy; that is, such a system should not give two distinguishable names to one and the same object. Recall, it is essential for Cantor's proof of the uncountability of **R** that two numbers which differ in at least one place in their decimal expansion are said to be unequal. In this case, 1.00000... and 0.999999... agree at no decimal place. However, within the bounds of our numerical nomenclature, it is

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[97] In following Hilbert, I use this term to refer to objects of mathematics which 'go beyond' the natural limits of finite formal objects.

[98] See Appendix, Axiom 2.

impossible to define an infinite decimal expansion that names a number which is both larger that 0.9999... and smaller than the unit. Hence we must understand the two as equal, otherwise the hope of modeling continuity with **R** is lost entirely. But it is evident from the nature of the names that we have 'denoted' two separate number concepts. It is impossible therefore to associate each real number with a unique decimal expansion without corrupting their compactness. In virtue of this antinomy, I reject the claim of 1-1 correspondence of infinite decimal expansions with geometric points.

In the words of Wittgenstein, what obtains in the instance of an irrational number is a 'process, not a result.'[99] The construction of an irrational real never results in a number at all, but rather a massive and plentitudinous collection of numbers, none of which satisfy the value in question. We may liken Russell's approach to the analogy of a child who asks her parent to buy a horse for her birthday. Given the immense expense and resources needed to maintain a horse, the parent may attempt to placate the child with a rich collection of presents, dolls, clothes and so forth, in the hope that she will accept this generous and plentiful substitution in lieu of the one gift she originally asked for. Russell's conception of irrational number presents a similar offer of 'plentitudinous substitution,' meant to resolve the irreconcilable absence of a number corresponding to the value in question. In the case of **R**, we never have the number desired. Rather, we accept the 'next best thing' - that is a bounded yet infinite collection of rationals in which the differences between magnitudes

[99]Wittgenstein, Ludwig. *Wittgenstein's Lectures, Cambridge 1932-1935*. Oxford: Blackwell, 1979. p221

represented by the members and the magnitude in question diminish limitlessly. Any desired 'closeness' to the value of interest is satisfied by some term in the collection. However, until some allowance of error is admitted, until we accept what is other than the magnitude of interest on the grounds that it is 'close enough,' we have no number at all. The name 'π' denotes no determinate number, but rather an infinite series of finite approximations, each better than its antecedents. But we must ask, what does it mean to approximate that which is not a number at all? To answer this, we must distinguish between a determinate magnitude and number.

There are two kinds of irrational number, usually referred to as the algebraic, and the transcendental respectively. The former, like the integers and rationals are countably infinite and arise in the context of algebra, as will come into greater relevance in Chapter 5. The latter however are of greater interest to us here. Transcendental numbers are irrational reals which solve no polynomial equation with algebraically constructible coefficients. In the absence of such an analytic connexion, such as the satisfaction of a finite algebraic expression, transcendental numbers originate primordially in reference to intuitions of the manifold. The circle is one such intuition, the relation of proportion between its distance around and its distance across being probably the most famous transcendental number, namely π. We speak about the existence of such 'numbers' on the grounds that the difference which subsists between any two determinate lengths must itself be determinate. Present and inseparable from one and the same intuition, the circumference and diameter of the circle are subject to such comparison. Given the determinate nature of this relation, we give it a single name: 'π'.

In the absence of a finite (algebraic) and incomplete structure such as a polynomial equation of which such a number may have been a solution, the grounds of calling π a number are even less stable than those of a radical number such as $\sqrt{2}$. Relations such as that of the diameter of a circle to its circumference are however both a quantum and are fixed.

Let us define the term *determinate magnitude* as something which (1) admits of an asymmetrical and transitive (i.e., ordinal) relation with an arbitrary second instance of its kind and (2) has a combinable capacity with an arbitrary second instance of its kind such that the two can be brought together to produce a third. This concept must be distinguished from number, the former being a genus of which the latter is a species.

Number is a kind of determinate magnitude, in which all order is traceable to that of the principle of mathematical induction, and its associated concept of heritage among the natural numbers. The requirement of combinable capacities is satisfied with ordinary numerical addition. Spatial quanta certainly meet the criteria of a determinate magnitude, as they are each comparable with one another in terms of length and they can be combined together to form new distances and figures in terms of concatenation. What is obtained in the instance of an irrational number is a determinate magnitude that does not fall into the schema of possible ratios, and therefore is not a number in the true sense. Whereas the algebraic or radical irrationals are partly traceable to the principle of mathematical induction in that they each solve an equation strictly in terms of rational coefficients and natural exponents, the transcendental numbers such as π have minimal, if any, original connexion to the natural numbers, and should not be understood as being derivative from them.

4.2 The Sense of the Unit Length

Russell's claim that space is not a unity, as Kant would suggest, but is rather an aggregate of points, is grounded upon the Dedekind-style analysis of the geometric line with the set of rational numbers described in Chapter 2. The first thirteen axioms of **R**[100] give an exhaustive description of the set of rational numbers in terms of their algebraic and ordinal[101] behavior; adding the fourteenth axiom extends the structure to those 'points' which correspond to irrational cuts.[102] As indicated in the Appendix, the fourteen axioms of **R** define the same instrument as that which 'atomizes' the line in Dedekind's, and therefore Russell's picture. Russell's claim that space can be reduced in this fashion, unavoidably, however tacitly, presupposes that the construction of such an instrument as **R**, insofar as it is to be structurally featured, is antecedently committed to the formal nature of extensive magnitude.

The application of the formal object **R** to space and time, however, rests entirely on the givenness of the unit length of the metric concerned. That is, **R** is nothing more than an elaborate game of logic, rendering no connexion with extensive magnitude at all, unless the unit length can be considered determinate. This crucial fact has largely been overlooked because of the arbitrary status of the unit; nothing changed significantly in our physics when

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[100]See appendix.

[101]'Ordinal' in the sense of 'that which is of order.' An allusion to Cantor's theory of ordinal numbers is not intended with the use of this term.

[102]This fact is illustrated in the details of the proof included in the Appendix: every *cut* constitutes a set which is bounded above; the irrational number corresponding to the cut is the least upper bound of its lower half, and every set which is bounded above implicitly constitutes a *cut*.

we replaced the yard with the meter. A foot, a centimeter and a mile are equally sufficient in giving a sense to the unit length. If the unit remains purely hypothetical however, having no further sense than the name '1' along with its associated algebraic and ordinal properties, then there is no real connexion between the ordering of the constituents of **R** and the orders of magnitude (distance) in space. This connexion must be posited, however, if such a Dedekind-based reduction as that which Russell attempts is to obtain. Russell differs minimally from the standard classical view on this point, and similarly overlooks the problem on the grounds that any given unit will satisfy the necessary role in the metric. This is reflective of the principle indicated earlier, that all lengths are considered to be mutually commensurable.

Let us consider the picture presented at the end of Chapter 3 concerning the class of all cuts on the set of rational numbers. It would seem that every possible spatial position is accounted for by either a rational or an irrational cut. Dedekind's analysis of the line involves pairing each irrational cut (as well as those which are rational) with the unit to be divided, such that a magnitude of 'proportion' obtains between them. In the suspension from this connexion with the unit length, no true length obtains. The original undifferentiated unit, that *which is to be divided,* is tacitly presupposed in order to give any particular cut the status which would constitute a magnitude. Given the length of the unit, which need not itself be compared with another so as to generate a magnitude, a cut c can be identified as a magnitude in term of its relative proportion to this '1.' In the case of an irrational cut, in which their is no determinate figure of natural numbers which constitutes the final value of the point, the wedding of 'place' with 'number' is inseparable from this relation of proportion to the unit. Here we might raise the question, 'In admitting this classification of all possible geometric points, do we not presuppose a primordial intuition that determines the *sense* of the unit length?' Consider the answer 'No, because *any* length is suitable for this purpose.' It doesn't matter where we locate the origin, nor how 'long' the unit length is. However, the arbitrary status of this length does not eliminate the question of how the sense of this unit length is made determinate, for if it were to be undetermined, then it could not perform the role required of the unit, insofar as **R** is supposed to model space. In order to carry out Dedekind's division of the line, however, we posit as basic that which the analysis was to explain; namely, the continuum. In order to realize this comprehensive application of the lattice of the rational numbers to the geometric line, we must account for the means by which the length of the unit is determined. Once selecting a point to be represented by '0,' how do we determine the difference between this position and that to be associated with '1'? It is true that any length will do, but the question of what the particular length of the unit is must be distinguished from that which conditions it. What determines this length? It must have a *sense*, the determination of which must be either presupposed as basic and inexplicable, or be exhaustively analyzed in all its features by the logical construction to which it is reduced.

The unit length can be identified with the distance of the initial terms 1 and 0, to which all elements of **R** can be reduced. By this, we mean to say that each real number is a logical construction based on processes and predications which are admissible under the axioms of **R**, which can be reduced to functions which take only 0 and 1 as arguments. In terms of the standard analysis, the *metric* of **R** is the mapping $(x, y) \to |x - y|$, where $|x| = x$ if $0 \leq x$, and $= -x$ if $x < 0$. The length of the difference of 0 and 1, under this nomenclature, is '1.'

Insofar as 0 and 1 are to be logically distinguishable, the fact that the terms constitute *two* proper names, and therefore refer to *two* abstract objects will suffice. Moreover, to serve as abstract algebraic entities, no additional features must be added to their status as proper names. However, the sense of their difference, as it is to be employed in any Dedekind-style analysis, is more than simple logical inequality. They must differ by a determinate length. This difference of distance clearly lies beyond the mere lack of logical identity. For example, it makes sense to ask, for four magnitudes *a*, *b*, *c*, and *d*, whether the difference of *a* and *b* is greater or less than that of *c* and *d*, whereas no such question could be meaningfully phrased strictly in terms of logical identity, negation and no other relations. To explain this difference of distance through the idea of the metric on **R** would involve a viciously circular presupposition, in that the sense of the unit length is the ultimate *foundation on which the sense of every other length is grounded.* The primordial unit, characterized solely as the difference between 0 and 1, and from which all other real valued quantities can be considered logically derivable, can only be apprehended as a *continuous* segment. That is, in the series of concepts essential to this application of the rationals to the geometric line, a continuum must be posited as a first principle, without which the series of rationals cannot be geometrically located, nor given any determinate spatial sense.

The sense of the unit length must be given by non-analytical means. The order of **R** is traceable to the relation of ancestry on the natural numbers; similarly, all algebraic combinations are derivative from the field axioms governing multiplication and addition together with the primordial elements 0 and 1. In every other respect, **R** is given analytically, as the formal axiomatic presentation included in the appendix indicates. However, these axioms define nothing more than a peculiar piece of logical architecture if the unit is to have a purely analytical sense. All elements of **R** are traceable to 1 in terms of a determinate difference and proportion. Therefore, in giving a sense to the unit length, it follows that any arbitrary real number itself has a well-defined sense. There is no logical structure that can adequately substitute for the sense of the unit length; the strong meaning[103] of Russell's maxim fails to apply in this case, as the original unit length is both continuous and left unexplained. Although he undermines the appeal to geometric intuition in our understanding of the real numbers, favoring a strictly analytic presentation of them, Russell's understanding of **R** underlies his theory of time as a system of indivisible dates, as well as his claim that space could be regarded as an aggregate rather than as a true unity. If too much attention were given to the problem of 'calibrating' the unit length, then the analytic reduction of the continuum would be seem significantly incomplete. However, as has been indicated above, the applicability of the formal structure **R** to space and time rests entirely on the unit length having a determinate sense, and this cannot be given analytically. Russell's reduction of

[103]Although Russell qualifies his maxim so as to substitute *wherever possible* logical constructions for inferred entities, it is evident both in his discussion of space in the *Principles of Mathematics* , as well as from his remarks on the nature of time, that he believes this to in fact be possible in the case of extensive magnitude. It is this view which is challenged here.

extensive magnitude to a logical construction is, in fact, incomplete.

The question 'How long is 'one foot'?' cannot be answered without some appeal to a sensible intuition, some primordial act of intuiting a continuous length. The particular length of the unit is arbitrary - in this sense, a foot is as 'good' as a metre. However, even in the arbitrary or hypothetical case, no unit can be given save as the determinate quantum associated with some primary intuition, such as 'the king's foot' or a 'the length of the edge of a one gram cube of water in the liquid phase.' Such denoting complexes obtain their extension only in the appeal to some instance of sense data. The sense of the unit length is essentially dependent on an act of sensible intuition, in which the said magnitude can be calibrated through association with an instance of a sensible object. This act is twofold, in that it firstly requires an act of perception, or some equivalent form of sensible interface with the manifold, and secondly requires the act of positing this non-number (determinate magnitude) as the representational counter-part of the value of 1 in **R**. There is no act of counting in the simple perception of a single object with an observable length - neither is there any natural connexion of magnitudes such as that of the proportion of the circumference of a circle to its diameter with number as such. The axiomatic configuration of **R** can be exhaustively explained with analysis. Its relation to real space however is necessarily mediated by an *act* of intuition.

This 'first,' *given* continuum that is the unit length is a necessary antecedent condition for the relation of **R** to actual space.[104] There is a vicious circle at work in Russell's reduction. The arbitrary status of the unit length indicates that it does not matter at all what particular unit we use in the application of **R**. This fact is misinterpreted in the classical tradition as indicating that there is no real issue at stake in the selection of a unit, since any unit will do. However the question of what the particular unit *is* must be distinguished from that regarding the condition in which it obtains, for to be a length is to be a *determinate* difference.[105] The former presents no difficulty, whereas the latter is both crucial and overlooked. In the absence of any determinate sense, the unit remains on the side of the purely analytical, rendering no connexion at all with extensive magnitude. If the totality of **R** can be identified with the formal nature of extensive magnitude (that is, continuity) in the exhaustive sense, then this structure too must be a necessary pre-condition of any instance of a sensible intuition. However, as we have seen, such an exhaustive identification requires that we account for some sense of a continuous unit length, and that in the absence of such a determinate relation of metric, **R** has little if anything at all to do with extensive magnitude. But such a *calibration* of the unit length is inseparable from an act of intuition, for which the coherence of space and time, together with their formal determinations (in particular, their continuity) are necessary antecedents. So, an act of intuition is prior to the calibration of the metric, the sense of the unit length is a necessary component of the formal representation of

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[104]Perhaps this is not far from what Kant meant when he said that space is a unity - any part of it implies the whole formality.

[105]The necessity of this assumption becomes all the more clear in the case of variables in ordinary algebra: the equation $(x^2 + 5x + 6) = (x + 2)(x + 3)$ presupposes that x is everywhere fixed, or self-identical.

extensive magnitude with **R**, and the coherence of both space and time (*quanta continua*) is an antecedent condition to any such act of intuition. But the formal character of extensive magnitude cannot presuppose some primordial instance of an act of intuition, for every such act holds this formal nature as its ground. The formal entity that is **R** therefore falls short of an exhaustive account of the nature of extensive continuity, in that its representative status holds as a necessary antecedent condition the very thing which was to be constructed.

Chapter 5: Freedom and the Continuum

In this Chapter, I shall argue that the continuum provides a crucial instance of a mathematical object which violates the most basic principles of mathematical realism, thereby giving grounds to abandon it as a foundational theory. Recall, the Logicist understanding of mathematical existence holds the entities of mathematics as utterly fixed. If the continuum is to be subsumed under this account, then the set of its divisions must in principle have some complete totality that is itself a fixed structure. Implicit in Russell's conception of **R**, particularly regarding the nature of irrational number, is a principle that, if taken in its complete sense provides a definition of a geometric point which no law like system such as **R** could account for. Before discussing this principle, however, it is best to introduce the notion of a Cauchy Sequence.

An infinite sequence $\alpha = a1, a2, ... an, ...$, in which each *an* is a mathematical object,[106] is said to be law-like if there exists an effective rule[107] which determines the general, or n^{th} term of the sequence. A law-like sequence of rational numbers is said to be a Cauchy

[106]For the non-Platonist, we must specify that each *an* is in principle constructible independently of the entire sequence. Even for Russell, in virtue of his 'vicious circle principle' (which shall be discussed in the upcoming pages), the sequence terms should rightly be considered as ranging over a domain of possible arguments, the respective definitions of which make no reference to any whole of which the argument is a part. In particular, the sequence terms must be definable without reference to the sequence as infinitely extended. However, this does not bar the possibility of defining or constructing new terms *an* using 'already established' pieces of the sequence. Such a function, in which the n^{th} term can be obtained solely by referring to the terms $a1, ..., a(n\text{-}1)$, is said to be *recursively* defined.

[107]That is, a rule which unambiguously determines each term of a sequence, in such a way that is subject-independent. In other words, the result of the rule does not in any way depend on the agent who is carrying it out.

sequence when the difference of the n^{th} and m^{th} terms approaches zero as n and m increase.[108] In other words, the difference among terms of the series diminishes limitlessly as the sequence proceeds. A real number which can be posited as the limit of a Cauchy Sequence is called a Cauchy real. It is the case that the set of Cauchy reals is an equivalent system to that which is given with the axioms of **R** and therefore to the set of all cuts on the rational numbers. Russell's conception of the continuum therefore implies the identification of certain points on the continuum with a derivation that essentially presupposes the point to be the limit of some Cauchy sequence.

A choice sequence is one in which the general term is not given by a law, but rather is determined by a free choice. In contrast with law-like sequences, the n^{th} term cannot be given in terms of a rule, but rather is entirely dependent on the choosing agent. However, the difference between both types of sequence is not absolute; it is possible to define a choice sequence in such a way that certain conditions may be met. In particular, we may restrict our 'choice' in such a way that for every $n > 1$, the difference between the n^{th} and the $(n + 1)^{th}$ terms must be strictly less than that of the $(n - 1)^{th}$ and the n^{th} terms. As there is no finite bound on the number of distinguishable rational values we can generate between any 2 rationals there is always room for choice no matter how small an interval in consideration. We could be even more specific, and still afford the room for free choice. Suppose we impose the restriction that the n^{th} choice may differ from the $(n-1)^{th}$ choice by less than the value of $1/2^n$ (notice with this, we still have not exhausted the infinite variety of possible

[108]See Marion: p. 193.

choices for any *n*).[109] In such a case, we have configured a sequence of rational numbers that is very much like a Cauchy Sequence, in terms of the 'shrinking' difference of each successive term with the last as the sequence progresses. They differ, we might say, in the principle that determines their individual terms. On the other hand, in terms of their possible 'convergence behavior[110],' the difference is insubstantial, in that we can restrict our choice in such a way as to meet Cauchy's criterion.

When a sequence satisfies Cauchy's criterion of convergence, we are merely assured that there is no contradiction in the supposition that it has a limit. Russell interprets this as sufficient grounds to assume that the 'limit' of the sequence exists. This inference remains, however, at the purely hypothetical level; it merely shows that if there is to be a limit for a sequence in question, then it must be unique. This is a different matter than the question of whether such an object *exists*.

As discussed in Chapter 4, there is no rational number to be found in the instance of a transcendental value, nor is there any finite algebraic expression that it will solve. In the case of a number like $\sqrt{2}$, the latter of the two previous conditions is satisfied. That is, it solves the equation

$$x^2 - 2 = 0$$

and hence can be essentially identified with this intentional content. In terms of Dedekind

[109]See Heyting pp.13-30.

[110]That is, as the sequence progresses, the difference between successive terms diminishes to zero.

cuts, we can think of the value √2 as identified with the cut $(\{x: x^2 < 2\}, \{x: 2 < x^2\})$ (where x ranges over the rational numbers). Notice, this cut is defined by *algebraic* means, and hence can be encapsulated in finite script. The question of the existence of a transcendental value, however, takes on a different meaning. There is no finite object (such as the expression $x^2 - 2 = 0$) on which the existence of a transcendental real number can be said to depend.

In the case of a 'transcendental cut,'[111] there is merely an infinity of rational approximations, none of which are equal to the value in question. It is said that they approach a *limit*. But this limit, taken as *number*, cannot be supposed to have the same derivative status as the rational numbers which approximate it. The rational numbers are exhaustively traceable to the natural numbers in terms of 'pairs' in that each pair of naturals, *m* and *n* define two different (algebraically reciprocal) ratios *m/n* and *n/m*. The limit is defined by and hence derivative from the sequence. The sequence, in turn, presupposes the existence of the rational numbers which constitute it. In the strict sense, to say that the sequence 'converges' is merely to predicate the terms of the sequence, and nothing else. This is not to be confused with some 'geometric' conception of convergence, in which there is an actual 'place' which an object in question converges to. Cauchy's criteria of convergence makes no reference to the limit value in question: it can be verified solely in terms of the relations among the sequence terms and the law that governs them. There is no limit intrinsic to a convergent sequence, nor does the judgement that a sequence converges necessitate the

[111]That is, a cut which cannot be defined by algebraic means.

assumption that there is one. Consider Cauchy's criterion of convergence on a sequence α:

$$\forall\, \varepsilon > 0\ \exists\, p \in \mathbf{N} : \forall\, m \in \mathbf{N},\ \forall\, n \in \mathbf{N}\ p < n < m \Rightarrow |am - an| < \varepsilon$$

For the current purposes, let us consider a simple example, in which $an = 1/n$.

Let $\varepsilon > 0$. Then, by Archimedes' principle,[112] there exists a natural number q such that $1/\varepsilon < q$. Set $p = q$. Note, in this case, an is always positive-valued quantity. If $m = n$, $|an - am| = 0 < \varepsilon$; so, without loss of generality, let $p < n < m$; note, since all values here are positive, $0 < 1/m < 1/n < 1/p$. Therefore $|an - am| = |1/n - 1/m| = (1/n - 1/m)$. Since $1/m$ is positive, $(1/n - 1/m) < 1/n < 1/p = \varepsilon$. The limit of α is 0. Differing from other analytical uses of the limit concept, the satisfaction of Cauchy's criterion can be proven without reference to the specific limit value. Notice, in the proof above, we do not at any point make use of the number 0, nor do we insist directly that the sequence terms have any specific relation to their 'limit.' One of the main advantages of this criterion is that it establishes convergence without first presupposing the existence of an entity to which the sequence converges. The lack of this reference to the *destination* of an infinitely proceeding sequence is neither a mechanical expediency, nor is it logically requisite in the discourse over real numbers. In fact, it allows us to construct new limits as entities thoroughly derivative from the sequences which converge to them. Cauchy's criterion enables us to produce definitions of new numbers, without presupposing the number to be defined in the sequence construction. Fidelity to

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[112] This principle states that for every real number r, there exists a natural number n such that $r<n$. There has been considerable development since the 'foundational crisis' of the early 20th century of non-Archimedean geometries and infinitesimal analysis. In the latter, the line is analyzed in terms of objects which resemble segments more so that indivisible points. For more on this, see J. Bell *A Primer of Infinitesimal Analysis*. Cambridge: Cambridge University Press, 1998. The above proof would not be valid in such an analysis of the real line.

such relations of constructive priority is not peculiar to mathematical intuitionism or constructivism; in fact, it lies at the heart of Russell's vicious circle principle,[113] and therefore his predicativism, along with the Theory of Types.

In contrast with such circular definitions (which seem to lie at the heart of most paradoxes), there is no disadvantage of thinking of the limit as obtaining its ground in the sequence. Russell's predicativism would not allow the appeal to an extreme Platonism that held the continuum to be purely extensional; for the extension of a function which takes infinitely many values must be grounded in some predicatively admissible law. Existentially speaking, Russell would only admit such extensions.

When asked 'What does this sequence converge to?' we cannot simply reply 'Its limit,' for such an answer would be circular. It is unwarranted to presuppose that in each case of a convergent sequence there exists an independent limit, the reality of which is in no fundamental way derivative from the sequence. On the contrary, that which it converges to is thoroughly determined by its associated sequence, and hence need not be presupposed to have an independent being as some pre-existing end towards which the sequence progresses. Moreover, the appeal to strict extensionalism, such as that of early Ramsey mentioned above, is entirely vulnerable to the paradoxes of set theory such as Russell's. The available objects

[113]Recall, this principle states that for any mathematical object x, x must be definable (or, constructible) in a way that makes no reference to a collection of which x is a part. This principle is obviously motivated by problems like the Russell paradox, in which the set $X = \{Y : Y \notin Y\}$ fail to satisfy the principle, since the predicate $Y \notin Y$ must be quantified over a class of objects to which X itself belongs. Otherwise, the question $X \in X$ does not make sense. This is exactly the point of Russell's vicious circle principle - to eliminate 'constructions' such as this one. Predicativism can be thought of as a theory of mathematical existence in which admissible object must be characterized as satisfying a non-circular, non-contradictory predicate.

of discourse implicit in such an extensionalism would result in the necessary admission of the coherence of objects like the 'set of all sets.' Predicativism could be thought of as the thinnest possible restriction on such objects; namely that each object be defined as possessing intelligible properties, and that these properties are well-formed only if they make reference to other objects, the respective definitions of which the object to be defined is not constitutive. We take this notion as our point of departure here, as the figure of main concern, namely Russell (the alleged Platonist) would concede to these restrictions. Under this predicativist view, Cauchy sequences *essentially* define their limit.

Suppose, by limit, we mean nothing more than the sequence itself (as in the case of the method of 'cuts,' Russell's strategy is not to refer to the individual rational numbers which identify a geometric *point*, but rather to a 'set' of such objects); but by what right do we call this entity a number? In the case of any irrational value, there is nothing to be found other than an infinite collection of disjunct atoms, none of which exactly represent the magnitude which they approximate. This is the crucial point: we have an idea of algebraic extension, in which fidelity to the algebraic and ordinal relations on the primary structure is the grounding principle. Hence, in the instance of an irrational number, we are 'inserting' an additional element to a field which already possesses a determinate algebraic structure. In such a case, the field can be extended to include such a number only if the algebraic and ordinal structure of the original field is not disrupted by such an addition. However, the ordinal characteristics of the 'inserted' value can only be determined through reference to the sequence which defines it. If x were such a value, then, given a class of rational numbers R, the question whether x exceeds the value of each member of R can only be determined in consultation of a the sequence which converges to it: let Π be a series of rational numbers which converges to x, and let $r \in R$. Then, $x \leq r$ if and only if $\exists\, p \in \mathbf{N} : \forall\, n \in \mathbf{N}, \Pi n \leq r$, and $r < x$ if and only if $\exists\, n \in \mathbf{N} : \Pi n > r$. The point is, such essential questions (as 'where' the point is on the line) are both satisfactorily, and *uniquely* answerable in terms of their definitions, or constructions. Insofar as a point could be on the line, its definitive characteristic would be *where it is located*. By this, we mean that any 'two' points which have the exact same place should rightly be called one and the same point, and that given any point, it must have a unique place (or alternatively, any two distinct places cannot correspond to a single point). Location is not merely an accidental property of a point, but rather serves to uniquely characterize it. However, insofar as a transcendental irrational value is concerned, the location of its limit is essentially dependent on the sequence which defines it. Geometrically speaking, a 'point' is nothing more than such a specific location. The question 'Where is the point?' has no determinate meaning in isolation from such a locating basis as that of a convergent sequence. To answer the question, we needn't mention the 'limit' at all, but rather appeal directly to the behavior of the convergent sequence. On the other hand, we could not satisfactorily answer the question if we spoke strictly of the limit, and made no reference to rational values which are close to it.

This is because: **Q**[114] is an ordered field, and we have to preserve its order in our

[114] **Q** shall refer to the rational numbers, taken in the algebraic sense. This structure is governed by axioms 1-13 in the Appendix.

extensions. That is, the ordinal determinations of extended values are antecedently committed to the preservation of the order of the rationals. This preservation characterizes the algebraic behavior of the extension with both rational values, as well as other extensions. For instance, $\pi + 1 > 207/50$, since $3.14 < \pi$. The coordinates of **R** are essentially algebraic and ordinal entities. Hence, what such values are, predicatively admissible or otherwise, must be characterized by their algebraic and ordinal behavior. Insofar as an irrational cut can be included under such an extension, both its algebraic and ordinal properties are definitively traceable to some sequence which converges to it. A 'point,' in any analysis such as Russell's must have a determinate place, therefore its comparison with other points must in principle be decidable, and its combination with other points (via the canonical 2-argument operators, $+$ and $\times$) must result in a unique third point. However, in the case of a value which cannot be considered as derivative of any finite structure, these characteristics are inseparable from the objects with which the value is defined. We might contrast this with the example of the 'circle' as representing the 'limit' of a sequence of regular polygons of increasing size. It is true that no polygon could be the 'best approximation,' however, the circle can be constructed, as in topology,[115] by methods which make little or no use of the idea of polygons. The same cannot be said, however, for a transcendental irrational value, as they are essentially (that is exhaustively, and up to isomorphism, uniquely) conditioned by nothing

[115]For an interesting account of this, consult J. Vick. *Homology Theory: An Introduction to Algebraic Topology*. New York: Springer, 1994. pp165-8. The development of algebraic topology, the founding of which is largely credited to Henri Poincaré, was largely motivated by the task of constructing a straight forward proof of the fact that the 'circle' divides the plane into two distinct regions: the inside and the outside. It seems that we can construct such objects as 'circles' in a variety of different ways, some of which use the limit concept, and others not.

other than an infinite bounded structure, for which the number is a limit.

In Dedekind's analysis, the 'point' is made *punctual* by the inexhaustibility of its rational approximations. That is, given a 'gap' in the series of rationals, we treat it like a point, rather than an interval with non-zero length, because the sequence terms force us to view the gap, if positively[116], as a point, rather than something with a length. Any such interval would be 'intruded upon' by some rational approximation. The same holds for Cauchy sequences. The punctuality of the cut must coincide with the limitless proximity which the sequence terms have to the place of the cut. However, the law together with the set of sequence terms do not together constitute a third entity, namely the object to which the rational numbers are to be algebraically extended. Dedekind cuts alone are not a sufficiently operable device; many of the results of modern analysis are possible only through the use of such a law-like instrument as that of Cauchy sequences. For our purposes, we could consider a Cauchy sequence as a sub-set of the lower half of a Dedekind cut. This sequence is (1) governed by a law, which need not be specified (unlike the cut) in terms of a 'place' and (2) like the cut it is a subset of, it approximates the 'place of the cut' without limit. Hence, we could think of the Cauchy sequence as a kind of 'operable Dedekind cut,' over which it is easier to both conduct analysis, and construct new numbers, in that such a sequence can be recognized as convergent even without knowledge of the 'place of the cut.' We access the point through the law; it is based on the law that we prove and judge propositions about the

[116]I.e., if we make the assumption that the gap constitutes a positive mathematical individual constitutive of the geometric line, as opposed to simply viewing it as the negation of such an entity as thus far (up to **Q**) constructed.

value the sequence converges to. But insofar as this value is an algebraic individual, subject to ordinal comparison (i.e., representing a definite place on the line), a third entity is required. I contend that we posit this third individual, for it is neither to be found as a sequence term, nor in the law which governs such a sequence. However, to demonstrate this view would be to settle the Platonist/Constructivist debate once and for all, an accomplishment not likely to occur in an essay of this humble scope. For now, it would serve our purpose to establish that such an act of positing is implicit in the predicativist view of mathematical existence, for the topic of the current chapter involves the question of what other kinds of point-values could be produced using the same presuppositions as that of Russell.

In the case of an irrational number, no sequence term (in that they are all rational) exactly coincides with the location in question. Insofar as 'number' is concerned, there is sheer difference where we seek identity, and sheer multiplicity where we seek unity. How then is the concept of an individual transcendental value to be formed? In the case of $\sqrt{2}$, the 'locus' of the cut in question can be defined in terms of an equation. An equation is finite, and hence gives a complete *rule* by which the algebraic and ordinal properties of the radical value can be determined. However, the set of all algebraic reals is countably infinite; it is only with the inclusion of all transcendental values that Cantor's continuum is obtained. In the absence of such a finitary basis as in the case of a radical number, there must be some other principle that *unifies* the convergent sequence associated with a transcendental value. Such a value can be accounted for, however, by a synthetic act of mind, in which its existence is traceable to the subject who performs the act of positing this entity as a real number. In other words, every law-like sequence defines a 'cut,' and therefore a place on the line.[117]However, the question of whether this cut constitutes a positive point is a matter of debate (as indicated in the previous footnote, it is no less than the debate over the validity of the law of the excluded middle in the context of the geometric line.) Moreover, even if such a spatial designation as could be viewed as implicit in the convergence of a sequence, such an individual would not constitute the algebraic entity required by Russell. Insofar as a real number can be thought of as an infinite decimal expansion (the only means by which we homogenize the transcendental value with rational numbers so that it can be an argument of computation,) this infinite entity is neither equal to any member of the sequence, nor would it be possible for it to coincide with any rational number lying outside the sequence.

Russell, as a predicativist, could not appeal to the continuum in strict extension: he has committed to the existence only of those cuts which admit of predicative definition.[118] Hence, the strong Platonist answer to the question outlined here, that of strict and exhaustive extensionalism, is neither presupposed, nor tacitly appealed to in Russell's analysis. To be

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[117]If we grant the availability of the unit length to begin with, Dedekind's 'imagery' becomes very convincing. The tacitly concealed problem in this is that the assumption that such a gap constitutes a point requires the admission of the law of excluded middle. Peirce pointed this out long before Brouwer. For more, see Peirce 1976 *The New Elements of Mathematics*, vol. III (ed. Eisele). New Jersey: Humanities Press. p. xiv. and J. Bell [1998]. p. 5. In the suspension of Aristotle's much debated 'law,' one could analyse the line in terms of the infinitesimal. Such methods of nonstandard analysis have been recently carried out with the help of category theory, and the algebra of 'nilpotents' (ring objects the square of which, and hence all higher powers, equal zero). The infinitesimal which is not quite 'zero,' yet is exceeded by every positive length, has traditionally been condemned by philosophers and mathematicians such as Berkley, Cantor and Russell, can be given an algebraically rigorous definition by way of the field extension of **Q** to include nilpotent elements.

[118]That is, one that could be defined in terms of the satisfaction of a predicate which does not violate the vicious circle principle.

a real number is to be both comparable and combinable with other reals. The sequence 'decides' the location, insofar as a transcendental limit number can only enter into comparison with other reals mediately through reference to the sequence. Yet, the algebraic individual to which **R'** is extended (let **R'** denote the 'reals in construction'; that is an object, inclusive of the rationals, to which we have already added new extensions, and are in the process of extending) is never to be found as a constituent of the sequence, nor can it be identified as merely the sequence itself, for two sequences can converge to the same limit. The limit must be considered as identifying an equivalence class of convergent sequences, all of which have the same limit (in virtue of the examples above, it is easy to imagine a Cauchy-style definition which would decide whether two sequences converge to the same 'limit' without reference to the value of the limit.) But, then we might ask, what unifies such a family of equivalence classes? 'Its characterizing family of predicates,' Russell would likely answer. It is true that a convergent sequence, is sufficient to define a single equivalence class of convergent sequences, thereby defining a real number value. However, if we extend such a question to the family of all such equivalence classes (i.e., **R**), the predicativist position runs into problems. A sequence can be given in extension, or can be specified by means of a general method of establishing the n^{th} term. Arbitrary infinite sequences, recall, are ruled out by the predicativist position. Hence, to consider **R** to be a set of equivalences classes as above, requires a (non-circular) second-order predicate which would, in general, be satisfied by every possible predicate which would generate a convergent sequence, and by nothing else. If all such sequences were given in extension (the strong Platonist view), such a second-order predicate would be plausible. However, the predicativist does not allow such an assumption.

The concept of a law-like sequence is more narrow than that of a mere sequence; indeed, there are infinitely proceeding sequences which obey no law.[119] From the predicativist view, Cauchy reals are admissible. The comparison of the a transcendental irrational with other reals is unambiguously defined in terms of its associated sequence. Convergence, however, is not to be found in the law alone - the law gives no limit. Through referring to the behavior of the sequence under its governing law, we confirm that the sequence satisfies the condition by which we might, without contradiction, establish a limit. This 'limit' amounts to nothing more than an algebraic extension of the rational numbers. Insofar as it is a strictly algebraic entity, it is nothing more than a proper name. Its 'location' on the number line is determined by referring to the sequence terms, which are rational. If the sequence terms do not exhibit the correct 'behavior,' the supposition of an unique limit would be contradictory. Cauchy's criterion affords us a way to check whether or not a sequence satisfies the conditions necessary for such a supposition. Sequences cannot be given to us strictly in terms of their extension.[120] So, by what right do we affirm that such an object has been defined? Firstly, we must have a method of determining each term. The simplest example of this would be a sequence defined by a computable rule on the number n (such as $an = 1/n$). Other law-like methods would involve reference to other infinite sequences (such

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[119]See Heyting, pp. 13-17.

[120]No infinite sequence could be written down. Whether all sequences exist in pure extension is a separate matter - as we are not able to 'survey' such an object, we must have some means by which to judge the behavior of the sequence in general.

as $an = 1 + 1/2^n + 1/3^n + ...$) or definition by recursion. The question of what type of *law* we use is superfluous to our considerations. It is merely a 'vessel,' a means by which we can verify that a sequence satisfies Cauchy's criterion. This satisfaction is the key element in the supposition of a limit; provided we are justified in saying that for each *n*, the meaning of *an* obtains, nothing more need be supposed in order to be able to speak of a limit. There is no reason, however, to limit our definitions to sequences that are law-like. A choice is equally as good to define a term as is law, or so much as the 'throw of a die.' A sequence is convergent if it meets Cauchy's criterion - that it be law-like is an entirely superfluous issue. However, if we restrict our choice, as that outlined above, then there is no question of whether Cauchy's criterion is satisfied. The choice could at least be considered empirically objective, and hence would serve to unambiguously establish the value of *an*, for any particular *n*. Hence it would seem that the construction of a real number by means of a choice sequence should in principle satisfy Russell's understanding.

Concerning the principle of unification over the rational terms of a sequence, we must either admit strict extensionalism, or appeal to some condition which constitutes the limit as something that can be algebraically combined with rational values. The mind of the acting mathematician serves as such a principle of unification. This infinite collection of serially ordered objects can be unified by a singular synthetic act undertaken by an agent who is fully self-conditioning, and is in this sense free. It is true that a criterion of convergence assures the mathematician that there is no contradiction in the presupposition of a limit, and perhaps may *inspire* the mathematician to 'insert' a point where there is none. However, these conditions do not, as we have discussed, account for the existence of a transcendental field-element, the third required entity, which is neither the set of sequence terms, nor the law governing their progression. As we have shown, to consider **R** as an uncountable set, we must accept the existence of non-law-like sequences (i.e., sequences for which there is no general rule which defines a map from *n* to *an*). The predicativist (i.e., anti-strict extensionalist) position therefore tacitly assumes that, in the instance of transcendental value, either we can, in one singular act, posit a limit for such a convergent sequence, or that we must content ourselves with a incomplete (that is, 'in becoming') set of objects (in particular, the above mentioned equivalence classes) which constitute the logically reducible points on the geometric line.

The spatially defined magnitude associated with π becomes a real number, i.e., an algebraically combinable element of **R**, only upon the instance of such a free synthetic (unifying) act. It *exists* insofar as we *posit* it. The existence of a transcendental value is really an act of algebraic extension, in which a free choice is made to include an additional constituent term in the system, thereby changing the system. Consider the fact that there is no simpler way to write $5 + \pi$ than as *such.*[121]

Real analysis and complex analysis are not mutually incoherent algebraic systems; in fact, one is isomorphic to a proper sub-field of the other. The subject of real analysis should

[121]Of course, this example demonstrates little, in that we might adopt a style of nomenclature which does not keep the numbers separate. However, the term 'complex number' indicates that in the case of $5 + i$, the sum simply does *not* fold into a single term. Such an algebraic entity is rightly considered to be complex.

not be condemned because complex numbers have been successfully defined without contradiction. In the instance of our study of the real numbers, we merely suspend this act of *extension* in the case of $\sqrt{-1}$. It is true that a necessary lattice of algebra proceeds from the act of positing a value which satisfies the equation $x^2 + 1 = 0$, and that the necessary structure of the complex numbers forming upon the inclusion of this value is universal. Nevertheless, this initial act of positing the value is free. It is possible to imagine the development of mathematics having carried on without the advent of the 'imaginary number' ever occurring. Of course, our mathematics would be significantly impoverished if this were the case. However, this does not provide grounds to suppose that the stipulation of this value was necessarily in the destiny of mankind. The mathematics of the ancient Greeks for instance was, in every effective sense, coherent and viable. It is true that Gauss had the insight to see that by adding such a value, we develop a field in which every quadratic equation can be solved. The existence of the number i is however not traceable to this insight, but rather to Gauss' free *act* of *positing* this entity. Here we must be clear as to our intended meaning of the word 'freedom.' Surely, there we are not free to posit an object to satisfy an arbitrary collection of conditions - for the conditions may contradict one another, making the existence of such an object an impossibility. The structural features of the lattice of algebra which ensues from the positing of i as $\sqrt{(-1)}$ could not be otherwise. However, the question asking 'could we have defined the complex numbers differently?' is distinguished from that of whether the number i exists. It seems as though the necessity of the structure which follows such an act is independent of the choice of taking it on, so in a sense, numbers like i can be thought of as 'discovered.' However, it is not that we 'find an object,' given as such, but rather our discovery is of the possibility of a construction, which if carried out in accordance with certain specifications, can only result in a certain structure (or a certain range of possible structures). What we have discovered is not an object which need be, nor should be considered as truly external to us, but rather as something that is intrinsic to our own constructive capacities.

In virtue of what has been outlined above, the admission of arbitrary transcendental values in the predicative concept of **R** contains all the suppositions which would enable us to construct choice sequences. To restate an important fact, the real numbers without the transcendental values are countably infinite. It is essential that the 'continuum' which Russell inherits from Dedekind and Cantor must include these values. Whether the existence of transcendental values could only be obtained through synthetic acts of mind is not the crucial question - rather, what is essential is that the subject can serve as a source of generating a sequence. I contend that the subject is a required condition for the grounding of each instance of an uncountably infinite collection of such objects, in that, each of the uncountably many transcendental values presuppose the coherence of a free synthetic act. However, admission of such a claim is not necessary to the admission of choice sequences. Cauchy's criterion of convergence necessitates that as the sequence grows, successive terms become more alike, according to certain restriction. The law governing the sequence in question is merely a 'gadget' by which we can verify such convergence. We can reverse the construction, and specify ahead of time that the n^{th} term of a sequence α will be determined by a free choice selected form the interval of rational values which differ from the n-1th term of the sequence by less than $1/2^n$. Not understood as being given in full extension, we need only specify a method by which the n^{th} can be determined. Insofar as a finite segment is concerned, any method for determining the n^{th} term will suffice: the throw of a die, a law, or a free choice all serve equally to establish a finite set of terms. If we restrict our choice to ever shrinking intervals as above, we have satisfied Cauchy's condition 'ahead of the question' - no reference need be made between the arbitrary values of *an*, since, by construction, we have already satisfied the requirement of convergence.

We know that the reals are not exhausted by the law-like sequences, and the free choice of a subject is one means by which we could characterize a sequence as non-law-like (if we appeal to throws of the die, the appeal to physics and mechanism could usurp the claim that such processes are not law-like). In the denial of extensionalism, the admission of a free act is likely the best grounds to legitimize a condition for the determination of a mathematical object such as a transcendental number. However, this need not be the case to admit this basis for mathematical construction, for, insofar as any finite segment is concerned, that 'free choices' can serve to establish a sequence can be verified by anyone with a pen, and spare time. We abstract in this case from the individual person, and appeal to an idea of a non-terminating series of choices, such that, regardless of the size of n, the n+1th term can be chosen.

If it were the case that no new values were defined by choice sequences, then perhaps the distinctions which I have drawn would bear little relevance to the situation. However, as in the case of Brouwer's famous pendulum number, it has been shown that choice sequences do indeed define new values which are not real numbers in the classical sense. The intensional character of choice sequence is not subsumed by the alleged 'comprehensiveness' of Cantor's continuum. We could only define such numbers by the appeal to free choice, and hence **R**, or any law-like system for that matter would omit such values. To accept Russell's reconciliation of the discontinuity of the rational numbers with the insertion of a point at each instance of an irrational cut however is to accept the basic presuppositions on which the construction of choice sequences rely. The grounds for constructing this crucial counter-example are implicit in the basic assumptions of the method of Russell's logical reduction of the irrational real numbers, and because of this we must reject Russell's conception of continuity, and therefore his analysis of extensive magnitude.

We must point out, before closing the current Chapter, that choice sequences apparently cannot be accounted for in Kant's epistemology either. In fact, the claim seems to be specifically denied in the 'Second Analogy of Experience' in the first *Critique* when Kant discusses causation as serial succession. Here, he explicitly claims, that in order for a sequence to be ***objective***, there must be a rule to be met with in it:

> "The objective succession will therefore consist in that order of the manifold of appearance according to which, *in conformity with a rule*, the apprehension of that which happens follows upon the apprehension of that which precedes. Thus only could I be justified in asserting, not merely of my apprehension, but of appearance itself, that a succession is to be met with in it. "[122]

For a thing to be considered as an object of mathematical discourse, it seems clear that it be

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[122]CPR: [A193/B238]

objective (i.e., the same for all parties who would contemplate it.) Here, Kant, more explicitly than Russell, rules out choice sequences. Hence, if we are prepared to admit choice sequences as legitimate mathematical objects, then we cannot conclude, overall, that Kant had a better understanding of the continuum than did Russell; we might at best consider him to have a better conception of the continuum insofar as it relates to its divisions. Whereas Russell can conduct his system without contradictions, most constructivists would say that the above quote, coupled with those above (in Chapter 2) constitutes an outright contradiction. It would seem that neither Russell, nor Kant had a perfect understanding of the nature of the continuous line. As for the exhaustibility of the points on the line, however, it would seem, in virtue of the discussion in Chapter 4, that Kant was closer to the truth than was Russell. We cannot attach too much glory to Kant for knowing this, however, as the mathematics of the early 20th century[123] suggests, for his own doctrine is apparently inconsistent on this point.

[123]I.e., Brouwer, Heyting, Troelstra: the intuitionists.

Chapter 6: Conclusion:

To summarize, let us review the steps of our discussion.

In the first Chapter, mathematical realism is characterized as an ontological position which holds the existence of mathematical entities to be utterly fixed and their totality as constituting a real and complete world, independent of both cognition and the acts of entities who cogitate. Russell's foundational program, Logicism, is one species of mathematical realism, in which (1) mathematical entities are considered to be reducible to logical entities, and (2) the existence of logical entities is held to be utterly fixed and mind independent. At the end of the first chapter, the conjecture is made that Russell's Logicism fails to give an adequate account of the geometric continuum and therefore is an untenable position regarding the foundations of mathematics.

In the second Chapter, the identification of continuity with the formal nature of extensive quantity in connexion with the philosophy of Immanuel Kant is established, followed by an exegesis of the development of the mathematical continuum in the 19^{th} Century with respect to Cantor's notion of the continuum as an infinite collection (and therefore as constituting a transfinite '*number*'). Succeeding this was an account of Dedekind's notion of continuity as an essentially ordinal *determination,* which can be characterized as a form of *series*.

The third Chapter begins with a description of an important epistemological divergence between Russell and Kant, in particular on the connexion of *a priori* truth with synthetic judgements (namely, that Kant held there to be synthetic judgments which are also necessary, whereas Russell held all *a priori* judgements to be analytic). After establishing this general theme in Russell's epistemology, an account of the technique by which Russell reduces the geometric continuum to logical principles is presented.

In the fourth Chapter, Russell's position is criticized on the grounds of the presuppositions and circularities tacitly contained in his argument. In particular, his logical reduction of the geometric continuum is argued to be circular in that it implicitly presupposes a determinate continuous line segment as a first principle. It is there argued that this necessarily primordial element is essentially obtained through an act of intuition, and can be given in no other way.

In the fifth Chapter, the presuppositions of Russell's predicative conception of a real number, (particularly in the case of a transcendental irrational) are examined. It is there argued that such presuppositions are grounds for the admission of real values that are defined by choice sequences which are necessarily lawless, and by definition usurp the authority which any law-like encapsulation of the continuum (such as the axiomatic conception of the real numbers of classical mathematics) might have. Furthermore, we have shown that the admission of such objects, as would seem to be necessitated by the acceptance of the anti-atomic picture of the continuum, is ruled out in Kant's own system. From this, we might infer that (if one is prepared to admit choice sequences as legitimate mathematical entities) Russell's theory is simply deficient, whereas Kant's is inconsistent. Finally we conclude from this that any law-like presentation of the real numbers in which a (transcendental) irrational value is defined in terms of a sequence of rational approximations either presupposes that all such objects can be given in extension (strong Platonism), or that free synthetic acts are admissible grounds to constitute the determination of a real-value. In other words, we cannot have a system which both rules out arbitrary extensions, and allows for transcendental real numbers without allowing free choice as a possible method for mathematical construction. Hence any project such as Russell's Logicism cannot, in principle, result in a fully law-like classification of all possible positions on the geometric line (in that such characterizations would result in a countably infinite set of points). The reduction of the transcendental reals to logical principles, insofar as they are to represent possible extensive determinations, opens the door to acts undertaken by the free synthetic subject as a basis for mathematical construction.

The main conjecture here is not that we should abandon the analytical theory of the reals numbers as such, for this would be to commit a grave disservice to the thinkers who have contributed to the development to what is perhaps the most central concept in all higher mathematics. The profundity and elegance of this instrument is in no way undermined by what has been said. What has been undertaken in the present essay is an elucidation of the primordial and irreducible connexion which this structure bears to the intuitively given continuum. It is his failure to grasp this point to which Russell's misapprehension of the nature of space and time can be traced. This failure is accounted for by the fact that a principle of intuition is missing in Russell's general epistemology.

Appendix

A relation R is said to be:

(1) *asymmetric* if any instance of aRb excludes the case of bRa (except in the trivial case where $a = b$)

(2) *transitive* if for every case in which aRb and bRc, implies the relation aRc.

If A and B are sets, then their cross product ($A \times B$) is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. A function, or map from a set A to a set B is a set $F \subseteq A \times B$ such that
$\forall\, a \in A\ \exists!\, b \in B : (a, b) \in F$

A binary operation is a map from the cross product of a set to itself ($F \subseteq (A \times A) \times A$). In other words, a binary operation is a rule associating with every pair of elements in a set a third element.

Group axioms[124]

Let G be a set and suppose '+' is a binary operation defined on G.
G is said to be a group if the first three of the following four axioms are satisfied:

1. $\exists!\, e \in G : \forall\, x \in G\ x + e = e + x = x$ (identity)
2. $\forall\, x \in G\ \exists\, y \in G : x + y = y + x = e$ (existence of the inverse)
3. $\forall\, x \in G, \forall\, y \in G, \forall\, z \in G : (x + y) + z = x + (y + z)$ (associativity)
4. $\forall\, x \in G, \forall\, y \in G : x + y = y + x$ (commutativity)

If axiom 4 is satisfied, G is said to be an Abelian, or commutative Group. For instance the integers under normal arithmetic 'addition' are such a structure:

-the sum of every two whole numbers is a whole number (closure)
-0 is an integer, and for any integer x, $x + 0 = x$; no integer other than zero has this property (unique identity element)
-the equation $(x+y)+z = x+(y+z)$ holds for all integers x,y,z (associativity)
-the integers are an Abelian group since the order of the summands is irrelevant to the summation of quantities (commutativity)

[124]See Issacs p.9.

A *ring*[125] is an *Abelian group* (R) with a second associative operation * (denoted by juxtaposition) that is distributive over addition:

1. $\forall x \in R, \forall y \in R, \forall z \in R : x(y + z) = (xy) + (xz)$.
2. $\forall x \in R, \forall y \in R, \forall z \in R : (x + y)z = (xz) + (yz)$.

A *field*[126] is a *ring* in which (i) the operation of multiplication is commutative (ii) there is a unique identity element of multiplication f such that $fx = x$ for all x in R, and (iii) for every non-zero b there exists a unique solution z to every equation $bz = f$ in R.

Note, the integers satisfy property (i) since $1x = x$ for all x, but not property (ii); eg, $4x = 1$ has no solution in the whole numbers. The rational numbers (that is the whole number fractions: a/b where a and b are integers and $b \neq 0$) do form a field:

$a/b + x/y = (ay + xb)/by$: the sum of any two rational numbers is also rational number
$(a/b)*(x/y) = ax/by$: the product of any two fractions is also a fraction
$(a/b)x = 1$ is solved when $x = b/a$

The Real Numbers (**R**) are defined by the following axioms, where standard numerals denote particular numbers, and lowercase Latin letters denote variables, or arbitrary values:

Axiom 1. $\exists\, 0 \in \mathbf{R} : \forall x \in \mathbf{R}\ 0 + x = x$
Axiom 2. $\exists\, 1 \in \mathbf{R} : \forall x \in \mathbf{R}\ 1x = x$
Axiom 3. $\forall x, y \in \mathbf{R}$[127] $: x + y = y + x$
Axiom 4. $\forall x, y \in \mathbf{R} : xy = yx$
Axiom 5. $\forall x, y, z \in \mathbf{R} : (x + y) + z = x + (y + z)$
Axiom 6. $\forall x, y, z \in \mathbf{R} : (xy)z = x(yz)$
Axiom 7. $\forall x, y, z \in \mathbf{R} : x(y + z) = xy + xz$
Axiom 8. $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R} : x + y = 0$
Axiom 9. $\forall x \in \mathbf{R} : x \neq 0\ \exists y \in \mathbf{R} : xy = 1$
Axiom 10. $\forall x, y \in \mathbf{R}\ (x = y) \vee (x < y) \vee (y < x)$[128]
Axiom 11. $\forall x, y, z \in \mathbf{R}\ (x < y) \wedge (y < z) \Rightarrow (x < z)$

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[125]See Issacs p.159.

[126]See Issacs pp.165-8.

[127]In the interest of presentational clarity, I have adopted the script $\forall x, y \in \mathbf{R}$ to denote $\forall x \in \mathbf{R}\ \forall y \in \mathbf{R}$.

[128]We should note that these three relational states are mutually exclusive; exactly one of the three states must subsist between any two values, x and y.

Axiom 12. $\forall x, y, z \in \mathbf{R}\ (x < y) \Rightarrow [(x + z) < (x + y)]$
Axiom 13. $\forall x, y, z \in \mathbf{R}\ (0 < z) \wedge (x < y) \Rightarrow (xz < yz)$

Let us define the relation '$\Subset$' as follows: given a real number c and an non-empty subset X of **R**, we write $X \Subset c$ if and only if $\forall x \in X : (x \leq c)$. If this is so, we say that c is an upper bound of X.

Axiom 14. $\forall S \subseteq \mathbf{R}$: $[\exists y \in \mathbf{R}: (S \Subset y)] \Rightarrow \exists!\, b \in \mathbf{R} : [(S \Subset b) \wedge (\forall x \in \mathbf{R}\ (S \Subset x) \Rightarrow (b \leq x))]$

We should note that axioms 1 through 13 are satisfied by the set of rational numbers. The main axiom concerned with continuity is number 14.

A cut on **R** is a pair of non-empty, mutually-exclusive, all-inclusive subsets of **R** such that $\forall\, a \in A, \forall\, b \in B\ a < b$. A cut on the rational numbers is defined in the same way.

Proof of the equivalence of Dedekind's criterion (for every cut (A, B), there exists a unique q such that $\forall\, a \in A, \forall\, b \in B\ a \leq q \leq b$) with the Supremum principle (Axiom 14):

($\Leftarrow$) Suppose axiom 14 is true of **R**. Let (A, B) be a cut on **R**. (i.e., $\forall\, a \in A, \forall\, b \in B\ a < b$, A and B are both nonempty)

Since B is non-empty, A must have an upper bound. Therefore, by axiom 14, there must exist a unique least upper bound r of A. By definition, $\forall\, a \in A$, $a \leq r$. It remains now to show that $\forall\ b \in B, r \leq b$. Let $b \in B$; by the definition of a cut, b must be an upper bound for A; however since r is the least upper bound of A, $r < b$ must be false: by axiom 10, $r \leq b$, which was to be proved.

($\Rightarrow$) Suppose Dedekind's Criterion is true. Let $X \subseteq \mathbf{R}$ be a non-empty subset that is bounded above.

In the case that X has a maximal element, the problem is trivial. Let us suppose there is no maximal element of X. Consider $G = \{y \in \mathbf{R}: \forall\, x \in X, x \leq y\}$, the set of all upper bounds of X; since X has no maximal element, we may equivalently write $G = \{y \in \mathbf{R}: \forall\, x \in X, x < y\}$. Since X is bounded above, G is non-empty. Consider $H = \mathbf{R}\text{-}G = \{y \in \mathbf{R}: \exists\, x \in X, y \leq x\}$: since X is non-empty, so is H. Therefore, G and H constitute a cut on **R**: neither is empty, they are mutually- exclusive and all-inclusive because H is the compliment of G in **R** and $\forall\, h \in H, \forall\, g \in G\ \ \exists\, x \in X$: $h \leq x$. By definition, $x < g$, therefore by axiom 11, $h < g$. By our assumed premise, there exists a unique q such that $\forall\, h \in H, \forall\, g \in G\ h \leq q \leq g$. It remains now to show that q is the least upper bound for X. The set $X \cap G$ is empty, therefore $X \subseteq H$. If p is an upper bound of X, then $p \in G$, therefore $q \leq p$: in other words, q is the least upper bound of X, which was to be proved.

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