CONSTRUCTION OF LEAVES AND EXCESSES WHEN $k=3,4$

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Construction of Leaves and Excesses when k=3,4

by

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Abstract

A *packing design*, or a PD(v, k, λ) is a family of k-subsets (called *blocks*), of a v-set S, such that every 2-subset (called a *pair*), of S is contained in at most λ blocks. The *packing number* P(v, k, λ) is the number of blocks in a PD(v, k, λ).

The edges in the multigraph λK_v not contained in the packing form the *leave* of the PD(v, k, λ), denoted by leave(v, k, λ). Generally we consider maximum packings (packings with maximum number of blocks) unless stated otherwise.

A *covering design*, or a CD(v, k, λ) is a family of k-subsets (called *blocks*), of a v-set S, such that every 2-subset (called a *pair*), of S is contained in at least λ blocks. The *covering number* C(v, k, λ) is the number of blocks in a CD(v, k, λ).

The extrone edges added to the multigraph λK_v in the covering form the *excess* of the CD(v, k, λ), denoted by excess(v, k, λ). Generally we consider minimum coverings (coverings with minimum number of blocks) unless stated otherwise.

In this thesis we give the direct constructions of the leaves and excesses for $k = 3, 4$. Some of them are from existing papers, some are the author's original work. This is the first time to put all the leaves and excesses for $k = 4$ and all λs together (with only few possible exceptions).

Dedication

This thesis is dedicated to the memory of my father.

Acknowledgment

I would like to thank my supervisor, Dr. N Shalaby, for his commitment and assistance over the past years. I learned from him not only the methods of mathematical research, but also the attitude towards life and people. I will forever remember his instructions and his warm heart.

Also I would like to thank Mr Robin Swain, for his help in Latex editing.

Contents

1 Introduction

In the thesis, Chapter I is an introduction to the basic concepts and known results.

Chapter II gives a complete and detailed description on the properties and constructions of leaves and excesses when $k = 3$. The table of leaves and excesses when $k = 3$ were put together by Shalaby.

Chapter III gives a near complete and detailed description on the properties and constructions of leaves and excesses when $k = 4$. This is the first time all the leaves and excesses when $k = 4$ has been put together (There are few unsolved cases).

Chapter IV is a brief discussion on nuclear designs. A theorem on nuclear designs when $k = 4$ is given in this chapter.

Chapter V is a conclusion of what we achieved in this thesis and what is still open for future research.

The Appendix contains the tables of the known spectrum of leaves and excesses when $k = 3, 4$, and the special cases where the general constructions does not work.

1.1 Basic Concepts

In 1782, Euler posed the first problem in combinatorial design theory, the "36 officers problem". Euler's problem was to arrange the 36 officers in a 6×6 array so that each row and each column contained a officer of six legions and a officer of six ranks [5].

Euler conjectured that there is no such $n \times n$ arrangement for $n = 6$ and for all $n \equiv 2 \pmod 4$. This began the research of mutually orthogonal Latin squares (MOLS or POLS).

In 1900, Tarry showed by exhaustive method that there are no two orthogonal Latin squares of order 6, hence proved Euler's conjecture for $n = 6$. In the 1990s, Stinson reproved this in a different way. Euler's conjecture on $n \equiv 2 \pmod 4$, $n > 6$ was finally proved to be wrong in 1960, by Bose, Shrikhandi and Parker [25]; they proved that there are pairs of orthogonal Latin square of order 10 and all $n \equiv 2$ (mod 4), $n \geq 10$.

In 1835, Plucker [54] in a study of algebraic curves, observed that given v elements, a family of subsets of size three in which every pair of elements occurs in exactly one of the subsets, will contain $\frac{1}{6}v(v-1)$ such subsets.

Such a system is called a balanced incomplete block design, or a BIBD$(v, 3, 1)$ now. Later Plucker conjectured (correctly) that $v \equiv 1, 3 \pmod 6$ is the necessary and sufficient condition for a BIBD$(v, 3, 1)$ to exist [55].

Definition 1. A *balanced incomplete block design*, or a BIBD(v, k, λ) is a collection of k-subsets (called *blocks*), of a v-set S, such that each 2-subset (called a *pair*) of S appears in exactly λ of the blocks. b is the number of blocks and r is the number of blocks containing a certain vertex.

EXAMPLE 1. A BIBD(7,3,1), the blocks are {1,2,4}, {2,3,5}, {3,4,6}, {4,5,7}, {5,6,1}, {6,7,2}, {7,1,3}. Each pair (edge) appears in exactly one block (see Figure 1).

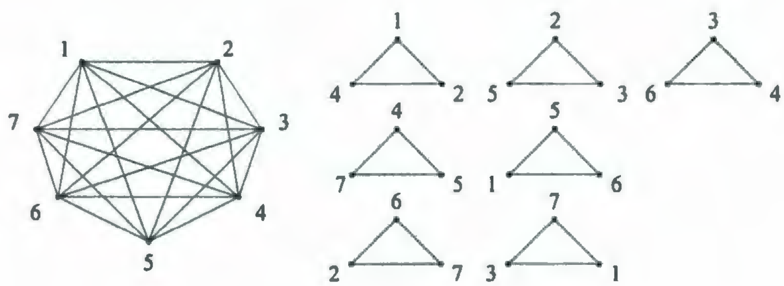

Figure 1: A BIBD(7,3,1)

Theorem 1. Some necessary conditions of the parameters of a BIBD(v, k, λ).
(1) $\lambda(v-1) = r(k-1)$;
(2) $bk = vr$.

Proof. A vertex x in the multigraph λK_v has degree $\lambda(v-1)$, and each block containing x takes $k-1$, so x is contained in $r = \lambda\frac{v-1}{k-1}$ blocks, so clearly $r(k-1) = \lambda(v-1)$.

Both bk and vr stand for the total appearance of the vertices in the design, so $bk = vr$.

Parameters satisfying (1) and (2) are said to be *admissible*.

Theorem 2. A necessary condition for a BIBD(v, k, λ) to exist is that $\lambda(v-1) \equiv 0 \pmod{(k-1)}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

Proof. If a BIBD(v, k, λ) exists, then $r = \lambda\frac{v-1}{k-1}$ and $b = \frac{vr}{k} = \lambda\frac{v}{k}\frac{v-1}{k-1}$ should be integers. So $\lambda(v-1) \equiv 0 \pmod{(k-1)}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

It is natural to ask what designs we could obtain if the necessary condition for BIBDs does not apply. We now define *packing* and *covering* designs.

Definition 2. A *packing design*, or a PD(v, k, λ) is a family of k-subsets (called *blocks*), of a v-set S, such that every 2-subset (called a *pair*), of S is contained in at most λ blocks. The *packing number* P(v, k, λ) is the number of blocks in a PD(v, k, λ).

The edges in the multigraph λK_v not contained in the packing form the *leave* of the PD(v, k, λ), denoted by leave(v, k, λ). Generally we consider maximum packings (packings with maximum number of blocks) unless stated otherwise.

EXAMPLE 2. A PD(6,3,1), the blocks are {1,5,6}, {2,4,6}, {1,3,4}, {2,3,5}, and the leave is {1,2}, {3,6}, {4,5} (see Figure 2).

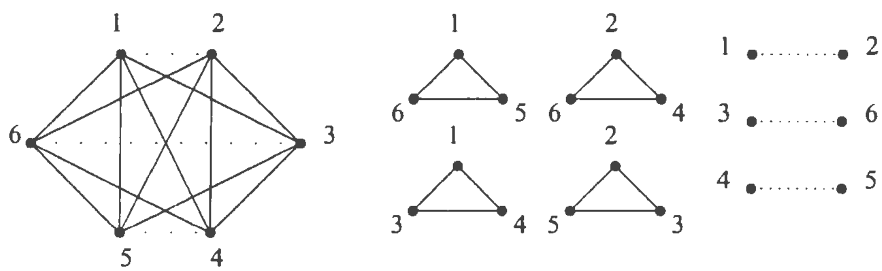

Figure 2: A PD(6,3,1) and its Leave

Definition 3. A *covering design*, or a CD(v, k, λ) is a family of k-subsets (called *blocks*), of a v-set S, such that every 2-subset (called a *pair*), of S is contained in at least λ blocks. The *covering number* C(v, k, λ) is the number of blocks in a CD(v, k, λ).

The extrone edges added to the multigraph λK_v in the covering form the *excess* of the CD(v, k, λ), denoted by excess(v, k, λ). Generally we consider minimum coverings (coverings with minimum number of blocks) unless stated otherwise.

EXAMPLE 3. A CD(6,3,1), the blocks are {1,2,4}, {2,3,5}, {3,4,6}, {5,6,2}, {1,3,6}, {1,4,5}, and the excess is {1,4}, {2,5}, {3,6} (see Figure 3).

Schonheim [58] [59] gave the following upper bound and lower bound for the number of blocks and edges in optimal packing and covering designs. These bounds are called the Schonheim bounds.

Theorem 3. An upper bound of number of blocks in a PD(v, k, λ) is $\Psi(v, k, \lambda) = \lfloor v/k \lfloor \lambda(v-1)/(k-1) \rfloor \rfloor$.

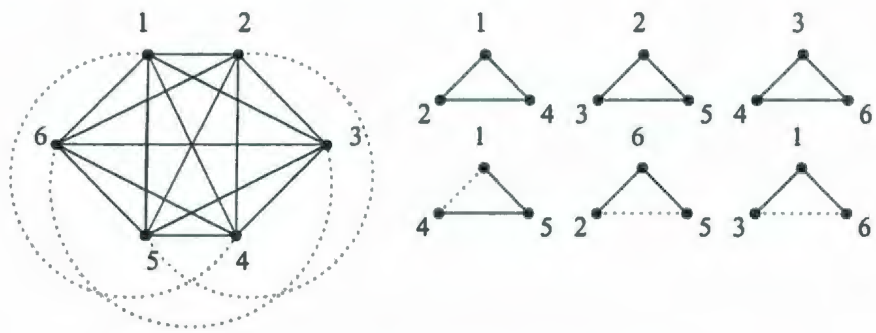

Figure 3: A CD(6,3,1) and its Excess

Proof. A vertex x in the complete multigraph λK_v has degree $\lambda(v-1)$, and each block containing x takes $k-1$, so x can be contained in no more than $\lfloor \lambda(v-1)/(k-1) \rfloor$ blocks. There are v vertices, so the total occurrence of the vertices in a PD(v, k, λ) is no more than $v\lfloor \lambda(v-1)/(k-1) \rfloor$.

Finally, each block has k vertices, so the number of blocks is no more than $\Psi(v, k, \lambda) = \lfloor v/k \lfloor \lambda(v-1)/(k-1) \rfloor \rfloor$.

Theorem 4. A lower bound of number of blocks in a CD(v, k, λ) is $\Phi(v, k, \lambda) = \lceil v/k \lceil \lambda(v-1)/(k-1) \rceil \rceil$.

Proof. A vertex x in the complete multigraph λK_v has degree $\lambda(v-1)$, and each block containing x takes $k-1$, so x can be contained in no less than $\lceil \lambda(v-1)/(k-1) \rceil$ blocks. There are v vertices, so the total occurrence of vertices in a CD(v, k, λ) is no less than $v\lceil \lambda(v-1)/(k-1) \rceil$.

Finally, each block will take k vertices, so the number of blocks is no less than $\Phi(v, k, \lambda) = \lceil v/k \lceil \lambda(v-1)/(k-1) \rceil \rceil$.

Below are two powerful types of designs used in constructing packings and coverings.

Definition 4. A *pairwise balanced design*, or a PBD(v, K, λ) is a collection of subsets (called *blocks*) of a v-set S, with sizes in K, such that every 2-subset (called a *pair*), of S appears in exactly λ blocks.

EXAMPLE 4. A PBD(6,{3,2},1), the blocks are {1,5,6}, {2,4,6}, {1,3,4}, {2,3,5}, {1,2}, {3,6}, {4,5} (see Figure 4).

Definition 5. A *group divisible design*, or a GDD(v, M, k, λ) is a collection of

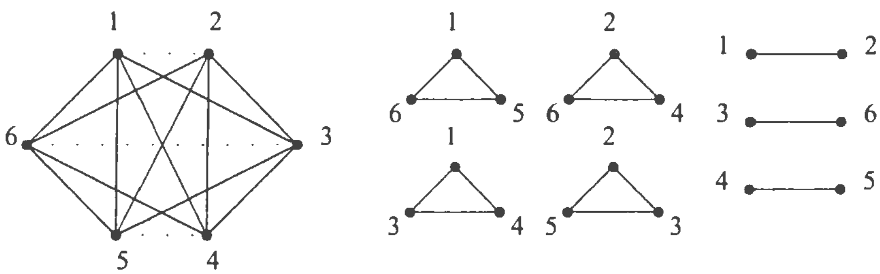

Figure 4: A PBD(6,{3,2},1)

two kinds of subsets of a v-set S: (i) disjoint subsets with sizes in M (called *groups*), whose union is S; and (ii) k-subsets (called *blocks*), such that

(i) each group and each block have at most one common element;

(ii) each pair $\{x, y\}$ in S where x and y belong to different groups, is contained in exactly λ blocks.

Note When $\lambda = 1$, we often describe a GDD by its type, i.e. a k-GDD of type $t_1^{u_1} t_2^{u_2} ... t_l^{u_l}$ if there are u_i groups of size t_i for $i = 1, 2, ..., l$ and the blocks are of size k. If $t_1, t_2, ..., t_l$ are the sizes of all groups (counting multiplicity), we can also denote the GDD by type $[t_1, t_2, ..., t_l]$.

EXAMPLE 5. A GDD(6,2,3,1), the groups are {1,2}, {3,6}, {4,5}, and blocks are {1,5,6}, {2,4,6}, {1,3,5}, {2,3,4}. It's a 3-GDD of type 2^3 or type [2,2,2] (see Figure 5).

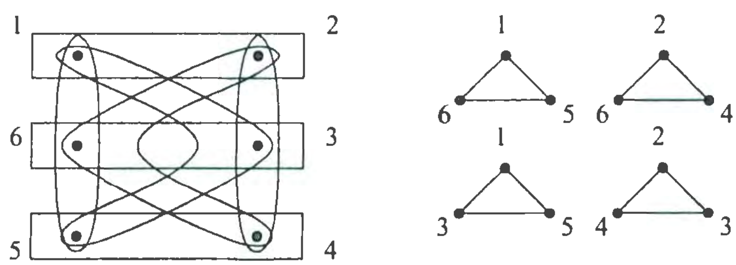

Figure 5: A GDD(6,2,3,1) or a 3-GDD of type 2^3 or type [2,2,2].

1.2 Known Results for BIBDs

Most results of the sufficient conditions of the BIBDs, PDs and CDs are obtained by a constructive method, either direct or recursive.

As early as in 1847, Kirkman [43] proved that $v \equiv 1, 3 \pmod 6$ is sufficient for a BIBD(v,3,1) to exist. In 1939 Bose [24] proved that $v \equiv 0, 1 \pmod 3$ is the sufficient condition for a BIBD$(v, 3, 2)$ to exist.

Hanani [38][39][40] found sufficient conditions for a BIBD(v, k, λ) to exist, when $k = 3, 4, 5$, with the exception of BIBD(15,5,2), which does not exist; when $k = 6$, $\lambda \neq 1$, with the exception of BIBD(21,6,2), which does not exist; and when $k = 7$, $\lambda \equiv 6, 7, 12, 18, 24, 30, 35, 36 \pmod{42}$ or $\lambda > 30$, and λ is not divisible by 2 or 3.

Mills [47] found sufficient conditions for the existence of a BIBD$(v, 6, 1)$, with some possible small exceptions, among which $v = 51$ is the smallest.

Abel and Greig [3] found sufficients condition for the existence of a BIBD$(v, 7, \lambda)$, when $\lambda > 2$, with the exception of $v = 253$, $\lambda = 4, 5$; when $\lambda = 2$, with the exception of BIBD(22,7,2), which doesn't exist, and some other 8 uncertain cases; when $\lambda = 1$, with the exception of BIBD(43,7,1), which doesn't exist, and some other 26 uncertain cases.

Du and Zhu [33] found the sufficient conditions for the existence of a BIBD$(v, 8, 1)$, with at most 223 possible exceptions, among which $v = 21897$ is the largest.

Abel, Bluskov and Greig [2] reduced the number of uncertain cases to 38, among which $v = 3753$ is the largest.

Abel, Bluskov and Greig [2] also found the sufficient conditions for the existence of a BIBD$(v, 8, \lambda)$, when $\lambda > 1$, with the definite exception of two values of v, and the possible exception of 7 further values of v, among which the largest is $v = 589$. In particular, they found the sufficient condition for a BIBD$(v, 8, \lambda)$, when $\lambda > 5$, and when $\lambda = 4$ and $v \neq 22$.

Finally, in the same paper Abel, Bluskov and Greig [2] found the sufficient conditions for the existence of a BIBD$(v, 9, \lambda)$, when $\lambda = 2, 4, 8$. The cases of higher λs are still open now.

In this thesis, we shall only study the cases of $k = 3, 4$.

Hanani (1961) [38] proved that

(1) A BIBD$(v, 3, \lambda)$ exists if and only if $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 6$.

i.e. a BIBD$(v, 3, \lambda)$ exists for the following cases
$\lambda \equiv 0 \pmod 6$ $\quad v \equiv 0, 1, 2, 3, 4, 5 \pmod 6$)
$\lambda \equiv 1 \pmod 6$ $\quad v \equiv 1, 3 \pmod 6$
$\lambda \equiv 2 \pmod 6$ $\quad v \equiv 0, 1, 3, 4 \pmod 6$
$\lambda \equiv 3 \pmod 6$ $\quad v \equiv 1, 3, 5 \pmod 6$
$\lambda \equiv 4 \pmod 6$ $\quad v \equiv 0, 1, 3, 4 \pmod 6$
$\lambda \equiv 5 \pmod 6$ $\quad v \equiv 1, 3 \pmod 6$

(2) A BIBD$(v, 4, \lambda)$ exists if and only if $\lambda(v-1) \equiv 0 \pmod 3$ and $\lambda v(v-1) \equiv 0 \pmod{12}$.

i.e. a BIBD$(v, 4, \lambda)$ exists for the following cases
$\lambda \equiv 0 \pmod 6$ $\quad v \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \pmod{12}$
$\lambda \equiv 1 \pmod 6$ $\quad v \equiv 1, 4 \pmod{12}$
$\lambda \equiv 2 \pmod 6$ $\quad v \equiv 1, 4, 7, 10 \pmod{12}$
$\lambda \equiv 3 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$
$\lambda \equiv 4 \pmod 6$ $\quad v \equiv 1, 4, 7, 10 \pmod{12}$
$\lambda \equiv 5 \pmod 6$ $\quad v \equiv 1, 4 \pmod{12}$

1.3 Known Results for PDs and CDs

Fort and Hedlund [35] first studied the covering numbers; and Hanani [40] first studied packing numbers.

Let C$(v, 3, \lambda)$ be the number of blocks in a CD$(v, 3, \lambda)$ and $\Phi(v, 3, \lambda)$ be the Schonheim lower bound; let P$(v, 3, \lambda)$ be the number of blocks in a PD$(v, 3, \lambda)$ and $\Psi(v, 3, \lambda)$ be the Schonheim upper bound. We have the following results.

Haggard [37], Hanani [40], Stanton and Rogers [62]:

$$\mathrm{C}(v,3,\lambda) = \begin{cases} \Phi(v,3,\lambda)+1, & v \equiv \lambda \equiv 2 \pmod 6; \\ \Phi(v,3,\lambda)+1, & v \equiv 5, \lambda \equiv 2 \pmod 6; \\ \Phi(v,3,\lambda)+1, & v \equiv \lambda \equiv 5 \pmod 6; \\ \Phi(v,3,\lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were by Fort and Hedlund [35].

Hanani [40], Stanton, Rogers, Quinn and Cowan [63]:

$$P(v,3,\lambda) = \begin{cases} \Psi(v,3,\lambda) - 1, & v \equiv 2, \lambda \equiv 4 \pmod 6; \\ \Psi(v,3,\lambda) - 1, & v \equiv 5, \lambda \equiv 1 \pmod 6; \\ \Psi(v,3,\lambda) - 1, & v \equiv 5, \lambda \equiv 4 \pmod 6; \\ \Psi(v,3,\lambda), & \text{otherwise.} \end{cases}$$

Let $C(v,4,\lambda)$ be the number of blocks in a $CD(v,4,\lambda)$ and $\Phi(v,4,\lambda)$ be the Schonheim lower bound; let $P(v,4,\lambda)$ be the number of blocks in a $PD(v,4,\lambda)$ and $\Psi(v,4,\lambda)$ be the Schonheim upper bound, we have the following results.

Assaf [6] proved:

$$C(v,4,\lambda) = \begin{cases} \Phi(v,4,\lambda) + 1, & v = 7, 9, 10, \lambda = 1; \\ \Phi(v,4,\lambda) + 2, & v = 19, \lambda = 1; \\ \Phi(v,4,\lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were done by Mills [45] [46], Horton, Mullin and Stanton [42].

Billington, Stanton and Stinson [23], Assaf [7]:

$$P(v,4,\lambda) = \begin{cases} \Psi(v,4,\lambda) - 1, & v = 7, 10 \pmod{12}, v \neq 10, 19, \lambda = 1; \\ \Psi(v,4,\lambda) - 1, & v = 9, 17, \lambda = 1; \\ \Psi(v,4,\lambda) - 2, & v = 8, 10, 11, \lambda = 1; \\ \Psi(v,4,\lambda) - 3, & v = 19, \lambda = 1; \\ \Psi(v,4,\lambda) - 1, & v = 9, \lambda = 2; \\ \Psi(v,4,\lambda) - 1, & v = 6, \lambda = 3; \\ \Psi(v,4,\lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were done by Brouwer [26], and $P(19,4,1)$ was due to Stinson [64]; the cases of $\lambda > 1$, $v \not\equiv 0 \pmod 3$ were done by Billington, Stanton and Stinson [23]; the remaining cases were done by Assaf [7], where some small vs were done by Hartman [41].

For $PD(v,5,\lambda)$s, Assaf and Hartman [15] solved the cases of $\lambda = 4$; Assaf and Shalaby [20] solved the cases of $\lambda = 8, 12, 16$; Assaf, Shalaby and Singh [22] solved the cases of $\lambda = 2$ with v even; Assaf [13] [14] solved the cases of $\lambda = 5$ and $7 \leq \lambda \leq 21$; Assaf, Shalaby and Singh [22] solved the case of $\lambda = 3, 6$; Yin [72] solved the cases of $v \equiv 0 \pmod 4$, $\lambda = 1$, with two definite exceptions and 13 possible exceptions.

For $CD(v,5,\lambda)$, Assaf and Shalaby [19], Assaf [8] solved the cases of $\lambda = 4$; Assaf [12] [10] [11] [9] solved the cases of $\lambda = 5, 6, 7$, $\lambda \equiv 0 \pmod 4$; Assaf and Singh [21] solved $11 \leq \lambda \leq 21$.

Little has been done on $k = 6$ and higher. Assaf, Hartman and Shalaby [16], Yin and Chen [73] gave a solution to P$(v, 6, 5)$. That is the only solved case for now.

1.4 Known Results for Leaves and Excesses when $k = 3, 4$

After we found the packings and coverings, we would like to study the leaves and excesses. Some leaves and excesses are already known.

Kirkman [43] solved the leave$(v, 3, 1)$s when constructing the packings. Schonheim [59] [60], Spencer [61] solved the same cases independently. Engel [34] and Stanton and Rogers [62] solved all leave$(v, 3, \lambda)$s.

Fort and Hedlund [35] solved the excess$(v, 3, 1)$s. Haggard [41], and Engel [34] solved all excess$(v, 3, \lambda)$s.

Mendelsohn, Shalaby and Shen [52] gave some necessary and sufficient conditions for a multigraph to be a leave or an excess (not necessarily maximum leave or minimum excess), they also gave a complete table for all leaves and excesses when $k = 3$.

Stinson [65] [66] gave the leave$(v, 4, 1)$s, and excess$(v, 4, 1)$s.

Some leaves and excesses of higher λs can be found or deduced from papers dealing with packing and covering designs, like in the papers of Assaf [6] [7], Billington, Stanton and Stinson [23], Brouwer [27], Stanton and Rogers [62].

The the author is filling up the gaps and making a spectrum of the leaves and excesses when $k = 3, 4$. We will give at least one construction for each v and λ, with few exceptions.

Mendelsohn, Shalaby and Shen [52] introduced the concept of nuclear design which connects leaves and excesses. If we can find the intersection of a PD(v, k, λ) and a CD(v, k, λ), then we can easily construct and index the table of packings and coverings with the help of small additional designs.

2 Leaves and Excesses when $k = 3$.

2.1 Necessary Conditions

Necessary conditions for a multigraph G to be a leave(v, k, λ) were given by Mendelsohn, Shalaby and Shen [52]. Let $|E|$ be the number of edges in G,

(1) $\lambda v(v-1)/2 - |E| \equiv 0 \pmod{k(k-1)/2}$;
(2) for all $x \in G$, deg$(x) \equiv \lambda(v-1) \pmod{k-1}$;
(3) for all $x \in G$, deg$(x) \leq \lambda(v-1)$.

The followings are the necessary conditions in the same artcle for a multigraph G to be an excess(v, k, λ). Let $|E|$ be the number of edges in G,

(1) $\lambda v(v-1)/2 + |E| \equiv 0 \pmod{k(k-1)/2}$;
(2) for all $x \in G$, deg$(x) \equiv \lambda(v-1) \pmod{k-1}$;
(3) for all $x \in G$, deg$(x) \geq \lambda(v-1)$.

It was proved in the same paper that all leaves and excesses for $k = 3$ are admissible, i.e. the necessary conditions are also sufficient.

2.2 Sufficient Conditions

One way to approach the question "sufficient conditions for a multigraph G to be a leave(v, k, λ) or excess(v, k, λ)" is to construct the full spectrum of all possible leaves and excesses. The goal of this thesis is to give as many as possible constructions of leaves and excesses. Some of the constructions are from existing papers, some are original work by the author.

Let C$(v, 3, \lambda)$ be the number of blocks in a CD$(v, 3, \lambda)$ and $\Phi(v, 3, \lambda)$ be the Schonheim lower bound. Let P$(v, 3, \lambda)$ be the number of blocks in a PD$(v, 3, \lambda)$ and $\Psi(v, 3, \lambda)$ be the Schonheim upper bound. Recall the following results.

Haggard [37], Hanani [40], Stanton and Rogers [62]:

$$\mathrm{C}(v,3,\lambda) = \begin{cases} \Phi(v,3,\lambda)+1, & v \equiv \lambda \equiv 2 \pmod 6; \\ \Phi(v,3,\lambda)+1, & v \equiv 5, \lambda \equiv 2 \pmod 6; \\ \Phi(v,3,\lambda)+1, & v \equiv \lambda \equiv 5 \pmod 6; \\ \Phi(v,3,\lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were proved by Fort and Hedlund [35].

Hanani [40], Stanton, Rogers, Quinn and Cowan [63]:

$$P(v,3,\lambda) = \begin{cases} \Psi(v,3,\lambda)-1, & v \equiv 2, \lambda \equiv 4 \pmod 6; \\ \Psi(v,3,\lambda)-1, & v \equiv 5, \lambda \equiv 1 \pmod 6; \\ \Psi(v,3,\lambda)-1, & v \equiv 5, \lambda \equiv 4 \pmod 6; \\ \Psi(v,3,\lambda), & \text{otherwise.} \end{cases}$$

The following lemmas are very obvious, but very useful in our constructions.

Lemma 1. If there exists a BIBD(v, k, λ_1), then the leave$(v, k, \lambda_1 + \lambda_2)$ can be the same as the leave(v, k, λ_2). The same applies to excesses.

Lemma 2. When we combine a leave(v, k, λ_1) and a leave(v, k, λ_2), sometimes we obtain blocks. If after removing the new blocks, the Schonheim bound is satisfied, then we obtain a leave$(v, k, \lambda_1 + \lambda_2)$. The same applies to excesses.

Lemma 3. When we add a graph E to a leave(v, k, λ), minus the newly obtained blocks, if the leave is consumed and the Schonheim bound is satisfied, then E is an excess(v, k, λ). The same applies to leaves.

Note that in the following figures, we will use solid lines for leaves and dotted lines for excesses. In the figures combining two leaves or excesses, a leave/excess will be in solid line and the other in dotted line.

Note that we use [1,2,3,4] for a 4-cycle (C_4) and {1,2,3,4} for a block of size 4 (K_4) in case of possible confusion.

2.3 The cases of $\lambda = 1$

2.3.1 Case 1: $v = 6i$

The leave is a 1-factor (1^{6i}); also is the excess (1^{6i}).

Leave. If the upper bound $\Psi(6i, 3, 1)$ is satisfied, then a PD$(6i, 3, 1)$ takes $\lfloor 6i\lfloor(6i-1)/2\rfloor/3\rfloor = 6i^2 - 2i$ blocks and $18i^2 - 6i$ edges. There are $\binom{6i}{2} = 18i^2 - 3i$ edges in K_{6i}, so the leave has $3i$ edges.

Every vertex x in K_{6i} has degree $6i - 1$, each block in the PD containing x takes two. So x has an odd degree in the leave, it must be 1, and the leave is a 1-factor.

A construction by Stanton and Rogers [62]. Delete a vertex 1 and all its edges from a BIBD$(6i + 1, 3, 1)$. The blocks not containing 1 form a PD$(6i, 3, 1)$; those containing 1 become $3i$ edges, the leave (see Figure 6).

Excess. If the lower bound $\Phi(6i, 3, 1)$ is satisfied, then a CD$(6i, 3, 1)$ takes

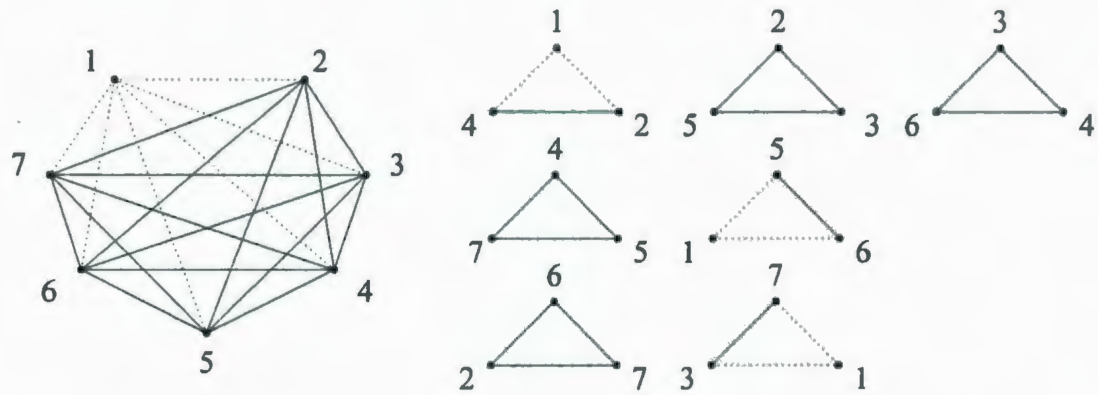

Figure 6: From BIBD(7,3,1) to PD(6,3,1). The leave is {2,4}, {5,6}, {3,7}.

$\lceil 6i\lceil (6i-1)/2\rceil /3\rceil = 6i^2$ blocks and $18i^2$ edges, so the excess has $3i$ edges. Similar to the leave, every vertex has degree 1 in the excess. So the excess is a 1-factor.

A construction by Shalaby and Zhong. By rearranging vertex labelling, we may assume that a PD$(6i, 3, 1)$ has blocks $\{6j-5, 6j-3, 6j-1\}$, $j = 1, \ldots i$; and has leave $L_1 = \{6j-5, 6j-4\}$, $\{6j-3, 6j-2\}$, $\{6j-1, 6j\}$, $j = 1, \ldots i$. (See for example Figure 7, where permutation (1734) is used).

Add edges $\{6j-4, 6j-3\}$, $\{6j-2, 6j-1\}$, $\{6j-5, 6j\}$ to L_1, we obtain new blocks $\{6j-5, 6j-4, 6j-3\}$, $\{6j-3, 6j-2, 6j-1\}$, $\{6j-1, 6j, 6j-5\}$, $j = 1, \ldots i$ (see Figure 8).

The leave L_1 is consumed, so the excess$(6i, 3, 1)$ is the added edges $\{6j-4, 6j-3\}$, $\{6j-2, 6j-1\}$, $\{6j-5, 6j\}$, $j = 1, \ldots i$.

2.3.2 Case 2: $v = 6i + 1$

There is a BIBD$(6i+1, 3, 1)$.

2.3.3 Case 3: $v = 6i + 2$

The leave is a 1-factor (1^{6i+2}); the excess is a 3-star and a 1-factor $(1^{6i+1}3^1)$.

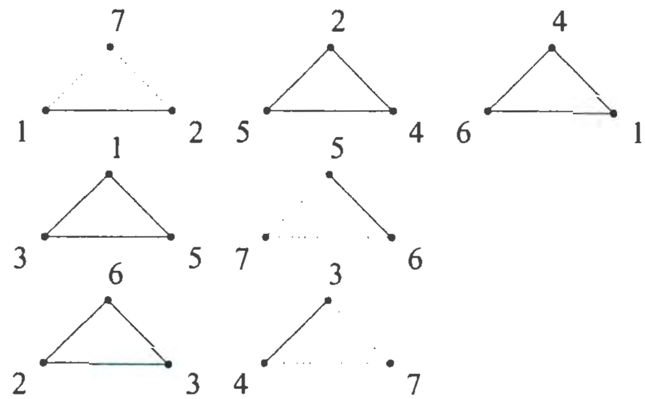

Figure 7: Permutation (1734) on the leave(6,3,1).

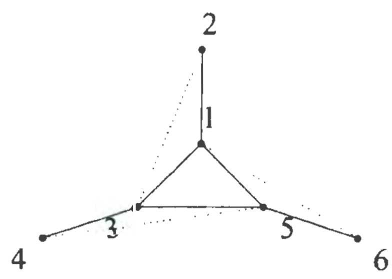

Figure 8: From leave(6,3,1) to excess(6,3,1).

Leave. By the same arithmetic as in **Case 1**, the leave$(6i + 2, 3, 1)$ has $3i + 1$ edges. Every vertex has degree 1 in the leave. So the leave is a 1-factor.

A construction by Stanton and Rogers [62]. Delete a vertex 1 and all its edges from a BIBD$(6i + 3, 3, 1)$. The blocks not containing 1 form a PD$(6i + 2, 3, 1)$; those containing 1 form a 1-factor, the leave$(6i + 2, 3, 1)$ (see Figure 9).

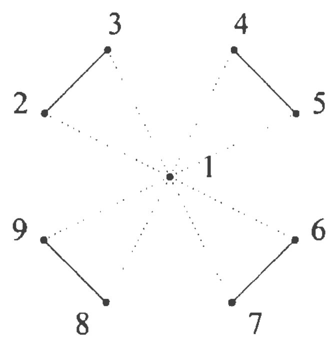

Figure 9: The leave(8,3,1).

Excess. By the same arithmetic as in **Case 1**, the excess$(6i + 2, 3, 1)$ has $3i + 2$ edges. Every vertex has an odd degree in the excess. So there is a vertex of degree 3, and $6i + 1$ vertices of degree 1. So the excess is a 3-star and a 1-factor.

A construction by Stanton and Rogers [62]. Add a vertex 0 and blocks $\{0, 2j, 2j + 1\}$, $j = 1, 2, ..., 3i$, and $\{0, 1, 2\}$ to a BIBD$(6i + 1, 3, 1)$. The Schonheim bound $6i^2 + 4i + 1$ is met and the repeated edges {1,2}{0,2}{2,3} (the 3-star), $\{2j, 2j + 1\}$, $j = 2, ..., 3i$ (the 1-factor) form the excess$(6i + 2, 3, 1)$ (see Figure 10).

2.3.4 Case 4: $v = 6i + 3$

There is a BIBD$(6i + 3, 3, 1)$.

Lemma 4. Wilson [69]: There exists a PBD$(v, \{3, 5^*\}, 1)$ for all $v \equiv 5 \pmod 6$, where 5^* means there is only one block of size 5.

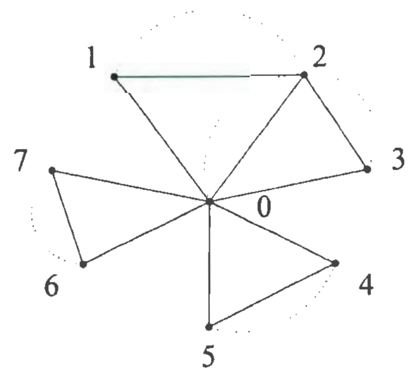

Figure 10: The excess(8,3,1).

2.3.5 Case 5: $v = 6i + 4$

The leave is a 3-star and a 1-factor $(3^1 1^{6i+3})$**; also is the excess** $(3^1 1^{6i+3})$**.**

Leave. By the same arithmetic, the leave$(6i + 4, 3, 1)$ has $3i + 3$ edges. Every vertex has an odd degree in the leave. So there is a vertex of degree 3, and $6i + 3$ vertices of degree 1. So the leave is a 3-star and a 1-factor.

A construction by Stinson [66]. Suppose the block of size 5 in a PBD$(6i + 5, \{3, 5^*\}, 1)$ is {1,2,3,4,5}. Delete the vertex 1 and all all its edges. Blocks containing 1 will be reduced to a 1-factor.

Remove block {2,3,4} from {1,2,3,4,5}, we obtain a 3-star {2,5}{3,5}{4,5}. The leave is the 3-star and the 1-factor (see Figure 11).

Excess. By the same arithmetic, the excess$(6i + 4, 3, 1)$ has $3i + 3$ edges. Every vertex has odd degree in the excess. So there is a vertex of degree 3, and $6i + 3$ vertices of degree 1. So the excess is a 3-star and a 1-factor.

A construction by Stanton and Rogers [62]. Add vertex 0 and the blocks $\{0, 1, 2\}$, $\{0, 2j, 2j + 1\}$, $j = 1, 2, ..., 3i + 1$, to a BIBD$(6i + 3, 3, 1)$. The Schonheim bound $6i^2 + 8i + 3$ is met, and the repeated edges {1,2}{0,2}{2,3} (the 3-star), $\{2j, 2j + 1\}$, $j = 2, ..., 3i + 1$ (the 1-factor) form the excess$(6i + 4, 3, 1)$ (see Figure 12).

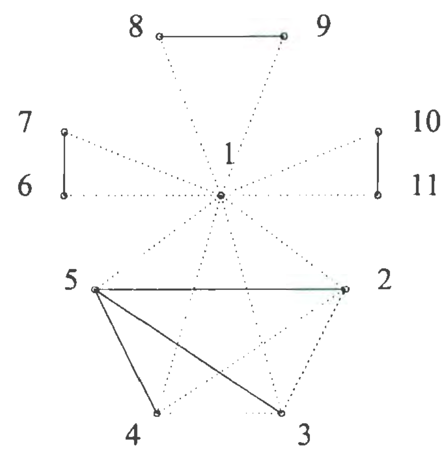

Figure 11: From PBD(11,{3,5*},1) to PD(10,3,1).

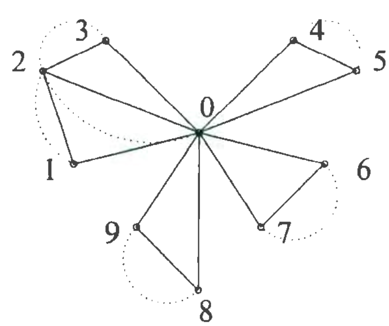

Figure 12: From BIBD(9,3,1) to CD(10,3,1).

2.3.6 Case 6: $v = 6i + 5$

The leave is a 4-cycle (2^4); the excess is a double edge (2^2).

Leave. By the same arithmetic, the leave$(6i + 5, 3, 1)$ has at least one edge. Every vertex has an even degree in the leave. Clearly one edge is not enough. So the leave has at least four edges. There is no double edge since $\lambda = 1$, so the leave is a 4-cycle.

A construction by Stinson [66]. Suppose the block of size 5 in a PBD$(6i + 5, \{3, 5^*\}, 1)$ is {1,2,3,4,5}, it can be decomposed into blocks {1,2,3}, {1,4,5} and a 4-cycle [2,4,3,5]. The leave$(6i + 5, 3, 1)$ is the 4-cycle (see Figure 13).

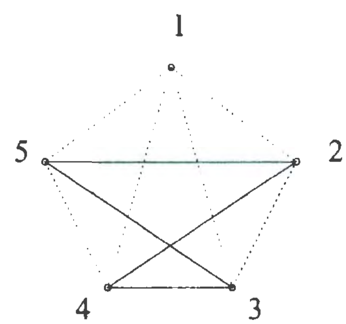

Figure 13: From K_5 to two blocks and a 4-cycle.

Excess. By the same arithmetic, the excess$(6i + 5, 3, 1)$ has at least two edges. Every vertex in the excess has an even degree, so the excess is a double edge.

A construction by Shalaby and Zhong. Suppose the leave$(6i + 5, 3, 1)$ is the 4-cycle [2,3,4,5]. Add edges {2,4}, {2,4} to it, we obtain blocks {2,3,4}, {2,4,5}, and the excess$(6i + 5, 3, 1)$ is the double edge {2,4}{2,4} (see Figure 14).

2.4 The cases of $\lambda = 2$

2.4.1 Case 7: $v = 6i$

There is a BIBD$(6i, 3, 2)$.

Figure 14: From leave(11,3,1) to excess(11,3,1).

2.4.2 Case 8: $v = 6i + 1$

There is a BIBD($6i + 1, 3, 2$).

2.4.3 Case 9: $v = 6i + 2$

The leave is a double edge (2^2); the excess is (1) a 4-cycle (2^4), (2) two independent double edges (2^4), or (3) two adjacent double edges (an "∞") ($2^2 4^1$).

Leave. By the same arithmetic, the leave($6i + 2, 3, 2$) has at least two edges. Every vertex has an even degree in the leave. So the leave could be a double edge.

A construction by Shalaby and Zhong. We know that the leave($6i + 2, 3, 1$) is a 1-factor (see **Case 3**). Combine two such 1-factors we obtain $3i$ new blocks as in **Case 1 Excess** and a double edge. The leave($6i + 2, 3, 2$) is the double edge (see Figure 15).

Excess. By the same arithmetic, the excess($6i + 2, 3, 2$) has one edge. But every vertex has an even degree in the excess, so the excess has four edges. It can be (1) a 4-cycle, (2) two independent double edges, (3) two adjacent double edges, or (4) a quadruple edge. Since $\lambda = 2$, (4) is illegal here.

A construction by Shalaby and Zhong. We know that the excess($6i + 2, 3, 1$) is a 3-star and a 1-factor (see **Case 3**). Combine two such excesses.

(1) Assume that the two 3-stars are {1,2}{1,3}{1,4} and {1,4}{2,4}{3,4}, and that the two excesses have edges {5,6}, {7,8}, {5,8}, {6,7}.
Those yield blocks {1,2,4}, {1,3,4} and the 4-cycle [5,6,7,8] (see Figure 16).

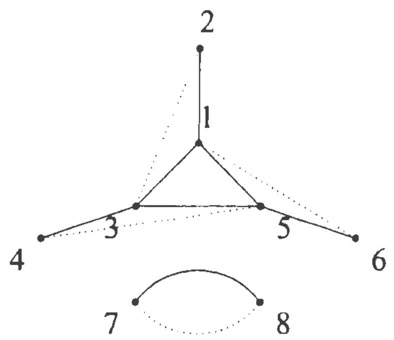

Figure 15: From leave$(8, 3, 1)$s to leave$(8, 3, 2)$.

The rest of the two 1-factors yield new blocks as in **Case 1 Excess**.
The excess$(6i + 2, 3, 2)$ is the 4-cycle.

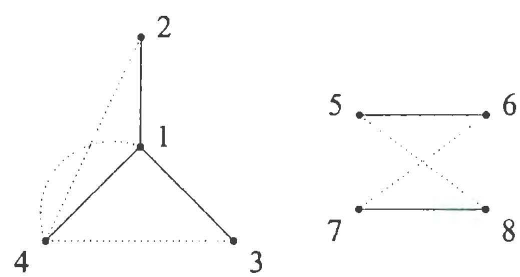

Figure 16: From excess$(8, 3, 1)$s to excess$(8, 3, 2)$, case (1).

(2) Assume that the two 3-stars are {1,2}{1,3}{1,4} and {1,4}{2,4}{3,4}, and that the two excesses have edges {5,6}, {7,8}, {5,6}, {7,8}.

Similar to (1), those yield blocks {1,2,4}, {1,3,4} and two double edges {5,6}{5,6}, {7,8}{7,8}.

The rest of the two 1-factors yield new blocks as in **Case 1 Excess**.

The excess$(6i + 2, 3, 2)$ is the double edges.

(3) Assume that one of the CD$(6i+2, 3, 1)$ has block {2,5,7}, and that the two 3-stars are {1,3}{1,4}{1,5} and {1,2}{1,3}{1,4}, and that the two excesses have edges {2,8}, {6,7}, {5,6}, {7,8}.

Those yield blocks {1,2,5}, {2,7,8}, {5,6,7} and two double edges {1,3}{1,3}, {1,4}{1,4} (the "∞") (see Figure 17).

The rest of the two 1-factors yield new blocks as in **Case 1 Excess**.

The excess$(6i + 2, 3, 2)$ is the "∞".

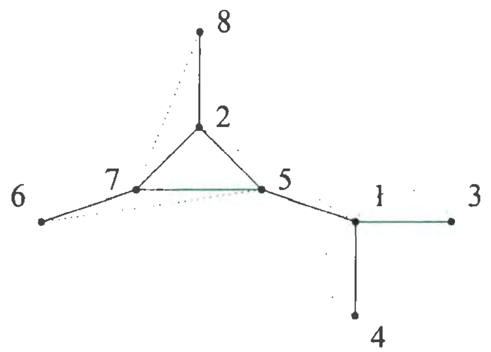

Figure 17: From excess$(8, 3, 1)$s to excess$(8, 3, 2)$, case (3).

Note A simpler way to construct the excess (1) and (3) by Shalaby and Zhong as follows.

We know that the leave$(6i + 2, 3, 2)$ is a double edge, say $L_1 = \{1,2\}\{1,2\}$.

(1) Add the 4-cycle [1,3,2,4] to L_1, we obtain blocks {1,2,3}, {1,2,4}. So the excess$(6i + 2, 3, 2)$ is the 4-cycle.

(3) Add edges {1,3}, {1,3}, {2,3}, {2,3} to L_1, we obtain blocks {1,2,3} (twice). So the excess$(6i + 2, 3, 2)$ is {1,3}{1,3}, {2,3}{2,3}.

Note When $\lambda \geq 8$, the excess$(6i+2, 3, 8)$ can be a quadruple edge. A construction by Shalaby and Zhong. We know that the excess$(6i + 2, 3, 4)$ is a double edge (see **Case 21**). Combine two such excesses, we can have a quadruple edge.

2.4.4 Case 10: $v = 6i + 3$

There is a BIBD$(6i + 3, 3, 2)$.

2.4.5 Case 11: $v = 6i + 4$

There is a BIBD$(6i + 4, 3, 2)$.

2.4.6 Case 12: $v = 6i + 5$

The leave is a double edge (2^2); the excess is (1) a 4-cycle (2^4), (2) two independent double edges (2^4), or (3) two adjacent double edges (an "∞") ($2^2 4^1$).

Leave. By the same arithmetic, the leave$(6i + 5, 3, 2)$ has at least two edges. Every vertex has an even degree in the leave. So the leave is a double edge.

A construction by Shalaby and Zhong. We know that the leave$(6i + 5, 3, 1)$ is a 4-cycle (see **Case 6**). Let L_1 = [1,2,3,4], L_2 = [1,2,4,3] be two such leaves.

Combine L_1 and L_2, we obtain blocks {1,2,3}, {1,2,4}, and the leave$(6i + 5, 3, 2)$ is {3,4}{3,4} (see Figure 18).

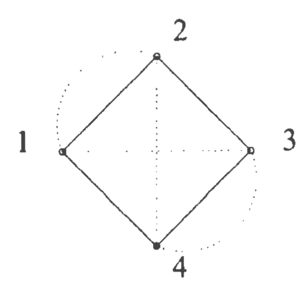

Figure 18: From leave$(11, 3, 1)$s to leave$(11, 3, 2)$.

Excess. By the same arithmetic, the excess$(6i + 5, 3, 2)$ has at least one edge. Every vertex has an even degree in the excess. So the leave has at least four edges. It can be (1) a 4-cycle, (2) two independent double edges, (3) two adjacent double edges or (4) a quadruple edge. Since $\lambda = 2$, (4) is illegal here.

A construction by Shalaby and Zhong. We know that the leave$(6i+5, 3, 2)$ (shown above) or the excess$(6i + 5, 3, 1)$ is a double edge (see **Case 6**).

To construct (1), let the leave$(6i + 5, 3, 2)$ be {1,2}{1,2}, add a 4-cycle [1,4,2,3] to it, we obtain new blocks as in **Case 6 Excess**. So the excess$(6i + 5, 3, 2)$ is the 4-cycle.

To construct (2) or (3), combine two excess$(6i+5, 3, 1)$s, nonadjacent or adjacent, respectively.

Note When $\lambda \geq 8$, the excess$(6i+5, 3, 8)$ can be a quadruple edge. A construction by Shalaby and Zhong. We know that the excess$(6i + 5, 3, 4)$ is a double edge (see

Case 24). Combine two such excesses, we can have a quadruple edge.

2.5 The cases of $\lambda = 3$

2.5.1 Case 13: $v = 6i$

The leave and the excess are the same as when $\lambda = 1$ (Case 1).

A construction by Stanton and Rogers [62]. Since there is a BIBD$(6i, 3, 2)$, the leave$(6i, 3, 3)$ and the excess$(6i, 3, 3)$ are the same as the leave$(6i, 3, 1)$ and excess$(6i, 3, 1)$, respectively.

2.5.2 Case 14: $v = 6i + 1$

There is a BIBD$(6i + 1, 3, 3)$.

2.5.3 Case 15: $v = 6i + 2$

The leave is (1) a graph {1,2}{1,3}{1,4}{3,5}{3,6}, (an "H") and a 1-factor $(3^2 1^{6i})$, (2) a 5-star and a 1-factor $(5^1 1^{6i+1})$, (3) two 3-stars and a 1-factor $(3^2 1^{6i})$, (4) a triple edge and a 1-factor $(3^2 1^{6i})$, or (5) a graph {5,6}{6,7}{6,7}{7,8}, (a "-0-") and a 1-factor $(3^2 1^{6i})$; also is the excess.

This time we begin with the **Excesses.** By the same arithmetic, the excess$(6i + 2, 3, 3)$ has $3i + 3$ edges. Every vertex has an odd degree in the excess. So there are two vertices of degree 3 or a vertex of degree 5, and the others of degree 1. There are 5 possibilities as above.

A construction by Shalaby and Zhong. We know that the excess$(6i + 2, 3, 1)$ is a 1-factor and a 3-star (see **Case 3**). Let E_1 = {1,2}{1,3}{1,4} (the 3-star), $\{2j + 1, 2j + 2\}$, $j = 2, ..., 3i$ (the 1-factor).

We also know that the excess$(6i + 2, 3, 2)$ can be a 4-cycle or an "∞" or two double edges (see **Case 9**).

(1) Let E_2 = [3,5,9,6] (a 4-cycle).

Combine E_1 and E_2, we obtain block {5,6,9}, and the excess$(6i + 2, 3, 3)$ is {1,2}{1,3}{1,4}{3,5}{3,6} (the "H"), $\{2j + 1, 2j + 2\}$, $j = 3, ..., 3i$ (the 1-factor) (see Figure 19).

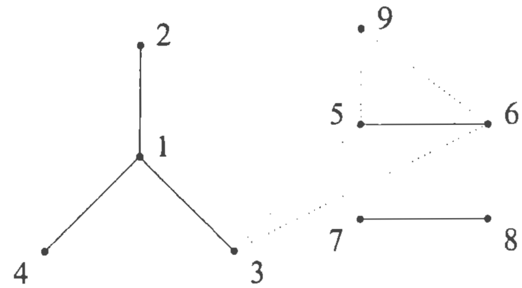

Figure 19: From excess$(8, 3, 1)$ and excess$(8, 3, 2)$ to excess$(8, 3, 3)$, case (1).

(2) Let $E_3 = \{1,5\}\{1,5\}\{1,6\}\{1,6\}$ (an "∞").

Combine E_1 and E_3, we obtain block $\{1,5,6\}$, and the excess$(6i + 2, 3, 3)$ is $\{1,2\}\{1,3\}\{1,4\}\{1,5\}\{1,6\}$ (the 5-star), $\{2j + 1, 2j + 2\}$, $j = 3, ..., 3i$ (the 1-factor) (see Figure 20).

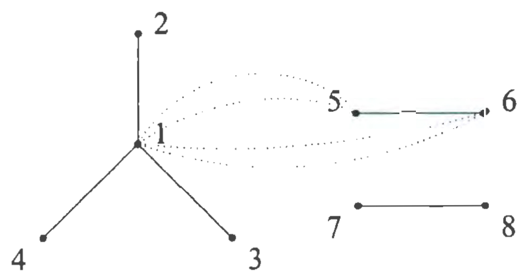

Figure 20: From excess$(8, 3, 1)$ and excess$(8, 3, 2)$ to excess$(8, 3, 3)$, case (2).

(3) Let $E_4 = \{5,7\}\{5,7\}\{5,8\}\{5,8\}$ (an "∞").

Combine E_1 and E_4, we obtain block $\{5,7,8\}$, and the excess$(6i + 2, 3, 3)$ is $\{1,2\}\{1,3\}\{1,4\}$, $\{5,6\}\{5,7\}\{5,8\}$ (two 3-stars), $\{2j + 1, 2j + 2\}$, $j = 4, ..., 3i$ (the 1-factor) (see Figure 21).

(4) Let $E_5 = \{2,3\}\{2,3\}\{1,4\}\{1,4\}$ (two double edges).

Combine E_1 and E_5, we obtain block $\{1,2,3\}$, and the excess$(6i + 2, 3, 3)$ is $\{1,4\}\{1,4\}\{1,4\}$ (the triple edge), $\{2,3\}$, $\{2j + 1, 2j + 2\}$, $j = 2, ..., 3i$ (the 1-factor) (see Figure 22).

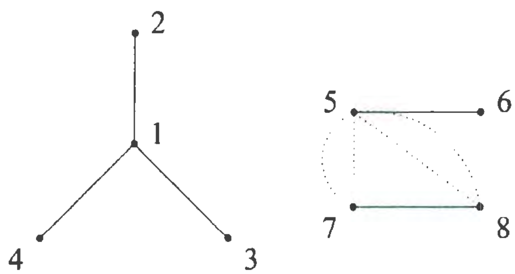

Figure 21: From excess(8, 3, 1) and excess(8, 3, 2) to excess(8, 3, 3), case (3).

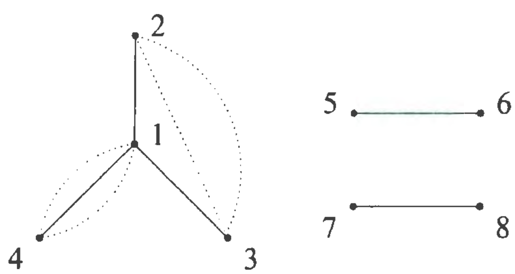

Figure 22: From excess(8, 3, 1) and excess(8, 3, 2) to excess(8, 3, 3), case (4).

(5) Let E_6 = {2,3}{2,3}{6,7}{6,7} (two double edges).

Combine E_1 and E_6, we obtain block {1,2,3}, and the excess$(6i + 2, 3, 3)$ is {5,6}{6,7}{6,7}{7,8} (the "-0-"), {1,4}, {2,3}, $\{2j + 1, 2j + 2\}$, $j = 4, ..., 3i$ (the 1-factor) (see Figure 23).

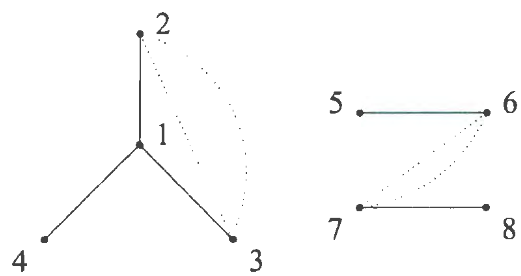

Figure 23: From excess$(8, 3, 1)$ and excess$(8, 3, 2)$ to excess$(8, 3, 3)$, case (5).

Leave. By the same arithmetic, the leave$(6i + 2, 3, 3)$ has $3i + 3$ edges. Every vertex has an odd degree in the leave. So there are two vertices of degree 3 or a vertex of degree 5, and the others of degree 1. There are 5 possibilities as well.

A construction by Shalaby and Zhong. We know that the excess$(6i+2, 3, 3)$ can be two 3-stars and a 1-factor (shown above). Let E_1 = {1,2}{1,3}{1,4}, {5,6}{5,7}{5,8} (the two 3-stars), $\{6j - 3, 6j - 2\}$, $\{6j - 1, 6j\}$, $\{6j + 1, 6j + 2\}$, $j = 2, ..., i$ (the 1-factor).

Add the 1-factor $\{6j - 2, 6j - 1\}$, $\{6j, 6j + 1\}$, $\{6j + 2, 6j - 3\}$, $j = 2, ..., i$ to E_1, we obtain blocks as in **Case 1 Excess**. So we have found the 1-factor of the leave$(6i + 2, 3, 3)$.

For the rest part of the leave$(6i + 2, 3, 3)$:

(1) Add edges {1,5}{1,3}{1,6}{3,2}{3,4} (an "H"), {7,8} to the two 3-stars of E_1, we obtain blocks {1,2,3}, {1,3,4}, {1,5,6}, {5,7,8}, and case (1) is realized (see Figure 24).

(2) Add edges {3,2}{3,1}{3,4}{3,5}{3,6} (a 5-star), {7,8} to the two 3-stars of E_1, we obtain blocks {1,2,3}, {1,3,4}, {3,5,6}, {5,7,8}, and case (2) is realized (see Figure 25).

(3) Add edges {4,1}{4,2}{4,3}, {8,6}{8,7}{8,5} (two 3-stars) to the two 3-stars of E_1, we obtain blocks {1,2,3}, {1,2,4}, {5,6,8}, {5,7,8}, and case (3) is realized (see

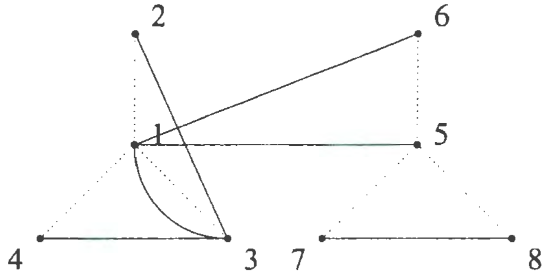

Figure 24: From excess(8, 3, 3) to leave(8, 3, 3), case (1).

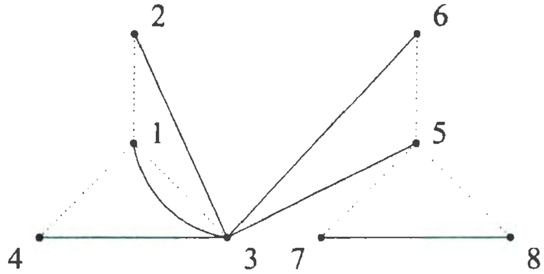

Figure 25: From excess(8, 3, 3) to leave(8, 3, 3), case (2).

Figure 26).

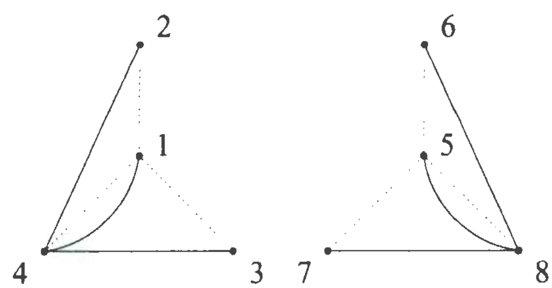

Figure 26: From excess$(8,3,3)$ to leave$(8,3,3)$, case (3).

To construct case (4) or (5), we know that the leave$(6i+2,3,1)$ is a 1-factor (see **Case 3**), and that the leave$(6i+2,3,2)$ is a double edge (see **Case 9**). Let $L_1 = \{2j-1, 2j\}$, $j = 1, \ldots, 3i+1$.

Let $L_2 = \{1,2\}\{1,2\}$, combine L_1 and L_2, case (4) is realized.

Let $L_2 = \{2,3\}\{2,3\}$, combine L_1 and L_2, case (5) is realized.

2.5.4 Case 16: $v = 6i + 3$

There is a BIBD$(6i+3,3,3)$.

2.5.5 Case 17: $v = 6i + 4$

The leave and the excess are the same as when $\lambda = 1$ (Case 5).

A construction by Stanton and Rogers [62]. Since there is a BIBD$(6i+4,3,2)$, the leave$(6i+4,3,3)$ and the excess$(6i+4,3,3)$ are the same as the leave$(6i+4,3,1)$ and excess$(6i+4,3,1)$, respectively.

2.5.6 Case 18: $v = 6i + 5$

There is a BIBD$(6i+5,3,3)$.

2.6 The cases of $\lambda = 4$

2.6.1 Case 19: $v = 6i$

There is a BIBD$(6i, 3, 4)$.

2.6.2 Case 20: $v = 6i + 1$

There is a BIBD$(6i + 1, 3, 4)$.

2.6.3 Case 21: $v = 6i + 2$

The leave is (1) a 4-cycle (2^4), (2) a quadruple edge (4^2), (3) two independent double edges (2^4), or (4) two adjacent double edges (an "∞") $(2^2 4^1)$; the excess is a double edge (2^2).

Leave. By the same arithmetic, the leave$(6i + 2, 3, 4)$ has at least one edge. But every vertex has an even degree in the leave. So the leave has at least four edges. There are 4 possibilities as above.

A construction by Shalaby and Zhong. (1) We know that a leave$(6i + 2, 3, 3)$ can be a "-0-" and a 1-factor (see **Case 15**); and that a leave$(6i + 2, 3, 1)$ is a 1-factor (see **Case 3**).

Combine two such leaves. And let the "-0-" be {1,2}{2,3}{2,3}{3,4}, and assume we have {5,6}, {7,8}, {5,7}, {6,8} in the two leaves.

Those form blocks {1,2,3}, {2,3,4} and a 4-cycle [5,6,8,7] (see Figure 27).

The rest of the two 1-factors form new blocks as in **Case 1 Excess**. Case (1) is realized.

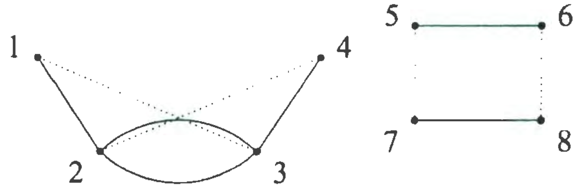

Figure 27: From leave$(8, 3, 1)$ and leave$(8, 3, 3)$ to leave$(8, 3, 4)$, case (1).

To construct case (2), (3) or (4), we know that the leave($6i + 2, 3, 2$) is a double edge (see **Case 9**). Combine two double edges, case (2), (3) or (4) is realized.

Excess. By the same arithmetic, the excess($6i + 2, 3, 4$) has at least two edges. Every vertex has an even degree in the excess, so the excess is a double edge.

A construction by Shalaby and Zhong. We know that the leave($6i + 2, 3, 4$) can be a 4-cycle (shown above), we can use the same construction as in **Case 6 Excess**.

2.6.4 Case 22: $v = 6i + 3$

There is a BIBD($6i + 3, 3, 4$).

2.6.5 Case 23: $v = 6i + 4$

There is a BIBD($6i + 4, 3, 4$).

2.6.6 Case 24: $v = 6i + 5$

The leave is (1) a 4-cycle (2^4), (2) a quadruple edge (4^2), (3) two independent double edges (2^4), or (4) two adjacent double edges (an "∞") ($2^2 4^1$); the excess is a double edge (2^2).

Leave. By the same arithmetic, the leave($6i + 5, 3, 4$) has at least one edge. But every vertex has an even degree in the leave. So the leave has at least 4 edges. There are 4 possibilities as above.

(1) A construction by Stanton and Rogers [62]. Since there is a BIBD($6i+5, 3, 3$), the leave($6i + 5, 3, 4$) can be the same as the leave($6i + 5, 3, 1$), a 4-cycle (see **Case 6**).

To construct case (2), (3) or (4), we know that the leave($6i + 5, 2, 2$) is a double edge (see **Case 12**). Combine two double edges, case (2), (3) or (4) is realized.

Excess. By the same arithmetic, the excess($6i + 5, 3, 4$) has 2 edges. Every vertex has an even degree in the excess, so the excess is a double edge.

A construction by Stanton and Rogers [62]. Since there is a BIBD($6i + 5, 3, 3$), the leave($6i + 5, 3, 4$) can be the same as the leave($6i + 5, 3, 1$), a double edge (see **Case 6**).

2.7 The cases of $\lambda = 5$

2.7.1 Case 25: $v = 6i$

The leave and the excess are the same as when $\lambda = 1$ (Case 1).

A construction by Stanton and Rogers [62]. Since there is a BIBD$(6i, 3, 4)$, the leave$(6i, 3, 5)$ and the excess$(6i, 3, 5)$ are the same as the leave$(6i, 3, 1)$ and excess$(6i, 3, 1)$, respectively.

2.7.2 Case 26: $v = 6i + 1$

There is a BIBD$(6i + 1, 3, 5)$.

2.7.3 Case 27: $v = 6i + 2$

The leave is a 3-star and a 1-factor $(3^1 1^{6i+1})$; the excess is a 1-factor (1^{6i+2}).

Leave. By the same arithmetic, the leave$(6i + 2, 3, 5)$ has $3i + 2$ edge. Every vertex has an odd degree in the leave. So there is a vertex of degree 3, and others of degree 1.

A construction by Shalaby and Zhong. We know that the leave$(6i+2, 3, 3)$ can be two 3-stars and a 1-factor (see **Case 15**); and that the leave$(6i + 2, 3, 2)$ is a double edge (see **Case 9**).

Combine two such leaves. And let one of the 3-stars be {4,1}{4,2}{4,3}, and the double edge be {1,2}{1,2}. We obtain block {1,2,4} and edges {1,2}, {3,4} (see Figure 28).

The 1-factor and the other 3-star of the leave$(6i+2, 3, 3)$ remain in the leave$(6i+2, 3, 5)$.

Excess. By the same arithmetic, the excess$(6i + 2, 3, 5)$ has $3i + 1$ edges. Every vertex has an odd degree in the excess, so the excess is a 1-factor.

A construction by Shalaby and Zhong. We know that the excesses$(6i + 2, 3, 1)$ can be a 3-star and a 1-factor (see **Case 3**); and that the excess$(6i + 2, 3, 4)$ can be a double edge (see **Case 21**).

Combine two such excesses. And let the 3-stars be {1,2}{1,3}{1,4}, and the double edge be {2,3}{2,3}. We obtain block {1,2,3} and edges {1,4}, {2,3} (see Figure 29).

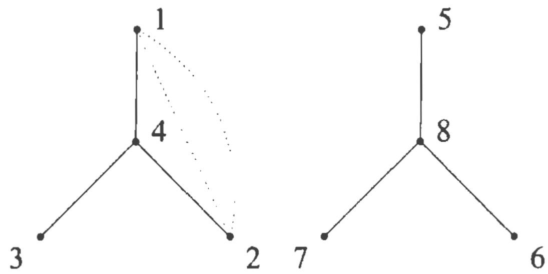

Figure 28: From leave$(8, 3, 2)$ and leave$(8, 3, 3)$ to leave$(8, 3, 5)$.

The rest of the excess$(6i + 2, 3, 3)$ remain in the excess$(6i + 2, 3, 5)$.

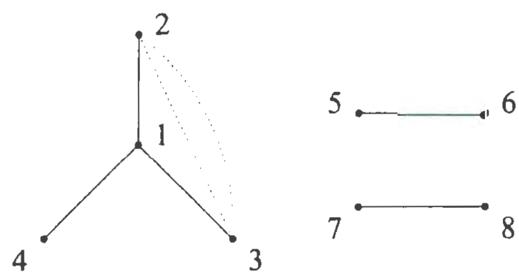

Figure 29: From excess$(8, 3, 1)$ and excess$(8, 3, 4)$ to excess$(8, 3, 5)$.

2.7.4 Case 28: $v = 6i + 3$

There is a BIBD$(6i + 3, 3, 5)$.

2.7.5 Case 29: $v = 6i + 4$

The leave and the excess are the same as when $\lambda = 1$ (Case 5).

A construction by Stanton and Rogers [62]. Since there is a BIBD$(6i + 4, 3, 4)$, the leave$(6i+4, 3, 5)$ and the excess$(6i+4, 3, 5)$ are the same as the leave$(6i+4, 3, 1)$ and excess$(6i + 4, 3, 1)$, respectively.

2.7.6 Case 30: $v = 6i + 5$

The leave and the excess are the same as when $\lambda = 2$ (Case 12), except that the excess can be a quadruple edge (4^2).

A construction by Stanton and Rogers [62]. Since there is a BIBD$(6i + 5, 3, 3)$, the leave$(6i + 5, 3, 5)$ or the excess$(6i + 5, 3, 5)$ is the same as the leave$(6i + 5, 3, 2)$ or the excess$(6i + 5, 3, 2)$, respectively.

Note that excess$(6i + 5, 3, 5)$ can be a quadruple edge, since $\lambda = 5 \geq 4$. Our construction is to combine the excess$(6i + 5, 3, 1)$ and the excess$(6i + 5, 3, 4)$, which are double edges (see **Case 6** and **Case 24**).

3 Leaves and Excesses when $k = 4$.

3.1 Necessary Conditions

Necessary conditions for a multigraph G to be a leave(v, k, λ), Mendelsohn, Shalaby and Shen [52]: let $|E|$ be the number of edges in G,

(1) $\lambda v(v-1)/2 - |E| \equiv 0 \pmod{k(k-1)/2}$;

(2) for all $x \in G$, $\deg(x) \equiv \lambda(v-1) \pmod{k-1}$;

(3) for all $x \in G$, $\deg(x) \leq \lambda(v-1)$.

Necessary conditions for a multigraph G to be an excess(v, k, λ): let $|E|$ be the number of edges in G:

(1) $\lambda v(v-1)/2 + |E| \equiv 0 \pmod{k(k-1)/2}$;

(2) for all $x \in G$, $\deg(x) \equiv \lambda(v-1) \pmod{k-1}$;

(3) for all $x \in G$, $\deg(x) \geq \lambda(v-1)$.

3.2 Sufficient Conditions

The following results on packing and covering numbers give the number of edges in the PDs and CDs, which is very useful in constructing the leaves and excesses. Some of the papers already constructed some PDs and CDs, hence the leaves and excesses can be easily obtained.

Let $\mathrm{C}(v, 4, \lambda)$ be the number of edges in a $\mathrm{CD}(v, 4, \lambda)$ and $\Phi(v, 4, \lambda)$ be the Schonheim lower bound; let $\mathrm{P}(v, 4, \lambda)$ be the number of blocks in a $\mathrm{PD}(v, 4, \lambda)$ and $\Psi(v, 4, \lambda)$ be the Schonheim upper bound, recall:

Assaf [6]:

$$\mathrm{C}(v, 4, \lambda) = \begin{cases} \Phi(v, 4, \lambda) + 1, & v = 7, 9, 10, \lambda = 1; \\ \Phi(v, 4, \lambda) + 2, & v = 19, \lambda = 1; \\ \Phi(v, 4, \lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were proved by Mills [45] [46], Horton, Mullin and Stanton [42].

Billington, Stanton and Stinson [23], Assaf [7]:

$$\mathrm{P}(v,4,\lambda) = \begin{cases} \Psi(v,4,\lambda)-1, & v = 7, 10 (\mathrm{mod}\ 12)\ , v \neq 10, 19, \lambda = 1; \\ \Psi(v,4,\lambda)-1, & v = 9, 17, \lambda = 1; \\ \Psi(v,4,\lambda)-2, & v = 8, 10, 11, \lambda = 1; \\ \Psi(v,4,\lambda)-3, & v = 19, \lambda = 1; \\ \Psi(v,4,\lambda)-1, & v = 9, \lambda = 2; \\ \Psi(v,4,\lambda)-1, & v = 6, \lambda = 3; \\ \Psi(v,4,\lambda), & \text{otherwise.} \end{cases}$$

Note The cases of $\lambda = 1$ were done by Brouwer [26], and $P(19, 4, 1)$ was due to Stinson [64]; the cases of $\lambda > 1$, $v \not\equiv 0$ (mod 3) were done by Billington, Stanton and Stinson [23]; the remaining cases were done by Assaf [7], where some small vs were done by Hartman [41].

Note that in the following figures, we will use solid lines for leaves and dotted lines for excesses. And in the figures combining two leaves or excesses, a leave/excess will be in solid line and the other in dotted line.

Note that we use [1,2,3,4] for a 4-cycle (C_4) and {1,2,3,4} for a block of size 4 (K_4) in case of possible confusion.

3.3 The cases of $\lambda = 1$

Lemma 5. Kreher and Stinson [44]: There exists a 4-GDD of type $6^{v/6}$ for all $v \equiv 0$ (mod 6), $v \geq 30$.

3.3.1 Case 1: $v = 12i$

The leave is a 2-factor (2^{12i}); the excess is a 1-factor(1^{12i}).

Leave. If the upper bound $\Psi(12i, 4, 1)$ is satisfied, then a PD$(12i, 4, 1)$ has $\lfloor 12i/4 \lfloor (12i-1)/3 \rfloor \rfloor = 12i^2 - 3i$ blocks and $72i^2 - 18i$ edges. There are $\binom{12i}{2} = 72i^2 - 6i$ edges in K_{12i}, so the leave$(12i, 4, 1)$ has $12i$ edges.

Every vertex x has degree $12i - 1$ in K_{12i}, and every block in the PD$(12i, 4, 1)$ containing x takes 3, so the degree of x in the leave is 2 (mod 3). The only possibility is that every vertex has degree 2, so the leave is a 2-factor.

There are many non isomorphic 2-factors; we give three constructions here.

(1) $4i$ triangles. A construction by Brouwer [27]. Delete a vertex $12i + 1$ from a BIBD$(12i + 1, 4, 1)$. The blocks containing $12i + 1$ reduce into triangles, and those not containing $12i + 1$ remain as the PD$(12i, 4, 1)$.

The Schonheim bound is satisfied and the triangles are the leave$(12i, 4, 1)$ (see Figure 30).

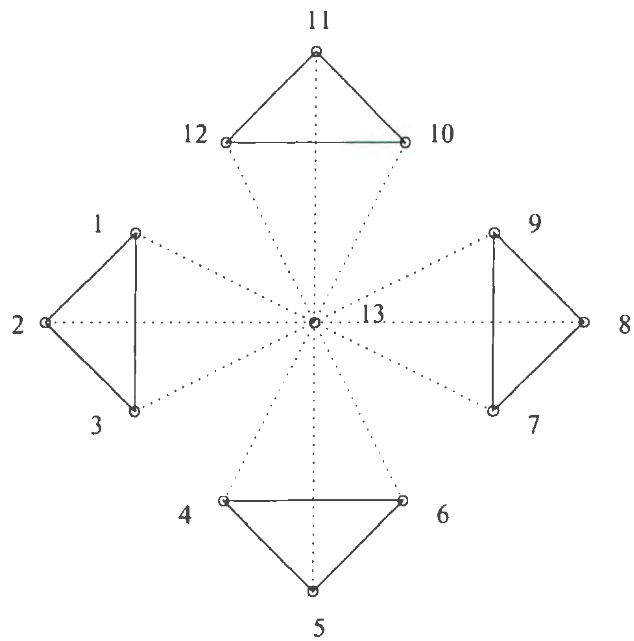

Figure 30: Leave(12,4,1) case (1).

(2) $3i$ squares. A construction by Shalaby and Zhong.

We know that the excess$(12i, 4, 1)$ is a 1-factor (shown below). Let $E = \{4j - 3, 4j - 1\}, \{4j - 2, 4j\}, j = 1, ..., 3i$.

Add the squares $[4j - 3, 4j - 2, 4j - 1, 4j]$ to E, we obtain blocks $\{4j - 3, 4j - 2, 4j - 1, 4j\}, j = 1, ..., 3i$.

So the leave$(12i, 4, 1)$ is these squares (see Figure 31).

(3) When $\lambda \geq 7$, we can have a more general constructionby Shalaby and Zhong.

Since there is a BIBD$(12i, 4, 3)$, the leave$(12i, 4, 7)$ can be the same as the leave$(12i, 4, 4)$, a (disjoint) union of even cycles (see **Case 37**).

Excess. By the same arithmetic, the excess$(12i, 4, 1)$ has $6i$ edges. Similar to the leave, every vertex in the excess has degree 1 (mod 3), so the excess is a 1-factor.

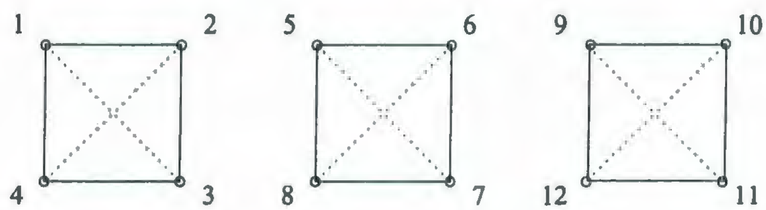

Figure 31: Leave(12,4,1) case (2).

A construction by Stinson [65]. Lemma 5 gives a 4-GDD of type $6^{v/6}$, except for $v = 12, 24$.

To construct a CD$(12i, 4, 1)$, fill in each group with a CD(6,4,1) (see Figure 32); and take the blocks of the GDD as well.

The union of the excess(6,4,1)s is the excess$(12i, 4, 1)$, a 1-factor.

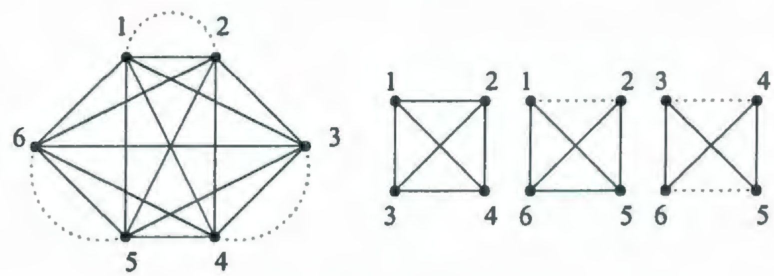

Figure 32: A CD(6,4,1).

3.3.2 Case 2: $v = 12i + 1$

There is a BIBD$(12i + 1, 4, 1)$.

Lemma 6. Kreher and Stinson [44]: There exists a 4-GDD of type $2^{v/2}$, for all $v \equiv 2 \pmod 6$, $v \geq 14$.

3.3.3 Case 3: $v = 12i + 2$

The leave is a 1-factor(1^{12i+2}); the excess is (1) a graph with two vertices

of degree 5 and other vertices with degree 2 ($5^2 2^{12i}$) or (2) a graph with a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+1}$).

Leave. By the same arithmetic, the leave$(12i + 2, 4, 1)$ has $6i + 1$ edges. Every vertex has degree 1 (mod 3) in the leave. So the leave is a 1-factor.

A construction by Kreher and Stinson [44]. Lemma 6 gives a 4-GDD of type $2^{v/2}$. The blocks form the packing; the groups of size 2 form the leave, a 1-factor.

Excess. By the same arithmetic, the excess$(12i+2, 4, 1)$ has $12i+5$ edges. Every vertex has a degree 2 (mod 3) in the excess. There are 2 solutions as above.

(1) $4i - 1$ triangles and a "Crown". A construction by Stinson [65]. Take a BIBD($12i$ + 1,4,1), add a new point 0 and new blocks {0,1,2,3}, {0,1,2,4}, $\{0, 3j - 1, 3j, 3j + 1\}$, $j = 2, ..., 4i$.

The excess is these repeated edges, the $4i - 1$ triangles and a graph {1,2}{1,2} {0,1}{0,2}{3,1}{3,2}{4,1} {4,2} (the "Crown") (see Figure 33).

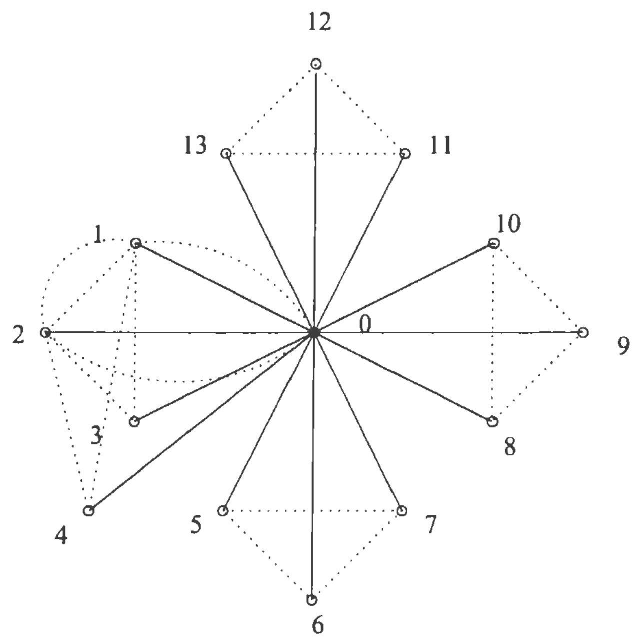

Figure 33: Excess(14,4,1) case (1).

(2) $3i - 1$ squares and a $K_6 \setminus K_4$. A construction by Shalaby and Zhong. We know that the leave$(12i + 2, 4, 1)$ is a 1-factor (shown above). Let $L = \{2j - 1, 2j\}$, $j = 1, ..., 6i + 1$.

Add the edges {1,3}{1,4}{2,3}{2,4}{3,4}{3,5}{3,6}{4,5}{4,6} (it's in fact a $K_6 \setminus K_4$) to the edges {1,2}, {3,4}, {5,6}, we obtain blocks {1,2,3,4}, {3,4,5,6}.

Add the squares $[4j - 1, 4j + 1, 4j, 4j + 2]$ to the rest of the 1-factor, we obtain new blocks as in **Case 1 Leave (2)**..

So the excess$(12i + 2, 4, 1)$ is the squares and the $K_6 \setminus K_4$ (see Figure 34).

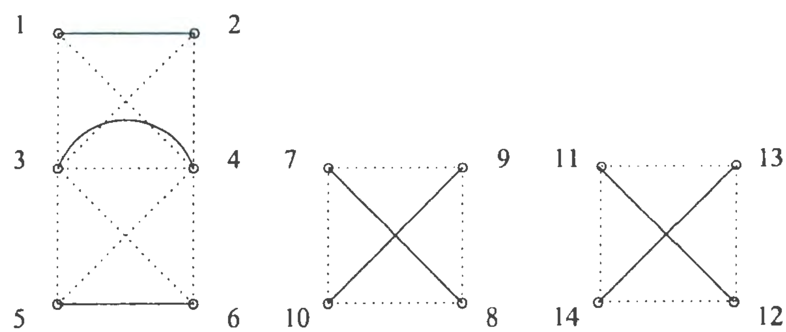

Figure 34: Excess(14,4,1) case (2)

Lemma 7. Kreher and Stinson [44]: There exists a 4-GDD of type $6^{(v-15)/6}15^1$, for all $v \equiv 3 \pmod 6$, $v \geq 15$.

3.3.4 Case 4: $v = 12i + 3$

The leave is a 2-factor (2^{12i+3}); the excess is a 1-factor and a 4-star ($4^1 1^{12i+2}$).

Leave. By the same arithmetic, the leave$(12i + 3, 4, 1)$ has $12i + 3$ edges. Every vertex has degree 2 (mod 3) in the leave. So the leave is a 2-factor.

(1) $4i + 1$ triangles. A construction by Brouwer [27]. Delete a vertex $12i + 4$ from a BIBD$(12i + 4, 4, 1)$. The blocks containing $12i + 4$ reduce to $4i + 1$ triangles, and form the leave$(12i + 3, 4, 1)$ (see Figure 35).

(2) $3i$ squares and a triangle. A construction by Shalaby and Zhong. We know that the excess$(12i + 3, 4, 1)$ is a 1-factor and a 4-star (shown below). Let $E =$

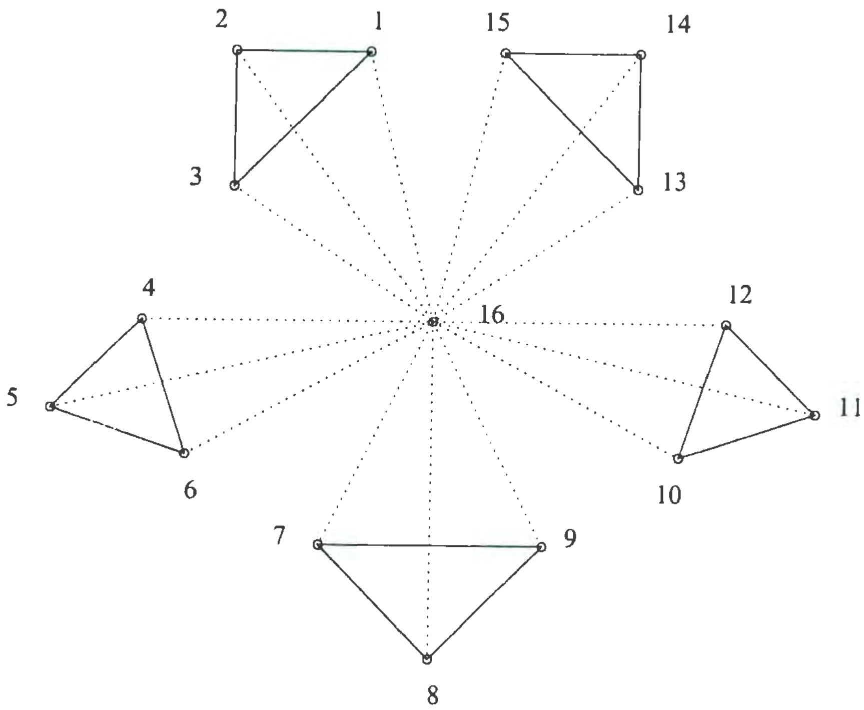

Figure 35: Leave(15,4,1) case (1).

{1,2}{1,3}{1,4}{1,5} (the 3-star), $\{4j, 4j+1\}$, $\{4j+2, 4j+3\}$, $j = 2, ..., 3i$ (the 1-factor).

Add the triangle {2,3,4} and the squares [1,6,5,7] to the 3-star and the edge {6,7} in E, we obtain blocks {1,2,3,4}, {1,5,6,7}.

Add the squares $[4j, 4j+2, 4j+1, 4j+3]$ to the rest of the 1-factor, we obtain new blocks as in **Case 1 Leave 2**.

So the leave$(12i+3, 4, 1)$ is the squares and the triangle (see Figure 36).

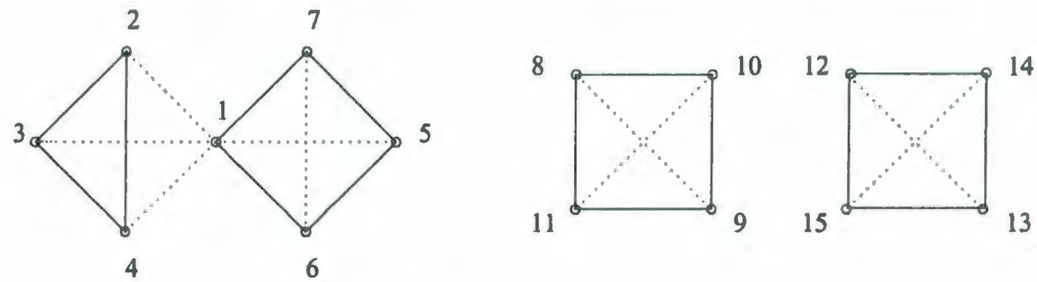

Figure 36: Leave(15,4,1) case (2).

Excess. By the same arithmetic, the excess$(12i+3, 4, 1)$ has $6i+3$ edges. Every vertex has degree 1 (mod 3) in the excess. So there is a vertex with degree 4 and other $12i+2$ vertices with degree 1, and the excess is a 1-factor and a 4-star.

A construction by Stinson [65]. Lemma 7 gives a 4-GDD of type $6^{(v-15)/6}15^1$ for $v \geq 15$.

Fill in each group of the GDD with a CD(6,4,1) (three edges, see Figure 32) or a CD(15,4,1) (a 4-star and a 1-factor, given in Brouwer [27], see Figure 37).

The excess$(12i+3, 4, 1)$ is the union of the excess$(6, 4, 1)$s and the excess$(15, 4, 1)$.

3.3.5 Case 5: $v = 12i + 4$

There is a BIBD$(12i + 4, 4, 1)$.

Lemma 8. Brouwer [27]: There exists a 4-GDD of type $2^{(v-5)/2}5^1$, for all $v \equiv 5$ (mod 6), $v \geq 23$.

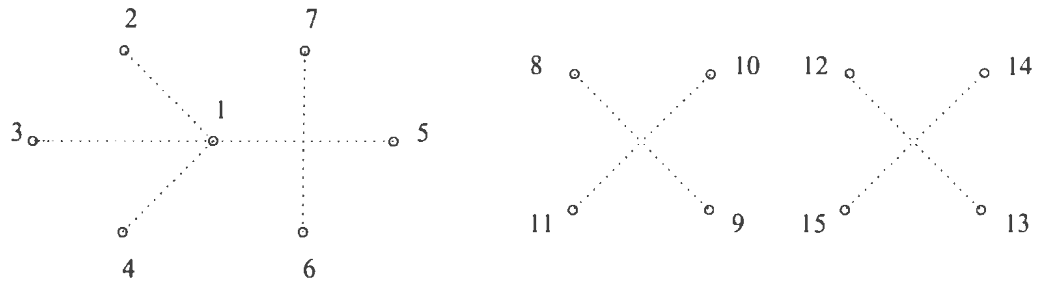

Figure 37: An Excess(15,4,1)

3.3.6 Case 6: $v = 12i + 5$

The leave is a 1-factor and a 4-star ($4^1 1^{12i+4}$); the excess is (1) a graph with two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+3}$), or (2) a graph with a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+4}$).

Leave. By the same arithmetic, the leave$(12i + 5, 4, 1)$ has $6i + 4$ edges. Every vertex has degree 1 (mod 3) in the leave. So there is a vertex of degree 4 and others of degree 1, and the leave is a 1-factor and an 4-star.

A construction by Brouwer [27]. Lemma 8 gives a 4-GDD of type $2^{6i}5^1$, except for $v = 17$. Suppose {1,2,3,4,5} is the unique group of size 5.

Take the blocks of the GDD, take the block {1,2,3,4} from the K_5 and the $K_5 \setminus K_4$ gives the 4-star; the groups of size 2 form the 1-factor (see Figure 38).

Excess. By the same arithmetic, the excess$(12i + 5, 4, 1)$ has has $12i + 8$ edges. Every vertex has degree 2 (mod 3) in the excess. There are two solutions as above.

(1) $4i$ triangles and a "Crown". A construction by Stinson [65]. Take a BIBD$(12i+4,4,1)$, add a new point 0 and new blocks {0,1,2,3}, {0,1,2,4}, $\{0, 3j - 1, 3j, 3j + 1\}$, $j = 2, ..., 4i + 1$.

The excess is the triangles and a graph {1,2}{1,2} {1,3} {2,3}{0,1}{0,2}{4,1}{4,2} (the "Crown") (see Figure 39).

(2) $3i - 1$ squares and a triangle and a $K_6 \setminus K_4$. A construction by Shalaby and Zhong. We know that the leave$(12i+5, 4, 1)$ is a 1-factor and a 4-star (shown above). Let $L = \{4j - 3, 4j - 2\}$, $\{4j - 1, 4j\}$, $j = 1, ..., 3i$ (the 1-factor), $\{12i + 2, 12i + 1\}\{12i + 2, 12i + 3\}\{12i + 2, 12i + 4\}\{12i + 2, 12i + 5\}$ (the 4-star).

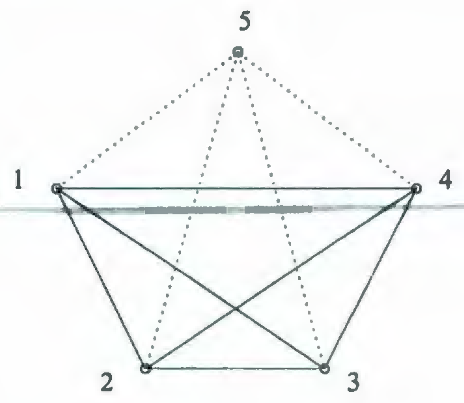

Figure 38: A $K_5 \setminus K_4$

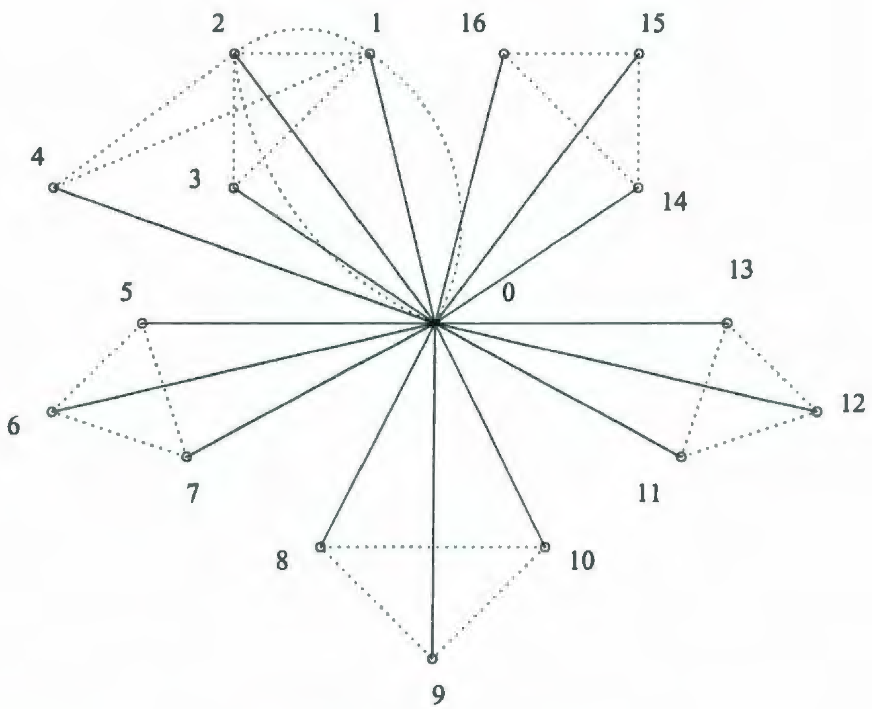

Figure 39: Excess(17,4,1) case (1).

Add the edges {1,3}{1,4}{2,3}{2,4}{3,4}{3,5}{3,6}{4,5}{4,6} (a $K_6 \setminus K_4$) to the edges {1,2}, {3,4}, {5,6}, we obtain blocks {1,2,3,4}, {3,4,5,6}.

Add the triangle $\{12i{+}3, 12i{+}4, 12i{+}5\}$ and the square $[12i{-}1, 12i{+}2, 12i, 12i{+}1]$ to the 3-star and the edge $\{12i{-}1, 12i\}$, we obtain blocks $\{12i{-}1, 12i, 12i{+}1, 12i{+}2\}$, $\{12i+2, 12i+3, 12i+4, 12i+5\}$.

Add the squares $[4j-1, 4j+1, 4j, 4j+2]$, $j = 1, ..., 3i$ to the rest of the 1-factor, we obtain new blocks as in **Case 1 Leave (2)**..

So the excess$(12i+5, 4, 1)$ is the $K_6 \setminus K_4$ and the squares and the triangle (see Figure 40).

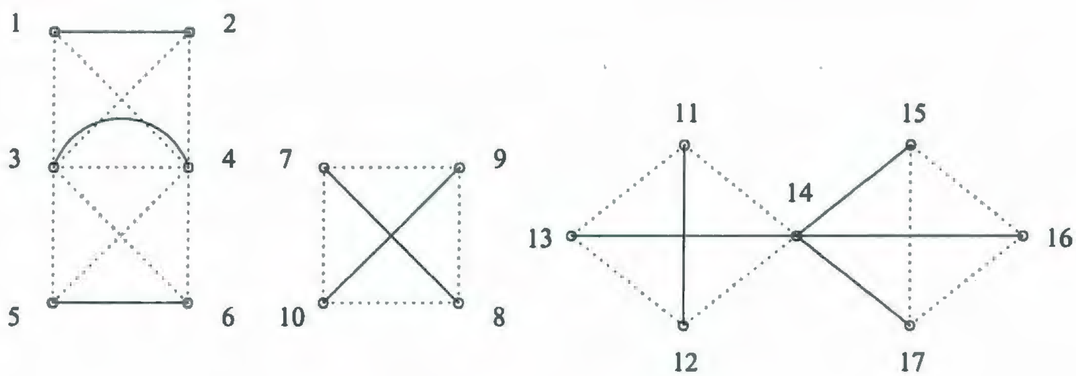

Figure 40: Excess(17,4,1) case (2).

(3) When $\lambda \geq 7$, we can have a more general construction.

A construction by Shalaby and Zhong. Since there is a BIBD$(12i+5, 4, 3)$, the excess$(12i{+}5, 4, 7)$ can be the same as the excess$(12i{+}5, 4, 4)$, the three constructions (see **Case 42**).

Lemma 9. Brouwer [27]: There exists a PBD(v,{4,7*},1) for all $v \equiv 7, 10 \pmod{12}$, with the exception of $v = 10, 19$.

3.3.7 Case 7: $v = 12i + 6$

The leave is (1) a graph with two vertices of degree 5 and others of degree 2 ($2^{12i+4}5^2$), or (2) a graph with a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+5}$); the excess is a 1-factor (1^{12i+6}).

Leave. By the same arithmetic, the leave$(12i + 6, 4, 1)$ has $12i + 9$ edges. Every vertex has degree 2 (mod 3) in the leave. There are two solutions as above.

(1) $4i$ triangles and a $K_6 \setminus K_4$. A construction by Brouwer [27]. Lemma 9 gives a PBD($12i$ + 7,{4,7*},1), except for $v = 19$. Suppose {1,2,3,4,5,6,7} is the unique block of size 7.

Delete vertex 7, the blocks containing 7 become triangles; remove block {1,2,3,4} from {1,2,3,4,5,6}, we obtain a $K_6 \setminus K_4$.

So the leave$(12i + 6, 4, 1)$ is the $4i$ triangles and the $K_6 \setminus K_4$ (see Figure 41).

Note that this figure is for illustration only, since there is no PBD(19,{4,7*},1).

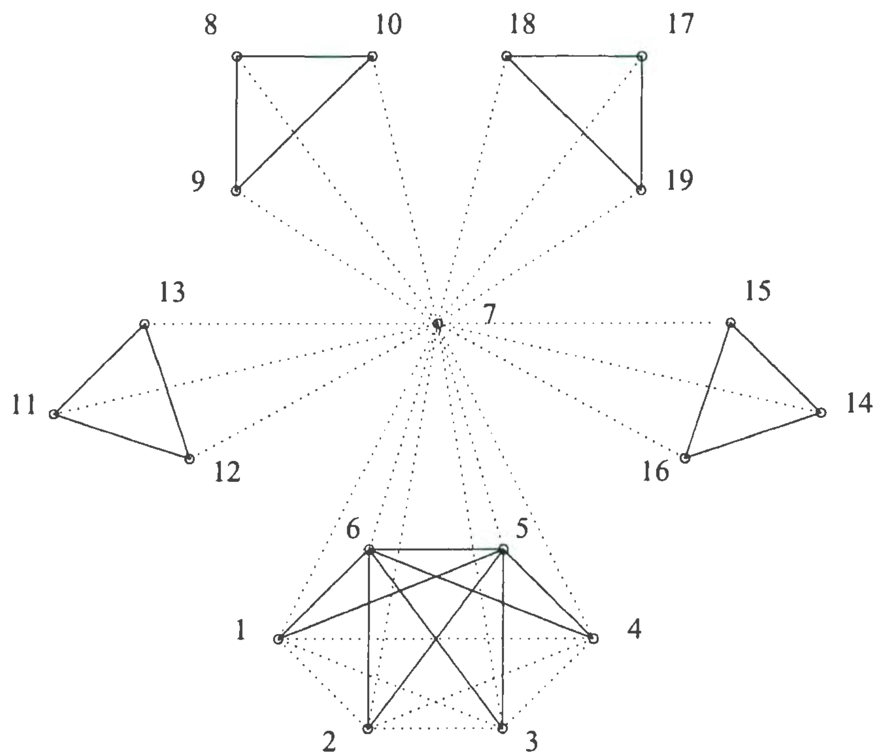

Figure 41: Leave(18,4,1) case (1).

(2) $3i$ squares and a $K_6 \setminus K_4$. A construction by Shalaby and Zhong. We know that the excess$(12i + 6, 4, 1)$ is a 1-factor (shown below). We can use the same construction as in **Case 3 Excess (2)**.

Excess. By the same arithmetic, the excess$(12i + 6, 4, 1)$ has $6i + 3$ edges. Every vertex has a degree 1 (mod 3) in the excess. The excess is a 1-factor.

A construction by Stinson [65]. Lemma 5 gives a 4-GDD of type $6^{v/6}$, except $v = 6, 18$. We can use the same construction as in **Case 1 Excess**.

Lemma 10. Stinson [65]: There exists a PBD(v,{4,22*},1) for all $v \equiv 7, 10 \pmod{12}$, with the exception of $v = 7, 10, 19$.

3.3.8 Case 8: $v = 12i + 7$

The leave is a $K_{3,3}$ (3^6) for $\lambda = 1$ and a triple edge (3^2) for $\lambda \geq 7$; the excess is a triple edge (3^2).

Leave. By the same arithmetic, the leave$(12i+7, 4, 1)$ has at least 3 edges. Every vertex has degree 0 (mod 3). So the leave would be a triple edge. But for $\lambda = 1$ a triple edge is illegal, so the leave has at least 9 edges, it could be a $K_{3,3}$.

(1) $K_{3,3}$ ($\lambda = 1$). A construction by Brouwer [27]. Lemma 9 gives a PBD($12i + 7$,{4,7*},1), except for $v = 19$. Suppose {1,2,3,4,5,6,7} is the unique block of size 7.
Take the blocks of size 4 from the PBD, and take the blocks {1,2,3,4} and {4,5,6,7} from {1,2,3,4,5,6,7}, we obtain a $K_{3,3}$, the leave$(12i + 7, 4, 1)$ (see Figure 42).

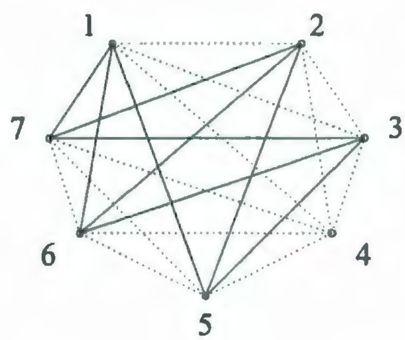

Figure 42: A $K_{3,3}$.

(2) Triple edge ($\lambda \geq 7$). A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD$(12i + 7, 4, 4)$, the leave$(12i + 7, 4, 7)$ is the same as the leave$(12i + 7, 4, 3)$, a triple edge (see **Case 32**).

Excess. By the same arithmetic, the excess$(12i + 7, 4, 1)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in the excess. The excess is a triple edge.

A construction by Stinson [65]. Lemma 10 gives a PBD$(12i + 7, \{4, 22^*\}, 1)$, except for $v = 7, 19$. Take the blocks of size 4 and replace the block of size 22 with a CD(22,4,1) (the excess(22,4,1) is a triple edge, given by Mills [46]).

3.3.9 Case 9: $v = 12i + 8$

The leave is a 1-factor (1^{12i+8}); the excess is a 2-factor (2^{12i+8}).

Leave. By the same arithmetic, the leave$(12i + 8, 4, 1)$ has $6i + 4$ edges. Every vertex has degree 1 (mod 3) in the leave. So the leave is a 1-factor.

A construction by Kreher and Stinson [44]. Lemma 6 gives a 4-GDD of type $2^{v/2}$, except for $v = 8$. The blocks form the packing; the groups of size 2 form the leave, a 1-factor.

Excess. By the same arithmetic, the excess$(12i+8, 4, 1)$ has $12i+8$ edges. Every vertex has degree 2 (mod 3). The excess is a 2-factor.

(1) $3i + 2$ squares. A construction by Shalaby and Zhong. We know that the leave$(12i + 8, 4, 1)$ is a 1-factor. We can use the same construction as in **Case 1 Leave 2**.
So the excess$(12i + 8, 4, 1)$ is the squares.

(2) When $\lambda \geq 7$, we can have a more general construction.
A construction by Shalaby and Zhong. Since there is a BIBD$(12i + 8, 4, 3)$, the excess$(12i+8, 4, 7)$ can be the same as the excess$(12i+8, 4, 4)$, a union of even cycles (see **Case 45**).

3.3.10 Case 10: $v = 12i + 9$

The leave is (1) a graph with two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+7}$), or (2) a graph with a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+8}$); the excess is a 1-factor and a 4-star ($4^1 1^{12i+8}$).

Leave. By the same arithmetic, the leave$(12i+9, 4, 1)$ has $12i+12$ edges. Every vertex has degree 2 (mod 3) in the leave. There are 2 solutions as above.

(1) $4i + 1$ triangles and a $K_6 \setminus K_4$. A construction by Brouwer [27]. Lemma 9 gives a PBD($12i+10$,{4,7*},1), except for $v = 10$. We can use the same construction as in **Case 7 Leave (1)**.

(2) $3i$ squares and a triangle and a $K_6 \setminus K_4$. A construction by Shalaby and Zhong. We know the excess$(12i + 9, 4, 1)$ is a 1-factor and a 4-star (shown below). We can use the same construction as in **Case 6 Excess 2**.

(3) When $\lambda \geq 7$, we can have a more general construction.
A construction by Shalaby and Zhong. Since there is a BIBD$(12i + 9, 4, 3)$, the excess$(12i+9, 4, 7)$ can be the same as the excess$(12i+9, 4, 4)$, the three constructions (see **Case 46**).

Excess. By the same arithmetic, the excess$(12i + 9, 4, 1)$ has $6i+6$ edges. Every vertex has degree 1 (mod 3). The excess is a 1-factor and a 4-star.

A construction by Stinson [65]. Lemma 7 gives a 4-GDD of type $6^{(v-15)/6}15^1$, except for $v = 9$. We can use the same construction as in **Case 4 Excess**.

3.3.11 Case 11: $v = 12i + 10$

The leave is a $\mathbf{K_{3,3}}$ (3^6) for $\lambda = 1$ and a triple edge (3^2) for $\lambda \geq 7$; the excess is a triple edge (3^2).

Proof. Same as in **Case 8**, leave$(12i+10, 4, 1)$ by Brouwer [27] and excess$(12i+10, 4, 1)$ by Stinson [65], with exception of $v = 10$.

3.3.12 Case 12: $v = 12i + 11$

The leave is a 1-factor and a 4-star ($4^1 1^{12i+10}$); the excess is a 2-factor (2^{12i+11}).

Leave. By the same arithmetic, the leave$(12i + 11, 4, 1)$ has $6i + 7$ edges. Every vertex has degree 1 (mod 3). So the leave is a 1-factor and a 4-star.

A construction by Brouwer [27]. Lemma 8 gives a 4-GDD of type $2^{(v-5)/2}5^1$, except for $v = 11$. We can use the same construction as in **Case 6 Leave**.

Excess. By the same arithmetic, the excess$(12i + 11, 4, 1)$ has $12i + 11$ edges. Every vertex has degree 2 (mod 3) in. The excess is a 2-factor.

A construction by Shalaby and Zhong, $3i + 2$ squares and a triangle. We know that the leave$(12i + 11, 4, 1)$ is a 1-factor and a 4-star (shown above). We can use the same construction as in **Case 4 Leave 2**.

3.4 The cases of $\lambda = 2$

3.4.1 Case 13: $v = 12i$

The leave is a 1-factor (1^{12i}); the excess is a 2-factor (2^{12i}).

Leave. The leave$(12i, 4, 2)$ has $6i$ edges. Every vertex has degree 1 (mod 3) in it. So the leave is a 1-factor.

A construction by Shalaby and Zhong. We know that the leave$(12i, 4, 1)$ can be a union of squires (see **Case 1**). Let $L_1 = [4j - 3, 4j - 2, 4j - 1, 4j]$, let $L_2 = [4j - 3, 4j - 2, 4j, 4j - 1]$, $j = 1, ..., 3i$.
Add them up, we obtain new blocks $\{4j - 3, 4j - 2, 4j - 1, 4j\}$, $j = 1, ..., 3i$ and the leave$(12i, 4, 2)$ is the 1-factor {1,2}, {3,4}, ..., $\{12i - 1, 12i\}$ (see Figure 43).

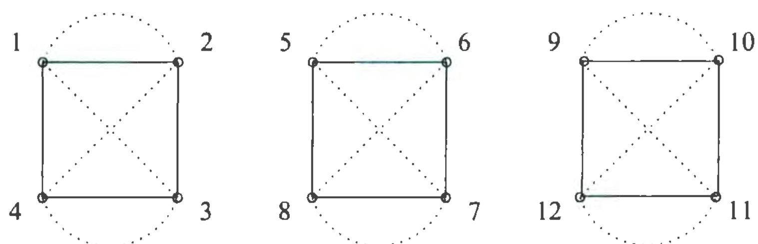

Figure 43: A leave(12,4,2).

Excess. The excess$(12i, 4, 2)$ has $12i$ edges. Every vertex has degree 2 (mod 3) in it. So the excess is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. We know that the excess$(12i, 4, 1)$ is a 1-factor (see **Case 1**). Add two such 1-factors up, we obtain a (disjoint) union of even cycles (see Figure 44).
Note that any union of even cycles is admissible, Figure 44 is just one possibility.

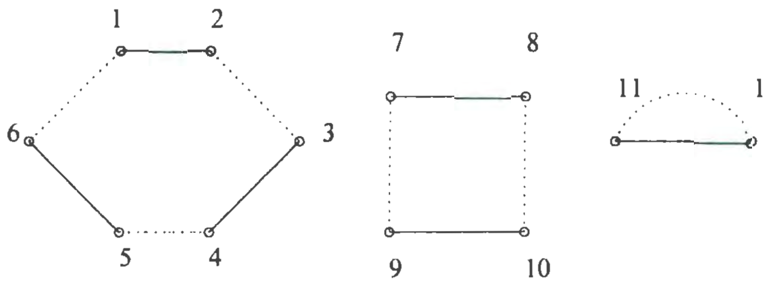

Figure 44: An excess(12,4,2).

3.4.2 Case 14: $v = 12i + 1$

There is a BIBD($12i + 1, 4, 2$).

3.4.3 Case 15: $v = 12i + 2$

The leave is a 2-factor (2^{12i+2}); the excess is (1) two vertices of degree 4 and others of degree 1 ($4^2 1^{12i}$), or (2) a vertex of degree 7 and others of degree 1 ($7^1 1^{12i+1}$).

Leave. The leave$(12i + 2, 4, 2)$ has $12i + 2$ edges. A vertex has degree 2 (mod 3) in it. So the leave is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. We know that the leave$(12i + 2, 4, 1)$ is a 1-factor (see **Case 3**). Add two such 1-factors up, we obtain a union of even cycles as in **Case 13 Excess**.

Excess. The excess$(12i + 2, 4, 2)$ has $6i + 4$ edges. A vertex has degree 1 (mod 3) in it. There are two solutions as above.

(1) A 1-factor and a "Candy". A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 1)$ can be $3i - 1$ squares and a $K_6 \setminus K_4$ (see **Case 3**).

Let $E_1 = \{1,3\}\{1,4\}\{2,3\}\{2,4\}\{3,4\}\{3,5\}\{3,6\}\{4,5\}\{4,6\}$ (the $K_6 \setminus K_4$), [7,8,9,10], [11,12,13,14], $[4j - 1, 4j, 4j + 1, 4j + 2]$, $j = 4, ..., 3j$ (the squares);

Let $E_2 = \{7,9\}\{7,10\}\{8,9\}\{8,10\}\{9,10\}\{9,11\}\{9,12\}\{10,11\}\{10,12\}$ (the $K_6 \setminus K_4$), [1,2,3,4], [5,6,13,14], $[4j - 1, 4j, 4j + 2, 4j + 1]$, $j = 4, ..., 3j$ (the squares).

Combine E_1 and E_2, the two $K_6 \setminus K_4$s and first four squares give blocks $\{1,2,3,4\}$, $\{3,4,5,6\}$, $\{7,8,9,10\}$, $\{9,10,11,12\}$, a graph $\{13,14\}\{13,14\}\{5,13\}\{11,13\}\{6,14\}\{12,14\}$

(the "Candy"), and edges {7,9}, {8,10}, {1,3}, {2,4} (see Figure 45).

The other squares in E_1 and E_2 give the rest of the 1-factor as in **Case 13 Leave**.

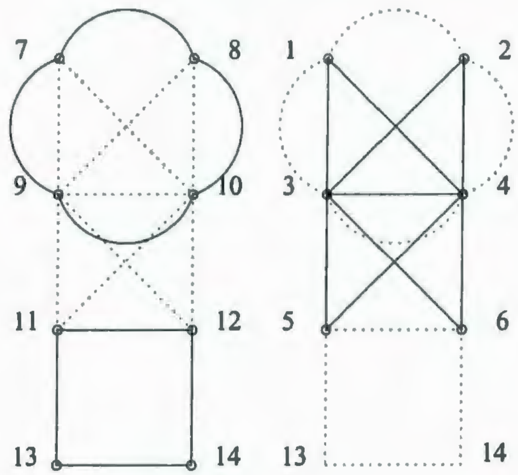

Figure 45: An excess(14,4,2).

(3) When $\lambda \geq 8$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**), and that the excess$(12i+2, 4, 5)$ is a 1-factor (see **Case 51**).

Combine two such excesses we can have the same construction as in **Case 39 Leave**.

3.4.4 Case 16: $v = 12i + 3$

The leave is (1) three vertices of degree 4 and others of degree 1 ($4^3 1^{12i}$), (2) a vertex of degree 7, a vertex of degree 4 and others of degree 1 ($7^1 4^1 1^{12i+1}$), or (3) a vertex of degree 10 and others of degree 1 ($10^1 1^{12i+2}$); the excess is (1) two vertices of degree 5 and others with degree 2 ($5^2 2^{12i+1}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+2}$).

Leave. The leave$(12i + 3, 4, 2)$ has $6i + 6$ edges. A vertex has degree 1 (mod 3) in it. There are two solutions as above.

(1) A 1-factor and a double triangle. A construction by Shalaby and Zhong. We know that the leave$(12i+3, 4, 1)$ can be $3i$ squares and a triangle (see **Case 4**). Add two such leave$(12i + 3, 4, 1)$s up, we obtain the leave$(12i + 3, 4, 2)$ (see Figure 46).

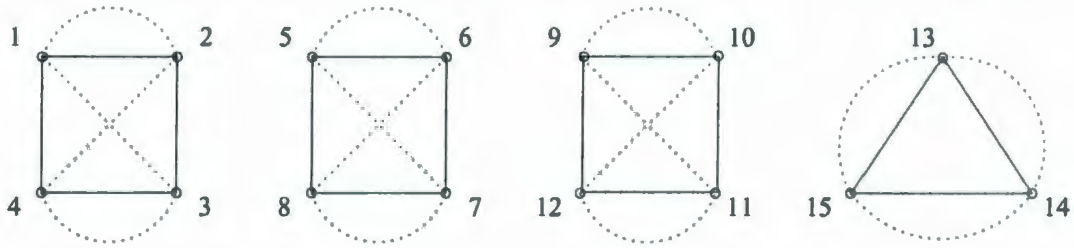

Figure 46: A Leave(15,4,2).

(2) When $\lambda \geq 8$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the leave$(12i + 3, 4, 3)$ is a triple edge (see **Case 28**), and that the leave$(12i + 3, 4, 5)$ is a 1-factor and a 4-star (see **Case 52**).

Combine two such leaves we can have the same construction as in **Case 40 Excess**.

Excess. The excess$(12i + 3, 4, 2)$ has $12i + 6$ edges. A vertex has degree 2 (mod 3) in it. There are two solutions as above.

A construction by Shalaby and Zhong. We know that the excess$(12i + 3, 4, 1)$ is a 4-star and a 1-factor (see **Case 4**). Add two such excess$(12i + 3, 4, 1)$s up, we can obtain three different constructions. They all have the union of even cycles as in **Case 13 Excess**. The rest of the excess$(12i + 3, 4, 2)$ can be:

(1) a graph {1,2}{1,2}{1,3}{1,4}{1,5}{2,3}{2,4}{2,5} (a "Crown").

Let the two 4-stars of E_1 and E_2 be {1,2}{1,3}{1,4}{1,5} and {1,2}{2,3}{2,4}{2,5}. We can have the "Crown" (see Figure 47).

Extension With the 1-factors of E_1 and E_2, the "Crown" can be extended into three even paths and two odd paths between vertices 1 and 2 (see Figure 48).

(2) a graph {1,2}{1,2}{1,3}{1,4}{1,5}{3,4}{2,5}{2,6}{2,7}{6,7} (a "Hat").

Let the two 4-stars of E_1 and E_2 be {1,2}{1,3}{1,4} {1,5} and {1,2}{2,5}{2,6}{2,7}, and assume we have edge {6,7} in E_1, edge {3,4} in E_2. We can have the "Hat" (see

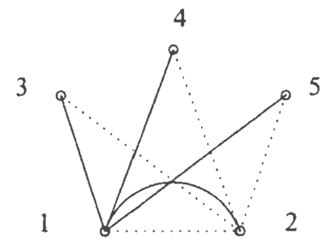

Figure 47: A "Crown".

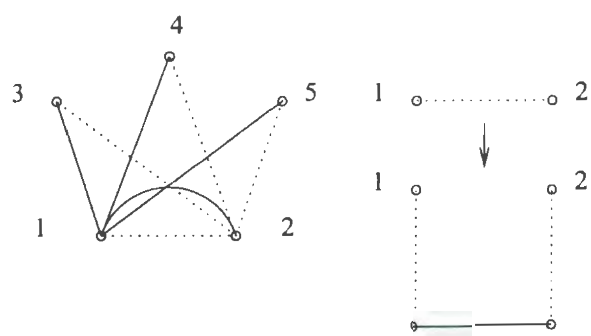

Figure 48: Extension of a "Crown".

Figure 49).

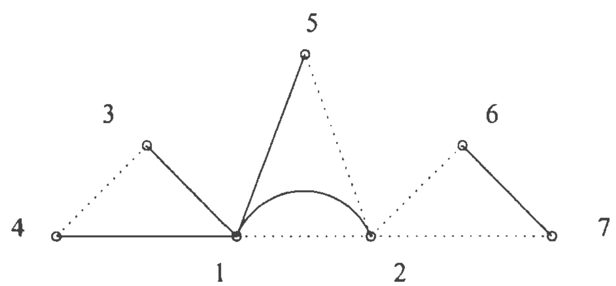

Figure 49: A "Hat".

Extension. With the 1-factors of E_1 and E_2, the "Hat" can be extended into a even paths and two odd paths between vertices 1 and 2, a odd cycle on 1 and another odd cycle on 2.

(3) a graph {1,2}{2,3}{1,3}{1,4}{4,5}{1,5}{1,6}{6,7}{1,7} {1,8}{8,9}{1,9} (a "Windmill").

Let the two 4-stars of E_1 and E_2 be {1,2}{1,3}{1,4} {1,5} and {1,6}{1,7}{1,8}{1,9}, and assume we have edges {6,7}, {8,9} in E_1, and edges {2,3}, {4,5} in E_2. We can have the "Windmill" (see Figure 50).

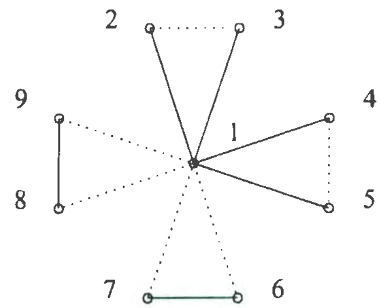

Figure 50: A "Windmill".

Extension. With the 1-factors of E_1 and E_2, the "Windmill" can be extended into four odd cycles on vertex 1.

3.4.5 Case 17: $v = 12i + 4$

There is a BIBD$(12i + 4, 4, 2)$.

3.4.6 Case 18: $v = 12i + 5$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+3}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+4}$); the excess is a 1-factor and a 4-star ($4^1 1^{12i+4}$).

Leave. The leave$(12i + 5, 4, 2)$ has $12i + 8$ edges. A vertex has degree 2 (mod 3) in it. There are 2 solutions as above.

A construction by Shalaby and Zhong. We know that the leave$(12i + 5, 4, 1)$ is a 4-star and a 1-factor (see **Case 6**). Add two such leave$(12i + 5, 4, 1)$s up, we can have three different constructions as in **Case 16 Excess.**

Excess. The excess$(12i + 5, 4, 2)$ has $6i + 4$ edges. A vertex has degree 1 (mod 3) in it. So the excess is a 1-factor and 4-star.

A construction by Shalaby and Zhong. We know that the leave$(12i + 5, 4, 2)$ can be a "Crown" and a union of squares (shown above). Let the "Crown" be {1,2}{1,2}{1,3}{1,4}{1,5} {2,3}{2,4}{2,5}.

Add the 4-star {1,4}{2,4}{3,4}{5,4} to the "Crown", we obtain new blocks {1,2,3,4}, {1,2,4,5}.

We obtain the 1-factor from the squares as in **Case 1 Leave 2** (see Figure 51).

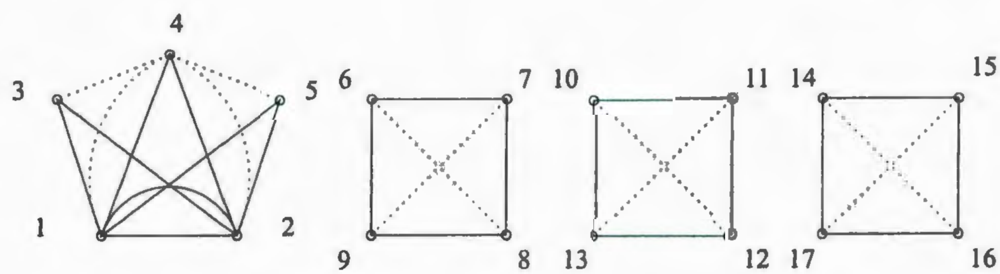

Figure 51: An excess(17,4,2).

3.4.7 Case 19: $v = 12i + 6$

The leave is (1) two vertices of degree 4 and others of degree 1 ($4^2 1^{12i+4}$); (2) a vertex of degree 7 and others of degree 1 ($7^1 1^{12i+5}$); the excess is a 2-factor (2^{12i+6}).

Leave. The leave$(12i + 6, 4, 2)$ has $6i + 6$ edges. A vertex has degree 1 (mod 3) in it. There are two solutions as above.

(1) A 1-factor and a "Candy". A construction by Shalaby and Zhong. We know that the leave$(12i + 6, 4, 1)$ can be $3i$ squares and a $K_6 \setminus K_4$ (see **Case 7**). Combine two such leaves we obtain a 1-factor and a "Candy" as in **Case 15 Excess**.

(2) When $\lambda \geq 8$, we can have a more general construction.
A construction by Shalaby and Zhong. We know that the leave$(12i + 6, 4, 3)$ is a triple edge (see **Case 31**), and that the leave$(12i + 6, 4, 5)$ is a 1-factor (see **Case 55**).
Combine two such excesses we can have the same construction as in **Case 39 Leave**.

Excess. The excess$(12i + 6, 4, 2)$ has $12i + 6$ edges. A vertex has degree 2 (mod 3) in it. So the excess is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. We know that the excess$(12i + 6, 4, 1)$ is a 1-factor (see **Case 7**). Add up two 1-factors, we obtain a union of even cycles as in **Case 13 Excess**.

3.4.8 Case 20: $v = 12i + 7$

There is a BIBD$(12i + 7, 4, 2)$.

3.4.9 Case 21: $v = 12i + 8$

The leave is a 2-factor (2^{12i+8}); the excess is a 1-factor (2^{12i+8}).

Leave. The leave$(12i + 8, 4, 2)$ has $12i + 8$ edges. A vertex has degree 2 (mod 3) in it. So the leave is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. We know that the leave$(12i + 8, 4, 1)$ is a 1-factor (see **Case 9**). Add up two 1-factors, we obtain a union of even cycles as in **Case 13 Excess**.

Excess. The excess$(12i + 8, 4, 2)$ has $6i + 4$ edges. A vertex has degree 1 (mod 3) in it. So the excess is a 1-factor.

A construction by Shalaby and Zhong. We know that the excess$(12i+8, 4, 1)$ can be a union of squares (see **Case 9**). Add up two such excesses, we obtain a 1-factor as in **Case 13 Leave**.

3.4.10 Case 22: $v = 12i + 9$

The leave is a 1-factor and a 4-star ($4^1 1^{12i+8}$); the excess is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+7}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+8}$).

Leave. The leave$(12i + 9, 4, 2)$ has $6i + 6$ edges. A vertex has degree 1 (mod 3) in it. So the leave is a 1-factor and a 4-star.

We know that the excess$(12i + 9, 4, 2)$ can be a "Crown" and a union of squares (shown below). We can use the same construction as in **Case 18 Excess**.

Excess. The excess$(12i+9, 4, 2)$ has $12i+12$ edges. A vertex has degree 2 (mod 3) in it. There are 2 solutions as above.

A construction by Shalaby and Zhong. We know that the excess$(12i + 9, 4, 1)$ is a 4-star and a 1-factor (see **Case 10**). Add up two excess$(12i + 9, 4, 1)$s, we can obtain three constructions as in **Case 16 Excess**.

3.4.11 Case 23: $v = 12i + 10$

There is a BIBD$(12i + 10, 4, 2)$.

3.4.12 Case 24: $v = 12i + 11$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+9}$) or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+10}$); the excess is (1) three vertices of degree 4 and others of degree 1 ($4^3 1^{12i+8}$), (2) a vertex of degree 7, a vertex of degree 4 and others of degree 1 ($7^1 4^1 1^{12i+9}$), or (3) a vertex of degree 10 and others of degree 1 ($10^1 1^{12i+10}$).

Leave. The leave$(12i + 11, 4, 2)$ has $12i + 14$ edges. A vertex has degree 2 (mod 3) in it. There are 2 solutions as above.

A construction by Shalaby and Zhong. We know that the leave$(12i + 11, 4, 1)$ is **a 4-star and a 1-factor (see Case 12)**. Add up two such leaves, we can obtain three constructions as in **Case 16 Excess**.

Excess. The excess$(12i+11, 4, 2)$ has $6i+10$ edges. A vertex has degree 1 (mod 3) in it. There are 2 solutions as above.

(1) A 1-factor and a double triangle. A construction by Shalaby and Zhong. We know that the excess$(12i+11, 4, 1)$ can be $3i+2$ squares and a triangle. Add up two such excesses, we can obtain a 1-factor and a double triangle as in **Case 16 Leave**.

(2) When $\lambda \geq 8$, we can have a more general construction.
A construction by Shalaby and Zhong. We know that the leave$(12i+11, 4, 3)$ is a triple edge (see **Case 36**), and that the leave$(12i+11, 4, 5)$ is a 1-factor and a 4-star (see **Case 60**).
Combine two such leaves we can have the same construction as in **Case 40 Excess**.

3.5 The cases of $\lambda = 3$

3.5.1 Case 25: $v = 12i$

There is a BIBD$(12i, 4, 3)$.

3.5.2 Case 26: $v = 12i + 1$

There is a BIBD$(12i + 1, 4, 3)$.

3.5.3 Case 27: $v = 12i + 2$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 2, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

A construction by Shalaby and Zhong. We know that the leave$(12i + 2, 4, 1)$ is a 1-factor (see **Case 3**). Add three 1-factors together, we have $3i$ new blocks, and the edge $\{12i + 1, 12i + 2\}$ three times (see Figure 52).

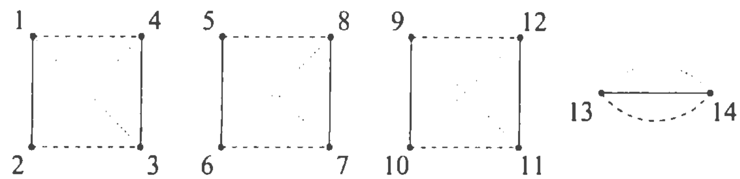

Figure 52: A leave(14,4,3).

Excess. The excess$(12i + 2, 4, 3)$ has 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

Direct constructions are not found, but there is a recursive construction by Assaf [6].

3.5.4 Case 28: $v = 12i + 3$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 3, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

Direct constructions are not found, but there exists a recursive construction by Assaf [7].

Excess. The excess$(12i + 3, 4, 3)$ has 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

A construction by Shalaby and Zhong. We know that the excess$(12i + 3, 4, 1)$ is a 1-factor and a 4-star (see **Case 4**). Take three such excesses and let the 4-stars be {1,2}{1,3}{1,4}{1,5}, {2,1}{2,3}{2,4}{2,5}, {4,1}{4,2}{4,3}{4,5}.

Those form blocks {1,2,3,4}, {1,2,4,5}. The 1-factors form new blocks and a triple edge as in **Case 27 Leave** (see Figure 53).

3.5.5 Case 29: $v = 12i + 4$

There is a BIBD$(12i + 4, 4, 3)$.

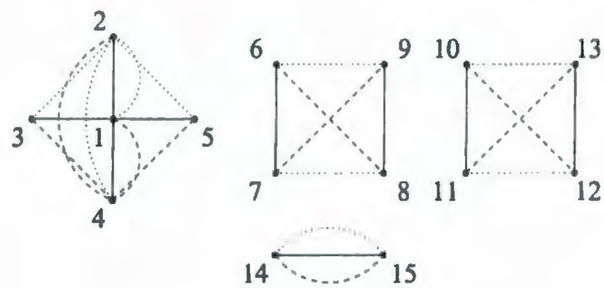

Figure 53: An excess(15,4,3).

3.5.6 Case 30: $v = 12i + 5$

There is a BIBD$(12i + 5, 4, 3)$.

3.5.7 Case 31: $v = 12i + 6$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 6, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

Direct constructions are not found, but there is a recursive construction by Assaf [7].

Excess. The excess$(12i + 6, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

A construction by Shalaby and Zhong. We know that the excess$(12i+6, 4, 1)$ is a 1-factor (see **Case 7**). Add three such excesses up, we have a triple edge as in **Case 27 Leave**.

3.5.8 Case 32: $v = 12i + 7$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 7, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

A construction by Billington, Stanton and Stinson [23]. Lemma 9 gives a PBD(v,{4,7*},1), except for $v = 19$. Let {1,2,3,4,5,6,7} be the unique block of size 7.

Take three copies of this block, and remove the blocks {1,2,3,6}, {1,2,4,6}, {1,2,5,7}, {1,3,4,7}, {1,3,5,6}, {1,4,5,7}, {2,3,4,7}, {2,3,5,7}, {2,4,5,6} and {3,4,5,6} from them. Also take the blocks of size 4 from the PBDs.

The leave is the edge {6,7} three times (see Figure 54).

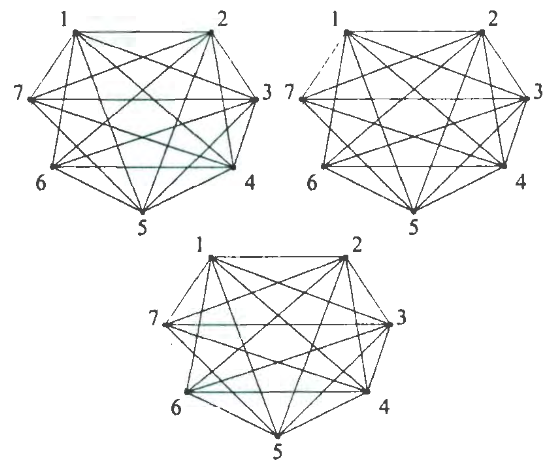

Figure 54: A leave(7,4,3).

Excess. The excess($12i + 7, 4, 3$) has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

A construction by Assaf [6]. Since there is a BIBD($12i + 7, 4, 2$), the excess($12i + 7, 4, 3$) is the same as the excess($12i + 7, 4, 1$), a triple edge (see **Case 8**).

3.5.9 Case 33: $v = 12i + 8$

There is a BIBD($12i + 8, 4, 3$).

3.5.10 Case 34: $v = 12i + 9$

There is a BIBD($12i + 9, 4, 3$).

3.5.11 Case 35: $v = 12i + 10$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 10, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

A construction by Billington, Stanton and Stinson [23]. Lemma 9 gives a PBD(v,{4,7*},1), except for $v = 10$. We can use the same construction as in **Case 32 Leave**.

Excess. The excess$(12i + 10, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

A construction by Assaf [6]. Since there is a BIBD$(12i+10, 4, 2)$, the excess$(12i+10, 4, 3)$ is the same as the excess$(12i + 10, 4, 1)$, a triple edge (see **Case 11**).

3.5.12 Case 36: $v = 12i + 11$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 11, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the leave is a triple edge.

A construction by Shalaby and Zhong. We know that the leave$(12i + 11, 4, 1)$ is a 1-factor and a 4-star (see **Case 12**). Add three such leaves together, we have a triple edge as in **Case 28 Excess**.

Excess. The excess$(12i + 11, 4, 3)$ has at least 3 edges. Every vertex has degree 0 (mod 3) in it. So the excess is a triple edge.

Direct constructions are not found, but there is a recursive construction by Assaf [6].

3.6 The cases of $\lambda = 4$

3.6.1 Case 37: $v = 12i$

The leave is a 2-factor (2^{12i}); the excess is a 1-factor (1^{12i}).

Leave. The leave$(12i, 4, 4)$ has $12i$ edges. Every vertex has degree 2 (mod 3). So the leave is a 2-factor.

(1) Union of even cycles. A construction by Shalaby and Zhong. We know that the leave$(12i, 4, 2)$ is a 1-factor (see **Case 13**). Add up two such leaves, we can obtain a union of even cycles as in **Case 13 Excess**.

(2) $4i$ triangles. A construction by Assaf [7]. Since there exists a BIBD$(12i, 4, 3)$, the leave$(12i, 4, 4)$ can be the same as leave$(12i, 4, 1)$ (see **Case 1**).

Excess. The excess$(12i, 4, 4)$ has $6i$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor.

A construction by Assaf [6]. Since there exists a BIBD$(12i, 4, 3)$, the excess$(12i, 4, 4)$ can be the same as excess$(12i, 4, 1)$ (see **Case 1**).

3.6.2 Case 38: $v = 12i + 1$

There is a BIBD$(12i + 1, 4, 4)$.

3.6.3 Case 39: $v = 12i + 2$

The leave is (1) two vertices of degree 4 and others of degree 1 $(4^2 1^{12i})$; (2) a vertex of degree 7 and others with degree 1 $(7^1 1^{12i+1})$; the excess is a 2-factor (2^{12i+2}).

Leave. The leave$(12i+2, 4, 4)$ has $12i+4$ edges. Every vertex has degree 1 (mod 3). There are two solutions as above.

(1) A 1-factor and a quadruple edge. A construction by Shalaby and Zhong. We know that the leave$(12i + 2, 4, 2)$ can be $3i$ squires and a double edge (see **Case 15**). Add two such leave$(12i + 2, 4, 2)$s up, the two double edges form the quadruple edge, the squares form the 1-factor as in **Case 13 Leave** (see Figure 55).

(2) A 1-factor and a "Flydisk".
We know that the leave$(12i + 2, 4, 2)$ can be $3i - 1$ squires and a 6-cycle (see **Case 15**). Add two such leave$(12i + 2, 4, 2)$s up, the two 6-cycles form a graph {3,4}{4,5}{4,5} {4,5}{5,6} (the "Flydisk") and the edge {1,2}; the squares form the rest of the 1-factor as in **Case 13 Leave** (see Figure 56).

Note A simpler construction by Shalaby and Zhong, but we need the leave$(12i + 2, 4, 3)$, for which we don't have a direct construction.

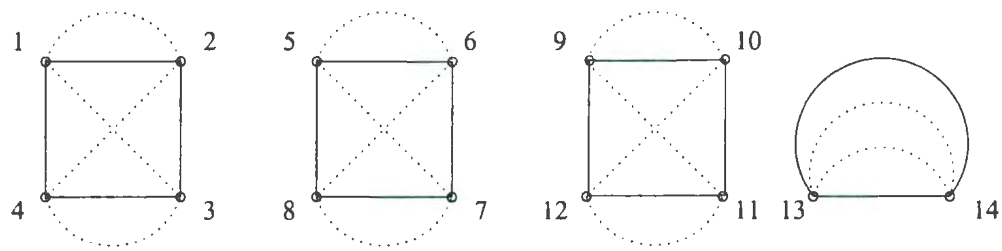

Figure 55: Leave(14,4,4) case 1.

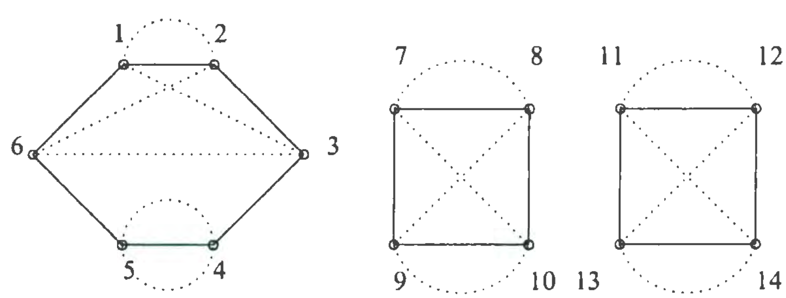

Figure 56: Leave(14,4,4) case 2.

We know that the leave$(12i + 2, 4, 1)$ is a 1-factor (see **Case 3**), and that the leave$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**). Add them up we can have (1) or (2). So the two constructions can be denoted 1F+3.

Excess. The excess$(12i + 2, 4, 4)$ has $12i + 2$ edges. Every vertex has degree 2 (mod 3). So the excess is a 2-factor.

(1) $4i$ triangles and a double edge. A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 1)$ can be $4i - 1$ triangles and a "Crown" (see **Case 3**); and that the excess$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**).

Let the "Crown" be {1,2}{1,2}{1,3}{1,4} {1,5}{2,3}{2,4}{4,5}; let the triple edge be {4,5}{4,5}{4,5}.

Add the triple edge to the Crown, we obtain the block {1,2,4,5}, a triangle {1,2,3} and a double edge {4,5}{4,5} (see Figure 57).

The $4i - 1$ triangles from the excess$(12i + 2, 4, 1)$ remain in the the excess$(12i + 2, 4, 4)$.

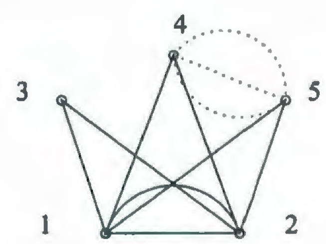

Figure 57: A Crown and a triple edge.

(2) $3i$ squares and a double edge. A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 1)$ can be $3i - 1$ squares and a $K_6 \setminus K_4$ (see **Case 3**); and that the excess$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**).

Let the $K_6 \setminus K_4$ be {1,3}{1,4}{2,3}{2,4} {3,4}{3,5}{3,6}{4,5}{4,6}; let the triple edge be {1,2}{1,2}{1,2}.

Add the triple edge to the $K_6 \setminus K_4$, we obtain the block {1,2,3,4}, a square [3,4,5,6] and a double edge {1,2}{1,2} (see Figure 58).

The $3i - 1$ squares from the excess$(12i + 2, 4, 1)$ remain in the the excess$(12i + 2, 4, 4)$.

(3) Union of even cycles (containing a 6-cycle). A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 2)$ can be a "Candy" and a 1-factor (see

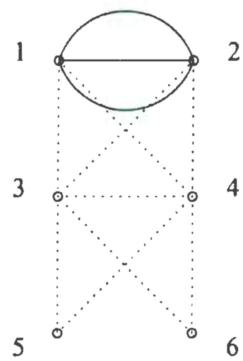

Figure 58: A $K_6 \setminus K_4$ and a triple edge.

Case 15). Take two such excesses and let the "Candies" be $\{1,2\}\{1,2\}\{1,3\}\{1,4\}\{2,5\}\{2,6\}$ and $\{3,5\}\{3,5\}\{3,2\}\{3,4\}\{5,1\}\{5,6\}$.

Add them up, we have a new block $\{1,2,5,3\}$ and a 6-cycle $(1,4,3,5,6,2)$ (see Figure 59).

The two 1-factors form the union of even cycles as in **Case 13 Excess**.

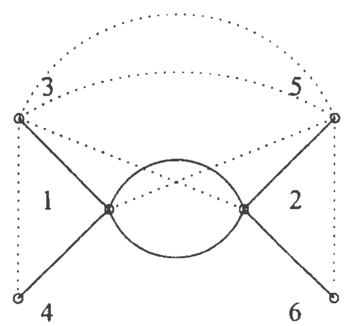

Figure 59: Two Candies.

(4) When $\lambda \geq 10$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the excess$(12i + 2, 4, 5)$ is a 1-factor (see **Case 51**).

Combine two such leaves we can have a union of even cycles as in **Case 13 Excess**.

3.6.4 Case 40: $v = 12i + 3$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+1}$) or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+2}$); the excess is (1) three vertices of degree 4 and others of degree 1 ($4^3 1^{12i}$), (2) a vertex of degree 7, a vertex of degree 4 and others of degree 1 ($7^1 4^1 1^{12i+1}$), or (3) a vertex of degree 10 and others of degree 1 ($10^1 1^{12i+2}$).

Leave. The leave$(12i+3, 4, 4)$ has $12i+6$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

(1) $3i$ squares and a "Parachute". A construction by Shalaby and Zhong. We know that the leave$(12i + 3, 4, 1)$ can be $3i$ squares and a triangle (see **Case 4**); and that the leave$(12i + 3, 4, 3)$ is a triple edge (see **Case 28**). Let the triangle be {1,2}{1,3}{2,3}; let the triple edge be {1,2}{1,2}{1,2}.

Add the triple edge to a triangle, we have a graph {1,2}{1,2}{1,2} {1,2}{1,3}{2,3} (a "Parachute") (see Figure 60), plus the $3i$ squares from the leave$(12i + 3, 4, 1)$.

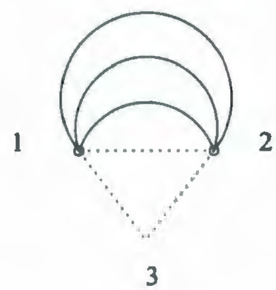

Figure 60: A Parachute.

(2) $3i - 1$ squares and a triangle and a "Mushroom". A construction by Shalaby and Zhong. Similar to (1) except that the triple edge is added to a square. We obtain a graph {1,2}{1,2}{1,2}{1,2}{1,3}{3,4}{4,2} (a "Mushroom") (see Figure 61), plus the $3i - 1$ squares and the triangle from the leave$(12i + 3, 4, 1)$.

(3) $4i$ triangles and a "Parachute". A construction by Shalaby and Zhong. We know that the leave$(12i + 3, 4, 1)$ can be $4i + 1$ triangles (see **Case 4**); and that the leave$(12i + 3, 4, 3)$ is a triple edge (see **Case 28**).

Similar to (1), Add the triple edge to a triangle, we obtain the "Parachute", and the rest $4i$ triangles from the leave$(12i + 3, 4, 1)$ remain.

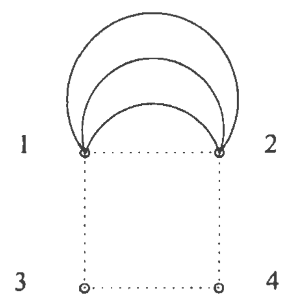

Figure 61: A Mushroom.

(4) When $\lambda \geq 10$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the excess$(12i + 3, 4, 5)$ is a 1-factor and a 4-star (see **Case 52**).

Combine two such leaves we can have three constructions as in **Case 16 Excess**.

Excess. The excess$(12i + 3, 4, 4)$ has $6i + 6$ edges. Every vertex has degree 1 (mod 3). There are three solutions as above.

A construction by Shalaby and Zhong. We know that the excess$(12i + 3, 4, 1)$ is a 1-factor and a 4-star (see **Case 4**); and that the excess$(12i + 3, 4, 3)$ is a triple edge (see **Case 28**).

Combine two such excesses, we can have many constructions similar to **Case 39 Leave**, except that we can also attached the triple edge to the 4-star, or use it to connect the 4-star and an edge in the 1-factor. These constructions can be denoted as $1FX + 3$.

For example, a 1-factor and a "Joker". Add the triple edge to the 4-star, we obtain a graph {1,2}{1,2}{1,2}{1,4}{2,4} {3,4}{4,5} (a "Joker") (see Figure 62), plus the 1-factor from the excess$(12i + 3, 4, 1)$.

3.6.5 Case 41: $v = 12i + 4$

There is a BIBD$(12i + 4, 4, 4)$.

3.6.6 Case 42: $v = 12i + 5$

The leave is a 1-factor and a 4-star $(4^1 1^{12i+4})$; the excess is (1) two

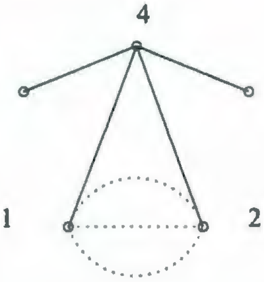

Figure 62: A Joker.

vertices of degree 5 and others of degree 2 ($5^2 2^{12i+3}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+4}$).

Leave. The leave($12i+5, 4, 4$) has $6i+4$ edges. Every vertex has degree 1 (mod 3). The leave is a 1-factor and a 4-star.

A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD($12i+5, 4, 3$), the leave($12i+5, 4, 4$) is the same as the leave($12i+5, 4, 1$) (see **Case 6**).

Excess. The excess($12i+5, 4, 4$) has $12i+8$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the excess($12i+5, 4, 2$) is a 1-factor and a 4-star (see **Case 18**). Add up two such excesses, we can obtain three constructions as in **Case 16 Excess**.

Note Another construction by Assaf [6]. Since there is a BIBD($12i+5, 4, 3$), the excess($12i+5, 4, 4$) can be the same as the excess($12i+5, 4, 1$), i.e. a "Crown" and $4i$ triangles; or a $K_6 \setminus K_4$ and $3i-1$ squares and a triangle (see **Case 6**).

3.6.7 Case 43: $v = 12i + 6$

The leave is a 2-factor (2^{12i+6}); the excess is (1) two vertices of degree 4 and others of degree 1 ($4^2 1^{12i+4}$), or (2) a vertex of degree 7 and others of degree 1 ($7^1 1^{12i+5}$).

Leave. The leave($12i+6, 4, 4$) has $12i+6$ edges. Every vertex has degree 2 (mod 3). So the leave is a 2-factor.

(1) $4i$ triangles and a square and a double edge. A construction by Shalaby and Zhong. We know that the leave$(12i+6,4,1)$ can be a $K_6 \setminus K_4$ and a union of triangles (see **Case 7**); and that the leave$(12i+6,4,3)$ is a triple edge (see **Case 31**).

Add the triple edge to the $K_6 \setminus K_4$, we have the same construction as in **Case 39 Excess 2**, except that the $3i$ squares are replaced by $4i$ triangles.

(2) $3i+1$ squares and a double edge. A construction by Shalaby and Zhong. We know that the leave$(12i+6,4,1)$ can be a $K_6 \setminus K_4$ and a union of squares (see **Case 7**); and that the leave$(12i+6,4,3)$ is a triple edge (see **Case 31**).

Add the triple edge to the $K_6 \setminus K_4$, we have the same construction as in **Case 39 Excess 2**.

(3) Union of even cycles (containing a 6-cycle). A construction by Shalaby and Zhong. We know that the leave$(12i+6,4,2)$ can be a "Candy" and a 1-factor (see **Case 19**).

Add two such leaves up, we have the same construction as in **Case 39 Excess 3**.

(4) When $\lambda \geq 10$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the excess$(12i+6,4,5)$ is a 1-factor (see **Case 54**).

Combine two such leaves we can have a union of even cycles as in **Case 13 Excess**.

Excess. The leave$(12i+6,4,4)$ has $6i+6$ edges. Every vertex has degree 1 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the excess$(12i+6,4,1)$ is a 1-factor (see **Case 7**); and that the excess$(12i+6,4,3)$ is a triple edge (see **Case 31**).

Add them up we can obtain the $1F+3$ as in **Case 39 Leave**.

3.6.8 Case 44: $v = 12i + 7$

There is a BIBD$(12i+7,4,4)$.

3.6.9 Case 45: $v = 12i + 8$

The leave is a 1-factor (1^{12i+8}); the excess is a 2-factor (2^{12i+8}).

Leave. The leave$(12i+8,4,4)$ has $6i+4$ edges. Every vertex has degree 1 (mod 3). So the leave is a 1-factor.

A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD$(12i+8,4,3)$, the leave$(12i+8,4,4)$ is the same as the leave$(12i+8,4,1)$ (see **Case 9**).

Excess. The leave$(12i+3,4,4)$ has $12i+8$ edges. Every vertex has degree 2 (mod 3). So the excess is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. We know that the excess$(12i+8,4,2)$ is a 1-factor (see **Case 21**). Add up two 1-factors, we can have a union of even cycles as in **Case 13 Excess**.

3.6.10 Case 46: $v = 12i+9$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+7}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+8}$); the excess is a 1-factor and a 4-star ($4^1 1^{12i+8}$).

Leave. The leave$(12i+9,4,4)$ has $12i+12$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the leave$(12i+9,4,2)$ is a 1-factor and a 4-star (see **Case 22**). Add up two such leaves, we can obtain three constructions as in **Case 13 Excess**.

Note Another construction by Assaf [7]. Since there is a BIBD$(12i+9,4,3)$, the leave$(12i+9,4,4)$ can be the same as the leave$(12i+9,4,1)$, i.e. a $K_6 \setminus K_4$ and $4i+1$ triangles; or a $K_6 \setminus K_4$ and $3i$ squares and a triangle (see **Case 10**).

Excess. The excess$(12i+9,4,4)$ has $6i+6$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor and a 4-star.

A construction by Assaf [6]. Since there is a BIBD$(12i+9,4,3)$, the excess$(12i+9,4,4)$ can be the same as the excess$(12i+9,4,1)$, i.e. a 1-factor and a 4-star (see **Case 10**).

3.6.11 Case 47: $v = 12i+10$

There is a BIBD$(12i+10,4,4)$.

3.6.12 Case 48: $v = 12i + 11$

The leave is (1) three vertices of degree 4 and others of degree 1 ($4^3 1^{12i+8}$), (2) a vertex of degree 7, a vertex of degree 4 and others of degree 1 ($7^1 4^1 1^{12i+9}$), or (3) a vertex of degree 10 and others of degree 1 ($10^1 1^{12i+10}$); the excess is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+9}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+10}$).

Leave. The leave$(12i + 11, 4, 4)$ has $6i + 10$ edges. Every vertex has degree 1 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the leave$(12i + 11, 4, 1)$ is a 1-factor and a 4-star (see **Case 12**); and that the leave$(12i + 11, 4, 3)$ is a triple edge (see **Case 36**).

Add them up, we have the same construction as in **Case 40 Excess**.

Excess. The excess$(12i + 11, 4, 4)$ has $12i + 14$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

(1) A construction by Shalaby and Zhong. We know that the excess$(12i+11, 4, 1)$ can be $3i$ squares and a triangle (see **Case 12**); and that the excess$(12i + 11, 4, 3)$ is a triple edge (see **Case 36**).

Add them up, we can obtain two constructions as in **Case 40 Leave (1) (2)**.

(2) When $\lambda \geq 10$, we can have a more general construction.

A construction by Shalaby and Zhong. We know that the excess$(12i + 11, 4, 5)$ is a 1-factor and a 4-star (see **Case 60**).

Combine two such leaves we can have three constructions as in **Case 16 Excess**.

3.7 The cases of $\lambda = 5$

3.7.1 Case 49: $v = 12i$

The leave is a 1-factor (1^{12i}); the excess is a 2-factor (2^{12i}).

Leave. The leave$(12i, 4, 5)$ has $6i$ edges. Every vertex has degree 1 (mod 3). So the leave is a 1-factor.

A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD$(12i, 4, 3)$, the leave$(12i, 4, 5)$ is the same as the leave$(12i, 4, 2)$ (see **Case 13**).

Excess. The excess$(12i, 4, 5)$ has $12i$ edges. Every vertex has degree 2 (mod 3). So the excess is a 2-factor.

Union of even cycles. A construction by Shalaby and Zhong. Since there is a BIBD$(12i, 4, 3)$, the excess$(12i, 4, 5)$ is the same as the excess$(12i, 4, 2)$ (see **Case 13**).

3.7.2 Case 50: $v = 12i + 1$

There is a BIBD$(12i + 1, 4, 5)$.

3.7.3 Case 51: $v = 12i + 2$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+1}$); the excess is a 1-factor (1^{12i+2}).

Leave. The leave$(12i+2, 4, 5)$ has $12i+5$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the leave$(12i + 2, 4, 2)$ can be an union of even cycles (see **Case 15**); and that the leave$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**).

Add the triple edge to a cycle, or use the triple edge to connect two cycles. Both way we obtain two vertices of degree 5, the rest vertices are of degree 2. We denote this construction by $\cup C_{2k} + 3$.

Excess. The excess$(12i + 2, 4, 5)$ has $6i + 1$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor.

A construction by Shalaby and Zhong ($\lambda = 11$). We know that the excess$(12i + 2, 4, 4)$ can be an union of squares and a 6-cycle (see **Case 39**); and that the excess$(12i + 2, 4, 3)$ is a triple edge (see **Case 27**).

Take the excess$(12i + 2, 4, 3)$ and two copies of the excess$(12i + 2, 4, 4)$. Let the 6-cycles be [1,2,4,6,5,3], [1,2,3,6,5,4], let the triple edge be {3,4} (three times).

Add them up, we obtain blocks {1,2,3,4}, {3,4,5,6} and edges {1,2}, {3,4}, {5,6}. We also obtain the rest of the 1-factor from the squares as in **Case 13 Leave** (see Figure 63).

Direct constructions for $\lambda = 5$ are not found, but there exists a recursive construction by Assaf [6].

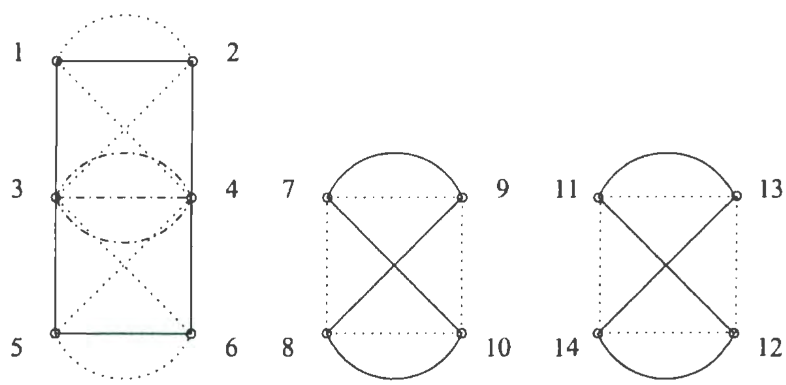

Figure 63: An excess(14,4,11).

3.7.4 Case 52: $v = 12i + 3$

The leave is a 1-factor and a 4-star ($4^1 1^{12i+2}$); the excess is a 2-factor (2^{12i+3}).

Leave. The leave$(12i+3, 4, 5)$ has $6i+3$ edges. Every vertex has degree 1 (mod 3). So the leave is a 1-factor and a 4-star.

Direct constructions are not found, but there exists a recursive construction by Assaf [7].

Excess. The excess$(12i+3, 4, 5)$ has $12i+3$ edges. Every vertex has degree 2 (mod 3). So the excess is a 2-factor.

A triangle, a double edge and a union of even cycles. A construction by Shalaby and Zhong. We know that the excess$(12i+3, 4, 2)$ can be a "Crown" and an union of even cycles (see **Case 16**); and that the excess$(12i+3, 4, 3)$ is a triple edge (see **Case 28**).

Add the triple edge to the "Crown", we obtain a triangle and a double edge as in **Case 39 Excess (1)** (see Figure 57), the union of even cycles remain in the excess$(12i+3, 4, 5)$.

3.7.5 Case 53: $v = 12i + 4$

There is a BIBD$(12i+4, 4, 5)$.

3.7.6 Case 54: $v = 12i + 5$

The leave is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+3}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+4}$); the excess is a 1-factor and a 4-star ($4^1 1^{12i+4}$).

Leave. The leave$(12i+5, 4, 5)$ has $12i+8$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. Since there is a BIBD$(12i + 5, 4, 3)$, the leave$(12i + 5, 4, 5)$ is the same as the leave$(12i + 5, 4, 2)$ (see **Case 18**).

Excess. The excess$(12i + 5, 4, 5)$ has $6i + 4$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor and a 4-star.

A construction by Assaf [6]. Since there is a BIBD$(12i + 5, 4, 3)$, the excess$(12i + 5, 4, 5)$ is the same as the excess$(12i + 5, 4, 2)$ (see **Case 18**).

3.7.7 Case 55: $v = 12i + 6$

The leave is a 1-factor (1^{12i+6}); the excess is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+4}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+5}$).

Leave. The leave$(12i+6, 4, 5)$ has $6i+3$ edges. Every vertex has degree 1 (mod 3). So the leave is a 1-factor.

A construction by Shalaby and Zhong. We know that the leave$(12i + 6, 4, 4)$ can be $3i + 1$ squares and a double edge (see **Case 43**); and that the leave$(12i + 6, 4, 1)$ can be $3i$ squares and a $K_6 \setminus K_4$ (see **Case 7**).

Let the $K_6 \setminus K_4$ be {1,3}{1,4}{2,3}{2,4}{3,4}{3,5}{3,6}{4,5}{4,6}, let the double edge be {1,2}{1,2}, and let one of the squares be [3,4,5,6].

Add them up we obtain blocks {1,2,3,4}, {3,4,5,6} and edge {1,2}. The rest squares form the rest of the 1-factor as in **Case 13 Leave** (see Figure 64).

Excess. The excess$(12i + 6, 4, 5)$ has $12i + 9$ edges. Every vertex has degree 2 (mod 3). There are two solutions as above.

A construction by Shalaby and Zhong. We know that the excess$(12i + 6, 4, 2)$ is an union of even cycles (see **Case 19**); and that the excess$(12i + 6, 4, 3)$ is a triple edge (see **Case 31**).

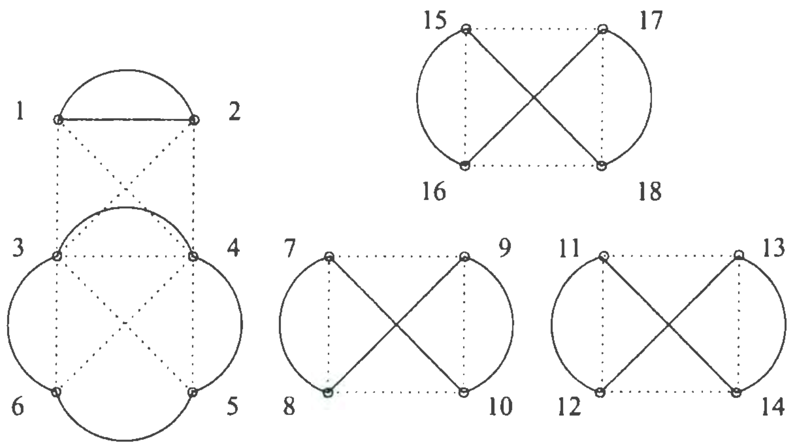

Figure 64: A leave(18,4,5).

Add the triple edge to a cycle, or use the triple edge to connect two cycles. We obtain two constructions as in **Case 51 Leave.**.

3.7.8 Case 56: $v = 12i + 7$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i + 7, 4, 5)$ has at least 3 edges. Every vertex has degree 0 (mod 3). So the leave is a triple edge.

A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD$(12i+7, 4, 2)$, the leave$(12i + 7, 4, 5)$ is the same as the leave$(12i + 7, 4, 3)$ (see **Case 32**).

Excess. the excess$(12i + 7, 4, 5)$ has at least 3 edges. Every vertex has degree 0 (mod 3). So the excess is a triple edge.

A construction by Assaf [6]. Since there is a BIBD$(12i + 7, 4, 2)$, the excess$(12i + 7, 4, 5)$ is the same as the excess$(12i + 7, 4, 3)$ (see **Case 32**).

3.7.9 Case 57: $v = 12i + 8$

The leave is a 2-factor (2^{12i+8}); the excess is a 1-factor (1^{12i+8}).

Leave. The leave$(12i+8,4,5)$ has $12i+8$ edges. Every vertex has degree 2 (mod 3). So the leave is a 2-factor.

Union of even cycles. A construction Shalaby and Zhong.
Since there is a BIBD$(12i+8,4,3)$, the leave$(12i+8,4,5)$ is the same as the leave$(12i+8,4,2)$ (see **Case 21**).

Excess. The excess$(12i+8,4,5)$ has $6i+4$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor.

A construction by Assaf [6]. Since there is a BIBD$(12i+8,4,3)$, the excess$(12i+8,4,5)$ is the same as the excess$(12i+8,4,2)$ (see **Case 21**).

3.7.10 Case 58: $v = 12i + 9$

The leave is a 1-factor and a 4-star ($4^1 1^{12i+8}$); the excess is (1) two vertices of degree 5 and others of degree 2 ($5^2 2^{12i+7}$), or (2) a vertex of degree 8 and others of degree 2 ($8^1 2^{12i+8}$).

Leave. The leave$(12i+9,4,5)$ has $6i+6$ edges. Every vertex has degree 1 (mod 3). The leave is a 1-factor and a 4-star.

A construction by Assaf [7]. Since there is a BIBD$(12i+9,4,3)$, the leave$(12i+9,4,5)$ is the same as the leave$(12i+9,4,2)$ (see **Case 22**).

Excess. The excess$(12i+9,4,5)$ has $12i+12$ edges. Every vertex has degree 2 (mod 3). There are two solutions: (1) two vertices of degree 5 and others of degree 2; (2) a vertex of degree 8 and others of degree 2.

A construction by Shalaby and Zhong. Since there is a BIBD$(12i+9,4,3)$, the excess$(12i+9,4,5)$ is the same as the excess$(12i+9,4,2)$, (see **Case 22**).

3.7.11 Case 59: $v = 12i + 10$

The leave is a triple edge (3^2); also is the excess (3^2).

Leave. The leave$(12i+10,4,5)$ has at least 3 edges. Every vertex has degree 0 (mod 3). So the leave is a triple edge.

A construction by Billington, Stanton and Stinson [23]. Since there is a BIBD$(12i+10,4,2)$, the leave$(12i+10,4,5)$ is the same as the leave$(12i+10,4,3)$ (see **Case 35**).

Excess. The excess$(12i + 10, 4, 5)$ has at least 3 edges. Every vertex has degree 0 (mod 3). So the excess is a triple edge.

A construction by Assaf [6]. Since there is a BIBD$(12i+10, 4, 2)$, the excess$(12i+10, 4, 5)$ is the same as the excess$(12i + 10, 4, 3)$ (see **Case 35**).

3.7.12 Case 60: $v = 12i + 11$

The leave is a 2-factor (2^{12i+11}); the excess is a 1-factor and a 4-star $(4^1 1^{12i+10})$.

Leave. The leave$(12i + 11, 4, 5)$ has $12i + 11$ edges. Every vertex has degree 2 (mod 3). So the leave is a 2-factor.

A triangle, a double edge and an union of even cycles. A construction by Shalaby and Zhong. We know that the leave$(12i + 11, 4, 2)$ can be a "Crown" and a union of even cycles (see **Case 24**); and that the leave$(12i + 11, 4, 3)$ is a triple edge (see **Case 36**).

Add the triple edge to the "Crown", we obtain the same construction as in **Case 52 Excess**.

Excess. The excess$(12i + 11, 4, 5)$ has $6i + 7$ edges. Every vertex has degree 1 (mod 3). So the excess is a 1-factor and a 4-star.

Direct constructions are not found, but there exists a recursive construction by Assaf [6].

4 Nuclear Designs

Definition 6. A *nuclear design*, or a ND(v, k, λ) is a collection of k-subsets (called *blocks*), of a v-set S, which is the largest intersection of a PD(v, k, λ) and a CD(v, k, λ).

PD(v, k, λ)-ND(v, k, λ) is the *packing supplement*, or P-N; CD(v, k, λ)-ND(v, k, λ) is the *covering supplement*, or C-N.

Theorem 6. Mendelsohn, Shalaby and Shen [52]. For all v, λ, ND$(v, 3, \lambda)$ exists and

(a) for v odd, the packing design is the nuclear design;

(b) for v even, the nuclear design is the packing design except for $\lfloor v/6 \rfloor$ blocks.

Now from Chapter III, we can conclude the following:

Theorem 7. In the following cases of v and λ, the ND$(v, 4, \lambda)$ is exactly the PD$(v, 4, \lambda)$:

$\lambda \equiv 0 \pmod 6$ $\quad v \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \pmod{12}$
$\lambda \equiv 1 \pmod 6$ $\quad v \equiv 0, 1, 2, 3, 4, 5, 6, 8, 9, 11 \pmod{12}$
$\lambda \equiv 2 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 7, 8, 9, 10 \pmod{12}$
$\lambda \equiv 3 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$
$\lambda \equiv 4 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 7, 8, 9, 10 \pmod{12}$
$\lambda \equiv 5 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$.

Proof. In the following cases, we have a BIBD$(v, 4, \lambda)$ by Hanani [38], so the ND$(v, 4, \lambda)$ is exactly the PD$(v, 4, \lambda)$

$\lambda \equiv 0 \pmod 6$ $\quad v \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \pmod{12}$
$\lambda \equiv 1 \pmod 6$ $\quad v \equiv 1, 4 \pmod{12}$
$\lambda \equiv 2 \pmod 6$ $\quad v \equiv 1, 4, 7, 10 \pmod{12}$
$\lambda \equiv 3 \pmod 6$ $\quad v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$
$\lambda \equiv 4 \pmod 6$ $\quad v \equiv 1, 4, 7, 10 \pmod{12}$
$\lambda \equiv 5 \pmod 6$ $\quad v \equiv 1, 4 \pmod{12}$

In the following cases, in Chapter III, we were able to construct the excess$(v, 4, \lambda)$ from the leave$(v, 4, \lambda)$, or the other way, without breaking existing blocks. That means, the excess$(v, 4, \lambda)$ contains all the blocks of the leave$(v, 4, \lambda)$, so the ND$(v, 4, \lambda)$ is exactly the PD$(v, 4, \lambda)$

$\lambda \equiv 1 \pmod 6$ $\quad v \equiv 0, 2, 3, 5, 6, 8, 9, 11 \pmod{12}$
$\lambda \equiv 2 \pmod 6$ $\quad v \equiv 0, 5, 8, 9 \pmod{12}$
$\lambda \equiv 4 \pmod 6$ $\quad v \equiv 0, 5, 8, 9 \pmod{12}$
$\lambda \equiv 5 \pmod 6$ $\quad v \equiv 0, 5, 8, 9 \pmod{12}$.

5 Conclusions

5.1 Summary

In this thesis we surveyed the necessary and sufficient conditions of packing designs and covering designs when $k = 3, 4$ and their leaves and excesses. We give the direct construction of all the leaves and excesses for $k = 4$ and all λs together (with only few exceptions).

The constructions are basically in the following ways:

(1) directly from PBDs and GDDs;
(2) from BIBDs and PBDs, by deleting or adding vertices;
(3) from PDs and CDs of lower λs;
(4) from leave to excess of the same order, and the other direction.

5.2 Incomplete Cases

λ (mod 6)	v (mod 12)	leave or excess	structure	construction	reference
3	2	excess	3	recursive	Assaf [6]
3	3	leave	3	recursive	Assaf [7]
3	6	leave	3	recursive	Assaf [7]
3	11	excess	3	recursive	Assaf [6]
5	2	excess	1F	recursive	Assaf [6]
5	3	leave	1FX	recursive	Assaf [7]
5	11	excess	1FX	recursive	Assaf [6]

Table 1: Cases where direct constructions are not found.

5.3 Further Research Points.

1. In some cases we can construct some particular 2-factors (2^v) as leaves and excesses on v vertices, but how to verify if an arbitrary 2-factor (2^v) can be constructed?

2. In some cases we can construct some particular (1) $5^2 2^{v-2}$, (2) $8^1 2^{v-1}$ as leaves and excesses on v vertices, but how to verify if an arbitrary (1) $5^2 2^{v-2}$, (2) $8^1 2^{v-1}$ can be constructed?

3. In some cases we can construct some particular (1) $4^2 1^{v-2}$, (2) $7^1 1^{v-1}$ as leaves and excesses on v vertices, but how to verify if an arbitrary (1) $4^2 1^{v-2}$, (2) $7^1 1^{v-1}$ can be constructed?

4. In some cases we can construct some particular (1) $4^3 1^{v-3}$, (2) $7^1 4^1 1^{v-1}$, (3) $10^1 1^{v-1}$ as leaves and excesses on v vertices, but how to verify if an arbitrary (1) $4^3 1^{v-3}$, (2) $7^1 4^1 1^{v-1}$, (3) $10^1 1^{v-1}$ can be constructed?

5. Can we find direct constructions of the cases in the Incomplete Cases table?

6. Can we find complete tables of the **Nuclear Designs**. The cases of $k = 3$ was solved by Mendelsohn E., Shalaby N. and Shen H. [52]. The results of this thesis provide a first step towards the cases of $k = 4$.

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6 Appendix.

6.1 Table of leaves and excesses when $k = 3$.

Below the table of known constructions of leaves and excesses when $k = 3$ put together by Shalaby [57]. Note that each row contains both leaves and excesses, separated by a semicolon.

	λ (mod 6)					
v (mod 6)	0	1	2	3	4	5
0	0;0	1F;1F	0;0	1F;1F	0;0	1F;1F
1	0;0	0;0	0;0	0;0	0;0	0;0
2	0;0	1F;1FY	2;4*	1F6;1F6	4^+;2	1FY;1F
3	0;0	0;0	0;0	0;0	0;0	0;0
4	0;0	1FY;1FY	0;0	1FY;1FY	0;0	1FY;1FY
5	0;0	C_4;2	2;4*	0;0	4^+;2	2;4^+

Table 2: Leaves and excesses when $k = 3$.

Legends:
1F: a 1-factor;
1FY: a 1-factor and a 3-star;
1FH: a 1-factor and a graph {1,2}{1,3}{1,4}{2,5}{2,6};
1F⋆: a 1-factor and a graph {1,2}{1,3}{1,4}{1,5}{1,6};
1FYY: a 1-factor and two 3-stars;
1F3: a 1-factor and a triple edge;
1F-0-: a 1-factor and a graph {1,2}{2,3}{2,3}{3,4};
1F6: (1)1FH (2)1F⋆ (3)1FYY (4)1F3 (5)1F-0-.

2: a double edge;
2+2: two (separate) double edges;
C_4: a 4-cycle;
∞: two adjacent double edges;
4: a quadruple edge;
$4^*: (1)C_4, (2)2+2, (3)\infty, (4)4(\lambda \geq 8)$;
$4^+: (1)C_4, (2)2+2, (3)\infty, (4)4$.

6.2 Tables of leaves and excesses when $k = 4$

Here we give the tables of known constructions of leaves and excesses when $k = 4$. Note that each row contains both leaves and excesses, with the leaves above and the excesses beneath.

$v \pmod{12}$	Property	Construction
0	2^{12i} 1^{12i}	$\Delta^{4i}; \square^{3i}; \cup C_{2k}(\lambda \geq 7)$ $1F$
1	0 0	0 0
2	1^{12i+2} $5^2 2^{12i}; 8^1 2^{12i+1}$	$1F$ $\Delta^{4i-1}C; \square^{3i-1}K_6 \setminus K_4$
3	2^{12i+3} $4^1 1^{12i+2}$	$\Delta^{4i+1}; \square^{3i}\Delta$ $1FX$
4	0 0	0 0
5	$4^1 1^{12i+4}$ $5^2 2^{12i+3}; 8^1 2^{12i+4}$	$1FX$ $\Delta^{4i}C; \square^{3i-1}\Delta K_6 \setminus K_4; \cup C_{2k}CHW(\lambda \geq 7)$
6	$5^2 2^{12i+4}; 8^1 2^{12i+5}$ 1^{12i+6}	$\Delta^{4i}K_6 \setminus K_4; \square^{3i}K_6 \setminus K_4$ $1F$
7	$3^6(\lambda = 1); 3^2(\lambda \geq 7)$ 3^2	$K_{3,3}(\lambda = 1); 3(\lambda \geq 7)$ 3
8	1^{12i+8} 2^{12i+8}	$1F$ $\square^{3i+2}; \cup C_{2k}(\lambda \geq 7)$
9	$5^2 2^{12i+7}; 8^1 2^{12i+8}$ $4^1 1^{12i+8}$	$\Delta^{4i+1}K_6 \setminus K_4; \square^{3i}\Delta K_6 \setminus K_4; \cup C_{2k}CHW(\lambda \geq 7)$ $1FX$
10	$3^6(\lambda = 1); 3^2(\lambda \geq 7)$ 3^2	$K_{3,3}\ (\lambda = 1); 3\ (\lambda \geq 7)$ 3
11	$4^1 1^{12i+10}$ 2^{12i+11}	$1FX$ $\square^{3i+2}\Delta$

Table 3: $\lambda = 1$.

v (mod 12)	Property	Construction
0	1^{12i} 2^{12i}	1F $\cup C_{2k}$
1	0 0	0 0
2	2^{12i+2} $4^2 1^{12i}; 7^1 1^{12i+1}$	$\cup C_{2k}$ $1FCa; 1F + 3(\lambda \geq 8)$
3	$4^3 1^{12i}; 7^1 4^1 1^{12i+1}; 10^1 1^{12i+2}$ $5^2 2^{12i+1}; 8^1 2^{12i+2}$	$1F\trianglerighteq; 1FX + 3(\lambda \geq 8)$ $\cup C_{2k}CHW$
4	0 0	0 0
5	$5^2 2^{12i+3}; 8^1 2^{12i+4}$ $4^1 1^{12i+4}$	$\cup C_{2k}CHW$ $1FX$
6	$4^2 2^{12i+4}; 7^1 2^{12i+5}$ 2^{12i+6}	$1FCa; 1F + 3(\lambda \geq 8)$ $\cup C_{2k}$
7	0 0	0 0
8	2^{12i+8} 1^{12i+8}	$\cup C_{2k}$ $1F$
9	$4^1 1^{12i+8}$ $5^2 2^{12i+7}; 8^1 2^{12i+8}$	$1FX$ $\cup C_{2k}CHW$
10	0 0	0 0
11	$5^2 2^{12i+9}; 8^1 2^{12i+10}$ $4^3 1^{12i+8}; 10^1 1^{12i+10}$	$\cup C_{2k}CHW$ $1F\trianglerighteq; 1FX + 3(\lambda \geq 8)$

Table 4: $\lambda = 2$.

$v \pmod{12}$	Property	Construction
0	0	0
	0	0
1	0	0
	0	0
2	3^2	3
	3^2	3
3	3^2	3
	3^2	3
4	0	0
	0	0
5	0	0
	0	0
6	3^2	3
	3^2	3
7	3^2	3
	3^2	3
8	0	0
	0	0
9	0	0
	0	0
10	3^2	3
	3^2	3
11	3^2	3
	3^2	3

Table 5: $\lambda = 3$.

$v \pmod{12}$	Property	Construction
0	2^{12i} 1^{12i}	$\cup C_{2k}; \Delta^{4i}$ $1F$
1	0 0	0 0
2	$4^2 1^{12i}; 7^1 1^{12i+1}$ 2^{12i+2}	$1F+3$ $\Delta^{4i}2; \square^{3i}2; \cup C_{2k}C_6; \cup C_{2k}(\lambda \geq 10)$
3	$5^2 2^{12i+1}; 8^1 2^{12i+2}$ $4^3 1^{12i}; 7^1 4^1 1^{12i+1}; 10^1 1^{12i+2}$	$\square^{3i}P; \square^{3i-1}\Delta M; \cup C_{2k}CHW(\lambda \geq 10)$ $1FX+3$
4	0 0	0 0
5	$4^1 1^{12i+4}$ $5^2 2^{12i+3}; 8^1 2^{12i+4}$	$1FX$ $\cup C_{2k}CHW; \Delta^{4i}C; \square^{3i-1}\Delta K_6 \setminus K_4$
6	2^{12i+6} $4^2 1^{12i+4}; 7^1 1^{12i+5}$	$\Delta^{4i}\square 2; \square^{3i+1}2; \cup C_{2k}C_6; \cup C_{2k}(\lambda \geq 10)$ $1F+3$
7	0 0	0 0
8	1^{12i+8} 2^{12i+8}	$1F$ $\cup C_{2k}$
9	$5^2 2^{12i+7}; 8^1 2^{12i+8}$ $1^4 1^{12i+8}$	$\cup C_{2k}CHW; \Delta^{4i+1}K_6 \setminus K_4; \square^{3i}\Delta K_6 \setminus K_4$ $1FX$
10	0 0	0 0
11	$4^3 1^{12i+8}; 7^1 4^1 1^{12i+9}; 10^1 1^{12i+10}$ $5^2 2^{12i+9}; 8^1 2^{12i+10}$	$1FX+3$ $\square^{3i+2}P; \square^{3i+1}\Delta M$

Table 6: $\lambda = 4$.

$v \pmod{12}$	Property	Construction
0	1^{12i} 2^{12i}	$1F$ $\cup C_{2k}$
1	0 0	0 0
2	$5^2 2^{12i}; 8^1 2^{12i+1}$ 1^{12i+2}	$\cup C_{2k} + 3$ $1F$
3	$4^1 1^{12i+2}$ 2^{12i+3}	$1FX$ $\cup C_{2k}\Delta 2$
4	0 0	0 0
5	$5^2 2^{12i+3}; 8^1 2^{12i+4}$ $4^1 1^{12i+4}$	$\cup C_{2k}CHW$ $1FX$
6	1^{12i+6} $5^2 2^{12i+4}; 8^1 2^{12i+5}$	$1F$ $\cup C_{2k} + 3$
7	3^2 3^2	3 3
8	2^{12i+8} 1^{12i+8}	$\cup C_{2k}$ $1F$
9	$4^1 1^{12i+8}$ $5^2 2^{12i+7}; 8^1 2^{12i+8}$	$1FX$ $\cup C_{2k}CHW$
10	3^2 3^2	3 3
11	2^{12i+11} $4^1 2^{12i+10}$	$\cup C_{2k}\Delta 2$ $1FX$

Table 7: $\lambda = 5$.

Legends:

$\alpha^i\beta^j$: a graph with i vertices of degree α and j vertices of degree β;

A^iB^j: a graph with i components isomorphic to A and j components isomorphic to B;

$1F$: a 1-factor;

$2F$: a 2-factor;

$1FX$: a 1-factor and a 4-star, i.e. a graph {1,2}{1,3}{1,4}{1,5};

$K_{3,3}$: a complete bipartite graph with 3 vertices on each side;

2: a double edge;

3: a triple edge;

Δ: a triangle;

$\square$: a square;

$K_6\backslash K_4$: a K_6 minus a K_4, i.e. a graph {1,3}{1,4}{2,3}{2,4}{3,4}{5,3}{5,4}{6,3}{6,4};

C: a Crown, i.e. a graph {1,2}{1,2}{1,3}{2,3}{1,4}{2,4}{1,5}{2,5};

$\trianglerighteq$: a double triangle i.e. a graph {1,2}{1,2}{2,3}{2,3}{1,3}{1,3};

$\cup C_{2k}$: a (disjoint) union of even cycles;

H: a Hat, i.e. a graph {1,3}{3,4}{4,1}{1,2}{1,2}{1,5}{5,2}{2,6}{6,7}{7,2};

W: a Windmill, i.e. a graph {1,2}{2,3}{3,1}{1,4}{4,5}{5,1}{1,6}{6,7}{7,1}{1,8}{8,9}{9,1};

$\cup C_{2k}CHW$: a union of even cycles and a Crown*, or Hat**, or Windmill***;

*: can be extended into three even paths and two odd paths between vertices u and v;

**: can be extended into a even paths and two odd paths between vertices u and v and a odd cycle on u and v each;

***: can be extended into four odd cycles on a vertex u;

Ca: a Candy, i.e. a graph {1,2}{1,2}{1,3}{1,4}{2,5}{2,6};

$1F + 3$: attaching a triple edge to a 1-factor;

$1FX + 3$: attaching a triple edge to a 1-factor and a 4-star;

P: a Parachute, i.e. a graph {1,2}{1,2}{1,2}{1,2}{1,3}{2,3};

M: a Mushroom, i.e. a graph {1,2}{1,2}{1,2}{1,2}{2,3}{3,4}{4,1};

C_6: a 6-cycle;

J: a Joker, i.e. a graph {1,2}{1,2}{1,2}{1,3}{3,4}{2,3}{3,5};

$\cup C_{2k}$+3: attaching a triple edge to a union of even cycles;

6.3 Special Cases

There are two types of special cases in this thesis: (I) the packing or covering numbers don't match the Schonheim bounds; (II) the packing or covering numbers match the Schonheim bounds, but the prerequisite structures in the general constructions don't exist. Below are the tables of the special cases in the thesis.

Note we don't take $v \equiv 7, 10 \pmod{12}$ as a special case here, since there are general constructions for them.

v	λ	leave or excess	reference
7	1	excess	Mills [46]
9	1	excess	Mills [45]
10	1	excess	Mills [46]
19	1	excess	Mills [46]
6	3	leave	Assaf [7]
8	1	leave	Brouwer [27]
9	1	leave	Brouwer [27]
9	2	leave	Assaf [7]
10	1	leave	Brouwer [27]
11	1	leave	Brouwer [27]
17	1	leave	Brouwer [27]
19	1	leave	Brouwer [27]

Table 8: Special Cases of Type I.

v	λ	leave or excess	reference
7	3	excess	Assaf [6]
7	5	excess	Assaf [6]
9	2	excess	Assaf [6]
9	4	excess	Assaf [6]
9	5	excess	Assaf [6]
10	3	excess	Assaf [6]
10	5	excess	Assaf [6]
19	3	excess	Assaf [6]
19	5	excess	Assaf [6]
8	2	leave	Assaf [7]
8	4	leave	Assaf [7]
8	5	leave	Assaf [7]
9	4	leave	Assaf [7]
9	5	leave	Assaf [7]
10	3	leave	Assaf [7]
10	5	leave	Assaf [7]
11	2	leave	Assaf [7]
11	3	leave	Assaf [7]
11	4	leave	Assaf [7]
11	5	leave	Assaf [7]
17	2	leave	Assaf [7]
17	4	leave	Assaf [7]
17	5	leave	Assaf [7]
19	3	leave	Assaf [7]
19	5	leave	Assaf [7]
12	1	excess	Mills [45]
12	≥ 2	excess	Assaf [6]
18	1	leave	Brouwer [27]
18	≥ 2	leave	Assaf [7]
24	1	excess	Mills [45]
24	≥ 2	excess	Assaf [6]

Table 9: Special Cases of Type II.