BEST CONSTANTS IN HIGHER-ORDER SOBOLEV INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

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BEST CONSTANTS IN HIGHER-ORDER SOBOLEV INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

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TO MY LOVELY PARENTS, MY BROTHERS, MY SISTERS, AND MY FRIENDS

Contents

Abstract

Let (M, g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold, G be a subgroup of the isometry group $Is(M, g)$. Assume that l is the minimum orbit dimension of G. For $1 < p < n/k$ with $k \in \mathbb{N}$, let $p^* = np/(n - kp)$, and $\tilde{p}^* = p(n-1)/(n-kp)$; and for $1 < p < (n-l)/k$ with $k \in \mathbb{N}$, let $q = (n-l)p/(n-l-kp)$, and $\tilde{q} = \frac{(n-l-1)p}{n-l-kp}$. On one hand, if (M, g) is without boundary, under some specific conditions, we find the best constants for the inequalities

$$\|f\|^p_{L^{p^*}(M)} \leq A \|\Lambda_{k,g} f\|^p_{L^p(M)} + B \|f\|^p_{L^p(M)} \quad \text{for all } f \in W^{k,p}(M),$$

$$\|f\|^p_{L^q_G(M)} \leq A \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + B \|f\|^p_{L^p_G(M)} \quad \text{for all } f \in W^{k,p}_G(M),$$

where $A, B \in \mathbb{R}$, and $\Lambda_{k,g} f = \nabla_g f$, $trace(\nabla^k_g f)$ if $k = 1$, $k > 1$, respectively. On the other hand, if (M, g) is with boundary, we establish the best constants for the inequality

$$\|f\|^p_{L^{\tilde{p}^*}(\partial M)} \leq A \|\Lambda_{k,g} f\|^p_{L^p(M)} + B \|f\|^p_{L^p(\partial M)} \quad \text{for all } f \in W^{k,p}(M).$$

Furthermore, if (M, g) is a G-invariant under the action of G, then we determine the best constants for the inequality

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq A \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + B \|f\|^p_{L^p_G(\partial M)} \quad \text{for all } f \in W^{k,p}_G(M).$$

The proofs of our results are based on the arguments used in [**24**], the techniques applied in [**9**]-[**10**], and the methods taken in [**30**].

Key words: Best constants, Sobolev spaces/ inequalities/ trace inequalities, Riemannian manifolds.

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Introduction

The best constants in Sobolev inequalities as well as Sobolev trace inequalities, of first and second orders, on Riemannian manifolds have been extensively studied by many authors. They are of vast scope and utility due to their powerful role in dealing with many significant problems arising in such parts of analysis as partial differential equations. In this work, we establish these best constants of higher-order on Riemannian manifolds under some certain conditions.

The thesis is organized as follows:

In Chapter 1, we introduce some necessary definitions and basic results concerning our work, which play an important role in proving our results.

In Chapter 2, we review some known results about the best constants in Sobolev inequalities as well as in Sobolev trace inequalities on compact Riemannian manifolds.

In Chapter 3, we find the best constants in higher-order for Sobolev inequalities on $3 \leq n$-dimensional smooth compact Riemannian manifolds, with or without boundary, under some specific conditions. More precisely, let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold. Suppose that $1 < p < n/k$ with $n > k \in \mathbb{N}$, and $p^* = np/(n-kp)$. Consider the space

$$F^{k,p}(M) = \begin{cases} W^{k,p}(M) & \text{if } M \text{ has no boundary,} \\ W_c^{k,p}(M) & \text{if } M \text{ has a boundary.} \end{cases}$$

Let

$$1/\mathcal{K}(n,p) = \inf_{f \in E^{k,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Lambda_k f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}},$$

where $E^{k,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{E^{k,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\Lambda_k f|^p \, dx\right)^{\frac{1}{p}},$$

and $\Lambda_k f = \nabla f$, $trace(\nabla^k f)$ if $k = 1$, $k > 1$, respectively. We prove that for any $\varepsilon > 0$ there exists $A \in \mathbb{R}$ such that for all $f \in F^{k,p}(M)$,

$$\|f\|^p_{L^{p^*}(M)} \le \left(\left(Vol_{(M,g)}\right)^{-pk/n} + \varepsilon\right) \|f\|^p_{L^p(M)} + A \|\Lambda_{k,g} f\|^p_{L^p(M)}, \tag{1}$$

and if for all $f \in W_c^{k,p}(B_{\delta'})$,

$$\|f\|_{L^{p^*}(B_{\delta'})} \le \mathcal{K}(n,p) \|\Lambda_{k,g} f\|_{L^p(B_{\delta'})},$$

where $B_{\delta'} = B(x_0, \delta')$ is a geodesic ball with radius δ', and δ' is a small number > 0, then there exists $B \in \mathbb{R}$ such that

$$\|f\|^p_{L^{p^*}(M)} \le B \|f\|^p_{L^p(M)} + \left(\mathcal{K}^p(n,p) + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} \tag{2}$$

for any $f \in F^{k,p}(M)$. Moreover, we show that $\left(Vol_{(M,g)}\right)^{-pk/n}$ is the best constant such that the inequality (1) holds for any $f \in F^{k,p}(M)$, and if the inequality (2) holds for all $f \in F^{k,p}(M)$, then $\mathcal{K}^p(n,p)$ is the best constant.

In Chapter 4, under some certain conditions, we establish the best constants in higher-order for Sobolev inequalities in the presence of symmetries on compact Riemannian manifolds without boundary. Specifically, assume that (M, g) is a smooth compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G is a subgroup of the isometry group $Is(M, g)$. Suppose that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$ and $q = \frac{(n-l)p}{n-l-kp}$. Then we prove that for any $\varepsilon > 0$, there are two real constants A_1 and B_1 such that for all $f \in W_G^{k,p}(M)$,

$$\|f\|^p_{L^q_G(M)} \leq \left(\left(Vol_{(M,g)}\right)^{-pk/(n-l)} + \varepsilon\right) \|f\|^p_{L^p_G(M)} + A_1 \|\Lambda_{k,g} f\|^p_{L^p_G(M)}, \tag{3}$$

and

$$\|f\|^p_{L^q_G(M)} \leq B_1 \|f\|^p_{L^p_G(M)} + \left(\frac{\mathcal{K}^p(n-l,p)}{V^{pk/(n-l)}} + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p_G(M)}. \tag{4}$$

Furthermore, $\left(Vol_{(M,g)}\right)^{-pk/(n-l)}$ is the best constant such that the inequality (3) is satisfied for any $f \in W_G^{k,p}(M)$, and $\frac{\mathcal{K}^p(n-l,p)}{V^{pk/(n-l)}}$ is the best constant when the inequality (4) holds for any $f \in W_G^{k,p}(M)$.

In Chapter 5, we determine, under some certain conditions, the best constants for Sobolev inequalities in the presence of symmetries on compact Riemannian manifolds. In particular, let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary, $1 < p < n/k$ with $n > k \in \mathbb{N}$, $\tilde{p}^* = p(n-1)/(n-kp)$, and

$$1/\tilde{\mathcal{K}}(n,p) = \inf_{f \in L^{\tilde{p}^*}(\partial \mathbb{R}^n_+) \setminus \{0\},\ \Lambda_k f \in L^p(\mathbb{R}^n_+)} \frac{\|\Lambda_k f\|_{L^p(\mathbb{R}^n_+)}}{\|f\|_{L^{\tilde{p}^*}(\partial \mathbb{R}^n_+)}}.$$

If for all $f \in W_c^{k,p}(B_{\delta'})$,

$$\|f\|_{L^{\tilde{p}^*}(B_{\delta'})} \leq \tilde{\mathcal{K}}(n,p) \|\Lambda_{k,g} f\|_{L^p(B_{\delta'})},$$

then for any $\varepsilon > 0$ there exists $C \in \mathbb{R}$ such that for all $f \in W^{k,p}(M)$,

$$\|f\|^p_{L^{p^*}(\partial M)} \leq C \|f\|^p_{L^p(\partial M)} + \left(\tilde{\mathcal{K}}^p(n,p) + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} . \tag{5}$$

In addition, suppose that (M, g) is a G-invariant under the action of a subgroup G of the isometry group $Is(M, g)$, and l, V are as above. Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$, and $\tilde{q} = \frac{(n-l-1)p}{n-l-kp}$. Then for any $\varepsilon > 0$ there exists a real constant $\tilde{C}$ such that

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq \tilde{C} \|f\|^p_{L^p_G(\partial M)} + \left(\frac{\tilde{\mathcal{K}}^p(n-l,p)}{V^{(kp-1)/(n-l-1)}} + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} \tag{6}$$

for any $f \in W^{k,p}_G(M)$. Further, the constants $\tilde{\mathcal{K}}^p(n,p)$, $\frac{\tilde{\mathcal{K}}^p(n-l,p)}{V^{(kp-1)/(n-l-1)}}$ are the best constants for the inequalities (5), (6), respectively.

Chapter 1

Preliminaries

The purpose of this chapter is to present some necessary definitions and basic results concerning our work. Throughout this section, Ω denotes an open subset of $\mathbb{R}^n$.

1.1 L^p spaces

In this section we review some facts that will be used in the proofs of our results. Also, we present some inequalities that play a powerful role in the sequel. We begin with the following definitions:

Definition 1.1.1 ([1]). *For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the class of all real valued measurable functions that satisfy*

$$\int_{\Omega} |f(x)|^p \, dx < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\underset{x\in\Omega}{ess\sup}\,|f(x)| = \inf\{K : |f(x)| \leq K \text{ a.e. } x \in \Omega\} < \infty \text{ for } p = \infty.$$

For $1 \leq p \leq \infty$, the space $L^p(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int\limits_\Omega |f(x)|^p\, dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \underset{x\in\Omega}{ess\sup}\,|f(x)|, & p = \infty. \end{cases}$$

Definition 1.1.2 ([23]). *For $1 \leq p \leq \infty$, we say that a real valued measurable function $f \in L^p_{loc}(\Omega)$ if and only if $f \in L^p(V)$ for each open $V \subset\subset \Omega$.*

The following three theorems are well-known.

Theorem 1.1.3 ([27] Hölder's inequality). *Let p and p' be non-negative extended real numbers with $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ with*

$$\int\limits_\Omega |f(x)\, g(x)|\, dx \leq \|f\|_{L^p(\Omega)}\, \|g\|_{L^{p'}(\Omega)}\,.$$

Moreover, the equality holds if and only if, for some non-zero constants A and B, we have $A|f|^p = B|g|^{p'}$ a.e. on Ω.

Theorem 1.1.4 ([43] Minkowski's inequality). *If f and g are in $L^p(\Omega)$ with $1 \leq p \leq \infty$, then so is $f + g$ and*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}\,.$$

Furthermore, If $1 < p < \infty$, then the equality holds only if there are non-negative constants α and β such that $\alpha f = \beta g$ a.e. on Ω.

Theorem 1.1.5 ([23] Fubini's theorem). *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty)$. If $f(x, y) \geq 0$ or $f(x, y) \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y)\, dx \right) dy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x, y)\, dy \right) dx.$$

1.2 Sobolev spaces

We now introduce the Sobolev spaces $W^{k,p}$ on open sets of $\mathbb{R}^n$, which were first used by S. Sobolev. The importance of these spaces lies in the fact that solutions of partial differential equations are naturally in Sobolev spaces rather than in the classical spaces of continuous functions, and with the derivatives understood in the classical sense. In order to be able to define the Sobolev spaces, we need to start by introducing the following definitions:

Definition 1.2.1 ([27]). *Let $f \in L^1_{loc}(\Omega)$. For a given n-index α, a function $g \in L^1_{loc}(\Omega)$ is called the α^{th} weak derivative of f if for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int_{\Omega} \varphi(x) g(x) dx = (-1)^{|\alpha|} \int_{\Omega} f(x)(\partial^{\alpha} \varphi)(x) dx,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and

$$(\partial^{\alpha} \varphi)(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \varphi(x).$$

It is clear that the α^{th} weak derivative of f is uniquely determined a.e., if it exists, then we write $g = \partial^{\alpha} f$.

Definition 1.2.2 ([27]). *For any function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, we say that f is a Hölder continuous function with exponent $0 < \alpha \leq 1$ if there exists a constant C such that*

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha}$$

for all $x,\ y \in \Omega$.

Similarly, for any $f \in C^k(\Omega)$, we say that $f \in C^{k,\alpha}(\Omega)$ with exponent $0 < \alpha \leq 1$ if there exists $C \in \mathbb{R}$ such that

$$\left|(\partial^{k} f)(x) - (\partial^{k} f)(y)\right| \leq C\,|x - y|^{\alpha}$$

for all $x,\ y \in \Omega$.

Definition 1.2.3 ([1]). *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by $W^{k,p}(\Omega)$, $W_c^{k,p}(\Omega)$ the k^{th}-order Sobolev spaces of $C^{\infty}(\Omega)$, $C_c^{\infty}(\Omega)$, respectively such that $\left|\partial^{\alpha} f\right| \in L^p(\Omega)$ with the norm*

$$\|f\|_{W^{k,p}(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha| \leq k} \left|\partial^{\alpha} f\right|^{p} dx\right)^{\frac{1}{p}} \cong \sum_{|\alpha| \leq k} \left\|\partial^{\alpha} f\right\|_{L^p(\Omega)} \quad for\ 1 \leq p < \infty.$$

For $p = \infty$, the Sobolev spaces $W^{k,\infty}(\Omega)$, $W_c^{k,\infty}(\Omega)$ are defined to be the Hölder spaces $C^{k-1,1}(\overline{\Omega})$, $C_c^{k-1,1}(\overline{\Omega})$, respectively.

Similarly, for $1 \le p < \infty$ and $k \in \mathbb{N}$, the k^{th}-order Sobolev spaces of $C^{\infty}(V)$ such that

$$\left|\partial^{\alpha} f\right| \in L^p(V) \qquad \text{for each open } V \subset\subset \Omega \tag{1.2.1}$$

is denoted by $W^{k,p}_{loc}(\Omega)$, and $W^{k,\infty}_{loc}(\Omega)$ is defined to be the Hölder spaces $C^{k-1,1}(\Omega)$.

Our next goal is to approximate a Sobolev function by a continuous function. To achieve this, we make the following definition:

Definition 1.2.4 ([23]). *For $x \in \Omega$, $f \in L^1_{loc}(\Omega)$, and $0 < \varepsilon < dist(x, \partial\Omega)$, the regularization of f, denoted by f^{ε}, is defined by*

$$f^{\varepsilon}(x) = (\eta_{\varepsilon} * f)(x) = \int_{\Omega} \eta_{\varepsilon}(x-y) f(y) dy,$$

where $\eta_{\varepsilon}(x) = \frac{1}{\varepsilon^n}\eta(x/\varepsilon)$ is the standard mollifier,

$$\eta(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & for\ |x| < 1, \\ 0 & for\ |x| \ge 1, \end{cases}$$

and c is chosen so that $\int_{\Omega} \eta(x)dx = 1$.

It is easy to prove that for $0 < \varepsilon < dist(x, \partial\Omega)$, $f^{\varepsilon} \in C^{\infty}(V)$ for any open $V \subset\subset \Omega$. Let us use mollification to approximate Sobolev functions by continuous functions.

Theorem 1.2.5 ([27]). *For $1 \le p < \infty$, let $f \in L^p_{loc}(\Omega)$, $L^p(\Omega)$, $W^{k,p}_{loc}(\Omega)$, $W^{k,p}(\Omega)$. Then f^{ε} converges to f in $L^p_{loc}(\Omega)$, $L^p(\Omega)$, $W^{k,p}_{loc}(\Omega)$, $W^{k,p}_{loc}(\Omega)$, respectively.*

The following theorem describes the interaction of weak derivatives and mollifiers.

Theorem 1.2.6 ([1]). *Suppose that $f \in L^1_{loc}(\Omega)$, α is a multi-index, and $\partial^\alpha f$ exists. Then if $0 < \varepsilon < dist(x, \partial\Omega)$, we have*

$$\partial^\alpha f^\varepsilon(x) = (\partial^\alpha f)^\varepsilon(x). \tag{1.2.2}$$

The previous two theorems tell us that if $f \in W^{1,p}(\Omega)$, then $\partial^\alpha f^\varepsilon$ goes to $\partial^\alpha f$ when ε approaches 0. This fact derives a global approximation result.

Theorem 1.2.7 ([27]). *Let $f \in W^{k,p}(\Omega)$ for $1 \leq p < \infty$. Then there exists a sequence $\{f_j\}_j^\infty \subset W^{k,p}(\Omega) \cap C^\infty(\Omega)$ such that*

$$f_j \longrightarrow f \quad in \quad W^{k,p}(\Omega).$$

Namely, $W^{k,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.

We are now going to use Whitney's Extension Theorem (see [23]) to show that if f is a Sobolev function, then it equals a C^1 function, $\tilde{f}$, except on a small set. In fact, we use the following theorem to reach our target.

Theorem 1.2.8 ([23]). *Let $f \in W^{1,p}(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Then for each $\varepsilon > 0$, there exists a Lipschitz function $\overline{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$\mathcal{L}^n \left\{ x \mid f(x) \neq \overline{f}(x) \quad or \quad \nabla f(x) \neq \nabla \overline{f}(x) \right\} \leq \varepsilon$$

and

$$\| f - \overline{f} \|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon,$$

where $\mathcal{L}^n$ is the n-Lebesgue measure, and $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$.

Theorem 1.2.9 ([23]). *Let* $f \in W^{1,p}(\mathbb{R}^n)$ *for* $1 \leq p < \infty$. *Then for any* $\varepsilon > 0$ *there exists a* C^1 *function* $\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that*

$$\mathcal{L}^n \left\{ x \mid f(x) \neq \tilde{f}(x) \;\; or \;\; \nabla f(x) \neq \nabla \tilde{f}(x) \right\} \leq \varepsilon$$

and

$$\left\| f - \tilde{f} \right\|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon.$$

In view of the above facts, we can approximate Sobolev functions by smooth functions, and consequently we can verify that some of the calculus rules hold for weak derivatives. For example, in general, if f and g are absolutely continuous functions, then the composition $f \circ g$ does not need to be absolutely continuous. However, a function f is absolutely continuous if and only if f is continuous, of bounded variation, and has the property $\mathcal{L}^n(f(E)) = 0$ whenever $\mathcal{L}^n(E) = 0$. So the problem that prevents $f \circ g$ from being absolutely continuous is that $f \circ g$ does not need to be of bounded variation. In fact, $f \circ g$ is absolutely continuous if and only if $(f' \circ g) \cdot g'$ is integrable (see for instance [**27**]). Next, we consider the case when the outer function, f, is Lipschitz.

Theorem 1.2.10 ([27]). *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function and* $g \in W^{1,p}(\Omega)$ *with* $p \geq 1$. *If* $f \circ g \in L^p(\Omega)$, *then* $f \circ g \in W^{1,p}(\Omega)$. *Moreover,*

$$\partial (f \circ g)(x) = f'[g(x)] \cdot \partial g(x)$$

for almost all $x \in \Omega$.

1.3 Riemannian manifolds

A manifold M of dimension n is a topological space in which every point of M possesses a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$. Throughout this section, we assume that M is a Hausdorff, a second countable, and a connected topological space. The principal object of this section is to introduce some basic facts concerning Riemannian manifolds.

Definition 1.3.1 ([15]). *We say-that M is a C^m n-dimensional manifold if and only if there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ on M and homeomorphisms $x_\alpha : U_\alpha \to x_\alpha(U_\alpha)$ onto open subsets of $\mathbb{R}^n$ such that for any α, $\beta \in I$,*

$$x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \to x_\alpha(U_\alpha \cap U_\beta)$$

is a C^m diffeomorphism whenever $U_\alpha \cap U_\beta \neq \phi$.

For $y \in U_\alpha$, the coordinators of $x_\alpha(y)$ in $\mathbb{R}^n$ are said to be the coordinators of y in (U_α, x_α). The pair (U_α, x_α) is called a chart, and the collection of charts $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$ such that $M = \bigcup_{\alpha\in I} U_\alpha$ is called an atlas.

It is easy to check that any subset $U \subseteq \mathbb{R}^n$ is a C^∞ n-dimensional manifold with a single chart $(U, 1_U)$. Furthermore, the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, and the torus $\mathbb{T}^n$ are considered classical examples of smooth manifolds. The following theorem demonstrates that $\mathbb{S}^{n-1}$ is a smooth $(n-1)$-dimensional manifold.

Theorem 1.3.2 ([15]). *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^m function, and $c \in \mathbb{R}$ be a regular value, that is, $\nabla f(x) \neq 0$, for all $x \in f^{-1}\{c\}$. Then $f^{-1}(c)$ is a C^m manifold.*

Applying this theorem to $f(x) = \|x\|^2$ and $c = 1$, we obtain that $\mathbb{S}^{n-1}$ is a smooth $(n-1)$-dimensional manifold.

The preceding analysis tends to shed some light on the following:

Definition 1.3.3 ([35]). *(i) Let $m \in M$. We say that a linear map $\zeta : C^\infty(M) \to \mathbb{R}$ is a tangent vector at m if and only if*

$$\zeta(fg) = f(m)\zeta(g) + \zeta(f)g(m)$$

for all f, $g \in C^\infty(M)$. The vector space of all tangent vectors at m is denote by M_m.
(ii) The disjoint union of the tangent spaces is called a tangent bundle of M, and it is denoted by $TM = \coprod_{m \in M} M_m$.

For any open set $U \subseteq \mathbb{R}^n$ and $m \in U$, it is easy to check that U_m is canonically isomorphic to $\mathbb{R}^n$ via

$$vf = df_m(v)$$

for any $v \in \mathbb{R}^n$. Moreover, for any $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, let $E = f^{-1}\{c\}$ be a regular set of f. Then E_m is a linear subspace of $\Omega_m \cong \mathbb{R}^n$. In fact,

$$E_m = \{v \in \mathbb{R}^n : v \perp \nabla f_m\}. \tag{1.3.1}$$

Next, we define how to differentiate maps between manifolds.

Definition 1.3.4 ([15]). *For a smooth map of manifolds $\phi : M \longrightarrow N$, the tangent map $d\phi_m : M_m \longrightarrow N_{\phi(m)}$ at $m \in M$ is the linear map defined by*

$$d\phi_m(\xi)f = \xi(f \circ \phi)$$

for any $\xi \in M_m$ and $f \in C^\infty(N)$.

Definition 1.3.5 ([15]). *A linear map $X : C^\infty(M) \to C^\infty(M)$ such that*

$$X(fg) = f(Xg) + g(Xf)$$

is called a vector field.

A useful geometry arises if we provide each tangent space, M_m, with an inner product. This procedure leads us to define Riemannian manifolds. In particular, with Riemannian metric g, a real differentiable manifold M, in which each tangent space is equipped with an inner product g, in a manner which varies smoothly from point to point, is called a Riemannian manifold (M, g). In fact, the following is the definition of Riemannian manifold.

Definition 1.3.6 ([11]). *A Riemannian metric g on M is an inner product g_m on each M_m such that for all vector fields X and Y, the function*

$$m \mapsto g_m(X_{|m}, Y_{|m})$$

is smooth. The pair (M, g) is called Riemannian manifold.

In particular, any open set $\Omega \subseteq \mathbb{R}^n$ is a Riemannian manifold. Indeed, for any $m \in \Omega \subseteq \mathbb{R}^n$ and for any $\nu, \upsilon \in \Omega_m$, define a Riemannian metric by

$$g_m(\nu, \upsilon) = \langle \nu, \upsilon \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual metric on $\mathbb{R}^n$. In addition, $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is also Riemannian manifold. To see this, use (1.3.1) to get that $\mathbb{S}^{n-1}_m \cong m^\perp \subset \mathbb{R}^n$. Therefore, for any $\nu, \upsilon \in \mathbb{S}^{n-1}$

$$g_m(\nu, \upsilon) = \langle \nu, \upsilon \rangle$$

defines a Riemannian metric.

Let us say a few words about the gradients of smooth functions.

Definition 1.3.7 ([30]). *For any smooth function $f : M \to \mathbb{R}$, we denote the gradient of f by the vector field $\nabla_g f$ such that*

$$g(\nabla_g f, Y) = Yf$$

for any vector field Y on M.

By simple calculations, we can prove that if (U, x) is a chart, and $\partial_1, ..., \partial_n$ are the corresponding vector fields on that chart, then

$$\nabla_g f = \sum_{i,j} g^{ij}(\partial_i f)\partial_j, \tag{1.3.2}$$

where (g^{ij}) is the matrix inverse to (g_{ij}) and $g_{ij} = g(\partial_i, \partial_j)$. Similarly, the divergence of any vector field X, denote by $div_g X \in C^\infty(M)$, is defined by

$$div_g X = \sum_i \left(\partial_i X_i + \sum_j \Gamma^i_{ij} X_j \right),$$

where Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$. Therefore, for any $f \in C^\infty(M)$,

$$\Delta_g f = div_g(\nabla_g f) = \frac{1}{\sqrt{det(g_{ij})}} \sum_{i,j} g^{ij}(\partial_i \partial_j f - \Gamma^k_{ij}\partial_k).$$

We quickly reviewed some information regarding the derivatives on Riemannian manifolds. Let us go over the integration concept on Riemannian manifolds.

Definition 1.3.8 ([35]). *Let (M, g) be a Riemannian manifold and (U, x) be a chart. Suppose that $f : U \longrightarrow \mathbb{R}$ is a measurable function. If (M, g) has a canonical measure $dv(g)$, then we define*

$$\int_U f dv(g) = \int_{x(U)} (f \circ x^{-1}) \sqrt{\det(g_{ij})} dx.$$

It is easy to observe, by using the change of variables formula, that this integral is well defined on the intersection of any two charts.

Since M is a second countable and locally compact space, then any open cover of M admits a locally finite refinement, which has a partition of unity. Therefore, we can define a global measure.

Definition 1.3.9 ([35]). *Let (M, g) be a Riemannian manifold of dimension n, and let $f : M \longrightarrow \mathbb{R}$ be a measurable function. Choose an appropriate locally finite cover of M by charts $\{(U_\alpha, x_\alpha)\}$ and a partition of unity $\{\phi_\alpha\}$ for $\{U_\alpha\}$, then we define*

$$\int_M f dv(g) = \sum_\alpha \int_{U_\alpha} \phi_\alpha f dv(g).$$

This definition, of course, is independent of all choices. Recall that M is connected, and so it is path connected. Thus, we can join any two elements x and y in M with a piece-wise C^1 path. Hence, we can define the distance between any two points on M.

Definition 1.3.10 ([21]). *Let x, $y \in M$. Then the distance between x and y is defined by*

$$d_g(x, y) = \inf \{L(\gamma) : \gamma : [a, b] \longrightarrow M \text{ is a path with } \gamma(a) = x,\ \gamma(b) = y\},$$

where $L(\gamma) = \int_a^b \left|\left(\frac{d\gamma}{dt}\right)\right| dt$ *is the the length of the path* γ.

For a given smooth Riemannian manifold (M, g), and its Levi-Civita connection D_t, a smooth curve $\gamma : [a, b] \to M$ is said to be geodesic if for all $t \in [a, b]$,

$$D_t(\frac{d\gamma}{dt}) = 0,$$

where $D_t(X) = \left(\left(\frac{d\gamma}{dt}\right)^{-1} D_g\right)_{\tilde{\partial}_1} X$, $X : [a, b] \to TM$ is a map such that $X(t) \in M_{\gamma(t)}$ for all $t \in [a, b]$, $\tilde{\partial}_1$ is a vector field on $[a, b]$, and $\left(\frac{d\gamma}{dt}\right)^{-1} D_g$ is the pull-back of D_g by $\frac{d\gamma}{dt}$ (see for instance [**15**]).

remark 1.3.11. *The Einstein summation convention is needed for expressions with indices. More precisely, if in any single term an index variable appears twice, once in an upper position and once in a lower position, it implies that term is assumed to be summed over all of its possible values of that index.*

We finish this section by presenting the Ricci curvature. The components of the Ricci curvature $Rc_{(M,g)}$ of M is given by

$$R_{ik} = g^{jl} R_{ijkl},$$

where

$$R_{ijkl} = g_{i\alpha}(\partial_k \Gamma^{\alpha}_{jl} - \partial_l \Gamma^{\alpha}_{jk} + \Gamma^{\alpha}_{k\beta}\Gamma^{\beta}_{jl} - \Gamma^{\alpha}_{l\beta}\Gamma^{\beta}_{jk}).$$

In fact, for any chart, we can check that

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij},$$

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0, \quad \textit{and} \quad R_{ij} = R_{ji}.$$

1.4 Sobolev embeddings

The main object of this section is to introduce Sobolev spaces on Riemannian manifolds, and to study the Sobolev embeddings on Riemannian manifolds as well as on open sets of $\mathbb{R}^n$. In fact, if (M,g) is Riemannian manifold, then for any given non-negative integer k, and for any $p \geq 1$, the space $C_k^p(M)$ equals the space $C^\infty(M)$, where

$$C_k^p(M) = \left\{ f \in C^\infty(M) : \textit{for each } i = 0, \ldots, k, \int_M \left|\partial_g^i f\right|^p dv(g) < \infty \right\},$$

$\partial_g^i f$ denotes the i^{th} covariant derivative of f (see for example [**29**]). We define the Sobolev spaces on compact Riemannian manifolds as:

Definition 1.4.1 ([**30**]). *Let (M,g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold. For $k \in \mathbb{N}$ and $p \geq 1$, the k^{th}-order Sobolev spaces $W^{k,p}(M)$, $W_c^{k,p}(M)$ are defined as the completion of $C^\infty(M)$, $C_c^\infty(M)$, respectively with respect to the norm*

$$\|f\|_{W^{k,p}(M)} = \left(\int_M \sum_{i=0}^{k} \left|\partial_g^i f\right|^p dv(g) \right)^{\frac{1}{p}},$$

where $dv(g) = \sqrt{\det(g_{ij})}\, dx$, and dx is the Lebesgue's volume element of $\mathbb{R}^n$.

Let us look at the space $W_c^{k,p}(M)$ from another side. Indeed, it is the closure of $C_c^\infty(M)$ in $W^{k,p}(M)$.

For the Euclidean space $\mathbb{R}^n$, it is well known that $W_c^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$; however, this fact can not be extended to all Riemannian manifolds. In fact, some assumptions are needed on manifolds to satisfy that fact. Indeed, [**3**] showed that if (M, g) is a complete $3 \leq n$-dimensional Riemannian manifold, then for any $p \geq 1$, $W_c^{1,p}(M) = W^{1,p}(M)$ (see also [**30**]).

Now, let us discuss Sobolev embeddings. For any two real numbers p and q with $1 \leq p < q$, and for any two integers k and m with $0 \leq m < k$. If $1/q = 1/p - (k-m)/n$, then $W^{k,p}(M) \subset W^{m,q}(M)$ is continuous; it means that a positive constant $C = C(M, p, q, m, k)$ exists such that for any $f \in W^{k,p}(M)$,

$$\|f\|_{W^{m,q}(M)} \leq C \|f\|_{W^{k,p}(M)} .$$

This result is referred to as the Sobolev embedding theorem as these embeddings were first proved by Sobolev for $\mathbb{R}^n$. This result is extendable to compact Riemannian manifolds.

Sobolev [**38**] applied a difficult lemma to prove that the Sobolev embeddings are valid for $\mathbb{R}^n$. Later, [**26**] and [**37**] proved the validity of these embeddings in a simple way as follows:

Theorem 1.4.2 ([**37**]). *For any* $f \in C_c^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n/n-1} \, dx \right)^{(n-1)/n} < \frac{1}{2} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_i} \right| dx \right)^{1/n} .$$

With this result, they proved the validity of Sobolev embeddings for $\mathbb{R}^n$. In fact, they proved the following:

Theorem 1.4.3 ([37]). *Let* $1 \leq p < n$ *and* $f \in W^{1,p}(\mathbb{R}^n)$. *Then*

$$\left(\int_{\mathbb{R}^n} |f|^q \, dx \right)^{\frac{1}{q}} \leq \frac{q(n-1)}{2n} \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{\frac{1}{p}},$$

where $q = np/(n-p)$. *In particular, for any real numbers* $1 \leq p < q$, *and any integers* $0 \leq m < k$ *satisfying* $1/q = 1/p - (k-m)/n$, *the embeddings* $W^{k,p}(\mathbb{R}^n) \subset W^{m,q}(\mathbb{R}^n)$ *are continuous.*

We present Sobolev embeddings for complete Riemannian n-manifolds as:

Theorem 1.4.4 ([30]). *Let* (M, g) *be a complete* $3 \leq n$*-dimensional Riemannian manifold. Suppose that* $W^{1,1}(M) \subset L^{n/(n-1)}(M)$ *is valid. Then, for any real numbers* $1 \leq p < q$ *and any integers* $0 \leq m < k$ *satisfying* $1/q = 1/p - (k-m)/n$, *the embeddings* $W^{k,p}(M) \subset W^{m,q}(M)$ *are continuous.*

Therefore, by Theorem 1.4.3, such embeddings $W^{k,p}(\mathbb{R}^n) \subset W^{m,q}(\mathbb{R}^n)$ hold for the Euclidean space; however, for complete Riemannian manifolds, such embeddings $W^{k,p}(M) \subset W^{m,q}(M)$ hold if the embeddings $W^{1,1}(M) \subset L^{n/(n-1)}(M)$ are valid. More precisely, the following is a counterexample that the above embeddings are generally not true for complete Riemannian n-manifolds.

Proposition 1.4.5 ([29]). *For any integer* $n \geq 2$, *there exist smooth complete* n-*Riemannian manifolds* (M, g) *such that for any* $1 \leq p < n$,

$$W^{1,p}(M) \not\subset L^q(M),$$

where $q = np/(n-p)$.

Our work focuses on the Sobolev embeddings when (M, g) is a compact Riemannian n-manifold. The following theorem plays an important role in the sequel.

Theorem 1.4.6 ([**30**]). *Let (M, g) be a compact Riemannian n-manifold. For any real numbers $1 \leq p < q$ and any integer numbers $0 \leq m < k$ satisfying $1/q = 1/p - (k-m)/n$, the embeddings $W^{k,p}(M) \subset W^{m,q}(M)$ are continuous.*

Let us now present the Rellich-Kondrakov theorem when (M, g) is compact, and $Vol_{(M,g)}$ is finite. For any $p_1 < p_2$, $L^{p_2}(M) \subset L^{p_1}(M)$. Therefore we can use Theorem 1.4.6 to get that for any integers $j \geq 0$ and $m \geq 1$, and for any real numbers p and q such that $1 \leq q < np/(n - mp)$, $W^{j+m,p}(M)$ is embedding in $W^{j,q}(M)$. The Rellich-Kondrakov theorem can be stated as follows (see for instance [**29**]):

Theorem 1.4.7 ([**29**]). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$. For any integers $j \geq 0$, $k \geq 1$; any real numbers $p \geq 1$; and any real number q such that $1 \leq q < np/(n - kp)$, the embedding of $W^{j+k,p}(M)$ in $W^{j,q}(M)$ is compact.*

In Theorem 1.4.7, take $j = 0$ and $k = 1$, we obtain the following corollary.

Corollary 1.4.8 ([**29**]). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$. For any $q \geq 1$ and $1 \leq p < n$ such that $1/q > 1/p - 1/n$, the embedding of $W^{1,p}(M)$ in $L^q(M)$ is compact.*

Bellow is the well-known Poincaré embedding theorem.

Theorem 1.4.9 ([**27**]). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$, and*

$1 \leq p < n$. *Then there exists* $C = C(M, g, p)$ *such that for any* $f \in W^{1,p}(M)$,

$$\left(\int_M |f - \overline{f}|^p \, dv(g) \right)^{\frac{1}{p}} \leq C \left(\int_M |\nabla_g f|^p \, dv(g) \right)^{\frac{1}{p}},$$

where $\overline{f} = \frac{1}{Vol_{(M,g)}} \int_M f dv(g)$.

A combination of the the Poincaré inequalities and the Sobolev inequalities related to the embeddings $W^{1,p}(M) \subset L^q(M)$ yields the Sobolev-Poincaré inequalities.

Theorem 1.4.10 ([29]). *Let* (M, g) *be a compact* $3 \leq n$*-dimensional Riemannian manifold. Let* $1 \leq p < n$ *and* $1/q = 1/p - 1/n$. *Then there exists* $C = C(M, g, p)$ *such that*

$$\left(\int_M |f - \overline{f}|^q \, dv(g) \right)^{\frac{1}{q}} \leq C \left(\int_M |\nabla_g f|^p \, dv(g) \right)^{\frac{1}{p}}$$

for any $f \in W^{1,p}(M)$.

Chapter 2

Best constants in Sobolev inequalities on Riemannian manifolds

This chapter presents some known results about the best constants in Sobolev inequalities on compact Riemannian manifolds, which are related to our work. Best constants in Sobolev inequalities on Riemannian manifolds play an important role in many fields such as analysis and partial differential equations (see [**9**], [**10**], [**16**], [**17**], and [**18**]). They have been extensively studied by many mathematicians. We refer the readers to [**3**], [**4**], [**6**], [**29**], and [**30**] for its history and significance, and [**25**], [**36**], [**40**], [**41**], and [**42**] for recent developments in this area.

2.1 Best constants in Sobolev inequalities on compact Riemannian manifolds

Let (M, g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold. For $k \in \mathbb{N}$ and $1 \leq p < n/k$, we denote by $E^{k,p}(\mathbb{R}^n)$ the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{E^{k,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\Lambda_k f|^p \, dx\right)^{\frac{1}{p}},$$

where $\Lambda_k f = \nabla f$, $trace(\nabla^k f)$ if $k = 1$, $k > 1$, respectively. Consider the space

$$F^{k,p}(M) = \begin{cases} W^{k,p}(M) & \text{if } M \text{ has no boundary,} \\ W_c^{k,p}(M) & \text{if } M \text{ has a boundary.} \end{cases}$$

For any $f \in F^{k,p}(M)$ and $p^* = np/(n - kp)$, the embedding $F^{k,p}(M) \subset L^{\tilde{p}}(M)$ is compact for any $\tilde{p} \in [1, p^*)$; however, the embedding $F^{k,p}(M) \subset L^{p^*}(M)$ is only continuous. Hence, real constants A_1, B_1 exist such that

$$\left(\int_M |f|^{p^*} dv(g)\right)^{1/p^*} \leq A_1 \left(\int_M |f|^p \, dv(g)\right)^{1/p} + B_1 \left(\int_M |\Lambda_{k,g} f|^p \, dv(g)\right)^{1/p}, \quad (I_p^1)$$

where $\Lambda_{1,g} f = \nabla_g f$, and $\Lambda_{k,g} f = trace(\nabla_g^k f)$ if $k > 1$ (see for example **[15]** and **[35]**). This is equivalent to the existence of constants A_p and B_p such that the following inequality holds for any $f \in F^{k,p}(M)$

$$\left(\int_M |f|^{p^*} dv(g)\right)^{p/p^*} \leq A_p \int_M |f|^p dv(g) + B_p \int_M |\Lambda_{k,g} f|^p dv(g). \qquad (I_p^p)$$

For $s \in \{1, p\}$, set

$$\mathcal{A}_p^s(M) = \inf\left\{A_s \in \mathbb{R} : \text{there exists } B_s \in \mathbb{R} \text{ such that } I_p^s \text{ is satisfied}\right\} \qquad (2.1.1)$$

and

$$\mathcal{B}_p^s(M) = \inf\left\{B_s \in \mathbb{R} : \text{there exists } A_s \in \mathbb{R} \text{ such that } I_p^s \text{ is satisfied}\right\}. \qquad (2.1.2)$$

If the best constant $\mathcal{A}_p^s(M)$ is attained, then there exists B_s such that

$$\|f\|^s_{L^{p*}(M)} \leq \mathcal{A}_p^s(M) \|f\|^s_{L^p(M)} + B_s \|\Lambda_{k,g} f\|^s_{L^p(M)} \qquad (J^s_{p,opt})$$

for any $f \in F^{k,p}(M)$. Similarly, if the best constant $\mathcal{B}_p^s(M)$ is attained, then there exists A_s such that for any $f \in F^{k,p}(M)$,

$$\|f\|^s_{L^{p*}(M)} \leq A_s \|f\|^s_{L^p(M)} + \mathcal{B}_p^s(M) \|\Lambda_{k,g} f\|^s_{L^p(M)}. \qquad (I^s_{p,opt})$$

It is obvious, by using the elementary inequality $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$ for any $p \geq 1$, that the existence of B_p, A_p implies the existence of B_1, A_1, respectively. Therefore, the validity of $(I^p_{p,opt})$ implies the validity of $(I^1_{p,opt})$, and the validity of $(J^p_{p,opt})$ implies the validity of $(J^1_{p,opt})$. However, **[19]** proved that the converse is generally not true. In fact, **[19]** proved the following theorem:

Theorem 2.1.1 ([19]). *Let (M,g) be a smooth compact Riemannian $2 \leq n$-manifold, and let $2 < p < n$ with $p^2 < n$. Assume that the scalar curvature of g is positive somewhere on M. Then the inequality $(I^p_{p,opt})$ for $k = 1$ is false on (M,g). Furthermore, if (M,g) is of non-negative Ricci curvature, $(I^p_{p,opt})$ is not valid as soon as $p > 4$, $p^2 < n$, and g is not flat.*

On the other hand, [33] showed that the inequality $(I^1_{p,opt})$ is true on the standard n-sphere $(\mathbb{S}^{n-1},h)$. Consequently, on $(\mathbb{S}^{n-1},h)$, $(I^1_{p,opt})$ is true while $(I^p_{p,opt})$ is false (at least when $p^2 < n$).

For the the best constant $\mathcal{B}^p_p(\mathbb{R}^n)$ in I^p_p, [4] proved that $\mathcal{B}^p_p(\mathbb{R}^n)$ equals $\mathcal{J}^p(n,p)$, where

$$\frac{1}{\mathcal{J}(n,p)} = \inf_{f \in L^{p^*}(\mathbb{R}^n)\setminus\{0\}, \nabla f \in L^p(\mathbb{R}^n)} \frac{\|\nabla f\|^p_{L^p(\mathbb{R}^n)}}{\|f\|^p_{L^{p^*}(\mathbb{R}^n)}}.$$

In addition, when $f \in W^{1,p}(M)$, $1 \leq p < n$, and (M,g) is a compact $3 \leq n$-dimensional Riemannian manifold without boundary, then the best constant $\mathcal{B}^1_p(M)$ in I^1_p is the same as the best constant for the Sobolev embedding for $\mathbb{R}^n$ under the Euclidean metric. Independently, [3] and [39] explicitly computed that

$$\begin{aligned}
\mathcal{J}(n,1) &= \frac{1}{n^{\frac{n-1}{n}}\omega_{n-1}^{1/n}}, \\
\mathcal{J}(n,p) &= \frac{p-1}{n-p}\left(\frac{n-p}{n(p-1)}\right)^{1/p}\left(\frac{\Gamma(n+1)}{\omega_{n-1}\Gamma(n/p)\Gamma(n+1-(n/p))}\right)^{1/n},
\end{aligned}$$

where ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. The same result is still true for any $f \in W^{1,p}_c(M)$, and (M,g) has a boundary. Furthermore, [16] established that the best constant $\mathcal{B}^1_p(M)$ equals $2^{1/n}\mathcal{J}(n,p)$ if $f \in W^{1,p}(M)$, and the compact Riemannian n-manifold (M,g) has a boundary. More precisely, [16] proved the following:

Theorem 2.1.2 ([16]). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary. Then*

(a) For any $\varepsilon > 0$ there exists a real constant A_ε such that

$$\|f\|_{L^{p^*}(M)} \leq A_\varepsilon \|f\|_{L^p(M)} + 2^{1/n}\mathcal{J}(n,p) \|\nabla_g f\|_{L^p(M)}$$

for $1 < p < n$.

(b) $2^{1/n}\mathcal{J}(n,p)$ is the best constant such that the above inequality holds for any $f \in W^{1,p}(M)$.

On one hand, for $p = 2$, $k = 1$, and (M, g) is without boundary, [30] proved the validity of $I^2_{2,opt}$ for any smooth compact Riemannian n-manifold with $n \geq 3$. On the other hand, [20] obtained that the $I^p_{p,opt}$ is valid for any smooth compact Riemannian 2-manifold with $1 \leq p < 2$.

For any compact $3 \leq n$-dimensional Riemannian manifold (M, g), with or without boundary, [30] proved the validity of $(J^1_{p,opt})$. Moreover, [30] found that $\mathcal{A}^1_p(M) = Vol^{-1/n}_{(M,g)}$ for any $f \in W^{1,p}(M)$. Further, [30] showed that $\mathcal{A}^1_p(M)$ mainly depends on an upper bound for the diameter, a lower bound for the Ricci curvature, and the lower bound for the volume.

Proposition 2.1.3 ([30]). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold. Suppose that its Ricci curvature, volume, and diameter satisfy*

$$Rc_{(M,g)} \geq kg, \quad Vol_{(M,g)} \geq v, \quad and \quad diam_{(M,g)} \leq d,$$

where k, v, and d are positive real numbers. Then there exists a positive constant $A = A_{n,p,k,d,v}$ such that

$$\|f\|_{L^{p^*}(M)} \leq A\,\|\nabla_g f\|_{L^p(M)} + \left(Vol_{(M,g)}\right)^{-1/n} \|f\|_{L^p(M)}$$

for any $f \in W^{1,p}(M)$.

Concerning the best constants in I_p^1 for $k = 2$ (see **[2]**, **[12]**, and **[18]**), the same approaches used in **[7]** and **[29]** implies that $\mathcal{A}_p^p(M)$ equals $\left(Vol_{(M,g)}\right)^{-2p/n}$. On the other hand, it is found in **[10]** that $\mathcal{B}_p^p(M) = \mathcal{I}^p(n,p)$, where

$$\frac{1}{\mathcal{I}(n,p)} = \inf_{f \in E^{2,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Lambda_2 f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}}$$

for any $f \in F^{2,p}(M)$ or $f \in W_c^{1,p}(M) \cap W^{2,p}(M)$.

Theorem 2.1.4 (**[10]**). *Let* (M, g) *be a smooth compact* $3 \leq n$*-dimensional Riemannian manifold, with or without boundary, and* $1 < p < n/2$. *Then for any* $\varepsilon > 0$ *there exists a real constant* $B = B_{\varepsilon, M, g}$ *such that*

$$\|f\|^p_{L^{p^*}(M)} \leq B\,\|f\|^p_{L^p(M)} + \left(\mathcal{I}^p(n,p) + \varepsilon\right) \|\Lambda_{2,g} f\|^p_{L^p(M)} \tag{2.1.3}$$

for any $f \in F^{2,p}(M)$ *or* $f \in W_c^{1,p}(M) \cap W^{2,p}(M)$. *Furthermore,* $\mathcal{I}^p(n,p)$ *is the best constant such that the inequality* (2.1.3) *holds.*

Subsequently, **[10]** applied the above result to solve the following equation: For any given a, b, $c \in C^0(M)$, if M has no boundary, **[10]** found solutions to the following equation:

$$\Delta_g\left(|\Delta_g f|^{p-2}\Delta_g f\right) - div_g\left(a(x)\,|\nabla_g f|^{p-2}\nabla_g f\right) + b(x)\,|f|^{p-2} f$$
$$= c(x)\,|f|^{p^*-2} f \quad \text{in } M. \qquad \textbf{(P1)}$$

If, however, M has a boundary, [**10**] determined solutions to the Dirichlet problem:

$$\begin{aligned} \Delta_g\left(|\Delta_g f|^{p-2}\Delta_g f\right) - div_g\left(a(x)|\nabla_g f|^{p-2}\nabla_g f\right) + b(x)|f|^{p-2}f & \\ = c(x)|f|^{p^*-2}f \qquad & \text{in } M, \\ f = \nabla_g f = 0 \qquad & \text{on } \partial M, \end{aligned} \tag{P2}$$

and to the Navier problem

$$\begin{aligned} \Delta_g\left(|\Delta_g f|^{p-2}\Delta_g f\right) - div_g\left(a(x)|\nabla_g f|^{p-2}\nabla_g f\right) + b(x)|f|^{p-2}f & \\ = c(x)|f|^{p^*-2}f \qquad & \text{in } M, \\ f = \Delta_g f = 0 \qquad & \text{on } \partial M. \end{aligned} \tag{P3}$$

2.2 Best constants in Sobolev inequalities in the presence of symmetries

The goal of this section is to show that Sobolev embeddings can be improved in the presence of symmetries. We devote this section to discus the best constants in Sobolev inequalities in the presence of symmetries, and to indicate some elementary definitions as well as results related to this concept.

In our work, we use the isometry group; it is the set of all isometries from the metric space onto itself, with the function composition as group operation.

In this section, let (M, g) be a $3 \leq n$-dimensional smooth compact Riemannian manifold without boundary. For any point x in M, and any compact subgroup G of the isometry group $Is(M,g)$, denote by

$$O_G^x = \{\sigma(x),\ \sigma \in G\}$$

the G-orbit of x under the action of G, and

$$S_G^x = \{\sigma \in G \mid \sigma(x) = x\}$$

the isotropy group of x. We now recall some standard results related to the action of compact subgroups of the G-orbit of $Is(M,g)$. The G-orbit of x, O_G^x, is smooth compact submanifold of M; the isotropy group of x, S_G^x, is a Lie subgroup of G; and the quotient manifold G/S_G^x exists. Moreover, the canonical map $\Phi_x : G/S_G^x \longrightarrow O_G^x$ is diffeomorphism.

Definition 2.2.1 ([24]). *Let x be an element in a group G. A homomorphism map $\Psi_x : G \longrightarrow G$ defined by $\Psi_x(y) = xyx^{-1}$ is called conjugate by x. Moreover, we say that G conjugates to G.*

Definition 2.2.2 ([25]). *A G-orbit O_G^x is said to be principal if and only if for any $y \in M$, S_G^y possesses a subgroup that conjugates to S_G^x.*

Theorem 2.2.3 ([13]). *Let Ω be the union of all principal orbits in M. Then Ω is a dense open subset of M.*

For any subgroup G of the isometry group $Is(M,g)$, let $C_G^\infty(M)$ be the set of all functions $f \in C^\infty(M)$ for which $f \circ \sigma = f$ for all $\sigma \in G$. Similarly, for $p \geqslant 1$ and $k \in \mathbb{N}$, let

$$W_G^{k,p}(M) = \left\{f \in W^{k,p}(M) \mid f \circ \sigma = f \text{ for all } \sigma \in G\right\}$$

be the subspace of $W^{k,p}(M)$ of all G-invariant functions.

Suppose that l is the minimum orbit dimension of G, V is the minimum of the volume of the l-dimensional orbits, $q = \frac{(n-l)p}{n-l-kp}$, and $1 < p < (n-l)/k$ with $k \in \mathbb{N}$. Similarly to what we have done above, for any $f \in W_G^{k,p}(M)$, and $q = \frac{(n-l)p}{n-l-kp}$, the continuity of the embedding $W_G^{k,p}(M) \subset L^q(M)$ implies that there exist two real numbers A and B such that

$$\|f\|^p_{L^q_G(M)} \leq A \|f\|^p_{L^p_G(M)} + B \|\Lambda_{k,g} f\|^p_{L^p_G(M)} \tag{2.2.1}$$

for any $f \in W_G^{k,p}(M)$.

The aim of this section is to introduce some results that have been worked in finding the best constants in Sobolev inequalities in the presence of symmetries. For $k = 1$, **[30]** established the following:

Theorem 2.2.4 ([**30**]). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and let G be a compact subgroup of the isometry group $Is(M, g)$. Let $N = \inf_{x \in M} Card(O_G^x)$. Then for any $\varepsilon > 0$, there exists $B \in \mathbb{R}$ such that for any $f \in W_G^{1,p}(M)$ with $1 \leq p < n$,*

$$\|f\|^p_{L^q_G(M)} \leq B \|f\|^p_{L^p_G(M)} + \left(\frac{\mathcal{K}^p(n,p)}{N^{p/n}} + \varepsilon \right) \|\nabla_g f\|^p_{L^p_G(M)}, \tag{2.2.2}$$

where $1/q = 1/p - 1/n$ and $\frac{\mathcal{K}^p(n,p)}{N^{p/n}} = 0$ if $N = \infty$. Moreover, $\frac{\mathcal{K}^p(n,p)}{N^{p/n}}$ is the best constant such that the inequality (2.2.2) holds for any $f \in W_G^{1,p}(M)$.

Furthermore, under some conditions, **[30]** showed that for $p = 2$ the inequality (2.2.2) is still true if $\varepsilon = 0$. Specifically, **[30]** proved the following:

Theorem 2.2.5 (**[30]**). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and let G be a compact subgroup G of the isometry group $Is(M, g)$ that possesses at least one finite orbit. Let $N = \min_{x \in M} Card(O_G^x)$. Then there exists $B \in \mathbb{R}$ such that for any $f \in W_G^{1,2}(M)$*

$$\|f\|^2_{L^q_G(M)} \leq B \|f\|^2_{L^2_G(M)} + \frac{\mathcal{K}^2(n,2)}{N^{2/n}} \|\nabla_g f\|^2_{L^2_G(M)},$$

where $1/q = 1/2 - 1/n$.

In addition, **[24]** found the explicit value of $\mathcal{K}(n,p)$ as follows:

Theorem 2.2.6 (**[24]**). *Let (M, g) be a $3 \leq n$-dimensional compact Riemannian manifold without boundary, and G be a subgroup of the isometry group $Is(M, g)$. Let l be the minimum orbit dimension of G, V the minimum volume of orbits of dimension l (if G has finite orbits, then $l = 0$ and $V = \min_{x \in M} Card(O_G^x)$). Assume that $1 \leq p < (n-l)$ and $q = \frac{(n-l)p}{n-k-p}$. Then*

(a) For any $\varepsilon > 0$, there exists a real constant B_ε such that for any $f \in W_G^{1,p}(M)$,

$$\|f\|^p_{L^q_G(M)} \leq B_\varepsilon \|f\|^p_{L^p_G(M)} + \left(\frac{\mathcal{K}^p(n-l,p)}{V^{p/(n-l)}} + \varepsilon\right) \|\nabla_g f\|^p_{L^p_G(M)},$$

where

$$\mathcal{K}(n,p) = \frac{p-1}{n-p}\left[\frac{n-p}{n(p-1)}\right]^{1/p}\left[\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\right]^{1/n}$$

for $1 < p < (n-l)$, and $\mathcal{K}(n,1) = \frac{1}{n}\left[\frac{n}{\omega_{n-1}}\right]^{1/n}$.

(b) $\frac{\mathcal{K}^p(n-l,p)}{V^{p/(n-l)}}$ is the best constant such that the above inequality holds for any $f \in W_G^{1,p}(M)$.

2.3 Best constants in Sobolev trace inequalities

The best constants in Sobolev trace inequalities are fundamental in the study of non-linear partial differential equations on manifolds, and they have received much attention from many authors (see for example **[9]**, **[17]**, **[22]**, **[28]**, **[34]**, and **[36]**). In this section, we present some results in the best constants in Sobolev trace inequalities on Riemannian manifolds with boundary.

Let (M, g) be a $3 \leq n$-dimensional compact Riemannian manifold with boundary, and let $1 < p < n/k$ with $n > k \in \mathbb{N}$. For any $f \in W^{k,p}(M)$ and $\tilde{p}^* = p(n-1)/(n-kp)$, the Sobolev trace embedding $W^{k,p}(M) \subset L^d(\partial M)$ is compact for any $1 \leq d < \tilde{p}^*$; nevertheless, the Sobolev trace embedding $W^{k,p}(M) \subset L^{\tilde{p}^*}(\partial M)$ is only continuous. Therefore, there exist two real constants A, B such that

$$\left(\int_{\partial M} |f|^{\tilde{p}^*} ds(g) \right)^{1/\tilde{p}^*} \leq A \left(\int_M |f|^p dv(g) \right)^{1/p} + B \left(\int_M |\Lambda_{k,g} f|^p dv(g) \right)^{1/p}.$$

For all $f \in W^{k,p}(M)$ there exists $C > 0$ such that

$$\|f\|^p_{L^p(M)} \leq C \left(\|f\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f\|^p_{L^p(M)} \right). \tag{2.3.1}$$

Thus, we conclude that there are two real numbers $\tilde{A}$, $\tilde{B}$ such that

$$\|f\|^p_{L^{\tilde{p}^*}(\partial M)} \leq \tilde{A} \, \|f\|^p_{L^p(\partial M)} + \tilde{B} \, \|\Lambda_{k,g} f\|^p_{L^p(M)} \tag{$\tilde{I}^p_{p,gen}$}$$

for any $f \in W^{k,p}(M)$. Set

$$\tilde{A}^p_p(M) = \inf \left\{ \tilde{A} \in \mathbb{R} : \text{there exists } \tilde{B} \in \mathbb{R} \text{ such that } \tilde{I}^p_{p,gen} \text{ holds} \right\}$$

and

$$\tilde{\mathcal{B}}_p^p(M) = \inf\left\{\tilde{B} \in \mathbb{R} : \text{there exists } \tilde{A} \in \mathbb{R} \text{ such that } \tilde{I}_{p,gen}^p \text{ holds}\right\}.$$

In a similar method to what we have said in Sections 2.2.1 and 2.2.2, we denote $(\tilde{I}_{p,gen}^p)$ by $(\tilde{I}_{p,opt}^p)$ if the best constant $\tilde{\mathcal{A}}_p^p(M)$ is achieved. However, if the best constant $\tilde{\mathcal{B}}_p^p(M)$ is attained, then we denote $(\tilde{I}_{p,gen}^p)$ by $(\tilde{J}_{p,opt}^p)$.

Easily, we can prove that the validity of $(\tilde{I}_{p,opt}^p)$ implies the validity of $(\tilde{I}_{p,opt}^1)$, and the validity of $(\tilde{J}_{p,opt}^p)$ yields the validity of $(\tilde{J}_{p,opt}^1)$. In general, [9] showed that the converse is not true. Indeed, [9] proved the following:

Theorem 2.3.1 ([9]). *Let (M,g) be a $3 \leq n$-dimensional compact Riemannian manifold with boundary, and let $1 < p < n$. Then $(\tilde{J}_{p,opt}^1)$ is valid with*

$$\tilde{\mathcal{A}}_p^1(M) = \left(Vol_{(\partial M,\tilde{g})}\right)^{-1/n},$$

where $\tilde{g} = g_{|\partial M}$. Furthermore, $(\tilde{J}_{p,opt}^p)$ is valid if and only if $n = 2$ and $1 < p < 2$, or if $n \geq 3$ and $1 < p \leq 2$.

Consequently, for any compact $3 \leq n$-dimensional Riemannian manifold with boundary with $2 < p < n$, the validity of $(\tilde{J}_{p,opt}^1)$ is satisfied while the validity of $(\tilde{J}_{p,opt}^p)$ is not.

Many mathematicians have studied the best constants in Sobolev trace inequalities. For example, [34] found that the best constant $\mathcal{B}_p^1(M)$ in Sobolev trace embedding for the Euclidean half-space $\mathbb{R}_+^n = \{(x',t) : x' \in \mathbb{R}^{n-1},\, t > 0\}$ equals $\tilde{\mathcal{J}}(n,p)$, where

$$\frac{1}{\tilde{\mathcal{J}}(n,p)} = \inf_{f \in L^{\tilde{p}^*}(\partial\mathbb{R}_+^n)\setminus\{0\},\ \nabla f \in L^p(\mathbb{R}_+^n)} \frac{\|\nabla f\|_{L^p(\mathbb{R}_+^n)}^p}{\|f\|_{L^{\tilde{p}^*}(\partial\mathbb{R}_+^n)}^p}.$$

For any $3 \leq n$-dimensional smooth compact Riemannian manifold with boundary, and $f \in W^{1,p}(M)$ with $1 < p < n$, **[9]** proved that the result of **[34]** still remains true. In fact, **[9]** established the following theorem:

Theorem 2.3.2 ([**9**]). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary, and $1 < p < n$. Then for any $\varepsilon > 0$ there exists $A = A_{\varepsilon,M,g} \in \mathbb{R}$ such that*

$$\|f\|^p_{L^{\bar{p}^*}(M)} \leq A \|f\|^p_{L^p(\partial M)} + \left(\tilde{\mathcal{J}}^p(n,p) + \varepsilon\right) \|\nabla_g f\|^p_{L^p(M)}$$

for any $f \in W^{1,p}(M)$. Moreover, if there exist real numbers A, B such that for any $f \in W^{1,p}(M)$,

$$\|f\|^p_{L^{\bar{p}^*}(\partial M)} \leq A \|f\|^p_{L^p(\partial M)} + B \|\nabla_g f\|^p_{L^p(M)},$$

then $B \geq \tilde{\mathcal{J}}^p(n,p)$.

Independently, **[8]** and **[22]** explicitly computed that

$$\tilde{\mathcal{J}}(n,2) = \frac{2}{(n-2)\,\omega_{n-1}^{1/(n-1)}}.$$

However, the problem to compute $\tilde{\mathcal{J}}(n,p)$ for $p \neq 2$ remains still open.

Under the same conditions of **[9]**, **[17]** showed that if (M, g) is a G-invariant under the action of a subgroup G of the isometry group $Is(M, g)$, then for any $\varepsilon > 0$, there exists a positive constant B_ε depending on p, G, and the geometry (M, g) such that for any $f \in W^{1,p}_G(M)$,

$$\|f\|^p_{L^{q'}_G(\partial M)} \leq B_\varepsilon \|f\|^p_{L^p_G(\partial M)} + \left(\frac{\tilde{\mathcal{J}}^p(n-l,p)}{V^{(p-1)/(n-l-1)}} + \varepsilon\right) \|\nabla_g f\|^p_{L^p_G(M)}, \tag{2.3.2}$$

where V is the minimum of the volume of the l-dimensional orbits, l is the minimum orbit dimension of G, $q' = \frac{(n-l-1)p}{n-l-p}$, and $1 < p < n - l$. Moreover, **[17]** applied the asymptotically sharp inequality (2.3.2) in solving the following equations for given a, b, c, $d \in C^0(M)$:

$$\triangle_p f + a(x) f^{p-1} = c(x) f^{q-1}, \quad f > 0 \text{ on } M \quad f\mid_{\partial M} = 0, \qquad (\tilde{\mathbf{P}}1)$$
$$\frac{2(n-l)}{n} < p < n-l, \quad q = \frac{(n-l)p}{n-l-p}$$

and

$$\triangle_p f + a(x) f^{p-1} = c(x) f^{q-1}, \quad f > 0 \text{ on } M,$$
$$|\nabla_g f|^{p-2} \frac{\partial f}{\partial n} + b(x) f^{p-1} = d(x) u^{\tilde{q}-1} \quad \text{on } \partial M, \qquad (\tilde{\mathbf{P}}2)$$
$$\frac{2(n-l)}{n} < p < n-l, \quad q = \frac{(n-l)p}{n-l-p}, \quad \tilde{q} = \frac{(n-l)p}{n-l-p},$$

where $\triangle_p f = -div_g \left(|\nabla_g f|^{p-2} \nabla_g f \right)$ is the $p-$ Laplacian operator.

Concerning the second best constant $\tilde{\mathcal{A}}_p^1(M)$, **[9]** established

$$\tilde{\mathcal{A}}_p^1(M) = \left(Vol_{(\partial M, \tilde{g})} \right)^{-1/n}.$$

In view of the results on Sobolev inequalities and Sobolev trace inequalities on compact Riemannian manifolds, a question arises naturally. For any $f \in W^{k,p}(M)$, can we obtain the best constants for any smooth compact $3 \leq n$-dimensional Riemannian manifold under certain conditions? Positively, we obtain an affirmative answer to this question.

Chapter 3

Best constants in higher-order Sobolev inequalities on smooth compact Riemannian manifolds

In this chapter, we find the best constants in higher order for Sobolev inequalities on smooth compact $3 \leq n$-dimensional Riemannian manifolds under some specific conditions.

Let (M, g) be a $3 \leq n$-dimensional smooth compact Riemannian manifold. For $1 \leq p < n/k$, $n > k \in \mathbb{N}$, and $p^* = np/(n-kp)$, the embedding $F^{k,p}(M) \subset L^{p^*}(M)$ is continuous. Thus, there exist two constants $A_p, B_p \in \mathbb{R}$ such that for any $f \in F^{k,p}(M)$,

$$\left(\int_M |f|^{p^*} dv(g)\right)^{p/p^*} \leq A_p \int_M |f|^p \, dv(g) + B_p \int_M |\Lambda_{k,g} f|^p \, dv(g). \qquad (I_p^p)$$

In view of the results on the first and second orders in Sobolev inequalities, we conjecture that we can establish the best constants in higher-order Sobolev inequalities on compact Riemannian manifolds under some certain conditions. In fact, we obtain an affirmative answer to this conjecture in the next sections.

3.1 Establishing the best constant $\mathcal{B}_p^p(M)$

The purpose of this section is to find the best constant $\mathcal{B}_p^p(M)$ for any $f \in F^{k,p}(M)$ under some specific conditions, and to prove some lemmas used in the sequel. In particular, the following theorem is the main result of this section.

Theorem 3.1.1. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold, and let $1 < p < n/k$ with $k \in \mathbb{N}$. Assume that*

$$\|f\|_{L^{p^*}(B_{\delta'})} \leq \mathcal{K}(n,p)\,\|\Lambda_{k,g} f\|_{L^p(B_{\delta'})} \tag{3.1.1}$$

for any $f \in W_c^{k,p}(B_{\delta'})$, where $B_{\delta'} = B(x_0, \delta')$ is a geodesic ball with radius δ', and δ' is a small number > 0. Then

$$\mathcal{B}_p^p(M) = \mathcal{K}^p(n,p)$$

for any $f \in F^{k,p}(M)$.

In order to prove Theorem 3.1.1, following **[10]**, it suffices to prove the following two lemmas.

Lemma 3.1.2. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold, (3.1.1) be true, and $1 < p < n/k$ with $k \in \mathbb{N}$. Let A, B be two real constants such that*

$$\|f\|^p_{L^{p*}(M)} \leq A\,\|f\|^p_{L^p(M)} + B\,\|\Lambda_{k,g} f\|^p_{L^p(M)} \tag{3.1.2}$$

for all $f \in F^{k,p}(M)$. Then $B \geq \mathcal{K}^p(n,p)$.

Proof. Suppose by contradiction that for any $f \in F^{k,p}(M)$, there exist real numbers A, B with $B < \mathcal{K}^p(n,p)$ such that (3.1.2) holds. Fix $x_0 \in M \setminus \partial M$. Given $\varepsilon_1 > 0$, let $B(x_0, \delta)$ be a geodesic ball of radius $\delta < \delta'$ and center x_0 such that in normal coordinators of $B(x_0, \delta)$ we have

$$\prod_{s=1}^{k} \left| g^{i_s j_s} - \delta^{i_s j_s} \right| \leq \varepsilon_1 \text{ for all } k = 1, 2, \ldots, n-1, \tag{3.1.3}$$

$$1 - \varepsilon_1 \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon_1 \text{ and } \left|\Gamma^l_{ij}\right| \leq \varepsilon_1, \tag{3.1.4}$$

where $(\delta^{i_s j_s})$ is the identity matrix.

For any $f \in C_c^\infty(B_\delta)$, and ε_1 small enough, two real numbers A, B exist with $B < \mathcal{K}^p(n,p)$ such that

$$\|f\|^p_{L^{p*}(B_\delta)} \leq A\,\|f\|^p_{L^p(B_\delta)} + (1+\varepsilon_2)B\,\|\Lambda_{k,g} f\|^p_{L^p(B_\delta)}, \tag{3.1.5}$$

where $\varepsilon_2 = \mathrm{O}(\varepsilon_1)$, and B_δ is the Euclidean ball with radius δ and center 0.

For $\alpha > 0$ and integer m greater than 1, using the following inequalities:

$$\int_{B_\delta} |\partial^m f|^p\,dx \;\leq\; C_{n,p} \int_{B_\delta} |\Lambda_m f|^p\,dx, \tag{3.1.6}$$

$$\int_{B_\delta} |\partial^{m-1} f|^p\,dx \;\leq\; \alpha \int_{B_\delta} |\partial^m f|^p\,dx + C_{\alpha,\delta} \int_{B_\delta} |f|^p\,dx\,, \tag{3.1.7}$$

plus the elementary inequality

$$(x+y)^p \leq (1+\varepsilon_3)\, x^p + C_{\varepsilon_3} y^p, \tag{3.1.8}$$

where $\varepsilon_3 = O(\varepsilon_1)$ and $C_{\alpha,\delta} = O(\delta^{-p})$, implies that there exists $B' < \mathcal{K}^p(n,p)$ such that

$$\|f\|^p_{L^{p^*}(B_\delta)} \leq B' \|\Lambda_k f\|^p_{L^p(B_\delta)} + C_{1,\delta} \|f\|^p_{L^p(B_\delta)}.$$

Using Hölder's inequality, we get that

$$\begin{aligned} \|f\|^p_{L^{p^*}(B_\delta)} &\leq B' \|\Lambda_k f\|^p_{L^p(B_\delta)} + C_{1,\delta} \left(|B_\delta|^{kp/n} \|f\|^p_{L^{p^*}(B_\delta)} \right) \\ &\leq \frac{B'}{1 - C_{1,\delta} |B_\delta|^{kp/n}} \|\Lambda_k f\|^p_{L^p(B_\delta)}. \end{aligned}$$

Choosing δ small enough such that $0 < C_{1,\delta} \left(|B_\delta|^{kp/n} \right) < 1$ is small enough, we obtain a real number $B_1' < \mathcal{K}^p(n,p)$ such that

$$\|f\|^p_{L^{p^*}(B_\delta)} \leq B_1' \|\Lambda_k f\|^p_{L^p(B_\delta)}.$$

For any $f \in C_c^\infty(\mathbb{R}^n)$, define $f^\mu = \mu^{n/p^*} f(\mu x)$. Then

$$\|f^\mu\|_{L^{p^*}(\mathbb{R}^n)} = \|f\|_{L^{p^*}(\mathbb{R}^n)} \quad \text{and} \quad \|\Lambda_k f^\mu\|_{L^p(\mathbb{R}^n)} = \|\Lambda_k f\|_{L^p(\mathbb{R}^n)}.$$

Hence, choosing μ large enough such that $f^\mu \in C_c^\infty(B_\delta)$, implies that

$$\|f\|^p_{L^{p^*}(\mathbb{R}^n)} < B_1' \|\Lambda_k f\|^p_{L^p(\mathbb{R}^n)},$$

contradicting the fact that $\mathcal{K}(n,p)$ is the best constant for the Sobolev inequality in $\mathbb{R}^n$.

This finishes the proof of the lemma. □

Lemma 3.1.3. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold, $\varepsilon > 0$, and $1 < p < n/k$. If (3.1.1) is true, then there exists $A = A_{M,g,\varepsilon} \in \mathbb{R}$ such that*

$$\|f\|^p_{L^{p^*}(M)} \leq A\,\|f\|^p_{L^p(M)} + \left(\mathcal{K}^p(n,p) + \varepsilon\right)\|\Lambda_{k,g} f\|^p_{L^p(M)}$$

for all $f \in F^{k,p}(M)$.

Proof. Throughout the proof of this lemma, we denote $\mathcal{K}^p(n,p)$ by $\mathcal{K}^p$.

Let $\varepsilon > 0$ be given. For some $\delta < \delta'$ small enough as we need, choose a finite covering of M by $\{B_j\}_{j=1}^{N_\delta}$, where B_j is geodesic ball of radius δ such that in normal geodesic coordinates of B_j's, and with any fixed $\varepsilon_1 > 0$,

$$\prod_{s=1}^{k} \left| g^{i_s j_s} - \delta^{i_s j_s} \right| \leq \varepsilon_1 \text{ for all } k = 1, 2, \ldots, n-1,$$

$$1 - \varepsilon_1 \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon_1 \quad \text{and} \quad |\Gamma^l_{ij}| \leq \varepsilon_1.$$

For each j, let $\{\varphi_j\}_{j=1}^{N_\delta}$ be a partition of unity associated to the covering $\{B_j\}$ such that $\varphi_j^{1/p} \in C^k_c(B_j)$. Minkowski's inequality gives that

$$\begin{aligned}
\|f\|^p_{L^{p^*}(M)} &= \left(\int_M \left(\sum_j \varphi_j^{p/p} |f|^p \right)^{p^*/p} dv(g) \right)^{\frac{p}{p^*}} \\
&\leq \sum_j \left(\int_M \left[\left(\varphi_j^{1/p} |f| \right)^p \right]^{p^*/p} dv(g) \right)^{\frac{p}{p^*}} \\
&= \sum_j \left\| \varphi_j^{1/p} f \right\|^p_{L^{p^*}(M)}.
\end{aligned}$$

Using (3.1.1), (3.1.6), (3.1.7), and (3.1.8) yields that

$$\begin{aligned}
\left\|\varphi_j^{1/p} f\right\|_{L^{p^*}(M)}^p &= \left(\int_M \left|\varphi_j^{1/p} f\right|^{p^*} dv(g)\right)^{\frac{p}{p^*}} \\
&\le (1+\varepsilon_1)^{p/p*} \left(\int_{B_j} \left|\varphi_j^{1/p} f\right|^{p^*} dx\right)^{\frac{p}{p^*}} \\
&\le (1+\varepsilon_2)\,\mathcal{K}^p \left\|\Lambda_{k,g}\left(\varphi_j^{1/p} f\right)\right\|_{L^p(B_j)}^p + C_{\varepsilon_1,\delta} \left\|\varphi_j^{1/p} f\right\|_{L^p(B_j)}^p \\
&\le (1+\varepsilon_3)\,\mathcal{K}^p \left\|\Lambda_{k,g}\left(\varphi_j^{1/p} f\right)\right\|_{L^p(M)}^p + C_{\varepsilon_1,\delta} \left\|f\right\|_{L^p(M)}^p,
\end{aligned}$$

where $\varepsilon_2 = \mathrm{O}(\varepsilon_1)$ and $\varepsilon_3 = \mathrm{O}(\varepsilon_1)$. Using (3.1.8) again, we obtain

$$\begin{aligned}
\|f\|_{L^{p^*}(M)}^p &\le (1+\varepsilon_4)\,\mathcal{K}^p \|\Lambda_{k,g} f\|_{L^p(M)}^p + C_{\varepsilon_1,\delta}\left(\sum_{i=2}^{k-1} \|\Lambda_{i,g} f\|_{L^p(M)}^p\right. \\
&\quad \left. + \left\|\nabla_g^1 f\right\|_{L^p(M)}^p + \|f\|_{L^p(M)}^p\right),
\end{aligned}$$

where $\varepsilon_4 = \mathrm{O}(\varepsilon_1)$. With ε_1 and δ, sufficiently small, we have

$$\begin{aligned}
\|f\|_{L^{p^*}(M)}^p &\le \left(\mathcal{K}^p + \frac{\varepsilon}{2}\right) \|\Lambda_{k,g} f\|_{L^p(M)}^p + d_{\varepsilon_1,\delta} \sum_{i=2}^{k-1} \|\Lambda_{i,g} f\|_{L^p(M)}^p \\
&\quad + d_{\varepsilon_1,\delta} \left\|\nabla_g^1 f\right\|_{L^p(M)}^p + d_{\varepsilon_1,\delta} \|f\|_{L^p(M)}^p. \qquad (3.1.9)
\end{aligned}$$

The L^p-theory of linear elliptic operations, (see for example [**27**]), gives that

$$\begin{aligned}
\int_{B_j} \left|\partial^2\left(\varphi_j^{1/p} f\right)\right|^p dx &\le C_\delta \int_{B_j} \left|\Lambda_{2,g}\left(\varphi_j^{1/p} f\right)\right|^p dx \\
&\le \frac{C_\delta}{1-\varepsilon_1}\left(\|\Lambda_{2,g} f\|_{L^p(M)}^p + \left\|\nabla_g^1 f\right\|_{L^p(M)}^p + \|f\|_{L^p(M)}^p\right).
\end{aligned}$$

The inequality

$$\left|\nabla_g^1 f\right| \leq (1+\varepsilon_1)\left|\partial f\right| \tag{3.1.10}$$

and (3.1.7) for $m=2$ imply that

$$\begin{aligned}
\left\|\nabla_g^1 f\right\|_{L^p(M)}^p &= \sum_j \left\|\varphi_j^{1/p}\left(\nabla_g^1 f\right)\right\|_{L^p(M)}^p \leq (1+\varepsilon_1)^{(p+1)} \sum_j \int_{B_j} \left|\varphi_j^{1/p} \partial f\right|^p dx \\
&\leq (1+\varepsilon_5) \sum_j \left(\int_{B_j} \left|\partial\left(\varphi_j^{1/p} f\right)\right|^p dx + C_{\varepsilon_1} \int_{B_j} \left|f \partial \varphi_j^{1/p}\right|^p dx \right) \\
&\leq \alpha_1 (1+\varepsilon_5) \sum_j \left(\left\|\partial^2\left(\varphi_j^{1/p} f\right)\right\|_{L^p(B_j)}^p + C_{\varepsilon_1,\alpha_1,\delta} \left\|f\right\|_{L^p(B_j)}^p \right),
\end{aligned}$$

where α_1 is as small as we want, and $\varepsilon_5 = O(\varepsilon_1)$. Thus,

$$\left\|\nabla_g^1 f\right\|_{L^p(M)}^p \leq C_{\varepsilon_1,\alpha_1,\delta} \left\|f\right\|_{L^p(M)}^p + \frac{\alpha_1 C_\delta (1+\varepsilon_6)}{1-\varepsilon_1} \left[\left\|\Lambda_{2,g} f\right\|_{L^p(M)}^p + \left\|\nabla_g^1 f\right\|_{L^p(M)}^p\right],$$

where $\varepsilon_6 = O(\varepsilon_1)$. Choosing ε_1 and α_1, sufficiently small, implies

$$\left\|\nabla_g^1 f\right\|_{L^p(M)}^p \leq \alpha_1 C_{\varepsilon_1,\delta} \left\|\Lambda_{2,g} f\right\|_{L^p(M)}^p + C_{\varepsilon_1,\alpha_1,\delta} \left\|f\right\|_{L^p(M)}^p. \tag{3.1.11}$$

Following the arguments we used above with the equivalence norm

$$\left\|f\right\|_{W^{2,p}(M)} \cong \left(\left\|\Lambda_{2,g} f\right\|_{L^p(M)}^p + \left\|f\right\|_{L^p(M)}^p\right)^{\frac{1}{p}},$$

we conclude

$$\left\|\Lambda_{2,g} f\right\|_{L^p(M)}^p \leq \alpha_2 C_{2,\varepsilon_1,\alpha_2,\delta} \left\|\Lambda_{3,g} f\right\|_{L^p(M)}^p + C_{\varepsilon_1,\alpha_2,\delta} \left\|f\right\|_{L^p(M)}^p,$$

where α_2 is chosen small enough. Similarly, for any positive α_i's with $i = 3, \cdots, k-1$,

$$\left\|\Lambda_{i,g} f\right\|_{L^p(M)}^p \leq \alpha_i C_{i,\varepsilon_1,\alpha_i,\delta} \left\|\Lambda_{i+1,g} f\right\|_{L^p(M)}^p + C_{\varepsilon_1,\alpha_i,\delta} \left\|f\right\|_{L^p(M)}^p.$$

Hence,

$$
\begin{aligned}
\|f\|^p_{L^{p^*}(M)} &\leq \left(\mathcal{K}^p + \frac{\varepsilon}{2}\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} + C_{k-2,\varepsilon_1,\alpha_{k-2},\delta} \|\Lambda_{k-1,g} f\|^p_{L^p(M)} \\
&\quad + \tilde{C}_{\varepsilon_1,\alpha_{k-2},\delta} \|f\|^p_{L^p(M)} \\
&\leq \left(\mathcal{K}^p + \frac{\varepsilon}{2}\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} + \alpha_{k-1}\tilde{C}_{k-1,\varepsilon_1,\alpha_{k-1},\delta} \|\Lambda_{k,g} f\|^p_{L^p(M)} \\
&\quad + C_{\varepsilon_1,\alpha_{k-1},\delta} \|f\|^p_{L^p(M)}.
\end{aligned}
$$

Choosing α_{k-1} small enough such that $\alpha_{k-1} \leq \varepsilon/(2\tilde{C}_{k\;1,\varepsilon_1,\alpha_{k-1},\delta})$ yields

$$
\|f\|^p_{L^{p^*}(M)} \leq \left(\mathcal{K}^p + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} + B \|f\|^p_{L^p(M)}
$$

for all $f \in F^{k,p}(M)$. □

As an immediate consequence of Lemmas 3.1.2 and 3.1.3, we obtain Theorem 3.1.1. Therefore, the best constant $\mathcal{B}^p_p(M)$ is $\mathcal{K}^p(n,p)$.

3.2 Establishing the best constant $\mathcal{A}^p_p(M)$

The main goal of this section is to establish the best constant $\mathcal{A}^p_p(M)$ for any $f \in W^{k,p}(M)$. Indeed, we will prove the following theorem:

Theorem 3.2.1. *Let (M,g) be a $3 \leq n$-dimensional compact Riemannian manifold without boundary, and let $1 < p < n/k$ with $n > k \in \mathbb{N}$. Then*

$$
\mathcal{A}^p_p(M) = \left(Vol_{(M,g)}\right)^{-pk/n}
$$

for any $f \in F^{k,p}(M)$.

The proof of Theorem 3.2.1 is similar to that proof of **[30, Proposition 4.1]**. More precisely, the proof is based on the following lemma:

Lemma 3.2.2. *Let* (M,g) *be a* $3 \leq n$*-dimensional Riemannian manifold without boundary, and let* $1 < p < n/k$ *with* $k \in \mathbb{N}$*. Then there exists* $C \in \mathbb{R}$ *such that for any* $f \in W^{k,p}(M)$ *with* $\|\Lambda_{k,g} f\|_{L^p(M)} \neq 0$ *one has*

$$\|f - (f)_M\|_{L^p(M)} \leq C \|\Lambda_{k,g} f\|_{L^p(M)}, \tag{3.2.1}$$

where $(f)_M = \frac{1}{Vol_{(M,g)}} \int\limits_M f dv(g)$*.*

Throughout this section, the letter C denotes a positive constant that may vary at each occurrence, and does not depend on the essential variables.

Proof. To prove the inequality (3.2.1), it is enough to show

$$\inf_{f \in \mathcal{H}^{k,p}(M)} \|\Lambda_{k,g} f\|^p_{L^p(M)} > 0, \tag{3.2.2}$$

where

$$\mathcal{H}^{k,p}(M) = \left\{ f \in W^{k,p}(M) : \|f\|_{L^p(M)} = 1 \text{ and } (f)_M = 0 \right\}.$$

Let $\{f_j\} \subset \mathcal{H}^{k,p}(M)$ be such that

$$\lim_{j \to \infty} \|\Lambda_{k,g} f_j\|^p_{L^p(M)} = \inf_{f \in \mathcal{H}^{k,p}(M)} \|\Lambda_{k,g} f\|^p_{L^p(M)}.$$

For $p > 1$, the Sobolev space $W^{k,p}(M)$ is reflexive. Hence, there is a subsequence $\{f_{j_m}\}$ of $\{f_j\}$, which converges weakly to h in $W^{k,p}(M)$.

Using Rellich-Kondrakov Theorem with $j = 0$ and $q = 1$, we get that $f_{j_m} \to h$ in $L^1(M)$. Applying Rellich-Kondrakov Theorem again with $j = 0$ and $p = q$, we obtain that $f_{j_m} \to h$ in $L^p(M)$. Thus, $f_{j_m} \to h$ in $L^p(M) \cap L^1(M)$. Furthermore,

$$\begin{aligned}
(h)_M = \frac{1}{Vol_{(M,g)}} \int_M h\,dv(g) &\le \left|\frac{1}{Vol_{(M,g)}}\right| \left|\int_M h\,dv(g)\right| \\
&\le \left|\frac{1}{Vol_{(M,g)}}\right| \left(\int_M |h - f_{j_m}|\,dv(g) + \left|\int_M f_{j_m}\,dv(g)\right|\right) \\
&\to 0 \text{ as } j_m \to \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left| \|h\|^p_{L^p(M)} - \|f_{j_m}\|^p_{L^p(M)} \right| &\le \|h - f_{j_m}\|^p_{L^p(M)} \\
&\to 0 \text{ as } j_m \to \infty.
\end{aligned}$$

So,

$$\|h\|_{L^p(M)} = \|f_{j_m}\|_{L^p(M)} = 1;$$

therefore, $h \in \mathcal{H}^{k,p}(M)$. Consequently,

$$\inf_{f \in \mathcal{H}^{k,p}(M)} \|\Lambda_{k,g} f\|^p_{L^p(M)} \ge \|\Lambda_{k,g} h\|^p_{L^p(M)} > 0.$$

This completes the proof of the lemma. □

Proof of Theorem 3.2.1. Setting $f = 1$ in (I^p_p) gives that $A \ge \left(Vol_{(M,g)}\right)^{-pk/n}$. Therefore, $\mathcal{A}^p_p(M) \ge \left(Vol_{(M,g)}\right)^{-pk/n}$. On one hand, if $\|\Lambda_{k,g} f\|^p_{L^p(M)} = 0$, then, by Hölder's inequality, we acquire

$$\|f\|^p_{L^{p^*}(M)} \le \left(Vol_{(M,g)}\right)^{-pk/n} \|f\|^p_{L^p(M)}.$$

On the other hand, if $\|\Lambda_{k,g}f\|^p_{L^p(M)} > 0$, then Hölder's inequality is used again to imply that

$$\begin{aligned}
\|(f)_M\|_{L^{p^*}(M)} &= \frac{1}{Vol_{(M,g)}}\left(\int_M \left|\int_M f dv(g)\right|^{p^*} dv(g)\right)^{1/p^*} \\
&\leq \left(Vol_{(M,g)}\right)^{1/p^*-1/p} \|f\|_{L^p(M)} \\
&= \left(Vol_{(M,g)}\right)^{-k/n} \|f\|_{L^p(M)}. \qquad (3.2.3)
\end{aligned}$$

Lemmas 3.1.3 and 3.2.2 give that

$$\begin{aligned}
\|f-(f)_M\|_{L^{p^*}(M)} &\leq A\|f-(f)_M\|_{L^p(M)} + B\|\Lambda_{k,g}f\|_{L^p(M)} \\
&\leq C\|\Lambda_{k,g}f\|_{L^p(M)}. \qquad (3.2.4)
\end{aligned}$$

Minkowski's inequality with (3.2.3) and (3.2.4) implies that

$$\begin{aligned}
\|f\|_{L^{p^*}(M)} &\leq \|f-(f)_M\|_{L^{p^*}(M)} + \|(f)_M\|_{L^{p^*}(M)} \\
&\leq C\|\Lambda_{k,g}f\|_{L^p(M)} + \left(Vol_{(M,g)}\right)^{-k/n} \|f\|_{L^p(M)}. \qquad (3.2.5)
\end{aligned}$$

Consequently, for any $\varepsilon > 0$, there exists $B \in \mathbb{R}$ such that

$$\|f\|^p_{L^{p^*}(M)} \leq \left(\left(Vol_{(M,g)}\right)^{-pk/n} + \varepsilon\right)\|f\|^p_{L^p(M)} + B\|\Lambda_{k,g}f\|^p_{L^p(M)}.$$

This finishes the proof of Theorem 3.2.1. Therefore, we conclude that the best constant $\mathcal{A}^p_p(M)$ is $\left(Vol_{(M,g)}\right)^{-pk/n}$.

Chapter 4

Best constants in Sobolev inequalities in the presence of symmetries on compact Riemannian manifolds

This chapter is devoted to finding the best constants in higher order for Sobolev inequalities in the presence of symmetries on compact Riemannian manifolds.

Let (M, g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G be a subgroup of the isometry group $Is(M, g)$. Assume that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$ and $q = \frac{(n-l)p}{n-l-kp}$. Set

$$\alpha_p^p(M) = \inf \{A \in \mathbb{R} : \text{ there exists } B \in \mathbb{R} \text{ such that the inequality (2.2.1) holds}\}$$

and

$$\beta_p^p(M) = \inf\{B \in \mathbb{R} : \text{ there exists } A \in \mathbb{R} \text{ such that the inequality (2.2.1) holds}\}.$$

In this chapter, we establish the best constants $\alpha_p^p(M)$ and $\beta_p^p(M)$ under some specific conditions.

4.1 Finding the best constant $\beta_p^p(M)$

To achieve our results, we need the following important lemma that was proved by Faget [24].

Lemma 4.1.1 ([**24**]). *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G be a compact subgroup of the isometry group $Is(M, g)$. Let x be in M with orbit of dimension $N < n$. Then there exists a chart (Ω, Ψ) around x such that the following properties are valid:*

(1) $\Psi(\Omega) = U_1 \times U_2$, where $U_1 \in \mathbb{R}^N$ and $U_2 \in \mathbb{R}^{n-N}$.

(2) $\Psi = \Psi_1 \times \Psi_2$ and Ψ_1, Ψ_2 can be chosen in the following way:

$\Psi_1 = \Phi_1 \circ \gamma \circ \Gamma_1$, γ defined from a neighborhood of Id in G to O_G^x, and $\gamma \circ \Gamma_1(\Omega) = \mathcal{V}_x$, where $\mathcal{V}_x$ is an open neighborhood of x in O_G^x.

$\Psi_2 = \Phi_2 \circ \Gamma_2$ with $\Gamma_2(\Omega) = \mathcal{W}_x$, where $\mathcal{W}_x$ is a submanifold of dimension $n - N$ orthogonal to O_G^x at x.

(3) (Ω, Ψ) is a normal chart of M around x, $(\mathcal{V}_x, \Phi_1)$ is a normal chart around x of the submanifold O_G^x, $(\mathcal{W}_x, \Phi_2)$ is a geodesic normal chart around x of the submani-

fold $\mathcal{W}_x$. In particular, for any $\varepsilon > 0$, (Ω, Ψ) can be chosen such that:

$$1-\varepsilon \leq \sqrt{\det(g_{ij})} \leq 1+\varepsilon \quad \text{on } \Omega, \text{ for } 1 \leq i,j \leq n,$$

$$1-\varepsilon \leq \sqrt{\det(\tilde{g}_{ij})} \leq 1+\varepsilon \quad \text{on } \mathcal{V}_x, \text{ for } 1 \leq i,j \leq N,$$

where $\tilde{g}$ denotes the metric induced by g on O_G^x.

Furthermore, $(1-\varepsilon)\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\delta_{ij}$ as bilinear forms, where δ_{ij} is the identity matrix.

(4) For any $f \in C_G^\infty$, $f \circ \Psi^{-1}$ depends only on U_2 variables.

Following the argument that [24] used in proving the above lemma, we get the same result plus the following properties:

(i) U_2 can be chosen as small as we want,

(ii)

$$\prod_{s=1}^{m} \left| g^{i_s j_s} - \delta^{i_s j_s} \right| \leq \varepsilon \quad \text{for all } m = 1,2,\ldots,n-1 \text{ and } \left|\Gamma_{ij}^l\right| \leq \varepsilon.$$

With this result, we have

Theorem 4.1.2. *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G be a subgroup of the isometry group $Is(M,g)$. Let l be the minimum orbit dimension of G, and V be the minimum of the volume of the l-dimensional orbits (if G has finite orbits, then $l = 0$ and $V = \min\limits_{x \in M} Card(O_G^x)$). Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$, and $q = \frac{(n-l)p}{n-l-kp}$. If (3.1.1) is true, then*

$$\beta_p^p(M) = \mathcal{K}_G^p = \frac{\mathcal{K}^p(n-l,p)}{V^{pk/(n-l)}}$$

for any $f \in W_G^{k,p}(M)$.

The techniques used in the proofs of [**29, Lemma 9.4**] and [**24, Lemma 2**] play a fundamental role in proving our results. In fact, the following lemmas are needed to obtain Theorem 4.1.2.

Lemma 4.1.3. *Let (M, g) and G be as in the above theorem, l be the minimum orbit dimension of G, and V be the minimum of the volume of the l-dimensional orbits. Assume that for any $1 < p < (n-l)/k$ with $k \in \mathbb{N}$ there exist real numbers A and B such that for all $f \in W_G^{k,p}(M)$,*

$$\|f\|^p_{L^q_G(M)} \leq A\,\|f\|^p_{L^p_G(M)} + B\,\|\Lambda_{k,g} f\|^p_{L^p_G(M)}\,. \tag{4.1.1}$$

Then $B \geq \mathcal{K}^p_G$.

Proof. The proof of this lemma is similar to the proof of Lemma 3.1.2. Assume by contradiction that there exist real numbers A, B with $B < \mathcal{K}^p_G$ such that (4.1.1) holds for any $f \in W_G^{k,p}(M)$. Fix $x_0 \in M$. Given $\varepsilon_1 > 0$; let $B(x_0, \delta)$ be a geodesic ball of radius $\delta < \delta'$ and center x_0 such that in normal coordinators of $B(x_0, \delta)$, (3.1.3) and (3.1.4) are satisfied.

As we did in Lemma 3.1.3, choosing ε_1 and δ small enough, we get a real number $B' < \mathcal{K}^p(n-l, p)$ such that for any $f \in C_c^\infty(B_\delta)$,

$$\|f\|^p_{L^q(B_\delta)} \leq B'\,\|\Lambda_k f\|^p_{L^p(B_\delta)}\,, \tag{4.1.2}$$

where B_δ is the Euclidean ball with radius δ and center 0.

For any $f \in C_c^\infty(\mathbb{R}^{n-l})$, define $f^\nu = \nu^{(l-n)/q} f(x/\nu)$. Then, choosing ν small enough such that $f^\nu \in C_c^\infty(B_\delta)$, we get that

$$\|f\|_{L^q(\mathbb{R}^{n-l})} = \|f^\nu\|_{L^q(\mathbb{R}^{n-l})} < B'\,\|\Lambda_k f^\nu\|_{L^p(\mathbb{R}^{n-l})} = B'\,\|\Lambda_k f\|_{L^p(\mathbb{R}^{n-l})}\,,$$

which contradicts the definition of $\mathcal{K}(n-l,p)$. $\square$

Lemma 4.1.4. *Let (M,g) and G be as in Lemma 1.4.2. Suppose that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. Then for any $\varepsilon > 0$ there exists real constant $A = A_{\varepsilon,M,g}$ such that for any $1 < p < (n-l)/k$ with $k \in \mathbb{N}$,*

$$\|f\|^{p}_{L^{q}_{G}(M)} \leq A\,\|f\|^{p}_{L^{p}_{G}(M)} + \left(\mathcal{K}^{p}_{G} + \varepsilon\right)\|\Lambda_{k,g} f\|^{p}_{L^{p}_{G}(M)}$$

for any $f \in W^{k,p}_{G}(M)$.

Proof. The proof of Lemma 4.1.4 depends on the proof of **[24, Theorem 1]** and the approaches used in **[32]**. Given $\varepsilon > 0$. Let $\delta > 0$ be taken as small as we wish. Fix $x \in M$; let O^{x}_{G} be its G-orbit, and (Ω, Ψ) be a chart around x such that the above properties are satisfied. Let $y \in O^{x}_{G}$, $\sigma \in G$ be such that $\sigma(x) = y$, and $(\sigma(\Omega), \Psi \circ \sigma^{-1})$ be a chart around y isometric to (Ω, Ψ). Then, O^{x}_{G} is covered by such charts. From that and due to the compactness of O^{x}_{G}, we say that $\{\Omega_m\}_1^L$ is a finite extract covering. Choose δ, depending on ε and x, small enough such that

$$(O^{x}_{G})_\delta = \{y \in M \mid d_g(y, O^{x}_{G}) < \delta\}$$

is covered by $\{\Omega_m\}_1^L$, and $\overline{(O^{x}_{G})_\delta}$ is a submanifold of M with boundary. Obviously, the manifold M is covered by $\bigcup_{x \in M} (O^{x}_{G})_\delta$; therefore, by the compactness of M, there exists a finite extract cover; say $\left\{(O^{x}_{G})_{i,\delta}\right\}_1^J$. Assume that (η_i) is a partition of unity relative to $(O^{x}_{G})_{i,\delta}$ such that $\eta_i \in C^{\infty}_{G}\left((O^{x}_{G})_{i,\delta}\right)$ for any i. Hence, $\eta_i f$ has a compact support in $(O^{x}_{G})_{i,\delta}$ for any $f \in W^{k,p}_{G}(M)$.

Furthermore, for each m we let $\alpha_m = \frac{\beta_m \circ \Psi_m}{\sum\limits_{m=1}^{L} (\beta_m \circ \Psi_m)}$, where $\beta_m \in C_c^\infty(U_{1m})$ and $\beta_m \geq 0$. Thus, (α_m) is a partition of unity relative to Ω_m's, which covers $(O_G^x)_{\iota,\delta}$. Notice that β_m is a function, defined on $U_{1m} \times U_{2m}$, depending only on U_{1m} variables; therefore, $\alpha_m \circ \Psi_m^{-1}$ depend only on U_{1m} variables.

For any integer $1 \leq i \leq J$, and for any $u \in W_{\varepsilon,G}^{k,\tilde{p}}((O_G^x)_{\iota,\delta})$ we have that

$$
\begin{aligned}
\int_M u\,dv(g) &= \sum_m \int_{\Omega_m} \alpha_m u\,dv(g) = \sum_m \int_{U_{1m} \times U_{2m}} \sqrt{det(g_{ij})}\alpha_m u \circ \Psi_m^{-1} dx dy \\
&\leq (1+\varepsilon) \sum_m \int_{U_{1m} \times U_{2m}} \alpha_m u \circ \Psi_m^{-1} dx dy \\
&= (1+\varepsilon) \sum_m \int_{U_{1m}} \alpha_m \circ \Psi_m^{-1} dx \int_{U_{2m}} u \circ \Psi_m^{-1} dy. \qquad (4.1.3)
\end{aligned}
$$

For simplicity, we assume for each m that $\alpha_{1m} = \alpha_m \circ \Psi_m^{-1}$, which is independent of U_{2m} variables, and $u_{2m} = u \circ \Psi_m^{-1}$, which is independent of U_{1m} variables.

As u is G-invariant, and as (Ω_m, Ψ_m) are isometric to each other, we conclude that $\int\limits_{U_{2m}} u_2$ does not depend on m. From this statement and the inequality (4.1.3), we obtain that

$$\begin{aligned}
\int_M u\,dv(g) &\le (1+\varepsilon)\int_{U_2} u_2dy \sum_m \int_{U_{1m}} \alpha_{1m}dx \\
&\le (1+\varepsilon)\int_{U_2} u_2dy \sum_m \frac{1}{1-\varepsilon}\int_{U_{1m}} \alpha_{1m}\sqrt{\det(\tilde{g}_{ij})}dx \\
&= \frac{1+\varepsilon}{1-\varepsilon}\int_{U_2} u_2dy \sum_m \int_{\mathcal{V}_x} \alpha_{1m}\circ\Phi_{1m}dv(\tilde{g}) \\
&= \frac{1+\varepsilon}{1-\varepsilon}\int_{U_2} u_2dy \int_{O_G^x}\sum_m \alpha_{1m}\circ\Phi_{1m}dv(\tilde{g}). \qquad (4.1.4)
\end{aligned}$$

Therefore,

$$\int_M udv(g) \le (1+\varepsilon_1)\,Vol(O_G^x)\int_{U_2} u_2dy, \qquad (4.1.5)$$

where $\varepsilon_1 = \mathrm{O}(\varepsilon)$. Similarly, we show that

$$\begin{aligned}
\int_M udv(g) &= \sum_m \int_{U_{1m}\times U_{2m}} \sqrt{\det(g_{ij})}\alpha_m u\circ\Psi_m^{-1}dxdy \\
&\ge (1-\varepsilon)\int_{U_2} u_2dy \sum_m \int_{U_{1m}} \alpha_{1m}dx \\
&\ge \frac{1-\varepsilon}{1+\varepsilon}\int_{U_2} u_2dy \int_{O_G^x}\sum_m \alpha_{1m}\circ\Phi_{1m}dv(\tilde{g}).
\end{aligned}$$

Thus,

$$\int\limits_M |u|^p \, dv(g) \geq C \int\limits_{U_2} |u_2|^p \, dy. \tag{4.1.6}$$

The techniques used in Section 3.1 and in [5], imply that for any $h_2 \in W_G^{k,p}(U_2)$,

$$\|h_2\|^p_{L^q(U_2)} \leq \left(\mathcal{K}^p(n-l,p)+\varepsilon\right) \|\Lambda_k h_2\|^p_{L^p(U_2)} + C\,\|h_2\|^p_{L^p(U_2)}. \tag{4.1.7}$$

For any $f \in W^{k,p}_{c,G}((O^x_G)_{i,\delta})$ we have that

$$\begin{aligned} Vol(O^x_G)\,\|\Lambda_k f_2\|^p_{L^p(U_2)} &\leq (1+\varepsilon_2)\sum_m \int\limits_{O^x_G} \alpha_{1m}\circ\Phi_{1m} dv(\tilde{g}) \int\limits_{U_{2m}} |\Lambda_k f_{2m}|^p \, dy \\ &= (1+\varepsilon_2)\sum_m \int\limits_{\mathcal{V}_x} \alpha_{1m}\circ\Phi_{1m} dv(\tilde{g}) \int\limits_{U_{2m}} |\Lambda_k f_{2m}|^p \, dy \\ &\leq (1+\varepsilon_3)\sum_m \int\limits_{U_{1m}} \alpha_{1m} dx \int\limits_{U_{2m}} |\Lambda_k f|^p \circ \Psi_m^{-1} dy, \end{aligned}$$

where $\varepsilon_2 = O(\varepsilon)$ and $\varepsilon_3 = O(\varepsilon)$. Applying what we did in Lemma 3.2.1, we get that

$$\begin{aligned} Vol(O^x_G)\,\|\Lambda_k f_2\|^p_{L^p(U_2)} &\leq (1+\varepsilon_4)\sum_m \int\limits_{U_{1m}} \alpha_{1m} dx \int\limits_{U_{2m}} |\Lambda_{k,g} f|^p \circ \Psi_m^{-1} dy \\ &+ C_\varepsilon \sum_m \int\limits_{U_{1m}} \alpha_{1m} dx \int\limits_{U_{2m}} |f|^p \circ \Psi_m^{-1} dy, \end{aligned} \tag{4.1.8}$$

where $\varepsilon_4 = O(\varepsilon)$. Substituting the value of α_{1m} in (4.1.8), we obtain that

$$\begin{aligned}
Vol(O_G^x)\left\|\Lambda_k f_2\right\|_{L^p(U_2)}^p &\le (1+\varepsilon_4)\sum_m \int_{U_{1m}\times U_{2m}} \alpha_m \left|\Lambda_{k,g} f\right|^p \circ \Psi_m^{-1} dxdy \\
&+ C_\varepsilon \sum_m \int_{U_{1m}\times U_{2m}} \alpha_m \left|f\right|^p \circ \Psi_m^{-1} dxdy \\
&\le \frac{1+\varepsilon_4}{1-\varepsilon}\sum_m \int_{U_{1m}\times U_{2m}} \sqrt{\det(g_{ij}^m)}\alpha_m \left|\Lambda_{k,g} f\right|^p \circ \Psi_m^{-1} dxdy \\
&+ \frac{C_\varepsilon}{1-\varepsilon}\sum_m \int_{U_{1m}\times U_{2m}} \sqrt{\det(g_{ij}^m)}\alpha_m \left|f\right|^p \circ \Psi_m^{-1} dxdy \\
&\le (1+\varepsilon_5)\left\|\Lambda_{k,g} f\right\|_{L_G^p(M)}^p + C_\varepsilon \left\|f\right\|_{L_G^p(M)}^p, \qquad (4.1.9)
\end{aligned}$$

where $\varepsilon_5 = O(\varepsilon)$. From the above and due to (4.1.5)-(4.1.7) we conclude that

$$\begin{aligned}
\left\|f\right\|_{L_G^q(M)}^p &\le (1+\varepsilon_6)\left(Vol(O_G^x)\right)^{q/p}\left\|f_2\right\|_{L^q(U_2)}^p \\
&\le (X+\varepsilon_6)\left\|\Lambda_k f_2\right\|_{L^p(U_2)}^p + C\left\|f_2\right\|_{L^p(U_2)}^p \\
&\le (X+\varepsilon_6)\left\|\Lambda_k f_2\right\|_{L^p(U_2)}^p + C\left\|f\right\|_{L_G^p(M)}^p \\
&\le \frac{(X+\varepsilon_7)}{Vol(O_G^x)}\left\|\Lambda_{k,g} f\right\|_{L_G^p(M)}^p + C\left\|f\right\|_{L_G^p(M)}^p,
\end{aligned}$$

where $X = (Vol(O_G^x))^{p/q}\mathcal{K}^p(n-l,p)$, $\varepsilon_6 = O(\varepsilon)$, and $\varepsilon_7 = O(\varepsilon)$.

On one hand, if (O_G^x) is of minimum dimension V, then

$$\left\|f\right\|_{L_G^q(M)}^p \le \left(\mathcal{K}_G^p+\varepsilon\right)\left\|\Lambda_{k,g} f\right\|_{L_G^p(M)}^p + C_\varepsilon \left\|f\right\|_{L_G^p(M)}^p. \qquad (4.1.10)$$

In the same manner as (4.1.6), we derive

$$\int_M \left|\Lambda_k f\right|^p dv(g) \ge C\int_{U_2} \left|\Lambda_k f_2\right|^p dy. \qquad (4.1.11)$$

On the other hand, if (O^x_G) is of minimum dimension $V^* > V$, then let U_2 be an open set of dimension $n - V^*$. Since $V^* > V$, we get that $\frac{(n-V^*)p}{n-V^*-kp} > \frac{(n-V)p}{n-V-kp}$. The compactness of the embedding $W^{k,p}(U_2) \subset L^q(U_2)$ plus the inequalities (4.1.5), (4.1.6), and (4.1.11) leads that for any $\varepsilon_0 > 0$ there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\left(\int_M |f|^q \, dv(g)\right)^{p/q} &\leq C\left(\int_{U_2} |f_2|^q \, dy\right)^{p/q} \\
&\leq C\left(\varepsilon_0 \int_{U_2} |\Lambda_k f_2|^p \, dy + C_1 \int_{U_2} |f_2|^p \, dy\right) \\
&\leq \varepsilon_0 C \int_M |\Lambda_k f|^p \, dv(g) + C_2 \int_M |f|^p \, dv(g) \\
&\leq \varepsilon_0 C \int_M |\Lambda_{k,g} f|^p \, dv(g) + C \int_M |f|^p \, dv(g).
\end{aligned}$$

Choose ε_0 small enough such that

$$\|f\|^p_{L^q_G(M)} \leq \left(\mathcal{K}^p_G + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + C \|f\|^p_{L^p_G(M)} \tag{4.1.12}$$

for any $f \in W^{k,p}_{c,G}((O^x_G)_{i,\delta})$. To end the proof of this lemma, it is sufficient to prove the following proposition:

Proposition 4.1.5. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and (O_i, η_i) be a partition of unity of M. Assume that for any $f \in W^{k,p}_G(M)$, $|\eta_i|^{1/p} f$ is in $W^{k,p}_{c,G}(O_i)$. Suppose that there exists C^* such that for any*

i there exists C_i such that

$$\|f\|^p_{L^q_G(M)} \leq C^* \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + C_i \|f\|^p_{L^p_G(M)}, \tag{4.1.13}$$

where p and q are as in the above lemma, then for any $\varepsilon > 0$ there exists real constant C_ε such that

$$\|f\|^p_{L^q_G(M)} \leq (C^* + \varepsilon) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + C_\varepsilon \|f\|^p_{L^p_G(M)}. \tag{4.1.14}$$

Proof. For any $f \in W^{k,p}_G(M)$, Minkowski's inequality with (4.1.13) yields that

$$\begin{aligned}
\|f\|^p_{L^q_G(M)} &= \left(\int_M \left(\sum_i \eta_i^{p/p} |f|^p \right)^{q/p} dv(g) \right)^{\frac{p}{q}} \\
&\leq \sum_i \left(\int_M \left[\left(\eta_i^{1/p} |f| \right)^p \right]^{q/p} dv(g) \right)^{\frac{p}{q}} \\
&= \sum_i \left\| \eta_i^{1/p} f \right\|^p_{L^q_G(M)} \\
&\leq \sum_i \left(C^* \left\| \Lambda_{k,g} (\eta_i^{1/p} f) \right\|^p_{L^p_G(M)} + C_i \left\| \eta_i^{1/p} f \right\|^p_{L^p_G(M)} \right).
\end{aligned}$$

In the manner used in Lemma 3.1.3, we obtain that for any $\varepsilon_0 > 0$

$$\|f\|^p_{L^q_G(M)} \leq (C^* + \varepsilon_0 + \varepsilon) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} + C \|f\|^p_{L^p_G(M)}.$$

Choosing $\varepsilon_0 \leq \varepsilon/2$, we achieve the inequality (4.1.14). □

With Proposition 4.1.5, and due to Lemmas 4.1.3 and 4.1.4, we finish the proof of the Theorem 4.1.2. Therefore, we conclude

$$\beta^p_p(M) = \mathcal{K}^p_G = \frac{\mathcal{K}^p(n-l,p)}{V^{pk/(n-l)}}.$$

$\square$

4.2 Finding the best constant $\alpha_p^p(M)$

In this section, we establish the best constant $\alpha_p^p(M)$. In fact, we prove the following:

Theorem 4.2.1. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G be a subgroup of the isometry group $Is(M, g)$. Let l be the minimum orbit dimension of G, V be the minimum of the volume of the l-dimensional orbits, and $1 < p < (n-l)/k$ with $k \in \mathbb{N}$. Then for any $f \in W_G^{k,p}(M)$,*

$$\alpha_p^p(M) = \left(Vol_{(M,g)}\right)^{-pk/(n-l)}.$$

Proof. Theorem 4.2.1 can be proved in the same manner as Theorem 3.2.1. On one hand, using Minkowski's inequality, Hölder's inequality, Lemma 3.2.2, and Lemma 4.1.4, we obtain that

$$\begin{aligned}
\|f\|_{L_G^q(M)} &\leq \|f-(f)_M\|_{L_G^q(M)} + \|(f)_M\|_{L_G^q(M)} \\
&\leq A\|f-(f)_M\|_{L_G^p(M)} + B\|\Lambda_{k,g} f\|_{L_G^p(M)} + \|(f)_M\|_{L_G^q(M)} \\
&\leq C\|\Lambda_{k,g} f\|_{L_G^p(M)} + \left(Vol_{(M,g)}\right)^{1/q-1/p} \|f\|_{L_G^p(M)} \\
&\leq C\|\Lambda_{k,g} f\|_{L_G^p(M)} + \left(Vol_{(M,g)}\right)^{-k/(n-l)} \|f\|_{L_G^p(M)}. \qquad (4.2.1)
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $B_1 \in \mathbb{R}$ such that

$$\|f\|_{L_G^q(M)}^p \leq \left(\left(Vol_{(M,g)}\right)^{-pk/(n-l)} + \varepsilon\right) \|f\|_{L_G^p(M)}^p + B_1 \|\Lambda_{k,g} f\|_{L_G^p(M)}^p. \qquad (4.2.2)$$

On the other hand, setting $f = 1$ in the inequality (2.2.1) gives that $A \geq \left(Vol_{(M,g)}\right)^{-pk/(n-l)}$.

Hence,

$$\alpha_p^{''}(M) \geq \left(Vol_{(M,g)}\right)^{-pk/(n-l)}. \tag{4.2.3}$$

By (4.2.2) and (4.2.3), we obtain that the best constant $\alpha_p^{''}(M)$ is $\left(Vol_{(M,g)}\right)^{-pk/(n-l)}$.

$\square$

Chapter 5

Best constants in Sobolev trace inequalities in smooth compact Riemannian manifolds

The focus of this chapter is to establish the best constants in higher order for Sobolev trace inequalities, in the presence of symmetries or not, on compact Riemannian manifolds under some certain conditions.

Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary. If $1 < p < n$, the Sobolev trace embeddings $W^{1,p}(M) \subset L^r(\partial M)$ are compact for any $r \in [1, \tilde{p}_1^*)$, where $\tilde{p}_1^* = p(n-1)/(n-p)$. However, the embeddings $W^{k,p}(M) \subset L^{\tilde{p}^*}(\partial M)$ are only continuous for any $1 < p < n/k$ and $\tilde{p}^* = p(n-1)/(n-kp)$.

Therefore, there exist real constants A_1, B_1 such that for any $f \in W^{k,p}(M)$,

$$\left(\int_{\partial M} |f|^{\tilde{p}^*}\, ds(g)\right)^{1/\tilde{p}^*} \leq A_1 \left(\int_M |f|^p\, dv(g)\right)^{1/p} + B_1 \left(\int_M |\Lambda_{k,g} f|^p\, dv(g)\right)^{1/p}.$$

For all $f \in W^{k,p}(M)$ there exists $C > 0$ such that

$$\|f\|^p_{L^p(M)} \leq C \left(\|f\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f\|^p_{L^p(M)}\right).$$

Hence, we conclude that there are real numbers $\tilde{A}$, $\tilde{B}$ such that

$$\|f\|^p_{L^{\tilde{p}^*}(\partial M)} \leq \tilde{A}\, \|f\|^p_{L^p(\partial M)} + \tilde{B}\, \|\Lambda_{k,g} f\|^p_{L^p(M)}. \qquad (\tilde{I}^p_{p,gen})$$

5.1 Determining the best constant $\tilde{\mathcal{B}}^p_p(M)$

The inequality $(\tilde{I}^p_{p,gen})$ is the main focus of our interest in this chapter. In fact, we find the best constant $\tilde{\mathcal{B}}^p_p(M)$ in the Sobolev trace inequalities under some certain conditions. More specifically, we prove the following:

Theorem 5.1.1. *Let (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary. Suppose that $1 < p < n/k$ with $n > k \in \mathbb{N}$, and*

$$1/\tilde{\mathcal{K}}(n,p) = \inf_{f \in L^{\tilde{p}^*}(\partial \mathbb{R}^n_+)\setminus\{0\},\ \Lambda_k f \in L^p(\mathbb{R}^n_+)} \frac{\|\Lambda_k f\|_{L^p(\mathbb{R}^n_+)}}{\|f\|_{L^{\tilde{p}^*}(\partial \mathbb{R}^n_+)}}. \qquad (5.1.1)$$

Assume

$$\|f\|_{L^{\tilde{p}^*}(B_{\delta'})} \leq \tilde{\mathcal{K}}(n,p)\, \|\Lambda_{k,g} f\|_{L^p(B_{\delta'})} \qquad (5.1.2)$$

for any $f \in W^{k,p}_c(B_{\delta'})$, where $B_{\delta'}$ is a geodesic ball with radius δ', and δ' is a small number > 0. Then $\tilde{\mathcal{B}}^p_p(M) = \tilde{\mathcal{K}}^p(n,p)$ for all $f \in W^{k,p}(M)$.

In the rest of this chapter, we denote $\tilde{\mathcal{K}}(n,p)$ by $\tilde{\mathcal{K}}$.

In order to show that $\tilde{\mathcal{K}}^p$ is the best constant $\tilde{\mathcal{B}}_p^p(M)$ for the inequality $(\tilde{I}_{p,gen}^p)$, it is enough to prove the following lemmas:

Lemma 5.1.2. *Let (M,g) be a $3 \leq n$-dimensional compact Riemannian manifold with boundary, (5.1.2) be true, and $1 < p < n/k$ with $k \in \mathbb{N}$. If there exist real constants A, B such that for any $f \in W^{k,p}(M)$,*

$$\|f\|^p_{L^{\tilde{p}^*}(\partial M)} \leq A \|f\|^p_{L^p(\partial M)} + B \|\Lambda_{k,g} f\|^p_{L^p(M)} . \tag{5.1.3}$$

Then $B \geq \tilde{\mathcal{K}}^p(n,p)$.

Proof. The proof of our result is based on the arguments used in [**9, Theorem 1**] and in [**10**]; and on the techniques used in Chapter 3. Suppose by contradiction that there exist constants A, $B \in \mathbb{R}$ with $B < \tilde{\mathcal{K}}^p$ such that (5.1.3) is satisfied for any $f \in W^{k,p}(M)$. Fix $x_0 \in \partial M$. Given $\varepsilon_1 > 0$, let $B(0,\delta) \equiv B_\delta \subset \mathbb{R}^n_+$ be the image through a chart of M of a convex neighborhood centered in x_0 such that in B_δ we have

$$\prod_{s=1}^{k} \left| g^{i_s j_s} - \delta^{i_s j_s} \right| \leq \varepsilon_1 \text{ for all } k = 1, 2, \ldots, n-1,$$

$$1 - \varepsilon_1 \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon_1 \text{ and } \left|\Gamma^l_{ij}\right| \leq \varepsilon_1 .$$

Following what we did in the proof of Lemma 3.1.2, we conclude that there exist real constants A_1, B_1, $C_{1,\delta}$ with $B_1 < \tilde{\mathcal{K}}^p$ such that for any $f \in C_c^\infty(B_\delta)$ and ε_1 small enough,

$$\|f\|^p_{L^{\tilde{p}^*}(\partial B_\delta)} \leq A_1 \|f\|^p_{L^p(\partial B_\delta)} + B_1 \|\Lambda_k f\|^p_{L^p(B_\delta)} + C_{1,\delta} \|f\|^p_{L^p(B_\delta)} \tag{5.1.4}$$

where $C_{1,\delta} = \mathrm{O}(\delta^{-p})$. Thereby, using Hölder's inequality and (5.1.4) we get

$$\begin{aligned} \|f\|^p_{L^{\tilde{p}^*}(\partial B_\delta)} \leq \; & A_1 \left(|\partial B_\delta|^{(kp-1)/(n-1)} \|f\|^p_{L^{\tilde{p}^*}(\partial B_\delta)} \right) \\ & + \left(B_1 + C_{1,\delta} \tilde{\mathcal{K}}^p \, |B_\delta|^{(kp-1)/(n-1)} \right) \|\Lambda_k f\|^p_{L^p(B_\delta)} . \end{aligned} \tag{5.1.5}$$

Choosing δ small enough, we derive that there exists $B' < \mathcal{K}^p$ such that

$$\|f\|^p_{L^{\tilde{p}^*}(\partial B_\delta)} \leq B' \, \|\Lambda_k f\|^p_{L^p(B_\delta)} .$$

For any $f \in C_c^\infty(\mathbb{R}^n_+)$, let $f^\lambda = \dfrac{f(x/\lambda)}{\lambda^{(n-1)/\tilde{p}^*}}$. In choosing λ small enough such that $f^\lambda \in C_c^\infty(B_\delta)$, we get that

$$\|f\|^p_{L^{\tilde{p}^*}(\partial \mathbb{R}^n_+)} < B' \, \|\Lambda_k f\|^p_{L^p(\mathbb{R}^n_+)}$$

contradicting the definition of $\tilde{\mathcal{K}}$. □

Lemma 5.1.3. *Let $\varepsilon > 0$, $1 < p < n/k$ with $k \in \mathbb{N}$, and (M, g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary. If (5.1.2) is true, then there exists $A = A_{M,g,\varepsilon} \in \mathbb{R}$ such that*

$$\|f\|^p_{L^{\tilde{p}^*}(\partial M)} \leq A \, \|f\|^p_{L^p(M)} + \left(\tilde{\mathcal{K}}^p + \varepsilon \right) \|\Lambda_{k,g} f\|^p_{L^p(M)} \tag{5.1.6}$$

for any $f \in W^{k,p}(M)$.

Proof. We apply the same approaches that we used in Lemma 3.1.3 to achieve Lemma 5.1.3. Given $\varepsilon_1 > 0$, we choose a finite covering of M by geodesic balls $B_j \equiv B_j(y_j)$ of radius $\delta < \delta'$ and center y_j such that: if $y_j \in M \setminus \partial M$, then $B_j \subset M \setminus \partial M$ and B_j is a normal geodesic neighborhood with normal geodesic coordinates $x_1, x_2, \cdots, x_n$; and if

$y_j \in \partial M$, then B_j is a Fermi neighborhood with Fermi coordinates $x_1, x_2, \cdots, x_{n-1}, t$. For ε_1 small enough, we have in these B_j's

$$\prod_{s=1}^{k} \left| g^{i_s j_s} - \delta^{i_s j_s} \right| \le \varepsilon_1 \text{ for all } k = 1, 2, \ldots, n-1,$$

$$1 - \varepsilon_1 \le \sqrt{\det(g_{ij})} \le 1 + \varepsilon_1 \text{ and } \left|\Gamma_{ij}^l\right| \le \varepsilon_1.$$

Let $\{\varphi_j\}$ be a partition of unity associated to the covering $\{B_j\}$ such that $\varphi_j^{1/p} \in C_c^k(B_j)$. Using arguments similar to those used in the proof of Lemma 3.1.3, and the inequality (5.1.2) plus the elementary inequality

$$(x+y)^p \le (1+\varepsilon')\, x^p + C_{\varepsilon'} y^p,$$

where $\varepsilon' = \mathrm{O}(\varepsilon_1)$, we obtain that

$$\begin{aligned}
\|f\|^p_{L^{\tilde{p}^*}(\partial M)} &\le \sum_j \left\| f\varphi_j^{1/p} \right\|^p_{L^{\tilde{p}^*}(\partial M)} \le (1+\varepsilon_2) \sum_j \left\| f\varphi_j^{1/p} \right\|^p_{L^{\tilde{p}^*}(\partial B_j)} \\
&\le (1+\varepsilon_3)\, \tilde{\mathcal{K}}^p \sum_j \left(\left\| \Lambda_{k,g}(f\varphi_j^{1/p}) \right\|^p_{L^p(B_j)} + C \|f\|^p_{L^p(B_j)} \right) \\
&\le (1+\varepsilon_4)\, \tilde{\mathcal{K}}^p \sum_j \left\| \Lambda_{k,g}(f\varphi_j^{1/p}) \right\|^p_{L^p(M)} + C \|f\|^p_{L^p(M)} \\
&\le (1+\varepsilon_5)\, \tilde{\mathcal{K}}^p \left\| \Lambda_{k,g} f \right\|^p_{L^p(M)} + C \left[\sum_{i=2}^{k-1} \left\| \Lambda_{i,g} f \right\|^p_{L^p(M)} \right. \\
&\quad \left. + \left\| \nabla_g^1 f \right\|^p_{L^p(M)} + \|f\|^p_{L^p(M)} \right],
\end{aligned}$$

where $\varepsilon_m = \mathrm{O}(\varepsilon_1)$ for $m = 2, 3, 4, 5$.

Following the same technique used in Lemma 3.1.3, we get

$$\left\|\nabla_g^1 f\right\|_{L^p(M)}^p \leq \alpha_1 C_{\varepsilon_1,\delta} \left\|\Lambda_{2,g} f\right\|_{L^p(M)}^p + C_{\varepsilon_1,\alpha_1,\delta} \left\|f\right\|_{L^p(M)}^p, \tag{5.1.7}$$

and for $i = 2, \cdots, k-1$ we have

$$\left\|\Lambda_{i,g} f\right\|_{L^p(M)}^p \leq \alpha_i C_{i,\varepsilon_1,\alpha_i,\delta} \left\|\Lambda_{i+1,g} f\right\|_{L^p(M)}^p + C_{\varepsilon_1,\alpha_i,\delta} \left\|f\right\|_{L^p(M)}^p, \tag{5.1.8}$$

where ε_1 and α_i's are sufficiently small. Thus,

$$\begin{aligned}
\left\|f\right\|_{L^{\tilde{p}^*}(\partial M)}^p &\leq \left(\tilde{\mathcal{K}}^p + \frac{\varepsilon}{2}\right) \left\|\Lambda_{k,g} f\right\|_{L^p(M)}^p + C_1 \left\|\Lambda_{k-1,g} f\right\|_{L^p(M)}^p + C_2 \left\|f\right\|_{L^p(M)}^p \\
&\leq \left(\tilde{\mathcal{K}}^p + \frac{\varepsilon}{2}\right) \left\|\Lambda_{k,g} f\right\|_{L^p(M)}^p + \alpha_{k-1}\tilde{C} \left\|\Lambda_{k,g} f\right\|_{L^p(M)}^p + C_3 \left\|f\right\|_{L^p(M)}^p.
\end{aligned}$$

We finish the proof of this Lemma by choosing α_{k-1} small enough such that $\alpha_{k-1} \leq \varepsilon/(2\tilde{C})$. Therefore, we obtain

$$\left\|f\right\|_{L^{\tilde{p}^*}(\partial M)}^p \leq \left(\tilde{\mathcal{K}}^p + \varepsilon\right) \left\|\Lambda_{k,g} f\right\|_{L^p(M)}^p + B \left\|f\right\|_{L^p(M)}^p$$

for all $f \in W^{k,p}(M)$. □

Theorem 5.1.4 (Concentration-compactness principle for manifolds with boundary). *For $1 < p < n/k$ with $n > k \in \mathbb{N}$, let $p^* = np/(n-kp)$ and $\tilde{p}^* = p(n-1)/(n-kp)$, and let $\mathcal{K}(n,p)$, $\tilde{\mathcal{K}}$ be as above. Suppose that (M,g) is a compact Riemannian n-manifold with boundary and let $f_j \rightharpoonup f$ in $W^{k,p}(M)$ satisfy*

$$\begin{aligned}
\left|\Lambda_{k,g} f_j\right|^p dv(g) &\rightharpoonup \mu, \\
\left|f_j\right|^{p^*} dv(g) &\rightharpoonup \nu, \\
\left|f_j\right|^{\tilde{p}^*} ds(g) &\rightharpoonup \pi.
\end{aligned}$$

where μ, ν, π are non-negative measures. Then there exist at most a countable set J, $\{x_j\}_{j\in J} \subset M$, and positive numbers $\{\alpha_j\}_{j\in J}$, $\{\beta_j\}_{j\in J}$, $\{\gamma_j\}_{j\in J}$ such that

$$\begin{aligned}
\mu &\geq |\Lambda_{k,g} f|^p \, dv(g) + \sum_{j\in J} \beta_j \delta_{x_j}, \\
\nu &= |f|^{p^*} \, dv(g) + \sum_{j\in J} \alpha_j \delta_{x_j}, \\
\pi &= |f|^{\bar{p}^*} \, ds(g) + \sum_{j\in J} \gamma_j \delta_{x_j}
\end{aligned}$$

with

$$\begin{aligned}
\alpha_j^{1/p^*} &\leq \mathcal{K} \beta_j^{1/p}, \\
\gamma_j^{1/\bar{p}^*} &\leq \tilde{\mathcal{K}} \beta_j^{1/p}.
\end{aligned}$$

Proof. Following the standard argument of Lions used in **[34]**, and using the inequalities

$$\|f\|^p_{L^{p^*}(M)} \leq A \|f\|^p_{L^p(M)} + \left(\mathcal{K}^p(n,p) + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)},$$

$$\|f\|^p_{L^{\bar{p}^*}(\partial M)} \leq B \|f\|^p_{L^p(M)} + \left(\tilde{\mathcal{K}}^p + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)},$$

we obtain Theorem 5.1.4. □

Next lemma completes the proof of Theorem 5.1.1.

Lemma 5.1.5. *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold with boundary, and let $1 < p < n/k$ with $n > k \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists $A = A_{M,g,\varepsilon} \in \mathbb{R}$ such that*

$$\|f\|^p_{L^{\bar{p}^*}(\partial M)} \leq A \|f\|^p_{L^p(\partial M)} + \left(\tilde{\mathcal{K}}^p + \varepsilon\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} \tag{5.1.9}$$

for any $f \in W^{k,p}(M)$.

Proof. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that for any $\tilde{A} > 0$ we can find $f \in W^{k,p}(M)$ with

$$\|f\|^p_{L^{\bar{p}^*}(\partial M)} > \left(\tilde{\mathcal{K}}^p + \varepsilon_0\right) \|\Lambda_{k,g} f\|^p_{L^p(M)} + \tilde{A} \|f\|^p_{L^p(\partial M)}.$$

This means that there exists $\varepsilon_0 > 0$ such that for any $A > 0$ we can find $f \in W^{k,p}(M)$ with the property

$$J_A = \frac{A \|f\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f\|^p_{L^p(M)}}{\|f\|^p_{L^{\bar{p}^*}(\partial M)}} < \frac{1}{\tilde{\mathcal{K}}^p + \varepsilon_0}. \tag{5.1.10}$$

For any $A > 1$ define

$$I_A = \inf_{f \in W^{k,p}(M) \setminus \{0\}} J_A < \frac{1}{\tilde{\mathcal{K}}^p + \varepsilon_0}. \tag{5.1.11}$$

As the quotient J_A is homogeneous, for any fixed A we can take a minimizing sequence $f_j \in W^{k,p}(M)$ satisfying $\|f_j\|^p_{L^{\bar{p}^*}(\partial M)} = 1$. As

$$A \|f_j\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f_j\|^p_{L^p(M)} \to I_A, \tag{5.1.12}$$

we conclude that $\Lambda_{k,g} f_j$ is bounded in $L^p(M)$, and f_j is bounded in $L^p(\partial M)$. Thereby, $\{f_j\}$ is bounded in $W^{k,p}(M)$. So we may assume, up to a subsequence, that $f_j \rightharpoonup f$ in $W^{k,p}(M)$ and $f_j \to f$ in $L^p(M)$ as well as in $L^p(\partial M)$. In particular, we have for some bounded, non-negative measures μ, ν, π that

$$\begin{aligned} |\Lambda_{k,g} f_j|^p \, dv(g) &\rightharpoonup \mu, \\ |f_j|^{\tilde{p}^*} \, ds(g) &\rightharpoonup \pi, \\ |f_j|^{p^*} \, dv(g) &\rightharpoonup \nu. \end{aligned} \tag{5.1.13}$$

Now, we are going to show that $f_j \to f$ in $L^{\tilde{p}^*}(\partial M)$ and $\|f\|^p_{L^{\tilde{p}^*}(\partial M)} = 1$. To achieve this result, we follow the manner [9] applied in the proof of Theorem 4.

Define $g : W^{k,p}(M) \to \mathbb{R}$ by

$$g(f) = \left(\int_{\partial M} |f|^{\tilde{p}^*} \, ds(g) \right)^{p/p^*}, \tag{5.1.14}$$

and also define $G : \mathcal{M} \to \mathbb{R}$ by

$$G(f) = \frac{1}{p} \int_M |\Lambda_{k,g} f|^p \, dv(g) + \frac{A}{p} \int_{\partial M} |f|^p \, ds(g), \tag{5.1.15}$$

where

$$\mathcal{M} = \left\{ f \in W^{k,p}(M) : g(f) = 1 \right\}.$$

The tangent space, $T_{f_j}\mathcal{M}$, consists exactly of those functions φ such that $Dg(f_j)\varphi = 0$. Therefore, we obtain that $\int_{\partial M} |f_j|^{\tilde{p}^*-2} f_j \varphi \, ds(g) = 0$. So, we can write

$$W^{k,p}(M) = T_{f_j}\mathcal{M} \oplus \langle f_j \rangle .$$

Thus, for any $\psi \in W^{k,p}(M)$, write

$$\psi = (\psi - \gamma f_j) + \gamma f_j \in T_{f_j}\mathcal{M} \oplus \langle f_j \rangle, \tag{5.1.16}$$

where

$$\gamma = \int_{\partial M} |f_j|^{\bar{p}^*-2} f_j \psi ds(g).$$

Take $x_k \in M$ in the support of the singular parts of μ, ν, and π. Let $\varphi \in C_c^\infty(M)$ satisfying $supp\,\varphi \subset B_{2\varepsilon}(x_k)$, $\varphi \equiv 1$ on $B_\varepsilon(x_k)$, and $|\Lambda_{k,g}\varphi| < 2/\varepsilon$, $\left|\nabla_g^i \varphi\right| < 2/\varepsilon$ for $i = 1, \cdots, k-1$.

As φf_j is bounded in $W^{k,p}(M)$, we have

$$\zeta_j = \left[\varphi - \int_{\partial M} |f_j|^{p^*} \varphi ds(g)\right] f_j \in T_{f_j}\mathcal{M}$$

is a bounded sequence in $W^{k,p}(M)$. On one hand, following the manner used in [**9, Theorem 4**], we get

$$\lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}\varphi dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j \varphi ds(g) \to 0. \qquad (5.1.17)$$

Writing $L = \int_{\partial M} |f_j|^{\bar{p}^*} \varphi ds(g)$, we have that

$$\begin{aligned}
&\lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}(\varphi f_j) dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j\,(\varphi f_j)\,ds(g) \\
&= \lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}(\zeta_j + L f_j) dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j(\zeta_j + L f_j) ds(g) \\
&= \lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}\zeta_j dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j\zeta_j ds(g) \\
&\quad + \lim_{j\to\infty}\left(\int_{\partial M} |f_j|^{p^*} \varphi ds(g)\right)\left[\int_M |\Lambda_{k,g} f_j|^{p} dv(g) + A\int_{\partial M} |f_j|^{p} ds(g)\right].
\end{aligned}$$

Therefore, by (5.1.16)-(5.1.17),

$$\begin{aligned}
&\lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}(\varphi f_j) dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j\,(\varphi f_j)\,ds(g) \\
&= \lim_{j\to\infty}\left(\int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j)\Lambda_{k,g}\zeta_j dv(g)\right) + A\int_{\partial M} |f_j|^{p-2} f_j\zeta_j ds(g) \\
&\quad + I_A \lim_{j\to\infty}\left(\int_{\partial M} |f_j|^{\bar{p}^*} \varphi ds(g)\right) \\
&= I_A \int_{\partial M} \varphi d\pi. \qquad (5.1.18)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& \lim_{j\to\infty}\left(\int_M \left|\Lambda_{k,g} f_j\right|^{p-2}(\Lambda_{k,g} f_j)\Lambda_{k,g}(\varphi f_j)dv(g)\right) + A\int_{\partial M} \left|f_j\right|^{p-2} f_j\,(\varphi f_j)\,ds(g) \\
= \; & \lim_{j\to\infty}\int_M \left|\Lambda_{k,g} f_j\right|^{p-2}(\Lambda_{k,g} f_j)\left[(\Lambda_{k,g} f_j)\varphi + \sum_{i=1}^{k-1}\binom{k}{i}\nabla_g^{k-i} f_j\cdot\nabla_g^{i}\varphi\right. \\
& \left. + f_j(\Lambda_{k,g}\varphi)\right] dv(g) + A\int_{\partial M} \left|f_j\right|^{p}\varphi ds(g) \\
= \; & \lim_{j\to\infty}(A\int_{\partial M}\left|f_j\right|^p \varphi ds(g) + \int_M \varphi d\mu + \sum_{i=1}^{k}\Upsilon_{i,j}) \\
= \; & I_A\int_{\partial M}\varphi d\pi,
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon_{i,j} &= \binom{k}{i}\int_M \left|\Lambda_{k,g} f_j\right|^{p-2}(\Lambda_{k,g} f_j)\left(\nabla_g^{k-i} f_j\cdot\nabla_g^{i}\varphi\right)dv(g) \text{ for } i = 1,\cdots,k-1, \\
\Upsilon_{k,j} &= \int_M \left|\Lambda_{k,g} f_j\right|^{p-2}(\Lambda_{k,g} f_j) f_j(\Lambda_{k,g}\varphi)dv(g).
\end{aligned}
$$

With $p^* = np/(n-kp)$, $s = np/(n-p)$, $r = \frac{p^*}{p}$, and $t = \frac{s}{p}$, we obtain that

$$\begin{aligned}
\lim_{j\to\infty} \Upsilon_{k,j} &= \lim_{j\to\infty} \int_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j) f_j (\Lambda_{k,g}\varphi) dv(g) \\
&\le \lim_{j\to\infty} \int_M |\Lambda_{k,g} f_j|^{p-1} |f_j| \, |\Lambda_{k,g}\varphi| \, dv(g) \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left(\int_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |f_j|^p \, |\Lambda_{k,g}\varphi|^p \, dv(g) \right)^{1/p} \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left\| |f_j|^p \right\|_{L^r(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))}^{1/p} \left\| |\Lambda_{k,g}\varphi|^p \right\|_{L^{r'}(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))}^{1/p} \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left(\int_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |f_j|^{p^*} \, dv(g) \right)^{1/p^*} \\
&\times \left(\int_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |\Lambda_{k,g}\varphi|^{n/k} \, dv(g) \right)^{k/n} \\
&\le C \lim_{j\to\infty} \left\{ \frac{1}{\varepsilon^{n/k}} Vol(B_{2\varepsilon}(x_k) \setminus B_\varepsilon(x_k)) \right\}^{k/n} \|f_j\|_{L^{p^*}(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))} .
\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} \Upsilon_{k,j} \to 0 \text{ as } \varepsilon \to 0.$$

Similarly, we derive that

$$
\begin{aligned}
\lim_{j\to\infty} \Upsilon_{i,j} &= \binom{k}{i} \lim_{j\to\infty} \int\limits_M |\Lambda_{k,g} f_j|^{p-2} (\Lambda_{k,g} f_j) \left(\nabla_g^{k-i} f_j \cdot \nabla_g^i \varphi\right) dv(g) \\
&\le C \lim_{j\to\infty} \int\limits_M |\Lambda_{k,g} f_j|^{p-1} \left|\nabla_g^{k-i} f_j \cdot \nabla_g^i \varphi\right| dv(g) \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} \left|\nabla_g^{k-i} f_j\right|^p \left|\nabla_g^i \varphi\right|^p dv(g) \right)^{1/p} \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left\| \left|\nabla_g^{k-i} f_j\right|^p \right\|_{L^t(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))}^{1/p} \left\| \left|\nabla_g^i \varphi\right|^p \right\|_{L^{t'}(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))}^{1/p} \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} \left|\nabla_g^{k-i} f_j\right|^s dv(g) \right)^{1/s} \\
&\times \left(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} \left|\nabla_g^i \varphi\right|^n dv(g) \right)^{1/n} \\
&\le C \lim_{j\to\infty} \|\Lambda_{k,g} f_j\|_{L^p(M)}^{p-1} \left\| \nabla_g^{k-i} f_j \right\|_{L^s(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k))},
\end{aligned}
$$

which approaches to 0 when ε goes to 0. Hence, sending ε to 0, we obtain that

$$
\beta_j = I_A \gamma_j \quad \text{and} \quad \tilde{\mathcal{K}} \beta_j^{1/p} \ge \gamma_j^{1/\tilde{p}^*}, \tag{5.1.19}
$$

which leads to

$$
\beta_j^{1-p/\tilde{p}^*} \ge \left(\frac{1}{I_A}\right)^{p/\tilde{p}^*} \frac{1}{\tilde{\mathcal{K}}^p}.
$$

Since $I_A < \frac{1}{\tilde{\mathcal{K}}^p}$, we get that $\left(\frac{1}{I_A}\right)^{p/\tilde{p}^*} > \tilde{\mathcal{K}}^{(p^2/\tilde{p}^*)}$, which yields

$$\beta_j > \frac{1}{\tilde{\mathcal{K}}^p}. \tag{5.1.20}$$

As μ is a bounded measure, the nonzero terms of β_j are only finite numbers. Moreover, if $\beta_j \neq 0$ for some $j \in \mathbb{N}$, then we obtain that

$$\begin{aligned}
\frac{1}{\tilde{\mathcal{K}}^p} &> \frac{1}{\left(\tilde{\mathcal{K}}^p + \varepsilon_0\right)} > I_A \\
&= \lim_{j\to\infty}\left(A\,\|f_j\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f_j\|^p_{L^p(M)}\right) \\
&\geq A\,\|f\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f\|^p_{L^p(M)} + \sum \beta_j \\
&\geq \sum \beta_j \geq \beta_j > \frac{1}{\tilde{\mathcal{K}}^p},
\end{aligned}$$

which is a contradiction. Therefore, $\beta_j = 0$ for all $j \in \mathbb{N}$; hence, $\gamma_j = 0$ for all $j \in \mathbb{N}$, which follows that

$$f_j \longrightarrow f \text{ in } L^{p^*}(\partial M) \text{ and } \|f\|^p_{p^*(\partial M)} = 1. \tag{5.1.21}$$

Indeed, we have that

$$\begin{aligned}
A\,\|f\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f\|^p_{L^p(M)} &\leq A\,\|f\|^p_{L^p(\partial M)} + \liminf_{j\to\infty} \|\Lambda_{k,g} f_j\|^p_{L^p(M)} \\
&\leq \liminf_{j\to\infty}\left(A\,\|f_j\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f_j\|^p_{L^p(M)}\right) \\
&= I_A.
\end{aligned} \tag{5.1.22}$$

In fact, (5.1.11) with (5.1.22) implies that f is a minimizing of I_A.

For any $A > 0$, let $f_A \in W^{k,p}(M)$ satisfies $\|f_A\|^p_{L^{p^*}(\partial M)} = 1$ and

$$I_A = A\|f_A\|^p_{L^p(\partial M)} + \|\Lambda_{k,g} f_A\|^p_{L^p(M)} \leq \frac{1}{\tilde{\mathcal{K}}^p + \varepsilon_0}. \tag{5.1.23}$$

On one hand, following the same arguments as above, we obtain the boundedness of f_A in $W^{k,p}(M)$. So, we can assume, up to a subsequence, that $f_A \rightharpoonup f$ in $W^{k,p}(M)$, and $f_A \to f$ in $L^p(M)$ as well as in $L^p(\partial M)$. On the other hand, Letting $A \to \infty$ in (5.1.23), we conclude

$$f = 0 \quad \text{on } \partial M. \tag{5.1.24}$$

For any $f \in W^{1,p}(M)$, **[9]** proved that

$$\nabla_g f_A \to \nabla_g f \;\; a.e. \text{ on } M. \tag{5.1.25}$$

Since $\nabla_g f_A \in W^{k-1,p}(M)$, we can assume, up to a subsequence, that $\nabla_g f_A \rightharpoonup h$ in $W^{k-1,p}(M)$, and $\nabla_g f_A \to h$ in $L^p(M)$ for some $h \in W^{k-1,p}(M)$. Hence, there exists a subsequence $\{f_{A_m}\} \subset \{f_A\}$ such that

$$\nabla_g f_{A_m} \to h \;\; a.e., \tag{5.1.26}$$

where $h = \nabla_g f$ *a.e.* Similarly, for $i = 1, ..., k$ we get

$$\nabla^i_g f_{A_s} \to h_i \text{ in } L^p(M) \quad \text{and} \quad \nabla^i_g f_{A_s} \to h_i \;\; a.e.,$$

where $h_i = \nabla_g^i f$ *a.e.* Thus,

$$\begin{aligned}
\left\|\Lambda_{k,g} f_{A_s} - \Lambda_{k,g} f\right\|_{L^p(M)} &\leq \left\|\Lambda_{k,g} f_{A_s} - \Lambda_{k,g} f\right\|_{L^p(M)} + \left\|f_{A_s} - f\right\|_{L^p(M)} \\
&\leq C \sum_{i=0}^{k} \left\|\nabla_g^i f_{A_s} - \nabla_g^i f\right\|_{L^p(M)} \\
&\leq C \sum_{i=0}^{k} \left\|\nabla_g^i f_{A_s} - h_i\right\|_{L^p(M)} + \left\|h_i - \nabla_g^i f\right\|_{L^p(M)} \to 0.
\end{aligned}$$

Therefore, there is a subsequence $\{f_{A_{s_i}}\}$ of $\{f_{A_s}\}$ such that

$$\Lambda_{k,g} f_{A_{s_i}} \to \Lambda_{k,g} f \quad a.e., \text{ on } M.$$

Lemma 5.1.3 plus [**14, Theorem 1**] gives that

$$\begin{aligned}
\frac{1}{\mathcal{K}^p + \varepsilon_0} &\geq I_A \geq \left\|\Lambda_{k,g} f_{A_{s_i}}\right\|_{L^p(M)}^p \\
&\geq \left\|\Lambda_{k,g}\left(f_{A_{s_i}} - f\right)\right\|_{L^p(M)}^p + o(1) \\
&\geq \frac{1}{\tilde{\mathcal{K}}^p + \varepsilon}\left(\left\|f_{A_{s_i}} - f\right\|_{L^{p^*}(\partial M)}^p - C_\varepsilon \left\|f_{A_{s_i}} - f\right\|_{L^p(M)}^p\right) + o(1)
\end{aligned}$$

for any $\varepsilon > 0$. Take $0 < \varepsilon < \varepsilon_0$ and let $A_{s_i} \to \infty$, we find, by using (5.1.24) and $\left\|f_{A_{s_i}}\right\|_{L^{p^*}(\partial M)}^p = 1$, that

$$\frac{1}{\mathcal{K}^p + \varepsilon_0} > \frac{1}{\mathcal{K}^p + \varepsilon},$$

which is a contradiction. Consequently, the inequality (5.1.9) is satisfied for any $f \in W^{k,p}(M)$. This completes the proof of Lemma 5.1.5. $\square$

Theorem 5.1.1 is obtained by a direct consequence of Lemmas 5.1.2 and 5.1.5.

5.2 Determining the best constant $\tilde{\beta}_p^p(M)$

In this section, under some specific conditions, we find the best constants in higher-order Sobolev trace inequalities on Riemannian manifolds with boundary in the presence of symmetries.

Let (M,g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold with boundary, G-invariant under the action of a subgroup G of the isometry group $Is(M,g)$. Assume that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. If $1 < p < (n-l)/k$ with $k \in \mathbb{N}$, and $\tilde{q} = \frac{(n-l-1)p}{n-l-kp}$, the embedding $W^{k,p}(M) \subset L^{\tilde{q}}(\partial M)$ is continuous. Thus, there exist real constants $\tilde{A}_1$, $\tilde{B}_1$ such that

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq \tilde{A}_1 \|f\|^p_{L^p_G(\partial M)} + \tilde{B}_1 \|\Lambda_{k,g} f\|^p_{L^p_G(M)} \qquad (\tilde{I}^p_{p,gen,G})$$

for any $f \in W^{k,p}(M)$. Let

$$\tilde{\alpha}_p^p(M) = \inf\left\{\tilde{A}_1 \in \mathbb{R} : \text{there exists } \tilde{B}_1 \in \mathbb{R} \text{ such that } \tilde{I}^p_{p,gen,G} \text{ is satisfied}\right\},$$

and

$$\tilde{\beta}_p^p(M) = \inf\left\{\tilde{B}_1 \in \mathbb{R} : \text{there exists } \tilde{A}_1 \in \mathbb{R} \text{ such that } \tilde{I}^p_{p,gen,G} \text{ is satisfied}\right\}.$$

Then we have the following:

Theorem 5.2.1. *Let (M,g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold with boundary, G-invariant under the action of a subgroup G of the isometry group $Is(M,g)$. Assume that l is the minimum orbit dimension of G, and V is the* *minimum of the volume of the l-dimensional orbits. Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$, and $\tilde{q} = \frac{(n-l-1)p}{n-l-kp}$. Then for any $\varepsilon > 0$ there exists real constant A such that*

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq A \|f\|^p_{L^p_G(\partial M)} + \left(\frac{\tilde{\mathcal{K}}^p(n-l,p)}{V^{(kp-1)/(n-l-1)}} + \varepsilon \right) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} \tag{5.2.1}$$

for any $f \in W^{k,p}_G(M)$.
Moreover, the constant $\tilde{\mathcal{K}}^p_G = \frac{\tilde{\mathcal{K}}^p(n-l,p)}{V^{(kp-1)/(n-l-1)}}$ is the best constant for this inequality.

Following arguments similar to those used in the proofs of Lemmas 4.1.3 and 5.1.2, we obtain that if there exist real constants A and B such that for any $f \in W^{k,p}_G(M)$,

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq A \|f\|^p_{L^p_G(\partial M)} + B \|\Lambda_{k,g} f\|^p_{L^p_G(M)}, \tag{5.2.2}$$

then $\tilde{\mathcal{K}}^p_G \geq B$.

To prove the validity of (5.2.1), we need the following propositions which the first three can be established via the arguments used in **[17]**.

Proposition 5.2.2. *Let (M,g), G, l, V be as above. Let $1 < p < (n-l)/k$ with $k \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists real constant $A = A_{\varepsilon,p,G,M}$ such that*

$$\|f\|^p_{L^{\tilde{q}}_G(\partial M)} \leq A \|f\|^p_{L^p_G(M)} + \left(\tilde{\mathcal{K}}^p_G + \varepsilon \right) \|\Lambda_{k,g} f\|^p_{L^p_G(M)} \tag{5.2.3}$$

for any $f \in W^{k,p}_G(M)$.

Proposition 5.2.3. *Let (M,g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold with boundary, and let $O_j = \{y \in M \mid d_g(y, O^x_G) < \delta\}$. Then for any $\varepsilon >$*

0 *there exists a positive constant* c *such that the following are valid for any* $\upsilon \in W_G^{k,p}(O_j \cap \partial M)$ *such that* $\upsilon \geq 0$:

(i) $(1 - c\varepsilon_0)V_j \int\limits_{\partial N} \upsilon_2 ds(\tilde{g}) \leq \int\limits_{\partial M} \upsilon ds(g) \leq (1 + c\varepsilon_0)V_j \int\limits_{\partial N} \upsilon_2 ds(\tilde{g})$,

(ii) $(1 - c\varepsilon_0)V_j \int\limits_{\partial\mathbb{R}_+^{n-l}} \upsilon_2 dy' \leq \int\limits_{\partial M} \upsilon ds(g) \leq (1 + c\varepsilon_0)V_j \int\limits_{\partial\mathbb{R}_+^{n-l}} \upsilon_2 dy'$,

where $V_j = Vol(O_j)$, $\upsilon_2 = \upsilon \circ \Psi^{-1}$, *and* $(N, \tilde{g})$ *is a compact submanifold of* (M, g) *of dimension* $n - l$ *as in* ***[17, Theorem 3.2]***.

Proposition 5.2.4. *Let* (M, g), O_j, N, υ_2, *and* V_j *be as above. Then for any* $\varepsilon > 0$ *there exists a positive constant* c *such that the following are valid:*

(i) $(1 - c\varepsilon_0)V_j \int\limits_N |\Lambda_{k,\tilde{g}}\upsilon_2|^p dv(\tilde{g}) \leq \int\limits_M |\Lambda_{k,g}\upsilon|^p dv(g) \leq (1 + c\varepsilon_0)V_j \int\limits_N |\Lambda_{k,\tilde{g}}\upsilon_2|^p dv(\tilde{g})$

(ii) $(1-c\varepsilon_0)V_j \int\limits_{\mathbb{R}_+^{n-l}} |\Lambda_k\upsilon_2|^p dy'dt \leq \int\limits_M |\Lambda_{k,g}\upsilon|^p dv(g) \leq (1+c\varepsilon_0)V_j \int\limits_{\mathbb{R}_+^{n-l}} |\Lambda_k\upsilon_2|^p dy'dt$.

for any $\upsilon \in W_G^{k,p}(O_j \cap M)$ with $\upsilon \geq 0$.

Proposition 5.2.5 ([25]). *Let* (M, g) *be a smooth compact* $3 \leq n$*-dimensional Riemannian manifold, and* G *be a compact subgroup of the isometry group of* M. *Then there exists an orbit of minimum dimension* l, *and of minimum volume.*

Proof of Theorem 5.2.1. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for any $A > 0$ we can find $f \in W_G^{k,p}(M)$ such that

$$\tilde{J}_A = \frac{A \|f\|_{L_G^p(\partial M)}^p + \|\Lambda_{k,g} f\|_{L_G^p(M)}^p}{\|f\|_{L_G^{\tilde{q}}(\partial M)}^p} < \frac{1}{\tilde{\mathcal{K}}_G^p + \varepsilon_0}. \tag{5.2.4}$$

Let, for any $A > 1$,

$$\tilde{I}_{A} = \inf_{f \in W_G^{k,p}(M)\setminus\{0\}} \tilde{J}_{A} < \frac{1}{\tilde{\mathcal{K}}_G^p + \varepsilon_0}. \tag{5.2.5}$$

As the quotient $\tilde{J}_{A}$ is homogeneous, for any fixed $A > 0$ we can take a minimizing sequence $\{\tilde{f}_j\} \subset W_G^{k,p}(M)$ satisfying $\left\|\tilde{f}_j\right\|^p_{L^{\bar{q}}(\partial M)} = 1$ such that

$$A\left\|\tilde{f}_j\right\|^p_{L^p_G(\partial M)} + \left\|\Lambda_{k,g}\tilde{f}_j\right\|^p_{L^p_G(M)} \to \tilde{I}_{A}. \tag{5.2.6}$$

Following the same manner used in the proof of Lemma 5.1.5, we derive that

$$\tilde{f}_j \to f \text{ in } L^{\bar{q}}_G(\partial M), \ \|f\|^p_{L^{\bar{q}}_G(\partial M)} = 1,$$

and

$$A\|f\|^p_{L^p_G(\partial M)} + \|\Lambda_{k,g}f\|^p_{L^p_G(M)} = \tilde{I}_{A}. \tag{5.2.7}$$

Moreover, we obtain that for any $A > 0$ there exists a function $\tilde{f}_{A} \in W_G^{k,p}(M)$ such that $\left\|\tilde{f}_{A}\right\|^p_{L^{\bar{q}}_G(\partial M)} = 1$ and

$$\tilde{I}_{A} = A\left\|\tilde{f}_{A}\right\|^p_{L^p_G(\partial M)} + \left\|\Lambda_{k,g}\tilde{f}_{A}\right\|^p_{L^p_G(M)} < \frac{1}{\left(\tilde{\mathcal{K}}_G^p + \varepsilon_0\right)}. \tag{5.2.8}$$

Furthermore, we find that there is a subsequence $\{\tilde{f}_{A_{s_i}}\}$ of $\{\tilde{f}_{A}\}$ such that $\Lambda_{k,g}\tilde{f}_{A_{s_i}} \in L^p_G(M)$ and

$$\Lambda_{k,g}\tilde{f}_{A_{s_i}} \to \Lambda_{k,g}\tilde{f} \ \ a.e., \text{ on } M.$$

Let $\mathcal{O} \in \bigcup_{j=1}^{N} O_j$ such that for all $j = 1, \ldots, N$,

$$Vol(\mathcal{O}) = min\, Vol(O_j) = V.$$

Assume that $\tilde{f}_{A_{s_i}} \in W_G^{k,p}(\mathcal{O})$. As the best constant of the Sobolev trace inequality in M has the same value with the best constant in the Sobolev trace inequality of the manifold $\mathcal{O}$ (c.f. Proposition 5.2.5), then by using propositions 5.2.3 and 5.2.4, we obtain

$$\begin{aligned} \frac{A\left\|\tilde{f}_{A_{s_i}}\right\|^p_{L^p(\partial M)} + \left\|\Lambda_{k,g}\tilde{f}_{A_{s_i}}\right\|^p_{L^p(M)}}{\left\|\tilde{f}_{A_{s_i}}\right\|^p_{L^{\bar{q}}(\partial M)}} &\geq (1 - c\varepsilon_0)\, V^{\frac{kp-1}{n-l-1}} \\ &\times \frac{A\left\|\tilde{f}_{2,A_{s_i}}\right\|^p_{L^p(\partial N)} + \left\|\Lambda_{k,g}\tilde{f}_{2,A_{s_i}}\right\|^p_{L^p(N)}}{\left\|\tilde{f}_{2,A_{s_i}}\right\|^p_{L^{\bar{q}}(\partial N)}}, \end{aligned}$$

where N is a submanifold of M of dimension $n - l$ as above. Therefore,

$$\begin{aligned} \frac{A\left\|\tilde{f}_{2,A_{s_i}}\right\|^p_{L^p(\partial N)} + \left\|\Lambda_{k,g}\tilde{f}_{2,A_{s_i}}\right\|^p_{L^p(N)}}{\left\|\tilde{f}_{2,A_{s_i}}\right\|^p_{L^{\bar{q}}(\partial N)}} &< \frac{1}{\tilde{\mathcal{K}}_G^p + \varepsilon_0} \times \frac{1}{V^{\frac{kp-1}{n-l-1}}} \\ &= \frac{1}{V^{\frac{kp-1}{n-l-1}}\tilde{\mathcal{K}}_G^p + V^{\frac{kp-1}{n-l-1}}\varepsilon_0} \\ &< \frac{1}{V^{\frac{kp-1}{n-l-1}}\tilde{\mathcal{K}}_G^p} = \frac{1}{\tilde{\mathcal{K}}^p(n-l,p)}, \end{aligned} \tag{5.2.9}$$

which contradicts what we did in Lemma 5.1.5. Consequently, the inequality (5.2.4) is false, and then Theorem 5.2.1 is proved.

Bibliography

[1] D. Adams and L. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag Berlin Heidelberg, 1996.

[2] M. Ahmedou, Z. Djadli, and A. Malchiodi, *Prescribed fourth order conformal invariant on the standard sphere*. Part I. A perturbation result. Commun. Contemp. Math. **3** (2002) 375-408.

[3] T. Aubin, *Espaces de Sobolev sur les variétés Riemanniennes*, Bull. Sc. Math. **100** (1976) 149-173.

[4] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differ. Geom. **11** (1976) 573-598.

[5] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, 1998.

[6] T. Aubin, Y. Li, *On the best Sobolev inequality*, J. Math. Pures Appl. **4** (1999) 353-387.

[7] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroups*, in: Lectures on Probability Theory, Saint-Flour (1992), in: Lecture Notes in Math. Springer-Verlag Berlin Heidelberg **1581** (1994) 1-114.

[8] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. **138** (1993) 213-242.

[9] R. Biezuner, *Best constants in Sobolev trace inequalities*, Nonlinear Analysis **54** (2003) 575-589.

[10] R. Biezuner and M. Montenegro, *Best constants in second-order Sobolev inequalities on Riemannian manifolds and applications*, J. Math. Pures Appl. **82** (2003) 457-502.

[11] M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Academic Press Inc., USA, 1975.

[12] T. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987) 199-291.

[13] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[14] W. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983) 486-490.

Spectral theory and geometry

[15] F. Burstall, *Basic Riemannian geometry*, in: Spectral theory and geometry: ICMS instructional conference, Edinburgh 1998, in Series: London Mathematical Society Lecture note series, Cambridge University Press **273** (1999) 1-29.

[16] P. Cherrier, *Meilleures constantes dans des inégalités relatives aux espaces de Sobolev*, Bull. Sci. Math. **108** (1984) 225-262.

[17] A. Cotsiolis and N. Labropoulos, *Best constants in Sobolev inequalities on manifolds with boundary in the presence of symmetries and applications*, Bull. Sci. Math. **132** (2008) 562-574.

[18] Z. Djadli, E. Hebey, and M. Ledoux, *Paneitz-type operators and applications.* Duke Math. J. **1** (2000) 129-169.

[19] O. Druet, *Best constants in Sobolev inequalities*, C. R. Acad. Sci. Paris **326** (1998) 965-969.

[20] O. Druet, *Optimal Sobolev inequalities of arbitrary order on compact Riemannian manifolds*, J. Funct. Anal. **159** (1998) 217-242.

[21] O. Druet, E. Hebey, and F. Robert, *Blow-up Theory for Elliptic PDEs in Riemannian Geometry*, Princeton University Press, 2004.

[22] J. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988) 687-698.

[23] L. Evans and R. Gariepy, *Measure Theory and Fine Prosperities of Functions*, CRC Press LLC, USA, 1992.

[24] Z. Faget, *Best constants in Sobolev inequalities on Riemannian manifolds in the presence of symmetries*, Potential Analysis **17** (2000) 105-124.

[25] Z. Faget, *Optimal constants in critical Sobolev inequalities on Riemannian manifolds in the presence of symmetries*, Annals of Global Analysis and Geometry **24** (2003) 161-200.

[26] E. Gagliardo, *Proprieta di alcune classi di funzioni in piu variabili*, Ric. Math. **7** (1958) 102-137.

[27] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, in: Grundlehren der Math. Wiss., pringer-Verlag Berlin Heidelberg **224** (1983).

[28] W. Han, *The best constant in a trace inequality*, J. Math. Anal. Appl. **163** (1992) 512-520.

[29] E. Hebey, *Nonlinear Analysis on Manifold: Sobolev Spaces and Inequalities*, in: Courant Lecture Notes in Math., Courant Institute of Mathematical Science, New York University, New York **5** (2000).

[30] E. Hebey, *Sobolev Spaces on Riemannian Manifold*, Lecture Notes in Math., **1635**, Springer-Verlag Berlin Heidelberg, 1996.

[31] E. Hebey and M. Vaugon, *Mielleures constants dans le théorème d'inclusion de Sobolev*, Ann. Inst. H. Pioncaré. Anal. Non Linéaire **13** (1996) 57-93.

[32] E. Hebey and M. Vaugon, *Sobolev Spaces in presence of symmetries*, J. Math. Pures Appl. **76** (1997) 859-881.

[33] E. Hebey and M. Vaugon, *The best constant problem in the Sobolev embedding for complete Riemannian manifolds*, Duke Math. J. **79** (1995) 235-279.

[34] P. Lions, *The concentration compactness principle in the calculus of variations,* The limit case, I Rev. Mat. Iberoamericana **1** (1985) 145-201.

[35] D. Martin, *Manifold Theory: An introduction for mathematical physicists*, Horwood Publishing Chichester, 2002.

[36] B. Nazaret, *Best constant in Sobolev trace inequalities on the half-space*, Nonlinear Analysis **65** (2006) 1977-1985.

[37] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. **13** (1959) 116-162.

[38] S. Sobolev. , *Sur un théorème d'analyse fonctionnelle*, Math. Sb. (N.S.) **46** (1938) 471-496.

[39] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3** (1976) 697-718.

[40] K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura, and H. Yamagishi, *The best constant of Sobolev inequality on a bounded interval*, J. Math. Anal. Appl. **340** (2008) 699-706.

[41] J. Xiao, *A sharp Sobolev trace inequality for the fractional-order derivatives*, Bull. Sci. Math. **130** (2006) 87-96.

[42] J. Xiao, *The sharp Sobolev and isoperimetric inequalities split twice*, Advances in Mathematics **211** (2007) 417-435.

[43] W. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag New York Inc., 1989.