GROUP GRADINGS ON SIMPLE LIE ALGEBRAS OF CARTAN AND MELIKYAN TYPE

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Group Gradings on Simple Lie Algebras of Cartan and Melikyan Type

by

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Abstract

In this thesis we explore the gradings by groups on the simple Cartan type Lie algebras and Melikyan algebras over algebraically closed fields of positive characteristic $p > 2$ ($p = 5$ for the Melikyan algebras).

We approach the gradings by abelian groups without p-torsion on a simple Lie algebra L by looking at the dual group action. This action defines an abelian semisimple algebraic subgroup (quasi-torus) of the automorphism group of L. A result of Platonov says that any quasi-torus of an algebraic group is contained in the normalizer of a maximal torus. We show that if L is a simple graded Cartan or Melikyan type Lie algebra, then any quasi-torus of the automorphism group of L is contained in a maximal torus. Thus all gradings by groups without p-torsion are, up to isomorphism, coarsenings of the eigenspace decomposition of a maximal torus in the automorphism group.

We also give examples of gradings by the cyclic group of order p which do

not follow the pattern of the general description of gradings by groups without p-torsion as well as describe gradings by arbitrary groups on the restricted Witt algebra $W(1;1)$.

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Introduction

The task of finding all gradings on simple Lie algebras by abelian groups in the case of algebraically closed fields of characteristic zero is almost complete (see [11] and also [5, 6, 7, 8, 9, 10, 12]). In the case of positive characteristic, a description of gradings on the classical simple Lie algebras, with certain exceptions, has been obtained in [1], [3], [4]. The gradings on the full matrix algebras over fields of characteristic $p \neq 2$ were important in the classification of classical Lie algebras. The gradings on the classical Lie algebras of type A_l, B_l, C_l and D_l (with exceptions for D_4), realised as matrix algebras, are the restrictions of gradings on the full matrix algebras. In the case of simple Cartan type Lie algebras, the gradings by $\mathbb{Z}$ have been described in [27]. All of them, up to isomorphism, fall into the category of what we call *standard* gradings (which are coarsenings of the canonical $\mathbb{Z}^k$-gradings) (see Section 2.2). The simple Cartan type Lie algebras can be viewed as subalgebras of the derivations of commutative algebras, $O(m; \underline{n})$, consisting of truncated polynomials in m variables. Using automorphism group schemes, the gradings on the restricted graded Witt and special type algebras are described in [2] as gradings induced by gradings on $O(m; \underline{n})$.

This paper will primarily deal with gradings on the simple graded Cartan or Melikyan type Lie algebras by arbitrary abelian groups without p-torsion in the case where the base field F (which is always assumed to be algebraically closed), has characteristic $p > 2$. Our main result is showing that all gradings by a group G, without p-torsion, on a simple graded Cartan or Melikyan type Lie algebra (with some exceptions when $p = 3$) are isomorphic to standard G-gradings. A standard G-grading is induced by a standard G-grading on $O(m; \underline{n})$. We will also give examples of gradings by groups with elements of order p.

The structure of this work is as follows. Chapter 1 contains basic definitions and describes the correspondence between the gradings on an algebra by finitely generated abelian groups without p-torsion and semisimple abelian algebraic subgroups (quasi-tori) of the automorphism group of this algebra. We also discuss the correspondence between the gradings on an algebra by finite elementary p-groups and semisimple derivations of L which is similar to the gradings by groups without p-torsion. Section 2.1 contains the definitions of the simple graded Cartan type Lie algebras and Melikyan algebras. The definitions of the standard gradings of these algebras are given in Section 2.2. The known results needed for the automorphism groups of these algebras are in Section 2.3 and the known results needed for the derivations are in Section 2.4. In Chapter 3 we show that a quasi-torus of the automorphism group of L is contained in a maximal torus (which is not true in general). Our main theorem is proven in Section 4.1 and we also give examples of gradings that are not standard when the grading group has order p. We finish by describing all gradings by arbitrary groups on the restricted Witt algebra $W(1;1)$ when $p > 3$ in Section 4.2. The description of the gradings by arbitrary groups on $W(1;1)$ is possible because for any non trivial $\mathbb{Z}_p$-grading, the homogeneous spaces are one dimensional.

We use the notation of [27], which is our standard reference for the background on Cartan type and Melikyan Lie algebras.

Chapter 1

Gradings and Actions

Throughout this work the base field of coefficients F is always algebraically closed and its characteristic is p. All vector spaces will be over F unless otherwise specified. Unless it is stated otherwise, we denote by i, j, k, l, q, s some integers. We also denote by $\mathbb{Z}_l$ the group of integers modulo l for $l > 0$.

1.1 Simple Lie Algebras

Definition 1.1.1. A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ and called the bracket or commutator of x and y, is called a *Lie algebra over F* if the following axioms are satisfied:

1. The bracket operation is bilinear.

2. $[x, x] = 0$, $x \in L$.

3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad x, y, z \in L$.

For a fixed $x \in L$, we denote by $\operatorname{ad} x$ the endomorphism of L which sends $y \in L$ to $[x, y]$.

Definition 1.1.2. A *derivation of an algebra* A is any endomorphism D satisfying $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. The set of all derivations of A is denoted by $\operatorname{Der}(A)$.

Remark 1.1.3. For any algebra A, the set $\operatorname{Der}(A)$ is closed under the operation $[D, E] = DE - ED$ so $\operatorname{Der}(A)$ is a Lie algebra. For a Lie algebra L, we have $\operatorname{ad} L \subset \operatorname{Der} L$.

Definition 1.1.4. An *ideal* I of a Lie algebra L is a subspace of L such that $[x, y] \in I$ for all $x \in L$ and $y \in I$. A Lie algebra L is said to be *simple* if $[L, L] \neq 0$ and L has no ideals other than L and 0.

Definition 1.1.5. For a Lie algebra L we define the *derived series* of L by setting $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 1$.

The following algebras in definitions 1.1.6, 1.1.7 and 1.1.8 are called the *classical simple Lie algebras over* $\mathbb{C}$. Let $\operatorname{Mat}(m, \mathbb{C})$ be the full $m \times m$ matrix algebra over $\mathbb{C}$. For $x, y \in \mathrm{Mat}(m, \mathbb{C})$ set $[x, y] := xy - yx$ and x^t the transpose of x.

Definition 1.1.6. Let $\mathfrak{sl}(m, \mathbb{C})$ be the Lie algebra consisting of all elements x in $\mathrm{Mat}(m, \mathbb{C})$ with trace 0. We call $\mathfrak{sl}(m, \mathbb{C})$ the *special linear algebra*. Lie algebras isomorphic to $\mathfrak{sl}(m, \mathbb{C})$ are called of type A_l where $l = m - 1$.

Definition 1.1.7. Let $\mathfrak{so}(m, \mathbb{C})$ be the Lie algebra consisting of all elements x in $\mathrm{Mat}(m, \mathbb{C})$ such that $x = -x^t$. We call $\mathfrak{so}(m, \mathbb{C})$ the *orthogonal algebra*. Lie algebras isomorphic to $\mathfrak{so}(m, \mathbb{C})$ are called of type B_l when $m = 2l + 1$ and of type D_l when $m = 2l$.

Definition 1.1.8. Let $m = 2l$, $P = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and $\mathfrak{sp}(m, \mathbb{C})$ be the Lie algebra consisting of all elements x in $\mathrm{Mat}(m, \mathbb{C})$ such that $Px = -x^t P$. We call $\mathfrak{sp}(m, \mathbb{C})$ the *symplectic linear algebra*. Lie algebras isomorphic to $\mathfrak{sp}(m, \mathbb{C})$ are called of type C_l.

The other types of simple Lie algebras over $\mathbb{C}$ are E_6, E_7, E_8, F_4 and G_2 (see [14]).

The following discussion on simple Lie algebras is a summary of [27, Section 4.1].

Let L be a simple finite-dimensional Lie algebra over $\mathbb{C}$, H a Cartan subalgebra, κ the Killing form and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base of simple roots. For all α, β in the root system R we set

$$\kappa(t_\alpha, h) := \alpha(h), \quad h \in H,$$

$$\langle \alpha, \beta \rangle := \frac{2\kappa(t_\alpha, t_\beta)}{\kappa(t_\beta, t_\beta)},$$

$$h_i := \frac{2t_\alpha}{\kappa(t_{\alpha_i}, t_{\alpha_i})}.$$

Theorem 1.1.9. [27, Theorem 4.1.1] Let L be a simple finite-dimensional Lie algebra over $\mathbb{C}$. Then there is a basis $\{x_\alpha, h_i \mid \alpha \in R,\ 1 \leq i \leq l\}$ of L such that

1. $[h_i, h_j] = 0, \quad 1 \leq i, j \leq l,$

2. $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha, \quad 1 \leq i \leq l, \quad \alpha \in R$

3. $[x_\alpha, x_{-\alpha}] = h_\alpha$ is a $\mathbb{Z}$-linear combination of $h_1, \ldots, h_l$.

4. If α, β are independent roots and $\beta - r\alpha, \ldots, \beta + q\alpha$ is the α-string through β, then $[x_\alpha, x_\beta] = 0$ if $q = 0$, while $[x_\alpha, x_\beta] = \pm(r+1)x_{\alpha+\beta}$ if $\alpha + \beta \in R$. □

We call the basis in Theorem 1.1.9 the *Chevalley basis*. Let $L_\mathbb{Z}$ be the $\mathbb{Z}$-span of the Chevalley basis. The tensor product $L_F = F \otimes_\mathbb{Z} L_\mathbb{Z}$ is a Lie algebra. If L is not of type A_l where p divides $l+1$, then L_F is simple. If L is of type A_l and p divides $l+1$ then L_F has a one dimensional center $C(L_F)$ and $L_F/C(L_F)$ is simple. The simple Lie algebras of the form $L_F/C(L_F)$ are called the *classical simple Lie algebras over F*.

For simple Lie algebras over fields of positive characteristic, one has, in addition to the classical simple Lie algebras, the simple Lie algebras of Cartan type, and in characteristic 5 also of Melikyan type, which will be introduced in Chapter 2.

1.2 Gradings

For an algebra A and a group G, we denote by $\operatorname{Aut} A$ and $\operatorname{Aut} G$ the automorphism groups of A and G respectively.

Definition 1.2.1. A *grading Γ on an algebra A by a group G*, also called a *G-grading*, is the decomposition of A as the direct sum of subspaces A_g,

$$\Gamma : A = \bigoplus_{g\in G} A_g,$$

such that $A_{g'}A_{g''} \subset A_{g'g''}$ for all g', $g'' \in G$. For $g \in G$, the subspace A_g is called the homogeneous space of degree g, and any nonzero element $y \in A_g$ is called *homogeneous of degree g*. A subspace U of A is called *a graded subspace* if

$$U = \bigoplus_{g \in G} (U \cap A_g).$$

If U is a subalgebra and $\Gamma' : U = \bigoplus_{g \in G}(U \cap A_g)$ we say that the grading Γ' is a *restriction of* Γ *to* U. We say that a homomorphism of A respects the grading Γ if the homogeneous spaces are invariant.

Example 1.2.2. Let $L = \mathfrak{sl}(2, \mathbb{C}) = \mathrm{Span}\{x_{\alpha_1}, x_{-\alpha_1}, h_1\}$ where $[x_{\alpha_1}, x_{-\alpha_1}] = h_1$, $[h_1, x_{\alpha_1}] = 2x_{\alpha_1}$ and $[h_1, x_{-\alpha_1}] = -2x_{-\alpha_1}$. Set $L_{-1} = \mathrm{Span}\{x_{-\alpha_1}\}$, $L_0 = \mathrm{Span}\{h_1\}$, $L_1 = \mathrm{Span}\{x_{\alpha_1}\}$ and $L_i = 0$ otherwise. Then $\Gamma : L = \bigoplus_{i \in \mathbb{Z}} L_i$ is a $\mathbb{Z}$-grading. This grading corresponds to a Cartan decomposition on L with respect to the Cartan subalgebra L_0 (see [14, Section 8]).

Definition 1.2.3. The set $\mathrm{Supp}\,\Gamma = \{g \in G \mid A_g \neq 0\}$ is called the *support* of the grading $\Gamma : A = \bigoplus_{g \in G} A_g$. By $\langle \mathrm{Supp}\,\Gamma \rangle$ we denote the subgroup of G generated by $\mathrm{Supp}\,\Gamma$.

Example 1.2.4. Let L and its $\mathbb{Z}$-grading Γ be as in Example 1.2.2. Then $\mathrm{Supp}\,\Gamma = \{-1, 0, 1\}$ and $\langle \mathrm{Supp}\,\Gamma \rangle = \mathbb{Z}$.

Remark 1.2.5. It is well known [7, Lemma 2.1], for a grading Γ by a group G on a simple Lie algebra, that $\langle \mathrm{Supp}\,\Gamma \rangle$ is an abelian group.

Remark 1.2.6. Let A be a finite-dimensional algebra, $\Gamma : A = \bigoplus_{g\in G} A_g$ a grading by a group G. Then the support is a finite set. This implies that $\langle \operatorname{Supp}\Gamma \rangle$ is finitely generated.

From now on, we will assume that the grading group G is abelian and finitely generated.

Definition 1.2.7. Let $\Gamma : A = \bigoplus_{g\in G} A_g$ be a grading by a group G on an algebra A and ϕ a group homomorphism of G to H. The *H-grading induced by ϕ* is the decomposition $A = \bigoplus_{h\in H} \overline{A}_h$ where

$$\overline{A}_h = \bigoplus_{g\in G,\ \phi(g)=h} A_g.$$

The above H-grading is also refered to as the *coarsening* of the G-grading (induced by ϕ). Conversely, a *refinement of an H-grading* $A = \bigoplus_{h\in H} \overline{A}_h$ is a G-grading $A = \bigoplus_{g\in G} A_g$ such that there exists a group homomorphism $\phi : G \to H$ such that $\overline{A}_h = \bigoplus\limits_{g\in G,\ \phi(g)=h} A_g$.

Definition 1.2.8. Two gradings $A = \bigoplus_{g\in G} A_g$ and $A = \bigoplus_{h\in G} A'_h$ by a group G on an algebra A are called *group-equivalent* if there exist $\Psi \in \operatorname{Aut} A$ and $\theta \in \operatorname{Aut} G$ such that $\Psi(A_g) = A'_{\theta(g)}$ for all $g \in G$. If θ is the identity, we call the gradings *isomorphic*.

We want to find all gradings on a Lie algebra up to group-equivalence (or, if possible, up to isomorphism) because the difference between equivalent gradings is a matter of relabelling.

There is also another type of equivalence that appears in the literature. For an explanation on the difference of the two concepts and a more general view of gradings (not only by groups) see [16, Section 3].

Definition 1.2.9. We will call two gradings $\Gamma : A = \bigoplus_{g\in G} A_g$ and $\Gamma' : A = \bigoplus_{h\in H} A'_h$ on an algebra A by groups G and H, respectively, *equivalent* if there exist $\Psi \in \operatorname{Aut} A$ and a bijection $\theta : \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma'$ such that $\Psi(A_g) = A'_{\theta(g)}$ for all $g \in G$.

The equivalence relation between gradings is weaker than the relations of group-equivalence since the map θ need only be a bijection between the supports, which is not necessarily the restriction of a group automorphism. In Example 1.2.10 we give group-equivalent gradings with $\theta \neq \mathrm{Id}$, which are, in fact isomorphic gradings. In Example 1.2.11 we give group-equivalent gradings that are not isomorphic. Finally in Example 1.2.12 we give a grading induced by a group homomorphism that is not equivalent to the original grading and another grading induced by a group homomorphism that is equivalent but not group-equivalent to the original grading.

Example 1.2.10. Let L and its $\mathbb{Z}$-grading Γ be as in Example 1.2.2. Let ϕ be the group automorphism of $\mathbb{Z}$ defined by $\phi(i) = -i$ for all $i \in \mathbb{Z}$. By setting $L'_{\phi(i)} = L_i$ we have another $\mathbb{Z}$-grading $L = \bigoplus_{i\in\mathbb{Z}} L'_i$, which is group-equivalent to Γ. It turns out that these two gradings are also isomorphic. Let Ψ be the automorphism of L defined by $\Psi(x_{\alpha_1}) = -x_{-\alpha_1}$, $\Psi(x_{-\alpha_1}) = -x_{\alpha_1}$ and $\Psi(h_1) = -h_1$. Then $\Psi(L_i) = L'_i$. Hence the gradings are isomorphic.

Example 1.2.11. Let $L = \mathrm{Span}\{y_{-1}, \ldots, y_3\}$ be a Lie algebra over a base field of characteristic 5 where $[y_i, y_j] = (j-i)y_{i+j}$, with the convention $y_k = 0$ if $-1 > k$ or $k > 3$ (L is $W(1;\underline{1})$ — see Definition 2.1.3). Setting $L_i = \mathrm{Span}\{y_i\}$ we have a $\mathbb{Z}$-grading on L. Let ϕ be the automorphism of $\mathbb{Z}$ defined by $\phi(i) = -i$. We get a $\mathbb{Z}$-grading $\Gamma' : L = \bigoplus_{i\in\mathbb{Z}} L'_i$ by setting $L'_{\phi(i)} = L_i$. The two gradings are group-equivalent but not isomorphic since they have different supports.

Example 1.2.12. Let $\Gamma : L = \bigoplus_{i\in\mathbb{Z}} L_i$ be as in Example 1.2.11 and let $\phi : \mathbb{Z} \to \mathbb{Z}_2$ be the homomorphism of groups defined by $\phi(i) = [i]_2$ where $[i]_2$ is i modulo 2. The $\mathbb{Z}_2$-grading induced by ϕ is $L = \overline{L}_{[0]_2} \oplus \overline{L}_{[1]_2}$ where $\overline{L}_{[0]_2} = \mathrm{Span}\{y_0, y_2\}$ and $\overline{L}_{[1]_2} = \mathrm{Span}\{y_{-1}, y_1, y_3\}$. The $\mathbb{Z}_2$-grading described has different homogeneous spaces than the $\mathbb{Z}$-grading. Hence the $\mathbb{Z}_2$-grading is not equivalent to the original $\mathbb{Z}$-grading (which implies that they are not group-equivalent). Now consider $\psi : \mathbb{Z} \to \mathbb{Z}_5$ defined by $\psi(i) = [i]_5$ where $[i]_5$ is i modulo 5. The $\mathbb{Z}_5$-grading induced by ψ is $L = \bigoplus_{[i]_5 \in \mathbb{Z}_5} L'_{[i]_5}$ where $L'_{[i]_5} = L_k$ for $-1 \leq k \leq 3$ and $[k]_5 = [i]_5$. The $\mathbb{Z}_5$-grading is equivalent to the $\mathbb{Z}$-grading but not group-equivalent.

Lemma 1.2.13. Let $G = G_1 \times G_2$ be the direct product of groups and $\pi_1 : G \to G_1$, $\pi_2 : G \to G_2$ the canonical homomorphisms defined by $\pi_i(g_1, g_2) = g_i$ for $i = 1, 2$. If $A = \bigoplus_{g \in G} A_g$ is a G-grading on an algebra A and $A = \bigoplus_{g_1 \in G_1} A'_{g_1}$, $A = \bigoplus_{g_2 \in G_2} A''_{g_2}$ are the gradings by G_1, G_2 induced by π_1, π_2 respectively then $A_{(g_1,g_2)} = A'_{g_1} \cap A''_{g_2}$. $\square$

1.3 Actions by Automorphisms and Derivations

From now on G will be a finitely generated abelian group. In this section we introduce the action by the dual group $\widehat{G}$ on an algebra A graded by a group G without p-torsion, so that the study of gradings by such abelian groups is equivalent to the study of actions of finitely generated abelian groups by automorphisms of A. We also introduce the action of an elementary p-subgroup of G by derivations if G has elements of order p.

We start by looking at the action by the dual group, $\widehat{G}$, when G is finite.

Definition 1.3.1. We call the group of homomorphisms $G \to F^{\times}$ the *dual group of* G and denote it by $\widehat{G}$. The elements of $\widehat{G}$ are called the multiplicative characters of G.

Let $\langle g_k \rangle$ be the cyclic group of order k generated by the element g.

Lemma 1.3.2. Let F be algebraically closed and G a finite abelian group. Assume G is without p-torsion if $p > 0$. We can express G and $\widehat{G}$ as

$$G = \langle g_1 \rangle_{k_1} \times \cdots \times \langle g_l \rangle_{k_l}, \tag{1.1}$$

$$\widehat{G} = \langle \chi_1 \rangle_{k_1} \times \cdots \times \langle \chi_l \rangle_{k_l}, \tag{1.2}$$

with $\chi_i : G \to F^{\times}$, $\chi_i(g_i) = \zeta_i$ and $\chi_j(g_i) = 1$ for $1 \leq i, j \leq l$, $i \neq j$ where ζ_i is a k_i-th primitive root of unity. □

Remark 1.3.3. In fields of positive characteristic p, the only p-th root of unity is 1 because $x^p - 1 \equiv (x-1)^p$ modulo p. Hence when $p > 0$, the dual group of $\mathbb{Z}_p$ is the trivial group of order one.

Lemma 1.3.4. The dual group of $\mathbb{Z}$ is

$$\widehat{\mathbb{Z}} = \{\chi_\alpha \mid \alpha \in F^{\times},\ \chi_\alpha(i) = \alpha^i,\ \forall i \in \mathbb{Z}\}$$

which is isomorphic to $F^{\times}$. □

Hence, the dual group of a finitely generated abelian group G is the direct product of a finite abelian group and an (algebraic) *torus* $(F^{\times})^k$ for some k. In particular, $\widehat{G}$ has a natural structure of an algebraic group.

We now define the action by $\widehat{G}$ on a G-graded algebra A. The following is well known.

Lemma 1.3.5. [16, Section 4] Let A be an algebra, G a finitely generated abelian group without p-torsion and $\Gamma : A = \bigoplus_{g \in G} A_g$ a G-grading on A. Then there is a homomorphism of $\widehat{G}$ into $\operatorname{Aut} A$ defined by the following action:

$$\chi * y = \chi(g)y, \quad \text{for all } y \in A_g, \quad g \in G, \quad \chi \in \widehat{G}.$$

The image of $\widehat{G}$ in $\operatorname{Aut} A$ is a semisimple abelian algebraic subgroup (*quasi-torus*). □

We will denote the homomorphism defined in Lemma 1.3.5 by $\eta_\Gamma : \widehat{G} \to \operatorname{Aut} A$, so

$$\eta_\Gamma(\chi)(y) = \chi * y, \quad \forall \chi \in \widehat{G}, \ y \in A. \tag{1.3}$$

Lemma 1.3.5 says that for a grading by a finitely generated group G without p-torsion there is a quasi-torus Q in $\operatorname{Aut} A$ isomorphic to $\widehat{G}$.

Remark 1.3.6. Given a quasi-torus Q in $\operatorname{Aut} A$, we obtain the eigenspace decomposition of A with respect to Q, which is a grading by the group of regular characters of the algebraic group Q, $G = \mathfrak{X}(Q)$.

Example 1.3.7. Let L and $\Gamma : L = \bigoplus_{i\in\mathbb{Z}} L_i$ be as in Example 1.2.11. Let η_Γ be as in (1.3). Lemma 1.3.5 says that for all $\alpha \in F^\times$ and $\chi_\alpha \in \widehat{\mathbb{Z}}$ defined by $\chi_\alpha(i) = \alpha^i$, $i \in \mathbb{Z}$, we have

$$\eta_\Gamma(\chi_\alpha)(y) = \alpha^j y, \quad \forall y \in L_j.$$

In other words, $\eta_\Gamma(\chi_\alpha)(y_i) = \alpha^i y_i$. Since

$$\begin{aligned}\eta_\Gamma(\chi_\alpha)([y_i, y_j]) &= \eta_\Gamma(\chi_\alpha)((j-i)y_{i+j}) = \alpha^{i+j}(j-i)y_{i+j} = \alpha^i\alpha^j[y_i, y_j] \\ &= [\alpha^i y_i, \alpha^j y_j] = [\eta_\Gamma(\chi_\alpha)(y_i), \eta_\Gamma(\chi_\alpha)(y_j)],\end{aligned}$$

it follows that $\eta_\Gamma(\chi_\alpha) \in \operatorname{Aut} L$, as expected.

If G is generated by the support of $\Gamma : A = \bigoplus_{g\in G} A_g$, then the homomorphism η_Γ is a closed embedding of algebraic groups.

Lemma 1.3.8. Let G, H be fintely generated groups without p-torsion, A an algebra, $\phi : G \to H$ a group homomorphism, $\Gamma : A = \bigoplus_{g\in G} A_g$ a G-grading and $\overline{\Gamma} : A = \bigoplus_{h\in H} \overline{A}_h$ the H-grading defined by $\overline{A}_h = \bigoplus_{g\in G,\ h=\phi(g)} A_g$. Then $\eta_{\overline{\Gamma}}(\widehat{H}) \subset \eta_\Gamma(\widehat{G})$ where the homomorphisms $\eta_\Gamma : \widehat{G} \to \operatorname{Aut} A$ and $\eta_{\overline{\Gamma}} : \widehat{H} \to \operatorname{Aut} A$ are defined by (1.3) using the gradings Γ and $\overline{\Gamma}$, respectively.

Proof. Let $\chi \in \widehat{H}$. For $y \in A_g$ we have $\eta_{\overline{\Gamma}}(\chi)(y) = \chi(\phi(g))y$ since $A_g \subset \overline{A}_{\phi(g)}$. Let $\zeta : G \to F^\times$ be the map defined by $\zeta(g) = \chi(\phi(g))$ for all $g \in G$. Then $\zeta \in \widehat{G}$. Furthermore, for all $y \in A_g$ we have

$$\eta_{\overline{\Gamma}}(\chi)(y) = \chi(\phi(g))y = \zeta(g)y = \eta_\Gamma(\zeta)(y).$$

Hence $\eta_{\overline{\Gamma}}(\chi) \in \eta_\Gamma(\widehat{G})$. □

Lemma 1.3.9. Let G, H be fintely generated abelian groups without p-torsion, A an algebra, $\Gamma : A = \bigoplus_{g\in G} A_g$ a G-grading and $\overline{\Gamma} : A = \bigoplus_{h\in H} \overline{A}_h$ an H-grading. Assume that G is generated by $\operatorname{Supp}\Gamma$ and H is generated by $\operatorname{Supp}\overline{\Gamma}$. If $\eta_{\overline{\Gamma}}(\widehat{H}) \subset \eta_\Gamma(\widehat{G})$ then $\overline{\Gamma}$ is an H-grading induced by a group epimorphism $\phi : G \to H$.

Proof. The eigenspaces of $\eta_\Gamma(\widehat{G})$ and $\eta_{\overline{\Gamma}}(\widehat{H})$ are A_g and $\overline{A}_h$ respectively for all $g \in \operatorname{Supp}\Gamma$ and $h \in \operatorname{Supp}\overline{\Gamma}$. Since $\eta_{\overline{\Gamma}}(\widehat{H}) \subset \eta_\Gamma(\widehat{G})$ we have an inclusion map $\iota : \eta_{\overline{\Gamma}}(\widehat{H}) \to \eta_\Gamma(\widehat{G})$. Also, for any $g \in \operatorname{Supp}\Gamma$ the eigenspace A_g of $\eta_\Gamma(\widehat{G})$ is contained in some eigenspace $\overline{A}_h$ of $\eta_{\overline{\Gamma}}(\widehat{H})$ for some $h \in \operatorname{Supp}\overline{\Gamma}$ where h is uniquely determined by g. Let $\phi : \operatorname{Supp}\Gamma \to \operatorname{Supp}\overline{\Gamma}$ be the map defined by $\phi(g) = h$. Let $\eta : \widehat{H} \to \widehat{G}$ be the homomorphism of algebraic groups determined by $\eta\eta_\Gamma = \eta_{\overline{\Gamma}}\iota$. Then by construction $\eta(\chi)(g) = \chi(\phi(g))$ for all $g \in \operatorname{Supp}\Gamma$. Hence ϕ extends to an epimorphism $G \to H$.

Thus $\overline{\Gamma}$ is the grading induced by ϕ. $\square$

We now look at gradings by finite elementary p-groups. From here on, F will be of positive characteristic p. A $\mathbb{Z}_p^l$-grading on an algebra A gives rise to the action of $\mathbb{Z}_p^l$ by derivations on A.

Lemma 1.3.10. Let A be an algebra, $G = \mathbb{Z}_p^l$ and $\Gamma : A = \bigoplus_{g\in G} A_g$ a G-grading on A. Then there is a homomorphism of G into $\operatorname{Der} A$ defined by the following action: For all $(g_1, \ldots, g_l)$, $(h_1, \ldots, h_l) \in \mathbb{Z}_p^l$ and $y \in A_{(h_1,\ldots,h_l)}$, set

$$(g_1, \ldots, g_l) * y = (g_1h_1 + \cdots + g_lh_l)y.$$

Conversely, given a set $\{D_j\}_{j=1}^l$ of commuting, linearly independent semisimple derivations with eigenvalues in $\mathbb{Z}_p \subset F$ we obtain the eigenspace decomposition of A with respect to these derivations, which is a grading by the group $\mathbb{Z}_p^l$. Specifically, $A_{(i_1,\ldots,i_l)} = \operatorname{Span}\{y \in A \mid D_j(y) = i_jy,\ 1 \leq j \leq l\}$ for $(i_1, \ldots, i_l) \in \mathbb{Z}_p^l$. $\square$

We will denote the map defined in Lemma 1.3.10 by $\upsilon_\Gamma : G \to \operatorname{Der} A$ when G is an elementary p-group, i.e.,

$$\upsilon_\Gamma(g)y = g * y, \quad \forall g, h \in G,\ \forall y \in A_h. \tag{1.4}$$

For convenience, we will refer to a set of commuting linearly independent semisimple derivations with eigenvalues in $\mathbb{Z}_p \subset F$ as a set of *toral elements.*

Example 1.3.11. Let $\Gamma' : L = \bigoplus_{[i]_5 \in \mathbb{Z}_5} L'_{[i]_5}$ be the $\mathbb{Z}_5$-grading in Example 1.2.12 and $\upsilon_{\Gamma'}$ as in (1.4). Lemma 1.3.10 says that

$$\upsilon_{\Gamma'}([1]_5)(y) = i(y), \qquad i \in \mathbb{Z},\ y \in L'_{[i]_5}.$$

More precisely, $\upsilon_{\Gamma'}([1]_5)(y_i) = iy_i$ for $i \in \mathbb{Z}$. Since

$$\begin{aligned} \upsilon_{\Gamma'}([1]_5)([y_i, y_j]) &= \upsilon_{\Gamma'}([1]_5)((j-i)y_{i+j}) \\ &= (i+j)(j-i)y_{i+j} = (i+j)[y_i, y_j] \\ &= [iy_i, y_j] + [y_i, jy_j] = [\upsilon_{\Gamma'}([1]_5)(y_i), y_j] + [y_i, \upsilon_{\Gamma'}([1]_5)(y_j)], \end{aligned}$$

it follows that $\upsilon_{\Gamma'}([1]_5) \in \operatorname{Der} L$.

Lemma 1.3.10 gives us a correspondence between gradings by elementary groups and sets of toral elements. For any abelian p-group G generated by l elements (l is minimal) there is a canonical mapping $\phi : G \to \mathbb{Z}_p^l$ which implies for all gradings by a p-group G there is a $\mathbb{Z}_p^l$-grading which is induced by ϕ. Hence Lemma 1.3.10 and ϕ gives us the following corollary.

Corollary 1.3.12. For any grading by an abelian p-group G generated by l elements in the support, where l is minimal, there is a set of l toral elements such that each homogeneous space of the G-grading is an eigenspace of the toral elements.

Proof. Let $\Gamma : A = \bigoplus_{g \in G} A_g$ be a G-grading of A. We can express $G = \langle g_1 \rangle_{p^{k_1}} \times \cdots \times \langle g_l \rangle_{p^{k_l}}$ where $g_1, \ldots, g_l \in \operatorname{Supp} \Gamma$. Let $\phi : G \to \mathbb{Z}_p^l$ be the homomorphism of groups defined by

$$\phi(g_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}_p^l$$

where the 1 is at the ith position. The $\mathbb{Z}_p^l$-grading $\overline{\Gamma} : A = \bigoplus_{h \in \mathbb{Z}_p^l} \overline{A}_h$, where $\overline{A}_h = \bigoplus_{g \in G,\ \phi(g) = h} A_g$, is the H-grading induced by ϕ. By Lemma 1.3.10 there exists a set $\{D_j\}_{j=1}^l$ of derivations of A such that $D_j(y) = i_j y$ for all $y \in \overline{A}_{(i_1, \ldots, i_l)}$, $(i_1, \ldots, i_l) \in \mathbb{Z}_p^l$. Let $g \in G$, $z \in A_g$ and $\phi(g) = (i_1, \ldots, i_l)$. Then $z \in \overline{A}_{(i_1, \ldots, i_l)}$ and $D_j(z) = i_j z$ for $1 \le j \le l$. □

Lemma 1.3.13. Let $\Gamma : A = \bigoplus_{g \in G} A_g$ be a G-grading on algebra A and $G = G_1 \times G_2$ a finitely generated group that is the direct product of groups where G_1 has no p-torsion and $G_2 = \mathbb{Z}_p^l$. If Γ_1 and Γ_2 are the gradings by G_1 and G_2 induced by the canonical homomorphisms $\pi_1 : G \to G_1$, $\pi_2 : G \to \mathbb{Z}_p^l$, respectively, then $\eta_{\Gamma_1}(\widehat{G_1})$ and $\upsilon_{\Gamma_2}(G_2)$ commute.

Proof. Let $\Gamma_1 : A = \bigoplus_{g_1 \in G_1} A'_{g_1}$ and $\Gamma_2 : A = \bigoplus_{g_2 \in G_2} A''_{g_2}$ be the gradings defined by $A'_{g_1} = \bigoplus_{(g_1, g_2) \in G} A_{(g_1, g_2)}$ and $A''_{g_2} = \bigoplus_{(g_1, g_2) \in G} A_{(g_1, g_2)}$ respectively. By Lemma 1.2.13 we have $A_{(g_1, g_2)} = A'_{g_1} \cap A''_{g_2}$. Any element $y \in A_{(g_1, g_2)}$ is an eigenvector of $\eta_{\Gamma_1}(\widehat{G_1})$ and $\upsilon_{\Gamma_2}(G_2)$ since A'_{g_1} is an eigenspace of $\eta_{\Gamma_1}(\widehat{G_1})$ and

A''_{g_2} is an eigenspace of $\upsilon_{\Gamma_2}(G_2)$. Hence $\eta_{\Gamma_1}(\widehat{G_1})$ and $\upsilon_{\Gamma_2}(G_2)$ commute. $\square$

The following corollary sums up Chapter 1.

Corollary 1.3.14. Let L be a finite-dimensional simple Lie algebra and G a group. If $\Gamma : L = \bigoplus_{g\in G} L_g$ is a G-grading and $\langle \operatorname{Supp}\Gamma \rangle = G$ then the following are true:

1) G is a finitely generated abelian group;
2) $G = G_1 \times G_2$ where G_1 is a group that has no p-torsion and G_2 is a p-group generated by l elements where l is minimal;
3) there exists an epimorphism $\phi : G_2 \to \mathbb{Z}_p^l$ and a set of l toral elements $\{D_i\}_{i=1}^l$ of L such that the subspaces $L''_h = \bigoplus_{g_1\in G_1,\ \phi(g_2)=h} L_{(g_1,g_2)}$, $h \in \mathbb{Z}_p^l$, are the eigenspaces with respect to $\{D_i\}_{i=1}^l$;
4) there exists a quasi-torus Q of $\operatorname{Aut} A$ isomorphic to $\widehat{G_1}$ and the subspaces $L'_{g_1} = \bigoplus_{g_2\in G_2} L_{(g_1,g_2)}$ are the eigenspaces of Q;
5) the elements of Q and the derivations D_i, $1 \le i \le l$, commute. $\square$

Chapter 2

Standard Gradings and Filtrations

In Section 2.1 we define the Witt, special, Hamiltonian, contact and Melikyan algebras. Section 2.2 is devoted to the natural gradings and filtrations of these simple Lie algebras. We will describe the automorphisms and the derivations of these algebras in Section 2.3 and 2.4. Unless it is stated otherwise, we denote by a and b some m-tuples of non-negative integers.

2.1 Cartan Type Algebras and Melikyan Algebras

In this section we will define some finite-dimensional simple Lie algebras which are generally of non-classical type and state the classification theorem of simple Lie algebras in characteristic $p \geq 5$ due to Premet and Strade in [26] which is the completion of the series of papers [21], [22], [23], [24] and [25]. The main reference for definitions and lemmas for this section is [27].

We start by defining the simple graded Cartan type Lie algebras $W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$. These algebras can be viewed as Lie subalgebras of the derivations of a commutative algebra $O(m;\underline{n})$.

For two tuples a, $b \in \mathbb{Z}^m$ we write $a \leq b$ if $a_i \leq b_i$ for $1 \leq i \leq m$. Similarly we write $a < b$ if $a_i \leq b_i$ for $1 \leq i \leq m$ and $a_j < b_j$ for at least one $1 \leq j \leq m$. Also let $|a| = \sum_{i=1}^{m} a_i$ for $a \in \mathbb{Z}^m$. When dealing with m-tuples, $\underline{1}$ will denote the m-tuple consisting of m ones, $\epsilon_i := (0, \ldots, 0, 1, 0 \ldots, 0)$, for $1 \leq i \leq m$, where the 1 is at the i-th position and $\underline{n} = (n_1, \ldots, n_m)$.

Definition 2.1.1. Let $O(m;\underline{n})$ be the commutative algebra

$$O(m;\underline{n}) := \left\{ \sum_{0 \leq a \leq \tau(\underline{n})} \alpha(a) x^{(a)} \mid \alpha(a) \in F \right\}$$

over a field F of characteristic p, where $\tau(\underline{n}) = (p^{n_1} - 1, \ldots, p^{n_m} - 1)$, with multiplication

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where we set $x^{(c)} = 0$ if $c_i < 0$ or $p^{n_i} - 1 < c_i$ for $1 \leq i \leq m$ and $\binom{a+b}{a} = \prod_{i=1}^{m} \binom{a_i + b_i}{a_i}$. For convenience, let $x_i = x^{(\varepsilon_i)}$ for $1 \leq i \leq m$.

The commutative algebra $O(m; \underline{n})$ has a built-in divided power structure. In [27, Definition 2.1.1] the divided power maps $f \mapsto f^{(i)}$ on $O_{(1)} = \mathrm{Span}\{x^{(a)} \in O(m; \underline{n}) \mid 0 < a \leq \tau(\underline{n})\}$ are defined.

Remark 2.1.2. The algebra $O(m; \underline{n})$ is generated by $V := \mathrm{Span}\{x_1, \ldots, x_m\}$ and divided power maps since $x^{(a)} = x_1^{(a_1)} \ldots x_m^{(a_m)}$ for $0 \leq a \leq \tau(\underline{n})$.

There are standard derivations on $O(m; \underline{n})$ defined by $\partial_i(x^{(a)}) = x^{(a-\varepsilon_i)}$ for $1 \leq i \leq m$.

Definition 2.1.3. Let $W(m; \underline{n})$ be the Lie algebra

$$W(m; \underline{n}) := \left\{ \sum_{1 \leq i \leq m} f_i \partial_i \mid f_i \in O(m; \underline{n}) \right\}$$

with the commutator defined by

$$[f\partial_i, g\partial_j] = f(\partial_i g)\partial_j - g(\partial_j f)\partial_i, \quad f, g \in O(m; \underline{n}).$$

We refer to the Lie algebras $W(m; \underline{n})$ as the *Witt algebras*.

Remark 2.1.4. The Lie algebra $W(m;\underline{n})$ is a subalgebra of $\operatorname{Der} O(m;\underline{n})$ where $(f\partial_i)(x^{(b)}) = f(\partial_i x^{(b)})$ for $f\partial_i \in W(m;\underline{n})$ and $x^{(b)} \in O(m;\underline{n})$. By definition, any element $D \in W(m;\underline{n})$ can be expressed as $D = \sum_{i=i}^{m} f_i\partial_i$ where $f_i \in O(m;\underline{n})$ for $1 \leq i \leq m$. Since

$$D(x_j) = \sum_{i=i}^{m} f_i(\partial_i x_j) = f_j,$$

D is uniquely defined by its action on V. Namely, for all $D \in W(m;\underline{n})$ we have $D = \sum_{i=1}^{m} D(x_i)\partial_i$.

Lemma 2.1.5. The dimension of $O(m;\underline{n})$ is $p^{|\underline{n}|}$ and the dimension of $W(m;\underline{n})$ is $mp^{|\underline{n}|}$. $\square$

The remaining simple graded Cartan type Lie algebras are subalgebras of $W(m;\underline{n})$. When dealing with the Hamiltonian and contact algebras in m variables (types $H(m;\underline{n})$ and $K(m;\underline{n})$ below), it is useful to introduce the following notation:

$$i' = \begin{cases} i+r, & \text{if } 1 \leq i \leq r, \\ i-r, & \text{if } r+1 \leq i \leq 2r; \end{cases}$$

$$\sigma(i) = \begin{cases} 1, & \text{if } 1 \leq i \leq r, \\ -1, & \text{if } r+1 \leq i \leq 2r; \end{cases}$$

where $m = 2r$ in the case of $H(m;\underline{n})$ and $2r+1$ in the case of $K(m;\underline{n})$. Note that we do not define m' or $\sigma(m)$ if $m = 2r+1$. We will also need the following differential forms (see [27, Section 4.2]).

$$\begin{aligned} \omega_S &:= dx_1 \wedge \cdots \wedge dx_m, && m \geq 3, \\ \omega_H &:= \sum_{i=1}^{r} dx_i \wedge dx_{i'}, && m = 2r, \\ \omega_K &:= dx_m + \sum_{i=1}^{2r} \sigma(i) x_i dx_{i'}, && m = 2r+1. \end{aligned}$$

Definition 2.1.6. We define the *special, Hamiltonian* and *contact algebras* as follows:

$$\begin{aligned} S(m;\underline{n}) &:= \{D \in W(m;\underline{n}) \mid D(\omega_S) = 0\}, && m \geq 3, \\ H(m;\underline{n}) &:= \{D \in W(m;\underline{n}) \mid D(\omega_H) = 0\}, && m = 2r, \\ K(m;\underline{n}) &:= \{D \in W(m;\underline{n}) \mid D(\omega_K) \in O(m;\underline{n})\omega_K\}, && m = 2r+1, \end{aligned}$$

respectively.

It is known that the Lie algebras $W(m;\underline{n})$ are simple, but $S(m;\underline{n})$ and $H(m;\underline{n})$ are not simple, and $K(m;\underline{n})$ are simple if and only if p does not divide $m+3$. The first derived algebras $S(m;\underline{n})^{(1)}$ and $K(m;\underline{n})^{(1)}$, and second derived algebras $H(m;\underline{n})^{(2)}$ are simple (see [27, Section 4.2]). We will refer to $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ and $K(m;\underline{n})^{(1)}$ as the special, Hamiltonian and contact algebras, respectively, from now on.

In order to explicitly describe the elements of $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ and $K(m;\underline{n})^{(1)}$ we will need the following maps.

Let div : $W(m;\underline{n}) \to O(m;\underline{n})$ be the map defined by

$$\operatorname{div}\left(\sum_{i=1}^{m} f_i \partial_i\right) = \sum_{i=1}^{m} \partial_i(f_i).$$

Let $D_{i,j}$, D_H and D_K be the linear maps from $O(m;\underline{n})$ to $W(m;\underline{n})$ defined by the following:

$$\begin{aligned}
D_{i,j}(f) : &= \partial_j(f)\partial_i - \partial_i(f)\partial_j, && f \in O(m;\underline{n}); \\
D_H(f) : &= \textstyle\sum_{i=1}^{2r} \sigma(i)\partial_i(f)\partial_{i'}, && m = 2r,\ f \in O(m;\underline{n}); \\
D_K(f) : &= \textstyle\sum_{i=1}^{2r} (\sigma(i)\partial_i + x_{i'}\partial_m)(f)\partial_{i'} && \\
& \quad + \left(2f - \textstyle\sum_{i=1}^{2r} x_i\partial_i(f)\right)\partial_m, && m = 2r+1 \text{ and } f \in O(m;\underline{n}).
\end{aligned}$$

Lemma 2.1.7. [27, p. 188] For $m \geq 3$,

$$\begin{aligned}
S(m;\underline{n}) &= \{D \in W(m;\underline{n}) \mid \operatorname{div}(D) = 0\}, \\
S(m;\underline{n}) &= \left(\textstyle\sum_{i=1}^{m-1} D_{i,m}(O(m;\underline{n}+\underline{1}))\right) \cap W(m;\underline{n}), \\
S(m;\underline{n})^{(1)} &= \operatorname{Span}\{D_{i,j}(f) \mid 1 \leq i < j \leq m,\ f \in O(m;\underline{n})\}.
\end{aligned}$$

Moreover, the dimension of $S(m;\underline{n})^{(1)}$ is $(m-1)(p^{|\underline{n}|}-1)$. □

Note that $S(m;\underline{n}) = \left(\sum_{i=1}^{m-1} D_{i,m}(O(m;\underline{n}+\underline{1}))\right) \cap W(m;\underline{n})$ follows from the proof of [27, Lemma 4.2.2].

Lemma 2.1.8. [27, p.189] For $m = 2r$,

$$H(m;\underline{n}) = \left\{ \sum_{i=1}^{2r} f_i\partial_i \in W(m;\underline{n}) \mid \sigma(i)\partial_{j'}(f_i) = \sigma(j)\partial_{i'}(f_j),\ 1 \leq i,j \leq m \right\},$$

$$H(m;\underline{n})^{(2)} = \mathrm{Span}\{D_H(x^{(a)}) \mid 0 < a < \tau(\underline{n})\}.$$

Moreover, the dimension of $H(m;\underline{n})^{(2)}$ is $p^{|\underline{n}|} - 2$. □

Lemma 2.1.9. [27, p.190] For $m = 2r + 1$,

$$K(m;\underline{n}) = D_K(O(m;\underline{n})).$$

If p is coprime to $m + 3$ then $K(m;\underline{n}) = K(m;\underline{n})^{(1)}$. If p divides $m + 3$ then

$$K(m;\underline{n})^{(1)} = \mathrm{Span}\{D_K(x^{(a)}) \mid 0 \leq a < \tau(\underline{n})\}.$$

Moreover, the dimension of $K(m;\underline{n})^{(1)}$ is $p^{|\underline{n}|} - 1$ if p divides $m + 3$ and $p^{|\underline{n}|}$ otherwise. □

There exist deformations of the special and Hamiltonian algebras which are simple but do not have a canonical $\mathbb{Z}$-grading as do the algebras defined above (see [27, Definition 4.2.4]). These deformations and the graded Cartan type Lie algebras are referred to collectively as the *simple Lie algebras of Cartan type*.

The next type of simple Lie algebras we will consider are the Melikyan algebras which are defined over fields of characteristic 5. Set $\widetilde{W}(2;\underline{n}) =$

$O(2;\underline{n})\widetilde{\partial}_1 + O(2;\underline{n})\widetilde{\partial}_2$ and

$$\widetilde{f_1\partial_1 + f_2\partial_2} := f_1\widetilde{\partial}_1 + f_2\widetilde{\partial}_2$$

for all $f_1,\ f_2 \in O(2;\underline{n})$.

Definition 2.1.10. Let $M(2;\underline{n}) := O(2;\underline{n}) \oplus W(2;\underline{n}) \oplus \widetilde{W}(2;\underline{n})$ be the algebra over a field F of characteristic 5 whose multiplication is defined by the following equations. For all $D \in W(2;\underline{n})$, $\widetilde{E} \in \widetilde{W}(2;\underline{n})$, $f, f_i, g_i \in O(2;\underline{n})$ we set

$$[D, \widetilde{E}] := \widetilde{[D, E]} + 2\operatorname{div}(D)\widetilde{E},$$
$$[D, f] := D(f) - 2\operatorname{div}(D)f,$$
$$[f, \widetilde{E}] := fE$$
$$[f_1, f_2] := 2(f_1\partial_1(f_2) - f_2\partial_1(f_1))\widetilde{\partial}_2 + 2(f_2\partial_2(f_1) - f_1\partial_2(f_2))\widetilde{\partial}_1.$$
$$[f_1\widetilde{\partial}_1 + f_2\widetilde{\partial}_2, g_1\widetilde{\partial}_1 + g_2\widetilde{\partial}_2] := f_1g_2 - f_2g_1.$$

We refer to the Lie algebras $M(2;\underline{n})$ as the *Lie algebras of Melikyan type.*

It turns out that $M(2;\underline{n})$ is a simple Lie algebra.

Remark 2.1.11. The dimension of $M(2;\underline{n})$ is $5^{|\underline{n}|+1}$. □

Theorem 2.1.12. [26, Theorem 1.1] If L is a finite-dimensional simple Lie algebra over a field of characteristic $p > 3$ then it is either of classical, Cartan or Melikyan type.

2.2 Standard Gradings and Filtrations

The Cartan type Lie algebras $W(m;\underline{n})$, $S(m;\underline{n})$, $H(m;\underline{n})$, $K(m;\underline{n})$ are called graded, because they have canonical $\mathbb{Z}$-gradings, and so do the Melikyan algebras $M(2;\underline{n})$. These gradings induce filtrations that are important in the study of the automorphism group of these algebras — see Section 2.3. For the graded Cartan type Lie algebras in question, the canonical gradings arise from the canonical grading on $W(m;\underline{n})$ that, in turn, arises from the canonical grading on $O(m;\underline{n})$.

Definition 2.2.1. Let $O_k = \operatorname{Span}\{x^{(a)} \in O(m;\underline{n}) \mid |a| = k\}$ for $k \in \mathbb{Z}$. We call the grading $O(m;\underline{n}) = \bigoplus_{k\in\mathbb{Z}} O_k$ the *canonical $\mathbb{Z}$-grading on $O(m;\underline{n})$*. We denote the degree with respect to this grading by $\deg_O$.

Definition 2.2.2. Set

$$\deg_W(x^{(a)}\partial_i) = \deg_O(x^{(a)}) - 1$$

for types $W(m;\underline{n})$, $S(m;\underline{n})$, $H(m;\underline{n})$, and

$$\deg_K(x^{(a)}\partial_i) = \deg_W(x^{(a)}\partial_i) + a_m - \delta_{i,m}$$

for type $K(m;\underline{n})$. For $k \in \mathbb{Z}$, the subspaces

$$X_k = \operatorname{Span}\{y \in X(m;\underline{n})^{(\infty)} \mid \deg_W(y) = k\},$$

where $X = W$, S, H, and $K_k = \mathrm{Span}\{y \in K(m;\underline{n}) \mid \deg_K(y) = k\}$ for $k \in \mathbb{Z}$ define the *canonical* $\mathbb{Z}$*-gradings on* $W(m;\underline{n})$, $S(m;\underline{n})$, $H(m;\underline{n})$ *and* $K(m;\underline{n})$, respectively, as well as on their derived subalgebras.

The difference between $\deg_W$ and $\deg_K$ for an element of $K(m;\underline{n})$ is that we "count" the degrees of x_m and ∂_m twice for $\deg_K$.

Remark 2.2.3. For $j \in \mathbb{Z}$ we have $W_j = \bigoplus_{i=1}^m O_{j+1}\partial_i$.

Example 2.2.4. For $(m;\underline{n}) = (2;(1,1))$,

$$O_2 = \mathrm{Span}\{x_1^{(2)}, x_1x_2, x_2^{(2)}\},$$

$$W_1 = O_2\partial_1 + O_2\partial_2 = \mathrm{Span}\{x_1^{(2)}\partial_1, x_1x_2\partial_1, x_2^{(2)}\partial_1, x_1^{(2)}\partial_2, x_1x_2\partial_2, x_2^{(2)}\partial_2\}.$$

Definition 2.2.5. Set

$$\begin{aligned}
\deg_M(x^{(a)}\partial_i) &:= 3\deg_W(x^{(a)}\partial_i),\\
\deg_M(x^{(a)}\tilde{\partial}_i) &:= 3\deg_W(x^{(a)}\partial_i) + 2,\\
\deg_M(x^{(a)}) &:= 3\deg_O(x^{(a)}) - 2.
\end{aligned}$$

The subspaces $M_k = \mathrm{Span}\{y \in M(2;\underline{n}) \mid \deg_M(y) = k\}$ for $k \in \mathbb{Z}$ define the *canonical* $\mathbb{Z}$*-grading on* $M(2;\underline{n})$.

Remark 2.2.6. For the canonical $\mathbb{Z}$-gradings on $W(2;\underline{n})$ and $M(2;\underline{n})$ we have $W_i = M_{3i}$. Hence $W(2;\underline{n}) = \bigoplus_{i\in\mathbb{Z}} M_{3i}$.

Definition 2.2.7. For any $\mathbb{Z}$-grading $\Gamma : A = \bigoplus_{k\in\mathbb{Z}} A_k$ on an algebra A with a finite support $\operatorname{Supp}\Gamma$ there is an *induced filtration*

$$A_s = A_{(s)} \subset A_{(s-1)} \subset \cdots A_{(q)} = A,$$

where $A_{(k)} = \bigoplus_{l\geq k} A_l$ for $k \in \mathbb{Z}$, $q = \min(\operatorname{Supp}\Gamma)$ and $s = \max(\operatorname{Supp}\Gamma)$. We call the induced filtrations of the canonical $\mathbb{Z}$-gradings on $O(m;\underline{n})$, $W(m;\underline{n})$, $S(m;\underline{n})$, $H(m;\underline{n})$, $K(m;\underline{n})$, $M(2;\underline{n})$ and their derived algebras the *canonical filtrations.*

Definition 2.2.8. For a filtration $\{A_{(i)}\}$ of an algebra A, we say that an endomorphism φ of A respects the filtration if all subspaces $A_{(i)}$ are invariant under φ.

Remark 2.2.9. It is not true in general that an endomorphism which respects an induced filtration of a $\mathbb{Z}$-grading also respects the $\mathbb{Z}$-grading.

The main contribution of this work is to show that all gradings by finitely generated groups without p-torsion on the Lie algebras in Section 2.1 are, up to isomorphism, induced by a group homomorphism from a canonical $\mathbb{Z}^k$-grading (where k depends on the type of algebra) by finding the maximal quasi-tori of the automorphism groups of the Lie algebras in question. In particular, the canonical $\mathbb{Z}$-gradings for the Lie algebras of Cartan and Melikyan type are induced by certain group homomorphisms $\phi : \mathbb{Z}^k \to \mathbb{Z}$. Now we describe the canonical $\mathbb{Z}^k$-gradings mentioned above.

Definition 2.2.10. The $\mathbb{Z}^m$-gradings on $O(m;\underline{n})$ and $W(m;\underline{n})$,

$$O(m;\underline{n}) = \bigoplus_{a\in\mathbb{Z}^m} O(m;\underline{n})_a,$$

$$W(m;\underline{n}) = \bigoplus_{a\in\mathbb{Z}^m} W(m;\underline{n})_a,$$

where

$$O(m;\underline{n})_a = \operatorname{Span}\{x^{(a)}\},$$

$$W(m;\underline{n})_a = \bigoplus_{i=1}^{m} O(m;\underline{n})_{a+\varepsilon_i}\partial_i = \operatorname{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid 1 \le i \le m\},$$

are called the *canonical $\mathbb{Z}^m$-gradings on $O(m;\underline{n})$* and $W(m;\underline{n})$, respectively. We denote the canonical $\mathbb{Z}^m$-gradings on $O(m;\underline{n})$ and $W(m;\underline{n})$ by Γ_O and Γ_W, respectively.

Note that the support Γ_W includes tuples with negative entries. For example $W(m;\underline{n})_{-\varepsilon_i} = \operatorname{Span}\{\partial_i\}$.

Lemma 2.2.11. Let $W(m;\underline{n}) = \bigoplus_{a\in\mathbb{Z}^m} W(m;\underline{n})_a$ be the canonical $\mathbb{Z}^m$-grading and $S(m;\underline{n})^{(l)}_a = S(m;\underline{n})^{(l)} \cap W(m;\underline{n})_a$ for $l = 0, 1$. Then $S(m;\underline{n})^{(l)} = \bigoplus_{a\in\mathbb{Z}^m} S(m;\underline{n})^{(l)}_a$ is a $\mathbb{Z}^m$-grading on $S(m;\underline{n})^{(l)}$.

Proof. We have to show that $S(m;\underline{n})$ is a graded subalgebra of $W(m;\underline{n})$. Then the result for $S(m;\underline{n})^{(1)}$ follows. By Lemma 2.1.7 we have that

$$S(m;\underline{n}) = \left(\sum_{i=1}^{m-1} D_{i,m}(O(m;\underline{n}+\underline{1})) \right) \cap W(m;\underline{n}).$$

The set $\{x^{(a)} \mid a \leq a \leq \tau(\underline{n}) + \underline{1}\}$ is a basis of $O(m;\underline{n}+\underline{1})$. Hence

$$S(m;\underline{n}) = \left(\mathrm{Span}\left\{D_{i,m}(x^{(a)}) \,|\, 0 \leq a \leq \tau(\underline{n}+\underline{1}),\ 1 \leq i \leq m\right\}\right) \cap W(m;\underline{n}).$$

Since

$$D_{i,m}(x^{(a)}) = \partial_m(x^{(a)})\partial_i - \partial_i(x^{(a)})\partial_m = x^{(a-\varepsilon_m)}\partial_i - x^{(a-\varepsilon_i)}\partial_m$$

is in $W(m;\underline{n}+\underline{1})_{a-\varepsilon_i-\varepsilon_m}$ it follows that $S(m;\underline{n}) = \bigoplus_{a\in\mathbb{Z}^m}(S(m;\underline{n}) \cap W(m;\underline{n})_a)$.

□

Let e_i be the $r+1$-tuple with all zeroes except for a one at the ith position and $\phi_H : \mathbb{Z}^{2r} \to \mathbb{Z}^{r+1}$ be the homomorphism defined by $\phi_H(\varepsilon_i) = e_i$ for $1 \leq i \leq r+1$ and $\phi(\varepsilon_i) = e_1 + e_{1+r} - e_{i'}$ for $r+1 < i \leq 2r$. Similarly let $\phi_K : \mathbb{Z}^{2r+1} \to \mathbb{Z}^{r+1}$ be the homomorphism defined by $\phi_K(\varepsilon_i) = e_i$ for $1 \leq i \leq r$, $\phi_K(\varepsilon_i) = e_{r+1} - e_{i'}$ for $r < i \leq 2r$ and $\phi_K(\varepsilon_m) = e_{r+1}$

The proof of the following Lemma 2.2.12 consists of showing that the gradings are induced by an abelian semisimple subgroup T_H of $\mathrm{Aut}\, H(m;\underline{n})^{(2)}$ defined in Section 2.3. The proof will be given after the T_H is defined.

Lemma 2.2.12. The Lie algebras $H(m;\underline{n})$, $H(m;\underline{n})^{(1)}$ and $H(m;\underline{n})^{(2)}$ are graded subalgebras of $W(m;\underline{n})$ with respect to the grading induced by ϕ_H from the canonical $\mathbb{Z}^m$-grading on $W(m;\underline{n})$. Namely,

$$H(m;\underline{n})^{(l)} = \bigoplus_{g\in\mathbb{Z}^{r+1}} (H(m;\underline{n})^{(l)} \cap W_g)$$

are $\mathbb{Z}^{r+1}$-gradings where $l = 0, 1, 2$ and $W_g = \bigoplus_{a\in\mathbb{Z}^m,\ \phi_H(a)=g} W_a$.

Lemma 2.2.13. The Lie algebras $K(m;\underline{n})$, $K(m;\underline{n})^{(1)}$ are graded subalgebras of $W(m;\underline{n})$ with respect to the grading induced by ϕ_K from the canonical $\mathbb{Z}^m$-grading on $W(m;\underline{n})$. That is, $K(m;\underline{n})^{(l)} = \bigoplus_{g\in\mathbb{Z}^{r+1}}(K(m;\underline{n})^{(l)} \cap W_g)$ are $\mathbb{Z}^{r+1}$-gradings where $l = 0, 1$ and $W_g = \bigoplus_{a\in\mathbb{Z}^m,\ \phi_K(a)=g} W_a$.

Proof. By Lemma 2.1.9 we have that $K(m;\underline{n}) = D_K(O(m;\underline{n}))$. Recall that

$$\begin{aligned}
D_K(x^{(a)}) &= \textstyle\sum_{i=1}^{2r}(\sigma(i)\partial_i + x_{i'}\partial_m)(x^{(a)})\partial_{i'} + \left(2x^{(a)} - \sum_{i=1}^{2r} x_i\partial_i(x^{(a)})\right)\partial_m \\
&= \textstyle\sum_{i=1}^{2r} \sigma(i)\partial_i(x^{(a)})\partial_{i'} \\
&\quad \textstyle+\sum_{i=1}^{m} x_{i'}\partial_m(x^{(a)})\partial_{i'} \\
&\quad +2x^{(a)}\partial_m \\
&\quad \textstyle-\sum_{i=1}^{2r} x_i\partial_i(x^{(a)})\partial_m.
\end{aligned}$$

Note that

$$\begin{aligned} \textstyle\sum_{i=1}^{2r} \sigma(i)\partial_i(x^{(a)})\partial_{i'} &\in \textstyle\sum_{i=1}^{r} W_{a-\varepsilon_i-\varepsilon_{i'}}, \\ \textstyle\sum_{i=1}^{m} x_{i'}\partial_m(x^{(a)})\partial_{i'} &\in W_{a-\varepsilon_m}, \\ x^{(a)}\partial_m &\in W_{a-\varepsilon_m}, \\ \textstyle\sum_{i=1}^{2r} x_i\partial_i(x^{(a)})\partial_m &\in W_{a-\varepsilon_m} \end{aligned}$$

and $\phi_K(a-\varepsilon_i-\varepsilon_{i'}) = \phi_K(a)-\phi_K(\varepsilon_i+\varepsilon_{i'}) = \phi_K(a)-\phi_K(\varepsilon_m)$. Hence $D_K(x^{(a)}) \in W_{\phi(a-\varepsilon_m)}$ and the claim follows. □

Definition 2.2.14. The $\mathbb{Z}^{r+1}$-gradings on $L = H(2r;\underline{n})$, $H(2r;\underline{n})^{(1)}$ and $H(2r;\underline{n})^{(2)}$ in Lemma 2.2.12 are called the *canonical $\mathbb{Z}^{r+1}$-gradings of L.* We denote the canonical $\mathbb{Z}^{r+1}$-grading on $H(m;\underline{n})^{(2)}$ by Γ_H.

Definition 2.2.15. The $\mathbb{Z}^{r+1}$-gradings on $L = K(2r+1;\underline{n})$ and $K(2r+1;\underline{n})^{(1)}$ in Lemma 2.2.13 are called the *canonical $\mathbb{Z}^{r+1}$-gradings of L.* We denote the canonical $\mathbb{Z}^{r+1}$-grading on $K(m;\underline{n})^{(1)}$ by Γ_K.

The proof of the following Lemma 2.2.16 is done by immediate calculations.

Lemma 2.2.16. Let $\overline{\Gamma}_M : M(2;\underline{n}) = \bigoplus_{(a_1,a_2)\in\mathbb{Z}^2} M_{(a_1,a_2)}$ where

$$\begin{aligned} M_{(3a_1,3a_2)} &:= \operatorname{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid 1 \le i \le 2\} \\ M_{(3a_1,3a_2)+\underline{1}} &:= \operatorname{Span}\{x^{(a+\varepsilon_i)}\widetilde{\partial}_i \mid 1 \le i \le 2\} \\ M_{(3a_1,3a_2)-\underline{1}} &:= \operatorname{Span}\{x^{(a)}\}. \end{aligned}$$

The decomposition above is a $\mathbb{Z}^2$-grading on $M(2;\underline{n})$. $\square$

Remark 2.2.17. The support of the $\mathbb{Z}^2$-grading $\overline{\Gamma}_M$ does not generate $\mathbb{Z}^2$. The support generates the subgroup $G = \langle (3i+j, j) \mid i, j \in \mathbb{Z} \rangle$, which is isomorphic to $\mathbb{Z}^2$. We can define a $\mathbb{Z}^2$-grading, Γ_M, for which the support generates $\mathbb{Z}^2$. Define $\phi_M : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $\phi_M((1,0)) = (3,0)$ and $\phi_M((0,1)) = (1,1)$. If we set $L_a = M_{\phi_M(a)}$ for $a \in \mathbb{Z}^2$ then $\Gamma_M : M(2;\underline{n}) = \bigoplus_{a \in \mathbb{Z}^2} L_a$ is a $\mathbb{Z}^2$-grading since $\phi_M(\mathbb{Z}^2) = G$. Also since $L_{(-1,0)} = M_{(-3,0)} = \mathrm{Span}\{\partial_1\}$ and $L_{(0,-1)} = M_{(-1,-1)} = F$ we have that the support of Γ_M generates $\mathbb{Z}^2$.

By Lemma 1.3.8 we have that $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2}) \subset \eta_{\Gamma_M}(\widehat{\mathbb{Z}^2})$. We will mainly work with the grading $\overline{\Gamma}_M$ and get results for Γ_M.

Definition 2.2.18. We call the $\mathbb{Z}^2$-grading Γ_M in Remark 2.2.17 the *canonical* $\mathbb{Z}^2$*-grading on* $M(2;\underline{n})$. Let $\deg_{\Gamma_M}(y)$ and $\deg_{\overline{\Gamma}_M}(y)$ be the degrees of y with respect to the $\mathbb{Z}^2$-gradings Γ_M and $\overline{\Gamma}_M$ respectively.

Remark 2.2.19. The canonical $\mathbb{Z}$-grading is the grading induced by the group homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ from $\overline{\Gamma}_M$, where $\phi(a_1, a_2) = a_1 + a_2$. Explicitly,

$$M_i = \bigoplus_{a_1+a_2=i} M_{(a_1,a_2)}.$$

Hence, the canonical $\mathbb{Z}$-grading on $M(2;\underline{n})$ is a grading induced by $\phi\phi_m$ from Γ_M.

Definition 2.2.20. Let G be an abelian group and $\varphi : \mathbb{Z}^2 \to G$ a homomorphism. The decomposition $M(2;\underline{n}) = \bigoplus_{g\in G} M_g$, given by

$$M_g = \mathrm{Span}\{y \in M(2;\underline{n}) \mid \varphi(\deg_{\Gamma_M}(y)) = g\},$$

is a G-grading on $M(2;\underline{n})$. We call such decomposition a *standard G-grading induced by φ* on $M(2;\underline{n})$. We will refer to a standard G-grading induced by φ as a *standard G-grading* when φ is not specified.

Remark 2.2.21. A standard G-grading on $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ or $M(2;\underline{n})$ is a grading induced by a homomorphism of groups $\phi : \mathbb{Z}^k \to G$ where k depends on $X(m;\underline{n})^{(\infty)}$. Hence for a standard G-grading we have that G has at most m generators when $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$ or $S(m;\underline{n})^{(1)}$, $r+1$ generators when $X(m;\underline{n})^{(\infty)} = H(m;\underline{n})^{(2)}$ or $K(m;\underline{n})^{(1)}$ and two generators when $X(m;\underline{n})^{(\infty)} = M(2;\underline{n})$.

2.3 Automorphisms

Corollary 1.3.14 shows the importance of semisimple abelian subgroups (quasi-tori) of an algebraic group $\mathrm{Aut}\, L$ for the determination of general gradings by groups on L. In Chapter 3 we will show that any quasi-torus in $\mathrm{Aut}\, L$ is contained in a maximal torus for a Lie algebra L of simple graded Cartan or

Melikyan type.

In this section we will show that the automorphism group of a simple graded Cartan type Lie algebra $X(m;\underline{n})^{(\infty)}$ is a subgroup of the automorphism group of $W(m;\underline{n})$, which in turn can be identified with a subgroup of the automorphism group of $O(m;\underline{n})$.

2.3.1 Automorphisms of the Cartan Type Lie Algebras

We start by describing an important subgroup of the automorphism group of $O(m;\underline{n})$.

Definition 2.3.1. Let $A(m;\underline{n})$ be the set of all m-tuples $(y_1,\dots,y_m)$ in $O(m;\underline{n})^m$ for which $\det(\partial_i(y_j))_{1\le i,j\le m}$ is invertible in $O(m;\underline{n})$ and also

$$y_i = \sum_{0<a\le\tau(\underline{n})} \alpha_i(a)x^{(a)} \quad \text{with } \alpha_i(p^l\epsilon_j)=0 \text{ if } n_i+l>n_j.$$

Lemma 2.3.2. [27, Theorem 6.3.2] For any $(y_1,\dots,y_m)\in A(m;\underline{n})$ the map $\psi: O(m;\underline{n})\to O(m;\underline{n})$ defined by setting

$$\psi\left(\sum_{0\le a\le\tau(\underline{n})} \alpha(a)x^{(a)}\right) = \sum_{0\le a\le\tau(\underline{n})} \alpha(a)\prod_{i=1}^{m} y_i^{(a_i)},$$

is an automorphism of $O(m;\underline{n})$. $\square$

Definition 2.3.3. We call an automorphism ψ of $O(m;\underline{n})$ as in Lemma 2.3.2

a *continuous automorphism* of $O(m;\underline{n})$. We denote the group of all continuous automorphisms of $O(m;\underline{n})$ as $\mathrm{Aut}_c\, O(m;\underline{n})$.

In [27, Section 7.3] it is shown that there is a map Φ from $\mathrm{Aut}_c\, O(m;\underline{n})$ to $\mathrm{Aut}\, W(m;\underline{n})$ defined on $\psi \in \mathrm{Aut}_c\, O(m;\underline{n})$ by

$$\Phi(\psi)(D) = \psi \circ D \circ \psi^{-1},$$

where $D \in W(m;\underline{n})$ and the elements of $W(m;\underline{n})$ are viewed as derivations of $O(m;\underline{n})$. Explicitly, by Remark 2.1.4 we have

$$\Phi(\psi)(D) = \sum_{i=1}^{m} \psi(D(\psi^{-1}(x_i)))\partial_i \tag{2.1}$$

for $D \in W(m;\underline{n})$.

Example 2.3.4. Let $p > 3$, $n_1 = n_2$ and $(y_1, y_2) = (x_2, x_1) \in O(2;\underline{n})^2$. The matrix $(\partial_i(y_j))_{1\leq i,j\leq 2}$ is

$$\begin{pmatrix} \partial_1(y_1) & \partial_1(y_2) \\ \partial_2(y_1) & \partial_2(y_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and its determinant is -1. It follows easily that $(y_1, y_2) \in A(2;\underline{n})$. Let ψ be the continuous automorphism defined by (y_1, y_2) as in Lemma 2.3.2. The automorphism $\Phi(\psi)$ of $W(2;\underline{n})$ is defined by $D \mapsto \psi \circ D \circ \psi^{-1}$ for all $D \in W(2;\underline{n})$. Let $E = \psi \circ D \circ \psi^{-1}$. The element $E(x_i)$ of $O(2;\underline{n})$ for $\{i, i'\} = \{1, 2\}$

is

$$E(x_i) = (\psi \circ D \circ \psi^{-1})(x_i) = (\psi \circ D)(x_{i'}) = \psi(D(x_{i'})).$$

Hence for $D = x_1\partial_2$ we have

$$\Phi(\psi)(x_1\partial_2) = \psi(D(x_2))\partial_1 + \psi(D(x_1))\partial_2 = \psi(x_1)\partial_1 + \psi(0)\partial_2 = x_2\partial_1.$$

Theorem 2.3.5. [27, Theorem 7.3.2] The map

$$\Phi : \operatorname{Aut}_c O(m;\underline{n}) \to \operatorname{Aut} W(m;\underline{n})$$

is an isomorphism of groups provided that $(m;\underline{n}) \neq (1,1)$ if $p = 3$. Also, except for the case of $H(m,(n_1;n_2))^{(2)}$ with $m = 2$ and $\min\{n_1, n_2\} = 1$ if $p = 3$,

$$\begin{aligned}
\operatorname{Aut} S(m;\underline{n})^{(1)} &= \Phi(\{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_S) \in F^{\times}\omega_S\}), \\
\operatorname{Aut} H(m;\underline{n})^{(2)} &= \Phi(\{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_H) \in F^{\times}\omega_H\}), \\
\operatorname{Aut} K(m;\underline{n})^{(1)} &= \Phi(\{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_K) \in O(m;\underline{n})^{\times}\omega_K\}),
\end{aligned}$$

where the groups on the right hand side of the equations are viewed as the restrictions of subgroups of $\operatorname{Aut} W(m;\underline{n})$ onto the corresponding subalgebras of $W(m;\underline{n})$. □

Corollary 2.3.6. Let $(m;\underline{n}) \neq (1,1)$ if $p = 3$. The automorphisms of $W(m;\underline{n})$ respect the canonical filtration $\{W_{(i)}\}$ of $W(m;\underline{n})$. □

Remark 2.3.7. The map of the tangent Lie algebras corresponding to Φ is a restriction of ad : $W \to \operatorname{Der} W$, and hence injective. It follows that Φ is an isomorphism of *algebraic* groups.

We will use the following notation

$$\begin{aligned}
\mathcal{S}(m;\underline{n}) &= \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_S) \in F^\times \omega_S\},\\
\mathcal{H}(m;\underline{n}) &= \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_H) \in F^\times \omega_H\},\\
\mathcal{K}(m;\underline{n}) &= \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_K) \in O(m;\underline{n})^\times \omega_K\}.
\end{aligned}$$

The above groups are subgroups of $\mathcal{W}(m;\underline{n}) := \operatorname{Aut}_c O(m;\underline{n})$. We will refer to them collectively by $\mathcal{X}(m;\underline{n})$ where $\mathcal{X} = \mathcal{W}$, $\mathcal{S}$, $\mathcal{H}$ or $\mathcal{K}$.

Definition 2.3.8. We denote by $\operatorname{Aut}_0 O(m;\underline{n})$ the subgroup of $\operatorname{Aut}_c O(m;\underline{n})$ consisting of all ψ such that $\psi(x_i) = \sum_{j=1}^{m} \alpha_{i,j} x_j$, $\alpha_{i,j} \in F$, $1 \le j \le m$. The group $\operatorname{Aut}_0 O(m;\underline{n})$ is canonically isomorphic to a subgroup of $\mathrm{GL}(V) = \mathrm{GL}(m)$, which we denote by $\mathrm{GL}(m;\underline{n})$.

If $n_i = n_j$ for $1 \le i,j \le m$ then $\mathrm{GL}(m;\underline{n}) = \mathrm{GL}(m)$, otherwise it is properly contained in $\mathrm{GL}(m)$. The condition for a tuple $(y_1,\ldots,y_n)$ to be in $A(m;\underline{n})$,

$$y_i = \sum_{0<a} \alpha_i(a) x^{(a)} \quad \text{with } \alpha_i(p^l \epsilon_j) = 0 \text{ if } n_i + l > n_j, \tag{2.2}$$

imposes a *flag* structure on the vector space $V = \operatorname{Span}\{x_1,\ldots,x_m\}$.

Definition 2.3.9. Given $\underline{n} = (n_1, \ldots, n_m)$, with $m > 0$, we set $\Xi_0 = \emptyset$ and then, inductively,

$$\Xi_i = \Xi_{i-1} \cup \{j \mid n_j = \max_{k \notin \Xi_{i-1}} \{n_k\}\}.$$

Set $V_i = \operatorname{Span}\{x_j \mid j \in \Xi_i\}$ for $i \geq 0$. Then $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ is a *flag* in V (i.e., an ascending chain of subspaces). We denote this flag by $\mathcal{F}(m; \underline{n})$.

Remark 2.3.10. Condition (2.2) implies that $\mathrm{GL}(m; \underline{n})$ consists of all elements of $\mathrm{GL}(m)$ that respect $\mathcal{F}(m; \underline{n})$.

In the case of special algebras we have to deal with the same subgroups of $\mathrm{GL}(m)$ as in the case of Witt algebras.

Lemma 2.3.11. [27, Section 7.3] $\operatorname{Aut}_0 O(m; \underline{n}) \cap \mathcal{S}(m; \underline{n}) = \operatorname{Aut}_0 O(m; \underline{n})$.

In the Hamiltonian case, $V = \operatorname{Span}\{x_1, \ldots, x_r\} \oplus \operatorname{Span}\{x_{1'}, \ldots, x_{r'}\}$, and ω_H induces a nondegenerate skew-symmetric form on V, given by $\langle x_i, x_j \rangle = \sigma(i)\delta_{i,j'}$, for all $i, j = 1, \ldots, 2r$.

Lemma 2.3.12. [27, Section 7.3] The image of $\operatorname{Aut}_0 O(m; \underline{n}) \cap \mathcal{H}(m; \underline{n})$ in $\mathrm{GL}(m)$, $m = 2r$, is the product of the subgroup of scalar matrices and the subgroup $\mathrm{Sp}(m; \underline{n}) := \mathrm{Sp}(m) \cap \mathrm{GL}(m; \underline{n})$. This product is almost direct: the intersection is $\{\pm \mathrm{Id}\}$.

For $\underline{t} = (t_1, \ldots, t_m) \in (F^\times)^m$ and $a \in \mathbb{Z}^m$ we define

$$\underline{t}^a := t_1^{a_1} \cdots t_m^{a_m}.$$

Lemma 2.3.13. [27, Section 7.4] The following groups are maximal tori of $\operatorname{Aut} W(m;\underline{n})$, $\operatorname{Aut} S(m;\underline{n})^{(1)}$, $\operatorname{Aut} H(m;\underline{n})^{(2)}$ and $\operatorname{Aut} K(m;\underline{n})^{(1)}$, respectively:

$$\begin{aligned}
T_W &= T_S = \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \Psi(x^{(a)}\partial_i) = \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m\},\\
T_H &= \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \Psi(x^{(a)}\partial_i) = \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m,\\
&\qquad t_i t_{i'} = t_j t_{j'},\ 1 \le i,j \le r\},\\
T_K &= \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m,\\
&\qquad t_i t_{i'} = t_j t_{j'} = t_m,\ 1 \le i,j \le r\}.
\end{aligned}$$

□

Corollary 2.3.14. Let $X = W$, S, H or K. The maximal tori T_X in $\operatorname{Aut} X(m;\underline{n})^{(\infty)}$ described in Lemma 2.3.13 correspond, under the algebraic group isomorphism Φ, to the following maximal tori in $\mathcal{X}(m;\underline{n})$:

$$\begin{aligned}
\mathcal{T}_W &= \mathcal{T}_S = \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times\},\\
\mathcal{T}_H &= \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times,\ t_i t_{i'} = t_j t_{j'},\},\\
\mathcal{T}_K &= \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times,\ t_i t_{i'} = t_j t_{j'} = t_m,\\
&\qquad 1 \le i,j \le r\}.
\end{aligned}$$

A convenient way to work with the elements of the above tori is to view them as m-tuples of nonzero scalars. Define $\lambda : (F^\times)^m \to \operatorname{Aut}_0 O(m;\underline{n})$ where $\lambda(\underline{t})(x_i) = t_i x_i$ for $1 \le i \le m$. Then $\lambda((F^\times)^m) = \mathcal{T}_W$.

Definition 2.3.15. We will say that $\underline{t} \in (F^\times)^m$ is *X-admissible* if $\lambda(\underline{t}) \in \mathcal{T}_X$, where $X = W$, S, H or K.

We now prove Lemma 2.2.12.

Proof. Let $m = 2r$ and $\overline{\Gamma}_H$ be the $\mathbb{Z}^{r+1}$-grading on $W(m;\underline{n})$ induced by ϕ_H with respect to the canonical $\mathbb{Z}^m$-grading Γ_W on $W(m;\underline{n})$. Then $\eta_{\overline{\Gamma}_H}(\mathbb{Z}^{r+1}) = T_H$.

We start by showing that $H(m;\underline{n})$ is invariant with respect to T_H which implies that T_H is a quasi-torus of $\operatorname{Aut} H(m;\underline{n})$. By definition $H(m;\underline{n})$ is the set of elements $D \in W(m;\underline{n})$ such that $D(\omega_H) = 0$. Now fix a $D \in H(m;\underline{n})$ and $\psi \in T_H$. By Theorem 2.3.5 and Corollary 2.3.14 there exists a $\lambda(\underline{t}) \in \mathcal{T}_H$ such that $\psi = \Phi_{\lambda(\underline{t})}$. Recall that $t_i t_{i'} = t_j t_{j'}$ if $\lambda(\underline{t}) \in \mathcal{T}_H$ and $\lambda(\underline{t}^{-1})(\omega_H) = (t_1 t_{1+r})^{-1}$. Then

$$(\psi(D))(\omega_H) = \lambda(\underline{t}) \circ D \circ \lambda(\underline{t})^{-1}(\omega_H) = (t_1 t_{1+r})^{-1} \lambda(\underline{t})(D(\omega_H)) = 0.$$

Hence $\psi(D) \in H(m;\underline{n})$. It then follows that $H(m;\underline{n})$ is a graded subalgebra of $W(m;\underline{n})$ with respect to $\overline{\Gamma}_H$.

Since the derived algebra $A^{(1)}$ of a graded algebra A is a graded subspace we have that the restriction of the $\overline{\Gamma}_H$ grading on $H(m;\underline{n})^{(1)}$ and $H(m;\underline{n})^{(2)}$ are gradings. The restriction of the $\overline{\Gamma}_H$ on $H(m;\underline{n})^{(2)}$ is Γ_H. $\square$

2.3.2 Automorphims of the Melikyan Algebras

Lemma 2.3.16. [17, p. 3921] Any automorphism Ψ of $M(2;\underline{n})$ respects the canonical filtration of $M(2;\underline{n})$.

The lemma below follows from the first paragraph on p.3920 of [17].

Lemma 2.3.17. For every automorphism ψ of $W(2;\underline{n})$ there exists an automorphism ψ_M of $M(2;\underline{n})$ which respects $W(2;\underline{n})$ and whose restriction to $W(2;\underline{n})$ is ψ. $\square$

Definition 2.3.18. Let $\operatorname{Aut}_W M(2;\underline{n}) = \{\Psi \in \operatorname{Aut} M(2;\underline{n});\, |\, \Psi(W(2;\underline{n})) = W(2;\underline{n})\}$ be the subgroup of automorphisms that leave $W(2;\underline{n})$ invariant and let $\pi : \operatorname{Aut}_W M(2;\underline{n}) \to \operatorname{Aut} W(2;\underline{n})$ be the restriction map.

Remark 2.3.19. The group of automorphisms that preserve the canonical $\mathbb{Z}^2$-grading is a subgroup of $\operatorname{Aut}_W M(2;\underline{n})$.

The $\mathbb{Z}^2$-grading $\overline{\Gamma}_M$ on $M(2;\underline{n})$ gives rise to a torus $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2})$ since $\widehat{\mathbb{Z}^2}$ is a connected group. We will show later that $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2})$ is actually a maximal torus. Let $\underline{t}^a := t_1^{a_1}t_2^{a_2}$ for all $\underline{t} = (t_1, t_2) \in (F^\times)^2$. We define $\Omega : (F^\times)^2 \to \operatorname{Aut} M(2;\underline{n})$

where

$$\begin{aligned}
\Omega(\underline{t})x^{(a)}\partial_i &:= \underline{t}^{3a-3\varepsilon_i}x^{(a)}\partial_i \\
\Omega(\underline{t})x^{(a)}\widetilde{\partial}_i &:= \underline{t}^{3a-3\varepsilon_i}\underline{t}^{\underline{1}}x^{(a)}\widetilde{\partial}_i \\
\Omega(\underline{t})x^{(a)} &:= \underline{t}^{3a}\underline{t}^{-\underline{1}}x^{(a)}.
\end{aligned}$$

For any element y in the homogeneous component M_a of $\overline{\Gamma}$ we have $\Omega(\underline{t})(y) = \underline{t}^a y$ which is the same as saying $\Omega(\underline{t})(y) = \underline{t}^{\deg_{\overline{\Gamma}_M}(y)}y$.

Lemma 2.3.20. Ω is a homomorphism of algebraic groups.

Proof. Since $\overline{\Gamma}_M$ is a grading of the Lie algebra $M(2;\underline{n})$, for $\underline{t} \in (F^\times)^2$ we have $\Omega(\underline{t}) \in \operatorname{Aut} M(2;\underline{n})$.

Now we show that Ω is a homomorphism. Let $\underline{s}$, $\underline{t} \in (F^\times)^2$ and y be a homogeneous element. Then

$$\Omega(\underline{s}\,\underline{t})y = (\underline{s}\,\underline{t})^{\deg(y)}y = \underline{s}^{\deg(y)}\underline{t}^{\deg(y)}y = \underline{s}^{\deg(y)}\Omega(\underline{t})(y) = \Omega(\underline{s})(\Omega(\underline{t})(y))$$

which shows that Ω is a homomorphism.

It is obvious that Ω is a regular map. $\square$

Let $T_M := \Omega((F^\times)^2)$. The kernel of Ω is $\{(t_1,t_2) \in (F^\times)^2 \,|\, t_1^3 = t_2^3 = 1,\ t_1t_2 = 1\}$. Since Ω is a homomorphism of algebraic groups we have that T_M is a torus.

Lemma 2.3.21. [17, Lemma 5] If $\Theta \in \mathrm{Aut}_W\, M(2;\underline{n})$ is such that $\pi(\Theta) = \mathrm{Id}_W$ then for $y \in M_i$, $i \in \mathbb{Z}$, there exists $\beta \in F^\times$ such that $\Theta(y) = \beta^i y$ and $\beta^3 = 1$.$\square$

We now fix β to be a primitive third root of unity and set $\Theta := \Omega(\beta^2, \beta^2)$. Note that $\Theta \in T_M$ and $\Theta(y) = \beta^i y$ for $y \in M_i$.

Corollary 2.3.22. Let Ψ and Υ be elements of $\mathrm{Aut}_W\, M(2;\underline{n})$. If $\pi(\Psi) = \pi(\Upsilon)$ then there exists an l such that $0 \leq l \leq 2$ and $\Psi = \Upsilon\Theta^l$.

Proof. If $\pi(\Psi) = \pi(\Phi)$ then $\pi(\Phi^{-1}\Psi) = \mathrm{Id}_W$. By Lemma 2.3.21 we have $\Phi^{-1}\Psi = \Theta^l$ for some $0 \leq l \leq 2$. $\square$

Lemma 2.3.23. Let $\eta_{\overline{\Gamma}_M} : \widehat{\mathbb{Z}^2} \to \mathrm{Aut}\, M(2;\underline{n})$ be the homomorphism defined by (1.3). Then $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2}) = T_M$.

Proof. First we show that $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2}) \subset T_M$. Let $\chi \in \widehat{\mathbb{Z}^2}$ and $\chi((1,0)) = t_1 \in F^\times$ and $\chi((0,1)) = t_2 \in F^\times$. For $y \in M_{(a_1,a_2)}$ we have

$$\begin{aligned}
\eta_{\overline{\Gamma}_M}(\chi)(y) &= \chi((a_1,a_2))y = \chi((a_1,0))\chi((0,a_2))y \\
&= \chi((1,0))^{a_1}\chi((0,1))^{a_2}y = \underline{t}^{\deg_{\overline{\Gamma}_M}(y)}y \\
&= \Omega(\underline{t})(y).
\end{aligned}$$

Hence $\eta_{\overline{\Gamma}_M}(\chi) \in T_M$ and we have $\eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2}) \subset T_M$.

Now we show that $T_M \subset \eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2})$. For $\underline{t} = (t_1, t_2) \in (F^\times)^2$ let $\chi_{\underline{t}}$ be the element of $\widehat{\mathbb{Z}^2}$ defined by $\chi_{\underline{t}}(a) = \underline{t}^a$ for all $a \in \mathbb{Z}^2$. For $y \in M_a$, $a \in \mathbb{Z}^2$ we have

$$\Omega(\underline{t})(y) = \underline{t}^a y = \chi_{\underline{t}}(a)y = \eta_{\overline{\Gamma}_M}(\chi_{\underline{t}})(y).$$

Hence $\Omega(\underline{t}) \in \eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2})$ and we have $T_M \subset \eta_{\overline{\Gamma}_M}(\widehat{\mathbb{Z}^2})$. $\square$

Lemma 2.3.24. The restriction of T_M to $W(2;\underline{n})$ is T_W. Moreover, for $\underline{t} \in (F^\times)^2$ we have $\pi(\Omega((t_1,t_2))) = \Phi(\lambda((t_1^3,t_2^3)))$.

Proof. We start by showing $T_W \subset \pi(T_M)$. For any $\psi \in T_W$ we have a pair $(s_1,s_2) \in (F^\times)^2$ such that $\psi(x^{(a)}\partial_i) = s_1^{a_1}s_2^{a_2}s_i^{-1}x^{(a)}\partial_i$. For any element u of $F^\times$ there is an element v such that $v^3 = u$ because F is algebraically closed. Hence there exist t_1 and t_2 in $F^\times$ such that $t_1^3 = s_1$ and $t_2^3 = s_2$. Computing $\Omega(\underline{t})$ on $x^{(a)}\partial_i$, we get

$$\Omega(\underline{t})(x^{(a)}\partial_i) = t_1^{3a_1}t_2^{3a_2}t_i^{-3}x^{(a)}\partial_i = s_1^{a_1}s_2^{a_2}s_i^{-1}x^{(a)}\partial_i.$$

This shows that $\psi = \pi(\Omega(\underline{t})) \in \pi(T_M)$, and we have established $T_W \subset \pi(T_M)$. The above calculation also shows that $\pi(\Omega(\underline{t})) = \lambda((t_1^3,t_2^3))$.

The inclusion $\pi(T_M) \subset T_W$ is obvious. $\square$

2.4 Derivations

Corollary 1.3.14 shows the importance of toral elements with regard to gradings by elementary p-groups on L. Hence the maximal Lie subalgebas of semisimple derivations in $\operatorname{Der} L$ are of importance with respect to the gradings on L.

In this section we will describe the derivations for all of the simple graded Cartan and Melikyan type Lie algebras. Also, we describe the maximal subalgebras of commuting semisimple derivations for $X(m;\underline{1})$ where $X = W$, S and H.

We will need the following Lie algebras to describe the derivations of the simple graded Cartan type Lie algebras.

Definition 2.4.1. Let

$$\begin{aligned} CS(m;\underline{n}) &:= \{D \in W(m;\underline{n}) \mid D(\omega_S) = F\omega_S\}, \quad m \geq 3, \\ CH(m;\underline{n}) &:= \{D \in W(m;\underline{n}) \mid D(\omega_H) = F\omega_H\}, \quad m = 2r. \end{aligned}$$

Lemma 2.4.2. [27, p. 188-189]

$$\begin{aligned} CS(m;\underline{n}) &:= S(m;\underline{n}) \oplus F x_1 \partial_1, & m \geq 3, \\ CH(m;\underline{n}) &:= H(m;\underline{n}) \oplus F \left(\sum_{i=1}^{m} x_i \partial_i \right), & m = 2r. \end{aligned}$$

Theorem 2.4.3. [27, Theorems 7.1.2 and 7.1.4] The map ad gives the following isomorphisms:

$$\begin{array}{lcll}
\operatorname{Der} W(m;\underline{n}) & \cong & W(m;\underline{n}) & \oplus \ \sum_{i=1}^{m} \sum_{0<j_i<n_i} F\partial_i^{p^{j_i}}, \\
\operatorname{Der} S(m;\underline{n})^{(1)} & \cong & CS(m;\underline{n}) & \oplus \ \sum_{i=1}^{m} \sum_{0<j_i<n_i} F\partial_i^{p^{j_i}}, \\
\operatorname{Der} H(m;\underline{n})^{(2)} & \cong & CH(m;\underline{n}) & \oplus \ \sum_{i=1}^{m} \sum_{0<j_i<n_i} F\partial_i^{p^{j_i}}, \\
\operatorname{Der} K(m;\underline{n})^{(1)} & \cong & K(m;\underline{n})^{(1)} & \oplus \ \sum_{i=1}^{m} \sum_{0<j_i<n_i} F\partial_i^{p^{j_i}}, \\
\operatorname{Der} M(2;\underline{n}) & \cong & M(2;\underline{n}) & \oplus \ \sum_{i=1}^{2} \sum_{0<j_i<n_i} F\partial_i^{p^{j_i}},
\end{array}$$

where m is even for the Hamiltonian case and odd for the contact case. □

Definition 2.4.4. For a Lie algebra L we call a Lie subalgebra A of L such that $\operatorname{ad} A$ consists of commuting semisimple derivations of L, a *toral subalgebra.* We call a toral subalgebra *maximal* it is not properly contained in another toral subalgebra.

Lemma 2.4.5. [27, Corollary 7.5.2] The maximal toral subalgebras of $W(m;\underline{1})$ are conjugate by automorphisms of $W(m;\underline{1})$ to

$$\mathfrak{T}_s := \left(\sum_{i=1}^{s} F(1+x_i)\partial_i\right) \oplus \left(\sum_{j=s+1}^{m} Fx_j\partial_j\right)$$

for some $0 \leq s \leq m$. □

Lemma 2.4.6. If $n_q = 1$ then the derivation $\operatorname{ad}((1+x_q)\partial_q)$ is a toral element of $\operatorname{Der} W(m;\underline{n})$, $\operatorname{Der} S(m;\underline{n})^{(1)}$ and $\operatorname{Der} H(m;\underline{n})^{(2)}$, with eigenvalues in $\mathbb{Z}_p \subset F$.

Proof. First we show that $\operatorname{ad}((1+x_q)\partial_q)$ is a derivation of $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ and then show that it is semisimple with respect to $W(m;\underline{n})$. It would then follow that $\operatorname{ad}((1+x_q)\partial_q)$ is semisimple on $S(m;\underline{n})^{(1)}$ and $H(m;\underline{n})^{(2)}$ since $\operatorname{ad}((1+x_q)\partial_q)$ would be semisimple on $W(m;\underline{n})$ and would preserve $S(m;\underline{n})^{(1)}$ and $H(m;\underline{n})^{(2)}$.

By Theorem 2.4.3 we have that $\operatorname{ad}(CS(m;\underline{n})) \subset \operatorname{Der}(S(m;\underline{n})^{(1)})$ and $\operatorname{ad}(CH(m;\underline{n})) \subset \operatorname{Der}(H(m;\underline{n})^{(2)})$. By Lemma 2.1.7 we have that $\partial_q \in S(m;\underline{n})$ since $\operatorname{div}(\partial_q) = 0$. By definition $x_q\partial_q \in CS(m;\underline{n})$.

By definition $x_q\partial_q \in CH(m;\underline{n})$. Also by Lemma 2.1.8 we have $\partial_q \in H(m;\underline{n})$ since $\sigma(i)\partial_{j'}(1) = 0$ for $1 \leq i, j \leq m$. It follows that $\operatorname{ad}((1+x_q)\partial_q) \in \operatorname{Der} H(m;\underline{n})^{(2)}$.

If $n_q = 1$ then

$$\{(1+x_q)^{a_0}x^{(a)} \mid a_q = 0,\ 0 \leq a_0 \leq p-1\}$$

is a basis of $O(m;\underline{n})$ and

$$\{(1+x_q)^{a_0}x^{(a)}\partial_i \mid a_q = 0,\ 0 \leq a_0 \leq p-1,\ 1 \leq i \leq m\}$$

is a basis of $W(m;\underline{n})$. Since $a_q = 0$ we have that

$$\operatorname{ad}((1+x_q)\partial_q)((1+x_q)^{a_0}x^{(a)}\partial_i) = (a_0 - \delta_{q,i})(1+x_q)^{a_0}x^{(a)}\partial_i.$$

It follows that $\mathrm{ad}((1+x_q)\partial_q)$ is a semisimple derivation of $W(m;\underline{n})$ with eigenvalues in $\mathbb{Z}_p \subset F$. $\square$

Lemma 2.4.7. If $n_q = 1$, then the derivation $\mathrm{ad}((1+x_q)\partial_q) : M(2;\underline{n}) \to M(2;\underline{n})$ is a toral element of $\mathrm{Der}\, M(2;\underline{n})$.

Proof. By Lemma 2.4.6 we have that $\mathrm{ad}((1+x_q)\partial_q)$ is a semisimple derivation on $W(2;\underline{n})$ with eigenvalues in $\mathbb{Z}_p \subset F$. Since $n_q = 1$ the set

$$\{(1+x_q)^{a_0}x^{(a)} \,|\, a_q = 0,\ 0 \le a_0 \le p-1\}$$

is a basis of $O(2;\underline{n})$ and

$$\{(1+x_q)^{a_0}x^{(a)}\widetilde{\partial}_i \,|\, a_q = 0,\ 0 \le a_0 \le p-1,\ 1 \le i \le 2\}$$

is a basis of $\widetilde{W}(2;\underline{n})$. Since $a_q = 0$ we have that

$$\begin{aligned}
\mathrm{ad}((1+x_q)\partial_q)((1+x_q)^{a_0}x^{(a)}\widetilde{\partial}_i) &= (a_0 - \delta_{q,i})(1+x_q)^{a_0}x^{(a)}\widetilde{\partial}_i \\
&\quad +2(\partial_q(1+x_q))(1+x_q)^{a_0}x^{(a)}\widetilde{\partial}_i \\
&= (a_0 - \delta_{q,i} + 2)(1+x_q)^{a_0}x^{(a)}\widetilde{\partial}_i
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ad}((1+x_q)\partial_q)((1+x_q)^{a_0}x^{(a)}) &= (a_0)(1+x_q)^{a_0}x^{(a)} \\
&\quad -2(\partial_q(1+x_q))(1+x_q)^{a_0}x^{(a)} \\
&= (a_0 - 2)(1+x_q)^{a_0}x^{(a)}.
\end{aligned}$$

It follows that $\mathrm{ad}((1+x_q)\partial_q)$ is a semisimple derivation of $M(2;\underline{n})$ with eigenvalues in $\mathbb{Z}_p \subset F$. $\square$

Chapter 3

Quasi-tori of the Automorphism Groups

In Section 3.1 we relate the maximal tori T_X of the automorphism groups of the simple graded Cartan type Lie algebras $X(m;\underline{n})^{(\infty)}$, $X = W$, S, H or K, to the canonical Γ_X-gradings on $X(m;\underline{n})^{(\infty)}$, as was done for the torus T_M in the case of Melikyan algebras in Lemma 2.3.23. We will show in Section 3.3 that T_M is a maximal torus. A theorem of Platonov says that any quasi-torus, in particular $\eta(\widehat{G})$ associated to a grading on the algebra in question, is inside the normalizer of a maximal torus. In Sections 3.2, 3.3, we find the normalizers of T_X, $X = W$, S, H, K or M and prove that up to conjugation by an automorphism, $\eta(\widehat{G})$ is in T_X. In other words T_X and its conjugates are the only maximal abelian semisimple subgroups of $\operatorname{Aut} L$ where L is of simple graded Cartan or Melikyan type. The main result is:

Proposition 3.0.8. Let $L = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ or $M(2;\underline{n})$ and $p \neq 3$ if $L = W(1;1)$ or $H(2;\underline{n})^{(2)}$ where $\min\{n_1, n_2\} = 1$. If Q is a maximal quasi-torus in $\operatorname{Aut} L$ then it is conjugate to the maximal torus T_X where $X = W$, S, H, K or M respectively.

Remark 3.0.9. Proposition 3.0.8 says that the maximal abelian diagonalizable (MAD) subgroup [19] are the maximal tori up to conjugation for the simple graded Cartan Lie algebras and Melikyan algebras.

3.1 Actions Associated to the Standard Gradings

In this section we show that $\eta_{\Gamma_X}(\widehat{\mathbb{Z}^k}) = T_X$ for the canonical $\mathbb{Z}^k$-gradings Γ_X on $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ and $K(m;\underline{n})^{(1)}$.

Lemma 3.1.1. The subgroup $\eta_{\Gamma_X}(\widehat{\mathbb{Z}^m})$ of $\operatorname{Aut} X(m;\underline{n})^{(\infty)}$ is T_W for $X = W, S$.

Proof. The dual group of $\mathbb{Z}^m$ is

$$\widehat{\mathbb{Z}^m} = \{\chi(\underline{t}) \mid \underline{t} \in (F^\times)^m\}$$

where $\chi(\underline{t})(a) = \underline{t}^a$ for $a \in \mathbb{Z}^m$. Then $\eta_{\Gamma_W}(\chi(\underline{t}))(x^{(a)}\partial_i) = \underline{t}^{a-\varepsilon_i}x^{(a)}\partial_i$ since $x^{(a)}\partial_i \in W(m;\underline{n})_{a-\varepsilon_i}$. Hence $\eta_{\Gamma_W}(\chi(\underline{t})) = \lambda(\underline{t})$, i.e. $T_W = \eta_{\Gamma_W}(\widehat{\mathbb{Z}^m})$. Since $S(m;\underline{n})^{(1)}$ is a graded subspace of Γ_W we have that $\eta_{\Gamma_S}(\widehat{\mathbb{Z}^m})$ is the restriction of $\eta_{\Gamma_W}(\widehat{\mathbb{Z}^m})$ on $S(m;\underline{n})^{(1)}$. Since T_S is the restriction of T_W, we have $\eta_{\Gamma_S}(\widehat{\mathbb{Z}^m}) = T_S$. $\square$

Lemma 3.1.2. The subgroup $\eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}})$ of $\operatorname{Aut} H(m;\underline{n})^{(2)}$ is T_H.

Proof. Recall that $m = 2r$ and Γ_H is the canonical $\mathbb{Z}^{r+1}$-grading on $H(m;\underline{n})^{(2)}$ induced by ϕ_H. We have

$$\widehat{\mathbb{Z}^{r+1}} = \{\chi(\underline{s}) \mid (s_1, \ldots, s_{r+1}) \in (F^\times)^{r+1}\}.$$

For $a \in \mathbb{Z}^m$, $\underline{s} \in (F^\times)^{r+1}$ we have

$$\begin{aligned} \phi_H(a) &= a_1e_1 + (a_2 - a_{r+2})e_2 + \cdots + (a_r - a_{2r})e_r + a_{r+1}e_{r+1} \\ &\quad + \left(\textstyle\sum_{i=2}^r a_{r+i}\right)(e_1 + e_{r+1}) \end{aligned}$$

and

$$\chi(\underline{s})(\phi(a)) = s_1^{a_1}s_2^{a_2-a_{r+2}}\cdots s_r^{a_r-a_{2r}}s_{r+1}^{a_{r+1}}(s_1s_{r+1})^{\sum_{i=2}^r a_{r+i}}.$$

Let $\kappa = \chi(\underline{s})$. Since $x^{(a)}\partial_i \in W_{\phi_H(a-\varepsilon_i)}$ we have

$$\eta_{\Gamma_H}(\kappa)(x^{(a)}\partial_i) = \kappa(\phi_H(a))(\kappa(\phi_H(\varepsilon_i)))^{-1}x^{(a)}\partial_i.$$

For $1 \leq i \leq r+1$ we have $\kappa(\phi(\varepsilon_i)) = \kappa(e_i) = s_i$. For $r+2 \leq i \leq 2r$ we have $\kappa(\phi_H(\varepsilon_i)) = \kappa(e_1 + e_{r+1} - e_i) = s_1 s_{r+1} s_{i'}^{-1}$.

Let $\underline{t} \in (F^\times)^m$ be H-admissible. That is, $t_1 t_{r+1} = t_i t_{i'}$ for $2 \leq i \leq r$. Then

$$\begin{aligned}
\lambda(\underline{t})(x^{(a)}\partial_i) &= \underline{t}^{a-\varepsilon_i}x^{(a)}\partial_i = t_1^{a_1}\cdots t_{2r}^{a_{2r}}t_i^{-1}x^{(a)}\partial_i \\
&= t_1^{a_1}\cdots t_{r+1}^{a_{r+2}}(t_1t_{r+1}t_2^{-1})^{a_{r+1}}\cdots(t_1t_{r+1}t_r^{-1})^{a_{2r}}t_i^{-1}x^{(a)}\partial_i \\
&= t_1^{a_1}t_2^{a_2-a_{r+2}}\cdots t_r^{a_r-a_{2r}}t_{r+1}^{a_{r+1}}(t_1t_{r+1})^{\sum_{i=2}^{r+1}a_{r+i}}t_i^{-1}x^{(a)}\partial_i.
\end{aligned}$$

Hence

$$\eta_{\Gamma_H}(\chi(t_1,\ldots,t_{r+1})) = \lambda((t_1,\ldots,t_m))$$

for every H-admissible $\underline{t}$, so $\eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}}) = T_H$. Since $H(m;\underline{n})^{(2)}$ is a graded subspace of $W(m;\underline{n})$ with the grading induced by ϕ_H, we have that $\eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}})$ is the restriction of $\eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}})$ on $H(m;\underline{n})^{(2)}$. Since $T_H = \eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}})$ we have $\eta_{\Gamma_H}(\widehat{\mathbb{Z}^{r+1}}) = T_H$. $\square$

Lemma 3.1.3. The subgroup $\eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}})$ of $\operatorname{Aut} K(m;\underline{n})^{(1)}$ is T_K.

Proof. Recall that $m = 2r+1$. Let Γ_K be the coarsening of the standard $\mathbb{Z}^m$-grading on $W(m;\underline{n})$ induced by ϕ_K and

$$\widehat{\mathbb{Z}^{r+1}} = \{\chi(\underline{s}) \mid (s_1,\ldots,s_{r+1}) \in (F^\times)^{r+1}\}.$$

For $a \in \mathbb{Z}^m$, $\underline{s} \in (F^\times)^{r+1}$ we have

$$\phi_K(a) = (a_1 - a_{r+1})e_1 + \cdots + (a_r - a_{2r})e_r + \left(\sum_{i=1}^{r+1} a_{r+i}\right) e_m$$

and

$$\chi(\underline{s})(\phi_K(a)) = s_1^{a_1 - a_{r+1}} \cdots s_r^{a_r - a_{2r}} (s_m)^{\sum_{i=1}^{r+1} a_{r+i}}.$$

Let $\kappa = \chi(\underline{s})$. Since $x^{(a)}\partial_i \in W_{\phi_K(a-\varepsilon_i)}$ we have

$$\eta_{\Gamma_K}(\kappa)(x^{(a)}\partial_i) = \kappa(\phi_K(a))(\kappa(\phi_K(\varepsilon_i)))^{-1} x^{(a)}\partial_i.$$

For $1 \leq i \leq r$ we have $\kappa(\phi_K(\varepsilon_i)) = \kappa(e_i) = s_i$. For $r+1 \leq i \leq 2r$ we have $\kappa(\phi_K(\varepsilon_i)) = \kappa(e_{r+1} - e_i) = s_{r+1}s_{i'}^{-1}$. Also, $\kappa(\phi_K(\varepsilon_m)) = s_{r+1}$.

Let $\underline{t} \in (F^\times)^m$ be K-admissible. That is, $t_m = t_i t_{i'}$ for $1 \leq i \leq r$. Then

$$\begin{aligned} \lambda(\underline{t})(x^{(a)}\partial_i) &= \underline{t}^{a-\varepsilon_i} x^{(a)}\partial_i = t_1^{a_1} \cdots t_m^{a_m} t_i^{-1} x^{(a)}\partial_i \\ &= t_1^{a_1} \cdots t_r^{a_r} (t_m t_1^{-1})^{a_{r+1}} \cdots (t_m t_r^{-1})^{a_{2r}} t_i^{-1} x^{(a)}\partial_i \\ &= t_1^{a_1 - a_{r+1}} \cdots t_r^{a_r - a_{2r}} (t_m)^{\sum_{i=1}^{r+1} a_{r+i}} t_i^{-1} x^{(a)}\partial_i. \end{aligned}$$

Hence

$$\eta_{\Gamma_K}(\chi(t_1, \ldots, t_{r+1})) = \lambda((t_1, \ldots, t_m))$$

for every K-admissible $\underline{t}$, so $\eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}}) = T_K$. Since $K(m;\underline{n})^{(1)}$ is a graded subspace of $W(m;\underline{n})$ with the grading induced by ϕ_K, we have that $\eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}})$ is the restriction of $\eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}})$ on $K(m;\underline{n})^{(1)}$. Since $T_K = \eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}})$ we have $\eta_{\Gamma_K}(\widehat{\mathbb{Z}^{r+1}}) = T_K$. $\square$

Since $\eta_{\Gamma_X}(\widehat{\mathbb{Z}^k}) = T_X$ for the canonical $\mathbb{Z}^k$-gradings Γ_X on $X(m;\underline{n})^{(\infty)}$, where $X(m;\underline{n})^{(\infty)}$ is a simple graded Cartan or Melikyan Lie algebra, the Lemmas 1.3.8 and 1.3.9 give us the following corollary.

Corollary 3.1.4. Let G be an abelian group without p-torsion and $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ or $M(2;\underline{n})$. Let $\Gamma : X(m;\underline{n})^{(\infty)} = \bigoplus_{g\in G} L_g$ be a G-grading. The grading Γ is a standard G-grading if and only if $\eta_\Gamma(\widehat{G}) \subset T_X$. $\square$

3.2 Normalizers of Tori for the Cartan Case

The goal of this section is to show that $\eta(\widehat{G})$ is conjugate in $\operatorname{Aut} X(m;\underline{n})^{(\infty)}$ to a subgroup of the maximal torus T_X where $X = W, S, H, K$. We are going to use an important general result following from [20, Corollary 3.28].

Proposition 3.2.1. Any quasi-torus of an algebraic group is contained in the normalizer of a maximal torus. $\square$

This brings us to the necessity of looking at normalizers of maximal tori in $\operatorname{Aut} X(m;\underline{n})^{(\infty)}$. Using the isomorphism Φ described in (2.3.1), we are going to work inside the groups $\mathcal{W}(m;\underline{n}) = \operatorname{Aut}_c O(m;\underline{n})$. An important subgroup which we use for the description of the normalizer $N_{\mathcal{X}(m;\underline{n})}(\mathcal{T}_X)$ for $X = W, S, H$ or K, is the subgroup $\mathcal{M}(m;\underline{n})$ of $\mathcal{W}(m;\underline{n})$.

Definition 3.2.2. Let $\mathcal{M}(m;\underline{n})$ be the subgroup of $\mathcal{W}(m;\underline{n})$ that consists of ψ such that, for each $1 \leq i \leq m$, we have $\psi(x_i) = \alpha_i x_{j_i}$ where $\alpha_i \in F^\times$ and $1 \leq j_i \leq m$.

Remark 3.2.3. Thus $\mathcal{M}(m;\underline{n}) \subset \mathrm{Aut}_0\, O(m;\underline{n})$ is isomorphic to the group of monomial matrices that respect the flag $\mathcal{F}(m;\underline{n})$.

Lemma 3.2.4. The subgroups $N_{\mathcal{W}(m;\underline{n})}(\mathcal{T}_W)$, $N_{\mathcal{S}(m;\underline{n})}(\mathcal{T}_S)$ and $N_{\mathcal{H}(m;\underline{n})}(\mathcal{T}_H)$ are contained in $\mathcal{M}(m;\underline{n})$.

Proof. We will show that $N_{\mathcal{W}(m;\underline{n})}(\mathcal{T}_X) \subset \mathcal{M}(m;\underline{n})$ for $X = W$, S, H. Since $\mathcal{X}(m;\underline{n}) \subset \mathcal{W}(m;\underline{n})$ we have $N_{\mathcal{X}(m;\underline{n})}(\mathcal{T}_X) \subset N_{\mathcal{W}(m;\underline{n})}(\mathcal{T}_X)$.

Let $\psi \in N_{\mathcal{W}(m;\underline{n})}(\mathcal{T}_X)$. For any $1 \leq i \leq m$ the element x_i is a common eigenvector of $\mathcal{T}_X$ so $\psi(x_i)$ is also a common eigenvector of $\mathcal{T}_X$. Also, since $\psi \in \mathrm{Aut}_c\, O(m;\underline{n})$, $\psi(x_i) = \sum\limits_{0<a\leq\tau(\underline{n})} \alpha_i(a) x^{(a)}$ where, among other conditions, $\alpha_i(\epsilon_{j_i}) \neq 0$ for some $1 \leq j_i \leq m$.

First we consider the case $X = W$ and $X = S$ (recall that $\mathcal{T}_W = \mathcal{T}_S$).

It is easy to see that the eigenspace decomposition of $O(m;\underline{n})$ with respect to $\mathcal{T}_W$ is the canonical $\mathbb{Z}^m$-grading on $O(m;\underline{n})$. The homogeneous space $O_a =$

$\mathrm{Span}\{x^{(a)}\}$ is the eigenspace with eigenvalue $\underline{t}^a := t_1^{a_1} \cdots t_m^{a_m}$ with respect to $\lambda(\underline{t}) \in \mathcal{T}_W$. It follows that $\psi(x_i) \in O_a$ for some $0 \le a \le \tau(\underline{n})$ since $\psi(x_i)$ is an eigenvector of $\mathcal{T}_X$. The condition that $\alpha_i(\epsilon_{j_i}) \neq 0$ for some $1 \le j_i \le m$ forces $a = \varepsilon_{j_i}$. Hence $\psi \in \mathcal{M}(m; \underline{n})$.

We continue with the case of $X = H$.

The torus $\mathcal{T}_H$ is contained in $\mathcal{T}_W$. In order for $\lambda(\underline{t}) \in \mathcal{T}_W$ to belong to $\mathcal{T}_H$, the m-tuple $\underline{t}$ must be H-admissible, i.e., satisfy $t_i t_{i'} = t_j t_{j'}$ for $1 \le i, j \le r$ where $m = 2r$. The eigenspace decomposition of $O(m; \underline{n})$ with respect to $\mathcal{T}_H$ is the grading by ϕ_H of the canonical $\mathbb{Z}^m$-grading. The eigenspace with eigenvalue $\underline{t}^a$ with respect to $\lambda(\underline{t}) \in \mathcal{T}_H$ is $Q_a := \bigoplus O_b$ where the direct sum is over the set of all b such that $\underline{t}^a = \underline{t}^b$ for all H-admissible $\underline{t}$. If $O_b \neq 0$ and $t_k = \underline{t}^b$ for all H-admissible $\underline{t}$, then $b = \varepsilon_k$ since all the entries in the m-tuple b are non-negative. This implies that $Q_{\varepsilon_k} = \mathrm{Span}\{x_k\}$. Now $\psi(x_i) \in Q_a$ for some $0 \le a \le \tau(\underline{n})$ since $\psi(x_i)$ is an eigenvector of $\mathcal{T}_H$. The condition that $\alpha_i(\epsilon_{j_i}) \neq 0$ for some $1 \le j_i \le 2r$ again forces $a = \varepsilon_{j_i}$. Hence $\psi \in \mathcal{M}(m; \underline{n})$. $\square$

For the case of contact algebras, similar arguments do not give us that $N_{W(m;\underline{n})}(\mathcal{T}_K)$ is in $\mathcal{M}(m; \underline{n})$.

Lemma 3.2.5. If $\psi \in N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)$, $m = 2r+1$ then

$$\psi(x_m) = \alpha_m(\varepsilon_m)x_m + \sum_{l=1}^{r} \alpha_m(\varepsilon_l + \varepsilon_{l'})x_l x_{l'} \tag{3.1}$$

and, for $1 \leq i \leq 2r$, we have $\psi(x_i) = \alpha_i(\varepsilon_{j_i})x_{j_i}$ where $1 \leq j_i \leq 2r$.

Proof. We will prove that any $\psi \in N_{\mathcal{W}(m;\underline{n})}(\mathcal{T}_K)$ has the form above. Since x_i is a common eigenvector of $\mathcal{T}_K$ we have that $\psi(x_i)$ is also a common eigenvector of $\mathcal{T}_K$. Since $\psi \in \mathrm{Aut}_c\, O(m;\underline{n})$, we have $\psi(x_i) = \sum_{0<a\leq\tau(\underline{n})} \alpha_i(a)x^{(a)}$ where, among other conditions, $\alpha_i(\epsilon_{j_i}) \neq 0$ for some $1 \leq j_i \leq m$.

The torus $\mathcal{T}_K$ is contained in $\mathcal{T}_W$. In order for a $\lambda(\underline{t})$ to be in $\mathcal{T}_K$, the m-tuple $\underline{t}$ must be K-admissible, i.e., $t_i t_{i'} = t_m$ for $1 \leq i \leq r$. The eigenspace decomposition of $O(m;\underline{n})$ with respect to $\mathcal{T}_K$ is the grading induced by ϕ_K with respect to the canonical $\mathbb{Z}^m$-grading. The eigenspace with eigenvalue $\underline{t}^a$ with respect to $\lambda(\underline{t}) \in \mathcal{T}_K$ is $R_a = \bigoplus O_b$ where the direct sum is over the set of b such that $\underline{t}^a = \underline{t}^b$ for all K-admissible $\underline{t}$. If $O_b \neq 0$ and $\underline{t}^b = t_k$ for all K-admissible $\underline{t}$, $1 \leq k \leq 2r$, then $b = \varepsilon_k$ since all entries of the m-tuple b are non-negative. This implies that $R_{\varepsilon_k} = \mathrm{Span}\{x_k\}$ for $1 \leq k \leq 2r$. If $O_b \neq 0$ and $\underline{t}^b = t_m$ for all K-admissible $\underline{t}$ then either $b = \varepsilon_m$ or $b = \varepsilon_i + \varepsilon_{i'}$ where $1 \leq i \leq r$. This implies that $R_{\varepsilon_m} = \mathrm{Span}\{x_m, x_i x_{i'} \mid 1 \leq i \leq r\}$.

Now $\psi(x_i) \in R_a$ for some $0 \leq a \leq \tau(\underline{n})$. The condition that $\alpha_i(\epsilon_{j_i}) \neq 0$ for some $1 \leq j_i \leq m$ forces $a = \varepsilon_{j_i}$. Note that the dimension of R_{ε_i} is 1 for

$1 \leq i \leq 2r$ and the dimension of R_{ε_m} is $r+1$. Hence, for $1 \leq i \leq 2r$, we have $\psi(x_i) = \alpha_i(\varepsilon_{j_i})x_{j_i}$ for some $1 \leq j_i \leq 2r$. Also, $\psi(x_m) \in R_{\varepsilon_m}$ which implies (3.1). $\square$

Proposition 3.2.6. Let $\mathcal{Q}$ be a quasi-torus contained in $N_{\mathcal{X}(m;\underline{n})}(\mathcal{T}_X)$ where $X = W$ or S. Then there exists $\psi \in \mathrm{GL}(m;\underline{n})$ such that $\psi\mathcal{Q}\psi^{-1} \subset \mathcal{T}_X$.

Proof. Recall the flag $\mathcal{F}(m;\underline{n})$,

$$V_0 \subset V_1 \subset \cdots \subset V.$$

Let $U_i = \mathrm{Span}\{x_j \mid x_j \in V_i, x_j \notin V_{i-1}\}$. Then $V_i = \bigoplus_{j=1}^{i} U_j$. Since $\mathcal{Q} \subset \mathcal{M}(m;\underline{n})$ we have $\mathcal{Q}(U_i) = U_i$. (Here, as before, we identify $\mathrm{Aut}_0\, O(m;\underline{n})$ with a subgroup of $\mathrm{GL}(m)$.)

Restricting the action of $\mathcal{Q}$ to U_i, we obtain a subgroup of $\mathrm{GL}(U_i)$. Since $\mathcal{Q}|_{U_i}$ is a quasi-torus, there exists a $P_i \in \mathrm{GL}(U_i)$ such that $P_i(\mathcal{Q}|_{U_i})P_i^{-1}$ is diagonal. We can extend the action of P_i to the whole space V by setting $P_i(y) = y$ for all $y \in U_j$, $i \neq j$. These extended P_i respect $\mathcal{F}(m;\underline{n})$. The product of these transformations, $P = P_1 \cdots P_l$, is an element of $\mathrm{Aut}_0\, O(m;\underline{n})$, which diagonalizes $\mathcal{Q}$. $\square$

In order to obtain the analog of Proposition 3.2.6 in the case of Hamiltonian algebras, we consider the canonical skew-symmetric inner product $\langle\, ,\rangle$

on V given by $\langle x_j, x_k \rangle = \sigma(j)\delta_{j,k'}$, for all $j, k = 1, \ldots, 2r$. Let $\mathrm{Sp}(m)$ be the symplectic group on V with respect to the skew-symmetric inner product $\langle\ ,\ \rangle$ and $\mathrm{Sp}(m; \underline{n}) = \mathrm{Sp}(m) \cap \mathrm{GL}(m; \underline{n})$.

Lemma 3.2.7. Let $\mathcal{Q}$ be a quasi-torus contained in $\mathrm{Sp}(m; \underline{n})$. Then there is a basis $\{f_i\}_{i=1}^{2r}$ of V such that $\langle f_j, f_k \rangle = \sigma(j)\delta_{j,k'}$, where all f_j are common eigenvectors of $\mathcal{Q}$, and $V_i = \mathrm{Span}\{f_j \mid j \in \Xi_i\}$ for all i.

Proof. We can decompose $V = \bigoplus V^\gamma$ where V^γ are the eigenspaces in V with respect to $\mathcal{Q}$, indexed by $\gamma \in \mathfrak{X}(\mathcal{Q})$ where $\mathfrak{X}(\mathcal{Q})$ is the group of characters of $\mathcal{Q}$. Since $\mathcal{Q}(V_i) = V_i$ for any i, there is a basis $\{y_j\}_{j=1}^{2r}$ such that y_j are eigenvectors of $\mathcal{Q}$ and each $V_i = \mathrm{Span}\{y_j \mid j \in \Xi_i\}$.

We have $\langle x_j, x_k \rangle = \sigma(j)\delta_{j,k'}$. We will show by induction on r that there is a basis $\{f_j\}_{j=1}^{2r}$ such that $\langle f_j, f_k \rangle = \sigma(j)\delta_{j,k'}$, all f_j are common eigenvectors of $\mathcal{Q}$, and $V_i = \mathrm{Span}\{f_j \mid j \in \Xi_i\}$. The base case $r = 1$ is obvious.

We have a basis $\{y_j\}_{j=1}^{2r}$ such that y_j are eigenvectors of $\mathcal{Q}$ and each $V_i = \mathrm{Span}\{y_j \mid j \in \Xi_i\}$. We apply a process similar to the Gram–Schmidt process to find a new basis of common eigenvectors for $\mathcal{Q}$ that satisfies the desired conditions.

Since $V_1 \neq 0$ and $\langle , \rangle$ is nondegenerate, $\langle V_1, V_l \rangle \neq 0$ for some l. Let l be minimal, i.e., $\langle V_1, V_l \rangle \neq 0$ and $\langle V_1, V_i \rangle = 0$ for $i < l$. So there exist $y_s \in V_1$

and $y_t \in V_l$ such that $\langle y_s, y_t \rangle \neq 0$ and $\langle y_s, V_i \rangle = 0$ if $i < l$ by the minimality of l. Let $\gamma_j \in \mathfrak{X}(\mathcal{Q})$ be the eigenvalue of y_j. Since $\mathcal{Q}$ consists of symplectic transformations, we have $\gamma_s = \gamma_t^{-1}$.

For $j \neq s, t$, let $z_j = \langle y_s, y_t \rangle y_j - \langle y_s, y_j \rangle y_t + \langle y_t, y_j \rangle y_s$. The z_j with y_s and y_t form a basis of V. They also satisfy

$$\langle y_s, z_j \rangle = \langle y_s, y_t \rangle \langle y_s, y_j \rangle - \langle y_s, y_j \rangle \langle y_s, y_t \rangle + \langle y_t, y_j \rangle \langle y_s, y_s \rangle = 0,$$

and similarly $\langle y_t, z_j \rangle = 0$.

We also have the property that $z_j \in V_i$ if and only if $y_j \in V_i$. Indeed, for $i < l$ and $y_j \in V_i$, we have $\langle y_s, y_j \rangle = 0$ by the minimality of l. This shows that z_j is in $V_l + V_i$ and hence $z_j \in V_i$. For $i \geq l$ and $y_j \in V_i$ we have $y_j, y_t, y_s \in V_i$ which implies $z_j \in V_i$. Finally, we want to verify that z_j are common eigenvectors of $\mathcal{Q}$. There are three cases to consider.

Case 1: $\gamma_j \neq \gamma_s^{\pm 1}$

Recall that $\gamma_s = \gamma_t^{-1}$. Since $\gamma_j \neq \gamma_s^{\pm 1}$, we have $\langle y_s, y_j \rangle = \langle y_t, y_j \rangle = 0$. This means $z_j = \langle y_s, y_t \rangle y_j$, which is an eigenvector with eigenvalue γ_j.

Case 2: $\gamma_j = \gamma_s$ or γ_s^{-1}, and $\gamma_s \neq \gamma_s^{-1}$.

Suppose $\gamma_j = \gamma_s$. Since $\gamma_j \neq \gamma_s^{-1}$, we have $\langle y_s, y_j \rangle = 0$. This means $z_j = \langle y_s, y_t \rangle y_j + \langle y_t, y_j \rangle y_s$, which is an eigenvector with eigenvalue $\gamma_j = \gamma_s$. A similar argument applies if $\gamma_j = \gamma_s^{-1}$.

Case 3: $\gamma_j = \gamma_s = \gamma_s^{-1}$.

Since $z_j = \langle y_s, y_t \rangle y_j - \langle y_s, y_j \rangle y_t + \langle y_t, y_j \rangle y_s$ and $\gamma_s = \gamma_s^{-1} = \gamma_t$, we see that z_j is an eigenvector with eigenvalue γ_j.

In order to use the induction hypothesis we relabel our basis as follows: Pick $x_q \in V_1$ with $\langle x_q, V_l \rangle \neq 0$. Since $\langle V_1, V_{l-1} \rangle = 0$, we have $x_{q'} \in V_l$ and $x_{q'} \notin V_{l-1}$. Set $w_q = y_s$, $w_{q'} = y_t$, $w_s = z_q$, $w_t = z_{q'}$ and $w_j = z_j$ for $j \neq q, q', s, t$. The relabelled basis still satisfies $V_i = \mathrm{Span}\{w_j \mid j \in \Xi_i\}$ since $z_{q'}, y_t \in V_l$, and $z_q, y_s \in V_1$. Also, $\langle w_q, w_j \rangle = \langle w_{q'}, w_j \rangle = 0$ for $j \neq q, q'$, $\langle w_q, w_{q'} \rangle \neq 0$, and w_j are eigenvectors of Q.

Let $V' = \mathrm{Span}\{w_j \mid 1 \leq j \leq 2r,\ j \neq q, q'\}$. Then V' is invariant under Q, and the quasi-torus $Q|_{V'}$ satisfies the conditions of the lemma with the flag

$$V'_i = \mathrm{Span}\{w_j \mid j \in \Xi_i \setminus \{q, q'\}\} = V_i \cap V'.$$

Since $\dim V' < \dim V$, we can apply the induction hypothesis and find a basis $\{f_j\}_{j \neq q,q'}$, $1 \leq j \leq 2r$, such that $\langle f_j, f_k \rangle = \sigma(j)\delta_{j,k'}$, where f_j are eigenvectors of $Q|_{V'}$ and $V'_i = \mathrm{Span}\{f_j \mid j \in \Xi_i \setminus \{q, q'\}\}$.

In order to have a complete basis for V we set $f_q = \dfrac{\sigma(q)}{\langle w_q, w_{q'} \rangle} w_q$ and $f_{q'} = w_{q'}$. Then the basis $\{f_j\}_{j=1}^{2r}$ is a basis with the desired properties, and the induction step is proven. □

In the following proofs we will deal with the differential forms ω_H and ω_K.

Proposition 3.2.8. Let $\mathcal{Q}$ be a quasi-torus contained in $N_{\mathcal{H}(m;\underline{n})}(\mathcal{T}_H)$. Then there exists $\psi \in \mathrm{Sp}(m;\underline{n})$ such that $\psi\mathcal{Q}\psi^{-1} \subset \mathcal{T}_H$.

Proof. By Lemma 3.2.4, $N_{\mathcal{H}(m;\underline{n})}(\mathcal{T}_H) \subset \mathcal{M}(m;\underline{n})$, which we regard as a subgroup of $\mathrm{GL}(m)$. Recall that any element of $\mathrm{Aut}_0\, O(m;\underline{n}) \cap \mathcal{H}(m;\underline{n})$ can be written as αS where $\alpha \in F^\times$ and $S \in \mathrm{Sp}(m;\underline{n})$. Let

$$\mathcal{Q}' = \{S \in \mathrm{Sp}(m;\underline{n}) \,|\, \text{there exists } \alpha \in F^\times \text{ such that } \alpha S \in \mathcal{Q}\}.$$

$\mathcal{Q}'$ is a quasi-torus since $\mathcal{Q}$ is a quasi-torus.

Let $\{f_j\}_{j=1}^m$ be a basis as in Lemma 3.2.7 with respect to $\mathcal{Q}'$, and define $\psi : V \to V$ by $\psi(x_j) = f_j$ for $1 \le j \le m$. Since $\langle\psi(x_j), \psi(x_k)\rangle = \langle f_j, f_k\rangle = \sigma(j)\delta_{j,k'} = \langle x_j, x_k\rangle$, we have $\psi \in \mathrm{Sp}(m)$. Since $V_i = \mathrm{Span}\{f_j \mid j \in \Xi_i\} = \mathrm{Span}\{x_j \mid j \in \Xi_i\}$, we have $\psi(V_i) = V_i$. Hence $\psi \in \mathrm{Sp}(m;\underline{n})$. Since f_j are common eigenvectors of $\mathcal{Q}'$, we have $\psi^{-1}\mathcal{Q}'\psi \subset \mathcal{T}_H$. Since every element of $\mathcal{Q}$ has the form αS with $\alpha \in F^\times$ and $S \in \mathcal{Q}'$, we have $\psi^{-1}\mathcal{Q}\psi \subset \mathcal{T}_H$. Replacing ψ with ψ^{-1}, we get the result. $\square$

In order to get a similar result for the contact algebras, we use the Hamiltonian algebras contained in them. Let $m = 2r + 1$, $\underline{n} = (n_1, \dots, n_{2r+1})$ and $\underline{n}' = (n_1, \dots, n_{2r})$.

Lemma 3.2.9. Let $\psi \in \mathrm{Aut}_0\, O(2r;\underline{n}')$. If $\psi(\omega_H) = \alpha\omega_H$ then there exists $\overline{\psi} \in \mathcal{K}(2r+1,\underline{n})$ such that $\overline{\psi}|_{O(2r;\underline{n}')} = \psi$ and $\overline{\psi}(x_{2r+1}) = \alpha x_{2r+1}$.

Proof. Suppose $\psi \in \mathrm{Aut}_0\, O(2r,\underline{n})$ given by $\psi(x_i) = \sum_{j=1}^{2r} \alpha_{i,j}x_j$ has the property $\psi(\omega_H) = \alpha\omega_H$. Since

$$\begin{aligned}
\psi(\omega_H) &= \psi\left(\sum_{i=1}^{r} dx_i \wedge dx_{i+r}\right) = \sum_{i=1}^{r} d(\psi(x_i)) \wedge d(\psi(x_{i+r})) \\
&= \sum_{i=1}^{r} d\left(\sum_{j=1}^{2r} \alpha_{i,j}x_j\right) \wedge d\left(\sum_{k=1}^{2r} \alpha_{i+r,k}x_k\right) \\
&= \sum_{1\le j<k\le 2r} \left(\sum_{i=1}^{r} \alpha_{i,j}\alpha_{i+r,k} - \alpha_{i,k}\alpha_{i+r,j}\right) dx_j \wedge dx_k
\end{aligned}$$

we have $\sum_{i=1}^{r} \alpha_{i,j}\alpha_{i+r,k} - \alpha_{i,k}\alpha_{i+r,j} = \delta_{k,j+r}\alpha$ for $1 \le j \le r$.

Define $\overline{\psi} \in \mathrm{Aut}_c\, O(2r+1,\underline{n})$ by setting $\overline{\psi}(x_i) = \psi(x_i)$, $1 \le i \le 2r$, and $\overline{\psi}(x_m) = \alpha x_m$. Then

$$\begin{aligned}
\overline{\psi}(\omega_K) &= \overline{\psi}\,(dx_m) + \overline{\psi}\left(\sum_{i=1}^{r} (x_i dx_{i+r} - x_{i+r}dx_i)\right) \\
&= d(\alpha x_m) + \sum_{i=1}^{r}\left(\left(\sum_{j=1}^{2r} \alpha_{i,j}x_j\right) d\left(\sum_{k=1}^{2r} \alpha_{i+r,k}x_k\right)\right. \\
&\qquad \left. - \left(\sum_{k=1}^{2r} \alpha_{i+r,k}x_k\right) d\left(\sum_{j=1}^{2r} \alpha_{i,j}x_j\right)\right)
\end{aligned}$$

$$
\begin{aligned}
\overline{\psi}(\omega_K) &= \alpha dx_m + \sum_{1\le j<k\le 2r}\left(\sum_{i=1}^{r}\alpha_{i,j}\alpha_{i+r,k}-\alpha_{i,k}\alpha_{i+r,j}\right)x_j dx_k \\
&\quad + \sum_{1\le j<k\le 2r}\left(\sum_{i=1}^{r}\alpha_{i,k}\alpha_{i+r,j}-\alpha_{i,j}\alpha_{i+r,k}\right)x_k dx_j \\
&= \alpha dx_m + \sum_{j=1}^{r}\alpha x_j dx_{j+r}+\sum_{j=1}^{r}(-\alpha)x_{j+r}dx_j \\
&= \alpha\omega_K
\end{aligned}
$$

Therefore, $\overline{\psi}\in\mathcal{K}(2r+1,\underline{n})$. $\square$

Proposition 3.2.10. Let $\mathcal{Q}$ be a quasi-torus contained in $N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)$. Then there exists $\psi\in\mathcal{K}(m;\underline{n})$ such that $\psi\mathcal{Q}\psi^{-1}\subset\mathcal{T}_K$.

Proof. Let $\mu\in\mathcal{Q}\subset N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)$. By Lemma 3.2.5, we have $\mu(x_i)=\alpha_i x_{j_i}$ for $1\le i\le 2r$ and $\mu(x_m)=\alpha_m x_m+\sum_{l=1}^{r}\beta_l x_l x_{l'}$. Since $\mu\in\mathcal{K}(m;\underline{n})$, we must have $\mu(\omega_K)\in\mathcal{O}(m;\underline{n})^{\times}\omega_K$. On the other hand,

$$
\begin{aligned}
\mu(\omega_K) &= \mu\left(dx_m+\sum_{i=1}^{r}(x_i dx_{i'}-x_{i'}dx_i)\right) \\
&= d\left(\alpha_m x_m+\sum_{l=1}^{r}\beta_l x_l x_{l'}\right)+\sum_{i=1}^{r}\alpha_i\alpha_{i'}(x_{j_i}dx_{j_{i'}}-x_{j_{i'}}dx_{j_i}) \\
&= \alpha_m dx_m+\sum_{l=1}^{r}\beta_l(x_l dx_{l'}+x_{l'}dx_l)+\sum_{i=1}^{r}\alpha_i\alpha_{i'}(x_{j_i}dx_{j_{i'}}-x_{j_{i'}}dx_{j_i})
\end{aligned}
$$

It follows that $\mu(\omega_K)=\alpha_m\omega_K$ since the only term with dx_m is $\alpha_m dx_m$.

We want to show that $\mu|_{\mathcal{O}(2r;\underline{n}')}$ belongs to $\mathcal{H}(2r;\underline{n}')$. Indeed,

$$\begin{aligned} d\omega_K &= d\left(dx_m + \sum_{i=1}^{r}(x_i dx_{i'} - x_{i'} dx_i)\right) = \sum_{i=1}^{r}(d(x_i dx_{i'}) - d(x_{i'} dx_i)) \\ &= \sum_{i=1}^{r}(dx_i \wedge dx_{i'} - dx_{i'} \wedge dx_i) = 2\omega_H, \end{aligned}$$

and hence

$$2\mu(\omega_H) = \mu(2\omega_H) = \mu(d\omega_K) = d\mu(\omega_K) = d(\alpha_m \omega_K) = 2\alpha_m \omega_H,$$

so $\mu|_{O(2r,\underline{n}')} \in \mathcal{H}(2r;\underline{n}')$.

We have shown that $N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)|_{O(2r;\underline{n}')} \subset \mathcal{H}(2r;\underline{n}')$. Moreover, since the restriction of $\mathcal{T}_K$ to $O(2r;\underline{n}')$ is $\mathcal{T}_H$, where $\mathcal{T}_H$ is the maximal torus in $\mathcal{H}(2r;\underline{n}')$, we have $N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)|_{O(2r,\underline{n}')} \subset N_{\mathcal{H}(2r;\underline{n}')}(\mathcal{T}_H)$. By Proposition 3.2.8, there exists $\psi \in \mathrm{Sp}(2r;\underline{n}')$ such that $\psi(\mathcal{Q}|_{O(2r,\underline{n}')})\psi^{-1} \subset \mathcal{T}_H$.

By Lemma 3.2.9, we can extend ψ to an automorphism $\overline{\psi}$ in $\mathcal{K}(m;\underline{n})$ such that $\overline{\psi}(x_i) = \psi(x_i)$ for $1 \leq i \leq 2r$ and $\overline{\psi}(x_m) = x_m$ (since $\psi(\omega_H) = \omega_H$). Let $\mu \in \mathcal{Q}$ as before and set $\rho = \overline{\psi}\mu\overline{\psi}^{-1}$. Then $\rho(x_i) = \gamma_i x_i$ for $1 \leq i \leq 2r$, $\gamma_i \in F^{\times}$, and $\rho(x_m) = \alpha_m x_m + y$ where $y = \psi\left(\sum_{l=1}^{r} \beta_l x_l x_{l'}\right)$. Furthermore,

$$\begin{aligned} \rho(\omega_H) &= \overline{\psi}\mu\overline{\psi}^{-1}(\omega_H) = \overline{\psi}\mu(\omega_H) = \overline{\psi}(\alpha_m\omega_H) = \alpha_m\omega_H \quad \text{and also} \\ \rho(\omega_H) &= \rho\left(\sum_{l=1}^{r} dx_l \wedge dx_{l'}\right) = \sum_{l=1}^{r} d\rho(x_l) \wedge d\rho(x_{l'}) = \sum_{l=1}^{r} \gamma_l\gamma_{l'} dx_l \wedge dx_{l'}. \end{aligned}$$

Hence we conclude that $\gamma_l\gamma_{l'} = \alpha_m$ for $1 \le l \le r$ and

$$\begin{aligned} \rho(\omega_K) &= \alpha_m dx_m + dy + \sum_{i=1}^{r} \gamma_i\gamma_{i'}(x_{j_i}dx_{j_{i'}} - x_{j_{i'}}dx_{j_i}) \\ &= \alpha_m x_m + dy + \alpha_m \sum_{l=1}^{r}(x_l dx_{l'} - x_{l'}dx_l) = \alpha_m\omega_K + dy. \end{aligned}$$

On the other hand, $\rho(\omega_K) = \overline{\psi}\mu\overline{\psi}^{-1}(\omega_K) = \overline{\psi}\mu(\omega_K) = \overline{\psi}(\alpha_m\omega_K) = \alpha_m\omega_K$. Since $dy = d\left(\psi\left(\sum_{l=1}^{r}\beta_l x_l x_{l'}\right)\right) = \psi\left(d\left(\sum_{l=1}^{r}\beta_l x_l x_{l'}\right)\right)$, we obtain:

$$0 = d\left(\sum_{l=1}^{r}\beta_l x_l x_{l'}\right) = \sum_{l=1}^{r}\beta_l(x_l dx_{l'} + x_{l'}dx_l),$$

which implies $\beta_l = 0$ for $1 \le l \le r$. It follows that $\rho \in \mathcal{T}_K$. □

3.3 Normalizers of Tori for the Melikyan Case

In this section we will use Proposition 3.2.1 to get a result similar to Propositions 3.2.6, 3.2.8 and 3.2.10 for the Melikyan algebras. We need a maximal torus of Aut $M(2;\underline{n})$. We will show that the torus T_M is a maximal torus by showing that it is its own centralizer.

Corollary 3.3.1. If $\Psi \in \mathrm{Aut}_W\, M(2;\underline{n})$ is such that $\pi(\Psi) \in T_W$ then $\Psi \in T_M$.

Proof. Lemma 1.3.13 shows that there exists $\Upsilon \in T_M$ such that $\pi(\Psi) = \pi(\Upsilon)$ and Corollary 2.3.22 says that $\Psi = \Upsilon\Theta^l$ for some $1 \le l \le 2$. Hence $\Psi \in T_M$. □

In order to describe the normalizers of T_M in $\operatorname{Aut} M(2;\underline{n})$ we introduce the automorphism ξ of $O(2;\underline{n})$ that induces an automorphism ς of $W(2;\underline{n})$ and finally we extend ς to an automorphism of $\operatorname{Aut} M(2;\underline{n})$. For $\underline{n} = (n_1, n_2)$ we define $\overline{a} := (a_2, a_1)$ for $a = (a_1, a_2) \in \mathbb{Z}^2$. For $n_1 = n_2$ let ξ and ς be the linear maps of $O(2;\underline{n})$ and $W(2;\underline{n})$ respectively, defined by $\xi(x^{(a)}) := x^{\overline{a}}$ and $\varsigma(D) := \xi D \xi^{-1}$ for $x^{(a)} \in O(2;\underline{n})$ and $D \in W(2;\underline{n})$. It follows from Lemma 2.3.2 and Theorem 2.3.5 that when $n_1 = n_2$, ς is an automorphism of $W(2;\underline{n})$. Let $U_M = \langle \varsigma_M \rangle$ when $n_1 = n_2$ and identity otherwise. The restriction of U_M to $W(2;\underline{n})$ is $M(2;\underline{n})$.

Proposition 3.3.2. The normalizer of T_M in $\operatorname{Aut} M$ is $T_M U_M$.

Proof. In [17, p. 3921] it is stated that we can decompose $\Psi \in \operatorname{Aut} M$ as the product of $\Psi = \Upsilon\Lambda$ where Υ, $\Lambda \in \operatorname{Aut} M(2;\underline{n})$ are such that for all $i \in \mathbb{Z}$ we have

$$\Upsilon(y) \in y + M_{(i+1)}, \quad \text{for } y \in M_i,$$

$$\Lambda(M_i) = M_i.$$

Let $\Psi \in N_{\operatorname{Aut} M(2;\underline{n})}(T_M)$. We will show that $\Upsilon = \operatorname{Id}_M$.

Let $y \in M_i$ be a nonzero eigenvector of T_M. Since $\Psi \in N_{\operatorname{Aut} M(2;\underline{n})}(T_M)$ we have $\Psi(y)$ is an eigenvector. The eigenspaces of T_M are the $M_{(a_1,a_2)}$, where

$(a_1, a_2) \in \mathbb{Z}^2$. It is easy to see that $M_l = \bigoplus_{a \in \mathbb{Z}^2,\ a_1+a_2=l} M_{(a_1,a_1)}$ for all $l \in \mathbb{Z}$. This gives us that $\Psi(y) \in M_k$ for some $k \in \mathbb{Z}$. Now we use the decomposition of $\Psi = \Upsilon\Lambda$. We have $\Lambda(y) = w \in M_i$ since $\Lambda(M_l) = M_l$ for all l.

$$\Psi(y) = \Upsilon\Lambda(y) = \Upsilon(w) \in w + M_{(i+1)}. \tag{3.2}$$

Since $w \neq 0$ the calculations above show that $\Psi(y) \in M_{(i)}$ and $\Psi(y) \notin M_{(i+1)}$.

The intersection of M_k and $M_{(i)} = \bigoplus_{j \geq i} M_j$ is zero if $k < i$. Since $\Psi(y) \in M_{(i)}$ by (3.2) and $0 \neq \Psi(y) \in M_k$ we have $k \geq i$. Since $\Psi(y) \notin M_{(i+1)}$ by (3.2) and $\psi(y) \in M_k$ we have $k \leq i$. Hence $k = i$. We have shown that $\Psi(M_i) = M_i$ for all i. Hence $\Upsilon = \mathrm{Id}_M$. Since $W(2;\underline{n}) = \bigoplus_{i \in \mathbb{Z}} M_{3i}$ we have $\Psi(W(2;\underline{n})) = W(2;\underline{n})$. We conclude that if $\Psi \in N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$ then Ψ preserves the standard $\mathbb{Z}$-grading of $M(2;\underline{n})$ and that $\pi(\Psi) \in N_{\mathrm{Aut}\, W(2;\underline{n})}(T_W)$ since $\pi(T_M) = T_W$ (Lemma 1.3.13).

It follows from Lemma 3.2.4 that the $N_{\mathrm{Aut}\, W(2;\underline{n})}(T_W) = T_W$ when $n_1 \neq n_2$ and $T_W\langle\varsigma\rangle$ if $n_1 = n_2$. By Corollary 3.3.1, the set of automorphisms of $M(2;\underline{n})$ which, when restricted to $W(2;\underline{n})$ are in T_W is T_M. For $n_1 = n_2$, Corollary 2.3.22 says that if $\Psi \in \mathrm{Aut}_W\, M(2;\underline{n})$ and $\pi(\Psi) = \rho\varsigma$, where $\rho \in T_W$ then there exists a $\Xi \in T_M$ such that $\pi(\Xi) = \rho$ and $\Psi = \Xi\varsigma_M\Theta^l$ for $0 \leq l \leq 2$. The automorphism Θ is in T_M since $\Theta = \lambda(\beta^2, \beta^2)$. Hence, $N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M) \subset T_M U_M$.

Conversely let $\Psi \in \mathrm{Aut}_W M(2;\underline{n})$ be such that $\pi(\Psi) \in N_{\mathrm{Aut}\, W(2;\underline{n})}(T_W)$. Then

$$\pi(\Psi\lambda(\underline{t})\Psi^{-1}) = \pi(\Psi)\pi(\lambda(\underline{t}))\pi(\Psi)^{-1} \in T_W.$$

By Corollary 3.3.1, we have that $\Psi\lambda(\underline{t})\Psi^{-1} \in T_M$ and hence it follows that $\Psi \in N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$. Since for $n_1 = n_2$ we have $\pi(\varsigma_m) = \varsigma \in N_{\mathrm{Aut}\, W(2;\underline{n})}(T_W)$ it follows that $\varsigma_M \in N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$. We have shown that $T_M U_M \subset N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$.

□

Corollary 3.3.3. The centralizer of T_M in $\mathrm{Aut}\, M(2;\underline{n})$ is T_M. Hence, T_M is a maximal torus.

Proof. The centralizer of T_M is contained in the normalizer of T_M. By Proposition 3.3.2, the normalizer of T_M is in $\mathrm{Aut}_W M(2;\underline{n})$. This implies that if $\Psi \in \mathrm{Aut}_W M(2;\underline{n})$ and Ψ is in the centralizer of T_M then $\pi(\Psi)$ must be in the centralizer of $\pi(T_M) = T_W$ (Lemma 3.2.4). Since T_W is a maximal torus and $\varsigma \notin T_W$ (for $n_1 = n_2$) we have that ς is not in the centralizer of T_W. Hence ς_M is not in the centralizer of T_M. We have shown that the centralizer of T_M in $\mathrm{Aut}\, M(2;\underline{n})$ is T_M. □

Proposition 3.3.4. Let Q be a quasi-torus in $\mathrm{Aut}\, M(2;\underline{n})$. There is an automorphism $\Psi \in \mathrm{Aut}\, M(2;\underline{n})$ such that $\Psi Q \Psi^{-1} \subset T_M$.

Proof. By Proposition 3.2.1, Q is inside the normalizer of a maximal torus. Up to conjugation we can assume $Q \subset N_{\operatorname{Aut} M(2;\underline{n})}(T_M)$. Then Q must preserve $W(2;\underline{n})$ since $N_{\operatorname{Aut} M(2;\underline{n})}(T_M) = T_M U_M$ (Proposition 3.3.2). Let $Q' = \pi(Q)$. It follows that $Q' \subset N_{\operatorname{Aut} W(2;\underline{n})}(T_W)$. By Proposition 3.2.6 there exists a $\psi \in \operatorname{Aut} W(2;\underline{n})$ such that $\psi Q' \psi^{-1} \subset T_W$. Proposition 3.2.8 says that there is $\Psi \in \operatorname{Aut} M(2;\underline{n})$ such that $\pi(\Psi) = \psi$. Hence $\pi(\Psi Q \Psi^{-1}) = \psi Q' \psi^{-1} \subset T_W$. Since $\pi(\Psi Q \Psi^{-1}) \subset T_W$, Corollary 3.3.1 gives us that $\Psi Q \Psi^{-1} \subset T_M$. $\square$

Now we can prove Proposition 3.0.8.

Proof. Let Q be a maximal quasi-torus of Aut L. By Propositions 3.2.6, 3.2.8, 3.2.10 and 3.3.4 there exists a $\psi \in \operatorname{Aut} L$ such that $\psi Q \psi^{-1} \subset T_X$. Hence Q is inside the torus $\psi^{-1} T_X \psi$. Since $\psi^{-1} T_X \psi$ is a quasi-torus and Q is not properly contained in a quasi-torus we have that $Q = \psi^{-1} T_X \psi$. $\square$

Chapter 4

Main Results

We start this chapter by proving our main theorem. We then show examples where the grading group has elements of order p and the gradings are not isomorphic to a standard grading. In Section 4.2 we describe all gradings by groups on $W(1;1)$.

4.1 General Results

Theorem 4.1.1. Let $L = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ or $M(2;\underline{n})$, $p = 5$ if $L = M(2;\underline{n})$ and $p \neq 3$ if $L = W(1;1)$ or $H(2;\underline{n})^{(2)}$ where $\min(n_1, n_2) = 1$. Suppose $\Gamma : L = \bigoplus_{g\in G} L_g$ is a G-grading where G is a group without p-torsion. Without loss of generality, we assume that the support of the grading generates G. Then the grading is isomorphic to a standard G-grading.

Proof. Let $\eta_\Gamma : \widehat{G} \to \operatorname{Aut} L$ be the corresponding embedding and $Q := \eta_\Gamma(\widehat{G})$. Then Q is a quasi-torus in $\operatorname{Aut} L$. By Proposition 3.0.8, we can conjugate Q by an automorphism Ψ of L so that $\Psi Q \Psi^{-1} \subset T_X$. It follows from Corollary 3.1.4 that the grading $L = \bigoplus_{g\in G} L'_g$, where $L'_g = \Psi(L_g)$, is a standard G-grading. Hence $L = \bigoplus_{g\in G} L_g$ is isomorphic to $L = \bigoplus_{g\in G} L'_g$. $\square$

We still have standard gradings by groups with elements of order p since there are non trivial homomorphisms from $\mathbb{Z}^k$ to groups with elements of order p for $k \geq 1$. For each abelian group G with an element of order p there is at least one semisimple derivation with eigenvalues in $\mathbb{Z}_p$ by Corollary 1.3.14. In Lemma 4.1.2 and Lemma 4.1.3 we deal with standard $\mathbb{Z}_p$-gradings on the simple graded Cartan and Melikyan type algebras, respectively. In Corollary 4.1.5 we give examples of non trivial $\mathbb{Z}_p$-gradings on $W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ and $M(2;\underline{n})$.

Lemma 4.1.2. Let $L = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ or $K(m;\underline{n})^{(1)}$ and $p \neq 3$ if $L = W(1;1)$ or $H(2;\underline{n})^{(2)}$ where $\min(n_1, n_2) = 1$. If Γ is a standard $\mathbb{Z}_p$-grading on L induced by a homomorphism $\phi : \mathbb{Z}^k \to \mathbb{Z}_p$ then Γ is the eigenspace decomposition on L of the derivation

$$\operatorname{ad}\left(\sum_{i=1}^{m}\phi(\varepsilon_i)x_i\partial_i\right).$$

Proof. Recall that the homogeneous spaces of a standard grading on $L = X(m;\underline{n})^{(\infty)}$ induced by a homomorphism ϕ are $X_{\phi(b)} = \operatorname{Span}\{x^{(a)}\partial_j \mid \phi(a-\varepsilon_j) = \phi(b)\} \cap L$. Since

$$\operatorname{ad}\left(\sum_{i=1}^{m}\phi(\varepsilon_i)x_i\partial_i\right)(x^{(a)}\partial_j) = \sum_{i=1}^{m}(a_i-\delta_{i,j})\phi(\varepsilon_i)x^{(a)}\partial_j = \phi(a-\varepsilon_j)x^{(a)}\partial_j$$

for all $a \in \mathbb{Z}^m$ we have that the eigenspaces of $\operatorname{ad}\left(\sum_{i=1}^{m}\phi(\varepsilon_i)x_i\partial_i\right)$ are $X_{\phi(b)}$. $\square$

Lemma 4.1.3. If Γ is a standard $\mathbb{Z}_5$-grading on $M(2;\underline{n})$ induced by a homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}_5$ then Γ is the eigenspace decomposition on $M(2;\underline{n})$ of the derivation

$$\operatorname{ad}\left(\phi((1,0))x_1\partial_1 + \phi((-1,3))x_2\partial_2\right).$$

Proof. Recall that with respect to the canonical $\mathbb{Z}^2$-grading

$$\begin{aligned}
\deg(x^{(a)}\partial_i) &= (a_1-\delta_{1,j})(1,0) + (a_2-\delta_{2,j})(-1,3),\\
\deg(x^{(a)}\widetilde{\partial}_i) &= (a_1-\delta_{1,j})(1,0) + (a_2-\delta_{2,j})(-1,3) + (0,1),\\
\deg(x^{(a)}) &= (a_1-\delta_{1,j})(1,0) + (a_2-\delta_{2,j})(-1,3) - (0,1).
\end{aligned}$$

Then for any standard $\mathbb{Z}_5$-grading $\overline{\Gamma}$ induced by ϕ we would have

$$\begin{aligned}
\deg_{\overline{\Gamma}}(x^{(a)}\partial_i) &= (a_1-\delta_{1,j})\phi((1,0))+(a_2-\delta_{2,j})\phi((-1,3)),\\
\deg_{\overline{\Gamma}}(x^{(a)}\widetilde{\partial}_i) &= (a_1-\delta_{1,j})\phi((1,0))+(a_2-\delta_{2,j})\phi((-1,3))+\phi((0,1)),\\
\deg_{\overline{\Gamma}}(x^{(a)}) &= a_1\phi((1,0))+a_2\phi((-1,3))-\phi((0,1)).
\end{aligned}$$

Let $y=\phi((1,0))x_1\partial_1+\phi((-1,3))x_2\partial_2$. Then

$$\begin{aligned}
\mathrm{ad}(y)(x^{(a)}\partial_i) &= \phi((1,0))[x_1\partial_1,x^{(a)}\partial_i]+\phi((-1,3))[x_2\partial_2,x^{(a)}\partial_i]\\
&= \phi((1,0))(a_1-\delta_{1,i})x^{(a)}\partial_i+\phi((-1,3))(a_2-\delta_{2,i})x^{(a)}\partial_i\\
&= ((a_1-\delta_{i,j})\phi((1,0))+(a_2-\delta_{i,j})\phi((-1,3)))x^{(a)}\partial_i\\
&= \deg_{\overline{\Gamma}}(x^{(a)}\partial_i)x^{(a)}\partial_i,
\end{aligned}$$

$$\begin{aligned}
\mathrm{ad}(y)(x^{(a)}\widetilde{\partial}_i) &= \phi((1,0))[x_1\partial_1,x^{(a)}\widetilde{\partial}_i]+\phi((-1,3))[x_2\partial_2,x^{(a)}\widetilde{\partial}_i]\\
&= \phi((1,0))(a_1-\delta_{1,i}+2)x^{(a)}\widetilde{\partial}_i\\
&\quad+\phi((-1,3))(a_2-\delta_{2,i}+2)x^{(a)}\widetilde{\partial}_i\\
&= ((a_1-\delta_{i,j})\phi((1,0))+2\phi((1,0))\\
&\quad+(a_2-\delta_{i,j})\phi((-1,3))+2\phi((-1,3)))x^{(a)}\widetilde{\partial}_i\\
&= ((a_1-\delta_{i,j})\phi((1,0))+(a_2-\delta_{i,j})\phi((-1,3))\\
&\quad+\phi((0,1)))x^{(a)}\widetilde{\partial}_i\\
&= \deg_{\overline{\Gamma}}(x^{(a)}\widetilde{\partial}_i)x^{(a)}\widetilde{\partial}_i,
\end{aligned}$$

$$\begin{aligned}
\mathrm{ad}(y)(x^{(a)}) &= \phi((1,0))[x_1\partial_1, x^{(a)}] + \phi((-1,3))[x_2\partial_2, x^{(a)}] \\
&= \phi((1,0))(a_1-2)x^{(a)} + \phi((-1,3))(a_2-2)x^{(a)} \\
&= ((a_1)\phi((1,0)) - 2\phi((1,0)) + (a_2)\phi((-1,3)) \\
&\quad -2\phi((-1,3)))x^{(a)} \\
&= (a_1\phi((1,0)) + a_2\phi((-1,3)) - \phi((0,1)))x^{(a)} \\
&= \deg_{\overline{\Gamma}}(x^{(a)})x^{(a)}.
\end{aligned}$$

□

Lemma 4.1.4. Let L be one of the algebras from Theorem 4.1.1. Let Γ be the $\mathbb{Z}_p^l$-grading which is the eigenspace decomposition of a set of l toral elements $\{D_i\}_{i=1}^l$ of $X(m;\underline{n})^{(\infty)}$. If for some $1 \leq j \leq l$ we have that $D_j = \mathrm{ad}(y)$ and $y \notin Y_{(0)}$, where $Y_{(0)} = M_{(0)}$ for $X(m;\underline{n})^{(\infty)} = M(2;\underline{n})$ and $Y_{(0)} = W_{(0)}$ otherwise, then Γ is not isomorphic to a standard $\mathbb{Z}_p^l$-grading.

Proof. Let Γ be as in the statement of the lemma, $D_j = \mathrm{ad}(y)$ and $y \notin Y_{(0)}$. Assume for contradiction that the grading Γ is isomorphic to a standard $\mathbb{Z}_p^l$-grading Γ' induced by $\phi : \mathbb{Z}^k \to \mathbb{Z}_p^l$. Let $\phi_j : \mathbb{Z}_p^l \to \mathbb{Z}_p$ be the homomorphism of groups defined by $\phi_j(i_1, \ldots, i_l) = i_j$ and $\overline{\Gamma}$, $\overline{\Gamma}'$ be the coarsenings induced by ϕ_j of the Γ, Γ'-gradings respectively. Note that $\overline{\Gamma}'$ is still a standard grading since it is induced by $\phi_j\phi : \mathbb{Z}^k \to \mathbb{Z}_p$. Also $\overline{\Gamma}$ is the eigenspace decomposition on $X(m;\underline{n})^{(\infty)}$ of $\mathrm{ad}(y)$. Let $\overline{\phi} = \phi_j\phi$, $\overline{\Gamma} : X(m;\underline{n})^{(\infty)} = \bigoplus_{i\in\mathbb{Z}_p} L_i$ and $\overline{\Gamma}'$:

$X(m;\underline{n})^{(\infty)} = \bigoplus_{i\in\mathbb{Z}_p} L'_i$.

By Lemma 4.1.2 and 4.1.3 we have that the $\mathbb{Z}_p$-grading $\overline{\Gamma}'$ is the eigenspace decomposition on $X(m;\underline{n})^{(\infty)}$ of a derivation $\mathrm{ad}(w)$ where $w \in W_0$.

For any $z \in X(m;\underline{n})^{(\infty)}$ we can express it as $z = \sum_{i\in\mathbb{Z}_p} z_i$ where $z_i \in L_i$. Since $\Psi^{-1}(z_i) \in L'_i$ it follows that

$$\begin{aligned}
\Psi\,\mathrm{ad}(w)\Psi^{-1}(z) &= \Psi\,\mathrm{ad}(w)\Psi^{-1}\left(\textstyle\sum_{i=0}^{p-1} z_i\right) = \Psi\,\mathrm{ad}(w)\left(\textstyle\sum_{i=0}^{p-1}\Psi^{-1}(z_i)\right)\\
&= \Psi\left(\textstyle\sum_{i=0}^{p-1} i\Psi^{-1}(z_i)\right) = \textstyle\sum_{i=0}^{p-1} iz_i = \sum_{i=0}^{p-1}\mathrm{ad}(y)(z_i)\\
&= \mathrm{ad}(y)(z).
\end{aligned}$$

We have shown that $\mathrm{ad}(y) = \Psi\,\mathrm{ad}(w)\Psi^{-1}$. For $z \in X(m;\underline{n})^{(\infty)}$ we have that

$$\begin{aligned}
\Psi\,\mathrm{ad}(w)\Psi^{-1}(z) &= \Psi\left([\mathrm{ad}(w),\Psi^{-1}(z)]\right) = [\Psi(\mathrm{ad}(w), z]\\
&= \mathrm{ad}(\Psi(w))(z).
\end{aligned}$$

Since $\mathrm{ad}\,X(m;\underline{n})^{(\infty)} \cong X(m;\underline{n})^{(\infty)}$ and $\mathrm{ad}(y) = \Psi\,\mathrm{ad}(w)\Psi^{-1} = \mathrm{ad}(\Psi(w))$ we have that $y = \Psi(w)$. This is a contradiction since Ψ respects the canonical filtration and $w \in Y_{(0)}$ but $y \notin Y_{(0)}$. $\square$

Corollary 4.1.5. Let $n_q = 1$, $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$, $K(m;\underline{n})^{(1)}$ or $M(2;\underline{n})$ and $p \neq 3$ if $X(m;\underline{n})^{(\infty)} = W(1;1)$ or $H(2;\underline{n})^{(2)}$ where $\min(n_1,n_2) = 1$. The derivation $\mathrm{ad}((1+x_q)\partial_q)$ induces a $\mathbb{Z}_p$-grading on $X(m;\underline{n})^{(\infty)} = W(m;\underline{n})$, $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ or $M(2;\underline{n})$ that is not isomorphic to a standard $\mathbb{Z}_p$-grading.

Proof. By Lemma 2.4.6 we have that $\mathrm{ad}((1+x_q)\partial_q)$ is semisimple on $X(m;\underline{n})^{(\infty)}$ with eigenvalues in $\mathbb{Z}_p \subset F$. Since $(1+x_q)\partial_q \notin Y_{(0)}$, where $Y_{(0)} = M_{(0)}$ for $X(m;\underline{n})^{(\infty)} = M(2;\underline{n})$ and $Y_{(0)} = W_{(0)}$ otherwise, we can use Lemma 4.1.4 to get our conclusion. □

4.2 Gradings on $W(1;1)$

Lemma 4.2.1. Let $p \neq 3$. A $\mathbb{Z}_p$-grading on $W(1;1)$ is either isomorphic to a standard $\mathbb{Z}_p$-grading or is isomorphic to the $\mathbb{Z}_p$-grading induced by $\mathrm{ad}(\alpha(1+x_1)\partial_1)$ where $\alpha \in \mathbb{Z}_p \subset F$. Moreover, the grading is either equivalent to the $\mathbb{Z}_p$-grading induced by $\mathrm{ad}((1+x_1)\partial_1)$ or the standard $\mathbb{Z}_p$-grading induced by $\mathrm{ad}(x_1\partial_1)$.

Proof. Let $\Gamma : W(1;1) = \bigoplus_{i\in\mathbb{Z}_p} L_i$ be a $\mathbb{Z}_p$-grading. By Lemma 1.3.10 there exists a semisimple derivation δ with eigenvalues in $\mathbb{Z}_p \subset F$ such that the homogeneous spaces are its eigenspaces. Theorem 2.4.3 says that the set of derivations of $W(1;1)$ is $\mathrm{ad}\, W(1;1)$. Lemma 2.4.5 implies that $\delta = \mathrm{ad}(\Psi(\alpha y)) = \Psi\,\mathrm{ad}(\alpha y)\Psi^{-1}$ where $y = x_1\partial_1$ or $(1+x_1)\partial_1$, $\alpha \in F$ and $\Psi \in \mathrm{Aut}\, W(1;1)$. Note that $\mathrm{ad}(\alpha y)$ has eigenvalues in $\mathbb{Z}_p \subset F$ if and only if $\alpha \in \mathbb{Z}_p \subset F$.

Let $\Gamma' : W(1;1) = \bigoplus_{i\in\mathbb{Z}_p} L'_i$ where $\Psi(L'_i) = L_i$. Then Γ' is a grading isomorphic to the grading Γ. For $z \in L'_i$ we have that

$$\operatorname{ad}(\alpha y)(z) = \Psi^{-1}\Psi \operatorname{ad}(\alpha y)\Psi^{-1}\Psi(z) = \Psi^{-1}\delta(\Psi(z)) = \Psi^{-1}(i\Psi(z)) = iz.$$

Hence Γ' is induced by the derivation $\operatorname{ad}(\alpha y)$. If $y = x_1\partial_1$ then Lemma 4.1.2 says that Γ' is the standard $\mathbb{Z}_p$-grading induced by the homomorphism $\rho : \mathbb{Z} \to \mathbb{Z}_p$ defined by $\rho(1) = \alpha$.

Let ϕ be the automorphism of $\mathbb{Z}_p$ defined by $\phi(1) = \alpha$, $\widetilde{L}_i = L'_{\phi(i)}$, $w = x_1$ if $y = x_1\partial_1$ and $w = (1+x_1)$ if $y = (1+x_1)\partial_1$. Recall that $L'_i = \operatorname{Span}\{w^{k+1}\partial_1 \mid -1 \leq k \leq p-2,\ k \text{ modulo } p = i\}$. Then

$$\begin{aligned}\widetilde{L}_i &= L'_{\phi(i)} = \operatorname{Span}\{w^{k+1}\partial_1 \mid \rho(i) = k\phi(1),\ -1 \leq k \leq p-2\} \\ &= \operatorname{Span}\{w^{k+1}\partial_1 \mid i\alpha = k\alpha,\ -1 \leq k \leq p-2\}.\end{aligned}$$

Hence for $-1 \leq i \leq p-2$ we have $\widetilde{L}_i = \operatorname{Span}\{w^{i+1}\partial_1\}$. For $-1 \leq i+2 \leq p-2$ we have

$$\operatorname{ad}(y)(w^{i+1}\partial_1) = \operatorname{ad}(w\partial_1)(w^{i+1}\partial_1) = [w\partial_1, w^{i+1}\partial_1] = iw^{i+1}\partial_1.$$

Hence $\widetilde{\Gamma} : W(1;1) = \bigoplus_{i\in\mathbb{Z}_p} \widetilde{L}_i$ is a $\mathbb{Z}_p$-grading equivalent to Γ'. Since Γ' is isomorphic to Γ we have that $\widetilde{\Gamma}$ is equivalent to Γ. □

Lemma 4.2.2. Let $p \neq 3$ and $\Gamma : W(1;1) = \bigoplus_{i\in\mathbb{Z}_p} L_i$ be the $\mathbb{Z}_p$-grading induced by $\operatorname{ad}(w\partial_1)$ where $w = x_1$ or $w = 1 + x_1$ and $-1 \leq i,j \leq p-2$. If

$-1 \leq i+j \leq p-2$ and $i \neq j$ then $[L_i, L_j] = L_{i+j}$. Moreover, if $w = x_1$ then $[L_i, L_j] = 0$ if and only if $i = j$ or $i+j \notin \{-1, \ldots, p-2\}$. Also for $w = (1+x_1)$ and $i \neq j$ we have $[L_i, L_j] = L_{i+j}$.

Proof. Let $L_i = \mathrm{Span}\{w^{k+1}\partial_1 \mid -1 \leq k \leq p-2,\ k \text{ modulo } p = i\}$. Note that $x_1^{k+1} = (k+1)!x_1^{(k+1)}$ and $(k+1)!x_1^{(k+1)} = 0$ if and only if $k \notin \{-1 \leq k \leq p-2\}$ and $(1+x_1)^k \neq 0$ for any $k \geq 0$. For $-1 \leq i, j \leq p-2$ we have

$$[x_1^{i+1}\partial_1, x_1^{j+1}\partial_1] = (j-i)x_1^{i+j+1}\partial_1$$

and

$$[(1+x_1)^{i+1}\partial_1, (1+x_1)^{j+1}\partial_1] = (j-i)(1+x_1)^{i+j+1}\partial_1.$$

Hence the claims follow. $\square$

Lemma 4.2.3. Let $p \neq 3$ and Γ be a grading on $W(1;1)$ by a group G such that G has an element h of order p, G is not isomorphic to $\mathbb{Z}_p$ and $\mathrm{Supp}\,\Gamma$ generates G. If the grading $\overline{\Gamma}$ induced by $\phi : G \to \langle h \rangle_p$ induces a $\mathbb{Z}_p$-grading such that $\mathrm{Supp}\,\overline{\Gamma}$ generates $\mathbb{Z}_p$ then G is cyclic and Γ is equivalent to a standard grading.

Proof. Let G, ϕ, $\Gamma : W(1;1) = \bigoplus_{g \in G} L_g$, $\overline{\Gamma} : W(1;1) = \bigoplus_{-1 \leq i \leq p-2} \overline{L}_{h^i}$ where $\overline{L}_{h^i} = \bigoplus_{g \in G,\ \phi(g) = h^i} L_g$ satisfy the conditions of the lemma. By Lemma 4.2.1

we have that the grading $\overline{\Gamma}$ is equivalent to the $\mathbb{Z}_p$-grading induced by $\mathrm{ad}(w\partial_1)$ where $w = x_1$ or $1 + x_1$. By Lemma 4.2.2 the dimension of $\overline{L}_{h^i}$ is 1 for $-1 \leq i \leq p-2$. Hence for $-1 \leq i \leq p-2$ there exists $g_i \in G$ such that $\overline{L}_{h^i} = L_{g_i}$.

Hence we can assume without loss of generality that $[\overline{L}_{h^i}, \overline{L}_{h^j}] = \overline{L}_{h^{i+j}}$ for $-1 \leq i, j \leq p-2$, $i \neq j$ and $-1 \leq i+j \leq p-2$. For $-1 \leq i, j \leq p-2$, $i \neq j$ and $-1 \leq i+j \leq p-2$ we have

$$[L_{g_i}, L_{g_j}] = [\overline{L}_{h^i}, \overline{L}_{h^j}] = \overline{L}_{h^{i+j}} = L_{g_{i+j}}$$

and hence $g_i g_j = g_{i+j}$. It follows that $g_0 g_i = g_i$ for $-1 \leq i \leq p-2$ and $i = 0$ as well as $g_{-1} g_1 = g_0$. Then $g_0 = \mathrm{Id}_G$ and $g_{-1} = g_1^{-1}$. Also, for $2 \leq i \leq p-2$ we have that $g_1^{-1} g_i = g_{-1} g_i = g_{i-1}$. By induction, $g_i = g_1^i$ for $-1 \leq i \leq p-2$.

Since

$$\operatorname{Supp}\Gamma = \{g_{-1}, \ldots, g_{p-2}\} = \{g_1^{-1}, \ldots, g_1^{p-2}\}$$

we have that G is cyclic. Since $\operatorname{Supp}\Gamma$ generates G and G is not isomorphic to $\mathbb{Z}_p$ we have that g_1 can not be of order p.

Assume for contradiction that $w = (1+x_1)$. Then by Lemma 4.2.2 we have $[L_2, L_{p-2}] = L_0$. It follows that $g_1^0 = g_2 g_{p-2} = g_1^2 g_1^{p-2} = g_1^p$ and g_1 is of order p. Since the elements of $\operatorname{Supp}\Gamma$ can be expressed as powers of g_1 we have that G is generated by g_1. Since g_1 is of order p we have a contradiction.

Hence $w = x_1$ and for $-1 \leq i \leq p-2$ we have $L_{g_i} = \text{Span}\{x^{i+1}\partial_1\}$. It follows that Γ is the standard grading induced by $\rho : \mathbb{Z} \to G$ given by $\rho(1) = g_1$. $\square$

The following theorem results from Theorem 4.1.1, Lemma 4.2.3 and Lemma 4.2.1.

Theorem 4.2.4. Let $p \neq 3$ and Γ be G-grading on $W(1;1)$ such that the support of the grading generates G. If G is not isomorphic to $\mathbb{Z}_p$ then Γ is isomorphic to a G-standard grading. If $G = \mathbb{Z}_p$ then Γ is either isomorphic to a standard $\mathbb{Z}_p$-grading or group-equivalent to the $\mathbb{Z}_p$-grading induced by $\text{ad}(\alpha(1+x_1)\partial_1)$ where $\alpha \in \mathbb{Z}_p \subset F$. Moreover, if $G = \mathbb{Z}_p$ then Γ is group-equivalent to a $\mathbb{Z}_p$-grading induced by either $\text{ad}(x_1\partial_1)$ or $\text{ad}(\alpha(1+x_1)\partial_1)$. $\square$

Remark 4.2.5. For any standard G-grading on $W(1;1)$ we have that G is cyclic.

Bibliography

[1] Bahturin, Y.; Kochetov, M. *Classification of group gradings on simple Lie algebras of types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.* Article in press in J. Algebra (2010).

[2] Bahturin, Y.; Kochetov, M. *Group Gradings on restricted Cartan Type Lie Algebras.* arXiv: 1001.0191 (2009).

[3] Bahturin, Y.; Kochetov, M. *Group Gradings on the Lie algebra $\mathfrak{psl}_n$ in positive characteristic.* J. Pure Appl. Algebra **213** (2009), no. 9, 1739–1749.

[4] Bahturin, Y.; Kochetov, M.; Montgomery, S. *Group Gradings on Simple Lie Algebras in Positive Characteristic.* Proc. AMS **137** (2009) no.4, 1245-1254.

[5] Bahturin, Y.; Shestakov, I. P.; Zaicev, M. V. *Gradings on simple Jordan and Lie algebras.* J. Algebra **283** (2005), no. 2, 849–868.

[6] Bahturin, Y.; Tvalavadze, M. *Group gradings on G_2*. Comm. Algebra **37** (2009), no. 3, 885–893.

[7] Bahturin, Y; Zaicev, M. *Group gradings on simple Lie algebras of type "A"*. J. Lie Theory **16** (2006), no. 4, 719–742.

[8] Draper, C.; Martín, C. *Gradings on $\mathfrak{g}_2$*. Linear Algebra Appl. **418** (2006), no. 1, 85–111.

[9] Draper, C.; Martín, C. *Gradings on the Albert algebra and on $\mathfrak{f}_4$*. Rev. Mat. Iberoamericana. **25** (2009), no. 3, 841–908.

[10] Draper, C.; Viruel, A. *Group gradings on* $\mathrm{o}(8,\mathbb{C})$. Rep. Math. Phys. **61** (2008), no. 2, 265–280.

[11] Elduque, A. *Fine gradings on simple classical Lie algebras.* arXiv: 0906.0655 (2009).

[12] Havlíček, M.; Patera, J.; Pelantová, E. *On Lie gradings II.* Linear Algebra App. **277** (1998), 97–125.

[13] Havlíček, M.; Patera, J.; Pelantová, E. *On Lie gradings. III. Gradings of the real forms of classical Lie algebras* Linear Algebra and its Applications, 314 (2000), no. 1-3, 1-47.

[14] Humphreys, J. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York, 1972.

[15] Humphreys, J. *Linear Algebraic Groups.* Springer-Verlag, New York-Heidelberg, 1975.

[16] Kochetov, M. *Gradings on Finite-Dimensional Simple Lie Algebras.* Acta Appl. Math. **108** (2009), 101-127.

[17] Kuznetsov, M. I.; Mulyar, O. A. *Automorphisms of exceptional simple Lie algebras.* Special issue dedicated to Alexei Ivanovich Kostrikin. Comm. Algebra **29** (2001), no. 9, 3919–3934.

[18] McGraw, J. *Gradings by Finite Groups of the Witt Algebra.* to appear in Communications in Algebra.

[19] Patera J.; Zassenhaus, H. *On Lie gradings.* Linear Algebra and its Applications, **112** (1989), 87-159.

[20] Platonov, V. P. *The Theory of Algebraic Linear Groups and Periodic Groups.* Trans Amer. Math. Soc. Series 2 **69** (1968), 61–159.

[21] Premet, A.; Strade, H. *Simple Lie algebras of small characteristic. I. Sandwich elements.* J. Algebra **189** (1997), no. 2, 419–480.

[22] Premet, A.; Strade, H. *Lie algebras of small characteristic. II. Exceptional roots.* J. Algebra **216** (1999), no. 1, 190–301.

[23] Premet, A.; Strade, H. *Simple Lie algebras of small characteristic. III. The toral rank 2 case.* J. Algebra **242** (2001), no. 1, 236–337.

[24] Premet, A.; Strade, H. *Simple Lie algebras of small characteristic. IV. Solvable and classical roots.* J. Algebra **278** (2004), no. 2, 766–833.

[25] Premet, A.; Strade, H. *Simple Lie algebras of small characteristic. V. The non-Melikian case.* J. Algebra **314** (2007), no. 2, 664–692.

[26] Premet, A.; Strade, H. *Simple Lie algebras of small characteristic. VI. Completion of the classification.* J. Algebra **320** (2008), no. 9, 3559–3604.

[27] Strade, H. *Simple Lie algebras over Fields of Positive Characteristic, I Structure Theory.* De Gruyter Expositions in Mathematics, 38. Walter de Gruyter & Co., Berlin, 2004. viii+540 pp.

[28] Waterhouse, W. C. *Automorphisms and twisted forms of generalized Witt Lie algebras.* Trans. Amer. Math. Soc., **327** (1991), no. 1, 185200.