COINCIDENCE NIELSEN NUMBERS FOR COVERING MAPS FOR ORIENTABLE AND NONORIENTABLE MANIFOLDS

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By

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TO MY LOVELY PARENTS

Contents

Abstract

Let $f, g : M \to N$ be maps between closed manifolds of the same dimension, and let $p : \widetilde{M} \to M$ and $p' : \widetilde{N} \to N$ be finite regular covering maps. If the manifolds M and N are orientable, then, under certain conditions, the Nielsen number $N(f, g)$ of f and g can be computed as a linear combination of the Nielsen numbers of the lifts of f and g. In the non-orientable case, using semi-index, we introduce two new Nielsen numbers. The first one is the Linear Nielsen number $N_L(f, g)$, which is a linear combination of the Nielsen numbers of the lifts of f and g. The second one is the Non-linear Nielsen number $N_{ED}(f, g)$. It is the number of certain essential classes whose inverse images by p are inessential Nielsen classes. In fact, $N(f, g) = N_L(f, g) + N_{ED}(f, g)$, where by abuse of notation, $N(f, g)$ denotes the coincidence Nielsen number defined using semi-index.

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Introduction

Let $f : X \longrightarrow X$ be a map on a topological space X, and let $\Phi(f) = \{x \in X|\ f(x) = x\}$ be the set of the fixed points of f. It is not always possible to find the set $\Phi(f)$ or even its cardinality $|\Phi(f)|$. One of the fundamental studies of this set has been to find an estimate for its cardinality. The most useful estimates are usually lower bounds of $|\Phi(f)|$. The closer to $|\Phi(f)|$ the lower bound is , the better the estimate. The Nielsen number [21, 22] is one method used to find such an estimate. It counts a special type of classes (called Nielsen classes) defined by an equivalence relation (called the Nielsen relation) on the elements of $\Phi(f)$. The importance of the Nielsen number arises from two facts. The first is that it is homotopy invariant. That is, homotopic maps have the same Nielsen number. The other fact is that it is equal, under certain conditions, to the minimum of the set $\{|\Phi(f_1)| \mid f_1 \text{ is homotopic to } f\}$. A drawback of the Nielsen number is that it is difficult to compute. For this reason, Nielsen Theorists search continuously for methods that help compute the Nielsen number.

Let X be a finite polyhedron, and H be a normal subgroup of $\pi_1(X)$ of finite index. Fix a covering $p : \widetilde{X} \longrightarrow X$ corresponding to H; that is, $p_{\#}(\pi_1(\widetilde{X})) = H$. If

$f_{\#}(H) \subseteq H$, then, f admits a lift $\widetilde{f}$, and hence we have the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ p\downarrow & & \downarrow p \\ X & \xrightarrow{f} & X \end{array} \tag{0.0.1}$$

In [15], J. Jezierski gave, under certain conditions, a method that computes the Nielsen number $N(f)$ of f as the following linear combination of the Nielsen numbers of its lifts.

$$N(f) = \sum_{i=1}^{r} (J_i/I_i) \cdot N(\widetilde{f}_i) , \tag{0.0.2}$$

where r denotes the number of the nonempty Reidemeister classes represented by the lifts $\widetilde{f}_i$ of f, and I_i and J_i are the order of specific subgroups of $\dfrac{\pi_1(X)}{H}$.

Let $f, g : M \longrightarrow N$ be maps from a topological space M to a topological space N, and let $\Phi(f,g) = \{x \in M|\ f(x) = g(x)\}$ be the set of coincidence points of f and g. Coincidence Theory is a natural extension of the Fixed Point Theory. The coincidence Nielsen number $N(f,g)$ of f and g is defined to be a homotopy invariant nonnegative integer which is a lower bound of the set

$$\{|\Phi(f_1, g_1)| \mid f_1 \text{ is homotopic to } f \text{ and } g_1 \text{ is homotopic to } g\} .$$

The Nielsen number $N(f,g)$ is homotopy invariant means that, if f_1 is a map homotopic to f and g_1 is a map homotopic to g, then $N(f_1, g_1) = N(f,g)$. In this thesis, we study two coincidence Nielsen numbers. The first one is the index Nielsen number [2] which is defined for orientable closed manifolds, and the second one is the semi-index Nielsen number [5] which is defined for smooth closed manifolds (the orientability condition is dropped). The Purpose of the thesis, given finite regular coverings for which the maps f and g admit lifts, is to generalize the formula 0.0.2 given above to formulas that compute the coincidence Nielsen numbers as linear combinations of lifts of the maps f and g. More precisely, under certain conditions, we show that the index Nielsen number is a linear combination of lifts of f and g (Chapter 2). The semi-index Nielsen number, however, is the sum of two Nielsen numbers (Chapter 3). The first number is called the Linear Nielsen number. It is a linear combination of lifts of f and g. The second one is called the Nonlinear Nielsen number. It counts special essential classes of f and g which their inverse image by the covering map are inessential classes of the lifts of f and g. The Nonlinear Nielsen number, as we will show in Chapter 3, equals to zero in the fixed point case or with orientable manifolds, where the index is defined in this case, are considered.

In Chapter 1, we give the necessary background for the thesis. In the first section, we give the definitions of the Nielsen classes and Reidemeister classes, along with their basic properties. In the second section, we present the main results of J. Jezierski in [15]. This gives a pattern to follow as we work toward generalizations of Formula 0.0.2 which is the main result in [15].

In Chapter 2, given finite regular coverings for which f and g admit lifts, we define, in the first section, the three numbers J_A, I_A, and S_A for a given Nielsen class A of f and g. They generalize J_A and I_A given in [15]. The number J_A is defined to be $|p^{-1}(x) \cap \widetilde{A}|$, where $x \in A$, and $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ is a Nielsen class such that $p(\widetilde{A}) = A$. We can compute J_A algebraically from the equation $J_A = j\,(C(f_\#, g_\#)_x)$, where $C(f_\#, g_\#)_x$ is the set of all $a \in \pi_1(M, x)$ such that $f_\#(a) = g_\#(a)$, and $j : \pi_1(M, x) \longrightarrow \dfrac{\pi_1(M,x)}{K(x)}$ is the canonical projection. The number I_A is defined to be $|p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})|$. We can also compute J_A algebraically from the equation $I_A = C(\bar{f}_\#, \bar{g}_\#)_x$, where $\bar{f}_\#$

and $\bar{g}_\#$ are the homomorphisms induced on $\dfrac{\pi_1(M,x)}{K(x)}$ by $f_\#$ and $g_\#$, respectively. The number S_A is the number of Nielsen classes $\widetilde{A} \subseteq \Phi(\widetilde{f},\widetilde{g})$ such that $p(\widetilde{A}) = A$. The numbers J_A, I_A, and S_A are independent of the orientabililty of the manifolds. They are used to compute the coefficients in the formulas given in Theorems 2.3.5 and 3.3.16, which generalize Theorem 4.2 of [15]. Also, we exhibit geometric and algebraic interpretations of these numbers, and give methods of computing them from the fundamental group $\pi_1(M)$. Also, in this section, we give results that help us to use a single coincidence point to compute the numbers J, I, and S for the $H-$Reidemeister representatives which appear in the formula given in Theorems 2.3.5 and 3.3.16. In the second section, where we assume the manifolds are orientable, we introduce the definition of index [26] and its properties. Then, we give the definition of the Lefschetz number and its relationship to the index. Next, we give the definition of a Nielsen number [16] and the index Nielsen number. In the third section, we give the relationship between the Nielsen classes in the base space M, and those of the covering space $\widetilde{M}$ through two equations (see Proposition 2.3.2) that link the indices of those Nielsen classes. In fact, the equations generalize to coincidences those given in Lemma 3.4 of [15] and in Theorem 3.7 of [25] for orientable manifolds. Then, we use the relationships between Nielsen classes to derive Theorem 2.3.5, which shows the index Nielsen number $N(f,g)$ as a linear combination of the Nielsen numbers of the lifts of f and g. Afterward, in the forth section, we give applications and examples of Theorem 2.3.5. In particular, we show how Theorem 2.3.5 generalizes Theorem 4.2 of [15], and explain how to use it to compute the Nielsen number.

In Chapter 3, we drop the orientability condition given in Chapter 2, but study only smooth manifolds. In the first section, we define the semi-index [5] (or [16]), and list its properties and its relationship to the index. We, then, define the semi-index Nielsen number. In the second section, we recall the notion of defective classes together with its properties. A relationship between the indices of the Nielsen classes of the base space and the covering space is developed in Propositions 3.2.12 and ??. In fact, these propositions generalize Lemma 3.4 of [15] and Theorem 3.7 of [25] to Coincidence Theory. In these two references, orientable and nonorientable manifolds respectively are considered. Next, using these propositions, we completely explain the relationship between Nielsen classes in the base space, and those in the covering space. This relationship is not straight forward, because the defective Nielsen classes exist for nonorientable manifolds and behave quite different from nondefective Nielsen classes. In the third section, we define our two new Nielsen numbers. The first is the Linear Nielsen number $N_L(f,g)$. We show that it is a well defined Nielsen number. It is a linear combination of the Nielsen numbers of lifts of f and g. The other number is called the Nonlinear Nielsen number $N_{ED}(f,g)$. It is the number of essential defective Nielsen classes of f and g for which J is even positive integer. We show that it is a well-defined Nielsen number, and give an example which shows that it can be nonzero. Next, we show that the semi-index Nielsen number is the sum of the Linear and the Nonlinear Nielsen numbers. In the forth section, we show how Theorem 3.3.16 generalizes both Theorem 2.4 of [15] and our 2.3.5. Finally, many special cases of Theorem 3.3.16 are discussed, and examples given.

Chapter 4 is divided into two parts. The first part, covered by Sections 4.1 and 4.2, gives a classification of $H-$Reidemeister classes. We show, under certain conditions, how to choose canonical $H-$Reidemeister representatives which appear in the formulas given in Theorems 2.3.5 and 3.3.16. The second part, covered by Section 4.3, includes many examples that illustrate the main results of the thesis. In more detail, in section 4.1, we study the general classification of the $H-$Reidemeister classes (or representatives) and give, under certain conditions, the exact number of $H-$Reidemeister representatives which appear in the formulas given in Theorems 2.3.5 and 3.3.16. In section 4.2, we study the special case of our classification when the cardinality of the group of covering transformations of both coverings is prime. We show, in this case, that the number of $H-$Reidemeister classes is either 1 or equal to the number of sheets of the covering space $(\widetilde{N}, p')$ of N. We also show that the coefficients, in our formulas, are either 1 or equal to the reciprocal of the number of sheets of the covering space $(\widetilde{M}, p)$ of M. This will give elegant versions of the formulas given in Theorems 2.3.5 and 3.3.16. Finally, in section 4.3, we give several examples which illustrate the main results of the thesis. The main example for nonorientable manifolds comes here in Example 4.3.18.

Chapter 1

Background

In this chapter, we give the background necessary for this thesis [4, 5, 15, 16, 18, 19, 26]. The first section presents Nielsen and Reidemeister coincidence classes and the relationship between them, as well as their basic properties. In the second section, we list the results for the fixed point case from [15] which were the motivation of this thesis.

1.1 Nielsen and Reidemeister Classes

In this section, we give the concepts of Nielsen class and Reidemeister coincidence classes along with their basic properties [4, 18].

Let M and N be path connected, locally path connected topological spaces, and $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be regular coverings corresponding to normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. The covering maps $p : \widetilde{M} \longrightarrow M$ and $p : \widetilde{N} \longrightarrow N$ are not the same in general. The context clearly shows which covering map is under consideration. In possible cases of confusion, we will give the covering maps different names. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f},\widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. Thus, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f},\widetilde{g}} & \widetilde{N} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{f,g} & N \end{array} \tag{1.1.1}$$

In what follows, $\mathcal{A}(\widetilde{M})$ and $\mathcal{A}(\widetilde{N})$ denote the groups of covering transformations of the corresponding covering spaces, and $Lift(f)$ and $Lift(g)$ denote the sets of lifts of f and g respectively.

Remark 1.1.1. *In case of fixed points, where $M = N$ and $g = 1_M$ is the identity on M, we assume $\widetilde{M} = \widetilde{N}$, and of course the covering maps are the same. Composition of functions f_1 and f_2 will be denoted by either $f_1 \circ f_2$ or $f_1 f_2$. In addition, if ω is a path in the domain of f, then the path $f \circ \omega$ is denoted for simplicity by $f(\omega)$.*

Lemma 1.1.2. *[19] Let M and $p : \widetilde{M} \longrightarrow M$ be as above, and let Y be a path connected space. Given two maps $\widetilde{f}_0, \widetilde{f}_1 : Y \longrightarrow \widetilde{M}$ such that $p\widetilde{f}_0 = p\widetilde{f}_1$, the set $\Phi(\widetilde{f}_0, \widetilde{f}_1)$ is either empty or equal to Y.* □

Lemma 1.1.3. *[19] Let M and $p : \widetilde{M} \longrightarrow M$ be as above and let $\widetilde{g}_0, \widetilde{g}_1 : [0,1] \longrightarrow \widetilde{M}$ be paths in $\widetilde{M}$ which have the same initial point. If $p\widetilde{g}_0$ is homotopic to $p\widetilde{g}_1$ relative endpoints, then $\widetilde{g}_0$ is homotopic to $\widetilde{g}_1$ relative endpoints; in particular, $\widetilde{g}_0$ and $\widetilde{g}_1$ have the same terminal points.* □

Definition 1.1.4. *We define $Lift(f,g)$ by*

$$Lift(f,g) = \{(\widetilde{f},\widetilde{g}) | \widetilde{f} \in Lift(f) \text{ and } \widetilde{g} \in Lift(g)\}\ .$$

Since the validity of our results requires a nonempty set of coincidences, we assume, without loss of generality, that the set $\Phi(f,g)$ of coincidence points of f and g is nonempty.

Lemma 1.1.5. *Let M and $p : \widetilde{M} \longrightarrow M$ be as above. Then, there are bijections $\mathcal{A}(\widetilde{M}) \longrightarrow p^{-1}(x)$ and $\dfrac{\pi_1(M,x)}{H(x)} \longrightarrow p^{-1}(x)$ for each $x \in M$.*

Proof. For each $x \in M$, fix $\widetilde{x}_0 \in p^{-1}(x)$. The function

$$\mathcal{A}(\widetilde{M}) \longrightarrow p^{-1}(x) \;\; : \;\; \gamma \longmapsto \gamma(\widetilde{x}_0)$$

is a well-defined bijection. On the other hand, for each $\alpha \in \pi_1(M,x)$, let $\widetilde{\alpha}$ be its lift at $\widetilde{x}_0$ (i.e., starting at $\widetilde{x}_0$). The function

$$\frac{\pi_1(M,x)}{H(x)} \longrightarrow p^{-1}(x) \;\; : \;\; \alpha\, H(x) \longmapsto \widetilde{\alpha}(1)$$

where $\alpha\, H(x)$ is the coset of $H(x)$ determined by α, is a well-defined bijection. $\square$

The following lemma is a special case of Lemma 6.1 of [19].

Lemma 1.1.6. *[19] Let M and $p : \widetilde{M} \longrightarrow M$ be as above, and let $\alpha_0, \alpha_1 \in \mathcal{A}(\widetilde{M})$. If $\Phi(\alpha_0, \alpha_1) \neq \emptyset$, then $\alpha_0 = \alpha_1$.* $\square$

The following is Schirmer's definition of coincidence Nielsen class.

Definition 1.1.7. *Let H be a normal subgroup of $\pi_1(N)$. Let $x, y \in \Phi(f,g)$. We say that x and y are in the same H-Nielsen class, and we write $x \sim_H y$, if there exists a path $\omega : x \to y$ in M such that $f(\omega)$ is homotopic to $g(\omega)$ relative endpoints (mod H), symbolically $f(\omega) \sim_H g(\omega)$, which means that $g(\omega)f(\omega)^{-1} \in H(f(x))$.*

If $H = 0$, then we say that x and y are in the same Nielsen class, and we write $x \sim_0 y$.

This relation is an equivalence relation, the equivalence classes are called H-Nielsen classes . For $x \in \Phi(f, g)$, we write $[x]_H$ for the $H-$Nielsen class of x.

Remark 1.1.8. *If $H = 0$ in Definition 1.1.7 the equivalence classes are called Nielsen classes. The symbol $[x]$ stands for the Nielsen class of the coincidence point x.*

Lemma 1.1.9. *$[x] \subseteq [x]_H$ for every $x \in \Phi(f, g)$. That is, each $H-$Nielsen class is a union of ordinary Nielsen classes.* □

Next, we introduce the definition of $H-$Reidemeister classes and an alternative description of the $H-$Nielsen classes in terms of the $H-$Reidemeister classes. We start with the following proposition.

Proposition 1.1.10. *[15] Let $\widetilde{f}$ and $\widetilde{\widetilde{f}}$ be lifts of f, then there exists a unique $\beta \in \mathcal{A}(\widetilde{N})$ such that $\widetilde{\widetilde{f}} = \beta\widetilde{f}$. In other words, if we fix a lift $\widetilde{f}$ of f, then the function*

$$\eta : \mathcal{A}(\widetilde{N}) \longrightarrow Lift(f) \quad : \quad \beta \longmapsto \beta\widetilde{f}$$

is a well-defined bijection. □

The group $\mathcal{A}(\widetilde{N})$ (resp. $\mathcal{A}(\widetilde{M})$) acts on $Lift(f)$ from the left (resp. from the right) by $\beta\widetilde{f} = \beta \circ \widetilde{f}$ (resp. $\widetilde{f}\alpha = \widetilde{f} \circ \alpha$) where $\widetilde{f} \in Lift(f)$ and $\beta \in \mathcal{A}(\widetilde{N})$ (resp. $\alpha \in \mathcal{A}(\widetilde{M})$).

Definition 1.1.11. *[5] Let $(\widetilde{f}, \widetilde{g}), (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \in Lift(f, g)$. We say $(\widetilde{f}, \widetilde{g})$ and $(\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}})$ are conjugate if there exist $\alpha \in \mathcal{A}(\widetilde{M})$ and $\beta \in \mathcal{A}(\widetilde{N})$ such that $(\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) = \beta(\widetilde{f}, \widetilde{g})\alpha := (\beta\widetilde{f}\alpha, \beta\widetilde{g}\alpha)$.*

The set of all conjugacy classes is called the set of H-Reidemeister classes, and is denoted by $\Re_H(f,g)$. The cardinality of $\Re_H(f,g)$ is denoted by $R_H(f,g)$ and is called the $H-$Reidemeister number of f and g.

Definition 1.1.12. *[18] Let $f : M \longrightarrow M$ be a map, and let $(\widetilde{M},p)$ be a covering space. Two lifts $\widetilde{f_1}$ and $\widetilde{f_2}$ of f are called conjugate in the fixed point sense if there exists $\gamma \in \mathcal{A}(\widetilde{M})$ such that $\widetilde{f_2} = \gamma\,\widetilde{f_1}\,\gamma^{-1}$.*

Proposition 1.1.13. *Let $f : M \longrightarrow M$ be a map, and let $(\widetilde{M},p)$ be a covering space. Two lifts $(\widetilde{f_1}, 1_{\widetilde{M}})$ and $(\widetilde{f_2}, 1_{\widetilde{M}})$ of $(f, 1_M)$ are conjugate (in the coincidence point sense) if and only if $\widetilde{f_1}$ and $\widetilde{f_2}$ are conjugate in the fixed point sense.*

Proof. Suppose $(\widetilde{f_1}, 1_{\widetilde{M}})$ and $(\widetilde{f_2}, 1_{\widetilde{M}})$ are conjugate. There exist α, $\beta \in \mathcal{A}(\widetilde{M})$ such that $(\widetilde{f_2}, 1_{\widetilde{M}}) = \beta\,(\widetilde{f_1}, 1_{\widetilde{M}})\,\alpha$. Thus,

$$(\widetilde{f_2}, 1_{\widetilde{M}}) = (\beta\,\widetilde{f_1}\,\alpha, \beta\,1_{\widetilde{M}}\,\alpha) = (\beta\,\widetilde{f_1}\,\alpha, \beta\,\alpha)\ .$$

Hence, $\widetilde{f_2} = \beta\,\widetilde{f_1}\,\alpha$ and $1_{\widetilde{M}} = \beta\,\alpha$. The later equation implies that $\alpha = \beta^{-1}$. Thus, $\widetilde{f_2} = \beta\,\widetilde{f_1}\,\beta^{-1}$. Therefore, $\widetilde{f_1}$ and $\widetilde{f_2}$ are conjugate in the fixed point sense.

Now, assume $\widetilde{f_1}$ and $\widetilde{f_2}$ are conjugate in the fixed point sense. There exists $\gamma \in \mathcal{A}(\widetilde{M})$ such that $\widetilde{f_2} = \gamma\,\widetilde{f_1}\,\gamma^{-1}$. If we let $\beta = \gamma$ and $\alpha = \gamma^{-1}$, we get that $(\widetilde{f_2}, 1_{\widetilde{M}}) = \beta\,(\widetilde{f_1}, 1_{\widetilde{M}})\,\alpha$. That is, $(\widetilde{f_1}, 1_{\widetilde{M}})$ and $(\widetilde{f_2}, 1_{\widetilde{M}})$ are conjugate in the coincidence point sense. □

The following corollary is an obvious consequence of Proposition 1.1.13.

Corollary 1.1.14. *Let $f : M \longrightarrow M$ be a map, and let $(\widetilde{M},p)$ be a finite covering space corresponding to the normal subgroup H. Then,*

$$R_H(f, 1_M) = R_H(f)\ .$$

That is, the coincidence $H-$Reidemeister number of the pair $(f, 1_M)$ is equal to the fixed point $H-$Reidemeister number of the map f.

Proposition 1.1.15. *[5] Assume we are given regular coverings as in diagram 1.1.1. Then,*

1. $\Phi(f,g) = \bigcup_{(\tilde{f},\tilde{g})} p\,\Phi(\widetilde{f},\widetilde{g})$ *where the index runs over all pairs of lifts.*
2. *The sets* $p\,\Phi(\widetilde{f},\widetilde{g})$ *and* $p\,\Phi(\widetilde{f'},\widetilde{g'})$ *are either equal or disjoint.*
3. $p\,\Phi(\widetilde{f},\widetilde{g}) = p\,\Phi(\widetilde{f'},\widetilde{g'})$ *if and only if* $(\widetilde{f},\widetilde{g})$ *and* $(\widetilde{f'},\widetilde{g'})$ *are conjugate.*
4. $\Phi(f,g) = \bigcup_{(\tilde{f},\tilde{g})} p\,\Phi(\widetilde{f},\widetilde{g})$ *is a disjoint union, where the union takes one* $(\widetilde{f},\widetilde{g})$ *from each conjugacy ($H-$Reidemeister) class.*

The following proposition generalizes the first part of Lemma 3.1 of [15].

Proposition 1.1.16. *Let $x, y \in \Phi(f,g)$. Then x and y belong to the same H-Nielsen class if and only if there exists a pair $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$ such that $x, y \in p\,\Phi(\widetilde{f},\widetilde{g})$. Moreover, $(\widetilde{f},\widetilde{g})$ is unique up to conjugacy.*

Proof. We know that

$$\Phi(f,g) = \bigcup_{(\tilde{f},\tilde{g}) \in Lift(f,g)} p\,\Phi(\widetilde{f},\widetilde{g})\,.$$

Since $x \in \Phi(f,g)$, there exists a lifting pair $(\widetilde{f},\widetilde{g})$ and $\widetilde{x} \in \Phi(\widetilde{f},\widetilde{g})$ such that $p(\widetilde{x}) = x$. Suppose x and y are in the same H-Nielsen class. Thus, there exists a path $\omega : x \longrightarrow y$ such that $g(\omega)f(\omega)^{-1} \in H(f(\widetilde{x})) = p_{\#}(\pi_1(\widetilde{N}, \widetilde{f}(\widetilde{x})))$. Let $\widetilde{\omega} : \widetilde{x} \longrightarrow \widetilde{y}$ be a lift of ω starting at $\widetilde{x}$ and ending at $\widetilde{y} \in p^{-1}(y)$. Then $g(\omega)f(\omega)^{-1} = p\,(\lambda)$ for some $\lambda \in$

$\pi_1(\widetilde{N}, \widetilde{f}(\widetilde{x})))$. So, $g(\omega)$ is homotopic to $p(\lambda)f(\omega)$ rel. endpoints. Now, $\lambda\widetilde{f}(\widetilde{\omega})$ is a path from $\widetilde{f}(\widetilde{x})$ to $\widetilde{f}(\widetilde{y})$, which is a lift of the path $p(\lambda)\,f(w)$. Similarly, $\widetilde{g}(\widetilde{\omega})$ is a path from $\widetilde{g}(\widetilde{x}) = \widetilde{f}(\widetilde{x})$ to $\widetilde{g}(\widetilde{y})$, which is a lift the path $g(\omega)$. Since these two lifts have the same initial point and $g(\omega)$ is homotopic to $p(\lambda)f(\omega)$ rel. endpoints, applying Lemma 1.1.3 gives that they are homotopic rel. endpoints, and, in particular, that $\widetilde{f}(\widetilde{y}) = \widetilde{g}(\widetilde{y})$. Thus, $\widetilde{y} \in \Phi(\widetilde{f}, \widetilde{g})$, and hence $y \in p\,\Phi(\widetilde{f}, \widetilde{g})$. The uniqueness of the pair $(\widetilde{f}, \widetilde{g})$ up to conjugacy follows from Proposition 1.1.15.

For the converse, assume $x,\, y \in p\,\Phi(\widetilde{f}, \widetilde{g})$. Let $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})$, $\widetilde{y} \in p^{-1}(y) \cap \Phi(\widetilde{f}, \widetilde{g})$, and $\widetilde{\omega} : \widetilde{x} \longrightarrow \widetilde{y}$ be a path in $\widetilde{M}$. Let $\omega = p(\widetilde{\omega})$. Then

$$\begin{aligned} g(\omega)f(\omega)^{-1} &= p\widetilde{g}(\widetilde{\omega})(p\widetilde{f}(\widetilde{\omega}))^{-1} \\ &= p\widetilde{g}(\widetilde{\omega})p(\widetilde{f}(\widetilde{\omega})^{-1}) \\ &= p\,(\widetilde{g}(\widetilde{\omega})(\widetilde{f}(\widetilde{\omega})^{-1}) \in H(f(x)). \end{aligned}$$

Therefore, x and y belong to the same H-Nielsen class. □

Corollary 1.1.17. *If $p\,\Phi(\widetilde{f}, \widetilde{g}) \neq \emptyset$ for a lift $(\widetilde{f}, \widetilde{g})$ of (f, g), then $p\,\Phi(\widetilde{f}, \widetilde{g}) = [x]_H$ for every $x \in p\,\Phi(\widetilde{f}, \widetilde{g})$.*

Proof. Apply Propoistion 1.1.16. □

Corollary 1.1.17 states that each $H-$Nielsen class is of the form $p\,\Phi(\widetilde{f}, \widetilde{g})$ for some lift $(\widetilde{f}, \widetilde{g})$ of (f, g). For nonempty $H-$Nieslen classes, this "covering form" of the definition coincides with Definition 1.1.7.

Proposition 1.1.18. *[10] If M is locally path connected and N is semilocally simply connected, then each Nielsen class is an open subset of $\Phi(f, g)$.* □

The following proposition generalizes the last part of Lemma 3.1 of [15].

Proposition 1.1.19. *If M and N are connected compact manifolds, and A is a Nielsen class, then the set $p^{-1}(A) \cap \Phi(\widetilde{f}, \widetilde{g})$, where $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$, is either empty, or splits into a finite union of nonempty Nielsen classes of $(\widetilde{f}, \widetilde{g})$.*

Proof. The proof is similar to that of Lemma 3.1 of [15]. □

Remark 1.1.20. *If the coverings are universal then the $H-$Nielsen classes are ordinary Nielsen classes, that is if $x \in \Phi(f, g)$ then $[x] = [x]_H$. Also, the $H-$Reidemeister classes are Reidemeister classes, and the $H-$Reidemeister number is the Reidemeister number.*

1.2 The Fixed Point Nielsen number of covering maps

The main results of this thesis generalize those of [15]. This section states these results of [15]. Concisely, [15] gives a method to compute $N(f)$ as a linear combination of the Nielsen number of lifts of f under certain conditions.

Let X be a finite polyhedron, and H be a normal subgroup of $\pi_1(X)$ of finite index. Fix a covering $p : \widetilde{X} \longrightarrow X$ corresponding to H, i.e., $p_\#(\pi_1(\widetilde{X})) = H$ (the covering p is finite). Let $f : X \longrightarrow X$ be a map such that $f_\#(H) \subseteq H$. Then, f admits a lift $\widetilde{f}$ and hence we have the commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\
p\downarrow & & \downarrow p \\
X & \xrightarrow{f} & X
\end{array}
\tag{1.2.1}
$$

Definition 1.2.1. *For a lift $\widetilde{f} \in Lift(f)$, a fixed point $x_0 \in \Phi(f)$ and element $b \in \pi_1(X, x_0)$ we define the subgroups*

$$L(\widetilde{f}) = \left\{\gamma \in \mathcal{A}(\widetilde{X}) |\ \widetilde{f} \circ \gamma = \gamma \circ \widetilde{f}\right\} \tag{1.2.2}$$

$$C(f_{\#}, x_0; b) = \{a \in \pi_1(X, x_0) |\ a\, b = b\, f_{\#}(a)\} \tag{1.2.3}$$

$$C_H(f_{\#}, x_0; b) = \{[a]_H \in \pi_1(X, x_0)/H(x_0) |\ a\, b = b\, f_{\#}(a) \ \ \text{modulo } H\}\ . \tag{1.2.4}$$

For $b = 1$ we write simply $C(f_{\#}, x_0)$ or $C_H(f_{\#}, x_0)$.

Lemma 1.2.2. *Let $\widetilde{A} \subseteq \Phi(\widetilde{f})$ be a Nielsen class of fixed lift $\widetilde{f}$ of f. Let us denote $A = p(\widetilde{A})$. Then,*

1. *Let $j : C(f_{\#}, x_0; b) \longrightarrow C_H(f_{\#}, x_0; b)$ be the homomorphism induced by the canonical projection $j : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)/H(x_0)$. Then, the restriction of the map $p : \widetilde{X} \longrightarrow X$ to the map $p : \widetilde{A} \longrightarrow A$ (the restriction is also denoted by p for simplicity) is a covering map, and the fiber is in bijective correspondence with the image $j(C(f_{\#}, x)) \subseteq \pi_1(X, x)/H(x)$ for $x \in A$.*

2. *The cardinality of the fiber (i.e., $|p^{-1}(x) \cap \widetilde{A}|$) does not depend on $x \in A$ and we denote it by J_A.*

3. *If $\widetilde{\acute{A}}$ is another Nielsen class of $\widetilde{f}$ such that $p(\widetilde{\acute{A}}) = p(\widetilde{A})$, then $J_A = |p^{-1}(x) \cap \widetilde{A}| = |p^{-1}(x) \cap \widetilde{\acute{A}}|$.* □

Lemma 1.2.3. *The map* $p : \widetilde{X} \longrightarrow X$ *restricts to a covering map* $p : \Phi(\widetilde{f}) \longrightarrow p\left(\Phi(\widetilde{f})\right)$. *The fiber over each point is in bijective correspondence with the subgroup* $L(\widetilde{f})$. $\square$

Remark 1.2.4. *It follows from Lemma 1.2.3 that if* $A_H = p\left(\Phi(\widetilde{f})\right)$ *is an* $H-$*Nielsen class that corresponds to a lift* $\widetilde{f}$ *of* f, *and* $x \in A_H$, *then the cardinality* $|p^{-1}(x) \cap \Phi(\widetilde{f})|$ *is independent of the choice of* $x \in A_H$. *That is, it depends only on the* $H-$*Nielsen class* A_H. *We denote it by* I_{A_H}. *Also, we write* $I_A := I_{A_H}$ *for each Nielsen class* $A \subseteq A_H$.

Lemma 1.2.5. *[15] Let* A *be a Nielsen class of* f *and* $\widetilde{A}$ *be a Nielsen class of* $\widetilde{f}$ *contained in* $p^{-1}(A)$. *Then, by Proposition 1.1.19* $A = p(\widetilde{A})$ *and moreover*

$$index(\widetilde{f}; p^{-1}(A) \cap \Phi(\widetilde{f})) = I_A \cdot index(f, A)$$

and

$$index(\widetilde{f}; \widetilde{A}) = J_A \cdot index(f, A)\ .$$

$\square$

To obtain a formula expressing $N(f)$ in terms of $N(\widetilde{f})$, we need the assumption that the numbers $J_A = J_{\acute{A}}$ for any Nielsen classes A and $\acute{A}$ that lie in the same $H-$Nielsen class of f. The next lemma gives a sufficient condition for such equality.

Lemma 1.2.6. *Let* $x \in p(\Phi(\widetilde{f}))$. *If the subgroups* $H(x)$ *and* $C(f, x)$ *commute in* $\pi_1(X, x)$, *that is,* $h \cdot a = a \cdot h$ *for every* $h \in H(x)$ *and* $a \in C(f, x)$, *then* $J_A = J_{\acute{A}}$ *for all Nielsen classes* $A, \acute{A} \subseteq p(\Phi(\widetilde{f}))$. $\square$

Remark 1.2.7. *The assumption in Lemma 1.2.6 holds if either of $H(x)$ or $C(f,x)$ is contained in the center of $\pi_1(X,x)$. On the other hand, if the subgroups $H(x)$ and $C(f,x)$ commute in $\pi_1(X,x)$ so do the corresponding groups at any point in $p(\Phi(\widetilde{f}))$.*

Now, we express the numbers I_A and J_A in terms of the homotopy group homomorphism $f_\# : \pi_1(X,x) \longrightarrow \pi_1(X,f(x))$ for $x \in \Phi(f)$. Let $\widetilde{f} \in Lift(f)$, and $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f})$. We fix the isomorphism $\pi_1(X,x)/H(x) \longrightarrow \mathcal{A}(\widetilde{X}) \quad : \quad [a]_H \mapsto \gamma_{[a]_H}$ where $\gamma_{[a]_H}(x) = \widetilde{a}(1)$ and $\widetilde{a}$ denotes the lift of a starting at $\widetilde{a}(0) = \widetilde{x}$ (see Lemma 1.1.5).

Lemma 1.2.8. *We have*

$$f \circ \gamma_{[a]_H} = \gamma_{[f(a)]_H} \circ \widetilde{f}\ .$$

□

Lemma 1.2.9. *Let $\widetilde{f} \in Lift(f)$, $A \subseteq p(\Phi(\widetilde{f}))$ be a Nielsen class of f, and $x \in A$. Then,*

$$I_A = |C_H(f_\#, x)|\ .$$

□

The next result follows directly from Lemmas 1.2.2 and 1.2.9.

Corollary 1.2.10. *Let $\widetilde{f} \in Lift(f)$, $A \subseteq p(\Phi(\widetilde{f}))$ be a Nielsen class of f, and $x \in A$. Then,*

$$\frac{I_A}{J_A} = \frac{|C_H(f_\#, x))|}{|j(C(f_\#, x))|}\ .$$

□

Fix a point $x \in \Phi(f)$. The following lemma shows how to express the cardinality of the subgroups $C(f_\#, x')$ and $C_H(f_\#, x')$ at $x' \in \Phi(f)$ in terms of the cardinality of subgroups of $\pi_1(X, x)$. It follows that the numbers I and J can be computed from a single fixed point of f. Let $\omega : x' \longrightarrow x$ be a path. We denote the isomorphism $\pi_1(X, x') \longrightarrow \pi_1(X, x)$ induced by ω by $\omega_\#$, i.e., $\omega_\#(b) = \omega^{-1} b\, \omega$ for each $b \in \pi_1(X, x')$. Further, the isomorphism $\omega_\#$ induces an isomorphism $\overline{\omega}_\#$ on $\pi_1(X, x)/H(x)$ defined in the natural way.

Lemma 1.2.11. *Let $\sigma = \omega^{-1} f(\omega)$. Then,*

$$\omega_\#(C(f_\#, x')) = \{a \in \pi_1(X, x) |\; a\, \sigma = \sigma\, f_\#(a)\}$$

and

$$\overline{\omega}_\#(C_H(f_\#, x')) = \{[a]_H \in \pi_1(X, x)/H(x) |\; a\, \sigma = \sigma\, f_\#(a) \textit{ modulo } H\} \ .$$

□

Lemma 1.2.12. *Let $A \subseteq p(\Phi(\widetilde{f}))$ be a Nielsen class of f. Then, $p^{-1}(A)$ contains exactly I_A/J_A fixed point classes of $\widetilde{f}$.* □

Fix lifts $\widetilde{f}_1, \ldots, \widetilde{f}_r$ representing all $H-$Nielsen classes of f, then $\Phi(f) = \bigcup_{i=1}^{i=r} \Phi(\widetilde{f}_i)$ is a disjoint union. If we assume that all the Nielsen classes that lie in the same $H-$Nielsen class have the same J number, then J depends only on the $H-$ Nielsen class. Thus, we let I_i and J_i denote the numbers corresponding to the $H-$ Nielsen class represented by $\widetilde{f}_i$, and $i = 1, \ldots, r$. By Lemmas 1.2.2 and 1.2.3 we have

$$I_i = |\mathrm{L}(\widetilde{f}_i)| = \left|\left\{\gamma \in \mathcal{A}(\widetilde{X}) |\; \gamma\, \widetilde{f}_i = \widetilde{f}_i\, \gamma\right\}\right|$$

and

$$J_i = |j(C(f_{\#}, x))| = |j(\{\gamma \in \pi_1(X, x_i)|\ f_{\#}(\gamma) = \gamma\})|$$

for any $x_i \in A_i$ and A_i is any Nielsen class in $p(\Phi(\widetilde{f}_i))$.

The following theorem gives $N(f)$ in terms of the Nielsen number of lifts of f.

Theorem 1.2.13. *Let X be a compact polyhedron, $p : \widetilde{X} \longrightarrow X$ be a finite regular covering corresponding to a normal subgroup H of $\pi_1(X)$, and let $f : X \longrightarrow X$: be a self-map admitting a lift $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$. We assume that for each two Nielsen classes $A, A' \subseteq \Phi(f)$, which represent the same Nielsen class modulo the subgroup H, the numbers $J_A = J_{A'}$. Then,*

$$N(f) = \sum_{i=1}^{r} (J_i/I_i) \cdot N(\widetilde{f}_i)\ , \tag{1.2.5}$$

where I_i and J_i denote the numbers defined above, and the lifts $\widetilde{f}_i$ represent all $H-$Reidemeister classes of f, corresponding to nonempty $H-$Nielsen classes. □

Corollary 1.2.14. *If moreover, under the assumptions of Theorem 1.2.13, $C = J_i/I_i$ does not depend on i, then*

$$N(f) = C \cdot \sum_{i=1}^{r} N(\widetilde{f}_i)\ .$$

□

Chapter 2

Computation of $N(f, g)$ for orientable manifolds

In this chapter, in Theorem 2.3.5, we generalize Theorem 1.2.13. We show that the coincidence Nielsen number of a pair of maps f and g can be presented as a linear combination of the Nielsen number of its pairs of lifts. There are three numbers J, I, and S are associated with an H-Reidemeister class, where H is a normal subgroup of the fundamental group of the co-domain space. Those numbers are used to compute the coefficients in the formulas given in Theorem 2.3.5, the main result in this chapter. As in the fixed point case (Theorem 1.2.13), our applications are limited to situations where the analogous numbers J are independent of certain choices. Furthermore, we show that we only need one coincidence point in each $H-$Nielsen class (in some cases, only one coincidence point in $\Phi(f, g)$ is needed) to make the computations.

2.1 The numbers J, I, and S.

In this section, we generalize the work in [15] related to the numbers I and J defined earlier in Section 1.2. More precisely, to each Nielsen class $A \subseteq \Phi(f,g)$, we assign three numbers namely J_A, I_A and S_A. Under the conditions given, all three numbers are always determined by A (In fact, I_A is determined by the H-Nielsen class containing A). They have both geometric and algebraic interpretations and are intimately inter-related. Moreover, they are the major ingredients in the computations of $N(f,g)$.

Let M and N be path connected, locally path connected topological spaces, and $(\widetilde{M},p)$ and $(\widetilde{N},p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. Let $(f,g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f},\widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. Thus, we have the same commutative diagram 1.1.1.

In [15], in the fixed point case, the author used covering spaces to define I_A and J_A for a Nielsen class $A \subseteq \Phi(f)$, and to investigate the relationship between the indices of the Nielsen classes in the base space and in the total space. He showed that essential classes in the total space are mapped onto essential classes in the base space. Our approach is rather to find a more complicated relationship between the essential classes in both the base and the total spaces.

Definition 2.1.1. *Let $A \subseteq \Phi(f,g)$ and $\widetilde{A} \subseteq \Phi(\widetilde{f},\widetilde{g})$ be Nielsen classes such that $p(\widetilde{A}) = A$, and let $x \in A$. Define J_A by*

$$J_A := |p^{-1}(x) \cap \widetilde{A}| \, .$$

In other words, J_A is the cardinality of the fiber of the map

$$p|\widetilde{A} : \widetilde{A} \longrightarrow A$$

at any point in A.

Remark 2.1.2. *In fact, it can be shown that the map $p|\widetilde{A}$ is a covering map with discrete fibers of cardinalities equal to J_A.*

Definition 2.1.3. *[15] Let $x \in \Phi(f, g)$. We define*

$$C(f_{\#}, g_{\#})_x = \{\gamma \in \pi_1(M, x) |\ f_{\#}(\gamma) = g_{\#}(\gamma)\}$$

The following proposition shows that J_A is well defined. Furthermore, it shows that J_A is the order of a specific subgroup of $\mathcal{A}(\widetilde{M}) \cong \dfrac{\pi_1(M)}{K}$.

Proposition 2.1.4. *Let A be a Nielsen class of f and g, and let $x \in A$. Then,*

$$J_A = |\, j(C(f_{\#}, g_{\#})_x) | \,,$$

where

$$j : \pi_1(M) \longrightarrow \frac{\pi_1(M)}{K}$$

is the natural epimorphism.

Proof. Let $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ be a Nielsen class such that $p(\widetilde{A}) = A$, and let $\widetilde{x}_0 \in p^{-1}(x) \cap \widetilde{A}$. For each $\lambda \in \pi_1(M, x)$, let $\widetilde{\lambda} : \widetilde{x}_0 \longrightarrow \widetilde{\lambda}(1)$ be the unique lift of λ which starts at $\widetilde{x}_0$. Consider the function

$$\varphi : j(C(f_{\#}, g_{\#})_x) \longrightarrow p^{-1}(x) \cap \widetilde{A} \ \ : \ \ \lambda \longmapsto \widetilde{\lambda}(1) \,.$$

• φ is well-defined:

(1) Let $\bar{\lambda}_1, \bar{\lambda}_2 \in j(C(f_\#, g_\#)_x)$. then

$$\begin{aligned}
\bar{\lambda}_1 = \bar{\lambda}_2 &\Rightarrow \lambda_1 \lambda_2^{-1} \in K(x) = p_\# \pi_1(\widetilde{M}, \tilde{x}_0) \\
&\Rightarrow \lambda_1 \lambda_2^{-1} = p(\widetilde{\lambda}) \text{ for some } \widetilde{\lambda} \\
&\Rightarrow \lambda_1 \sim p(\widetilde{\lambda})\, \lambda_2 \text{ rel. endpoints} \\
&\Rightarrow p(\widetilde{\lambda}_1) \sim p(\widetilde{\lambda}) p(\widetilde{\lambda}_2) \text{ rel. endpoints} \\
&\Rightarrow p(\widetilde{\lambda}_1) \sim p(\widetilde{\lambda}\, \widetilde{\lambda}_2) \text{ rel. endpoints} \\
&\Rightarrow \widetilde{\lambda}_1(1) = \widetilde{\lambda}\, \widetilde{\lambda}_2(1) = \widetilde{\lambda}_2(1) \\
&\Rightarrow \varphi(\bar{\lambda}_1) = \varphi(\bar{\lambda}_2)\ .
\end{aligned}$$

(2) Let $\lambda \in\ C(f_\#, g_\#)_x$. Then, $f_\#(\lambda) = g_\#(\lambda)$. Thus,

$$\begin{aligned}
f(\lambda) = g(\lambda) &\Rightarrow f(p\,(\widetilde{\lambda})) = g(p\,(\widetilde{\lambda})) \\
&\Rightarrow p\,(\widetilde{f}(\widetilde{\lambda})) = p\,(\widetilde{g}(\widetilde{\lambda})) \\
&\Rightarrow \widetilde{f}(\widetilde{\lambda}) \sim \widetilde{g}(\widetilde{\lambda}) \text{ rel. endpoints} \\
&\Rightarrow \widetilde{\lambda}(1) \in \widetilde{A} \\
&\Rightarrow \widetilde{\lambda}(1) \in \widetilde{A} \cap p^{-1}(x)\ .
\end{aligned}$$

• φ is injective: Let $\bar{\lambda}_1, \bar{\lambda}_2 \in j(C(f_\#, g_\#)_x)$. Then

$$\begin{aligned}
\varphi(\bar{\lambda}_1) = \varphi(\bar{\lambda}_2) &\Rightarrow \widetilde{\lambda}_1(1) = \widetilde{\lambda}_2(1) \\
&\Rightarrow \widetilde{\lambda}_1\, \widetilde{\lambda}_2^{-1} \in \pi_1(\widetilde{M}, \tilde{x}_0) \\
&\Rightarrow \lambda_1\, \lambda_2^{-1} = p\,(\widetilde{\lambda}_1) p\,(\widetilde{\lambda}_2^{-1}) = p\,(\widetilde{\lambda}_1 \widetilde{\lambda}_2^{-1}) \in p_\# \pi_1(\widetilde{M}, \tilde{x}_0) \\
&\Rightarrow \bar{\lambda}_1 = \bar{\lambda}_2
\end{aligned}$$

• φ is surjective: Let $\widetilde{x} \in p^{-1}(x) \cap \widetilde{A}$, $\widetilde{\omega} : \widetilde{x}_0 \longrightarrow \widetilde{x}$ be a path such that $\widetilde{f}(\widetilde{\omega}) \sim \widetilde{g}(\widetilde{\omega})$ rel. endpoints (which exists since $\widetilde{x}_0, \widetilde{x} \in \widetilde{A}$), and let $\omega = p(\widetilde{\omega}) \in \pi_1(M, x)$. Then

$$\begin{aligned} \widetilde{f}(\widetilde{\omega}) \sim \widetilde{g}(\widetilde{\omega}) \quad &\Rightarrow \quad f(\omega) = g(\omega) \\ &\Rightarrow \quad \omega \in C(f_\#, g_\#)_x \\ &\Rightarrow \quad \varphi(\bar{\omega}) = \widetilde{\omega}(1) = \widetilde{x}. \end{aligned}$$

Consequently, $J_A = |\, j(C(f_\#, g_\#)_x)|$. □

Remark 2.1.5. *If we change the base point $\widetilde{x}_0 \in p^{-1}(x) \cap \widetilde{A}$ and follow the same argument as above, we find that J_A is independent of $\widetilde{x}_0 \in p^{-1}(x) \cap \widetilde{A}$.*

Proposition 2.1.6. *Let $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ be a Nielsen class such that $p(\widetilde{A}) = A$, then J_A is independent of the choice of $x \in A$.*

Proof. By the definition of normal subgroup K, if z is another point in A and $\delta : x \longrightarrow z$ is a path in M such that $f(\delta)$ is homotopic to $g(\delta)$ rel. endpoints, then we have the commutative diagram

$$\begin{array}{ccc} \pi_1(M, x) & \xrightarrow{\delta_\#} & \pi_1(M, z) \\ j \downarrow & & \downarrow j \\ \dfrac{\pi_1(M, x)}{K(x)} & \xrightarrow{\overline{\delta}_\#} & \dfrac{\pi_1(M, z)}{K(z)} \end{array}$$

where $\delta_\#$ is the isomorphism induced by the path δ and $\overline{\delta}_\#$ is the isomorphism induced by $\delta_\#$ on the quotient groups. Moreover, the restriction $\delta_\# : C(f_\#, g_\#)_x \longrightarrow$

$C(f_{\#}, g_{\#})_z$ is also an isomorphism. Hence, the commutativity of the diagram ensures that $j(C(f_{\#}, g_{\#})_x)$ and $j(C(f_{\#}, g_{\#})_z)$ are isomorphic. This yields that J_A is independent of the choice of $x \in A$. $\square$

Remark 2.1.7. *Since the above argument is the same for each Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ with $p(\widetilde{A}) = A$, we conclude that J_A depends only on the Nielsen class A. That is, if $\widetilde{A}$, $\widetilde{B} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ are Nielsen classes such that $p(\widetilde{A}) = p(\widetilde{B}) = A$, then*

$$|p^{-1}(x) \cap \widetilde{A}| = |p^{-1}(x) \cap \widetilde{B}| \,.$$

Definition 2.1.8. *Let $A \subseteq \Phi(f, g)$ and $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ be Nielsen classes such that $p(\widetilde{A}) = A$, and let $x \in A$. Define I_A by*

$$I_A := |p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})|$$

In other words, I_A is the cardinality of the fiber of the map

$$p|\Phi(\widetilde{f}, \widetilde{g}) : \Phi(\widetilde{f}, \widetilde{g}) \longrightarrow p\left(\Phi(\widetilde{f}, \widetilde{g})\right)$$

at any point of $A \cap \Phi(f, g)$.

Remark 2.1.9. *In fact, it can be shown that the map $p|\Phi(\widetilde{f}, \widetilde{g})$ is a covering map with discrete fibers of cardinalities equal to I_A.*

Let $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g) and $\alpha \in \mathcal{A}(\widetilde{M})$. Since $\widetilde{f}\alpha$ and $\widetilde{g}\alpha$ are lifts of f and g respectively, there are unique elements β, $\acute{\beta} \in \mathcal{A}(\widetilde{N})$ (Proposition 1.1.10) that satisfy

$$\widetilde{f} \circ \alpha = \beta \circ \widetilde{f} \quad \text{and} \quad \widetilde{g} \circ \alpha = \acute{\beta} \circ \widetilde{g}\,.$$

Definition 2.1.10. *We define the number* $\delta\,(\widetilde{f},\widetilde{g}\,;\,\alpha)$ *by*

$$\delta\,(\widetilde{f},\widetilde{g}\,;\,\alpha) = \begin{cases} 0 & \mathit{if}\ \beta \neq \acute{\beta} \\ 1 & \mathit{if}\ \beta = \acute{\beta}\,. \end{cases}$$

Definition 2.1.11. *We define the set* $L(\widetilde{f},\widetilde{g})$ *by*

$$L(\widetilde{f},\widetilde{g}) = \left\{\alpha \in \mathcal{A}(\widetilde{M}) |\, \delta\,(\widetilde{f},\widetilde{g}\,;\,\alpha) = 1\right\}\,.$$

The following Proposition gives some facts about $\mathrm{L}(\widetilde{f},\widetilde{g})$.

Proposition 2.1.12. *Let* $(\widetilde{f},\widetilde{g})$ *be a lift of* (f,g) *and* $\beta \in \mathcal{A}(\widetilde{N})$. *Then,*

1. $L(\widetilde{f},\widetilde{g})$ *is a subgroup of* $\mathcal{A}(\widetilde{M})$.

2. $L(\widetilde{f},\widetilde{g}) = L(\beta\,\widetilde{f},\beta\,\widetilde{g})$.

Proof. (1) By definition, $1_{\widetilde{M}} \in \mathrm{L}(\widetilde{f},\widetilde{g})$. Let $\alpha_1\,,\,\alpha_2\,,\alpha \in \mathrm{L}(\widetilde{f},\widetilde{g})$. Then,

$$\begin{aligned} \widetilde{f}(\alpha_1\,\alpha_2) &= (\widetilde{f}\,\alpha_1)\,\alpha_2 \\ &= (\beta_1\,\widetilde{f})\,\alpha_2 \quad \text{for some} \quad \beta_1 \in \mathcal{A}(\widetilde{N}) \\ &= \beta_1\,(\widetilde{f}\,\alpha_2) \\ &= \beta_1\,(\beta_2\,\widetilde{f}) \quad \text{for some} \quad \beta_2 \in \mathcal{A}(\widetilde{N}) \\ &= (\beta_1\,\beta_2)\,\widetilde{f} \end{aligned}$$

Similarly, we get that $\widetilde{g}(\alpha_1\,\alpha_2) = (\beta_1\,\beta_2)\,\widetilde{g}$. Therefore, $\alpha_1\,\alpha_2 \in \mathrm{L}(\widetilde{f},\widetilde{g})$.
On the other hand,

$$\left.\begin{array}{l}\widetilde{f}\,\alpha=\beta\,\widetilde{f}\\ \widetilde{g}\,\alpha=\beta\,\widetilde{g}\end{array}\right\}\quad\Rightarrow\quad\left.\begin{array}{l}\beta^{-1}\widetilde{f}=\widetilde{f}\,\alpha^{-1}\\ \beta^{-1}\widetilde{g}=\widetilde{g}\,\alpha^{-1}\end{array}\right\}\quad\Rightarrow\quad\alpha^{-1}\in \mathrm{L}(\widetilde{f},\widetilde{g}).$$

So, $\mathrm{L}(\widetilde{f},\widetilde{g})$ is a subgroup of $\mathcal{A}(\widetilde{M})$.

(2) Let $\alpha\in \mathrm{L}(\widetilde{f},\widetilde{g})$ and $\gamma\in\mathcal{A}(\widetilde{N})$ such that $\widetilde{f}\,\alpha=\gamma\,\widetilde{f}$ and $\widetilde{g}\,\alpha=\gamma\,\widetilde{g}$. Then

$$\left.\begin{array}{l}(\beta\,\widetilde{f})\,\alpha=\beta\,(\widetilde{f}\,\alpha)=\beta\gamma\,\widetilde{f}\\ (\beta\,\widetilde{g})\,\alpha=\beta\,(\widetilde{g}\,\alpha)=\beta\gamma\,\widetilde{g}\end{array}\right\}\quad\Rightarrow\quad\left.\begin{array}{l}(\beta\,\widetilde{f})\,\alpha=\beta\gamma\,\beta^{-1}\,\beta\,\widetilde{f}\\ (\beta\,\widetilde{g})\,\alpha=\beta\gamma\,\beta^{-1}\,\beta\,\widetilde{g}\end{array}\right\}\quad\Rightarrow\quad\left.\begin{array}{l}(\beta\,\widetilde{f})\,\alpha=\beta\gamma\,\beta^{-1}\,(\beta\,\widetilde{f})\\ (\beta\,\widetilde{g})\,\alpha=\beta\gamma\,\beta^{-1}\,(\beta\,\widetilde{g})\end{array}\right\}$$

$\Rightarrow\quad \alpha\in \mathrm{L}(\beta\,\widetilde{f},\beta\,\widetilde{g})$.

Hence, we get that $\mathrm{L}(\widetilde{f},\widetilde{g})\subseteq \mathrm{L}(\beta\,\widetilde{f},\beta\,\widetilde{g})$.

Since the above argument holds for every $\beta\in\mathcal{A}(\widetilde{N})$ and $(\widetilde{f},\widetilde{g})\in Lift(f,g)$, we get

$$L(\beta\,\widetilde{f},\beta\,\widetilde{g})\subseteq L(\beta^{-1}\,\beta\,\widetilde{f},\beta^{-1}\,\beta\,\widetilde{g})=\mathrm{L}(\widetilde{f},\widetilde{g})$$

and (2) follows. $\square$

The next proposition shows that the number I_A is well-defined, i.e., it depends only on the $H-$Nielsen class that contains A. Further, it shows that I_A is equal to the order of a particular subgroup of $\mathcal{A}(\widetilde{M})\cong\dfrac{\pi_1(M)}{K}$.

Proposition 2.1.13. *Let $A\subseteq\Phi(f,g)$ be a Nielsen class and $x\in A$. Then*

$$I_A=\left|L\,(\widetilde{f},\widetilde{g})\right|.$$

Proof. Fix a point $\widetilde{x}_0\in p^{-1}(x)\cap\Phi(\widetilde{f},\widetilde{g})$. Consider the bijection

$$\xi:p^{-1}(x)\longrightarrow\mathcal{A}(\widetilde{M})$$

given by $\xi(\widetilde{x}) = \alpha$, where α is the unique covering transformation in $\mathcal{A}(\widetilde{M})$ with $\alpha(\widetilde{x}_0) = \widetilde{x}$. It follows that the restriction (for simplicity we call it ξ too)

$$\xi : p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g}) \longrightarrow \xi\left(p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})\right) \subseteq \mathcal{A}(\widetilde{M})$$

is also a bijection. We claim that

$$\xi\left(p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})\right) = \mathrm{L}(\widetilde{f}, \widetilde{g}).$$

Let $\alpha \in \xi\left(p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})\right)$, then there exists an $\widetilde{x} \in \widetilde{M}$ such that $\alpha(\widetilde{x}_0) = \widetilde{x}$, with $p(\widetilde{x}) = x$, and $\widetilde{f}(\widetilde{x}) = \widetilde{g}(\widetilde{x})$. Hence,

$$\begin{aligned}
\widetilde{f}(\widetilde{x}) = \widetilde{g}(\widetilde{x}) \quad &\Rightarrow \quad \widetilde{f}(\alpha(\widetilde{x}_0)) = \widetilde{g}(\alpha(\widetilde{x}_0)) \\
&\Rightarrow \quad \widetilde{f} \circ \alpha(\widetilde{x}_0) = \widetilde{g} \circ \alpha(\widetilde{x}_0) \\
&\Rightarrow \quad \beta\, \widetilde{f}(\widetilde{x}_0) = \beta'\, \widetilde{g}(\widetilde{x}_0) = \beta'\, \widetilde{f}(\widetilde{x}_0) \\
&\Rightarrow \quad \beta = \beta' \\
&\Rightarrow \quad \delta(\widetilde{f}, \widetilde{g}\,; \alpha) = 1.
\end{aligned}$$

Thus $\alpha \in \mathrm{L}(\widetilde{f}, \widetilde{g})$.

Now, let $\alpha \in \mathrm{L}(\widetilde{f}, \widetilde{g})$. Then,

$$\begin{aligned}
\delta(\widetilde{f}, \widetilde{g} : \alpha) = 1 \quad &\Rightarrow \quad \widetilde{f}\,\alpha = \beta \widetilde{f} \quad \text{and} \quad \widetilde{g}\,\alpha = \beta\, \widetilde{g} \ \text{ for some } \beta \in \mathcal{A}(\widetilde{N}). \\
&\Rightarrow \quad \widetilde{f}\,\alpha(\widetilde{x}_0) = \beta \widetilde{f}(\widetilde{x}_0) = \beta\, \widetilde{g}(\widetilde{x}_0) = \widetilde{g}\,\alpha(\widetilde{x}_0) \\
&\Rightarrow \quad \alpha(\widetilde{x}_0) \in \Phi(\widetilde{f}, \widetilde{g}).
\end{aligned}$$

Since $\widetilde{x} = \alpha(\widetilde{x}_0) \in p^{-1}(x)$, we get $\alpha \in \xi\left(p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})\right)$.

Consequently, $\xi\left(p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})\right) = \mathrm{L}(\widetilde{f}, \widetilde{g})$, and our result follows. $\square$

Remark 2.1.14. *In the case that $M = N$ and $g = 1_M$ is the identity on M, it is easy to see that*

$$L(\widetilde{f}, 1_{\widetilde{M}}) = \left\{\alpha \in \mathcal{A}(\widetilde{M}) | \, \alpha \widetilde{f} = \widetilde{f}\alpha\right\} = L(\widetilde{f}),$$

where $L(\widetilde{f})$ is defined in Definition 1.2.1. On the other hand, the proof of Proposition 2.1.13 is independent of the choice of $\widetilde{x}_0 \in \Phi(\widetilde{f}, \widetilde{g})$. Since $|L(\widetilde{f}, \widetilde{g})|$ is independent of the choice of $\widetilde{x}_0 \in \Phi(\widetilde{f}, \widetilde{g})$, we have that I_A is independent of the Nielsen class contained in $\Phi(\widetilde{f}, \widetilde{g})$. Therefore, for any pair of lifts $(\widetilde{f}, \widetilde{g})$ of (f, g) we can put $I(\widetilde{f}, \widetilde{g}) = I_A$ for any $A \in \Phi(\widetilde{f}, \widetilde{g})$.

Let x_0 be a coincidence point of f and g and $y_0 = f(x_0)$. Since $f_{\#}(K(x_0)) \cup g_{\#}(K(x_0)) \subseteq H(f(x_0))$, then $f_{\#}$ and $g_{\#}$ induce homomorphisms $\overline{f}_{\#}$ and $\overline{g}_{\#}$ which are defined such that the following diagram is commutative:

$$\begin{array}{ccc}
\pi_1(M, x_0) & \xrightarrow{f_{\#}, g_{\#}} & \pi_1(N, f(x_0)) \\
j \downarrow & & \downarrow j \\
\dfrac{\pi_1(M, x_0)}{K(x_0)} & \xrightarrow{\overline{f}_{\#}, \overline{g}_{\#}} & \dfrac{\pi_1(N, f(x_0))}{H(f(x_0))}
\end{array}.$$

Definition 2.1.15. *Define $C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}$ by*

$$C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0} = \left\{\overline{a} \in \frac{\pi_1(M, x_0)}{K(x_0)} \mid \overline{f}_{\#}(\overline{a}) = \overline{g}_{\#}(\overline{a})\right\}.$$

Let $A \subseteq p\,\Phi(\widetilde{f}, \widetilde{g})$ be a Nielsen class and let $x_0 \in A$. We show that I_A is equal to the order of the subgroup $C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}$. Fix $\widetilde{x}_0 \in p^{-1}(x_0) \cap \Phi(\widetilde{f}, \widetilde{g})$. Let $f(x_0) = g(x_0) = y_0$

and $\widetilde{f}(\widetilde{x_0}) = \widetilde{g}(\widetilde{x_0}) = \widetilde{y_0}$. Define bijections

$$\frac{\pi_1(M, x_0)}{K(x_0)} \xrightarrow{v} \mathcal{A}(\widetilde{M}) \quad : \quad \overline{a} \longmapsto \alpha_{\overline{a}}$$

where $\alpha_{\overline{a}}(\widetilde{x_0}) = \widetilde{a}(1)$, and $\widetilde{a}$ is the lift of a at $\widetilde{x_0}$,

and

$$\frac{\pi_1(M, y_0)}{H(y_0)} \xrightarrow{v} \mathcal{A}(\widetilde{N}) \quad : \quad \overline{b} \longmapsto \beta_{\overline{b}}\ ,$$

where $\beta_{\overline{b}}(\widetilde{y_0}) = \widetilde{b}(1)$, and $\widetilde{b}$ is the lift of b at $\widetilde{y_0}$.

Lemma 2.1.16. *Let $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g). Then,*

$$\widetilde{f}\,\alpha_{\overline{a}} = \beta_{\overline{f(a)}}\,\widetilde{f} \ \ \textit{and} \ \ \widetilde{g}\,\alpha_{\overline{a}} = \beta_{\overline{g(a)}}\,\widetilde{g}\ .$$

Proof. Since $\widetilde{f}(\widetilde{a})$ is a lift of $f(a)$, we have

$$\beta_{\overline{f(a)}}(\widetilde{y_0}) = \widetilde{f}(\widetilde{a})(1)\ .$$

Thus,

$$\widetilde{f}\,\alpha_{\overline{a}}(\widetilde{x_0}) = \widetilde{f}\,(\alpha_{\overline{a}}(\widetilde{x_0})) = \widetilde{f}\,(\widetilde{a}\,(1)) = \widetilde{f}(\widetilde{a})(1) = \beta_{\overline{f(a)}}(\widetilde{y_0}) = \beta_{\overline{f(a)}}(\widetilde{f}\,(\widetilde{x_0}))\ .$$

Since $\widetilde{f}\,\alpha_{\overline{a}}$ and $\beta_{\overline{f(a)}}\,\widetilde{f}$ are lifts of f, we get $\widetilde{f}\,\alpha_{\overline{a}} = \beta_{\overline{f(a)}}\,\widetilde{f}$.

Similarly, $\widetilde{g}\,\alpha_{\overline{a}} = \beta_{\overline{g(a)}}\,\widetilde{g}$. □

Proposition 2.1.17. *Let $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g). Then, there is a bijection between $L(\widetilde{f}, \widetilde{g})$ and $C(\overline{f}_\#, \overline{g}_\#)_{x_0}$.*

Proof. The restriction of the isomorphism υ, which we call it υ too, given just before Lemma 2.1.16

$$C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0} \xrightarrow{\upsilon} \upsilon\left(C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}\right) \subseteq \mathcal{A}(\widetilde{M})$$

is also an isomorphism. We claim that $\upsilon\left(C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}\right) = \mathrm{L}(\widetilde{f}, \widetilde{g})$.

Let $\overline{a} \in C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}$. We have $\widetilde{f}\,\upsilon(\overline{a}) = \upsilon(\overline{f(a)})\,\widetilde{f}$ and $\widetilde{g}\,\upsilon(\overline{a}) = \upsilon(\overline{g(a)})\,\widetilde{g}$. Since $\overline{f(a)} = \overline{g(a)}$, we have $\upsilon(\overline{f(a)}) = \upsilon(\overline{g(a)})$, i.e., $\delta(\widetilde{f}, \widetilde{g}\,; \upsilon(\overline{a})) = 1$, which yields $\upsilon(\overline{a}) \in \mathrm{L}(\widetilde{f}, \widetilde{g})$.

On the other hand, assume $\alpha \in \mathrm{L}(\widetilde{f}, \widetilde{g})$. Hence there exists $\overline{a} \in \dfrac{\pi_1(M, x_0)}{K(x_0)}$, such that $\alpha = \upsilon(\overline{a})$. Thus, $\widetilde{f}\,\upsilon(\overline{a}) = \upsilon(\overline{f(a)})\,\widetilde{f}$ and $\widetilde{g}\,\upsilon(\overline{a}) = \upsilon(\overline{g(a)})\,\widetilde{f}$ with $\upsilon(\overline{f(a)}) = \upsilon(\overline{g(a)})$. Because υ is an isomorphism, $\overline{f(a)} = \overline{g(a)}$ or $\overline{a} \in C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}$. Therefore, $\alpha \in \upsilon\left(C(\overline{f}_{\#}, \overline{g}_{\#})_{x_0}\right)$. □

In [15], Jezierski used the fact that every covering map is a local homeomorphism to exhibit the relationship between the indices of the fixed point classes A and $\tilde{A}$ (Lemma 1.2.5). He used this to derive the formula in Equation 1.2.5. The relationship bewteen the coincidence semi-indices of the analogous coincidence classes A and $\tilde{A}$ turns out to be far more complicated than in the fixed point case. In fact, it turns out that there are cases where A can be essential even when $\tilde{A}$ is not. This will ultimately lead to the definition of our non-linear Nielsen number. Nevertheless, there are several useful results that, under appropriate conditions, do effectively generalize the fixed point case. The complete picture is given in the following sequence of lemmas. The result for index (Proposition 2.3.2) combines Proposition 2.1.21 and Lemma 2.3.1. For semi-index Proposition 3.2.12 is a modified version of Theorem 3.2.10, and Proposition 3.2.13 combines Propositions 2.1.21 and Proposition 3.2.12.

Lemma 2.1.18. *Let $(\widetilde{f},\widetilde{g})$ be a lift of (f,g) and $p_\Phi : \Phi(\widetilde{f},\widetilde{g}) \longrightarrow p\Phi(\widetilde{f},\widetilde{g})$ be the restriction of p to $\Phi(\widetilde{f},\widetilde{g})$. Then,*

1. *We have $p_\Phi^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right) = \Phi(\widetilde{f},\widetilde{g})$.*

2. *We have $p^{-1}(\Phi(f,g)) \cap \Phi(\widetilde{f},\acute{\beta}\cdot\widetilde{g}) = \Phi(\widetilde{f},\acute{\beta}\cdot\widetilde{g})$, for every $\acute{\beta} \in \mathcal{A}(\widetilde{N})$.*

Proof. (1) It is obvious that $p_\Phi^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right) \subseteq \Phi(\widetilde{f},\widetilde{g})$. On the other hand, $\Phi(\widetilde{f},\widetilde{g}) \subseteq p^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right)$, which implies that $\Phi(\widetilde{f},\widetilde{g}) \subseteq p^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right) \cap \Phi(\widetilde{f},\widetilde{g}) = p_\Phi^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right)$. Therefore, $\Phi(\widetilde{f},\widetilde{g}) = p_\Phi^{-1}\, p\left(\Phi(\widetilde{f},\widetilde{g})\right)$.

(2) Follows directly from the facts that

$$p^{-1}\left(\Phi(f,g)\right) = \bigcup_{\beta\in\mathcal{A}(\widetilde{N})} \Phi(\widetilde{f},\beta\cdot\widetilde{g})\ ,$$

and $\Phi(\widetilde{f},\beta\cdot\widetilde{g}) \cap \Phi(\widetilde{f},\acute{\beta}\cdot\widetilde{g}) = \emptyset$ if and only if $\beta \neq \acute{\beta}$. $\square$

Lemma 2.1.19. *Assume we are given finite regular coverings as in Diagram 1.1.1. Then the following are equivalent*

1. *$\Phi(f,g)$ is finite.*

2. *$\Phi(\widetilde{f},\beta\cdot\widetilde{g})$ is finite, for each $\beta \in \mathcal{A}(\widetilde{N})$.*

3. *$\Phi(\widetilde{f},\widetilde{g})$ is finite, for each lift $(\widetilde{f},\widetilde{g})$ of (f,g).*

Proof. (1) implies (2): Assume that $\Phi(f,g)$ is finite. Let $\beta \in \mathcal{A}(\widetilde{N})$. By Lemma 2.1.18, we have

$$\begin{aligned}\Phi(\widetilde{f},\beta\cdot\widetilde{g}) &= p^{-1}(\Phi(f,g))\cap\Phi(\widetilde{f},\beta\cdot\widetilde{g}) = \left(\bigcup_{x\in\Phi(f,g)} p^{-1}(x)\right)\cap\Phi(\widetilde{f},\beta\cdot\widetilde{g})\\ &= \bigcup_{x\in\Phi(f,g)}\left(p^{-1}(x)\cap\Phi(\widetilde{f},\beta\cdot\widetilde{g})\right).\end{aligned}$$

Since the coverings are finite, $p^{-1}(x)\cap\Phi(\widetilde{f},\beta\cdot\widetilde{g})$ is finite. Since $\Phi(f,g)$ is finite, we obtain that $\Phi(\widetilde{f},\beta\cdot\widetilde{g})$ is finite.

(2) implies (3) directly from the facts, firstly that every lift $(\widetilde{f},\widetilde{g})$ of (f,g) is conjugate to a lift $(\widetilde{f},\beta\cdot\widetilde{g})$ for some $\beta\in\mathcal{A}(\widetilde{N})$, and secondly that coincidence sets corresponding to conjugate lifts have the same cardinality.

(3) implies (1) since $\displaystyle\bigcup_{\beta\in\mathcal{A}(\widetilde{N})} p\,\Phi(\widetilde{f},\beta\cdot\widetilde{g}) = \Phi(f,g)$, and $|\mathcal{A}(\widetilde{N})|$ is finite. □

Definition 2.1.20. *Assume we are given finite regular coverings as in Diagram 2.1.1. Let $A\subseteq p\,\phi(\widetilde{f},\widetilde{g})$ be a Nielsen class. We define the number S_A to be the number of Nielsen classes $\widetilde{A}\subseteq\phi(\widetilde{f},\widetilde{g})$ such that $p(\widetilde{A})=A$.*

The following proposition gives important relationships among $|\widetilde{A}|$, $|A|$, $|p_{\widetilde{\Phi}}^{-1}(A)|$, J_A, I_A, and S_A .

Proposition 2.1.21. *Assume that $\Phi(f,g)$ is finite. Let $A \subseteq \Phi(f,g)$ and $\widetilde{A} \subseteq \Phi(\widetilde{f},\widetilde{g})$ be Nielsen classes such that $p(\widetilde{A}) = A$. Then,*

$$(1) \qquad |\widetilde{A}| = J_A \cdot |A| \,. \tag{2.1.1}$$

$$(2) \quad |p_{\Phi}^{-1}(A)| = I_A \cdot |A| \,. \tag{2.1.2}$$

$$(3) \qquad S_A = \frac{I_A}{J_A} \,. \tag{2.1.3}$$

Proof. (1) Since the family $\left\{\widetilde{A} \cap p^{-1}(x) | \text{ for all } x \in A\right\}$ is a partition of $\widetilde{A}$ and $J_A = |p^{-1}(x) \cap \widetilde{A}|$, we have

$$|\widetilde{A}| = \sum_{x\in A} |\widetilde{A} \cap p^{-1}(x)| = \sum_{x \in A} J_A = J_A \cdot |A| \,.$$

(2) By Lemma 2.1.18, we get

$$\begin{aligned} p_{\Phi}^{-1}(A) &= p_{\Phi}^{-1}\left(A \cap p\left(\Phi(\widetilde{f},\widetilde{g})\right)\right) = p_{\Phi}^{-1}(A) \cap p_{\Phi}^{-1}\left(p\left(\Phi(\widetilde{f},\widetilde{g})\right)\right) = p_{\Phi}^{-1}(A) \cap \Phi(\widetilde{f},\widetilde{g}) \\ &= \bigcup_{x\in A} p_{\Phi}^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g}) \,. \end{aligned}$$

Hence,

$$|p_{\Phi}^{-1}(A)| = \sum_{x\in A} \underbrace{|p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})|}_{I_A} = \sum_{x\in A} I_A = I_A \cdot |A| \,.$$

(3) Let $p_{\Phi}^{-1}(A) = \bigcup_{j=1}^{j=S_A} \widetilde{A}_j$, where $\widetilde{A}_j$ is a Nielsen class of $\widetilde{f}$ and $\widetilde{g}$ such that $p(\widetilde{A}_j) = A$, for every j with $1 \leq j \leq S_A$. Using the same notations as in Lemma 2.1.18, by (1) we have

$$|p_{\Phi}^{-1}(A)| = \sum_{j=1}^{S_A} |\widetilde{A}_j| = \sum_{j=1}^{S_A} J_A \cdot |A| = S_A \cdot J_A \cdot |A|$$

or

$$S_A = \frac{|p_\Phi^{-1}(A)|}{J_A \cdot |A|}$$

which implies by (2) that

$$S_A = \frac{I_A}{J_A}\,.$$

□

Corollary 2.1.22. *If $A \subseteq p\,\Phi(\widetilde{f},\widetilde{g})$ is a Nielsen class, then*

$$S_A = \frac{|\,L(\widetilde{f},\widetilde{g})|}{|j\,(C(f_\#,g_\#)_{x_0})\,|} = \frac{|C(\overline{f}_\#,\overline{g}_\#)_{x_0}|}{|j\,(C(f_\#,g_\#)_{x_0})\,|} \qquad (2.1.4)$$

Proof. Apply Propositions 2.1.13, 2.1.17, and 2.1.21. □

The above results show that the numbers J, I, and S depend only on the Nielsen class or the H-Nielsen class. Next, we show that one coincidence point is sufficient to compute those numbers, for all Nielsen classes (and of course for all H-Nielsen classes).

Definition 2.1.23. *Let x and z be coincidence points and $\omega : x \to z$ be a path. We denote the loop $g(\omega)f(\omega)^{-1}$ by h_ω, and define $C(f_\#^{h_\omega}, g_\#)_x$ by*

$$\begin{aligned} C(f_\#^{h_\omega}, g_\#)_x &= \{\lambda \in \pi_1(M,x) |\; f_\#^{h_\omega}(\lambda) = g_\#(\lambda)\} \\ &= \{\lambda \in \pi_1(M,x) |\; h_\omega\, f(\lambda) = g(\lambda)\, h_\omega\}\,, \end{aligned}$$

where $f_\#^{h_\omega} = h_\omega \circ f_\# \circ h_\omega^{-1}$. We also define $C(\overline{f_\#^{h_\omega}}, \overline{g}_\#)_x$ by

$$\begin{aligned} C(\overline{f_\#^{h_\omega}}, \overline{g}_\#)_x &= \left\{\overline{\lambda} \in \frac{\pi_1(M,x)}{K(x)} |\; \overline{f_\#^{h_\omega}}(\overline{\lambda}) = \overline{g}_\#(\overline{\lambda})\right\} \\ &= \left\{\overline{\lambda} \in \frac{\pi_1(M,x)}{K(x)} |\; \overline{h_\omega}\,\overline{f(\lambda)} = \overline{g(\lambda)}\,\overline{h_\omega}\right\}\,. \end{aligned}$$

where $\overline{f_{\#}^{h_\omega}}$ *and* $\overline{g}_{\#}$ *are the homomorphisms induced by* $f_{\#}^{h_\omega}$ *and* $g_{\#}$.
It is obvious that $C(f_{\#}^{h_w}, g_{\#})_x$ *and* $C(\overline{f_{\#}^{h_w}}, \overline{g}_{\#})_x$ *are subgroups of* $\pi_1(M,x)$ *and* $\dfrac{\pi_1(M,x)}{K(x)}$ *respectively.*

Lemma 2.1.24. *Let* $x_0 \in \Phi(f,g)$. *Then,*

$$f_{\#}\left(C(f_{\#}, g_{\#})_{x_0}\right) = g_{\#}\left(C(f_{\#}, g_{\#})_{x_0}\right)$$

$\square$

The following Proposition generalizes Lemma 1.2.11. It also shows that $C(f_{\#}, g_{\#})$ and $C(\overline{f}_{\#}, \overline{g}_{\#})$, and hence I, J and S, can be computed using a single coincidence point.

Proposition 2.1.25. *Let* x_0 *and* x *be coincidence points and* $\omega : x_0 \to x$ *be a path. Then,*

1. $\omega_{\#}\left(C(f_{\#}^{h_w}, g_{\#})_{x_0}\right) = C(f_{\#}, g_{\#})_x$.

2. $C(f_{\#}^{h_w}, g_{\#})_{x_0} = C(f_{\#}, g_{\#})_{x_0}$ *if and only if* h_w *commutes with* $f_{\#}\left(C(f_{\#}, g_{\#})_{x_0}\right)$.

3. $\overline{\omega}_{\#}\left(C(\overline{f_{\#}^{h_\omega}}, \overline{g}_{\#})_{x_0}\right) = C(\overline{f}_{\#}, \overline{g}_{\#})_x$.

where $\omega_{\#}(\lambda) = \omega^{-1}\lambda\omega$ *for each* $\lambda \in \pi_1(M, x_0)$ *and* $\overline{\omega}_{\#}$, *is the isomorphisms induced by* $\omega_{\#}$.

Proof. (1) Recall that $\omega_\#$ is an isomorphism from $\pi_1(M, x_0)$ to $\pi_1(M, x)$. Let $\sigma \in C(f_\#, g_\#)_x$ and $\lambda \in \pi_1(M, x_0)$ such that $\sigma = \omega_\#(\lambda)$. Then,

$$\begin{aligned}
\sigma \in C(f_\#, g_\#)_x &\Leftrightarrow f(\sigma) = g(\sigma) \\
&\Leftrightarrow f(\omega^{-1}\lambda\omega) = g(\omega^{-1}\lambda\omega) \\
&\Leftrightarrow f(\omega^{-1})\, f(\lambda)\, f(\omega) = g(\omega^{-1})\, g(\lambda)\, g(\omega) \\
&\Leftrightarrow g(\omega)\, f(\omega^{-1})\, f(\lambda) = g(\lambda)\, g(\omega)\, f(\omega^{-1}) \\
&\Leftrightarrow h_\omega\, f(\lambda) = g(\lambda)\, h_\omega \\
&\Leftrightarrow \lambda \in C(f_\#^{h_\omega}, g_\#)_{x_0} \\
&\Leftrightarrow \sigma \in \omega_\# \left(C(f_\#^{h_\omega}, g_\#)_{x_0} \right) .
\end{aligned}$$

(2) Assume that $C(f_\#^{h_\omega}, g_\#)_{x_0} = C(f_\#, g_\#)_{x_0}$. Let $\lambda \in C(f_\#, g_\#)_{x_0}$. Then,

$$h_\omega\, f(\lambda) = g(\lambda)\, h_\omega = f(\lambda)\, h_\omega .$$

Thus, h_ω commutes with $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$.

Now suppose that h_ω commutes with $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$. We have,

$$\begin{aligned}
\lambda \in \left(C(f_\#, g_\#)_{x_0}\right) &\Leftrightarrow f(\lambda) = g(\lambda) \\
&\Leftrightarrow f(\lambda)\, h_\omega = g(\lambda)\, h_\omega \\
&\Leftrightarrow h_\omega\, f(\lambda) = g(\lambda)\, h_\omega \\
&\Leftrightarrow \lambda \in C(f_\#^{h_\omega}, g_\#)_{x_0} .
\end{aligned}$$

Therefore, $C(f_\#^{h_\omega}, g_\#)_{x_0} = C(f_\#, g_\#)_{x_0}$.

(3) Let $\overline{a} \in \dfrac{\pi_1(M, x_0)}{K(x_0)}$ and e be the identity of $H(f(x_0))$. Since $\overline{\omega}_\#^{-1} = \overline{\omega_\#^{-1}}$, we

have

$$\bar{b} = \bar{\omega}_\#(\bar{a}) \text{ and } \bar{a} \in C(\overline{f_\#^{h_\omega}}, \bar{g}_\#)_{x_0} \Leftrightarrow \bar{b} = \bar{\omega}_\#(\bar{a}) \text{ and } \overline{h_\omega}\,\bar{f}_\#(\bar{a}) = \bar{g}_\#(\bar{a})\,\overline{h_\omega}$$
$$\Leftrightarrow \overline{h_\omega}\,\bar{f}_\#\left(\bar{\omega}_\#^{-1}(\bar{b})\right) = \bar{g}_\#\left(\bar{\omega}_\#^{-1}(\bar{b})\right)\,\overline{h_\omega}$$
$$\Leftrightarrow \overline{h_\omega}\,\bar{f}_\#\left(\overline{\omega_\#^{-1}(b)}\right) = \bar{g}_\#\left(\overline{\omega_\#^{-1}(b)}\right)\,\overline{h_\omega}$$
$$\Leftrightarrow \overline{h_\omega}\,\overline{f_\#\left(\omega_\#^{-1}(b)\right)} = \overline{g_\#\left(\omega_\#^{-1}(b)\right)}\,\overline{h_\omega}$$
$$\Leftrightarrow \overline{g(\omega)\,f(\omega^{-1})}\;\overline{f\,(\omega\, b\, \omega^{-1})} = \overline{g\,(\omega\, b\, \omega^{-1})}\;\overline{g(\omega)\,f(\omega^{-1})}$$
$$\Leftrightarrow \overline{g(\omega)\,f(\omega^{-1})}\;\overline{f(\omega)\,f(b)\,f(\omega^{-1})} = \overline{g(\omega)\,g(b)\,g(\omega^{-1})}\;\overline{g(\omega)\,f(\omega^{-1})}$$
$$\Leftrightarrow \overline{g(\omega)\,f(\omega^{-1})\,f(\omega)\,f(b)\,f(\omega^{-1})} = \overline{g(\omega)\,g(b)\,g(\omega^{-1})\,g(\omega)\,f(\omega^{-1})}$$
$$\Leftrightarrow \overline{g(\omega)\,f(b)\,f(\omega^{-1})} = \overline{g(\omega)\,g(b)\,f(\omega^{-1})}$$
$$\Leftrightarrow \overline{(g(\omega)\,g(b)\,f(\omega^{-1}))^{-1}\;\, g(\omega)\,f(b)\,f(\omega^{-1})} = e$$
$$\Leftrightarrow \overline{f(\omega)\,g(b)^{-1}\,g(\omega^{-1})\,g(\omega)\,f(b)\,f(\omega^{-1})} = e$$
$$\Leftrightarrow \overline{f(\omega)\,g(b)^{-1}\,f(b)\,f(\omega^{-1})} = e$$
$$\Leftrightarrow f(\omega)\,g(b)^{-1}\,f(b)\,f(\omega^{-1}) = h \in H(f(x_0))$$
$$\Leftrightarrow g(b)^{-1}\,f(b) = f(\omega)^{-1}\,h\,f(\omega) \in H(f(x))$$
$$\Leftrightarrow \overline{g(b)} = \overline{f(b)}$$
$$\Leftrightarrow \bar{g}_\#(\bar{b}) = \bar{f}_\#(\bar{b})$$
$$\Leftrightarrow \bar{b} \in C(\bar{f}_\#, \bar{g}_\#)_x\,.$$

□

Remark 2.1.26. *If, in Proposition 2.1.25, x and x_0 belong to the same H-Nielsen class, then $h_\omega = 1$, and hence*

$$C(\overline{f_\#^{h_\omega}}, \bar{g}_\#)_{x_0} = C(\bar{f}_\#, \bar{g}_\#)_{x_0}\,.$$

The following definition allows us to change from the covering space approach to the fundamental group approach.

Definition 2.1.27. *Let $\widetilde{\Phi}(f,g)$ be the set of all nonempty Nielsen classes of f and g. Fix $x_0 \in \Phi(f,g)$. For every $x \in \Phi(f,g)$, define ω and h_ω as in Definition 2.1.23. Consider the injection*

$$\rho : \widetilde{\Phi}(f,g) \longrightarrow \Re(f_\#, g_\#) \quad : \quad [x] \longmapsto [h_\omega] \, .$$

We define

$$J_{[h_\omega]} = |\, j\,(C(f_\#, g_\#)_x)\,| = |\, j\,(\omega_\#\,(C(f_\#, g_\#)_{x_0}))\,|$$

and

$$I_{[h_\omega]} = |\, C(\overline{f_\#}, \overline{g}_\#)_x\,| = |\,\overline{\omega}_\#\left(C(\overline{f_\#^{h_\omega}}, \overline{g}_\#)_{x_0}\right)| \, .$$

Corollary 2.1.28. *Let x_0 and x be coincidence points and $\omega : x_0 \to x$ be a path. Then $\omega_\#\,(C(f_\#, g_\#)_{x_0}) = C(f_\#, g_\#)_x$ if and only if h_ω commutes with $f_\#\,(C(f_\#, g_\#)_{x_0})$.*

Proof. Suppose $\omega_\#\,(C(f_\#, g_\#)_{x_0}) = C(f_\#, g_\#)_x$. Thus, $C(f_\#, g_\#)_{x_0} = \omega_\#^{-1}\,(C(f_\#, g_\#)_x)$. By (1) of Proposition 2.1.25, $C(f_\#, g_\#)_{x_0} = C(f_\#^{h_\omega}, g_\#)_{x_0}$. By (2) of Proposition 2.1.25, we obtain that h_ω commutes with $f_\#\,(C(f_\#, g_\#)_{x_0})$.

Now assume that h_ω commutes with $f_\#\,(C(f_\#, g_\#)_{x_0})$. Part (2) of Proposition 2.1.25 implies that $C(f_\#, g_\#)_{x_0} = C(f_\#^{h_\omega}, g_\#)_{x_0}$. By (1) of Proposition 2.1.25, we have $C(f_\#, g_\#)_{x_0} = \omega_\#^{-1}\,(C(f_\#, g_\#)_x)$. Thus, $\omega_\#\,(C(f_\#, g_\#)_{x_0}) = C(f_\#, g_\#)_x$. □

In order to give applications of the main result, Theorem 3.3.16 of the next chapter, we need to impose the condition the number J be the same for all Nielsen classes that lie in the same $H-$Nielsen class. Proposition 2.1.31 below gives a sufficient condition for this to hold. Actually, Proposition 2.1.31 is a generalization of Lemma 1.2.6.

Lemma 2.1.29. *Let x_0 and x be coincidence points and $\omega : x_0 \to x$ be a path. Then,*

$$h_{\omega^{-1}} = (f(\omega))_{\#}\,(h_\omega^{-1}) = (g(\omega))_{\#}\,(h_\omega^{-1})$$

Proof. For the first equality,

$$\begin{aligned}
(f(\omega))_{\#}\,(h_\omega) &= f(\omega)^{-1}\,h_\omega\,f(\omega) \\
&= f(\omega)^{-1}\,g(\omega)\,f(\omega)^{-1}\,f(\omega) \\
&= f(\omega)^{-1}\,g(\omega) \\
&= \left(g(\omega)^{-1}\,f(\omega)\right)^{-1} \\
&= h_{\omega^{-1}}^{-1}
\end{aligned}$$

Since $(f(\omega))_{\#}$ is a homomorphism, we get that

$$h_{\omega^{-1}} = (f(\omega))_{\#}\,(h_\omega^{-1})\,.$$

Similarly we can show that

$$h_{\omega^{-1}} = (g(\omega))_{\#}\,(h_\omega^{-1})\,.$$

□

Corollary 2.1.30. *Let x_0 and x be coincidence points and $\omega : x_0 \to x$ be a path. Then, h_ω commutes with $f_{\#}\,(C(f_{\#}, g_{\#})_{x_0})$ if and only if $h_{\omega^{-1}}$ commutes with $f_{\#}\,(C(f_{\#}, g_{\#})_x)$.*

Proof. The proof depends on Corollary 2.1.28, where we saw that

h_ω commutes with $f_{\#}\,(C(f_{\#}, g_{\#})_{x_0}) \Leftrightarrow \omega_{\#}\,(C(f_{\#}, g_{\#})_{x_0}) = C(f_{\#}, g_{\#})_x$

$\Leftrightarrow \omega_{\#}^{-1}\,(C(f_{\#}, g_{\#})_x) = C(f_{\#}, g_{\#})_{x_0}$

$\Leftrightarrow h_{\omega^{-1}}$ commutes with $f_{\#}\,(C(f_{\#}, g_{\#})_x)$. □

Proposition 2.1.31. *Let x_0 and x be in the same H-Nielsen class, and $\omega : x_0 \to x$ be a path that establishes the H-Nielsen relation. If h_ω commutes with $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$ then $J_{[x_0]} = J_{[x]}$.*

Proof. By Corollary 2.1.28, $\omega_\# : C(f_\#, g_\#)_{x_0} \longrightarrow C(f_\#, g_\#)_x$ is an isomorphism. This isomorphism induces

$$\overline{\omega}_\# : j\left(C(f_\#, g_\#)_{x_0}\right) \longrightarrow j\left(C(f_\#, g_\#)_x\right) \quad : \quad \overline{a} \longmapsto \overline{\omega_\#(a)}\,.$$

We show that $\overline{\omega}_\#$ is a well defined isomorphism.

• It is a well defined injection: let $\overline{a}, \overline{b} \in j\left(C(f_\#, g_\#)_{x_0}\right)$. Since $\omega_\# \left(K(x_0)\right) = K(x)$, we have

$$\begin{aligned} \overline{a} = \overline{b} \;&\Leftrightarrow\; a\,b^{-1} \in K(x_0) \\ &\Leftrightarrow\; \omega_\#(a\,b^{-1}) \in K(x) \\ &\Leftrightarrow\; \omega_\#(a)\,\omega_\#(b^{-1}) \in K(x) \\ &\Leftrightarrow\; \overline{\omega_\#(a)} = \overline{\omega_\#(b)} \end{aligned}$$

• $\overline{\omega}_\#$ is onto since $\omega_\#$ is onto.

Consequently, by Proposition 2.1.4 we get $J_{[x_0]} = J_{[x]}$. □

In what follows, two subgroups of a given group are said to commute if each element in the former commutes with each element of the latter.

Proposition 2.1.32. *Let x_0 and x be in the same H-Nielsen class. Then, $H(f(x_0))$ commutes with $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$ if and only if $H(f(x))$ commutes with $f_\# \left(C(f_\#, g_\#)_x\right)$*

Proof. Let $\omega : x_0 \to x$ be a path that establishes the H-Nielsen relation. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(M, x_0) & \xrightarrow{\omega_\#} & \pi_1(M, x) \\ f_\# \downarrow & & \downarrow f_\# \\ \pi_1(N, f(x_0)) & \xrightarrow{(f(\omega))_\#} & \pi_1(N, f(x)) \end{array} \tag{2.1.5}$$

We need only to show that if $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$ commutes with $H(f(x_0))$ then $H(f(x))$ commutes with $f_\# \left(C(f_\#, g_\#)_x\right)$. Assume that $f_\# \left(C(f_\#, g_\#)_{x_0}\right)$ commutes with $H(f(x_0))$. Let $h \in H(f(x))$ and $\delta \in C(f_\#, g_\#)_x$. Then,

$$\begin{aligned} h\, f(\delta) &= f(\omega)^{-1} f(\omega)\, h\, f(\omega)^{-1} f(\omega)\, f(\delta)\, f(\omega)^{-1} f(\omega) \\ &= f(\omega)^{-1} \underbrace{f(\omega)\, h\, f(\omega)^{-1}}_{\in H(f(x_0))} \quad \underbrace{f(\omega\, \delta\, \omega^{-1})}_{\in f_\#\left(C(f_\#, g_\#)_{x_0}\right)} \quad f(\omega) \\ &= f(\omega)^{-1} f(\omega\, \delta\, \omega^{-1})\, \left(f(\omega)\, h\, f(\omega)^{-1}\right)\, f(\omega) \\ &= f(\delta)\, h \end{aligned}$$

where $\omega\, \delta\, \omega^{-1} \in C(f_\#, g_\#)_{x_0}$ by Corollary 2.1.28. Thus, $H(f(x))$ commutes with $f_\# \left(C(f_\#, g_\#)_x\right)$.

The converse is done similarly. $\square$

Remark 2.1.33. *The previous lemma states the property that $H(f(x))$ and $f_\# \left(C(f_\#, g_\#)_x\right)$ commute with each other, is independent of the choice of x within its H-Nielsen class.*

Corollary 2.1.34. *Let x_0 belong to some H-Nielsen class $p\,\Phi(\widetilde{f},\widetilde{g})$. If $H(f(x_0))$ commutes with $f_\#(C(f_\#,g_\#)_{x_0})$ then $J_A = J_B$ for all Nielsen classes A, $B \subseteq p\,\Phi(\widetilde{f},\widetilde{g})$. In other words, the number J_A depends only on the H-Nielsen class $p\,\Phi(\widetilde{f},\widetilde{g})$. In this case we write $J_A = J(\widetilde{f},\widetilde{g})$.*

Proof. Apply Proposition 2.1.31 and Lemma 2.1.32. □

Remark 2.1.35. *Since the number I depends only on the H-Nielsen class, by Proposition 2.1.21, J only depends on the H-Nielsen class if and only if the number $S = \dfrac{I}{J}$ does. In this case, we write $J_A = J(\widetilde{f},\widetilde{g})$ and $S_A = \dfrac{I_A}{J_A} = S(\widetilde{f},\widetilde{g})$, where A is any Nielsen class in $p\,\Phi(\widetilde{f},\widetilde{g})$.*

2.2 Index, the Lefschetz number, and the Nielsen number.

In this section, we recall the notion of index which is defined for maps on orientable manifolds and describe its properties. After that, we define the Nielsen coincidence number. The material on index can be found in [26].

Let M and N be oriented connected closed manifolds (compact manifolds without boundary) of the same dimension n. Denote the diagonal subset of $M \times M$ by $\Delta(M)$ and the pair $(M \times M,\ M \times M - \Delta(M))$ by $(M^\times)$; similarly for N. Let $z_M \in H_n(M)$ and $z_N \in H_n(N)$ be the respective fundamental classes [26], and $U_M \in H^n(M^\times)$ and $U_N \in H^n(N^\times)$ be the respective Thom classes [26]. Let W be an open subset of

M, and $f, g : W \longrightarrow N$ be maps such that $\Phi(f,g)$ is a compact subset of W. By normality of M, there exists an open set V in M such that $\Phi(f,g) \subseteq V \subseteq \overline{V} \subseteq W$. Consider the composition of the homomorphisms

$$\mathbf{Z} \cong H_n(M) \to H_n(M, M-V) \xrightarrow{excision} H_n(W, W-V) \xrightarrow{(f,g)_*} H_n(N^\times) \overset{h}{\cong} \mathbf{Z}\,.$$

where excision is an isomorphism, $(f,g) : W \longrightarrow N \times N$ is defined by $(f,g)(x) = (f(x), g(x))$, and the isomorphism h sends a homology class a to the integer $\langle U_N, a\rangle$ where $\langle , \rangle$ is the Kronecker index [26].

Definition 2.2.1. *The integer given by the image of the fundamental class z_M by the above composition of homomorphisms is called the index of the pair (f,g) on W and is denoted by index$(f,g;W)$.*

The proof of the following proposition is found in [26]:

Proposition 2.2.2. *Under the above hypotheses, the following properties of index hold:*

1. *Definition 2.2.1 is independent of the choice of the open set V.*
2. *The index is local: If $\acute{W}$ is another open subset of M, and $\acute{f}, \acute{g} : \acute{W} \longrightarrow N$ are maps such that $f = \acute{f}$ and $g = \acute{g}$ on $W \cap \acute{W}$, and if $\Phi(\acute{f},\acute{g}) = \Phi(f,g) \subseteq W \cap \acute{W}$, then*
$$index(\acute{f},\acute{g};\acute{W}) = index(f,g;W)\ .$$
3. *The index is additive: Let $W = W_1 \cup \ldots \cup W_k$ be a disjoint union of open subsets of M such that $\Phi(f,g)$ is a compact subset in W. Then,*
$$index(f,g;W) = index(f|W_1, g|W_1; W_1) + \ldots + index(f|W_k, g|W_k; W_k)\ .$$

4. *If $\Phi(f,g) = \emptyset$, then index$(f,g;W) = 0$. In particular, if index$(f,g;W) \neq 0$, then $\Phi(f,g) \neq \emptyset$.*

5. *The index is homotopy invariant: Suppose $f_t, g_t : W \longrightarrow N$, $0 \leq t \leq 1$, are homotopies. If $\bigcup_{0\leq t\leq 1} \Phi(f_t, g_t; W)$ is a compact subset of W, then*

$$index(f_0, g_0; W) = index(f_1, g_1; W) .$$

Definition 2.2.3. *Let $W \cap \Phi(f,g) = A$ be a Nielsen class of f and g. We define the index of f and g at A by*

$$index(f,g;A) = index(f,g;W) .$$

Remark 2.2.4. *If $M = N$ and $g = 1_W$, then $\Phi(f,g) = \Phi(f)$ is the set of the fixed points of f in W, and the coincidence index agrees with the fixed point index of f on W. Notice that the fixed point index is defined under more general conditions (see section 3, chapter 1, [18]).*

It is convenient to introduce here the concept of the Lefschetz number which will need later. Let M and N be connected closed oriented manifolds of the same dimension n, with fundamental classes z_M and z_N, respectively. Let $f, g : M \longrightarrow N$ be maps. Using the coefficient homomorphism $\mathbf{Z} \xrightarrow{\epsilon} \mathbf{Q}$, we denote by $\overline{z}_M$ and $\overline{z}_N$ the images of z_M and z_N in the rational homology. That is, $\overline{z}_M = \epsilon_*(z_M) \in H_n(M, \mathbf{Q})$ and $\overline{z}_N = \epsilon_*(z_N) \in H_n(N, \mathbf{Q})$. Consider the following diagram:

$$\begin{array}{ccc} H_k(M,\mathbf{Q}) & \xrightarrow{f_*} & H_k(N,\mathbf{Q}) \\ D(M)\downarrow & & \downarrow D(N) \\ H^{n-k}(M,\mathbf{Q}) & \xleftarrow{g^*} & H^{n-k}(N,\mathbf{Q}) \end{array}$$

where $D(M)$ and $D(N)$ are the corresponding Poincare duality isomorphisms. Define $\Theta_k : H_k(M, \mathbf{Q}) \longrightarrow H_k(M, \mathbf{Q})$ by $\Theta_k = D(M)^{-1} \circ g^* \circ D(N) \circ f_*$.

Definition 2.2.5. *The Lefschetz number of the pair (f, g) is defined to be the rational number*

$$L(f,g) = \sum_{k=0}^{n} (-1)^k tr(\Theta_k)$$

where $tr(\Theta_k)$ denotes the trace of the linear transformation Θ_k .

Remark 2.2.6. *If $M = N$ and $g = 1_M$, then we write $L(f, g) = L(f)$ and is called the Lefschetz fixed point number.*

The following theorem gives the basic properties of the Lefschetz number. The proof is found in [26].

Theorem 2.2.7. *Under the hypotheses just before Definition 2.2.5, we have that:*

1. *Lefschetz Coincidence Theorem: $L(f, g) = index(f, g; M)$. Thus, if $L(f, g) \neq 0$ then $\Phi(f, g) \neq \emptyset$.*

2. *Lefschetz Fixed Point Theorem: If $M = N$, then $L(f) = index(f; M)$. Thus, if $L(f) \neq 0$, then $\Phi(f) \neq \emptyset$.*

3. *If $L(f, g) \neq 0$, $\acute{f}$ is homotopic to f and $\acute{g}$ is homotopic to g, then $\acute{f}$ and $\acute{g}$ have a coincidence point.*

Now, we give our notion of a Nielsen number.

Definition 2.2.8. *[21, 22] Let $f, g : M \longrightarrow N$ be maps between topological spaces. A Nielsen number $N(f, g)$ of f and g is a number that has the following properties:*

1. *The number $N(f, g)$ is nonnegative integer.*
2. *The number $N(f, g)$ is homotopy invariant. That is, if $(\acute{f}, \acute{g})$ is a another pair in the category which is homotopic to (f, g) then $N(f, g) = N(\acute{f}, \acute{g})$.(The homotopy usually is compatible with the category under consideration; for instance, the homotopy in the category of pair of spaces is a homotopy of pairs,...etc).*
3. *$N(f, g) \leq |\Phi(f, g)|$.*
4. *$N(f, g)$ is computable in some situations.*

Remark 2.2.9. *The number $N(f, g) \equiv 0$ is a Nielsen number based on Definition 2.2.8. In practice, we will not use a Nielsen number $N(f, g)$ which is everywhere zero.*

The concept of the Nielsen number is usually related to the notion of essentiality. However, there are several definitions of essentiality [2, 5, 18]. We focus in this chapter on the definition that is related to index.

Definition 2.2.10. *Let $(f, g) : M \longrightarrow N$ be maps between oriented connected closed manifolds. A Nielsen class is said to be essential if it has a <u>nonzero</u> index.*

The Nielsen number $N(f, g)$ of f and g is defined to be the number of essential classes.

Remark 2.2.11. *The number $N(f, g)$ is usually called the coincidence Nielsen number of f and g. If $M = N$ and $g = 1_M$, then $N(f, g) = N(f)$ is the fixed point Nielsen number, see [3, 20].*

2.3 Computation of $N(f,g)$

Let $(f,g) : M \to N$ be maps between connected orientable closed manifolds of the same dimension. In this section we show that if the number J is the same for all Nielsen classes that lie in the same H-Nielsen class, then $N(f,g)$ is a linear combination of the Nielsen numbers of the lifts of f and g.

Let M and N be connected orientable closed manifolds of the same dimension n, and $(\widetilde{M},p)$ and $(\widetilde{N},p)$ orientable regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. We assume the coverings are finite, i.e., that $[\pi_1(M) : K] < \infty$ and $[\pi_1(N) : H] < \infty$. Let $(f,g) : M \longrightarrow N$ be a pair of maps which admits a pair of lifts $(\widetilde{f},\widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f,g} & \widetilde{N} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{\widetilde{f},\widetilde{g}} & N \end{array} \tag{2.3.1}$$

Since we can homotop the pair (f,g) to a pair with finite set of coincidences (see Theorem 2 of [23]), without loss of generality, we may assume that $\Phi(f,g)$ is finite. By Lemma 2.1.19, each coincidence point of either (f,g) or $(\widetilde{f},\widetilde{g})$ is isolated. We refer to the proof of Corollary 5.7, [20] for the proof of the following Lemma.

Lemma 2.3.1. *[20] Let $x \in \Phi(f,g)$ and $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})$. Then*

$$index(\widetilde{f},\widetilde{g};\widetilde{x}) = index(f,g;x)\,.$$

□

The following proposition explains the relationship between the indices of the Nielsen classes in the total space and those in the base space. It generalizes Lemma 1.2.5.

Proposition 2.3.2. *Let $(f, g) : M \to N$ be maps between the given orientable manifolds, and let $A \subseteq \Phi(f, g)$ and $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ be Nielsen classes such that $p(\widetilde{A}) = A$. Then*

1. $index(\widetilde{f}, \widetilde{g}; \widetilde{A}) = J_A \cdot index(f, g; A)$.
2. $index\left(\widetilde{f}, \widetilde{g}; p^{-1}(A) \cap \Phi(\widetilde{f}, \widetilde{g})\right) = I_A \cdot index(f, g; A)$.

Proof. 1. Recall that

$$\widetilde{A} = \bigcup_{x \in A} p^{-1}(x) \cap \widetilde{A}$$

is a disjoint union. Since both A and $\widetilde{A}$ consist of isolated coincidence points, we get that

$$\begin{aligned} index(\widetilde{f}, \widetilde{g}; \widetilde{A}) &= \sum_{\widetilde{x} \in \widetilde{A}} index(\widetilde{f}, \widetilde{g}; \widetilde{x}) \\ &= \sum_{x \in A} \sum_{\widetilde{x} \in p^{-1}(x) \cap \widetilde{A}} index(\widetilde{f}, \widetilde{g}; \widetilde{x}) \end{aligned}$$

$$= \sum_{x\in A} \sum_{\tilde{x}\in p^{-1}(x)\cap\widetilde{A}} index(f,g;x) \quad \text{(by Lemma 2.3.1)}$$

$$= \sum_{x\in A} J_A \cdot index(f,g;x)$$

$$= J_A \cdot \sum_{x\in A} index(f,g;x)$$

$$= J_A \cdot index(f,g;A).$$

2. We have that

$$p^{-1}(A) \cap \Phi(\widetilde{f},\widetilde{g}) = \bigcup_{i=1}^{S_A} \widetilde{A}_i$$

where for each $1 \le i \le S_A$, $\widetilde{A}_i$ is a Nielsen class of the pair $(\widetilde{f},\widetilde{g})$ and $p(\widetilde{A}_i) = A$. Thus,

$$\begin{aligned}
index(\widetilde{f},\widetilde{g};p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g})) &= \sum_{i=1}^{S_A} index(\widetilde{f},\widetilde{g};\widetilde{A}_i) \\
&= \sum_{i=1}^{S_A} J_A \cdot index(f,g;A) \quad \text{(by part (1))} \\
&= J_A \cdot \sum_{i=1}^{S_A} index(f,g;A) \\
&= J_A \cdot S_A \cdot index(f,g;A) \\
&= I_A \cdot index(f,g;A)\ .
\end{aligned}$$

□

Corollary 2.3.3. *Let A be a Nielsen class of (f, g) and $\widetilde{A}$ a Nielsen class of $(\widetilde{f}, \widetilde{g})$ such that $p(\widetilde{A}) = A$. Then, A is essential if and only if $\widetilde{A}$ is essential.*

Proof. Apply part (1) of Proposition 2.3.2 noting that J_A is a positive integer. □

Let $(\widetilde{f}_1, \widetilde{g}_1), \ldots, (\widetilde{f}_{R_H(f,g)}, \widetilde{g}_{R_H(f,g)})$ be the representatives of the H-Reidemeister classes of the pair (f, g), and Let r be the number of nonempty H-Nielsen classes of f and g. Thus, $r \leq R_H(f, g)$. Without lose of generality, let $(\widetilde{f}_1, \widetilde{g}_1), \ldots, (\widetilde{f}_r, \widetilde{g}_r)$ be the representatives of the H-Reidemeister classes of the pair (f,g) corresponding to the nonempty H-Nielsen classes. On the other hand, let $\widetilde{\Phi}(f, g)$ be the set of the Nielsen classes of the corresponding pair, and let $\widetilde{\Phi}_E(f, g)$ be the set of the essential Nielsen classes of the corresponding pair. Also, let $\widetilde{p\,\Phi}(\widetilde{f}, \widetilde{g})$ denote the set of Nielsen classes in the H-Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$, and $\widetilde{p\,\Phi}_E(\widetilde{f}, \widetilde{g})$ the set of the essential Nielsen classes that lie in the $H-$Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$. We are ready now to prove our main theorem of this chapter which shows that $N(f, g)$ is a linear combination of the Nielsen numbers of the lifts of (f, g).

Remark 2.3.4. *In the case where $J_A = J_B$ for all Nielsen classes A and B that lie in the same H-Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$, we write $J_A = J(\widetilde{f}, \widetilde{g})$ and $S_A = S(\widetilde{f}, \widetilde{g})$, for every Nielsen classes A that lies in the H-Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$.*

Theorem 2.3.5. *Let M and N be connected closed orientable manifolds of the same dimension, and $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be finite regular coverings which correspond to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$, respectively. Let $f, g : M \longrightarrow N$ be maps for which there exist lifts $\widetilde{f}, \widetilde{g} : \widetilde{M} \longrightarrow \widetilde{N}$, respectively. Suppose the number J_A*

is the same for all Nielsen classes A of f and g lying in the same H-Nielsen class. Then,

$$N(f,g)=\sum_{i=1}^{r}\frac{N(\widetilde{f}_i,\widetilde{g}_i)}{S(\widetilde{f}_i,\widetilde{g}_i)}. \tag{2.3.2}$$

Proof. Define the function $\chi:\mathbf{Z}\to\{0,1\}$ by

$$\chi(m)=\begin{cases}0 & \text{if } m=0,\\ 1 & \text{otherwise}.\end{cases}$$

The number of the essential Nielsen classes that lie in the $H-$Nielsen class $p\,\Phi(\widetilde{f}_i,\widetilde{g}_i)$can be given by

$$|\widetilde{p\,\Phi_E}(\widetilde{f}_i,\widetilde{g}_i)|=\sum_{A\in\widetilde{p\,\Phi_E}(\widetilde{f}_i,\widetilde{g}_i)}\chi\,(index(f,g;A))=\sum_{A\in\widetilde{p\,\Phi}(\widetilde{f}_i,\widetilde{g}_i)}\chi\,(index(f,g;A))\ .$$

Since $J_A=J_B$ for all Nielsen classes $A,B\subseteq\Phi(\widetilde{f}_i,\widetilde{g}_i)$, then $S_A=S_B$ for such classes.

So, we write $S_A = S(\widetilde{f}_i, \widetilde{g}_i)$ for every Nielsen class $A \subseteq p\,\Phi(\widetilde{f}_i, \widetilde{g}_i)$. Thus,

$$\begin{aligned}
N(\widetilde{f}_i, \widetilde{g}_i) &= \sum_{\widetilde{A} \in \widetilde{\Phi}(\widetilde{f}_i, \widetilde{g}_i)} \chi(index(\widetilde{f}_i, \widetilde{g}_i; \widetilde{A})) \\
&= \sum_{A \in \widetilde{p\Phi}(\widetilde{f}_i, \widetilde{g}_i)} \; \sum_{\widetilde{A} \subseteq p^{-1}(A) \cap \Phi(\widetilde{f}_i, \widetilde{g}_i)} \chi(index(\widetilde{f}_i, \widetilde{g}_i; \widetilde{A})) \\
&= \sum_{A \in \widetilde{p\Phi}(\widetilde{f}_i, \widetilde{g}_i)} S_A \cdot \chi(index(f, g; A)) \quad \text{(by Proposition 2.3.2)} \\
&= \sum_{A \in \widetilde{p\Phi}(\widetilde{f}_i, \widetilde{g}_i)} S(\widetilde{f}_i, \widetilde{g}_i) \cdot \chi(index(f, g; A)) \\
&= S(\widetilde{f}_i, \widetilde{g}_i) \cdot \sum_{A \in \widetilde{p\Phi}(\widetilde{f}_i, \widetilde{g}_i)} \chi(index(f, g; A)) \\
&= S(\widetilde{f}_i, \widetilde{g}_i) \cdot |\widetilde{p\Phi}_E(\widetilde{f}_i, \widetilde{g}_i)| \, .
\end{aligned}$$

Therefore,

$$|\widetilde{p\Phi}_E(\widetilde{f}_i, \widetilde{g}_i)| = \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} \, . \tag{2.3.3}$$

for each $1 \leq i \leq r$. Now, we have

$$N(f, g) = \sum_{i=1}^{r} |\widetilde{p\Phi}_E(\widetilde{f}_i, \widetilde{g}_i)| = \sum_{i=1}^{r} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}$$

as required. □

Remark 2.3.6. *For an empty $H-$Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$, we have $N(\widetilde{f}, \widetilde{g}) = 0$ and $|\widetilde{p\Phi}(\widetilde{f}, \widetilde{g})| = 0$. If we define $S(\widetilde{f}, \widetilde{g}) = 1$, Then , Equation 2.3.3 still holds for empty $H-$Nielsen classes. Hence, we can replace r in Equation 2.3.2 by $R_H(f, g)$.*

Corollary 2.3.7. *If in Theorem 2.3.5 we further have the condition "$S(\widetilde{f},\widetilde{g})$ is equal to a constant number q for every lifting pair of (f,g)" Then,*

$$N(f,g) = \frac{1}{q} \cdot \sum_{i=1}^{r} N(\widetilde{f}_i, \widetilde{g}_i) . \tag{2.3.4}$$

2.4 Applications and Examples

In this section, we give applications and examples for Theorem 2.3.5. We show that our theorem generalizes the fixed point case [15].

We re-write the following results from [3, 6, 7, 20] in the notation of this thesis.

Lemma 2.4.1. *[6, 7, 20] If M is a compact orientable manifold, A is an isolated fixed-point set for $f : M \to M$, then $index(f, 1_M; A) = index(f; A)$, where $index(f; A)$ is the fixed-point index of A.*

Theorem 2.4.2. *[3] If M is a compact orientable manifold, then $N(f, 1_M) = N(f)$ for all $f : M \to M$.*

In Chapter 3, we can delete the condition of orientability in Theorem 2.4.2. The following result shows that Theorem 2.3.5 generalizes Theorem 1.2.13 to the Coincidence Theory on orientable manifolds.

Theorem 2.4.3. *Let M be a connected closed orientable manifold, $(\widetilde{M}, p)$ be a finite regular covering of M, and $f : M \to M$ be a map for which there exists a lift*

$\widetilde{f} : \widetilde{M} \longrightarrow \widetilde{M}$. *Assume that all the Nielsen fixed point classes that lie in the same H-Nielsen class have the same number J. Then,*

$$N(f) = \sum_{i=1}^{r} \frac{J(\widetilde{f}_i)}{I(\widetilde{f}_i)} N(\widetilde{f}_i) .$$

where r is the number of nonempty H-Reidemeister classes of f, and $\widetilde{f}_i$ is a collection of one representative from each of these classes .

Proof. Apply Theorem 2.3.5 and Theorem 2.4.2 for $M = N$, $(\widetilde{M}, p) = (\widetilde{N}, p)$, and $g = 1_M$. □

Next, we list some special cases of Theorem 2.3.5. For the definitions of Jiang space and of pseudo Jiang maps, we refer the reader to [11, 18]. The first theorem is a part of Theorem 2.7 of [11].

Theorem 2.4.4. *[11] Let $f, g : M \to N$ be maps between connected closed orientable manifolds of the same dimension. If N is a Jiang space or if f and g are pseudo Jiang, then all nonempty Nielsen classes have the same index, and hence*

$$N(f, g) = \begin{cases} 0 & \text{if } L(f, g) = 0, \\ |Coker(g_\# - f_\#)| & \text{if } L(f, g) \neq 0 . \end{cases}$$

Theorem 2.4.5. *Suppose $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ are orientable coverings. $\widetilde{N}$ is a Jiang space or $(\widetilde{f}_i, \widetilde{g}_i)$ are pseudo Jiang for all $i = 1, \ldots, r$, where r is the number of nonempty H-Reidemeister classes, and all Nielsen classes that lie in the same H-Nielsen class of f and g have the same number J. Since the order of the lifts $(\widetilde{f}_i, \widetilde{g}_i)$ in*

Equation 2.3.2 does not affect the value of $N(f,g)$, without loss of generality we can assume $L(\widetilde{f}_i, \widetilde{g}_i) \neq 0$ for each $i = 1, \ldots, t$ and $1 \leq t \leq r$, and $L(\widetilde{f}_i, \widetilde{g}_i) = 0$ otherwise. Then,

$$N(f,g) = \sum_{i=1}^{t} \frac{|Coker(\widetilde{g}_{i\#} - \widetilde{f}_{i\#})|}{S(\widetilde{f}_i, \widetilde{g}_i)} .$$

Proof. Apply Theorems 2.3.5 and 2.4.4. □

Some of the details in the following example will be illustrated in Section 4.2.

Example 2.4.6. Let $f, g : S^1 \longrightarrow S^1$ be maps defined by $f(z) = z^6$ and $g(z) = z^3$ for every $z \in S^1$. Let $p, \acute{p} : S^1 \longrightarrow S^1$ be the covering maps defined by $p(z) = z^2$ and $\acute{p}(z) = z^3$. Both coverings are regular ($\pi_1(S^1, 1)$ is abelian). The maps f and g admit lifts $\widetilde{f}$ and $\widetilde{g}$ on S^1 defined by $\widetilde{f}(z) = z^4$, and $\widetilde{g}(z) = z^2$ respectively, where $z \in S^1$. We have the commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\widetilde{f}, \widetilde{g}} & S^1 \\ \downarrow p & & \acute{p} \downarrow \\ S^1 & \xrightarrow{f, g} & S^1 \end{array} \tag{2.4.1}$$

We have $K = 2\mathbf{Z}$ and $H = 3\mathbf{Z}$. Thus, $\mathcal{A}(S^1, p) = \{1_{S^1}, -1_{S^1}\}$ and $\mathcal{A}(S^1, \acute{p}) = \{1_{S^1}, \omega, \omega^2\}$ where ω is the third primitive root of unity. Let $\alpha = -1_{S^1}$. Then,

$$\widetilde{f}\alpha(z) = \widetilde{f}(-z) = z^4 = \widetilde{f}(z)$$

and

$$\widetilde{g}\,\alpha(z) = \widetilde{g}(-z) = z^2 = \widetilde{g}(z) .$$

That is $\left[\widetilde{f}, \alpha\right] = [\widetilde{g}, \alpha] = 1_{S^1}$ and hence $\delta(f, g) = 1$ (that is, $\delta(\widetilde{f}, \widetilde{g}, \alpha) = 1$ for every $\alpha \in \mathcal{A}(\widetilde{M})$, and $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$). Notice that the H-Nielsen classes are equal to the Nielsen classes. Moreover, since the fundamental group of S^1 is abelian, all the numbers J are uniform. Pick $z = 1 \in \Phi(f,g)$ as an initial point. We have $(g_\# - f_\#)(a) = g_\#(a) - f_\#(a) = 3a - 6a = -3a$ for every $a \in \pi_1(S^1, 1)$. Thus, $(g_\# - f_\#)(a) = 0$ if and only if $a = 0$. This means that $C(f_\#, g_\#)_1 = Ker(g_\# - f_\#) = 0$. Hence, $C(f_\#, g_\#)_1 = 0 \subseteq K(1)$. This implies that the number S equals 2 for all $H-$Nielsen classes, and $(\widetilde{f}, \widetilde{g})$, $(\widetilde{f}, \omega\widetilde{g})$, and $(\widetilde{f}, \omega^2\widetilde{g})$ are the representatives of the Reidemeister classes we seek (Theorem 4.2.16). Hence,

$$N(f,g) = \frac{1}{2}\left(N(\widetilde{f},\widetilde{g}) + N(\widetilde{f},\omega\widetilde{g}) + N(\widetilde{f},\omega^2\widetilde{g})\right).$$

Since the pairs $(\widetilde{f}, \widetilde{g})$, $(\widetilde{f}, \omega\widetilde{g})$, and $(\widetilde{f}, \omega^2\widetilde{g})$ are homotopic, S^1 is an orientable Jiang space, and $L(\widetilde{f}, \widetilde{g}) = 2 - 4 = -2 \neq 0$, we have

$N(\widetilde{f},\widetilde{g}) = N(\widetilde{f},\omega\widetilde{g}) = N(\widetilde{f},\omega^2\widetilde{g}) = R(\widetilde{f},\widetilde{g}) = |Coker(\widetilde{g}_\# - \widetilde{f}_\#)| = \left|\frac{\mathbf{Z}}{\mathbf{Z}_2}\right| = 2.$

Therefore, $N(f,g) = \frac{6}{2} = 3$.

Of course, the result $N(f,g) = 3$ can be obtained more simply in the usual way (see [11]). However, we felt it is important to illustrate the concepts being studied, and to at least once show that our methods give the same results as more conventional methods.

More examples will be given in Chapter 4. In that Chapter, we talk about the enumeration, which leads in some cases to the classification, of the representatives of the Reidemeister classes needed to appear in Equation 2.3.2.

Chapter 3

Computation of $N(f,g)$ for smooth manifolds

In this chapter, we generalize Theorem 1.2.13 to Coincidences (Theorem 3.3.16). Using the semi-index on smooth manifolds, new Nielsen numbers $N_L(f,g)$ (the linear Nielsen number) and $N_{ED}(f,g)$ (the non-linear Nielsen number) on non-orientable manifolds are introduced. The number $N_L(f,g)$ satisfies $N_L(f,g) \leq N(f,g)$, and is easier to compute than $N(f,g)$ being a linear combination of the lifts of (f,g). The numbers J, I, and S associated to the H-Reidemeister classes are also used here to compute the coefficients in the formula given by Equation 3.4.6 which represents the main result in this chapter. The applications of Theorem 3.3.16 is limited to the situation where the numbers J are independent of certain choices.

3.1 The semi-index and the Nielsen number.

In this section, we recall the notion of the semi-index of a pair of maps defined on smooth closed manifolds. We also describe its properties. After that, we define the Nielsen number in terms of the semi-index. The main references for semi-index can be found in [5] and [16].

Definition 3.1.1. *[5] Let $f, g : M \longrightarrow N$ be maps between smooth manifolds. The pair (f, g) is called transverse if the maps are smooth, and for any coincidence point $x \in \Phi(f, g)$ the difference of the tangent maps $T_x f - T_x g : T_x M \longrightarrow T_{f(x)} N$ is an epimorphism (or isomorphism when M and N have the same dimension).*

The following proposition gives a characterization of transversality, which is equivalent to Definition 3.1.1, when we consider smooth closed manifolds [16].

Proposition 3.1.2. *[16] Let $f, g : M \longrightarrow N$ be maps between smooth closed manifolds. The pair (f, g) is called transverse if the maps are smooth in a neighborhood of $\Phi(f, g)$ and the map $M \ni x \mapsto (f(x), g(x)) \in N \times N$ is transverse to the diagonal $\Delta(N) = \{(x, x) | \, x \in N\} \subset N \times N$.* □

Example 3.1.3. [17] Let us show that the pair $(f_1, g_1) : S^2 \rightarrow S^2$ defined by $f_1(x, y, z) = (-x, -y, z)$ and $g_1 = 1_{S^2}$ is a transverse pair.

We have $\Phi(f_1, g_1) = \Phi(f_1) = \{p, q\}$ where $p = (0, 0, 1)$ is the north pole and $q = (0, 0, -1)$ is the south pole. We need to show that f_1 and g_1 are transverse at p and q. We show the transversality at p. The transversality at q is quite similar. The tangent plane $T_p S^2$ at p is given by $T_p S^2 = \{(x, y, 0) | \, x, \, y \in \mathbf{R}\}$. The tangent map $T_p g_1 = 1_{T_pS^2}$ is the identity isomorphism on T_pS^2, while the tangent map $T_p f_1 : T_pS^2 \longrightarrow T_pS^2$ is defined by $T_p f_1 = -1_{T_pS^2}$. This implies that $T_p f_1 - T_p g_1 = -2 \cdot 1_{T_pS^2}$ which is an isomorphism. Thus, f_1 and g_1 are transverse at p.

Example 3.1.4. [17] Let M be a non-orientable connected manifold of dimension 2. It may be regarded as a CW-complex with a unique 2-cell (see section 8 of chapter III of [1] and chapter 1 of [19]). Let $f_2 : M \to S^2$ be a map which sends the 1-skeleton to a point $y_1 \in S^2$ and the interior of the 2-cell diffeomorphically to $S^2 - y_1$. Let $g_2 : M \to S^2$ denote the constant map with $g_2(M) = y_0 \neq y_1$. First, M and S^2 have the same dimension 2. Now, let x_0 be the unique point in the interior of M such that $f_2(x_0) = y_0$. Thus, $\Phi(f_2, g_2) = \{x_0\}$. To show transversality at x_0, we have g_2 is constant, so $T_{x_0} g_2 = 0$ is the trivial homomorphism. Since f_2 is diffeomorphism near x_0, $T_{x_0} f_2 : T_{x_0}M \longrightarrow T_{y_0}S^2$ is isomorphism. Thus the difference $T_{x_0} f_2 - T_{x_0} g_2$ is isomorphism. Therefore, the pair (f_2, g_2) is a transverse pair.

Example 3.1.5. [17] Consider the maps $f_1 \times f_2,\ g_1 \times g_2 : S^2 \times M \longrightarrow S^2 \times S^2$, where f_1 and g_1 are given in Example 3.1.3, and f_2 and g_2 are given in Example 3.1.4. We have $\Phi\,(f_1 \times f_2,\ g_1 \times g_2) = \{(p, x_0),\ (q, x_0)\}$. Let us prove the transversality of the maps $f_1 \times f_2$ and $g_1 \times g_2$. The transversality of the maps $f_1 \times f_2$ and $g_1 \times g_2$ at (p, x_0) follows from the following facts:

- The tangent space of the product of two smooth manifolds is isomorphic to the external direct sum of the tangent spaces.

- $T_{(p,x_0)} f_1 \times f_2 = T_p f_1 \times T_{x_0} f_2$. Similarly for $T_{(p,x_0)} g_1 \times g_2$,

- $(T_p f_1 \times T_{x_0} f_2) - (T_p g_1 \times T_{x_0} g_2) = (T_p f_1 - T_p g_1) \times (T_{x_0} f_2 - T_{x_0} g_2)$, and

- $(T_p f_1 - T_p g_1) \times (T_{x_0} f_2 - T_{x_0} g_2)$ is an isomorphism because the product of isomorphisms is an isomorphism.

The transversality at the other coincidence point is proved similarly.

Proposition 3.1.6. *[14] Any pair (f, g) of maps between smooth closed manifolds is homotopic to a transverse pair (f', g').* □

We call the pair (f', g') a transverse approximation to (f, g).

Let $(f, g) : M \longrightarrow N$ be a transverse pair between smooth closed $n-$manifolds. Then, $\Phi(f, g)$ is finite and hence any coincidence point is isolated (see [14]). Let $x, y \in \Phi(f, g)$ be in the same Nielsen class, and let $\omega : x \to y$ be a path that establishes the Nielsen relation. Let α_x be an orientation of the vector space $T_x M$ and α_y be the orientation of the vector space $T_y M$ obtained under the shift of α_x along ω. Let β_x be the orientation of the vector space $T_{f(x)}N$ obtained as the image of the isomorphism $T_x f - T_x g$ and β_y be the orientation of the vector space $T_{f(y)}N$ obtained by the image of the isomorphism $T_y f - T_y g$. Let γ_y be the orientation of the vector space $T_{f(y)}N$ obtained by shifting the orientation β_x along $f(\omega)$ (or equivalently $g(\omega)$) at $f(y)$.

Definition 3.1.7. *The point x is said to reduce to y if the orientations β_y and γ_y are opposite to each other. Also, the path ω is said to establish the reducibility between x and y.*

Remark 3.1.8. *Notice that reducibility need not to be an equivalence relation.*

Example 3.1.9. [17] In Example 3.1.3, the points p and q lie in the same Nielsen class since S^2 is simply connected. However, they do not reduce to each other because S^2 is orientable.

Example 3.1.10. [17] In Example 3.1.4, $\Phi(f_2, g_2) = \{x_0\}$. Thus, the point x_0 is the only Nielsen class of f_2 and g_2. The coincidence point x_0 is a self reducible coincidence point, that is it reduces to itself. In fact, there exists a smooth loop δ in the non-orientable manifold M that reverses orientation. Let x_1 be a point on δ and σ be a smooth path from x_0 to x_1 such that $\sigma - \{x_1\}$ lies in the interior of the 2-cell of M. Then, the loop $\sigma\,\delta\,\sigma^{-1}$ is a loop at x_0 that reverses orientation. Since S^2 is orientable, the loop $\sigma\,\delta\,\sigma^{-1}$ satisfies the conditions that make the point x_0 self reducing.

Example 3.1.11. [17] In this example we show that the points (p, x_0) and (q, x_0) given in Example 3.1.5 reduce to each other. Since $S^2 \times S^2$ is simply connected, these points belong to the same Nielsen class. Since $S^2 \times M$ is non-orientable and $S^2 \times S^2$ is orientable, a similar argument to that in Example 3.1.10 leads to the reducibility of these two points to each other, and moreover to themselves.

Now, Let (f, g) be a transverse pair, and let A be a subset of $\Phi(f, g)$. Let us present A as

$$A = \{x_1, y_1, \ldots, x_k, y_k; z_1, \ldots, z_s\} ,$$

where x_i reduces to y_i for each $i = 1, \ldots, k$, but z_i reduces to no z_j for each $j \neq i$, where $1 \leq i, j \leq s$. Such a presentation is called a decomposition of A. The elements $z_1, \ldots, z_s$ are called free in this decomposition of A. One may check that the number of free elements is the same for all decompositions of A ([5]).

Definition 3.1.12. *The semi-index of f and g at A is defined to be the number of the free elements of A, and we write $|ind|(f, g; A) = s$.*

Definition 3.1.13. *Let W be an open subset of M. We define $|ind|(f, g; W)$ by*

$$|ind|(f, g; W) = |ind|(f, g; W \cap \Phi(f, g)) .$$

Example 3.1.14. [17] From Example 3.1.9, let $W_p = S^2 - q$ and $W_q = S^2 - p$. Then W_p and W_q are open neighborhoods of p and q respectively. Thus, $|ind|(f_1, g_1; W_p) = |ind|(f_1, g_1; \{p\}) = 1$, $|ind|(f_1, g_1; W_q) = |ind|(f_1, g_1; \{q\}) = 1$, and $|ind|(f_1, g_1; \{p, q\}) = 2$.

Example 3.1.15. [17] In Example 3.1.10, it follows that $|ind|(f_2, g_2; \{x_0\}) = 1$.

Example 3.1.16. [17] Let $W_1 = W_p \times M$ and $W_2 = W_q \times M$, where W_p and W_q are the open neighborhoods of p and q respectively given in Example 3.1.14. From Example 3.1.11, we have that W_1 and W_2 are open neighborhoods of (p, x_0) and (q, x_0) respectively. Therefore, $|ind|(f_1 \times f_2, g_1 \times g_2; W_1) = |ind|(f_1 \times f_2, g_1 \times g_2; \{(p, x_0)\}) = 1$, $|ind|(f_1 \times f_2, g_1 \times g_2; W_2) = |ind|(f_1 \times f_2, g_1 \times g_2; \{(q, x_0)\}) = 1$, and $|ind|(f_1 \times f_2, g_1 \times g_2; \{(p, x_0), (q, x_0)\}) = 0$.

Lemma 3.1.17. *[5] Let (f_0, g_0) and (f_1, g_1) be transverse pairs and let $F, G : M \times [0,1] \longrightarrow N$ be homotopies between them. Let $A_0 \subseteq \Phi(f_0, g_0)$ be a Nielsen class which corresponds to the Nielsen class $A_1 \subseteq \Phi(f_1, g_1)$. Then,*

$$|ind|(f_0, g_0; A_0) = |ind|(f_1, g_1; A_1)$$

Note that, we do not require the homotopies in Lemma 3.1.17 to be transverse. Lemma 3.1.17, therefore, allows us to extend the definition of coincidence semi-index to any arbitrary pair of maps.

Definition 3.1.18. *Let $(f, g) : M \longrightarrow N$ be a pair of maps between two closed smooth $n-$manifolds, and let $A \subset \Phi(f, g)$ be a Nielsen class. Let $(\acute{f}, \acute{g})$ be a transverse approximation of (f, g) and $\acute{A} \subset \Phi(\acute{f}, \acute{g})$ be the corresponding class of A. We define*

$$|ind|(f, g; A) = |ind|(\acute{f}, \acute{g}; \acute{A})$$

The following proposition lists the properties of semi-index. We refer to [5] for proofs.

Proposition 3.1.19. *[5] Let $(f, g) : M \longrightarrow N$ be a pair of maps between two closed smooth $n-$manifolds. Then,*

1. *Definition 3.1.18 is independent of the transverse approximation of (f, g).*

2. *The semi-index is subadditive: if W_1, W_2 are open subsets (or disjoint open subsets) of M such that $W_i \cap \Phi(f, g)$ is compact for $i = 1, 2$, then*

$$|ind|(f, g; W_1 \cup W_2) \leq |ind|(f, g; W_1) + |ind|(f, g; W_2) .$$

3. *If W is open subset of M and $|ind|(f, g; W) \neq 0$ then $\Phi(f, g) \cap W \neq \emptyset$.*

4. *The semi-index is homotopy invariant: Let $(F, G) : M \times [0, 1] \longrightarrow N$ be a homotopy between pairs (f_0, g_0) and (f_1, g_1). Let $W \subseteq M \times [0, 1]$ be an open subset such that $W \cap \Phi(F, G)$ is compact. Let $W_t = \{x \in M | (x, t) \in W\}$. Then,*

$$|ind|(f_0, g_0; W_0) = |ind|(f_1, g_1; W_1) .$$

Remark 3.1.20. *The semi-index is not local, since the reducibility depends on the behavior of the maps on all M.*

Example 3.1.21. [5] To show that semi-index may not be strictly additive, let W_1 and W_2 be the open subsets of $S^2 \times M$ given in Example 3.1.16, and let $f = f_1 \times f_2$ and $g = g_1 \times g_2$. Then,

$$|ind|(f, g; W_1 \cup W_2) = 0 < 2 = 1 + 1 = |ind|(f, g; W_1) + |ind|(f, g; W_2) .$$

Proposition 3.1.22. [5] If $W_1 \cap \Phi(f, g) = A_1$ and $W_2 \cap \Phi(f, g) = A_2$ are different Nielsen classes, then

$$|ind|(f, g; W_1 \cup W_2) = |ind|(f, g; W_1) + |ind|(f, g; W_2) .$$

□

Example 3.1.23. [5] Let $f : S^1 \longrightarrow S^1$ be the flip map (the standard map of degree -1), and $g = 1_{S^1}$. Then, $\Phi(f) = \{p, q\}$ where $p = (0,1)$ and $q = (0,-1)$. Let $W_p = S^1 - q$ and $W_q = S^1 - p$. Then, W_p and W_q are open neighborhoods of p and q respectively. Moreover, $W_p \cap \Phi(f) = \{p\}$ and $W_q \cap \Phi(f) = \{q\}$ are different Nielsen classes. Thus,

$$\begin{aligned} |ind|(f, g; W_p \cup W_q) &= |ind|(f, g; S^1) = |ind|(f, g; \{p, q\}) \\ &= 2 = 1 + 1 = |ind|(f, g; W_p) + |ind|(f, g; W_q)\,. \end{aligned}$$

The following example shows that additivity might occur even if the necessary conditions of Proposition 3.1.22 are not satisfied.

Example 3.1.24. [5] In Example 3.1.14, $W_p \cap \Phi(f_1, g_1) = \{p\}$ and $W_q \cap \Phi(f_1, g_1) = \{q\}$ are not Nielsen classes. However,

$$\begin{aligned} |ind|(f_1, g_1; W_p \cup W_q) &= |ind|(f_1, g_1; S^2) = |ind|(f_1, g_1; \{p, q\}) \\ &= 2 = 1 + 1 = |ind|(f_1, g_1; W_p) + |ind|(f_1, g_1; W_q)\,. \end{aligned}$$

Next, we give the relationship between the index and semi-index.

Proposition 3.1.25. *[16] Let $(f, g) : M \longrightarrow N$ be a pair of maps between two oriented closed smooth $n-$manifolds. Then,*

1. *If (f, g) is a transverse pair, then $index(f, g; x) = \pm 1$ for every $x \in \Phi(f, g)$.*
2. *If $x, y \in \Phi(f, g)$ are in the same Nielsen class, then x reduces to y if and only if $index(f, g; x) = -index(f, g; y)$. Thus,*
3. *If $A \subseteq \Phi(f, g)$ is a Nielsen class, then $|ind|(f, g; A) = |index(f, g; A)|$.*

Next, we introduce the notion of essential Nielsen classes, as well as the concept of the Nielsen number in the context of semi index.

Definition 3.1.26. *[5] Let $(f,g) : M \longrightarrow N$ be maps between connected closed smooth manifolds. A Nielsen class is said to be essential if it has a <u>nonzero</u> semi-index.*

Definition 3.1.27. *[5] We define the semi-index Nielsen number $N(f,g)$ of the pair (f,g) to be the number of the essential classes.*

Part (3) of Proposition 3.1.25 leads to the following corollary.

Corollary 3.1.28. *[5] For orientable manifolds, the semi-index Nielsen number is equal to the usual index Nielsen number.*

3.2 Defective and Non-defective Nielsen classes

In this section, we give the notion of defective classes along with several properties. An important relationship between the Nielsen classes in the base space and in the total space of covering spaces, is given at the end of this section. Some of the results here are found in [5, 15, 25]. Others are generalizations or modifications of results in Section 1.2.

Let M and N be closed connected smooth manifolds of the same dimension n, and let $(\widetilde{M},p)$ and $(\widetilde{N},p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. We assume the coverings are finite; that is, $[\pi_1(M) : K] < \infty$ and $[\pi_1(N) : H] < \infty$. Let $(f,g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f},\widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f},\widetilde{g}} & \widetilde{N} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{f,g} & N \end{array} \tag{3.2.1}$$

Lemma 3.2.1. *[5] Let $x, y \in \Phi(f, g)$ be such that x reduces to y. Then, there exists a bijection $\varphi : p^{-1}\{x\} \cap \Phi(\widetilde{f},\widetilde{g}) \longrightarrow p^{-1}\{y\} \cap \Phi(\widetilde{f},\widetilde{g})$ such that $\widetilde{x}$ reduces to $\varphi(\widetilde{x})$ for every $\widetilde{x} \in p^{-1}\{x\} \cap \Phi(\widetilde{f},\widetilde{g})$. In other words, the set $p^{-1}\{x, y\} \cap \Phi(\widetilde{f},\widetilde{g})$ splits into pairs reducing themselves.*

Remark 3.2.2. *In the case where M and N are orientable manifolds, such x and y do not exist.*

Definition 3.2.3. *[17] A Nielsen class is called defective if it contains a self reducible point.*

Example 3.2.4. The coincidence Nielsen classes $\{x_0\}$ given in Example 3.1.10, and $\{(p, x_0), (q, x_0)\}$ given in Example 3.1.11 are defective since each of them contains a self reducible coincidence point. However, the coincidence Nielsen class $\{p, q\}$ in Example 3.1.9 is not defective, since neither of p nor q is self reducible point. In fact if either p or q is self reducible, then p reduces to q, which contradicts that p and q do not reduce to each other (see Lemma 3.2.9).

The following Lemma is an obvious geometric characterization of self reducibility.

Lemma 3.2.5. *Let $x \in \Phi(f,g)$ and $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$ be such that $p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g}) \neq \emptyset$. Then the following are equivalent*

1. *x reduces to itself .*

2. *There exist points in $p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})$ not necessarily distinct, which reduce to each other.*

3. *$p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})$ splits into pairs reducing each other if $|p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})|$ is even, or splits into pairs reducing each other together with a single self reducible point, if $|p^{-1}(x) \cap \Phi(\widetilde{f},\widetilde{g})|$ is odd .*

Proof. Apply Lemma 3.2.1. □

The following lemma is an algebraic characterization of self reducibility.

Lemma 3.2.6. *[17] Let $f,g : M \longrightarrow N$ be transverse maps, $x \in \Phi(f,g)$ and O_M and O_N be the subgroups of $\pi_1(M)$ and $\pi_1(N)$ respectively, each of which consists of orientation-preserving elements. The following are equivalent*

1. *x reduces to itself.*

2. *$C(f_{\#}, g_{\#})_x \cap O_M \neq C(f_{\#}, g_{\#})_x \cap f_{\#}^{-1}(O_N)$*

3. *There exists $\gamma \in \pi_1(M,x)$ such that $f(\gamma) = g(\gamma)$, and exactly one of the loops γ or $f(\gamma)$ is orientation-preserving.*

□

The characterization of self reducibility given in Lemma 3.2.6 is restricted to transverse pairs of maps. So, using Lemma 3.2.6 for any pairs of maps requires a transverse approximation (See Proposition 3.1.6), which is in practice difficult to obtain. The following proposition, which generalizes Lemma 3.2.6 to any pair of maps, allows us, in most cases, to ignore the transversality condition.

Proposition 3.2.7. *Let $(f, g) : M \longrightarrow N$ be a pair of maps homotopic to a transverse pair $(\acute{f}, \acute{g}) : M \longrightarrow N$ by the homotopy-pair $(F, G) : M \times [0, 1] \longrightarrow N$. Let $x \in \Phi(f, g)$ and $\acute{x} \in \Phi(\acute{f}, \acute{g})$ be F, G-related coincidence points. Then, the following are equivalent*

1. *$\acute{x}$ reduces to itself.*

2. *$C(f_\#, g_\#)_x \cap O_M \neq C(f_\#, g_\#)_x \cap f_\#^{-1}(O_N)$*

3. *There exists $\gamma \in \pi_1(M, x)$ such that $f(\gamma) = g(\gamma)$ and exactly one of the loops γ or $f(\gamma)$ is orientation-preserving.*

Proof. The equivalence between (2) and (3) is easily proved. We show the equivalence between (1) and (3).

Suppose that $\acute{x}$ reduces to itself. By Lemma 3.2.6, there exists $\acute{\gamma} \in \pi_1(M, \acute{x})$ such that $\acute{f}_\#(\acute{\gamma}) = \acute{g}_\#(\acute{\gamma})$ (for simplicity, we write $\acute{f}(\acute{\gamma})$ and $\acute{g}(\acute{\gamma})$ for $\acute{f}_\#(\acute{\gamma})$ and $\acute{g}_\#(\acute{\gamma})$ respectively), and exactly one of the loops $\acute{\gamma}$ or $\acute{f}_\#(\acute{\gamma})$ is orientation-preserving. Since x and $\acute{x}$ are F, G-related, there exists a path $u : x \longrightarrow \acute{x}$ such that $F(u) \sim_0 G(u)$. i.e., $F(u)$ is homotopic to $G(u)$ rel. endpoints, where $F(u), G(u) : [0, 1] \longrightarrow N$ are paths defined respectively by $F(u)(t) = F(u(t), t)$ and $G(u)(t) = G(u(t), t)$ for every $t \in [0,1]$. Define the path $F(x) : [0,1] \longrightarrow N$ by $F(x)(t) = F(x,t)$ for every $t \in [0,1]$ ($F(\acute{x})$, $G(x)$, and $G(\acute{x})$ are defined similarly). Then, the loop $\gamma = u\,\acute{\gamma}\,u^{-1}$ at x establishes the Nielsen relation between x and itself since

$$\begin{aligned}
\acute{f}(\acute{\gamma}) = \acute{g}(\acute{\gamma}) \quad &\Leftrightarrow \quad F(\acute{x})^{-1}\, f(\acute{\gamma})\, F(\acute{x}) = G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x}) \\
&\Leftrightarrow \quad f(\acute{\gamma}) = F(\acute{x})\, G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x})\, F(\acute{x})^{-1} \\
&\Leftrightarrow \quad f(u)\, f(\acute{\gamma})\, f(u)^{-1} = f(u)\, F(\acute{x})\, G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x})\, F(\acute{x})^{-1}\, f(u)^{-1} \\
&\Leftrightarrow \quad f(u\,\acute{\gamma}\,u^{-1}) = (f(u)\, F(\acute{x}))\; G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x})\; \left(F(\acute{x})^{-1}\, f(u)^{-1}\right) \\
&\Leftrightarrow \quad f(\gamma) = F(u)\, G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x})\, F(u)^{-1} \\
&\Leftrightarrow \quad f(\gamma) = G(u)\, G(\acute{x})^{-1}\, g(\acute{\gamma})\, G(\acute{x})\, G(u)^{-1} \\
&\Leftrightarrow \quad f(\gamma) = g(u)\, g(\acute{\gamma}) g(u)^{-1} \\
&\Leftrightarrow \quad f(\gamma) = g(u\,\acute{\gamma} u^{-1}) = g(\gamma)\ .
\end{aligned}$$

Now, suppose, without loss of generality, that $\acute{\gamma}$ preserves orientation at $\acute{x}$ and $\acute{f}(\acute{\gamma})$ reserves orientation at $\acute{f}(\acute{x})$. We show that γ preserves orientation at x, while $f(\gamma)$ reverses orientation at $f(x)$.

To see that the loop γ preserves orientation at x, let σ be an orientation at x which is translated by u to the orientation μ at $\acute{x}$. If we write the last statement symbolically as $\sigma \overset{u}{\rightarrowtail} \mu$. Then

$$\sigma \overset{u}{\rightarrowtail} \mu \overset{\acute{\gamma}}{\rightarrowtail} \mu \overset{u^{-1}}{\rightarrowtail} \sigma\ .$$

That is, the loop $\gamma = u\,\acute{\gamma}\,u^{-1}$ preserves orientation at x.

Note also that the loop $f(\gamma)$ reverses the orientation at $f(x)$ because $f(\acute{\gamma}) = F(\acute{x})\,\acute{f}(\acute{\gamma})\,F(\acute{x})^{-1}$, and if τ and ϵ are orientations at $f(\acute{x})$ and $\acute{f}(\acute{x})$ respectively such that $\tau \overset{F(\acute{x})}{\rightarrowtail} \epsilon$ then

$$\tau \overset{F(\acute{x})}{\rightarrowtail} \epsilon \overset{\acute{f}(\acute{\gamma})}{\rightarrowtail} -\epsilon \overset{F(\acute{x})^{-1}}{\rightarrowtail} -\tau \,.$$

Hence, the path $f(\acute{\gamma})$ reverses orientation. Thus, if η and ϱ are orientations at $f(x)$ and $f(\acute{x})$ respectively such that $\eta \overset{f(u)}{\rightarrowtail} \varrho$, then

$$\eta \overset{f(u)}{\rightarrowtail} \varrho \overset{f(\acute{\gamma})}{\rightarrowtail} -\varrho \overset{f(u)^{-1}}{\rightarrowtail} -\eta \,.$$

That is, the loop $f(\gamma) = f(u\,\acute{\gamma}\,u^{-1})$ reverses orientation at $f(x)$.

Similarly, if $\acute{\gamma}$ reverses orientation at $\acute{x}$, so does γ at x, and if $\acute{f}(\acute{\gamma})$ preserves orientation at $\acute{f}(\acute{x})$, so does $f(\gamma)$ at $f(x)$.

Therefore (3) holds.

For the converse, if (3) holds, the same argument as above shows that Lemma 3.2.6 (3) holds. By that same lemma this implies that $\acute{x}$ reduces to itself. $\square$

Remark 3.2.8. *Proposition 3.2.7 allows us to generalize the definition of self reducibility (defective class) to include coincidence points (Nielsen classes) of any pair of maps.*

Lemma 3.2.9. *[17] If A is a defective class, then any two points in A reduce to each other. Consequently,*

$$|ind|(f,g;\, A) = \begin{cases} 0 & \text{if } |A| \text{ is even.} \\ 1 & \text{if } |A| \text{ is odd.} \end{cases}$$

where $|A|$ denotes the cardinality of A.

We return to the context of Lemma 3.2.1.

Theorem 3.2.10. *[25] Let $\widetilde{A}$ be a coincidence class of the pair $(\widetilde{f}, \widetilde{g})$, then $p(\widetilde{A}) = A$ is a coincidence class of the pair (f, g) and*

$$|ind|(\widetilde{f}, \widetilde{g}\,;\, \widetilde{A}) = \begin{cases} s\,.\,k \pmod 2 & \text{if } A \text{ is defective.} \\ s\,.\,k & \text{if } A \text{ is not defective.} \end{cases}$$

where $s = |ind|(f, g;\, A)$, $k = |\, j(C(f_{\#}, g_{\#}))_{x_0}\,|$, and $x_0 \in A$. □

Remark 3.2.11. *The homotopy invariance of the number k in Theorem 3.2.10 follows by Proposition 3.3.12 (where $k = J$ there). Lemma 3.2.9 and Theorem 3.2.10 give sufficient information for us to complete the analysis of this thesis.*

The following proposition is a simple, but useful, modification of Theorem 3.2.10. It will be useful in the proof of Proposition 3.2.13.

Proposition 3.2.12. *Let $\widetilde{A}$ be a coincidence class of the pair $(\widetilde{f}, \widetilde{g})$, then $p(\widetilde{A}) = A$ is a coincidence class of the pair (f, g) and*

$$|ind|(\widetilde{f}, \widetilde{g}\,;\, \widetilde{A}) = \begin{cases} \frac{1-(-1)^{J_A}}{2} \cdot |ind|(f, g; A) & \text{if } A \text{ is defective,} \\ J_A \cdot |ind|(f, g; A) & \text{if } A \text{ is not defective.} \end{cases}$$

The following proposition generalizes Lemma 1.2.5 to semi-index Coincidence Theory.

Proposition 3.2.13. *Let $A \subseteq p\,\Phi(\widetilde{f},\widetilde{g})$ be a Nielsen class of the pair f, g. Then,*

$$|ind|(\widetilde{f},\widetilde{g}\,;\,p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g})) = \begin{cases} S_A \cdot \frac{1-(-1)^{J_A}}{2} \cdot |ind|\,(f,g;A) & \textit{if } A \textit{ is defective;} \\ I_A \cdot |ind|(f,g;A) & \textit{otherwise.} \end{cases}$$

Proof. As in Proposition 2.1.21, we have

$$p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g}) = \bigcup_{i=1}^{S_A} \widetilde{A}_i\,,$$

where $\widetilde{A}_i$ is a Nielsen class of $(\widetilde{f},\widetilde{g})$ such that $p(\widetilde{A}_i) = A$, for every $i = 1,\ldots,S_A$. Thus,

$$|ind|\left(\widetilde{f},\widetilde{g};p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g})\right) = \sum_{i=1}^{S_A} |ind|\left(\widetilde{f},\widetilde{g};\widetilde{A}_i\right)\,.$$

If A is not defective, then by Proposition 3.2.12

$$\begin{aligned} |ind|\left(\widetilde{f},\widetilde{g};p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g})\right) &= \sum_{i=1}^{S_A} J_A \cdot |ind|\,(f,g;A) \\ &= S_A \cdot J_A \cdot |ind|\,(f,g;A) \\ &= I_A \cdot |ind|\,(f,g;A)\,. \end{aligned}$$

If A is defective, by Proposition 3.2.12 we have

$$\begin{aligned} |ind|\left(\widetilde{f},\widetilde{g};p^{-1}(A)\cap\Phi(\widetilde{f},\widetilde{g})\right) &= \sum_{i=1}^{S_A} \frac{1-(-1)^{J_A}}{2} \cdot |ind|\,(f,g;A) \\ &= S_A \cdot \frac{1-(-1)^{J_A}}{2} \cdot |ind|\,(f,g;A)\,. \end{aligned}$$

□

Remark 3.2.14. *Notice that if $S_A = 1$ in Proposition 3.2.13, then $I_A = J_A$, and in this case, Proposition 3.2.12 and Proposition 3.2.13 coincide.*

The next proposition gives the complete relationship between the Nielsen classes in the base space and those in the total space.

Proposition 3.2.15. *Let $A \subseteq \Phi(f, g)$ be a Nielsen class. Then,*

1. *If J_A is odd and A is defective, then A is essential if and only if $\widetilde{A}$ is essential for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ with $p(\widetilde{A}) = A$.*

2. *If J_A is even and A is defective, then $\widetilde{A}$ is inessential, i.e., $|ind|(\widetilde{f}, \widetilde{g}; \widetilde{A}) = 0$ for every $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ such that $p(\widetilde{A}) = A$.*

3. *If A is not defective, neither is $\widetilde{A}$ for any $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ for which $p(\widetilde{A}) = A$. Hence, when A is not defective, A is essential if and only if $\widetilde{A}$ is essential for every $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ with $p(\widetilde{A}) = A$.*

Proof. The proof follows directly from Theorem 3.2.10 and Lemma 3.2.9. □

Corollary 3.2.16. *Let $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g), A be a Nielsen class of f and g such that J_A is even, and let $\widetilde{A}$ be a Nielsen class of $\widetilde{f}$ and $\widetilde{g}$ such that $p(\widetilde{A}) = A$. Then,*

1. *If $\widetilde{A}$ is essential, then $\widetilde{A}$ is not defective.*

2. *$\widetilde{A}$ is essential if and only if A is essential and not defective.*

3. *If $\widetilde{A}$ is essential, then all the other classes in $p_{\Phi}^{-1}(A)$ are essential and not defective.*

Proof. (1) Assume that $\widetilde{A}$ is essential. Since $|\widetilde{A}|$ is even, then $\widetilde{A}$ cannot be defective.
(2) Assume that $\widetilde{A}$ is essential. Since J_A is even, by (2), Proposition 3.2.15 we get A is not defective. Thus, by (3) of Proposition 3.2.15, we have A is essential. The converse follows immediately from part (3) of Proposition 3.2.15.
(3) Assume that $\widetilde{A}$ is essential. By (2), A is essential and not defective. By Proposition 3.2.12, every class in $p_{\Phi}^{-1}(A)$ is essential and not defective. □

Remark 3.2.17. *Note that, if A is a defective class for which J_A is even, then A is not necessarily essential or inessential. This fact is illustrated in Examples 3.2.18 and 3.2.20.*

Example 3.2.18. Let M be a nonorientable closed smooth manifold of dimension 2, and let $\chi : S^2 \longrightarrow \mathbf{RP^2}$ be the quotient map, where $\mathbf{RP^2}$ is the real projective plane. For every $(x, y, z) \in S^2$, we write $\chi(x, y, z) = [(x, y, z)]$. The maps $\widetilde{f}_0, \widetilde{g}_0, \widetilde{f}_1, \widetilde{g}_1, f_0, g_0 : S^2 \longrightarrow S^2$, $f_1, g_1 : \mathbf{RP^2} \longrightarrow \mathbf{RP^2}$, and $\widetilde{f}_2, \widetilde{g}_2, f_2, g_2 : M \longrightarrow S^2$ are defined as follows

- $\widetilde{f}_0(x, y, z) = \widetilde{f}_1(x, y, z) = f_0(x, y, z) = (-x, -y, z)$, for every $(x, y, z) \in S^2$,
- $\widetilde{g}_0(x, y, z) = \widetilde{g}_1(x, y, z) = g_0(x, y, z) = (x, y, z)$, for every $(x, y, z) \in S^2$,
- $f_1([(x, y, z)]) = [(-x, -y, z)]$, for every $[(x, y, z)] \in \mathbf{RP^2}$,
- $g_1([(x, y, z)]) = [(x, y, z)]$, for every $[(x, y, z)] \in \mathbf{RP^2}$.

- $\widetilde{f}_2$ maps the 1-skeleton to a point $\mathbf{y_1} = (x_1, y_1, z_1) \in S^2$ and the interior of the 2-cell diffeomorphically to $S^2 - (x_1, y_1, z_1)$.

- $\widetilde{g}_2$ is the constant map with $\widetilde{g}_2(M) = \mathbf{y_0} = (x_0, y_0, z_0) \neq (x_1, y_1, z_1)$.

- $f_2 = \widetilde{f}_2$, and

- $g_2 = \widetilde{g}_2$.

Notice that f_1 is well-defined since it is an odd function. That is, it maps antipodal points to antipodal points. We define the maps $\widetilde{f}, \widetilde{g} : S^2 \times S^2 \times M \longrightarrow S^2 \times S^2 \times S^2$ and $f, g : S^2 \times \mathbf{RP^2} \times M \longrightarrow S^2 \times \mathbf{RP^2} \times S^2$ by $\widetilde{f} = \widetilde{f}_0 \times \widetilde{f}_1 \times \widetilde{f}_2$, $\widetilde{g} = \widetilde{g}_0 \times \widetilde{g}_1 \times \widetilde{g}_2$, $f = f_0 \times f_1 \times f_2$ and $g = g_0 \times g_1 \times g_2$. We have the commutative diagram which represents a 2-fold covering

$$\begin{tikzcd} S^2 \times S^2 \times M \arrow[r, "\widetilde{f}, \widetilde{g}"] \arrow[d, "1_{S^2} \times \chi \times 1_M"'] & S^2 \times S^2 \times S^2 \arrow[d, "1_{S^2} \times \chi \times 1_{S^2}"] \\ S^2 \times \mathbf{RP^2} \times M \arrow[r, "f, g"] & S^2 \times \mathbf{RP^2} \times S^2 \end{tikzcd}$$

Let $p = (0, 0, 1)$ and $q = (0, 0, -1)$, and let $\mathbf{x_0} \in M$ be such that $\widetilde{f}_2(\mathbf{x_0}) = \widetilde{g}_2(\mathbf{x_0}) = \mathbf{y_0}$. Then,

$$\Phi(\widetilde{f}, \widetilde{g}) = \Phi(\widetilde{f}_0, \widetilde{g}_0) \times \Phi(\widetilde{f}_1, \widetilde{g}_1) \times \Phi(\widetilde{f}_2, \widetilde{g}_2) = \{p, q\} \times \{p, q\} \times \{\mathbf{x_0}\} .$$

Since $S^2 \times S^2 \times S^2$ is simply connected, $\Phi(\widetilde{f}, \widetilde{g})$ consists of a single Nielsen class of $\widetilde{f}$ and $\widetilde{g}$. Moreover, since $[p] = [q]$, we have

$$A := 1_{S^2} \times \chi \times 1_M \left(\Phi(\widetilde{f}, \widetilde{g}) \right) = \{(p, [p], \mathbf{x_0}), (q, [p], \mathbf{x_0})\}$$

is a Nielsen class of f and g. A similar argument to that of Example 3.1.11 shows that the point $(p, [p], \mathbf{x_0})$ is self reducible. Hence, A is defective. On the other hand,

$$J_A = |\,(1_{S^2} \times \chi \times 1_M)^{-1}\,((p, [p], \mathbf{x_0})) \cap \Phi(\widetilde{f}, \widetilde{g})| = |\{(p, p, \mathbf{x_0}), (p, q, \mathbf{x_0})\}| = 2\ .$$

Since $|A| = 2$, and A is defective, $|ind|(f, g; A) = 0$. In other words, A is an inessential defective class for which J_A is even.

The following version of Proposition 3.2.15 is useful.

Corollary 3.2.19. *Let $A \subseteq \Phi(f, g)$ be a Nielsen class. Then,*

- *If A is defective, then*
  - *A is inessential (equivalently $|\,A|$ is even) implies that $\widetilde{A}$ is inessential for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ such that $p\,(\widetilde{A}) = A$.*
  - *A is essential (equivalently $|\,A|$ is odd), and*
    - *J_A is even implies that $\widetilde{A}$ is inessential for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ such that $p\,(\widetilde{A}) = A$.*
    - *J_A is odd implies that $\widetilde{A}$ is essential and defective for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ such that $p\,(\widetilde{A}) = A$.*
- *If A is not defective, neither is $\widetilde{A}$ for any $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ with $p\,(\widetilde{A}) = A$. Hence, when A is not defective, A is essential if and only if $\widetilde{A}$ is essential for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})$ such that $p\,(\widetilde{A}) = A$.*

Proof. The proof follows directly from Proposition 3.2.15. $\square$

The following Example shows that there exist essential defective Nielsen classes with even J. In other words, this example shows that there exist maps f and g such that their nonlinear Nielsen number $N_{ED}(f,g)$, which counts the essential defective Nielsen classes of f and g for which J is even, <u>is not zero</u>. This case only appears when <u>nonorientable manifolds</u> are involved.

Example 3.2.20. Let $M = \mathbf{RP^2}$. It is well-known that $\mathbf{RP^2}$ is a nonorientable smooth manifold with finite cyclic fundamental group of order 2. Let $\widetilde{f}_1$, $\widetilde{g}_1$, $\widetilde{f}_2$, $\widetilde{g}_2$, f_1, f_2, g_1, and g_2 as given in Example 3.2.18. Let $\widetilde{f} := \widetilde{f}_1 \times \widetilde{f}_2$, $\widetilde{g} := \widetilde{g}_1 \times \widetilde{g}_2$, $f := f_1 \times f_2$, and $g := g_1 \times g_2$. Let $\chi : S^2 \longrightarrow \mathbf{RP^2}$ be the quotient map.

In what follows, the commutativity of diagrams (1) and (2) implies the commutativity of diagram (3). Also, all the covering spaces are regular since the fundamental groups of the involved manifolds are abelian. Moreover, each of the coverings is finite where diagrams (1), (2), and (3) represent 2- fold, 1-fold, and 2-fold covering, respectively.

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\widetilde{f}_1, \widetilde{g}_1} & S^2 \\
\chi \downarrow & (1) & \downarrow \chi \\
\mathbf{RP^2} & \xrightarrow{f_1, g_1} & \mathbf{RP^2}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{RP^2} & \xrightarrow{\widetilde{f}_2, \widetilde{g}_2} & S^2 \\
1_{\mathbf{RP^2}} \downarrow & (2) & \chi \downarrow \\
\mathbf{RP^2} & \xrightarrow{f_2, g_2} & \mathbf{RP^2}
\end{array}
\qquad
\begin{array}{ccc}
S^2 \times \mathbf{RP^2} & \xrightarrow{\widetilde{f}, \widetilde{g}} & S^2 \times S^2 \\
\chi \times 1_{\mathbf{RP^2}} \downarrow & (3) & \downarrow \chi \times \chi \\
\mathbf{RP^2} \times \mathbf{RP^2} & \xrightarrow{f, g} & \mathbf{RP^2} \times \mathbf{RP^2}
\end{array}
$$

Now, since the pair $(\widetilde{f}_1, \widetilde{g}_1)$ is transverse at the points p and q, the commutativity of diagrams (1) implies that the pair (f_1, g_1) is transverse at the point $[p]$. However,

(f_1, g_1) is not transverse at each point in $\Phi(f_1, g_1) = \{[p]\} \cup \{[(x, y, 0)]|\, x^2 + y^2 = 1\}$ because $\left(-\widetilde{f}_1, \widetilde{g}_1\right)$, the other Reidemeister representative, is not transverse. This is due to the fact that the set $\widetilde{B} := \Phi\left(-\widetilde{f}_1, \widetilde{g}_1\right) = \{(x, y, 0)|\, x^2 + y^2 = 1\}$ is homeomorphic to S^1 which is a 1-manifold (the pair $\left(-\widetilde{f}_1, \widetilde{g}_1\right)$ must be transverse on a discrete submanifold). Transversality of (f_2, g_2) follows from commutativity of Diagram (2). The commutativity of diagram (3) together with a similar argument as was given for the pair $(\widetilde{f}_1, \widetilde{g}_1)$ shows that (f, g) is transverse only at the point $([p], x_0)$. On the other hand, let $\widetilde{A} = \{(p, x_0), (q, x_0)\}$, then $\widetilde{A}$ is a defective Nielsen class of $\widetilde{f}$ and $\widetilde{g}$ (see Example 3.2.4) with $|ind|(\widetilde{f}, \widetilde{g}; \widetilde{A}) = 0$ (see Example 3.1.16). Thus, $A := (\chi \times 1_{\mathbf{RP^2}})\,(\widetilde{A}) = \{([p], x_0)\}$ consists of a single coincidence point, since p and q are antipodal and so they are identified to each other by χ. This implies that $|ind|(f, g; A) = 1$, so A is an essential class. By Lemma 3.2.5, the self reducibility of (p, x_0) implies the self reducibility of the point $([p], x_0))$. That is, A is a defective Nielsen class. Since $J_A = \left|(\chi \times 1_{\mathbf{RP^2}})^{-1}\,([p], x_0) \cap \Phi(\widetilde{f}_1, \widetilde{g}_1)\right| = |\{(p, x_0), (q, x_0)\}| = 2$, we get that A is an essential defective class with even J.

Notice that the covering map $\chi \times 1_{\mathbf{RP^2}}$ maps the nonessential Nielsen class $\widetilde{A}$ of the lift $(\widetilde{f}, \widetilde{g})$, to the essential Nielsen class A of (f, g). As we will see in the next section, this example implies that the nonlinear Nielsen number of the pair (f, g) (see Definition 3.3.1) $N_{ED}(f, g)$ is greater than or equal to 1. That is, $N_{ED}(f, g) \neq 0$.

3.3 Computation of $N(f, g)$.

In this section, we generalize Theorem 1.2.13 to Theorem 3.3.16. This generalization computes the coincidence Nielsen number $N(f,g)$ as a linear combination of the coincidence Nielsen numbers of the lifts of the pair (f,g). It is convenient to think of there being three Nielsen numbers. The first is the ordinary Nielsen number $N(f,g)$. We call the second one the linear Nielsen number $N_L(f,g)$. It is defined using a linear combination of the Nielsen numbers of the lifts of (f,g). The third one is called the non-Linear Nielsen number $N_{ED}(f,g)$. It is the number of the essential defective classes of (f,g) with even J. In fact, $N(f,g) = N_L(f,g) + N_{ED}(f,g)$. The main difficulty in the computation of $N(f,g)$ appears while computing $N_{ED}(f,g)$. As we will see, it cannot be computed in the same way we computed other Nielsen numbers, since it is related to the inessential classes of the lifts of (f,g). However, we do give a procedure for the computation of $N_{ED}(f,g)$.

Definition 3.3.1. *The number $N_{ED}(f,g)$ is defined to be the number of essential defective classes of f and g for which J is even. It is called the nonlinear Nielsen number of f and g.*

Example 3.3.2. In Example 3.2.20, $N_{ED}(f_1, g_1) = 0$. To see this, we have $\mathcal{A}(S^2) = \{1_{S^2}, -1_{S^2}\}$. Moreover, $\widetilde{f}_1 \circ (-1_{S^2}) = -1_{S^2} \circ \widetilde{f}_1$ and $\widetilde{g}_1 \circ (-1_{S^2}) = -1_{S^2} \circ \widetilde{g}_1$. This implies that there are two Reidemeister representatives of the pair (f_1, g_1), namely $(\widetilde{f}_1, 1_{S^2})$ and $(\widetilde{f}_1, -1_{S^2})$. Moreover, $\Phi(\widetilde{f}_1, 1_{S^2}) = \{p, q\}$ and $\Phi(\widetilde{f}_1, -1_{S^2})$ is homeomorphic to S^1.

Now, since $\widetilde{A} := \{p, q\}$ is the unique Nielsen class of the pair (f_1, g_1) and p does not reduce to q, the unique Nielsen class $A := \chi(\widetilde{A}) = \{[p]\}$ of (f_1, g_1) is not defective by Lemma 3.2.5. On the other hand, $\widetilde{B} := \Phi(\widetilde{f}_1, -1_{S^2})$ is a compact 1-manifold, and the pair $(\widetilde{f}_1, -1_{S^2})$ is not transverse on it. We have

$$\begin{aligned} index(\widetilde{f}_1, -1_{S^2}; \widetilde{B}) &= index(\widetilde{f}_1, -1_{S^2}; S^2) = L(\widetilde{f}_1, -1_{S^2}) \\ &= deg(-1_{S^2}) + (-1)^2 deg(\widetilde{f}_1) = -1 + 1 = 0. \end{aligned}$$

Thus, the Nielsen class $\widetilde{B}$ is inessential. Hence, $B = \chi(\widetilde{B})$ is either inessential or defective. Since this example considers the fixed point case, the defective classes do not exist. So, B is not defective and hence inessential. Therefore, $N_{ED}(f_1, g_1) = 0$.

Example 3.3.3. An argument analogous to that in Example 3.3.2 applied to the pair of maps (f_2, g_2) and their lifts given in Example 3.2.20 gives that the unique Nielsen class $A := \{x_0\}$ of (f_2, g_2) is essential defective with $J_A = 1$. This implies that $N_{ED}(f_2, g_2) = 0$.

Before we move to the next example, we give a formula for the semi-index of the product maps $f \times g$. We know that for the usual index, the index of the product maps is the product of the their indices. This is not always true for semi-index when defective classes are considered. For instance, in Example 3.2.20

$$\begin{aligned} |ind|(f_1 \times f_2, g_1 \times g_2; \{p, q\} \times \{x_0\}) &= |ind|(f_1 \times f_2, g_1 \times g_2; \{(p, x_0), (q, x_0)\}) \\ &= 0 \neq 2 = 2 \cdot 1 \\ &= |ind|(f_1 g_1; \{p, q\}) \cdot |ind|(f_2 g_2; \{x_0\}) \, . \end{aligned}$$

However, our formula of the semi-index of product maps extends the index formula when non defective classes are involved. We start with the following Definition.

Definition 3.3.4. *[5] Let $E = E_1 \oplus E_2$ be a real vector space of finite dimension, and let $\alpha_1 = [(e_1, \ldots, e_k)]$ and $\alpha_2 = [(e'_1, \ldots, e'_t)]$ be orientations of E_1 and E_2 respectively. We define $\alpha_1 \wedge \alpha_2$ to be the orientation of E determined by the ordered basis $(e_1, \ldots, e_k, e'_1, \ldots, e'_t)$.*

Definition 3.3.5. *[5] Let $\phi : E \longrightarrow E'$ be a linear transformation (isomorphism) between real vector spaces of finite dimension, and let $\alpha = [(e_1, \ldots, e_n)]$ be an orientation of E_1. Then the orientation of E_2 determined by ϕ is defined by $\phi(\alpha) = [(\phi(e_1), \ldots, \phi(e_n))]$.*

Proposition 3.3.6. *We have*

1. *The operation $\wedge$ is associative; that is, $(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3)$ [5].*
2. *$-(\alpha_1 \wedge \alpha_2) = -\alpha_1 \wedge \alpha_2 = \alpha_1 \wedge -\alpha_2$.*
3. *If $\phi : E \longrightarrow E'$ is a linear transformation, and $E = E_1 \oplus E_2$, then $\phi(\alpha_1 \wedge \alpha_2) = \phi(\alpha_1) \wedge \phi(\alpha_2)$.*

Proof. The proof of (2) depends on the fact that if M is a real square matrix and k is a real number, then $\det(k \cdot M) = k^n \cdot \det(M)$. The proof of (3) is easy since $\phi(E_1 \oplus E_2) = \phi(E_1) \oplus \phi(E_2)$ □

Lemma 3.3.7. *Let $(f_1, g_1) : M_1 \longrightarrow N_1$ and $(f_2, g_2) : M_2 \longrightarrow N_2$ be transverse pairs of maps between smooth closed manifolds of the same dimension, and Let a, a_1, $a_2 \in \Phi(f_1, g_1)$, and b, b_1, $b_2 \in \Phi(f_2, g_2)$. Then, with respect to $(f_1 \times f_2, g_1 \times g_2)$, we have*

1. If a_1 reduces to a_2, then (a_1, b) reduces to (a_2, b).

2. If b_1 reduces to b_2, then (a, b_1) reduces to (a, b_2).

3. If a_1 does not reduce to a_2, and b_1 does not reduce to b_2, then (a_1, b_1) does not reduce to (a_2, b_2).

Proof. 1. Suppose that a_1 reduces to a_2. Let C_b be the constant loop at b, and γ be an orientation at b, and let $\sigma : a_1 \mapsto a_2$ be a path that establishes the reducibility between a_1 and a_2. Let $\widetilde{\alpha}$ be an orientation at a_1 which is shifted by σ to the orientation $\widetilde{\beta}$ at a_2 (symbolically, $\widetilde{\alpha} \overset{\sigma}{\rightarrowtail} \widetilde{\beta}$). Let $g_{1*}^{a_1} - f_{1*}^{a_1}(\widetilde{\alpha}) = \alpha$, and $g_{1*}^{a_2} - f_{1*}^{a_2}(\widetilde{\beta}) = \beta$. Since a_1 reduces to a_2, we get that $\alpha \overset{f_1(\sigma)}{\rightarrowtail} -\beta$. On the other hand, we have $\widetilde{\gamma} \overset{C_b}{\rightarrowtail} \widetilde{\gamma}$. If we let $g_{2*}^{b} - f_{2*}^{b}(\widetilde{\gamma}) = \gamma$, then, $\gamma \overset{f_2(C_b)}{\rightarrowtail} \gamma$.

Now, the loop $\sigma \times C_b$ shifts the orientation $\widetilde{\alpha} \wedge \widetilde{\gamma}$ at (a_1, b) to the orientation $\widetilde{\beta} \wedge \widetilde{\gamma}$ at (a_2, b). Since

$$\begin{aligned}
(g_1 \times g_2)_*^{(a_1,b)} - (f_1 \times f_2)_*^{(a_1,b)} (\widetilde{\alpha} \wedge \widetilde{\gamma}) &= (g_{1*} \times g_{2*})^{(a_1,b)} - (f_{1*} \times f_{2*})^{(a_1,b)} (\widetilde{\alpha} \wedge \widetilde{\gamma}) \\
&= (g_{1*}^{a_1} - f_{1*}^{a_1}) \times \left(g_{2*}^{b} - f_{2*}^{b}\right) (\widetilde{\alpha} \wedge \widetilde{\gamma}) \\
&= (g_{1*}^{a_1} - f_{1*}^{a_1}) (\widetilde{\alpha}) \wedge \left(g_{2*}^{b} - f_{2*}^{b}\right) (\widetilde{\gamma}) = \alpha \wedge \gamma ,
\end{aligned}$$

and

$$(g_1 \times g_2)_*^{(a_2,b)} - (f_1 \times f_2)_*^{(a_2,b)} (\widetilde{\beta} \wedge \widetilde{\gamma}) = \beta \wedge \gamma ,$$

we have that

$$\alpha \wedge \gamma \overset{f_1(\sigma) \times f_2(C_b)}{\rightarrowtail} -\beta \wedge \gamma = -(\beta \wedge \gamma) .$$

Thus, (a_1, b) reduces to (a_2, b).

2. Similar to (1).

3. Assume that a_1 does not reduce to a_2, and b_1 does not reduce to b_2. Let $\delta : (a_1, b_1) \mapsto (a_2, b_2)$ be a path such that $f_1 \times f_2(\delta) \approx g_1 \times g_2(\delta)$ rel. endpoints. We can write $\delta = \sigma_1 \times \sigma_2$, where $\sigma_1 = \pi_1(\delta)$ is a path in M_1 from a_1 to a_2, and $\sigma_2 = \pi_2(\delta)$ is a path in M_2 from b_1 to b_2 (here π_1 and π_2 are the projections on the first and the second coordinates, respectively). Moreover, $f_i(\sigma_i) \approx g_i(\sigma_i)$ rel. endpoints, for $i = 1, 2$. Let $\widetilde{\alpha}$ be an orientation at (a_1, b_1). We can write $\widetilde{\alpha} = \widetilde{\alpha}_1 \wedge \widetilde{\alpha}_2$ where $\widetilde{\alpha}_1$ is an orientation at a_1 and $\widetilde{\alpha}_2$ is an orientation at b_1. Assume that $\widetilde{\alpha}_i$ is shifted by σ_i to the orientation $\widetilde{\beta}_i$, for $i = 1, 2$. Thus, $\widetilde{\alpha} \overset{\delta}{\rightarrowtail} \widetilde{\beta} = \widetilde{\beta}_1 \wedge \widetilde{\beta}_2$. If we let $g_{1*}^{a_1} - f_{1*}^{a_1}(\widetilde{\alpha}_1) = \alpha_1$, $g_{1*}^{a_2} - f_{1*}^{a_2}(\widetilde{\beta}_1) = \beta_1$, $g_{2*}^{b_1} - f_{2*}^{b_1}(\widetilde{\alpha}_2) = \alpha_2$, and $g_{2*}^{b_2} - f_{2*}^{b_2}(\widetilde{\beta}_2) = \beta_2$, then,

$$(g_1 \times g_2)_*^{(a_1,b_1)} - (f_1 \times f_2)_*^{(a_1,b_1)} (\widetilde{\alpha}_1 \wedge \widetilde{\alpha}_2) = \alpha_1 \wedge \alpha_2 \,,$$

and

$$(g_1 \times g_2)_*^{(a_2,b_2)} - (f_1 \times f_2)_*^{(a_2,b_2)} (\widetilde{\beta}_1 \wedge \widetilde{\beta}_2) = \beta_1 \wedge \beta_2 \,.$$

Now, since a_1 does not reduce to a_2, and b_1 does not reduce to b_2, $\alpha_1 \overset{f_1(\sigma_1)}{\not\rightarrowtail} \beta_1$, and $\alpha_2 \overset{f_2(\sigma_2)}{\not\rightarrowtail} \beta_2$. Hence, $\alpha_1 \wedge \alpha_2 \overset{f_1(\sigma_1)\times f_2(\sigma_2)}{\not\rightarrowtail} \beta_1 \wedge \beta_2$. This means that any path between (a_1, b_1) and (a_2, b_2) cannot establish the reducibility between them. Therefore, (a_1, b_1) does not reduce to (a_2, b_2). $\square$

Corollary 3.3.8. *Let $(f_1, g_1) : M_1 \longrightarrow N_1$ and $(f_2, g_2) : M_2 \longrightarrow N_2$ be transverse pairs of maps between smooth closed manifolds of the same dimension, and let A and B be Nielsen classes of (f_1, g_1) and (f_2, g_2), respectively. Then $A \times B$ is defective if and only if either A or B is defective.*

Proof. Assume that $A \times B$ is defective. Thus, there exists a coincidence point (a, b) in $A \times B$ that reduces to itself. By part (3) of Lemma 3.3.7, either a or b reduces to itself. That is, either A or B is defective.

Now, assume that either A or B is defective. Let us assume that A is defective and that $a \in A$ reduces to itself. Let $b \in B$. By part (1) of Lemma 3.3.7, $(a, b) \in A \times B$ reduces to itself. Therefore, $A \times B$ is defective. The case where B is defective is done similarly. □

The next proposition gives the semi-index formula for the product maps.

Proposition 3.3.9. *Let $(f_1, g_1) : M_1 \longrightarrow N_1$ and $(f_2, g_2) : M_2 \longrightarrow N_2$ be pairs of maps between smooth closed manifolds of the same dimension, and Let A and B be Nielsen classes of (f_1, g_1) and (f_2, g_2), respectively. Then*

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = \begin{cases} |ind|(f_1, g_1; A) \cdot |ind|(f_2, g_2; B), & \textit{if neither } A \textit{ nor } B \\ & \textit{are defective.} \\ \dfrac{1 - (-1)^{|ind|(f_1, g_1; A) \cdot |ind|(f_2, g_2; B)}}{2}, & \textit{otherwise} . \end{cases} \tag{3.3.1}$$

Proof. Since the semi-index is homotopy invariant, without lose of generality assume that (f_1, g_1) and (f_2, g_2) are transverse pairs. Firstly, suppose that both A and B are not defective. Let

$$A = \{a_1, a_2, \ldots a_s; z_1, \ldots, z_r\} \tag{3.3.2}$$

and

$$B = \{b_1, b_2, \ldots, b_t; y_1, \ldots, y_k\} \tag{3.3.3}$$

be decompositions of A and B, respectively. It is easy to see that $A \times B$ is a Nielsen class for $(f_1 \times f_2, g_1 \times g_2)$. Since the numbers s and t are even, Lemma 3.3.7 allows $A \times B$ to have the following decomposition

$$\begin{aligned} A \times B \;=\; & \{(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_t), (a_2, b_1), (a_2, b_2), \ldots, (a_2, b_t), \ldots, (a_s, b_1), \\ & (a_s, b_2), \ldots, (a_s, b_t), (z_1, b_1), (z_1, b_2), \ldots, (z_1, b_t), \ldots, (z_r, b_1), (z_r, b_2), \ldots, \\ & (z_r, b_t), (a_1, y_1), (a_2, y_1), \ldots, (a_s, y_1), \ldots, (a_1, y_k), (a_2, y_k), \ldots, (a_s, y_k); \\ & (z_1, y_1), (z_1, y_2), \ldots, (z_1, y_k), \ldots, (z_r, y_1), (z_r, y_2), \ldots, (z_r, y_k)\} \end{aligned}$$

Thus,

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = r \cdot s = |ind|(f_1, g_1; A) \cdot |ind|(f_2, g_2; B)\,.$$

Next, suppose that either A or B is defective. By Corollary 3.3.8, $A \times B$ is defective. We give A and B the decompositions given in Equations 3.3.2 and 3.3.3, respectively. Without lose of generality, let us assume that A is defective and that a_1 is self reducible. Then $r = 0$ or $r = 1$. We have the following cases

1. Suppose $|ind|(f_1, g_1; A) = 0$. Then $|A|$ is even. This implies that $|A \times B| = |A| \cdot |B|$ is even. Thus,

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = 0 = \frac{1-1}{2} = \frac{1-(-1)^{|ind|(f_1, g_1; A) \cdot |ind|(f_2, g_2; B)}}{2}\,.$$

2. Suppose $|ind|(f_1, g_1; A) = 1$, and B is defective with $|ind|(f_2, g_2; B) = 0$. This case is similar to the previous case.

3. Suppose $|ind|(f_1, g_1; A) = 1$, and B is defective with $|ind|(f_2, g_2; B) = 1$. Thus, $|A|$ and $|B|$ are odd and hence $|A \times B|$ is odd. Therefore,

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = 1 = \frac{1+1}{2} = \frac{1-(-1)^{|ind|(f_1,g_1;A)\cdot|ind|(f_2,g_2;B)}}{2}.$$

4. Suppose $|ind|(f_1, g_1; A) = 1$, and B is not defective. We have the following sub cases:

- If $|ind|(f_2, g_2; B)$ is even, the fact that the difference $|B| - |ind|(f_2, g_2; B)$ being always even gives that $|B|$ is even. Hence, $|A \times B|$ is even. Thus,

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = 0 = \frac{1-1}{2} = \frac{1-(-1)^{|ind|(f_1,g_1;A)\cdot|ind|(f_2,g_2;B)}}{2}.$$

- If $|ind|(f_2, g_2; B)$ is odd, the fact that the difference $|B| - |ind|(f_2, g_2; B)$ being always even gives that $|B|$ is odd. Hence, $|A \times B|$ is odd. Thus,

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = 1 = \frac{1+1}{2} = \frac{1-(-1)^{|ind|(f_1,g_1;A)\cdot|ind|(f_2,g_2;B)}}{2}.$$

Consequently, we get that

$$|ind|(f_1 \times f_2, g_1 \times g_2; A \times B) = \frac{1-(-1)^{|ind|(f_1,g_1;A)\cdot|ind|(f_2,g_2;B)}}{2}.$$

□

Now, we give an example where $N_{ED}(f, g) = 1$.

Example 3.3.10. In Example 3.2.20, we showed that $N_{ED}(f, g) > 0$. We show here that $N_{ED}(f, g) = 1$. The Nielsen class $B \times \{x_0\}$ is inessential. In fact, In Example

3.3.2, we have shown that B is inessential. Thus, $|ind|(f_1, g_1; B) = 0$. Since $\{x_0\}$ is defective, by Proposition 3.3.9

$$
\begin{aligned}
|ind|(f_1 \times f_2, g_1 \times g_2; B \times \{x_0\}) &= \frac{1 - (-1)^{|ind|(f_1, g_1; B) \cdot |ind|(f_2, g_2; \{x_0\})}}{2} \\
&= \frac{1 - (-1)^{0 \cdot 1}}{2} = \frac{0}{2} = 0 .
\end{aligned}
$$

Thus $\{([p], x_0)\}$ is the only essential defective such that J is even. Therefore, $N_{ED}(f, g) =$ 1.

The following proposition gives a procedure for the computation of $N_{ED}(f, g)$.

Proposition 3.3.11. *The number $N_{ED}(f, g)$ can be computed using the following procedure:*

1. *Fix a lift $(\widetilde{f}, \widetilde{g})$ of (f, g), then apply Remark 4.1.9 to generate the H-Reidemeister classes. Pick a representative of each H-Reidemeister class of the form $(\widetilde{f}, \beta\widetilde{g})$ and $\beta \in \mathcal{A}(\widetilde{N})$(we will explain in Chapter 4 why we focus on such representatives).*

2. *Choose a coincidence point x_0 of f and g, and use it to compute $J = |j(C(f_\#, g_\#)_{x_0})|$.*

3. *Select those representatives of the H-Reidemeister classes for which there exist Nielsen classes of even J.*

4. *Apply Lemma 3.2.5 or Proposition 3.2.7 to find the essential defective classes within the H-Nielsen classes corresponding to the representatives determined in the previous step.*

5. *For each of these representatives, find the essential defective Nielsen classes with even J that lie inside the corresponding H-Nielsen classes.*

6. *Count the Nielsen classes in the last step for each of these certain H-Reidemeister classes and denote the resulting number by ED. Then, add these ED's up to get the desired number $N_{ED}(f,g)$.*

Next, we show that $N_{ED}(f,g)$ is a Nielsen number. We start by showing that the three numbers I, J, and S are homotopy invariant.

Proposition 3.3.12. *The numbers J, I, and S are homotopy invariant.*

Proof. Assume $(f,g) : M \longrightarrow N$ is homotopic to a pair $(\acute{f},\acute{g}) : M \longrightarrow N$ by the homotopy-pair $(F,G) : M \times [0,1] \longrightarrow N$. Let $x \in \Phi(f,g)$ and $\acute{x} \in \Phi(\acute{f},\acute{g})$ be F,G-related coincidence points. Let $u : x \longrightarrow \acute{x}$ be a path such that $F(u) \sim_0 G(u)$.

(1) J is homotopy invariant: As in the proof of Proposition 3.2.7, the isomorphism

$$u_{\#} : \pi_1(M,x) \longrightarrow \pi_1(M,\acute{x})$$

restricts to the isomorphism

$$u_{\#} : C(f_{\#},g_{\#})_x \longrightarrow C(\acute{f}_{\#},\acute{g}_{\#})_{\acute{x}} \ .$$

Consider the diagram

$$\begin{array}{ccc} C(f_{\#},g_{\#})_x & \xrightarrow{u_{\#}} & C(\acute{f}_{\#},\acute{g}_{\#})_{\acute{x}} \\ j\downarrow & & \downarrow j \\ j\left(C(f_{\#},g_{\#})_x\right) & \xrightarrow{\overline{u}_{\#}} & j(C(\acute{f}_{\#},\acute{g}_{\#})_{\acute{x}}) \end{array},$$

where $\overline{u}_{\#}$ is the homomorphism induced by $u_{\#}$ on the given groups. The diagram is commutative and hence $\overline{u}_{\#}$ is an isomorphism. Therefore, by Proposition 2.1.4 we get that $J_{[x]} = J_{[\acute{x}]}$. In other words, the number J is homotopy invariant.

(2) I is homotopy invariant: First, Let us recall

- In regular coverings, f admits a lift if and only if $\acute{f}$ does. In other words,

$$f_{\#}(K(x)) \subseteq H(f(x)) \Leftrightarrow \acute{f}_{\#}(K(x)) \subseteq H(\acute{f}(x)),$$

  for all $x \in M$.

- The isomorphism

$$\overline{u}_{\#} : \frac{\pi_1(M,x)}{K(x)} \longrightarrow \frac{\pi_1(M,\acute{x})}{K(\acute{x})}$$

  induces the isomorphism

$$\overline{u}_{\#} : C(\overline{f}_{\#}, \overline{g}_{\#})_x \longrightarrow \overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right) .$$

We claim that

$$\overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right) = C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}} .$$

Let $\overline{b} \in \overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right)$. Then,

$\overline{b} = \overline{u}_{\#}(\overline{a})$ and $\overline{f}_{\#}(\overline{a}) = \overline{g}_{\#}(\overline{a}) \Rightarrow \overline{b} = \overline{u_{\#}(a)}$ and $\overline{f_{\#}(a)} = \overline{g_{\#}(a)}$

$\Rightarrow b\, u_{\#}(a)^{-1} \in K(\acute{x})$ and $\overline{f(a)} = \overline{g(a)}$

$\Rightarrow b\, u_{\#}(a)^{-1} = k \in K(\acute{x})$ and $\overline{f(a)} = \overline{g(a)}$

$\Rightarrow k^{-1} b = u_{\#}(a)$ and $f(a) = h\, g(a)$ for some $h \in H(f(x))$

$\Rightarrow k^{-1} b = u_{\#}(a)$ and $F(x)\, \acute{f}(a) F(x)^{-1} = h\, g(a)$

$\Rightarrow k^{-1} b = u^{-1}\, a\, u$ and $\acute{f}(a) = F(x)^{-1}\, h\, g(a)\, F(x)$

$$\Rightarrow a = u\, k^{-1}\, b\, u^{-1} \text{ and } \acute{f}(a) = F(x)^{-1}\, h\, g(a)\, F(x)$$

$$\Rightarrow \acute{f}(u\, k^{-1}\, b\, u^{-1}) = F(x)^{-1}\, h\, g(u\, k^{-1}\, b\, u^{-1})\, F(x)$$

$$\Rightarrow \acute{f}(u)\, \acute{f}(k^{-1})\, \acute{f}(b)\, \acute{f}(u^{-1}) = F(x)^{-1}\, h\, g(u)\, g(k^{-1})\, g(b)\, g(u^{-1})\, F(x)$$

$$\Rightarrow \acute{f}(b) = \acute{f}(k)\, \underbrace{\acute{f}(u^{-1})\, F(x)^{-1}}_{F(u)^{-1}}\, h\, g(u)\, g(k^{-1})\, g(b)\, g(u^{-1})\, \underbrace{F(x)\, \acute{f}(u)}_{F(u)}$$

$$\Rightarrow \acute{f}(b) = \acute{f}(k)\, G(u)^{-1}\, h\, g(u)\, g(k)^{-1}\, g(b)\, g(u)^{-1}\, G(u)$$

$$\Rightarrow \acute{f}(b) = \acute{f}(k)\, G(u)^{-1}\, h\, G(u)\, \underbrace{G(u)^{-1}\, g(u)}_{G(\acute{x})^{-1}}\, g(k)^{-1}\, g(b)\, \underbrace{g(u)^{-1}\, G(u)}_{G(\acute{x})}$$

$$\Rightarrow \acute{f}(b) = \underbrace{\acute{f}(k)}_{\in H(\acute{f}(\acute{x}))}\ \underbrace{G(u)^{-1}\, h\, G(u)}_{\in H(\acute{f}(\acute{x}))}\, \underbrace{G(\acute{x})^{-1}\, g(k)^{-1}\, G(\acute{x})}_{\in H(\acute{f}(\acute{x}))}\, \underbrace{G(\acute{x})^{-1}\, g(b)\, G(\acute{x})}_{\acute{g}(b)}$$

$$\Rightarrow \overline{\acute{f}(b)} = \overline{\acute{g}(b)}$$

$$\Rightarrow \overline{\acute{f}}_{\#}(\overline{b}) = \overline{\acute{g}}_{\#}(\overline{b})$$

$$\Rightarrow \overline{b} \in C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}} \,.$$

Therefore,

$$\overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right) \subseteq C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}} \,.$$

Similarly,

$$\overline{u}_{\#}^{-1}\left(C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}}\right) \subseteq C(\overline{f}_{\#}, \overline{g}_{\#})_x$$

which implies that

$$C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}} \subseteq \overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right) \,.$$

Consequently,

$$\overline{u}_{\#}\left(C(\overline{f}_{\#}, \overline{g}_{\#})_x\right) = C(\overline{\acute{f}}_{\#}, \overline{\acute{g}}_{\#})_{\acute{x}} \,.$$

Now, by Proposition 2.1.17, and the definition of the number I, we obtain $I_{[x]} = I_{[\acute{x}]}$, i.e., the number I is homotopy invariant.

(3) S is homotopy invariant: Since both J and I are homotopy invariant, Proposition 2.1.21 gives that S is homotopy invariant. $\square$

Corollary 3.3.13. *The number $N_{ED}(f,g)$ is homotopy invariant. In particular, $N_{ED}(f,g)$ is a Nielsen number.*

Proof. Proposition 3.2.7 states that "being defective" is homotopy invariant as is "being essential". Hence, by Proposition 3.3.12 we get that $N_{ED}(f,g)$ is homotopy invariant. Since it is also non-negative and a lower bound of $\Phi(f,g)$ we get that $N_{ED}(f,g)$ is a Nielsen number. $\square$

Now we define the Linear Nielsen number $N_L(f,g)$ and show that it is indeed a Nielsen number.

Definition 3.3.14. *The Linear Nielsen number $N_L(f,g)$ of the pair (f,g) is defined to be*

$$N_L(f,g) = N(f,g) \; - \; N_{ED}(f,g) \; .$$

Proposition 3.3.15. *The Linear Nielsen number $N_L(f,g)$ of a pair (f,g) is a Nielsen number of f and g.*

Proof. Obviously, $N_L(f,g)$ is a nonnegative integer. Since $N(f,g)$ is homotopy invariant, by Corollary 3.3.13 we obtain that $N_L(f,g)$ is homotopy invariant. Also, it is a lower bound of the set $\Big\{|\,\Phi(\acute{f},\acute{g})| \;\mid\; f \sim \acute{f} \;\text{ and }\; g \sim \acute{g}\Big\}$. $\square$

Again, let $(\widetilde{f}_1, \widetilde{g}_1), \ldots, (\widetilde{f}_{R_H(f,g)}, \widetilde{g}_{R_H(f,g)})$ be representatives of the H-Reidemeister classes of the pair (f, g), and Let r be the number of nonempty H-Nielsen classes of f and g. Without lose of generality, assume that $(\widetilde{f}_1, \widetilde{g}_1), \ldots, (\widetilde{f}_r, \widetilde{g}_r)$ are the representatives of the H-Reidemeister classes of the pair (f,g) corresponding to the nonempty H-Nielsen classes. We let $\widetilde{p\,\Phi}_E(\widetilde{f}, \widetilde{g})$ denote the set of essential classes in the H-Nielsen class $p\,\Phi(\widetilde{f}, \widetilde{g})$. We are ready now to prove the main theorem of this chapter which shows that $N_L(f, g)$ is a linear combination of the Nielsen number of the lifts of (f, g).

Theorem 3.3.16. *Let M and N be connected closed smooth manifolds of the same dimension, $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be finite regular coverings which correspond to normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$, respectively. Let $f, g : M \longrightarrow N$ be maps for which there exist lifts $\widetilde{f}, \widetilde{g} : \widetilde{M} \longrightarrow \widetilde{N}$ respectively. Suppose the number J_A is the same for all Nielsen classes A of f and g that lie in the same H-Nielsen class. Then,*

$$N_L(f, g) = N(f, g) \; - \; N_{ED}(f, g) = \sum_{i=1}^{R_H(f,g)} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}. \tag{3.3.4}$$

Proof. Without lose of generality, assume the pairs $(\widetilde{f}_1, \widetilde{g}_1), \ldots, (\widetilde{f}_t, \widetilde{g}_t)$ have odd J and the pairs $(\widetilde{f}_{t+1}, \widetilde{g}_{t+1}), \ldots, (\widetilde{f}_r, \widetilde{g}_r)$ have even J, where $t \leq r$. Then,

$$N(f, g) = \sum_{i=1}^{i=t} |\, \widetilde{p\,\Phi}_E(\widetilde{f}_i, \widetilde{g}_i) \,| + \sum_{i=t+1}^{i=r} |\, \widetilde{p\,\Phi}_E(\widetilde{f}_i, \widetilde{g}_i) \,| \,.$$

The assumptions yield that the number S is the same for all Nielsen classes in the same H-Nielsen class. Hence, by (1) and (3) of Proposition 3.2.15,

$$N(\widetilde{f}_i, \widetilde{g}_i) = |\, \widetilde{p\,\Phi}_E(\widetilde{f}_i, \widetilde{g}_i) \,| \cdot S(\widetilde{f}_i, \widetilde{g}_i)$$

for each $i = 1, \ldots, t$. Thus,

$$|\widetilde{p\Phi}_E(\widetilde{f}_i, \widetilde{g}_i)| = \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}$$

for each $i = 1, \ldots, t$.

On the other hand, for each $i = t+1, \ldots, r$, let $ED(\widetilde{f}_i, \widetilde{g}_i)$ denote the number of essential defective classes in $p\,\Phi(\widetilde{f}_i, \widetilde{g}_i)$, and $END(\widetilde{f}_i, \widetilde{g}_i)$ denote the number of essential non-defective classes in $p\,\Phi(\widetilde{f}_i, \widetilde{g}_i)$. It follows from Lemma 3.2.16 that

$$END(\widetilde{f}_i, \widetilde{g}_i) \cdot S(\widetilde{f}_i, \widetilde{g}_i) = N(\widetilde{f}_i, \widetilde{g}_i)\ .$$

or

$$END(\widetilde{f}_i, \widetilde{g}_i) = \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}\ ,$$

for $i = t+1, \ldots, r$. Thus,

$$|\widetilde{p\Phi}_E(\widetilde{f}_i, \widetilde{g}_i)| = END(\widetilde{f}_i, \widetilde{g}_i) + ED(\widetilde{f}_i, \widetilde{g}_i) = \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + ED(\widetilde{f}_i, \widetilde{g}_i)\ .$$

Now,

$$N(f,g) = \sum_{i=1}^{i=t} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + \sum_{i=t+1}^{i=r} \Big(END(\widetilde{f}_i, \widetilde{g}_i) + ED(\widetilde{f}_i, \widetilde{g}_i) \Big)$$

$$= \sum_{i=1}^{i=t} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + \sum_{i=t+1}^{i=r} \left(\frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + ED(\widetilde{f}_i, \widetilde{g}_i) \right) \text{ or}$$

$$N(f,g) = \sum_{i=1}^{i=t} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + \sum_{i=t+1}^{i=r} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + \sum_{i=t+1}^{i=r} ED(\widetilde{f}_i, \widetilde{g}_i)\ .$$

Finally,

$$N(f,g) = \sum_{i=1}^{i=r} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} + N_{ED}(f,g)\ .$$

Since $N(\widetilde{f},\widetilde{g}) = 0$ for the representatives corresponding to empty H-Nielsen classes and inessential H-Nielsen classes, we get

$$N_L(f,g) = N(f,g) \; - \; N_{ED}(f,g) = \sum_{i=1}^{R_H(f,g)} \frac{N(\widetilde{f}_i,\widetilde{g}_i)}{S(\widetilde{f}_i,\widetilde{g}_i)}.$$

□

Corollary 3.3.17. *Let M and N be connected closed smooth manifolds of the same dimension, $(\widetilde{M},p)$ and $(\widetilde{N},p)$ be finite regular coverings which correspond to normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$, respectively. Let $f,g : M \longrightarrow N$ be maps for which there exist lifts $\widetilde{f},\widetilde{g} : \widetilde{M} \longrightarrow \widetilde{N}$ respectively. Suppose the number J_A is the same for all Nielsen classes A of f and g that lie in the same H-Nielsen class. Then,*

$$N(f,g) = N_L(f,g) \; + \; N_{ED}(f,g) = \sum_{i=1}^{R_H(f,g)} \frac{N(\widetilde{f}_i,\widetilde{g}_i)}{S(\widetilde{f}_i,\widetilde{g}_i)} \; + \; N_{ED}(f,g)\,.$$

Example 3.3.18. From Example 3.3.2, $I_A = J_A = 2$. Thus, $S(\widetilde{f}_1,\widetilde{g}_1) = S_A = 1$. Therefore, $N(f_1,g_1) = N_L(f_1,g_1) \; + \; N_{ED}(f_1,g_1) = N(\widetilde{f}_1,\widetilde{g}_1) + 0 = 1 + 0 = 1$. This result agrees with the fact that A is the unique essential class of (f_1,g_1).

Example 3.3.19. From Example 3.3.3, $I_A = J_A = 1$. Hence, $S(\widetilde{f}_2,\widetilde{g}_2) = S_A = 1$. Therefore, $N(f_2,g_2) = N_L(f_2,g_2) \; + \; N_{ED}(f_2,g_2) = N(\widetilde{f}_2,\widetilde{g}_2) + 0 = 1 + 0 = 1$. Again, this result agrees with the fact that A is the unique essential class of (f_2,g_2).

Example 3.3.20. From Example 3.3.10, $I_A = J_A = 2$. Thus, $S(\widetilde{f}, \widetilde{g}) = S_A = 1$. Therefore, $N(f,g) = N_L(f,g) + N_{ED}(f,g) = N(\widetilde{f}, \widetilde{g}) + 1 = 0 + 1 = 1$. This result agrees with the fact that A is the unique essential class of (f,g).

Remark 3.3.21. *Since all Nielsen numbers are strictly non-negative, it follows trivially from the definition that the the linear Nielsen number $N_L(f,g)$ acts as a lower bound for $N(f,g)$, that is $N_L(f,g) \leq N(f,g)$. The point of the remark of course is, as usual with lower bounds, that they are easier to compute. In fact since $N_L(f,g)$ is a linear combination of the Nielsen numbers of the lifts of the pair (f,g), the computations are identical with those in Chapter 2. The comparative ease of computation of $N_L(f,g)$ over $N(f,g)$ is emphasized by the fact that we do not have a direct method for the computation of $N_{ED}(f,g)$. In the next section we will, among other things, discuss cases and give examples where $N_L(f,g) = N(f,g)$.*

3.4 Applications and More Examples of Theorem 3.3.16

This section contains some special cases of Theorem 3.3.16, and some examples. The results in this section agree with those in Section 2.4 when orientable manifolds are considered.

Let M and N be closed connected smooth manifolds of the same dimension n, and let $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. We assume the coverings are finite, i.e., $[\pi_1(M) : K] < \infty$ and $[\pi_1(N) : H] < \infty$. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f}, \widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}, \widetilde{g}} & \widetilde{N} \\ p \downarrow & & \downarrow p \\ M & \xrightarrow{f, g} & N \end{array} \tag{3.4.1}$$

The following result follows directly from Theorem 3.3.16.

Corollary 3.4.1. *Let M and N be connected closed smooth manifolds of the same dimension, and $f, g : M \longrightarrow N$ be smooth maps that admit lifts $\widetilde{f}, \widetilde{g} : \widetilde{M} \longrightarrow \widetilde{N}$ respectively. Suppose the number J is the same for all Nielsen classes of f and g that lie in the same H-Nielsen class. Suppose in addition that all essential Nielsen classes corresponding to even J are non-defective. Then,*

$$N(f, g) = N_L(f, g) = \sum_{i=1}^{R_H(f,g)} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}.$$

Proof. If the essential Nielsen classes corresponding to even J are non-defective, then $N_{ED}(f, g) = 0$. The rest follows by applying Theorem 3.3.16. □

We begin with the fixed point case. The following proposition shows that there do not exist defective classes when considering fixed points.

Proposition 3.4.2. *Suppose $M = N$, $(\widetilde{M}, p) = (\widetilde{N}, p)$, and $g = 1_M$. Then, the fixed point classes of f are non-defective.*

Proof. Any defective class must have x reduce to itself by Proposition 3.2.7. So, let σ be a path establishing this reducibility. Then, $f(\sigma) \sim_0 \sigma$. Thus, σ and $f(\sigma)$ induce the same effect on orientations by Proposition 3.2.7, and we cannot have the mismatch required by self reducibility. □

The following Theorem gives the same formula for computing $N(f)$ given in Theorem 1.2.13, and illustrates why Theorem 3.3.16 is a generalization of Theorem 1.2.13.

Theorem 3.4.3. *Suppose $M = N$, $(\widetilde{M}, p) = (\widetilde{N}, p)$, $f = 1_M$, and all the Nielsen fixed point classes that lie in the same H-Nielsen class have the same number J. Then,*

$$N(g) = N_L(g) = \sum_{i=1}^{r} \frac{J(\widetilde{g}_i)}{I(\widetilde{g}_i)} N(\widetilde{g}_i) .$$

where r is the number of nonempty H-Reidemeister classes of g.

Proof. By Proposition 3.4.2, we have $N_{ED}(1_M, g) = 0$. The rest follows by applying Proposition 2.1.21 and Theorem 3.3.16. □

Remark 3.4.4. *In Theorem 3.4.3, if we put $g = 1_M$, then we get the formula*

$$N(f) = N_L(f) = \sum_{i=1}^{r} \frac{J(\widetilde{f}_i, \alpha_i)}{I(\widetilde{f}_i, \alpha_i)} N(\widetilde{f}_i, \alpha_i)$$

where $\alpha_i \in \mathcal{A}(\widetilde{M})$. It allows us to compute the fixed point Nielsen number $N(f)$ in terms of the coincidence Nielsen numbers of a fixed lift of f and the covering transformations, which are the lifts of $g = 1_M$.

Next, we consider the orientable case. The following lemma shows that there are no defective classes when the involved manifolds are orientable.

Lemma 3.4.5. *Let $(f, g) : M \longrightarrow N$ be a pair of maps between two oriented closed smooth $n-$manifolds. Then, there do not exist defective Nielsen classes of (f, g).*

Proof. Without lose of generality, assume (f, g) is a transverse pair. We give proof by contradiction. Accordingly, assume there exists a self reducible point x. By (2) of Proposition 3.1.25, $index(f, g; x) = -index(f, g; x)$. Hence, $index(f, g; x) = 0$ which, under our assumptions, contradicts the fact that $index(f, g; x) = \pm 1$ (see part (1) of Proposition 3.1.25). □

Theorem 3.4.6 states that when orientable manifolds are involved, then Theorem 2.3.5 and Theorem 3.3.16 coincide.

Theorem 3.4.6. *Let $(f, g) : M \longrightarrow N$ be a pair of maps between two oriented closed smooth $n-$manifolds. Suppose the number J is the same for all Nielsen classes that lie in the same $H-$Nielsen class. Then,*

$$N(f, g) = N_L(f, g) = \sum_{i=1}^{R_H(f,g)} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)}.$$

Proof. Apply Corollary 3.4.1 and Lemma 3.4.5. □

Next, we consider the case of a universal covering.

Lemma 3.4.7. *Assume that $\widetilde{M}$ and $\widetilde{N}$ are orientable manifolds, and that $(\widetilde{N}, p)$ is universal. Therefore, $\Phi(\widetilde{f}, \widetilde{g})$ is a single Nielsen class, and there is only one J value. Let $p\,\Phi(\widetilde{f}, \widetilde{g})$ be an essential Nielsen class of f and g. Then, $p\,\Phi(\widetilde{f}, \widetilde{g})$ is defective with even J if and only if $L(\widetilde{f}, \widetilde{g}) = 0$.*

Proof. Assume $p\,\Phi(\widetilde{f},\widetilde{g})$ is defective with even J. Hence, $\Phi(\widetilde{f},\widetilde{g})$ is inessential and $|ind|(\widetilde{f},\widetilde{g};\,\Phi(\widetilde{f},\widetilde{g}))=0$ which implies $L(\widetilde{f},\widetilde{g})=0$.

Conversely, assume that $L(\widetilde{f},\widetilde{g})=0$. Now, $\Phi(\widetilde{f},\widetilde{g})$ is inessential. Hence, $p\,\Phi(\widetilde{f},\widetilde{g})$ is defective (otherwise, since $p\,\Phi(\widetilde{f},\widetilde{g})$ is essential, we get $\Phi(\widetilde{f},\widetilde{g})$ is essential). Now, $p\,\Phi(\widetilde{f},\widetilde{g})$ is defective and essential, so it has an odd cardinality. If $J(\widetilde{f},\widetilde{g})$ is odd, then $|\Phi(\widetilde{f},\widetilde{g})|$ has odd cardinality from the equation

$$|\Phi(\widetilde{f},\widetilde{g})| = J(\widetilde{f},\widetilde{g})\cdot|p\,\Phi(\widetilde{f},\widetilde{g})|$$

given in Proposition 2.1.21. Hence, $\Phi(\widetilde{f},\widetilde{g})$ is essential and this is a contradiction. Therefore, $J(\widetilde{f},\widetilde{g})$ must be even. □

Theorem 3.4.8. *If $\widetilde{M}$ and $\widetilde{N}$ are orientable manifolds, and $(\widetilde{M},p)$ and $(\widetilde{N},p)$ are universal, then*

- $N_L(f,g)=\sum_{i=1}^{r} N(\widetilde{f}_i,\widetilde{g}_i)$ *= the number of Reidemeister classes of representatives that have non-zero Lefschetz number.*

- $N_{ED}(f,g)$ *= the number of Reidemeister classes each with a representative $(\widetilde{f},\widetilde{g})$ such that $L(\widetilde{f},\widetilde{g})=0$ and $p\,\Phi(\widetilde{f},\widetilde{g})$ is essential.*

Proof. Since the coverings are universal, both f and g can be lifted. Moreover, the $H-$Nielsen classes are equal to the Nielsen classes. So, there is no uniformity requirement for the number J. Let $(\widetilde{f}_i,\widetilde{g}_i)$ be a representative of a Reidemeister class of f and g for all $i=1,\ldots,r$, where r is the number of non-empty Reidemeister classes.

Without lose of generality assume $L(\widetilde{f}_i, \widetilde{g}_i) \neq 0$ for each $i = 1, \ldots, t$ and $1 \leq t \leq r$, and $L(\widetilde{f}_i, \widetilde{g}_i) = 0$ otherwise.

(1) Fix i. Since $(\widetilde{N}, p)$ is universal, $\Phi(\widetilde{f}_i, \widetilde{g}_i)$ is the only Nielsen class for $\widetilde{f}_i$ and $\widetilde{g}_i$. The following are equivalent

1. $\Phi(\widetilde{f}_i, \widetilde{g}_i)$ is essential.

2. $N(\widetilde{f}_i, \widetilde{g}_i) \neq 0$

3. $L(\widetilde{f}_i, \widetilde{g}_i) \neq 0$.

where $L(\widetilde{f}_i, \widetilde{g}_i)$ denotes the Lefschetz number of the pair $(\widetilde{f}_i, \widetilde{g}_i)$. Thus, $N(\widetilde{f}_i, \widetilde{g}_i) = 1$ for all $i = 1, \ldots, t$. Moreover, $p_{\widetilde{\phi}}^{-1}\, p\, \Phi(\widetilde{f}_i, \widetilde{g}_i) = \Phi(\widetilde{f}_i, \widetilde{g}_i)$ which implies $S(\widetilde{f}_i, \widetilde{g}_i) = 1$, for each $i = 1, \ldots, r$. Since the number S is the same (for all Nielsen classes lying in the same H-Nielsen class), so is the number J. Thus, by Theorem 3.3.16 we have

$$N_L(f, g) = \sum_{i=1}^{r} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} = \sum_{i=1}^{t} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{1} = \sum_{i=1}^{t} 1 = t\,.$$

(2) We have

$$\begin{aligned} N_{ED}(f, g) &= \sum_{1 \leq i \leq r}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i) \\ &= \sum_{1 \leq i \leq t}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i) + \sum_{t+1 \leq i \leq r}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i)\,. \end{aligned}$$

By Lemma 3.4.7. if J_i is even

$$ED(\widetilde{f}_i, \widetilde{g}_i) = \begin{cases} 0 & 1 \leq i \leq t \\ 1 & t+1 \leq i \leq r \end{cases}$$

Therefore. $N_{ED}(f, g)$ equals to the number of Reidemeister classes each of which is of a representative $(\widetilde{f}, \widetilde{g})$ such that $L(\widetilde{f}, \widetilde{g}) = 0$ and $p\,\Phi(\widetilde{f}, \widetilde{g})$ is essential.

□

Corollary 3.4.9. *Assume that M and N are orientable closed connected manifolds, and the coverings are orientable closed connected manifolds such that $(\widetilde{N}, p)$ is universal, then*

$$N(f, g) = N_L(f, g) = t\ ,$$

where t is given as in Theorem 3.4.8.

Proof. Apply Lemma 3.4.5 and Theorem 3.4.8. □

We turn now to the case that the covering space is of Jiang type.

Lemma 3.4.10. *Suppose that $\widetilde{M}$ and $\widetilde{N}$ are orientable coverings. Assume $\widetilde{N}$ is a Jiang space or $(\widetilde{f}_i, \widetilde{g}_i)$ is pseudo Jiang for all $i = 1, \ldots, r$, where r is the number of non-empty H-Reidemeister classes. Then, $L(\widetilde{f}, \widetilde{g}) = 0$ if and only if all the essential Nielsen classes in $p\,\Phi(\widetilde{f}, \widetilde{g})$ are defective with even J.*

Proof. Assume $L(\widetilde{f},\widetilde{g}) = 0$. Thus, the class of $\widetilde{f}$ and $\widetilde{g}$ are inessential. Let $A \subseteq p\,\Phi(\widetilde{f},\widetilde{g})$ be an essential Nielsen classes. Hence, A is defective, and $|\,A|$ is odd by Proposition 3.2.15. If J_A is odd, by Proposition 2.1.21, $|\,\widetilde{A}|$ is odd for every Nielsen class $\widetilde{A}$ of $\widetilde{f}$ and $\widetilde{g}$ such that $p(\widetilde{A}) = A$ and hence essential. This is a contradiction, so therefore, J_A must be even.

For the converse, assume that every essential Nielsen class A in $p\,\Phi(\widetilde{f},\widetilde{g})$ is defective with even J. Consider the following two cases:
Case 1: A is essential. Since J_A is even and A is defective, we get from [2, Proposition 3.2.15] that $\widetilde{A}$ is inessential for every Nielsen class $\widetilde{A}$ of $\widetilde{f}$ and $\widetilde{g}$ such that $p(\widetilde{A}) = A$.
Case 2: A is inessential. It follows by Corollary 3.2.19 that $\widetilde{A}$ is inessential.
Consequently, there are no essential Nielsen classes for $\widetilde{f}$ and $\widetilde{g}$ which again implies that $L(\widetilde{f},\widetilde{g}) = 0$. □

The general lines of the proof of the next theorem are quite similar to those of Theorem 3.4.8.

Theorem 3.4.11. *Suppose $\widetilde{M}$ and $\widetilde{N}$ are orientable manifolds, $\widetilde{N}$ is a Jiang space or $(\widetilde{f}_i,\widetilde{g}_i)$ is pseudo Jiang for all $i = 1,\ldots,r$, where r is the number of non-empty H-Reidemeister classes, and all Nielsen classes that lie in the same H-Nielsen class of f and g have the same number J. Without lose of generality, assume $L(\widetilde{f}_i,\widetilde{g}_i) \neq 0$ for each $i = 1,\ldots,t$ and $1 \leq t \leq r$, and $L(\widetilde{f}_i,\widetilde{g}_i) = 0$ otherwise. Then,*

$$N_L(f,g) = \sum_{i=1}^{t} \frac{N(\widetilde{f}_i,\widetilde{g}_i)}{S(\widetilde{f}_i,\widetilde{g}_i)} = \sum_{i=1}^{t} \frac{|\widetilde{\Phi}(\widetilde{f}_i,\widetilde{g}_i)|}{S(\widetilde{f}_i,\widetilde{g}_i)}$$

and

$$N_{ED}(f,g) = \sum_{i=t+1}^{r} ED(\widetilde{f}_i, \widetilde{g}_i) = \sum_{i=t+1}^{r} \frac{|\widetilde{\Phi}(\widetilde{f}_i, \widetilde{g}_i)|}{S(\widetilde{f}_i, \widetilde{g}_i)} .$$

Proof. For simplicity, we set $J_i = J(\widetilde{f}_i, \widetilde{g}_i)$. By our assumptions, we have $L(\widetilde{f}_i, \widetilde{g}_i) \neq 0$ if and only if $N(\widetilde{f}_i, \widetilde{g}_i) \neq 0$ for all $i = 1, \ldots, r$. That is, either all the non-empty classes are simultaneously essential or simultaneously inessential. Thus,

$$N_L(f,g) = \sum_{i=1}^{r} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} = \sum_{i=1}^{t} \frac{N(\widetilde{f}_i, \widetilde{g}_i)}{S(\widetilde{f}_i, \widetilde{g}_i)} = \sum_{i=1}^{t} \frac{|\widetilde{\Phi}(\widetilde{f}_i, \widetilde{g}_i)|}{S(\widetilde{f}_i, \widetilde{g}_i)} .$$

On the other hand,

$$\begin{aligned} N_{ED}(f,g) &= \sum_{1 \leq i \leq r}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i) \\ &= \sum_{1 \leq i \leq t}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i) + \sum_{t+1 \leq i \leq r}^{Ji \text{ is even}} ED(\widetilde{f}_i, \widetilde{g}_i) . \end{aligned}$$

Let $1 \leq i \leq t$ and suppose J_i is even. Since $L(\widetilde{f}_i, \widetilde{g}_i) \neq 0$, by Lemma 3.4.10 $p\,\Phi(\widetilde{f}_i, \widetilde{g}_i)$ contains an essential non-defective class A. So, for every Nielsen class $\widetilde{A} \subseteq \Phi(\widetilde{f}_i, \widetilde{g}_i)$ such that $p(\widetilde{A}) = A$, we have by Theorem 3.2.12 that

$$|ind|(\widetilde{f}_i, \widetilde{g}_i;\, \widetilde{A}) = J_i \cdot |ind|(f, g;\, A) \geq J_i \cdot 1 = J_i \geq 2 .$$

However, by our assumptions, all the classes of $(\widetilde{f}_i, \widetilde{g}_i)$ have the same semi-index (in fact they have the same index, but on orientable manifolds index and semi-index agree). Thus, all of them are non-defective classes since they have a semi-index greater or equal to 2. Hence, every essential Nielsen class $B \subseteq p\,\Phi(\widetilde{f}_i, \widetilde{g}_i)$ is non-defective, because if there exists an essential defective Nielsen class B and since J_i is even, then $|ind|(\widetilde{f}_i, \widetilde{g}_i;\, \widetilde{B}) = 0$ for every Nielsen class $\widetilde{B} \subseteq \Phi(\widetilde{f}_i, \widetilde{g}_i)$ such that $p\,(\widetilde{B}) = B$ which is a contradiction. Consequently, we have got if $L(\widetilde{f}_i,\widetilde{g}_i)\neq 0$ and J_i is even, then all the essential Nielsen classes in $p\,\Phi(\widetilde{f}_i,\widetilde{g}_i)$ are non-defective. Thus, $ED(\widetilde{f}_i,\widetilde{g}_i)=0$ for all $1\le i\le t$ and J_i is even. Therefore, by Lemma 3.4.10, we have

$$N_{ED}(f,g)=\sum_{t+1\le i\le r}^{Ji \text{ is even}} ED(\widetilde{f}_i,\widetilde{g}_i)=\sum_{i=t+1}^{r}\frac{|\widetilde{\Phi}(\widetilde{f}_i,\widetilde{g}_i)|}{S(\widetilde{f}_i,\widetilde{g}_i)}\,.$$

□

Corollary 3.4.12. *Suppose $\widetilde{M}$ and $\widetilde{N}$ are orientable manifolds, $\widetilde{N}$ is a Jiang space or $(\widetilde{f}_i,\widetilde{g}_i)$ is pseudo Jiang for all $i=1,\dots,r$, where r is the number of non-empty H-Reidemeister classes, and all Nielsen classes that lie in the same H-Nielsen class of f and g have the same number J. Then,*

$$N(f,g)=\sum_{i=1}^{r}\frac{|\widetilde{\Phi}(\widetilde{f}_i,\widetilde{g}_i)|}{S(\widetilde{f}_i,\widetilde{g}_i)}\,.$$

Proof. By Theorem 3.4.11,

$$N(f,g)=N_L(f,g)+N_{ED}(f,g)=\sum_{i=1}^{r}\frac{|\widetilde{\Phi}(\widetilde{f}_i,\widetilde{g}_i)|}{S(\widetilde{f}_i,\widetilde{g}_i)}\,.$$

□

Corollary 3.4.13. *Suppose M, N, $\widetilde{M}$ and $\widetilde{N}$ are orientable smooth closed manifolds, $\widetilde{N}$ is a Jiang space or $(\widetilde{f}_i,\widetilde{g}_i)$ is pseudo Jiang for all $i=1,\dots,r$, where r is the number of non-empty H-Reidemeister classes, and all Nielsen classes that lie in the same H-Nielsen class of f and g have the same number J. Without lose of generality, assume $L(\widetilde{f}_i,\widetilde{g}_i)\neq 0$ for each $i=1,\dots,t$ and $1\le t\le r$, and $L(\widetilde{f}_i,\widetilde{g}_i)=0$ otherwise. Then,*

$$N(f,g)=\sum_{i=1}^{t}\frac{N(\widetilde{f}_i,\widetilde{g}_i)}{S(\widetilde{f}_i,\widetilde{g}_i)}\,.$$

Proof. Apply Lemma 3.4.5 and Theorem 3.4.11. □

The following example illustrates some of the results above. Since we are considering the fixed point case, we could, of course, use Theorem 3.4.3, however we preferred to use Theorem 3.4.8 to illustrate the general method of computing the coincidence Nielsen number. For the definition of Lens Spaces, see Example 2.43 of [12].

Example 3.4.14. Consider the commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\widetilde{f}} & S^3 \\ \downarrow p & & p\downarrow \\ L(5,1) & \xrightarrow{f} & L(5,1) \end{array} \tag{3.4.2}$$

where p represents the quotient map. Notice that the covering (S^3, p) is universal. We have $\mathcal{A}(S^3) \cong \mathbf{Z}_5$. Let ω be the primitive fifth root of unity. Let $f : L(5,1) \longrightarrow L(5,1)$ defined by $f\left[r\,e^{i\theta},\, \rho\,e^{i\varphi}\right] = \left[r\,e^{i6\theta},\, \rho\,e^{i\varphi}\right]$. Then, f is a well-defined map which admits the lift $\widetilde{f}$ on S^3 defined by $\widetilde{f}\left(r\,e^{i\theta},\, \rho\,e^{i\varphi}\right) = \left(r\,e^{i6\theta},\, \rho\,e^{i\varphi}\right)$. In fact, since $\widetilde{f}$ is an equivariant $\mathbf{Z}_5$-map (as we will see soon), f is well-defined.

- Since $\widetilde{f}(-1,0) = (1,0) \neq \omega^t\,(-1,0)$ for any $t = 0,\, 1,\, 2,\, 3,\, 4$, we have $f \neq 1_{L(5,1)}$ where $1_{L(5,1)}$ is the identity map on $L(5,1)$.

- $L(\widetilde{f}) = 1 - 6 = -5 \neq 0$.

- Let $q_t : S^3 \longrightarrow S^3$ be the map defined by $q_t(z_1,\, z_2) = (\omega^t\, z_1,\, \omega^t\, z_2)$. Then, $q_t^5 = 1_{S^3}$. Thus, $deg(q_t)^5 = deg(q_t^5) = deg(1_{S^3}) = 1$. Hence, $deg(q_t) = 1$. This implies that $L(\widetilde{f}, q_t) = -5 \neq 0$, for any $t = 0,\, 1,\, 2,\, 3,\, 4$.

- The number J depends only on the $H-$Nielsen class, since each $H-$Nielsen class consists of one Nielsen class (or the fundamental group of $L(5,1)$ is abelian).

- Since $L(\widetilde{f}, q_t) \neq 0$ for each t, Theorem 3.4.8 implies that $N_{ED}(f, 1_{L(5,1)}) = 0$ and

$$N(f) = \sum_{t=0}^{4} N(\widetilde{f}, q_t) = 5 ,$$

which is the same result that we obtain if we apply [Theorem 2.5, [5]], the usual Jiang space methods for the fixed point case, or Theorem 1.2.13 by J. Jezierski, [15].

- It can be shown that

$$\Phi(\widetilde{f}, q_t) = \left\{ \left(\omega^k \cdot e^{\frac{i2\pi t}{25}}, 0 \right) \mid k = 0, 1, 2, 3, 4 \right\}$$

and

$$\Phi(\widetilde{f}) = \left\{ (r\,\omega^k, z) \mid r \in \mathbf{R}^+, z \in \mathbf{C}, k = 0, 1, 2, 3, 4; \text{ and } r^2 + |z|^2 = 1 \right\} .$$

Notice that $|\Phi(\widetilde{f}, q_t)| = 5$, while $|\Phi(\widetilde{f})| = \infty$.

Chapter 4

Classification of $H-$Reidemeister classes, applications, and examples

In this chapter, we give a method that in some cases will classify the representatives of the Reidemeister classes which appear in Equation 2.3.2 or Equation 3.3.4. We also give more examples which illustrate the results of previous chapters. Unless otherwise stated, the work in this chapter does not require orientability for the considered spaces.

4.1 Classification of Reidemeister classes

In this section, we discuss to what extent we can characterize the representative lifting pairs in Equation 2.3.2 or Equation 3.3.4 in Theorem 3.3.16.

Let M and N be path connected, locally path connected topological manifolds, $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of finite index o M and N respectively. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f}, \widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$.

For the rest of this chapter, we write $\widetilde{f}\alpha$ for $\widetilde{f} \circ \alpha$, and $\beta\widetilde{f}$ for $\beta \circ \widetilde{f}$, where

$\alpha \in \mathcal{A}(\widetilde{M})$ and $\beta \in \mathcal{A}(\widetilde{N})$.

Definition 4.1.1. *Let* $\alpha \in \mathcal{A}(\widetilde{M})$ *and* $\beta \in \mathcal{A}(\widetilde{N})$ *such that* $\widetilde{f}\alpha = \beta\widetilde{f}$. *We write* $[\widetilde{f} : \alpha] = \beta$.

Proposition 4.1.2. *The notation* $[\widetilde{f} :\ ]$ *defines a homomorphism from* $\mathcal{A}(\widetilde{M})$ *to* $\mathcal{A}(\widetilde{N})$. *Moreover,* $[\beta\widetilde{f} : \alpha] = \beta \cdot [\widetilde{f} : \alpha] \cdot \beta^{-1}$ *for every* $\alpha \in \mathcal{A}(\widetilde{M})$ *and* $\beta \in \mathcal{A}(\widetilde{N})$.

Proof. We have that $[\widetilde{f} : \alpha]$ is uniquely determined by $\widetilde{f}$ and α. That is, $[\widetilde{f} :\ ]$ is well-defined. Clearly, $[\widetilde{f} : 1_{\mathcal{A}(\widetilde{M})}] = 1_{\mathcal{A}(\widetilde{N})}$. Let $\alpha_1, \alpha_2 \in \mathcal{A}(\widetilde{M})$. Then,

$$\begin{aligned}
\widetilde{f} \cdot (\alpha_1 \cdot \alpha_2) &= (\widetilde{f} \cdot \alpha_1) \cdot \alpha_2 \\
&= ([\widetilde{f} : \alpha_1] \cdot \widetilde{f}) \cdot \alpha_2 \\
&= [\widetilde{f} : \alpha_1] \cdot (\widetilde{f} \cdot \alpha_2) \\
&= [\widetilde{f} : \alpha_1] \cdot ([\widetilde{f} : \alpha_2] \cdot \widetilde{f}) \\
&= ([\widetilde{f} : \alpha_1] \cdot [\widetilde{f} : \alpha_2]) \cdot \widetilde{f} .
\end{aligned}$$

Thus,

$$[\widetilde{f} : \alpha_1 \cdot \alpha_2] = [\widetilde{f} : \alpha_1] \cdot [\widetilde{f} : \alpha_2] .$$

Now, let $\alpha \in \mathcal{A}(\widetilde{M})$ and $\beta \in \mathcal{A}(\widetilde{N})$. Then

$$\begin{aligned}
(\beta \cdot \widetilde{f}) \cdot \alpha &= \beta \cdot (\widetilde{f} \cdot \alpha) \\
&= \beta \cdot ([\widetilde{f} : \alpha] \cdot \widetilde{f}) \\
&= \beta \cdot [\widetilde{f} : \alpha] \cdot \beta^{-1} \cdot \beta \cdot \widetilde{f} \\
&= \beta \cdot [\widetilde{f} : \alpha] \cdot \beta^{-1} \cdot (\beta \cdot \widetilde{f}) .
\end{aligned}$$

Thus,

$$[\beta \cdot \widetilde{f} : \alpha] = \beta \cdot [\widetilde{f} : \alpha] \cdot \beta^{-1} .$$

□

Definition 4.1.3. *We define* <u>*the group*</u> $G(\widetilde{f})$ *to be the image of the homomorphism* $[\widetilde{f} : \ ]$ *in* $\mathcal{A}(\widetilde{N})$. *That is,*

$$G(\widetilde{f}) = \left\{ [\widetilde{f} : \alpha] \mid \alpha \in \mathcal{A}(\widetilde{M}) \right\} .$$

From now on, we fix a lift $(\widetilde{f}, \widetilde{g})$ of (f, g). The following proposition is the key to our classification.

Proposition 4.1.4. *The lift* $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ *is conjugate to* $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$ *if and only if there exists* $\alpha \in \mathcal{A}(\widetilde{M})$ *such that*

$$\beta_1 = [\widetilde{f} : \alpha] \cdot \beta_2 \cdot [\widetilde{g} : \alpha^{-1}] . \tag{4.1.1}$$

Proof. Suppose that $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$. Thus, there exist $\alpha \in \mathcal{A}(\widetilde{M})$ and $\gamma \in \mathcal{A}(\widetilde{N})$ such that

$$\begin{cases} \gamma \cdot \widetilde{f} \cdot \alpha = \widetilde{f}, \\ \gamma \cdot \beta_1 \cdot \widetilde{g} \cdot \alpha = \beta_2 \cdot \widetilde{g} . \end{cases}$$

Hence,

$$\begin{cases} \gamma \cdot [\widetilde{f} : \alpha] \cdot \widetilde{f} = \widetilde{f}, \\ \gamma \cdot \beta_1 \cdot [\widetilde{g} : \alpha] \cdot \widetilde{g} = \beta_2 \cdot \widetilde{g} . \end{cases} \tag{4.1.2}$$

The first line in Equation 4.1.2 implies that

$$\gamma = [\widetilde{f} : \alpha^{-1}] . \tag{4.1.3}$$

Thus, the second line in Equation 4.1.2 along with Equation 4.1.3 implies that

$$\beta_1 = \gamma^{-1} \cdot \beta_2 \cdot [\widetilde{g} : \alpha]^{-1} = [\widetilde{f} : \alpha] \cdot \beta_2 \cdot [\widetilde{g} : \alpha^{-1}] .$$

For the converse, we need to show that Equation 4.1.1 implies that $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$. Actually,

$[\widetilde{f} : \alpha] \cdot \left(\widetilde{f}, \beta_2 \cdot \widetilde{g}\right) \cdot \alpha^{-1} = \left([\widetilde{f} : \alpha] \cdot \widetilde{f} \cdot \alpha^{-1}, [\widetilde{f} : \alpha] \cdot \beta_2 \cdot \widetilde{g} \cdot \alpha^{-1}\right)$

$= \left([\widetilde{f} : \alpha] \cdot [\widetilde{f} : \alpha^{-1}] \cdot \widetilde{f}, [\widetilde{f} : \alpha] \cdot \beta_2 \cdot [\widetilde{g} : \alpha^{-1}] \cdot \widetilde{g}\right)$

$= (\widetilde{f}, \beta_1 \cdot \widetilde{g})$.

Therefore, $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$. □

Remark 4.1.5. *Proposition 4.1.4 states that the set of the covering transformations $[\widetilde{f} : \alpha] \cdot \beta_2 \cdot [\widetilde{g} : \alpha^{-1}]$, for $\alpha \in \mathcal{A}(\widetilde{M})$, is closely related to the set of the lifts $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ that lie in the Reidemeister class represented by (i.e., conjugate to) $(\widetilde{f}, \beta_2 \widetilde{g})$ and vise versa. As we will see, it is not necessarily that the two sets be in one to one correspondence with each other. So, our next job is to farther investigate the relationship between them.*

Definition 4.1.6. *Let $\beta \in \mathcal{A}(\widetilde{N})$. For a fixed pair of lifts $(\widetilde{f}, \widetilde{g})$ of f and g, the set $\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g}) \subseteq \mathcal{A}(\widetilde{N})$ is defined by*

$$\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g}) = \left\{ [\widetilde{f} : \alpha] \cdot \beta \cdot [\widetilde{g} : \alpha^{-1}] \mid \alpha \in \mathcal{A}(\widetilde{M}) \right\} .$$

Lemma 4.1.7. *Let $\beta_1, \beta_2 \in \mathcal{A}(\widetilde{N})$. Then, $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$ if and only if $\widehat{G}(\widetilde{f}, \beta_1 \cdot \widetilde{g}) \cap \widehat{G}(\widetilde{f}, \beta_2 \cdot \widetilde{g}) \neq \emptyset$.*

Proof. Suppose $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$. By Proposition 4.1.4, there exists $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\beta_1 = [\widetilde{f} : \alpha] \cdot \beta_2 \cdot [\widetilde{g} : \alpha^{-1}]$. Thus, $\beta_1 \in \widehat{G}(\widetilde{f}, \beta_2 \cdot \widetilde{g})$. Since $\beta_2 \in \widehat{G}(\widetilde{f}, \beta_2 \cdot \widetilde{g})$, we get that $\widehat{G}(\widetilde{f}, \beta_1 \cdot \widetilde{g}) \cap \widehat{G}(\widetilde{f}, \beta_2 \cdot \widetilde{g}) \neq \emptyset$.

Conversely, assume that $\widehat{G}(\widetilde{f}, \beta_1 \cdot \widetilde{g}) \cap \widehat{G}(\widetilde{f}, \beta_2 \cdot \widetilde{g}) \neq \emptyset$. This means that there exist $\alpha_1, \alpha_2 \in \mathcal{A}(\widetilde{M})$ such that

$$[\widetilde{f} : \alpha_1] \cdot \beta_1 \cdot [\widetilde{g} : \alpha_1^{-1}] = [\widetilde{f} : \alpha_2] \cdot \beta_2 \cdot [\widetilde{g} : \alpha_2^{-1}] .$$

Hence,

$$\begin{aligned} \beta_1 &= [\widetilde{f} : \alpha_1^{-1}] \cdot [\widetilde{f} : \alpha_2] \cdot \beta_2 \cdot [\widetilde{g} : \alpha_2^{-1}] \cdot [\widetilde{g} : \alpha_1]. \\ &= [\widetilde{f} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta_2 \cdot [\widetilde{g} : \alpha_2^{-1} \cdot \alpha_1]. \\ &= [\widetilde{f} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta_2 \cdot [\widetilde{g} : \left(\alpha_1^{-1} \cdot \alpha_2\right)^{-1}] . \end{aligned}$$

By Proposition 4.1.4, $(\widetilde{f}, \beta_1 \cdot \widetilde{g})$ is conjugate to $(\widetilde{f}, \beta_2 \cdot \widetilde{g})$. $\square$

Definition 4.1.8. *Let $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g) and $\beta \in \mathcal{A}(\widetilde{N})$. We define the subset $\Delta(\beta)$ of $Lift(f, g)$ by*

$$\Delta(\beta) = \left\{ \mu\,(\widetilde{f}, \beta\,\widetilde{g}) = (\mu\,\widetilde{f}, \mu\,\beta\,\widetilde{g}) \in Lift(f, g) \,|\, \mu \in \mathcal{A}(\widetilde{N}) \right\} .$$

Lemma 4.1.9. *Fix a lift $(\widetilde{f}, \widetilde{g})$ of (f, g). Then*

1. *$\Delta(\beta) = \Delta(\acute{\beta})$ if and only if $\beta = \acute{\beta}$. Moreover, $\Delta(\beta) \cap \Delta(\acute{\beta}) = \emptyset$ if and only if $\beta \neq \acute{\beta}$.*

2. $Lift(f,g) = \bigcup_{\beta\in\mathcal{A}(\widetilde{N})} \Delta(\beta)$. *Thus, the family* $\Delta = \left\{\Delta(\beta) |\ \beta \in \mathcal{A}(\widetilde{N})\right\}$ *is a partition of* $Lift(f,g)$.

3. *The set* $\Delta(\beta)$ *is a subset of the conjugacy class which includes* $(\widetilde{f}, \beta\,\widetilde{g})$. *Furthermore, each conjugacy class is a union of some of these* $\Delta(\beta)$*'s.*

4. $|R_H(f,g)| \leq \left|\mathcal{A}(\widetilde{N})\right| = [\pi_1(N) : H]$.

Proof. (1) Let β, $\acute{\beta} \in \mathcal{A}(\widetilde{N})$. Then,

$$\begin{aligned}
\Delta(\beta) = \Delta(\acute{\beta}) \ &\Rightarrow\ (\widetilde{f}, \beta\,\widetilde{g}) \in \Delta(\acute{\beta}) \\
&\Rightarrow\ (\widetilde{f}, \beta\,\widetilde{g}) = (\mu\,\widetilde{f}, \mu\acute{\beta}\,\widetilde{g}) \ \text{ for some } \mu \in \mathcal{A}(\widetilde{N}). \\
&\Rightarrow\ \widetilde{f} = \mu\,\widetilde{f} \ \text{ and } \ \beta\,\widetilde{g} = \mu\,\acute{\beta}\,\widetilde{g} \\
&\Rightarrow\ 1_{\widetilde{N}} = \mu, \ \text{ and hence } \ \beta\,\widetilde{g} = \acute{\beta}\,\widetilde{g} \\
&\Rightarrow\ \beta = \acute{\beta}\,.
\end{aligned}$$

The converse is trivial.

On the other hand, assume that $\Delta(\beta) \cap \Delta(\acute{\beta}) \neq \emptyset$. Then, there exists $\mu \in \mathcal{A}(\widetilde{N})$ such that $(\mu\,\widetilde{f}, \mu\,\beta\,\widetilde{g}) \in \Delta(\acute{\beta})$. Hence, there exists $\acute{\mu} \in \mathcal{A}(\widetilde{N})$ such that $(\mu\,\widetilde{f}, \mu\,\beta\,\widetilde{g}) = (\acute{\mu}\,\widetilde{f}, \acute{\mu}\,\acute{\beta}\,\widetilde{g})$. Thus, $\mu\,\widetilde{f} = \acute{\mu}\,\widetilde{f}$, and $\mu\,\beta\,\widetilde{g} = \acute{\mu}\,\acute{\beta}\,\widetilde{g}$. Hence, $\mu = \acute{\mu}$ and $\mu\,\beta = \acute{\mu}\,\acute{\beta}$. So, $\beta = \acute{\beta}$.
It is obvious that if $\beta = \acute{\beta}$, then $\Delta(\beta) \cap \Delta(\acute{\beta}) \neq \emptyset$.

(2) Let $(\widetilde{f_1}, \widetilde{g_1})$ be a lift of (f,g). Then, there exist β_1, $\beta_2 \in \mathcal{A}(\widetilde{N})$ such that $\widetilde{f_1} = \beta_1\,\widetilde{f}$, and $\widetilde{g_1} = \beta_2\,\widetilde{g}$. Thus,

$$(\widetilde{f_1}, \widetilde{g_1}) = (\beta_1\,\widetilde{f}, \beta_2\,\widetilde{g}) = \beta_1\,(\widetilde{f}, \beta_1^{-1}\,\beta_2\,\widetilde{g})\,.$$

So, $(\widetilde{f}_1, \widetilde{g}_1) \in \Delta(\beta_1^{-1}\beta_2) \subseteq \bigcup_{\beta \in \mathcal{A}(\widetilde{N})} \Delta(\beta)$. Since $\bigcup_{\beta \in \mathcal{A}(\widetilde{N})} \Delta(\beta) \subseteq Lift(f, g)$, we get that $Lift(f, g) = \bigcup_{\beta \in \mathcal{A}(\widetilde{N})} \Delta(\beta)$. Moreover, by (1), the family $\left\{\Delta(\beta) | \ \beta \in \mathcal{A}(\widetilde{N})\right\}$ is a partition of $Lift(f, g)$.

(3) The proof follows from Definitions 1.1.11 and 4.1.8.

(4) By (3), $|R_H(f, g)| \leq |\Delta| = \left|\mathcal{A}(\widetilde{N})\right| = [\pi_1(N) : H]$. □

Remark 4.1.10. *Fix a lift $(\widetilde{f}, \widetilde{g})$ of (f, g). Then any other lift $(\widetilde{f}_1, \widetilde{g}_1)$ is conjugate to $(\widetilde{f}, \beta\widetilde{g})$ for some $\beta \in \mathcal{A}(\widetilde{N})$. If we define the action of $\mathcal{A}(\widetilde{M})$ on the set $\left\{\Delta(\beta) | \ \beta \in \mathcal{A}(\widetilde{N})\right\}$ from the right by*

$$\Delta(\beta) \cdot \alpha = \left\{\mu\,(\widetilde{f}, \beta\widetilde{g})\,\alpha = \mu\,(\widetilde{f}\alpha, \beta\widetilde{g}\alpha) = (\mu\,\widetilde{f}\alpha, \mu\,\beta\widetilde{g}\alpha) | \ \mu \in \mathcal{A}(\widetilde{N})\right\} ,$$

then the union of the elements of each orbit, under this action, is a conjugacy class.

Definition 4.1.11. *A set $\Omega' \subseteq Lift(f, g)$ is said to be a set of Reidemeister representatives, if each conjugacy class is represented exactly once in Ω'.*

Proposition 4.1.12. *Let $\Omega = \left\{(\widetilde{f}, \beta \cdot \widetilde{g}) \mid \beta \in \mathcal{A}(\widetilde{N})\right\}$, and let Ω' be a subset of Ω. Then, Ω' is the set of Reidemeister representatives, which appear in Equation 2.3.2 or in Equation 3.3.4, if and only if Ω' satisfies the following conditions:*

1. *Any two distinct pairs in Ω' are not conjugate.*

2. *If we add any* $(\widetilde{f}, \beta' \cdot \widetilde{g}) \notin \Omega'$ *from* Ω *to* Ω'*, then* $\Omega' \cup \{(\widetilde{f}, \beta' \cdot \widetilde{g})\}$ *is not pairwise non conjugate; that is,* $(\widetilde{f}, \beta' \cdot \widetilde{g})$ *must be conjugate to some pair in* Ω'.

Proof. Apply Lemma 4.1.9. □

So then, Proposition 4.1.12 implies that we can make a suitable choice of $\beta \in \mathcal{A}(\widetilde{N})$, and use this choice to determine a set of Reiremeister representatives.

Definition 4.1.13. *From now on, we will assume that we have chosen an appropriate set of Reidemeister representatives. We use the the notation* $\Lambda \subseteq \mathcal{A}(\widetilde{N})$ *to denote the corresponding choice of* β*'s.*

The following theorem allows us to move one step closer to enumerate the $H-$ Reidemeister representatives.

Theorem 4.1.14. *We have*

$$\mathcal{A}(\widetilde{N}) = \bigcup_{\beta \in \Lambda} \widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g}) \,. \tag{4.1.4}$$

where the union is a disjoint union.

Proof. It is enough to show that $\mathcal{A}(\widetilde{N}) \subseteq \bigcup_{\beta \in \Lambda} \widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})$. Let $\acute{\beta} \in \mathcal{A}(\widetilde{N})$. By the definition of Λ, $(\widetilde{f}, \acute{\beta} \cdot \widetilde{g})$ belongs to the Reidemeister class represented by $(\widetilde{f}, \beta \cdot \widetilde{g})$ for some $\beta \in \Lambda$. This implies that $\acute{\beta} = [\widetilde{f} : \alpha] \cdot \beta \cdot [\widetilde{g} : \alpha^{-1}]$ for some $\alpha \in \mathcal{A}(\widetilde{M})$. Thus, $\acute{\beta} \in \widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})$. Therefore, $\acute{\beta} \in \bigcup_{\beta \in \Lambda} \widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})$. The union is disjoint by Lemma 4.1.7 and the definition of Λ. □

Corollary 4.1.15. *The following equation holds*

$$|\mathcal{A}(\widetilde{N})| = \sum_{\beta \in \Lambda} |\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})| \,.$$

□

We define next another set $G(\widetilde{f}, \beta \cdot \widetilde{g})$ which is related to $\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})$. The new set is also related to $\dot{\mathrm{L}}(\widetilde{f}, \beta \cdot \widetilde{g})$ as we shall see. Moreover, under certain conditions it is a subgroup of $\mathcal{A}(\widetilde{N})$. These facts will be useful in computing $|\Lambda|$ in some special cases.

Definition 4.1.16. *Let $\beta \in \mathcal{A}(\widetilde{N})$. The set $G(\widetilde{f}, \beta \cdot \widetilde{g})$ is defined by*

$$G(\widetilde{f}, \beta \cdot \widetilde{g}) = \left\{ [\widetilde{f} : \alpha] \cdot [\beta \cdot \widetilde{g} : \alpha^{-1}] \mid \alpha \in \mathcal{A}(\widetilde{M}) \right\} = \widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g}) \cdot \beta^{-1} \,.$$

Lemma 4.1.17. *Let $\beta \in \mathcal{A}(\widetilde{N})$. Then,*

$$|G(\widetilde{f}, \beta \cdot \widetilde{g})| = |\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g})| \,.$$

Proof. It is easy to see that the function $\widehat{G}(\widetilde{f}, \beta \cdot \widetilde{g}) \to G(\widetilde{f}, \beta \cdot \widetilde{g})$ defined by $[\widetilde{f} : \alpha] \cdot \beta \cdot [\widetilde{g} : \alpha^{-1}] \mapsto [\widetilde{f} : \alpha] \cdot \beta \cdot [\widetilde{g} : \alpha^{-1}] \cdot \beta^{-1}$ is bijective. □

Corollary 4.1.18. *We have the following equation*

$$|\mathcal{A}(\widetilde{N})| = \sum_{\beta \in \Lambda} |G(\widetilde{f}, \beta \cdot \widetilde{g})| \,. \tag{4.1.5}$$

Proof. Apply Corollary 4.1.15 and Lemma 4.1.17. □

Remark 4.1.19. *The set $G(\widetilde{f}, \beta \cdot \widetilde{g})$ is not necessarily a subgroup of $\mathcal{A}(\widetilde{N})$ since it is not always closed under the multiplication, nor does the inverse of an element necessarily belong to $G(\widetilde{f}, \beta \cdot \widetilde{g})$. However, the identity $1_{\widetilde{N}}$ always belongs to this set.*

The main result of this section is the following theorem. For the notation of $I(\widetilde{f}, \widetilde{g})$, where $(\widetilde{f}, \widetilde{g})$ is a lift of (f, g), see Remark 2.1.14.

Theorem 4.1.20. *If $L(\widetilde{f}, \beta \cdot \widetilde{g})$ is a normal subgroup of $\mathcal{A}(\widetilde{M})$, for each $\beta \in \mathcal{A}(\widetilde{N})$, then*

$$|\mathcal{A}(\widetilde{N})| = |\mathcal{A}(\widetilde{M})| \cdot \sum_{\beta \in \Lambda} \frac{1}{I(\widetilde{f}, \beta \cdot \widetilde{g})}. \tag{4.1.6}$$

Proof. Define $\varphi : \dfrac{\mathcal{A}(\widetilde{M})}{\mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})} \to G(\widetilde{f}, \beta \widetilde{g})$ by $\varphi(\alpha \cdot \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})) = [\widetilde{f} : \alpha] \cdot \beta \cdot [\widetilde{g} : \alpha^{-1}] \cdot \beta^{-1}$.

Let α_1, $\alpha_2 \in \mathcal{A}(\widetilde{M})$. Then

$$\begin{aligned}
\alpha_1 \cdot \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g}) = \alpha_2 \cdot \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g}) \quad &\Leftrightarrow \quad \alpha_1^{-1} \cdot \alpha_2 \in \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g}) \\
&\Leftrightarrow \quad [\widetilde{f} : \alpha_1^{-1} \cdot \alpha_2] = \beta \cdot [\widetilde{g} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta^{-1} \\
&\Leftrightarrow \quad [\widetilde{f} : \alpha_1^{-1}] \cdot [\widetilde{f} : \alpha_2] = \beta \cdot [\widetilde{g} : \alpha_1^{-1}] \cdot [\widetilde{g} : \alpha_2] \cdot \beta^{-1} \\
&\Leftrightarrow \quad [\widetilde{f} : \alpha_2] \cdot \beta \cdot [\widetilde{g} : \alpha_2^{-1}] = [\widetilde{f} : \alpha_1] \cdot \beta \cdot [\widetilde{g} : \alpha_1^{-1}] \\
&\Leftrightarrow \quad \varphi\left(\alpha_1 \cdot \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})\right) = \varphi\left(\alpha_2 \cdot \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})\right).
\end{aligned}$$

Thus, φ is a well-defined injection. Since it is obvious that φ is onto, we get that φ is bijective. Hence, we get that

$$|G(\widetilde{f}, \beta \widetilde{g})| = [\mathcal{A}(\widetilde{M}) : \mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})].$$

Since $\mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})$ is normal, we further have

$$|G(\widetilde{f}, \widetilde{g})| = \frac{|\mathcal{A}(\widetilde{M})|}{|\mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})|} = \frac{|\mathcal{A}(\widetilde{M})|}{I(\widetilde{f}, \beta \cdot \widetilde{g})}.$$

By Corollary 4.1.18, we get

$$|\mathcal{A}(\widetilde{N})| = |\mathcal{A}(\widetilde{M})| \cdot \sum_{\beta \in \Lambda} \frac{1}{I(\widetilde{f}, \beta \cdot \widetilde{g})}.$$

□

The next corollary gives conditions for a classification for the Reidemeister classes. It follows directly from Theorem 4.1.20.

Corollary 4.1.21. *Suppose $L(\widetilde{f}, \beta \cdot \widetilde{g})$ is a normal subgroup of $\mathcal{A}(\widetilde{M})$, for each $\beta \in \mathcal{A}(\widetilde{N})$, and I is the same for all $H-$Nielsen classes. Then,*

1. *We have*
$$|\Lambda| = \frac{|\mathcal{A}(\widetilde{N})| \cdot I}{|\mathcal{A}(\widetilde{M})|}. \tag{4.1.7}$$

2. *if $|\mathcal{A}(\widetilde{N})| = |\mathcal{A}(\widetilde{M})|$, then $|\Lambda| = I$.*

3. *if $|\mathcal{A}(\widetilde{N})|$ and $|\mathcal{A}(\widetilde{M})|$ are prime numbers and not equal, then $I = |\mathcal{A}(\widetilde{M})|$ and $|\Lambda| = |\mathcal{A}(\widetilde{N})|$.*

Next we give sufficient conditions under which $\mathrm{L}(\widetilde{f}, \beta \cdot \widetilde{g})$ is a normal subgroup of $\mathcal{A}(\widetilde{M})$, for each $\beta \in \mathcal{A}(\widetilde{N})$.

Proposition 4.1.22. *The following hold true*

1. *If $G(\widetilde{g})$ commutes with $G(\widetilde{f})$, then $G(\widetilde{f},\widetilde{g})$ is a subgroup of $\mathcal{A}(\widetilde{N})$.*

2. *If $G(\widetilde{g}) \subseteq Z(\mathcal{A}(\widetilde{N}))$, then $G(\widetilde{f},\beta\cdot\widetilde{g}) = G(\widetilde{f},\widetilde{g})$, for every $\beta \in \mathcal{A}(\widetilde{N})$. However, If $G(\widetilde{f}) \subseteq Z(\mathcal{A}(\widetilde{N}))$, then $G(\widetilde{f},\beta\cdot\widetilde{g})$ is a subgroup which is conjugate to $G(\widetilde{f},\widetilde{g})$ by β, for every $\beta \in \mathcal{A}(\widetilde{N})$.*

3. *If $G(\widetilde{g}) \subseteq Z(\mathcal{A}(\widetilde{N}))$ or $G(\widetilde{f}) \subseteq Z(\mathcal{A}(\widetilde{N}))$, then $L(\widetilde{f},\beta\cdot\widetilde{g}) = L(\widetilde{f},\widetilde{g})$, for every $\beta \in \mathcal{A}(\widetilde{N})$, and $L(\widetilde{f},\widetilde{g})$ is a normal subgroup of $\mathcal{A}(\widetilde{M})$.*

Proof. (1) Let α, α_1, $\alpha_2 \in \mathcal{A}(\widetilde{M})$. Then,

$$\Big([\widetilde{f}:\alpha_1]\cdot[\widetilde{g}:\alpha_1^{-1}]\Big)\cdot\Big([\widetilde{f}:\alpha_2]\cdot[\widetilde{g}:\alpha_2^{-1}]\Big) = [\widetilde{f}:\alpha_1]\cdot\Big([\widetilde{g}:\alpha_1^{-1}]\cdot[\widetilde{f}:\alpha_2]\Big)\cdot[\widetilde{g}:\alpha_2^{-1}]$$
$$= [\widetilde{f}:\alpha_1]\cdot\Big([\widetilde{f}:\alpha_2]\cdot[\widetilde{g}:\alpha_1^{-1}]\Big)\cdot[\widetilde{g}:\alpha_2^{-1}]$$
$$= \Big([\widetilde{f}:\alpha_1]\cdot[\widetilde{f}:\alpha_2]\Big)\cdot([\widetilde{g}:\alpha_1^{-1}]\cdot[\widetilde{g}:\alpha_2^{-1}])$$
$$= \Big([\widetilde{f}:\alpha_1]\cdot[\widetilde{f}:\alpha_2]\Big)\cdot([\widetilde{g}:\alpha_2^{-1}]\cdot[\widetilde{g}:\alpha_1^{-1}])$$
$$= [\widetilde{f}:\alpha_1\cdot\alpha_2]\cdot[\widetilde{g}:\alpha_2^{-1}\cdot\alpha_1^{-1}]$$
$$= [\widetilde{f}:\alpha_1\cdot\alpha_2]\cdot[\widetilde{g}:(\alpha_1\cdot\alpha_2)^{-1}] \in G(\widetilde{f},\widetilde{g}).$$

On the other hand, it is easy to see that $[\widetilde{f}:\alpha]\cdot[\widetilde{g}:\alpha^{-1}]$ has $[\widetilde{f}:\alpha^{-1}]\cdot[\widetilde{g}:\alpha] \in G(\widetilde{f},\widetilde{g})$ as an inverse.

(2) Apply Proposition 4.1.2 and Proposition 4.1.22.

(3) Assume $G(\widetilde{g}) \subseteq Z(\mathcal{A}(\widetilde{N}))$ or $G(\widetilde{f}) \subseteq Z(\mathcal{A}(\widetilde{N}))$. It is easy to see that $\mathrm{L}(\widetilde{f},\beta\cdot\widetilde{g}) = \mathrm{L}(\widetilde{f},\widetilde{g})$ for every $\beta \in \mathcal{A}(\widetilde{N})$. Now, Let $\alpha \in \mathrm{L}(\widetilde{f},\widetilde{g})$. Hence, $[\widetilde{f}:\alpha] = [\widetilde{g}:\alpha] \in$

$G(\widetilde{f}) \cap G(\widetilde{g})$. Let $\lambda \in \mathcal{A}(\widetilde{M})$. We need to show that $\lambda \cdot \alpha \cdot \lambda^{-1} \in \mathrm{L}(\widetilde{f}, \widetilde{g})$. In fact,

$$\begin{aligned}
[\widetilde{f} : \lambda \cdot \alpha \cdot \lambda^{-1}] &= [\widetilde{f} : \lambda] \cdot [\widetilde{f} : \alpha] \cdot [\widetilde{f} : \lambda^{-1}] \\
&= [\widetilde{f} : \lambda] \cdot [\widetilde{f} : \lambda^{-1}] \cdot [\widetilde{f} : \alpha] \\
&= [\widetilde{f} : \alpha] \\
&= [\widetilde{g} : \alpha] \\
&= [\widetilde{g} : \lambda] \cdot [\widetilde{g} : \lambda^{-1}] \cdot [\widetilde{g} : \alpha] \\
&= [\widetilde{g} : \lambda] \cdot [\widetilde{g} : \alpha] \cdot [\widetilde{g} : \lambda^{-1}] \\
&= [\widetilde{g} : \lambda \cdot \alpha \cdot \lambda^{-1}]
\end{aligned}$$

Therefore, $\lambda \cdot \alpha \cdot \lambda^{-1} \in \mathrm{L}(\widetilde{f}, \widetilde{g})$. □

4.2 The case where $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers

In this section, unless otherwise stated, we study the case where $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. This gives a simpler version of formula 3.4.6. This section generalizes [5]. Its flow is similar to that of [5], however we use the notion of $\delta(f, g)$ rather than the notion of even and odd lifts introduced in [5]. Also, we generalize [Theorem 2.5, [5]] by giving sufficient and necessary conditions for our desired formula to hold.

Let M and N be path connected, locally path connected topological spaces, $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. We assume the coverings are finite, and unless otherwise stated that $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are **prime numbers**. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f}, \widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$.

Recall from Definition 2.1.10, if $\alpha \in \mathcal{A}(\widetilde{M})$, then $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 1$ provided that $[\widetilde{f} : \alpha] = [\widetilde{g} : \alpha]$; otherwise, $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 0$.

In what follows, we list some geometric and algebraic characterizations for f and g to satisfy that $\delta(f, g) = 1$. The next lemma does not require that $\mathcal{A}(\widetilde{M})$ and $\mathcal{A}(\widetilde{N})$ have a prime order or even are cyclic.

Lemma 4.2.1. *Let $\alpha \in \mathcal{A}(\widetilde{M})$. If $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 1$, then $\delta(\widetilde{f}, \widetilde{g}; \sigma) = 1$ for all $\sigma \in \langle \alpha \rangle$, where $\langle \alpha \rangle$ is the cyclic subgroup of $\mathcal{A}(\widetilde{M})$ generated by α.*

Proof. By Proposition 4.1.2, we have $[\widetilde{f} : \alpha^k] = [\widetilde{f} : \alpha]^k$ for every $\alpha \in \mathcal{A}(\widetilde{M})$ and every integer k. Hence,

$$\begin{aligned} \delta(\widetilde{f}, \widetilde{g}; \alpha) = 1 \ &\Leftrightarrow\ [\widetilde{f} : \alpha] = [\widetilde{g} : \alpha] \\ &\Leftrightarrow\ [\widetilde{f} : \alpha]^k = [\widetilde{g} : \alpha]^k \\ &\Leftrightarrow\ [\widetilde{f} : \alpha^k] = [\widetilde{g} : \alpha^k] \\ &\Leftrightarrow\ \delta(\widetilde{f}, \widetilde{g}; \alpha^k) = 1\ . \end{aligned}$$

□

Proposition 4.2.2. *Let $\alpha \in \mathcal{A}(\widetilde{M})$. Then,*

1. *We have $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 1$ if and only if $\delta(\widetilde{f}, \widetilde{g}; \sigma) = 1$ for every $\sigma \in \mathcal{A}(\widetilde{M})$.*

2. *If $(\widetilde{f}_0, \widetilde{g}_0)$ is another lifting pair of (f, g), then $[\widetilde{f}_0 : \alpha] = [\widetilde{f} : \alpha]$ and $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 1$ if and only if $\delta(\widetilde{f}_0, \widetilde{g}_0; \alpha) = 1$.*

Proof. (1) Since $\mathcal{A}(\widetilde{M})$ has a prime order, we have $\mathcal{A}(\widetilde{M}) = \langle\alpha\rangle = \langle\sigma\rangle$. By Lemma 4.2.1, part (1) holds.

(2) Assume $\delta(\widetilde{f}, \widetilde{g}; \alpha) = 1$ and $[\widetilde{f} : \alpha] = [\widetilde{g} : \alpha] = \beta$ for some $\beta \in \mathcal{A}(\widetilde{N})$. Let $(\widetilde{f}_0, \widetilde{g}_0)$ be another lift of (f, g). Since $\mathcal{A}(\widetilde{N})$ has a prime order and by Remark 4.1.9 there exist integers k and l such that $\widetilde{f}_0 = \beta^k \widetilde{f}$ and $\widetilde{g}_0 = \beta^l \widetilde{g}$. Hence,

$$\widetilde{f}_0 \, \alpha = \beta^k \, \widetilde{f} \, \alpha = \beta^k \, \beta \, \widetilde{f} = \beta^{k+1} \widetilde{f} = \beta \, \beta^k \, \widetilde{f} = \beta \, \widetilde{f}_0 \, .$$

Similarly, we get $\widetilde{g}_0 \, \alpha = \beta \, \widetilde{g}_0$. Thus, $[\widetilde{f}_0 : \alpha] = [\widetilde{g}_0 : \alpha] = \beta$. Therefore, $\delta(\widetilde{f}_0, \widetilde{g}_0; \alpha) = 1$.

The converse can be proved in a similar way, since there are no restrictions on the lifting pairs involved. □

Proposition 4.2.2 emphasizes that when $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers, the value of $[\widetilde{f} : \alpha]$ is independent of the selected lift $\widetilde{f}$ of f, and the value of $\delta(\widetilde{f}, \widetilde{g}; \alpha)$ is independent of the chosen lift $(\widetilde{f}, \widetilde{g})$ of (f, g) or $\alpha \in |\mathcal{A}(\widetilde{M})|$. Equivalently, the values of $[\widetilde{f} : \alpha]$ and $\delta(\widetilde{f}, \widetilde{g}; \alpha)$ depend only on f and g. So, Proposition 4.2.2 allows us to generalize Definition 2.1.10 and Notation 4.1.1 as follows.

Definition 4.2.3. *Define $[f : \alpha]$ by $[f : \alpha] = [\widetilde{f} : \alpha]$, and $\delta(f, g)$ by $\delta(f, g) = \delta(\widetilde{f}, \widetilde{g}; \alpha)$, where $\widetilde{f}$ and $\widetilde{g}$ are any lifts of f and g respectively, and $\alpha \in \mathcal{A}(\widetilde{M})$.*

The next proposition gives a classification of the Reidemeister classes for the case where $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers.

Proposition 4.2.4. *Let $\beta \in \mathcal{A}(\widetilde{N}) - \{1_{\widetilde{N}}\}$. Then, there exist exactly $|\mathcal{A}(\widetilde{N})|^{\delta(f,g)}$ H-Reidemeister classes each of which can be represented by a pair of lifts of f and g of the form $(\widetilde{f}, \beta^i \widetilde{g})$ where $0 \le i \le |\mathcal{A}(\widetilde{N})|^{\delta(f,g)} - 1$.*

Proof. By Remark 4.1.9, the action of $\mathcal{A}(\widetilde{M})$ on the sets $\Delta(\beta^i)$, where $0 \le i \le |\mathcal{A}(\widetilde{N})| - 1$, places them in their conjugacy classes. The number of these classes depends on the value of $\delta(f,g)$. So, we differentiate between two cases.

In the first case, we assume that $\delta(f,g) = 1$. Let $\alpha \in \mathcal{A}(\widetilde{M})$, $0 \le i \le |\mathcal{A}(\widetilde{N})| - 1$, and $(\acute{\beta}\,\widetilde{f}, \acute{\beta}\,\beta^i\widetilde{g}) \in \Delta(\beta^i)$. Then,

$$\begin{aligned}
(\acute{\beta}\,\widetilde{f}, \acute{\beta}\,\beta^i\widetilde{g}) \cdot \alpha &= \left((\acute{\beta}\,\widetilde{f}) \cdot \alpha, (\acute{\beta}\,\beta^i\widetilde{g}) \cdot \alpha\right) = \left(\acute{\beta}\,(\widetilde{f}\,\alpha), \acute{\beta}\,\beta^i(\widetilde{g}\,\alpha)\right) \\
&= \left(\acute{\beta}\,[\widetilde{f} : \alpha]\,\widetilde{f}, \acute{\beta}\,\beta^i\,[\widetilde{g} : \alpha]\,\widetilde{g}\right) = \left(\acute{\beta}\,[\widetilde{f} : \alpha]\,\widetilde{f}, \acute{\beta}\,\beta^i\,[\widetilde{f} : \alpha]\,\widetilde{g}\right) \\
&= \left([\widetilde{f} : \alpha]\,\acute{\beta}\,\widetilde{f}, [\widetilde{f} : \alpha]\,\acute{\beta}\,\beta^i\widetilde{g}\right) = [\widetilde{f} : \alpha]\,\acute{\beta}\,\left(\widetilde{f}, \beta^i\widetilde{g}\right) \in \Delta(\beta^i)\ .
\end{aligned}$$

That is, the action of $\mathcal{A}(\widetilde{M})$ on $\Delta(\beta^i)$ carries it back on to itself, i.e., the elements of $\Delta(\beta^i)$ are conjugate only to themselves for each respective i. Hence, in this case, we have $|\mathcal{A}(\widetilde{N})|$ conjugacy classes (namely $\Delta(\beta^i)$, where $0 \le i \le |\mathcal{A}(\widetilde{N})| - 1$). That is, the number of H-Reidemeister classes is $|\mathcal{A}(\widetilde{N})|$, and each H-Reidemeister class has $(\widetilde{f}, \beta^i\widetilde{g})$ as a representative for some i.

In the second case, let us assume $\delta(f,g) = 0$. Let $\alpha \in \mathcal{A}(\widetilde{M})$, $0 \le i \le |\mathcal{A}(\widetilde{N})| - 1$, and $(\acute{\beta}\,\widetilde{f}, \acute{\beta}\,\beta^i\widetilde{g}) \in \Delta(\beta^i)$. Suppose $[\widetilde{g} : \alpha] = [\widetilde{f} : \alpha]^t$ where $t > 1$. Then,

$$\begin{aligned}
(\acute{\beta}\,\widetilde{f}, \acute{\beta}\,\beta^i\widetilde{g}) \cdot \alpha &= \Big((\acute{\beta}\,\widetilde{f}) \cdot \alpha, (\acute{\beta}\,\beta^i\widetilde{g}) \cdot \alpha\Big) = \Big(\acute{\beta}\,(\widetilde{f}\,\alpha), \acute{\beta}\,\beta^i(\widetilde{g}\,\alpha)\Big) \\
&= \Big(\acute{\beta}\,[\widetilde{f} : \alpha]\,\widetilde{f}, \acute{\beta}\,\beta^i\,[\widetilde{g} : \alpha]\,\widetilde{g}\Big) = \Big(\acute{\beta}\,[\widetilde{f} : \alpha]\,\widetilde{f}, \acute{\beta}\,\beta^i\,[\widetilde{f} : \alpha]^t\,\widetilde{g}\Big) \\
&= \Big(\acute{\beta}\,[\widetilde{f} : \alpha]\,\widetilde{f}, \acute{\beta}\,[\widetilde{f} : \alpha]^t\,\beta^i\widetilde{g}\Big) \\
&= \acute{\beta}\,[\widetilde{f} : \alpha]\,\Big(\widetilde{f}, [\widetilde{f} : \alpha]^{t-1}\,\beta^i\widetilde{g}\Big) \in \Delta([\widetilde{f} : \alpha]^{t-1}\,\beta^i)\,.
\end{aligned}$$

That is, the action of α maps $\Delta(\beta^i)$ bijectively onto $\Delta([f : \alpha]^{t-1}\,\beta^i)$, i.e., the elements of $\Delta(\beta^i)$ and $\Delta([f : \alpha]^{t-1}\,\beta^i)$ are conjugate to each other. Since $[f : \alpha]^{t-1}$ is fixed and $\mathcal{A}(\widetilde{N})$ is cyclic , if i runs over the set $\Big\{0, 1, \ldots, |\mathcal{A}(\widetilde{N})| - 1\Big\}$, then each of the elements of $\displaystyle\bigcup_{i=0}^{|\mathcal{A}(\widetilde{N})|-1}\Delta(\beta^i)$ is conjugate to the others. Thus, in this case we have only one conjugacy class (namely $\displaystyle\bigcup_{i=0}^{|\mathcal{A}(\widetilde{N})|-1}\Delta(\beta^i)$), and hence one H-Reidemeister class, which of course can be represented by $(\widetilde{f}, \widetilde{g})$. □

Next, we present many characterizations for which a pair of maps (f, g) satisfies the condition $\delta(f, g) = 1$. Afterward, we collect our results in Corollary 4.2.10.

For each $x \in \Phi(f, g)$, we have the following diagram:

$$
\begin{tikzcd}
C(f_\#, g_\#)_x \arrow[r, "\iota"] \arrow[d, "\Theta_M"'] \arrow[dr, phantom, "(1)"] & \pi_1(M, x) \arrow[r, "g_\# f_\#^{-1}"] \arrow[d, "\Theta_M"'] \arrow[dr, phantom, "(2)"] & \pi_1(N, f(x)) \arrow[r, "j"] \arrow[d, "\Theta_N"'] \arrow[dr, phantom, "(3)"] & \dfrac{\pi_1(N, f(x))}{H(f(x))} \arrow[d, "1"] \\
\Theta_M\left(C(f_\#, g_\#)_x\right) \arrow[r, "\bar{\iota}"] & H_1(M) \arrow[r, "\bar{g}_\# - \bar{f}_\#"] & H_1(N) \arrow[r, "\bar{j}"] & \dfrac{\pi_1(N, f(x))}{H(f(x))} \arrow[d, "\vartheta^{-1}"', "\cong"] \\
 & & & \mathcal{A}(\widetilde{N})
\end{tikzcd}
\tag{4.2.1}
$$

where

- ι and $\bar{\iota}$ are the inclusion homomorphism on the corresponding groups.

- Θ_M and Θ_N are the abelianizations on the corresponding groups.

- 1 in diagram 4.2.1 denotes the identity.

- The function $g_\# f_\#^{-1}$ is defined by $g_\# f_\#^{-1}(a) = g_\#(a)\, f_\#(a)^{-1}$ for every $a \in \pi_1(M, x)$.

The next lemma does not require that $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ be prime.

Lemma 4.2.5. *If $\mathcal{A}(\widetilde{N})$ is abelian, then Diagram 4.2.1 is commutative with $Ker\,\bar{j} = \Theta_N(H(f(x)))$ and $\Theta_M\left(C(f_\#, g_\#)_x\right) \subseteq Ker(\bar{g}_\# - \bar{f}_\#)$.*

Proof. Let F_M and F_N be the commutator subgroups of $\pi_1(M,x)$ and $\pi_1(N,f(x))$ respectively.

• Commutativity of box (1): it is obvious that $\bar{\iota}$ is well-defined. Let $a \in C(f_\#, g_\#)_x$. Then,

$$\bar{\iota} \circ \Theta_M(a) = \bar{\iota}\left(\Theta_M(a)\right) = \Theta_M(a) = \Theta_M\left(\iota(a)\right) = \Theta_M \circ \iota(a) \ .$$

• Commutativity of box (2): first, $\overline{f}_\#$ is defined such that the diagram

$$\begin{array}{ccc} \pi_1(M,x) & \xrightarrow{f_\#} & \pi_1(N,f(x)) \\ \Theta_M \downarrow & & \downarrow \Theta_N \\ H_1(M) & \xrightarrow{\overline{f}_\#} & H_1(N) \end{array} \tag{4.2.2}$$

is commutative. What we need to show is that $\overline{f}_\#$ is well defined, which is true since $f_\#(F_M) \subseteq F_N$. The same is true for $\overline{g}_\#$. Therefore, the homomorphism $\overline{g}_\# - \overline{f}_\#$ is well defined. Now, Let $a \in \pi_1(M,x)$. Then,

$$\begin{aligned} \Theta_N \circ g_\# \, f_\#^{-1}(a) &= \Theta_N\left(g_\#(a)\, f_\#(a)^{-1}\right) \\ &= \Theta_N\left(g_\#(a)\right) + \Theta_N\left(f_\#(a)^{-1}\right) \\ &= \Theta_N\left(g_\#(a)\right) - \Theta_N\left(f_\#(a)\right) \\ &= \overline{g}_\#\left(\Theta_M(a)\right) - \overline{f}_\#\left(\Theta_M(a)\right) \\ &= \overline{g}_\# - \overline{f}_\#\left(\Theta_M(a)\right) \\ &= \overline{g}_\# - \overline{f}_\# \circ \Theta_M(a) \ . \end{aligned}$$

• Commutativity of box (3): we have $\overline{j}$ is defined such that box (3) commutes. To show it is well-defined, it is sufficient to notice that since $\dfrac{\pi_1(N, f(x))}{H(f(x))}$ is abelian, $F_N \subseteq H(f(x))$.

- Let $b \in \pi_1(N, f(x))$. Then,

$$\begin{aligned}
\Theta_N(b) \in Ker(\overline{j}) \;&\Leftrightarrow\; \overline{j}(\Theta_N(b)) = 0 \\
&\Leftrightarrow\; j(b) = 0 \\
&\Leftrightarrow\; b \in H(f(x)) \\
&\Leftrightarrow\; \Theta_N(b) \in \Theta_N(H(f(x))) .
\end{aligned}$$

- Let $a \in C(f_\#, g_\#)_x$. Then,

$$\begin{aligned}
f_\#(a) = g_\#(a) \;&\Rightarrow\; \Theta_N(f_\#(a)) = \Theta_N(f_\#(a)) \\
&\Rightarrow\; \overline{f}_\#(\Theta_M(a)) = \overline{g}_\#(\Theta_M(a)) \\
&\Rightarrow\; \overline{g}_\#(\Theta_M(a)) - \overline{f}_\#(\Theta_M(a)) = 0 \\
&\Rightarrow\; \overline{g}_\# - \overline{f}_\#(\Theta_M(a)) = 0 \\
&\Rightarrow\; \Theta_M(a) \in Ker(\overline{g}_\# - \overline{f}_\#) .
\end{aligned}$$

Therefore,

$$\Theta_M\left(C(f_\#, g_\#)_x\right) \subseteq Ker(\overline{g}_\# - \overline{f}_\#) .$$

$\square$

The first characterization, for which a pair (f, g) satisfies that $\delta(f, g) = 1$, is geometric. The condition characterizes the fact $\delta(f, g) = 1$ through the action of $\mathcal{A}(\widetilde{M})$ on the coincidence set of every pair of lifts $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$.

Proposition 4.2.6. *Assume $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. The following are equivalent:*

1. $\delta(f, g) = 1$.

2. *For every $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$, $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$, we have $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$.*

3. *There exist $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$, $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$.*

Proof. $\bullet(1) \Rightarrow (2)$: Assume $\delta(f, g) = 1$. Let $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$ such that $\Phi(\widetilde{f}, \widetilde{g}) \neq \phi$, $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$. Then,

$$\widetilde{f}(\alpha(\widetilde{x})) = \widetilde{f}\,\alpha(\widetilde{x}) = [\widetilde{f} : \alpha]\,\widetilde{f}(\widetilde{x}) = [\widetilde{g} : \alpha]\,\widetilde{g}(\widetilde{x}) = \widetilde{g}\,\alpha(\widetilde{x}) = \widetilde{g}(\alpha(\widetilde{x}))\ .$$

That is, $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$.

$\bullet(2) \Rightarrow (3)$: Trivial.

$\bullet(3) \Rightarrow (1)$: Suppose there exist $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$, $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$. Then,

$$[\widetilde{f} : \alpha]\,\widetilde{f}(\widetilde{x}) = \widetilde{f}\,\alpha(\widetilde{x}) = \widetilde{f}(\alpha(\widetilde{x})) = \widetilde{g}(\alpha(\widetilde{x})) = \widetilde{g}\,\alpha(\widetilde{x}) = [\widetilde{g} : \alpha]\,\widetilde{g}(\widetilde{x}) = [\widetilde{g} : \alpha]\,\widetilde{f}(\widetilde{x})\ .$$

Thus, $[\widetilde{f} : \alpha] = [\widetilde{g} : \alpha]$ and hence $\delta(f, g) = 1$. $\square$

The second characterization, for which a pair (f, g) satisfies that $\delta(f, g) = 1$, is algebraic. It characterizes the fact $\delta(f, g) = 1$ through relations of the fundamental groups of the considered spaces .

Proposition 4.2.7. *Assume $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. The following are equivalent:*

1. $\delta(f, g) = 1$.

2. *For every $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$ and $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$, we have*

$$g_{\#}\, f_{\#}^{-1}\,(\pi_1(M, p\,(\widetilde{x}))) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

3. *There exist a lift $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$ and $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$ such that*

$$g_{\#}\, f_{\#}^{-1}\,(\pi_1(M, p\,(\widetilde{x}))) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

Proof. $\bullet(1) \Rightarrow (2)$: Assume that $\delta(f, g) = 1$. Let $(\widetilde{f}, \widetilde{g}) \in Lift(f, g)$ and $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$. Put $p\,(\widetilde{x}) = x$. Let $a \in \pi_1(M, x)$ and $\widetilde{a}$ be the lift of a at $\widetilde{x}$. Since the covering is regular, there exists $\alpha \in \mathcal{A}(\widetilde{M})$ such that $a(1) = \alpha(\widetilde{x})$. By Proposition 4.2.6, $\widetilde{f}(\alpha(\widetilde{x})) = \widetilde{g}(\alpha(\widetilde{x}))$. Hence, $\widetilde{g}(\widetilde{a})\, \widetilde{f}(\widetilde{a})^{-1} \in \pi_1(\widetilde{N}, \widetilde{f}(\widetilde{x}))$. Therefore,

$$\begin{aligned} g_{\#}\, f_{\#}^{-1}(a) &= g(a)\, f(a^{-1}) = g(p\,(\widetilde{a}))\, f(p\,(\widetilde{a}^{-1})) = p\,(\widetilde{g}(\widetilde{a}))\, p\,(\widetilde{f}(\widetilde{a}^{-1})) \\ &= p\,\left(\widetilde{g}(\widetilde{a})\, \widetilde{f}(\widetilde{a}^{-1})\right) \in H(\widetilde{f}(\widetilde{x}))\ . \end{aligned}$$

That is,

$$g_{\#}\, f_{\#}^{-1}\,(\pi_1(M, p\,(\widetilde{x}))) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

$\bullet(2) \Rightarrow (3)$: Trivial.

$\bullet(3) \Rightarrow (1)$: Assume

$$g_{\#}\, f_{\#}^{-1}\,(\pi_1(M, p\,(\widetilde{x}))) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

for some lift $(\widetilde{f}, \widetilde{g})$ of (f, g) and $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$. Let $\alpha \in \mathcal{A}(\widetilde{M})$ and $\widetilde{a} : \widetilde{x} \longrightarrow \alpha(\widetilde{x})$ be a path in $\widetilde{M}$. We have $p\,(\widetilde{a})$ is a loop in M at $p\,(\widetilde{x})$. Thus, there exists $b \in \pi_1(\widetilde{N}, \widetilde{f}(\widetilde{x}))$ such that $g_{\#}\, f_{\#}^{-1}(p\,(\widetilde{a})) = p\,(b)$. Hence,

$$\begin{aligned} g(p\,(\widetilde{a}))\, f(p\,(\widetilde{a}^{-1})) = p\,(b) \quad &\Rightarrow \quad g(p\,(\widetilde{a})) = p\,(b)\, f(p\,(\widetilde{a})) \\ &\Rightarrow \quad p\,(\widetilde{g}(\widetilde{a})) = p\,(b)\, p\,(\widetilde{f}(\widetilde{a})) \\ &\Rightarrow \quad p\,(\widetilde{g}(\widetilde{a})) = p\,(b\,\widetilde{f}(\widetilde{a}))\ . \end{aligned}$$

However, $\widetilde{g}(\widetilde{a})$ and $b\,\widetilde{f}(\widetilde{a})$ are lifts to the same path and having the same initial point $\widetilde{f}(\widetilde{x})$. Thus, they are homotopic relative endpoints and have the same end point, i.e., $\widetilde{f}(\alpha(\widetilde{x})) = \widetilde{g}(\alpha(\widetilde{x}))$ or $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$. By Proposition 4.2.6, we get $\delta(f, g) = 1$. $\square$

The third characterization, for which a pair (f, g) satisfies $\delta(f, g) = 1$, is also algebraic. It characterizes the fact $\delta(f, g) = 1$ through a sequence of homological groups and homomorphisms of the involved spaces .

Proposition 4.2.8. *Assume $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. The following are equivalent:*

1. $\delta(f,g) = 1$.

2. The sequence

$$\Theta_M\left(C(f_\#, g_\#)\right) \xrightarrow{\imath_\#} H_1(M) \xrightarrow{\overline{g}_\# - \overline{f}_\#} H_1(N) \xrightarrow{\overline{\jmath}} \mathcal{A}(\widetilde{N}) \tag{4.2.3}$$

is a chain complex.

Proof. $\bullet(1) \Rightarrow (2)$: Suppose $\delta(f,g) = 1$. Let $x \in \Phi(f,g)$. By Lemma 4.2.5, $\Theta_M\left(C(f_\#, g_\#)_x\right) \subseteq Ker(\overline{g}_\# - \overline{f}_\#)$. Moreover, let $\Theta_N(b) \in (\overline{g}_\# - \overline{f}_\#)(H_1(M))$. Then,

$$\begin{aligned}
\Theta_N(b) &= (\overline{g}_\# - \overline{f}_\#)(\Theta_M(a)) \text{ for same } a \in \pi_1(M,x)\ . \\
&= \overline{g}_\#(\Theta_M(a)) - \overline{f}_\#(\Theta_M(a)) \\
&= \Theta_N(g_\#(a)) - \Theta_N(f_\#(a)) \\
&= \Theta_N(g_\#(a)\, f_\#(a)^{-1}) \\
&= \Theta_N(g_\#\, f_\#^{-1}(a))\ .
\end{aligned}$$

Since $g_\#\, f_\#^{-1}(\pi_1(M,x)) \in H(f(x))$, we get that $\Theta_N(b) \in \Theta_N(H(f(x))) = Ker\overline{\jmath}$. Therefore, the sequence 4.2.3 is a chain complex.

$\bullet(2) \Rightarrow (1)$: Suppose the sequence 4.2.3 is a chain complex for some (or for every) $x \in \Phi(f,g)$. Hence, $(\overline{g}_\# - \overline{f}_\#)(H_1(M)) \subseteq \Theta_N(H(f(x)))$. Let $b = g_\#\, f_\#^{-1}(a)$ and $a \in \pi_1(M,x)$ Then,

$$\Theta_N(b) = \Theta_N(g_\#\, f_\#^{-1}(a)) = (\overline{g}_\# - \overline{f}_\#)(\Theta_M(a))\ .$$

Thus, $\Theta_N(b) \in \Theta_N(H(f(x)))$. Since $F_N \subseteq H(f(x))$, where F_N is the commutator subgroup of $\pi_1(N)$, we have $b \in H(f(x))$. That is, $g_\#\, f_\#^{-1}(\pi_1(M,x)) \subseteq H(f(x))$. It follows from Proposition 4.2.7 that $\delta(f,g) = 1$. $\square$

Remark 4.2.9. *In Proposition 4.2.8, we did not mention the coincidence point of f and g at which the sequence 4.2.3 is applied because the proposition is true whatever the coincidence point of f and g is.*

Now, we summarize the previous characterizations in the following corollary.

Corollary 4.2.10. *Assume $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. The following are equivalent:*

1. $\delta(f,g) = 1$.

2. *For every $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$, $\widetilde{x} \in \Phi(\widetilde{f},\widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$, we have $\alpha(\widetilde{x}) \in \Phi(\widetilde{f},\widetilde{g})$.*

3. *There exist $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$, $\widetilde{x} \in \Phi(\widetilde{f},\widetilde{g})$, and $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\alpha(\widetilde{x}) \in \Phi(\widetilde{f},\widetilde{g})$.*

4. *There exist a lift $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$ and $\widetilde{x} \in \Phi(\widetilde{f},\widetilde{g})$ such that*

$$g_{\#}\, f_{\#}^{-1}\left(\pi_1(M, p\left(\widetilde{x}\right))\right) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

5. *For every $(\widetilde{f},\widetilde{g}) \in Lift(f,g)$ and $\widetilde{x} \in \Phi(\widetilde{f},\widetilde{g})$, we have*

$$g_{\#}\, f_{\#}^{-1}\left(\pi_1(M, p\left(\widetilde{x}\right))\right) \subseteq H(\widetilde{f}(\widetilde{x}))\ .$$

6. *The sequence*

$$\Theta_M\left(C(f_\#, g_\#)\right) \xrightarrow{i_\#} H_1(M) \xrightarrow{\overline{g}_\# - \overline{f}_\#} H_1(N) \xrightarrow{\overline{j}} \mathcal{A}(\widetilde{N})$$

is a chain complex for every $x \in \Phi(f, g)$.

7. *The sequence*

$$\Theta_M\left(C(f_\#, g_\#)\right) \xrightarrow{i_\#} H_1(M) \xrightarrow{\overline{g}_\# - \overline{f}_\#} H_1(N) \xrightarrow{\overline{j}} \mathcal{A}(\widetilde{N})$$

is a chain complex for some $x \in \Phi(f, g)$.

Proof. Apply Propositions 4.2.6, 4.2.7, and 4.2.8. □

The following corollary generalizes part (3) of Corollary 4.1.21.

Corollary 4.2.11. *For every nonempty Nielsen class A of f and g, we have $I_A = |\mathcal{A}(\widetilde{M})|^{\delta(f,g)}$.*

Proof. Suppose $\delta(f, g) = 1$ and let A be a nonempty Nielsen class of f and g. Let $x \in A$ and $(\widetilde{f}, \widetilde{g})$ be a lift of (f, g) such $A \subseteq p\Phi(\widetilde{f}, \widetilde{g})$. By (2), Corollary 4.2.10, if $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})$, then $\alpha(\widetilde{x}) \in \Phi(\widetilde{f}, \widetilde{g})$ for every $\alpha \in \mathcal{A}(\widetilde{M})$. That is, $p^{-1}(x) \subseteq \Phi(\widetilde{f}, \widetilde{g})$. Therefore, $I_A = |p^{-1}(x)| = |\mathcal{A}(\widetilde{M})|$.

Suppose now $\delta(f, g) = 0$. Then, $\Phi(f, g) = p\,\Phi(\widetilde{f}, \widetilde{g})$. Let $x \in \Phi(f, g)$ and $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})$. By Proposition 4.2.6, $\alpha(\widetilde{x})$ does not belong to $\Phi(\widetilde{f}, \widetilde{g})$. Thus,

$|p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})| = 1$. This means that for any nonempty Nielsen class A of f and g, we have $I_A = 1$. $\square$

We now prepare for the main results in this section, Theorems 4.2.16, 4.2.18, and 4.2.19. For each $x \in \Phi(f, g)$, Consider the diagram:

$$\begin{array}{ccccc}
C(f_{\#}, g_{\#})_x & \xrightarrow{\iota} & \pi_1(M, x) & \xrightarrow{j} & \dfrac{\pi_1(M, x)}{K(x)} \\
\downarrow \Theta_M & & \downarrow \Theta_M & & \downarrow 1 \\
\Theta_M\left(C(f_{\#}, g_{\#})_x\right) & \xrightarrow{\bar{\iota}} & H_1(M) & \xrightarrow{\bar{j}} & \dfrac{\pi_1(M, x)}{K(x)} \\
 & & & & \vartheta^{-1} \downarrow \cong \\
 & & & & \mathcal{A}(\widetilde{M})
\end{array} \tag{4.2.4}$$

where the homomorphisms are as in 4.2.1.

Lemma 4.2.12. *Let x be a coincidence point of f and g. Then, diagram 4.2.4 commutes, and $ker\bar{j} = \Theta_M(K(x))$.*

Proof. The proof is quite similar to Lemma 4.2.5. $\square$

Lemma 4.2.13. *Let x be a coincidence point of f and g. In diagram 4.2.4, the first horizontal sequence is a chain complex if and only if the second horizontal sequence is a chain complex.*

Proof. Assume $C(f_\#, g_\#)_x \subseteq ker(j) = K(x)$. By Lemma 4.2.12, we have

$$\Theta_M\left(C(f_\#, g_\#)_x\right) \subseteq \Theta_M(K(x)) = ker\bar{j}\ .$$

Conversely, suppose $\Theta_M\left(C(f_\#, g_\#)_x\right) \subseteq ker\bar{j} = \Theta_M(K(x))$. Since $F_M \subseteq K(x)$, we get that $C(f_\#, g_\#)_x \subseteq K(x)$. ☐

Lemma 4.2.14. *Let $\widetilde{x}, \widetilde{y} \in \Phi(\widetilde{f}, \widetilde{g})$ be in the same Nielsen class, and let $x = p(\widetilde{x})$ and $y = p(\widetilde{y})$ Then,*

$$C(f_\#, g_\#)_x \subseteq K(x) \quad \textit{if and only if} \quad C(f_\#, g_\#)_y \subseteq K(y)\ .$$

Proof. It is sufficient to show that if $C(f_\#, g_\#)_x \subseteq K(x)$ then $C(f_\#, g_\#)_y \subseteq K(y)$. Let $\widetilde{\omega} : \widetilde{x} \longrightarrow \widetilde{y}$ be a path that establishes the Nielsen relation between $\widetilde{x}$ and $\widetilde{y}$. Put $\omega = p(\widetilde{\omega})$. We have the commutative diagram

$$\begin{array}{ccc}
\pi_1(\widetilde{M}, \widetilde{x}) & \xrightarrow{\widetilde{\omega}_\#} & \pi_1(\widetilde{M}, \widetilde{y}) \\
\downarrow p_\# & & p_\# \downarrow \\
\pi_1(M, x) & \xrightarrow{\omega_\#} & \pi_1(M, y) \\
f_\# \downarrow g_\# & & f_\# \downarrow g_\# \\
\pi_1(N, f(x)) & \xrightarrow{f(\omega)_\#} & \pi_1(N, f(y))\ .
\end{array}$$

Notice that $f(\omega)_\# = g(\omega)_\#$. Let $a \in \pi_1(M, y)$. Then,

$$\begin{aligned}
a \in C(f_\#, g_\#)_y \ &\Rightarrow\ \omega_\#^{-1}(a) \in C(f_\#, g_\#)_x \subseteq K(x) \\
&\Rightarrow\ \omega_\#^{-1}(a) = p_\#(\widetilde{d}) \ \text{ for some } \widetilde{d} \in \pi_1(\widetilde{M}, \widetilde{x}). \\
&\Rightarrow\ a = \omega_\# p_\#(\widetilde{d}) = p_\#(\widetilde{\omega}_\#(\widetilde{d})) \\
&\Rightarrow\ a \in p_\#\left(\pi_1(\widetilde{M}, \widetilde{y})\right) = K(y).
\end{aligned}$$

☐

Let $\alpha \in \mathcal{A}(\widetilde{M})$ and $\widetilde{x} \in \Phi(\widetilde{f}, \widetilde{g})$. We define the set $\alpha \cdot [\widetilde{x}]$ by $\{\alpha \cdot \widetilde{y} \mid \ \widetilde{y} \in [\widetilde{x}]\}$. The following proposition is a generalization of a part of [Theorem 2.5, [5]].

Proposition 4.2.15. *Assume $\delta(f,g) = 1$. Let $x \in \Phi(f,g)$ and $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \widetilde{g})$. Then, the family $\left\{\alpha \cdot [\widetilde{x}] \mid \ \alpha \in \mathcal{A}(\widetilde{M})\right\}$ is pairwise disjoint if and only if $C(f_{\#}, g_{\#})_x \subseteq K(x)$.*

Proof. Assume the family $\left\{\alpha \cdot [\widetilde{x}] \mid \ \alpha \in \mathcal{A}(\widetilde{M})\right\}$ is pairwise disjoint. Let $a \in C(f_{\#}, g_{\#})_x$ and $\widetilde{a} : \widetilde{x} \longrightarrow \widetilde{y}$ be the lift of a at $\widetilde{x}$. Hence, $\widetilde{y} \in p^{-1}(x)$. So, there exists $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\widetilde{y} = \alpha(\widetilde{x})$. Moreover,

$$\begin{aligned} f(a) = g(a) \ &\Rightarrow \ f(p(\widetilde{a})) = g(p(\widetilde{a})) \\ &\Rightarrow \ p(\widetilde{f}(\widetilde{a})) = p(\widetilde{g}(\widetilde{a})) \end{aligned}$$

Since $\widetilde{x}$, $\widetilde{y} \in \Phi(\widetilde{f}, \widetilde{g})$, we have $\widetilde{f}(\widetilde{a}) \sim_0 \widetilde{g}(\widetilde{a})$, i.e., $[\widetilde{y}] = [\widetilde{x}]$. However, $\widetilde{y} = \alpha(\widetilde{x})$. By the assumptions, we get $\alpha = 1_{\widetilde{M}}$ and $\widetilde{y} = \widetilde{x}$. Thus, $a = p(\widetilde{a}) \in p_{\#}(\pi_1(\widetilde{M}, \widetilde{x})) = K(x)$. Consequently, $C(f_{\#}, g_{\#})_x \subseteq K(x)$.

For the converse, suppose $C(f_{\#}, g_{\#})_x \subseteq K(x)$. Assume, for contrary, that there exists $\alpha \in \mathcal{A}(\widetilde{M})$ such that $\alpha \neq 1_{\widetilde{M}}$ and $[\widetilde{x}] \cap \alpha \cdot [\widetilde{x}] \neq \phi$. Thus, there exists an open path $\widetilde{a} : \widetilde{x} \longrightarrow \alpha(\widetilde{x})$ (that is, the endpoints of the path are different) in $\widetilde{M}$ such that $\widetilde{f}(\widetilde{a}) \sim_0 \widetilde{g}(\widetilde{a})$. So, we get $p(\widetilde{f}(\widetilde{a})) \sim_0 p(\widetilde{g}(\widetilde{a}))$ or $f(p(\widetilde{a})) \sim_0 g(p(\widetilde{a}))$. That is, $p(\widetilde{a}) \in C(f_{\#}, g_{\#})_x \subseteq K(x)$. Hence, there exists $\widetilde{d} \in \pi_1(\widetilde{M}, \widetilde{x})$ such that $p(\widetilde{a}) = p(\widetilde{d})$. The last statement implies that $\alpha(\widetilde{x}) = \widetilde{a}(1) = \widetilde{d}(1) = \widetilde{x}$ and hence $\alpha = 1_{\widetilde{M}}$ which is a contradiction. Therefore, $[\widetilde{x}] \cap \alpha \cdot [\widetilde{x}] = \phi$ for every $\alpha \in \mathcal{A}(\widetilde{M}) - \{1_{\widetilde{M}}\}$, and this in turn yields the information that the family $\left\{\alpha \cdot [\widetilde{x}] \mid \ \alpha \in \mathcal{A}(\widetilde{M})\right\}$ is pairwise disjoint. $\square$

The next two theorems generalize Theorem 2.5, [5].

Theorem 4.2.16. *Assume $\delta(f,g)=1$. Let $\beta \in \mathcal{A}(\widetilde{N})-\{1_{\widetilde{N}}\}$. Then,*

$$N(f,g)=\frac{1}{|\mathcal{A}(\widetilde{M})|}\cdot\sum_{i=0}^{|\mathcal{A}(\widetilde{N})|-1}N(\widetilde{f},\beta^i\widetilde{g}) \tag{4.2.5}$$

if and only if $\widetilde{A}\cap\left(\alpha\cdot\widetilde{A}\right)=\emptyset$ for every $\alpha\in\mathcal{A}(\widetilde{M})-\{1_{\widetilde{M}}\}$, $\beta^i\in\mathcal{A}(\widetilde{N})$ for all i, and for all $\widetilde{A}\in\widetilde{\Phi}(\widetilde{f},\beta^i\widetilde{g})$ for which $p(\widetilde{A})$ is an essential Nielsen class of f and g.

Proof. Set $|\mathcal{A}(\widetilde{M})|=P$ and $|\mathcal{A}(\widetilde{N})|=Q$. Assume $\widetilde{A}\cap\alpha\cdot\widetilde{A}=\emptyset$ for every $\alpha\in\mathcal{A}(\widetilde{M})$, $\beta^i\in\mathcal{A}(\widetilde{N})$, and $\widetilde{A}\in\widetilde{\Phi}(\widetilde{f},\beta^i\widetilde{g})$ for which $p(\widetilde{A})$ is an essential Nielsen class of f and g. Let us assume first that $f \pitchfork g$ (i.e,. f is transverse to g). Let $A=\{x_0,\dots,x_s\}$ be an essential Nielsen class of f and g. Then, there exists $0\le i\le Q-1$ such that $A\subseteq p\,\Phi(\widetilde{f},\beta^i\widetilde{g})$. Let $x_0\in A$ and $\widetilde{x}_0\in p^{-1}(x_0)\cap\Phi(\widetilde{f},\beta^i\widetilde{g})$. Let $\omega_j: x_0\longrightarrow x_j$ be a path in M which establishes the Nielsen relation between x_0 and x_j and $j=0,\dots,s$. Let $\widetilde{\omega}_j$ be the lift of ω_j at $\widetilde{x}_0$. Since the homotopy between $f(\omega_j)$ and $g(\omega_j)$ lifts to a homotopy between $\widetilde{f}(\widetilde{\omega}_j)$ and $\beta^i\widetilde{g}(\widetilde{\omega}_j)$, we get that the points $\widetilde{x}_0,\widetilde{\omega}_1(1),\dots,\widetilde{\omega}_s(1)$ lie in the same Nielsen class of $\widetilde{f}$ and $\beta^i\widetilde{g}$. On the other hand, since $\delta(f,g)=1$, $\alpha\cdot[\widetilde{x}_0]$ is also a Nielsen class of $\widetilde{f}$ and $\beta^i\widetilde{g}$ for each $\alpha\in\mathcal{A}(\widetilde{M})$; so by the assumptions, the family $\left\{[\widetilde{x}_0],\alpha\cdot[\widetilde{x}_0],\dots,\alpha^{P-1}\cdot[\widetilde{x}_0]\right\}$ is mutually disjoint. We show that the union of this family is $p^{-1}(A)$. Obviously, $\bigcup_{k=0}^{P-1}\alpha^k\cdot[\widetilde{x}_0]\subseteq p^{-1}(A)$. Let $\widetilde{x}\in p^{-1}(A)$. Let $\sigma: x_0\longrightarrow p(\widetilde{x})$ be a path that establishes the Nielsen relation between x_0 and $p(\widetilde{x})$, and $\widetilde{\sigma}$ be its lift in $\widetilde{M}$ at $\widetilde{x}_0$. Since $\widetilde{\sigma}(1)\in p^{-1}(p(\widetilde{x}))$, there exists $0\le k\le P-1$ such that $\widetilde{\sigma}(1)=\alpha^k(\widetilde{x})$. Thus, $\alpha^k(\widetilde{x})\in[\widetilde{x}_0]$ which implies $\widetilde{x}\in\alpha^{P-k}\cdot[\widetilde{x}_0]$. Therefore,

$$p^{-1}(A) \subseteq \bigcup_{k=0}^{P-1} \alpha^k \cdot [\widetilde{x}_0].$$

Consequently, we have $S_A = P$. That is, the number S is fixed for all Nielsen classes of f and g and equal to P. Since Corollary 4.2.11 implies that I is also equal to P for all Nielsen classes of f and g, we get that $J_A = 1$ for all Nielsen classes of f and g. Thus, $N_{ED}(f,g) = 0$ and hence by Theorem 3.3.16, and Lemma 4.2.4 we get

$$N(f,g) = N_L(f,g) = \sum_{i=0}^{Q-1} \frac{N(\widetilde{f}, \beta^i \widetilde{g})}{P}$$

or

$$N(f,g) = \frac{1}{P} \cdot \sum_{i=0}^{Q-1} N(\widetilde{f}, \beta^i \widetilde{g}) .$$

For the converse, assume

$$N(f,g) = \frac{1}{P} \cdot \sum_{i=0}^{Q-1} N(\widetilde{f}, \beta^i \widetilde{g}) .$$

Let $[x]$ be a nonempty Nielsen class of f and g. Pick $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \beta^i \widetilde{g})$ for some suitable i. As in the above argument we have $p^{-1}([x]) = \bigcup_{j=0}^{P-1} \alpha^j \cdot [\widetilde{x}]$. We claim that this union is disjoint. Let $\Psi([x]) = \left\{ [\widetilde{x}], \alpha \cdot [\widetilde{x}], \ldots, \alpha^{P-1} \cdot [\widetilde{x}] \right\}$ and $\Psi = \left\{ \Psi([x]) \mid [x] \in \widetilde{\Phi}_E(f,g) \right\}$, where $\widetilde{\Phi}_E(f,g)$ is the set of all essential Nielsen classes of f and g. Define the function

$$\widetilde{\Phi}_E(f,g) \longrightarrow \Psi \quad : \quad [x] \mapsto \Psi([x]) .$$

We show that this function is a well-defined bijection.

- Let $[x], [y] \in \widetilde{\Phi}_E(f, g)$ such that $[x] = [y]$. Let $\omega : x \longrightarrow y$ be a path that establishes the Nielsen relation and $\widetilde{\omega} : \widetilde{x} \longrightarrow \widetilde{y}$ be its lift at $\widetilde{x}$, where $\widetilde{y} \in p^{-1}(y)$. Then, $[\widetilde{x}] = [\widetilde{y}]$ and hence $\Psi([x]) = \Psi([y])$. Therefore, the function is well-defined.
- Suppose $\Psi([x]) = \Psi([y])$ and let $\widetilde{x} \in p^{-1}(x) \cap \Phi(\widetilde{f}, \beta^{i_1}\widetilde{g})$ and $\widetilde{y} \in p^{-1}(y) \cap \Phi(\widetilde{f}, \beta^{i_2}\widetilde{g})$. Thus, $i_1 = i_2$. Furthermore, there exists j with $0 \leq j \leq P-1$ such that $[\widetilde{y}] = \alpha^j \cdot [\widetilde{x}]$. Thus, $[y] = p([\widetilde{y}]) = p(\alpha^j \cdot [\widetilde{x}]) = p([\widetilde{x}]) = [x]$. This implies that the function is one to one.
- Surjectivity is obvious.

Since the function is bijective we get that $|\Psi| = N(f, g)$. Now, Let r denote the number of essential classes $[x] \in \Phi(f, g)$ such that $[\widetilde{x}] = \alpha^j \cdot [\widetilde{x}]$ for some j with $0 \leq j \leq P-1$ (and hence for all j with $0 \leq j \leq P-1$). In other words, r is the number of essential classes $[x]$ such that $|\Psi([x])| = 1$. So, there exist $N(f, g) - r$ elements in Ψ each of which has a cardinality of P. Hence,

$$\begin{aligned}
N(f,g) - r &= \frac{1}{P} \cdot \left[\left(\sum_{i=0}^{Q-1} N(\widetilde{f}, \beta^i \widetilde{g}) \right) - r \right] \\
&= \frac{1}{P} \cdot \left(\sum_{i=0}^{Q-1} N(\widetilde{f}, \beta^i \widetilde{g}) \right) - \frac{r}{P} \\
&= N(f, g) - \frac{r}{P} \qquad \text{(by the assumptions)}
\end{aligned}$$

which yields that $r = 0$. That is, $[\widetilde{x}] \cap \alpha \cdot [\widetilde{x}] = \emptyset$ for every $\alpha \in \mathcal{A}(\widetilde{M})$, $\beta^i \in \mathcal{A}(\widetilde{N})$, and $[\widetilde{x}] \in \widetilde{\Phi}(\widetilde{f}, \beta^i \widetilde{g})$ such that $p([\widetilde{x}])$ is an essential Nielsen class of f and g. $\square$

Notice that, although Theorem 4.2.16 gives a necessary and sufficient condition for Equation 4.2.5 to hold, it has a drawback. It uses the set of essential classes of f and g the very thing we are supposed to count. The following corollary helps us to get around this .

Corollary 4.2.17. *Assume $\delta(f,g) = 1$. Let $\beta \in \mathcal{A}(\widetilde{N}) - \{1_{\widetilde{N}}\}$. If $\widetilde{A} \cap \left(\alpha \cdot \widetilde{A}\right) = \emptyset$ for every $\alpha \in \mathcal{A}(\widetilde{M}) - \{1_{\widetilde{M}}\}$, $\beta^i \in \mathcal{A}(\widetilde{N})$ for all i, and for all $\widetilde{A} \in \widetilde{\Phi}(\widetilde{f}, \beta^i\widetilde{g})$, then*

$$N(f,g) = \frac{1}{|\mathcal{A}(\widetilde{M})|} \cdot \sum_{i=0}^{|\mathcal{A}(\widetilde{N})|-1} N(\widetilde{f}, \beta^i\widetilde{g})$$

Proof. Apply Theorem 4.2.16. □

Theorem 4.2.18. *If $\delta(f,g) = 0$, then $N(f,g) = N(\widetilde{f},\widetilde{g})$.*

Proof. Assume that $\delta(f,g) = 0$. By Corollary 4.2.11, the number $I = 1$. Since $J \cdot S = I$, we get that $J = S = 1$ for every nonempty Nielsen class of f and g. Since $J = 1$ is odd, and the same for all classes, we have that $N_{ED}(f,g) = 0$. So Theorem 3.3.16 holds implying that

$$N(f,g) = N_L(f,g) = \sum_{i=0}^{|\mathcal{A}(\widetilde{N})|^0-1} \frac{N(\widetilde{f}, \beta^i\widetilde{g})}{S(\widetilde{f}, \beta^i\widetilde{g})} = \frac{N(\widetilde{f},\widetilde{g})}{1} = N(\widetilde{f},\widetilde{g}) .$$

□

We sum up Theorem 4.2.16 and Corollary 4.2.17 in the following theorem.

Theorem 4.2.19. *Let M and N be connected closed smooth manifolds of the same dimension, and let $(\widetilde{M}, p)$ and $(\widetilde{N}, p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of M and N respectively. Assume the coverings are finite and that $|\mathcal{A}(\widetilde{M})|$ and $|\mathcal{A}(\widetilde{N})|$ are prime numbers. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f}, \widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. If either $\delta(f, g) = 0$ or $\delta(f, g) = 1$ with $C(f_{\#}, g_{\#})_{p(\widetilde{x})} \subseteq K(p(\widetilde{x}))$ for every nonempty Nielsen class $[\widetilde{x}]$ of $(\widetilde{f}, \beta^i\widetilde{g})$ with $0 \leq i \leq |\mathcal{A}(\widetilde{N})| - 1$, then*

$$N(f, g) = \frac{1}{|\mathcal{A}(\widetilde{M})|^{\delta(f,g)}} \cdot \sum_{i=0}^{|\mathcal{A}(\widetilde{N})|^{\delta(f,g)}-1} N(\widetilde{f}, \beta^i\widetilde{g}) \,. \tag{4.2.6}$$

Proof. Apply Theorem 4.2.18 and Corollary 4.2.17 along with Lemma 4.2.14. □

4.3 Examples

This section is devoted to examples. Before giving them, we do some necessary preparatory work. The main preparation is to give an explicit formula of the suspension homeomorphism between the $(n+1)-$sphere S^{n+1} and the suspension ΣS^n.

Notation 4.3.1. *We set the following notation which is necessary in this section:*

- *The nth unit sphere S^n defined by*

$$\begin{aligned} S^n &= \left\{ (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{i=1}^{n+1} |x_i|^2 = 1 \right\} \\ &= \left\{ (\eta, t) \mid \eta \in \mathbf{R}^n, t \in [-1, 1], \text{ and } |\eta|^2 + t^2 = 1 \right\} . \end{aligned}$$

- *The upper nth hemisphere* S^{n+} *and the lower nth hemisphere* S^{n-}*:*

$$\begin{aligned} S^{n+} &= \left\{(\eta, t) \mid \eta \in \mathbf{R}^n, t \in [0,1], \text{ and } |\eta|^2 + t^2 = 1\right\} \\ S^{n-} &= \left\{(\eta, t) \mid \eta \in \mathbf{R}^n, t \in [-1,0], \text{ and } |\eta|^2 + t^2 = 1\right\} . \end{aligned}$$

Notice that $S^n = S^{n+} \cup S^{n-}$ *and* $S^{n-1} = S^{n+} \cap S^{n-}$.

- *The nth unit disc* D^n *defined by*

$$D^n = \left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid \sum_{i=1}^{n} |x_i|^2 \leq 1\right\} .$$

- *The unreduced suspension* ΣX *of the topological space* X, *is obtained from* $X \times [-1,1]$ *by identifying each of* $X \times \{-1\}$ *and* $X \times \{1\}$ *to a single point (the points are different). The elements of* ΣX *are denoted by* $[x, t]$ *where* $x \in X$ *and* $t \in [-1,1]$.

- *We let*

$$\begin{aligned} (\Sigma X)^+ &= \{[x,t] \mid x \in X \text{ and } t \in [0,1]\} , \\ (\Sigma X)^- &= \{[x,t] \mid x \in X \text{ and } t \in [-1,0]\} , \end{aligned}$$

and

$$x_+ = [x, 1] \text{ and } x_- = [x, -1] \text{ for all } x \in X .$$

Notice that $\Sigma X = (\Sigma X)^+ \cup (\Sigma X)^-$ *and* $(\Sigma X)^+ \cap (\Sigma X)^- = \{[x, 0] \mid x \in X\} \cong X$.

- $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ *the suspension of the map* $f : X \longrightarrow Y$.

- *Regarding* $\mathbf{R}^{n+1}$ *as* $\mathbf{R}^n \times \mathbf{R}$, *we define*

- $\pi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \quad : \quad (x, t) \mapsto x$ *to be the projection of* $\mathbf{R}^{n+1}$ *on* $\mathbf{R}^n$.

- $\pi_+ : S^{n+} \longrightarrow D^n$ *to be the restriction of* π *on* S^{n+}.

- $\pi_- : S^{n-} \longrightarrow D^n$ *to be the restriction of* π *on* S^{n-}.

- $T_+ : (\Sigma S^n)^+ \longrightarrow D^{n+1}$ *by* $T_+[z, t] = (1-t)\, z$.

- $T_- : (\Sigma S^n)^- \longrightarrow D^{n+1}$ *by* $T_-[z, t] = (1+t)\, z$.

where $(1-t)\, z$ *is the scalar multiplication of* z *by* $1-t$. *Similarly for* $(1+t)\, z$.

- $sgn(t) = \begin{cases} t/|t| & \text{if } t \in \mathbf{R} - \{0\}; \\ 0 & \text{if } t = 0 \end{cases}$ *is the well known signon function from Real Analysis.*

Lemma 4.3.2. *The maps* π_+, π_-, T_+, *and* T_- *are homeomorphisms. Moreover,*

- $\pi_+^{-1} : D^n \longrightarrow S^{n+}$ *defined by* $\pi_+^{-1}(z) = (z, \sqrt{1-|z|^2})$.

- $\pi_-^{-1} : D^n \longrightarrow S^{n-}$ *defined by* $\pi_-^{-1}(z) = (z, -\sqrt{1-|z|^2})$.

- $T_+^{-1} : D^{n+1} \longrightarrow (\Sigma S^n)^+$ *by* $T_+^{-1}(z) = \begin{cases} x_+ & ;\ \text{if } z = 0 \\ [z/|z|\, , 1 - |z|\,] & ;\ \text{if } z \neq 0 \end{cases}$.

- $T_-^{-1} : D^{n+1} \longrightarrow (\Sigma S^n)^-$ *by* $T_-^{-1}(z) = \begin{cases} x_- & ;\ \text{if } z = 0 \\ [z/|z|\, , |z| - 1\,] & ;\ \text{if } z \neq 0 \end{cases}$.

Proof. It is not difficult to show that the mentioned maps are homeomorphisms. We only point that as z approaches 0, the element $[z/|z|\,,1-|z|]$ approaches to x_+ and $[z/|z|\,,|z|-1]$ approaches to x_- although $z/|z|$ diverges. This confirms the continuity of T_+^{-1} and T_-^{-1} at $z=0$. $\square$

Remark 4.3.3. *Since $[x,\mp 1]=x_\mp$ for all $x\in S^n$, without loss of generality we write $T_-^{-1}(z)=[z/|z|\,,|z|-1]$ and $T_+^{-1}(z)=[z/|z|\,,1-|z|]$ for all $z\in D^n$.*

Consider the figure

$$\Sigma S^n = \left\{ \begin{array}{c} (\Sigma S^n)^+ \xrightarrow{T_+} D^{n+1} \xrightarrow{\pi_+^{-1}} (S^{n+1})^+ \\ \\ (\Sigma S^n)^- \xrightarrow{T_-} D^{n+1} \xrightarrow{\pi_-^{-1}} (S^{n+1})^- \end{array} \right\} = S^{n+1} .$$

The following proposition gives an explicit formula for the homeomorphism between ΣS^n and S^{n+1} and its inverse.

Proposition 4.3.4. *Let $h:\Sigma S^n \longrightarrow S^{n+1}$ be a map defined by $h|\,(\Sigma S^n)^+=\pi_+\circ T_+$ and $h|\,(\Sigma S^n)^-=\pi_-\circ T_-$. Then, h is a homeomorphism and it is given explicitly by*

$$h[z,t]=\Big((1-|t|)z,\, sgn(t)\cdot\sqrt{1-(1-|t|)^2}\Big)$$

for all $[z,t]\in\Sigma S^n$, and h^{-1} is given by

$$h^{-1}(z,t)=\begin{cases} x_{sgn(t)} & ;\ if\ \ t=\mp 1 \\ \\ \left[z/|z|\,,sgn(t)\cdot\dfrac{t^2}{1+|z|}\right] & ;\ if\ \ -1<t<1 \end{cases},$$

or by Remark 4.3.3

$$h^{-1}(z,t) = \left[z/|z|\,,sgn(t) \cdot \frac{t^2}{1+|z|} \right]$$

for all $(z,t) \in S^{n+1}$.

Proof. It is not difficult to prove our proposition if we recall the following properties of $sgn(t)$:

1. $(sgn(t))^2 = 1$ for every $t \in \mathbf{R} - \{0\}$.

2. $sgn(t) \cdot |t| = t$ for every $t \in \mathbf{R}$.

3. $sgn\,(r(t).sgn(t)) = sgn(t)$ for every $t \in \mathbf{R}$, where r is a nonnegative real valued function such that $r(t) = 0$ if and only if $t = 0$.

□

Now we apply Proposition 4.3.4 to create maps on S^n of any degree we wish. We start with the following proposition.

Proposition 4.3.5. *The map* $g : S^{n+1} \longrightarrow S^{n+1}$ *given by*

$$g(z,\eta) = \begin{cases} (z^k/|z|^{k-1}, \eta) & ; \textit{if } \; z \in \mathbf{C},\ \eta \in \mathbf{R}^{n-1},\ |z|^2 + |\eta|^2 = 1, \;\; \textit{and} \;\; z \neq 0 \\ (0,\eta) & ; \textit{if } \; z = 0 \end{cases},$$

or $g(|z|\,e^{i\theta}, \eta) = (|z|\,e^{ik\theta}, \eta)$ *for every* $(z,\eta) \in S^{n+1}$ *has the degree* k*, where* $k \in \mathbf{Z}$.

Proof. Let $f : S^1 \longrightarrow S^1$ be the map defined by $f(z) = z^k$. Then, $deg(f) = k$. Since the suspension map $\Sigma^n f = \underbrace{\Sigma \ldots \Sigma}_{n\text{-times}} f$ preserves the degree, we need only to show that $h \circ \Sigma^n f \circ h^{-1} = g$ where h the homeomorphism given in Proposition 4.3.4; this is done by the induction on n.

For $n = 1$, we have the map $\Sigma f : \Sigma S^1 \longrightarrow \Sigma S^1$ is given by

$$\Sigma f\,[z,t] = \begin{cases} \left[z^k/|z|^{k-1}, t\right] & ; \text{if } z \neq 0 \\ [0,t] & ; \text{if } z = 0 \ (t = \mp 1) \end{cases} .$$

Let $h : \Sigma S^1 \longrightarrow S^2$ be the homeomorphism given in Proposition 4.3.4 and $(z,t) \in S^2$ such that $z \neq 0$. Then,

$$h \circ \Sigma f \circ h^{-1}(z,t) = h \circ \Sigma f \left[z/|z|,\ sgn(t) \cdot \frac{t^2}{1+|z|}\right] = h\left[z^k/|z|^k,\ sgn(t) \cdot \frac{t^2}{1+|z|}\right]$$

$$= \left((1 - \frac{t^2}{1+|z|})\, z^k/|z|^k,\ sgn\left(sgn(t) \cdot \frac{t^2}{1+|z|}\right) \cdot \sqrt{1 - (1 - \frac{t^2}{1+|z|})^2}\right)$$

$$= \left(z^k/|z|^{k-1}, t\right).$$

If $z = 0$, then $t = \mp 1$. Hence,

$$h \circ \Sigma f \circ h^{-1}(0, \mp 1) = h \circ \Sigma f(x_{\mp}) = h \circ \Sigma f\,[0, \mp 1] = h\,[0, \mp 1] = h(x_{\mp}) = (0, \mp 1)\ .$$

Consequently, $g = h \circ \Sigma f \circ h^{-1}$ which implies that $g_{*,2} = h_{*,2} \circ \Sigma f_{*,2} \circ h^{-1}_{*,2}$ where $g_{*,2}$, $h_{*,2}$, $\Sigma f_{*,2}$ and $h^{-1}_{*,2}$ are the homomorphisms induced by the corresponding maps on the suitable homological groups. Let ξ and ς be the generators of the homological groups $H_2(S^2)$ and $H_2(\Sigma S^1)$ respectively. Hence,

$$\begin{aligned} g_{*,2}(\xi) &= h_{*,2} \circ \Sigma f_{*,2} \circ h_{*,2}^{-1}(\xi) = h_{*,2} \circ \Sigma f_{*,2}((-1)^l \varsigma) \\ &= h_{*,2}(k\,(-1)^l \varsigma) = (-1)^l\, k\,(-1)^l\, \xi = k\,\xi \end{aligned}$$

where $l = \mp 1$. This means that $deg(g) = k$.

Suppose our claim is true for $n-1$, i.e., the map $f : S^n \longrightarrow S^n$ given by

$$f(z, \acute{\eta}) = \begin{cases} (z^k/|z|^{k-1}, \acute{\eta}) & ; \text{if } z \neq 0 \\ (0, \acute{\eta}) & ; \text{if } z = 0 \end{cases},$$

for every $(z, \acute{\eta}) \in S^n$, where $z \in \mathbf{C}$ and $\acute{\eta} \in \mathbf{R}^{n-1}$, has a degree k. Let $(\mu, t) \in S^{n+1}$ (use Notation 4.3.1). Write $\mu = (z, \acute{\eta})$ where $z \in \mathbf{C}$ and $\acute{\eta} \in \mathbf{R}^{n-1}$. Assume firstly that $\mu \neq 0$. Then $\mu/|\mu| \in S^n$, $\mu/|\mu| = (z/|\mu|, \acute{\eta}/|\mu|)$, and $|\mu|^2 = |z|^2 + |\acute{\eta}|^2$. We have two cases:

- Case 1: If $z \neq 0$. Then,

$$h \circ \Sigma f \circ h^{-1}(\mu, t) = h \circ \Sigma f \left[(z/|\mu|, \acute{\eta}/|\mu|),\ sgn(t) \cdot \frac{t^2}{1+|\mu|}\right]$$

$$= h \left[f(z/|\mu|, \acute{\eta}/|\mu|),\ sgn(t) \cdot \frac{t^2}{1+|\mu|}\right]$$

$$= h \left[(\frac{z^k/|\mu|^k}{|z/|\mu||^{k-1}}, \acute{\eta}/|\mu|),\ sgn(t) \cdot \frac{t^2}{1+|\mu|}\right]$$

$$= h \left[(\frac{z^k}{|\mu| \cdot |z|^{k-1}}, \acute{\eta}/|\mu|),\ sgn(t) \cdot \frac{t^2}{1+|\mu|}\right]$$

$$= h \left[\frac{1}{|\mu|}(\frac{z^k}{|z|^{k-1}}, \acute{\eta}),\ sgn(t) \cdot \frac{t^2}{1+|\mu|}\right]$$

$$= \left(\left(1 - \left|sgn(t) \cdot \frac{t^2}{1+|\mu|}\right|\right) \frac{1}{|\mu|}(\frac{z^k}{|z|^{k-1}}, \acute{\eta}),\ sgn\left(sgn(t) \cdot \frac{t^2}{1+|\mu|}\right) \cdot\right.$$

$$\sqrt{1-\left(1-\left|sgn(t)\cdot\frac{t^2}{1+|\mu|}\right|\right)^2}\Bigg) = \left((\frac{z^k}{|z|^{k-1}}, \acute{\eta}), t\right) = \left(\frac{z^k}{|z|^{k-1}}, \eta\right),$$

where without loss of generality, $\eta = (\acute{\eta}, t) \in \mathbf{R}^n$.

- Case 2: If $z = 0$. Then,

$h \circ \Sigma f \circ h^{-1}(\mu, t) = h\left[f(0, \acute{\eta}/|\mu|), sgn(t)\cdot\frac{t^2}{1+|\mu|}\right]$

$$= h\left[(0, \acute{\eta}/|\mu|), \underbrace{sgn(t)\cdot\frac{t^2}{1+|\mu|}}_{\lambda(t)}\right]$$

$= \left((1-|\lambda(t)|)\,(0, \acute{\eta}/|\mu|), sgn\,(\lambda(t))\cdot\sqrt{1-(1-|\lambda(t)|)^2}\right)$
$= ((0, \acute{\eta}), t) = (0, \eta)$.

Secondly, we assume $\mu = 0$. Then, $z = 0$, $\acute{\eta} = 0$, and $t = \mp 1$. Hence, $h \circ \Sigma f \circ h^{-1}(0, \mp 1) = h \circ \Sigma f(x_{\mp}) = h \circ \Sigma f[y, \mp 1] = h[f(y), \mp 1] = (0, \mp 1)$, where $y \in S^n$.

Consequently, we get that $h \circ \Sigma f \circ h^{-1} = g$. Moreover, since $deg(\Sigma f) = deg(f) = k$, a similar argument shows that $deg(g) = k$.

Finally, by mathematical induction, our result holds for each positive integer n. □

Corollary 4.3.6. *Let $f^{(k,l)} : S^3 \longrightarrow S^3$ be the map defined by*

$$f^{(k,l)}(r\, e^{i\theta}, \rho\, e^{i\varphi}) = (r\, e^{i\,k\,\theta}, \rho\, e^{i\,l\,\varphi})\,.$$

Then, $deg(f^{(k,l)}) = k\,l$.

Proof. Firstly, we show that $f^{(1,l)}$ is of degree l. Let $q : S^3 \longrightarrow S^3$ be the map defined by $q(z_1, z_2) = (z_2, z_1)$. We have $q \circ f^{(1,l)} = f^{(l,1)}$. Thus, $deg(q) \cdot deg(f^{(1,l)}) = deg(q \circ f^{(1,l)}) = deg(f^{(l,1)})$. Since $deg(f^{(l,1)}) = l$ by Proposition 4.3.5 and $deg(q) = 1$, we get $deg(f^{(1,l)}) = l$.

Secondly, $f^{(k,l)} = f^{(k,1)} \circ f^{(1,l)}$. Hence, $deg(f^{(k,l)}) = deg(f^{(k,1)} \circ f^{(1,l)}) = deg(f^{(k,1)}) \cdot deg(f^{(1,l)}) = k\,l$. □

The following proposition is useful for computing the Lefschetz number of a pair of maps on the sphere S^n:

Proposition 4.3.7. *Let $(f, g) : S^n \longrightarrow S^n$ be a pair of maps. Then, the Lefschetz number of f and g is given by*

$$L(f, g) = deg(g) + (-1)^n\, deg(f)\ .$$

Proof. We have

$$H_i(S^n) \cong \begin{cases} \mathbf{Z}, & \text{if } i = 0, n\,; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D_i : H^i(S^n) \longrightarrow H_{n-i}(S^n)$ be the Poincare′ isomorphism. Let a_i and a^i be the generators of $H_i(S^n)$ and $H^i(S^n)$ respectively such that $D_i(a^i) = a_{n-i}$ for each $i = 0, n$. Consider the sequence

$$H_i(S^n) \xrightarrow{f_*} H_i(S^n) \xrightarrow{D_{n-i}^{-1}} H^{n-i}(S^n) \xrightarrow{g^*} H^{n-i}(S^n) \xrightarrow{D_{n-i}} H_i(S^n)$$

Denote the composition $D_{n-i} \circ g^* \circ D_{n-i}^{-1} \circ f_*$ by Θ_i. By the definition of the Lefschetz number of a pair of maps, we can easily see that

$$L(f,g) = tr(\Theta_0) + (-1)^n \, tr(\Theta_n) \, ,$$

where $tr(\Theta_i)$ is the trace of the linear isomorphism Θ_i. Now,

$$\begin{aligned}
\Theta_0(a_0) &= D_n \, g^* \, D_n^{-1} \, f_* \, (a_0) \\
&= D_n \, g^* \, D_n^{-1} \, (a_0) \\
&= D_n \, g^* \, (a^n) \\
&= D_n \, (deg(g) \, a^n) \\
&= deg(g) \, D_n \, (a^n) \\
&= deg(g) \, a_0
\end{aligned}$$

Thus, $tr(\Theta_0) = deg(g)$. Similarly, we get that $tr(\Theta_n) = deg(f)$. Substituting these traces in the above formula we obtain the required result. □

We are now ready to give the examples that show the usefulness of our results in the previous work.

Example 4.3.8. As in Example 3.4.14, define the maps $f,\, g : L(5,1) \longrightarrow L(5,1)$ by $f\left[\rho\, e^{i\theta}, z\right] = \left[\rho\, e^{i6\theta}, z\right]$ and $g[z_1,\, z_2] = [z_2,\, z_1]$. Both maps are well-defined, differ from the identity map and admit lifts $\widetilde{f}$ and $\widetilde{g}$ defined in the natural way. In addition, $\widetilde{f}$ and $\widetilde{g}$ are equivariant under the action of $\mathcal{A}(S^3) \cong \mathbf{Z}_5$ on S^3. Since $\omega^t \cdot \widetilde{g}$ is homotopic to $\widetilde{g}$ for all t, we have $L(\widetilde{f},\, \omega^t \cdot \widetilde{g}) \neq 0$ for all $0 \leq t \leq 4$. By Theorem 3.4.8, $N_{ED}(f,g) = 0$ and $N(f,g) = 5$.

- It can be shown that

$$\Phi(\widetilde{f},\, \omega^t \cdot \widetilde{g}) = \left\{ \left(\frac{\omega^k}{\sqrt{2}}\, e^{\frac{i4\pi t}{25}},\, \frac{\omega^{k+t}}{\sqrt{2}}\, e^{\frac{i4\pi t}{25}} \right) \mid k = 0,\, 1,\, 2,\, 3,\, 4 \right\}$$

for all $0 \leq t \leq 4$. Hence, $|\Phi(\widetilde{f},\, \omega^t \cdot \widetilde{g})| = 5$ and $|\Phi(f, g)| = 5 \times 5 = 25$.

- Notice that, although $|\mathcal{A}(S^3)| = 5$, which is prime, and $\delta(f, g) = 1$ since $[f : \omega] = \omega = [g : \omega]$, we cannot apply Theorem 4.2.16 to compute $N(f, g)$ in this example because the Nielsen classes $\Phi(\widetilde{f}, \widetilde{g})$ and $\omega \cdot \Phi(\widetilde{f}, \widetilde{g})$ are not disjoint.

The following example shows that our formulas sometimes give the best estimation of the minimum number of coincidence points in the homotopy classes of the considered maps.

Example 4.3.9. We use the same spaces given in Example 3.4.14. Define the maps f and g by $f\left[\rho_1\, e^{i\theta_1},\, \rho_2\, e^{i\theta_2}\right] = \left[\rho_1\, e^{i4\theta_1},\, \rho_2\, e^{i4\theta_2}\right]$ and $g[z_1,\, z_2] = [z_2,\, z_1]$ respectively. Both f and g are well-defined smooth maps which admit the lifts $\widetilde{f}$ and $\widetilde{g}$ defined in the natural way.

- We have $\delta(f, g) = 0$ since $\widetilde{g}\,(\omega\,(z_1, z_2)) = \omega\,\widetilde{g}(z_1, z_2)$ and $\widetilde{f}\,(\omega\,(z_1, z_2)) = \omega^4\,\widetilde{f}(z_1, z_2)$, or equivalently since $[f : \omega] = \omega^4 \neq \omega = [g : \omega]$. Thus, by Theorem 4.2.18 we have $N(f, g) = N(\widetilde{f}, \widetilde{g})$.

- Since $deg(\widetilde{f}) = 16$ and $deg(\widetilde{g}) = 1$, we get that $L(\widetilde{f},\, \widetilde{g}) = 1 - 16 = -15 \neq 0$. Hence, $N(f, g) = 1$.

- Notice that $\Phi(\widetilde{f},\, \widetilde{g}) = \left\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \right\}$. Thus, $|\Phi(f, g)| = 1 = N(f, g)$.

The next example shows that our method might not be good enough to estimate the number of coincidence points of the maps under consideration.

Example 4.3.10. We use the same spaces given in Example 3.4.14. Define the maps f and g by $f\,[z_1,\, z_2] = [\overline{z}_1,\, \overline{z}_2]$ and $g[z_1,\, z_2] = [z_2,\, z_1]$ respectively. Both f and g are well-defined smooth maps which admit lifts $\widetilde{f}$ and $\widetilde{g}$ defined in the natural way.

- We have $\delta(f, g) = 0$ since $\widetilde{g}\,(\omega\,(z_1, z_2)) = \omega\,\widetilde{g}(z_1, z_2)$ and $\widetilde{f}\,(\omega\,(z_1, z_2)) = \overline{\omega}\,\widetilde{f}(z_1, z_2)$. Thus, by Theorem 4.2.18 we have $N(f, g) = N(\widetilde{f}, \widetilde{g})$.
- Since $deg(\widetilde{f}) = deg(\widetilde{g}) = 1$, we get that $L(\widetilde{f},\, \widetilde{g}) = 0$. Hence, $N(f, g) = 0$.
- Notice that $\Phi(\widetilde{f},\, \widetilde{g}) = \left\{(z,\, \overline{z}) \mid |z| = \dfrac{1}{\sqrt{2}}\right\}$. Thus, $|\Phi(f, g)| = \infty$.

The following Lemma is useful in finding the $H-$Reidemeister representatives. We will use it in later examples.

Lemma 4.3.11. *Assume the hypotheses given in the beginning of Section 4.1. Let* $\beta \in \mathcal{A}(\widetilde{N})$. *Then,* $(\widetilde{f}, \widetilde{g})$ *and* $(\widetilde{f}, \beta\,\widetilde{g})$ *belong to the same* $H-$*Reidemeister class if and only if there exist* $\alpha \in \mathcal{A}(\widetilde{M})$ *and* $\acute{\beta} \in \mathcal{A}(\widetilde{N})$ *such that* $(\widetilde{f}, \widetilde{g})\,\alpha = \acute{\beta}\,(\widetilde{f}, \beta\,\widetilde{g})$. *Moreover, fixing* α, *such a* $\acute{\beta}$ *is unique.*

Proof. The proof is quite direct from definition of conjugate lifts. □

The following example includes covering spaces with covering transformation groups of non-prime cardinality. Of course, since this is a more general case, more work is required to compute the Nielsen numbers. Although the spaces involved are orientable and we could calculate the Nielsen number by easier methods, we use our method to demonstrate the advantages of our methods.

Example 4.3.12. Let $f, g : S^1 \longrightarrow S^1$ be maps defined by $f(z) = z^3$ and $g(z) = z^6$ for every $z \in S^1$. Let $p\,, \acute{p} : S^1 \longrightarrow S^1$ be the covering maps defined by $p\,(z) = z^4$ and

$\acute{p}(z) = z^6$. The maps f and g admit lifts $\widetilde{f}$ and $\widetilde{g}$ on S^1 defined by $\widetilde{f}(z) = z^2$ and $\widetilde{g}(z) = z^4$ respectively, where $z \in S^1$. We have the commutative diagram 2.4.1. Let ω be the 4th primitive root of unity and μ be the 6th primitive root of unity. Then, $\mathcal{A}(S^1, p) = \langle \omega \rangle$ and $\mathcal{A}(S^1, \acute{p}) = \langle \mu \rangle$. Since S^1 is a Jiang space, the number J is fixed for all $H-$Nielsen classes. By Theorem 2.3.5

$$N_L(f,g) = \sum \frac{N(\widetilde{f},\widetilde{g})}{S(\widetilde{f},\widetilde{g})},$$

where the sum runs over all $H-$Reidemeister representatives. The next step is to chose the $H-$Nielsen classes representatives from among $(\widetilde{f},\widetilde{g}), (\widetilde{f},\mu\,\widetilde{g}), \ldots, (\widetilde{f},\mu^5\,\widetilde{g})$ (some of them may lie in the same $H-$Reidemeister class). We use Lemma 4.3.11.

• We start with the pair $(\widetilde{f},\widetilde{g})$: applying the action of $\mathcal{A}(S^1,p)$ from the right on this pair leads us to the following:

- Assume $(\widetilde{f},\widetilde{g})\,\omega = \mu^k\,(\widetilde{f},\mu^j\,\widetilde{g})$ for some j and k with $0 \le k, j \le 5$. Then, for every $z \in S^1$ we have

$$\begin{aligned}
(\widetilde{f},\widetilde{g})\,\omega\,(z) = \mu^k\,(\widetilde{f},\mu^j\,\widetilde{g})\,(z) \quad &\Rightarrow \quad (\omega^2\,z^2, \omega^4\,z^4) = (\mu^k\,z^2, \mu^{k+j}\,z^4)\\
&\Rightarrow \quad \mu^k = \omega^2 \quad \text{and} \quad \mu^{k+j} = 1 \quad \text{(for example put } z = 1)\\
&\Rightarrow \quad \mu^k = \omega^2 \quad \text{and} \quad k + j = 0 \quad (mod\,6)\\
&\Rightarrow \quad k = j = 3.
\end{aligned}$$

The uniqueness in Lemma 4.3.11 guarantees that such k and j are unique.

- Similarly, we have $(\widetilde{f},\widetilde{g})\,\omega^2 = (\widetilde{f},\widetilde{g})$ and $(\widetilde{f},\widetilde{g})\,\omega^3 = (\widetilde{f},\widetilde{g})\,\omega = \mu^3\,(\widetilde{f},\mu^3\widetilde{g})$ as shown before. Thus, we deduce that $\underline{(\widetilde{f},\widetilde{g}) \text{ and } (\widetilde{f},\mu^3\,\widetilde{g})}$ belong to the same $H-$Nielsen class.

• A similar argument applied to $(\widetilde{f},\mu\,\widetilde{g})$ and $(\widetilde{f},\mu^2\,\widetilde{g})$ leads us to deduce that $\underline{(\widetilde{f},\mu\,\widetilde{g})}$

and $(\widetilde{f}, \mu^4 \widetilde{g})$ belong to the same $H-$Reidemeister class and also that $(\widetilde{f}, \mu^2 \widetilde{g})$ and $(\widetilde{f}, \mu^5 \widetilde{g})$ belong to the same $H-$Reidemeister class, and there are no other equivalences.

We chose $(\widetilde{f}, \widetilde{g})$, $(\widetilde{f}, \mu \widetilde{g})$, and $(\widetilde{f}, \mu^2 \widetilde{g})$ as our Reidemeister representatives. Since $\pi_1(S^1, 1)$ is abelian, we get by Proposition 2.1.25 that

$$N(f,g) = \sum_{i=0}^{i=2} \frac{N(\widetilde{f}, \mu^i \widetilde{g})}{S(\widetilde{f}, \mu^i \widetilde{g})} = 3 \times \frac{N(\widetilde{f}, \widetilde{g})}{S(\widetilde{f}, \widetilde{g})}\,.$$

So, now we want to compute both $S(\widetilde{f}, \widetilde{g})$ and $N(\widetilde{f}, \widetilde{g})$. Since S^1 is a Jiang space and $deg\left(\mu^k \widetilde{g}\right) = deg(\widetilde{g})$ for all k with $0 \leq k \leq 5$, then $\mu^k \widetilde{g}$ is homotopic to $\widetilde{g}$ and hence $L(\widetilde{f}, \mu^k \widetilde{g}) = L(\widetilde{f}, \widetilde{g}) = 4 - 2 = 2 \neq 0$ for all k with $0 \leq k \leq 5$. Thus, we get that $N(\widetilde{f}, \widetilde{g}) = \left|deg(\widetilde{g}) - deg(\widetilde{f})\right| = |4-2| = 2$. On the other hand, $I(\widetilde{f}, \widetilde{g}) = \left|\mathrm{L}(\widetilde{f}, \widetilde{g})\right|$ where $\mathrm{L}(\widetilde{f}, \widetilde{g}) = \left\{\alpha \in \mathcal{A}(S^1, p) |\ \delta(\widetilde{f}, \widetilde{g}; \alpha) = 1\right\}$. The work done before while searching for the representatives shows that $\left[\widetilde{f} : \omega\right] = \mu^3 \neq 1 = [\widetilde{g} : \omega]$ and $\left[\widetilde{f} : \omega^2\right] = 1 = [\widetilde{g} : \omega^2]$. Since $\omega \notin \mathrm{L}(\widetilde{f}, \widetilde{g})$, we have that $\omega^3 = \omega^{-1} \notin \mathrm{L}(\widetilde{f}, \widetilde{g})$ either (recall that $\mathrm{L}(\widetilde{f}, \widetilde{g})$ is a subgroup of $\mathcal{A}(S^1, p)$). Thus, we get that $I(\widetilde{f}, \widetilde{g}) = 2$. Next, let $z = 1 \in \Phi(\widetilde{f}, \widetilde{g})$, then $p\,(1) = 1 \in \Phi(f, g)$. We know that $J(\widetilde{f}, \widetilde{g}) = |j\,(C(f_\#, g_\#)_{z=1})|$, but as in the previous example $C(f_\#, g_\#)_1 = Ker(g_\# - f_\#) = 0$ since $g_\# - f_\#$ is a monomorphism. Hence, $J(\widetilde{f}, \widetilde{g}) = 1$. Finally,

$$S(\widetilde{f}, \widetilde{g}) = \frac{I(\widetilde{f}, \widetilde{g})}{J(\widetilde{f}, \widetilde{g})} = \frac{2}{1} = 2\,,$$

and hence $N(f,g) = \dfrac{3 \times 2}{2} = 3$. This, of course, is the same result obtained if we used the well known computation methods applied to the Jiang spaces.

The next example is one of the main examples in this chapter. We start with some lemmas from abstract algebra.

Lemma 4.3.13. *Let G_1 and G_2 be abelian groups, H_i and K_i be subgroups of G_i, and $\varphi_i, \psi_i : G_i \longrightarrow G_i$ be homomorphisms for $i = 1, 2$. Then,*

1. *We have $\dfrac{G_1 \oplus G_2}{H_1 \oplus H_2} \cong \dfrac{G_1}{H_1} \oplus \dfrac{G_2}{H_2}$.*
2. *Define $\varphi_1 \times \varphi_2, \psi_1 \times \psi_2 : G_1 \oplus G_2 \longrightarrow G_1 \oplus G_2$ by $\varphi_1 \times \varphi_2(g_1, g_2) = (\varphi_1(g_1), \varphi_2(g_2))$ and $\psi_1 \times \psi_2(g_1, g_2) = (\psi_1(g_1), \psi_2(g_2))$ respectively, where $g_i \in G_i$ for $i = 1, 2$. Then,*
$$C(\varphi_1 \times \varphi_2, \psi_1 \times \psi_2) = C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2) .$$
3. *Let $j_i : G_i \longrightarrow \dfrac{G_i}{H_i}$ and $j : G_1 \oplus G_2 \longrightarrow \dfrac{G_1 \oplus G_2}{H_1 \oplus H_2}$ be the natural homomorphisms, where $i = 1, 2$ and $j_1 \times j_2 : G_1 \oplus G_2 \longrightarrow \dfrac{G_1}{H_1} \oplus \dfrac{G_2}{H_2}$ is the homomorphism defined by $j_1 \times j_2(g_1, g_2) = (j_1(g_1), j_2(g_2))$ where $g_i \in G_i$ and $i = 1, 2$. Then,*
$$j\left(C(\varphi_1 \times \varphi_2, \psi_1 \times \psi_2)\right) \cong j_1(C(\varphi_1, \psi_1)) \oplus j_2(C(\varphi_2, \psi_2)) .$$
4. *Assume $\varphi_i(K_i) \cup \psi_i(K_i) \subseteq H_i$ for $i = 1, 2$. Let $\overline{\varphi}_i, \overline{\psi}_i : \dfrac{G_i}{K_i} \longrightarrow \dfrac{G_i}{H_i}$ be the homomorphisms induced by φ_i and ψ_i in the natural way respectively, for $i = 1, 2$. Then, $\varphi_1 \times \varphi_2$ and $\psi_1 \times \psi_2$ induce homomorphisms $\overline{\varphi_1 \times \varphi_2}$ and $\overline{\psi_1 \times \psi_2}$ from $\dfrac{G_1 \oplus G_2}{K_1 \oplus K_2}$ to $\dfrac{G_1 \oplus G_2}{H_1 \oplus H_2}$ defined in the natural way, and*
$$C(\overline{\varphi_1 \times \varphi_2}, \overline{\psi_1 \times \psi_2}) \cong C(\overline{\varphi}_1 \times \overline{\varphi}_2, \overline{\psi}_1 \times \overline{\psi}_2) = C(\overline{\varphi}_1, \overline{\psi}_1) \oplus C(\overline{\varphi}_2, \overline{\psi}_2) .$$

Proof. 1. It is not difficult to see that $j_1 \times j_2$ is an epimorphism and $ker\,(j_1 \times j_2) = H_1 \oplus H_2$. The rest follows by the First Isomorphism Theorem.

2. Let $(g_1, g_2) \in G_1 \oplus G_2$. Then,

$(g_1, g_2) \in C(\varphi_1 \times \varphi_2, \psi_1 \times \psi_2) \Leftrightarrow \varphi_1 \times \varphi_2(g_1, g_2) = \psi_1 \times \psi_2(g_1, g_2)$

$\Leftrightarrow (\varphi_1(g_1), \varphi_2(g_2)) = (\psi_1(g_1), \psi_2(g_2))$

$\Leftrightarrow \varphi_1(g_1) = \psi_1(g_1) \quad \text{and} \quad \varphi_2(g_2) = \psi_2(g_2)$

$\Leftrightarrow g_1 \in C(\varphi_1, \psi_1) \quad \text{and} \quad g_2 \in C(\varphi_2, \psi_2)$

$\Leftrightarrow (g_1, g_2) \in C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2)$.

Thus, 2 follows.

3. It is easy to show that the following diagram commutes:

$$\begin{tikzcd} G_1 \oplus G_2 \arrow[rr, "j"] \arrow[dr, "j_1 \times j_2"'] & & \dfrac{G_1 \oplus G_2}{H_1 \oplus H_2} \arrow[dl, "\epsilon"] \\ & \dfrac{G_1}{H_1} \oplus \dfrac{G_2}{H_2} & \end{tikzcd}$$

where ϵ is the isomorphism induced by $j_1 \times j_2$. By part 2, we have

$j\,(C(\varphi_1 \times \varphi_2, \psi_1 \times \psi_2)) = j\,(C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2)) \cong \epsilon\,(j\,(C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2)))$

$= \epsilon \circ j\,(C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2)) = j_1 \times j_2\,(C(\varphi_1, \psi_1) \oplus C(\varphi_2, \psi_2))$

$= j_1\,(C(\varphi_1, \psi_1)) \oplus j_2\,(C(\varphi_2, \psi_2))$,

which means that part 3 is true.

4. The equality $C(\overline{\varphi}_1 \times \overline{\varphi}_2, \overline{\psi}_1 \times \overline{\psi}_2) = C(\overline{\varphi}_1, \overline{\psi}_1) \oplus C(\overline{\varphi}_2, \overline{\psi}_2)$ is proved by applying part 2 to the correspondent groups, subgroups and homomorphisms. On the other hand, $C(\overline{\varphi_1 \times \varphi_2}, \overline{\psi_1 \times \psi_2}) \cong C(\overline{\varphi}_1 \times \overline{\varphi}_2, \overline{\psi}_1 \times \overline{\psi}_2)$ holds by the commutativity of the

diagram:

$$\begin{array}{ccc}
\dfrac{G_1 \oplus G_2}{K_1 \oplus K_2} & \xrightarrow{\overline{\varphi_1 \times \varphi_2},\, \overline{\psi_1 \times \psi_2}} & \dfrac{G_1 \oplus G_2}{H_1 \oplus H_2} \\
\epsilon \downarrow & & \downarrow \epsilon \\
\dfrac{G_1}{K_1} \oplus \dfrac{G_2}{K_2} & \xrightarrow{\overline{\varphi}_1 \times \overline{\varphi}_2,\, \overline{\psi}_1 \times \overline{\psi}_2} & \dfrac{G_1}{H_1} \oplus \dfrac{G_2}{H_2}
\end{array}$$

□

For orientable manifolds, we recall the following results.

Proposition 4.3.14. *[26] A compact connected $n-$manifold M without boundary is orientable if and only if $H_n(M)$ is isomorphic to $\mathbf{Z}$ (the group of integers).* □

Corollary 4.3.15. *The real projective plane $\mathbf{RP^2}$ is not orientable, but the Lens space $L(5,1)$ is orientable.*

Proof. The proof follows directly from Proposition 4.3.14, since $H_2(\mathbf{RP^2})$ is isomorphic to $\mathbf{Z_2}$, and $H_3(L(5,1))$ is isomorphic to $\mathbf{Z}$. □

Proposition 4.3.16. *[9] The product of two manifolds is orientable if and only if each of them is orientable.* □

Example 4.3.17. Let $f_1, g_1 : L(5,1) \longrightarrow L(5,1)$ and $f_2, g_2 : S^1 \longrightarrow S^1$ be maps defined by $f_1\left[r_1\, e^{i\theta_1}, r_2\, e^{i\theta_2}\right] = \left[r_1\, e^{i6\theta_1}, r_2\, e^{i\theta_2}\right]$, $g_1\left[z_1, z_2\right] = \left[z_1, z_2\right]$, $f_2(e^{i\varphi}) = e^{i6\varphi}$, and $g_2(e^{i\varphi}) = e^{i3\varphi}$. Let $p, \acute{p} : S^1 \longrightarrow S^1$ be the covering maps defined by $p(z) = z^2$ and $\acute{p}(z) = z^3$ respectively, and $q : S^3 \longrightarrow L(5,1)$ be the quotient map that defines the lens space. Define $f, g : L(5,1) \times S^1 \longrightarrow L(5,1) \times S^1$ by $f = f_1 \times f_2$ and $g = g_1 \times g_2$. We have that $q \times p\, , q \times \acute{p} : S^3 \times S^1 \longrightarrow L(5,1) \times S^1$ are covering maps. Both f and g admit lifts $\widetilde{f} = \widetilde{f_1} \times \widetilde{f_2}$ and $\widetilde{g} = \widetilde{g}_1 \times \widetilde{g}_2$ where $\widetilde{f_1}\left(r_1\, e^{i\theta_1}, r_2\, e^{i\theta_2}\right) = \left(r_1\, e^{i6\theta_1}, r_2\, e^{i\theta_2}\right)$, $\widetilde{f_2}(z) = z^4$, $\widetilde{g}_1 = 1_{S^3}$, and $\widetilde{g}_2(z) = z^2$, for $z \in S^1$. Consider the commutative diagrams

$$\begin{array}{ccc} S^3 & \xrightarrow{\widetilde{f_1}, \widetilde{g}_1} & S^3 \\ q \downarrow & & \downarrow q \\ L(5,1) & \xrightarrow{f_1, g_1} & L(5,1) \end{array} \qquad \begin{array}{ccc} S^1 & \xrightarrow{\widetilde{f_2}, \widetilde{g}_2} & S^1 \\ \downarrow p & & \acute{p} \downarrow \\ S^1 & \xrightarrow{f_2, g_2} & S^1 \end{array} \qquad \begin{array}{ccc} S^3 \times S^1 & \xrightarrow{\widetilde{f}, \widetilde{g}} & S^3 \times S^1 \\ q \times p \downarrow & & \downarrow q \times \acute{p} \\ L(5,1) \times S^1 & \xrightarrow{f, g} & L(5,1) \times S^1 \end{array}$$

Notice that the space $L(5,1) \times S^1$ is a orientable connected closed smooth manifold and all maps considered are smooth. Moreover, the coverings are regular since the fundamental groups of $L(5,1)$, S^1 and $L(5,1) \times S^1$ are abelian. Our goal is to compute $N(f, g)$.

• By Lemma 4.3.13, $\mathcal{A}(S^3 \times S^1, q \times p) \cong \mathcal{A}(S^3, q) \oplus \mathcal{A}(S^1, p) \cong \mathbf{Z_5} \oplus \mathbf{Z_2} \cong \mathbf{Z_{10}}$. Similarly, $\mathcal{A}(S^3 \times S^1, q \times \acute{p}) \cong \mathcal{A}(S^3, q) \oplus \mathcal{A}(S^1, \acute{p}) \cong \mathbf{Z_5} \oplus \mathbf{Z_3} \cong \mathbf{Z_{15}}$. Let ω, λ, and μ be the 5th, the 3rd, and the square primitive roots of unity respectively. Then, we can write that $\mathcal{A}(S^3 \times S^1, q \times p) = \langle(\omega, \mu)\rangle$ and $\mathcal{A}(S^3 \times S^1, q \times \acute{p}) = \langle(\omega, \lambda)\rangle$. We choose the $H-$Reidemeister representatives among the members of the set $\left\{\left(\widetilde{f}, \left(\omega^l, \lambda^t\right) \widetilde{g}\right) \mid 0 \le l \le 4 \text{ and } 0 \le t \le 2\right\}$. Let $0 \le k, l \le 4$ and $0 \le t \le 2$.

Since $\widetilde{f}_2\,\mu = \widetilde{f}_2$ and $\widetilde{g}_2\,\mu = \widetilde{g}_2$, we have

$$\begin{aligned}
\left(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g}\right)\left(\omega^k, \mu\right) &= \left(\widetilde{f}_1 \times \widetilde{f}_2\left(\omega^k, \mu\right), \left(\omega^l, \lambda^t\right)\widetilde{g}_1 \times \widetilde{g}_2\left(\omega^k, \mu\right)\right) \\
&= \left(\widetilde{f}_1\,\omega^k \times \widetilde{f}_2\,\mu\,, \omega^l\,\widetilde{g}_1\,\omega^k \times \lambda^t\,\widetilde{g}_2\,\mu\right) \\
&= \left(\widetilde{f}_1\,\omega^k \times \widetilde{f}_2\,, \omega^l\,\widetilde{g}_1\,\omega^k \times \lambda^t\,\widetilde{g}_2\right) = \left(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g}\right)\left(\omega^k, 1_{S^1}\right) \\
&= \left(\omega^k\,\widetilde{f}_1 \times \widetilde{f}_2\,, \omega^{l+k}\,\widetilde{g}_1 \times \lambda^t\,\widetilde{g}_2\right) \\
&= \left(\omega^k, 1_{S^1}\right)\left(\widetilde{f}_1 \times \widetilde{f}_2\,, \left(\omega^l, \lambda^t\right)\widetilde{g}_1 \times \widetilde{g}_2\right) \\
&= \left(\omega^k, 1_{S^1}\right)\left(\widetilde{f}\,, \left(\omega^l, \lambda^t\right)\widetilde{g}\right)\,.
\end{aligned}$$

So we must choose 15 $H-$Reidemeister representatives $\left(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g}\right)$ for $0 \le l \le 4$ and $0 \le t \le 2$. The $H-$Reidemeister classes are $\Delta\left(\omega^l, \lambda^t\right)$ (see Definition 4.1.8 and Lemma 4.1.9), where $0 \le l \le 4$ and $0 \le t \le 2$.

• Since The fundamental group of $L(5,1) \times S^1$ is abelian, the number J only depends on the $H-$Nielsen class. Thus,

$$N_L(f,g) = \sum_{l=0}^{4}\sum_{t=0}^{2} \frac{N(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g})}{S(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g})}\,.$$

Since $(\widetilde{f}, \left(\omega^l, \lambda^t\right)\widetilde{g})$ is homotopic to $(\widetilde{f}, \widetilde{g})$ and the fundamental group of $L(5,1) \times S^1$ is abelian, it follows by Proposition 2.1.25 that

$$N_L(f,g) = 15 \times \frac{N(\widetilde{f}, \widetilde{g})}{S(\widetilde{f}, \widetilde{g})}\,. \tag{4.3.1}$$

• To compute $S(\widetilde{f}, \widetilde{g})$, parts (3) and (4) of Lemma 4.3.13 imply that

$$J(\widetilde{f}, \widetilde{g}) = J(\widetilde{f}_1, \widetilde{g}_1) \times J(\widetilde{f}_2, \widetilde{g}_2) = 5 \times 1 = 5\,,$$

and

$$I(\widetilde{f}, \widetilde{g}) = I(\widetilde{f}_1, \widetilde{g}_1) \times I(\widetilde{f}_2, \widetilde{g}_2) = 5 \times 2 = 10\,.$$

Thus,

$$S(\widetilde{f},\widetilde{g}) = \frac{10}{5} = 2\ . \tag{4.3.2}$$

The computation of $N(\widetilde{f},\widetilde{g})$ and $N_{ED}(f,g)$ in this example depends on the Lefschetz number of $(\widetilde{f},\widetilde{g})$ which we compute next. We refer the reader to Chapter 4 of [26] for more information about the tensor product, the external product, and the cap product.

- By the Kunneth Formula for free chain complexes that

$$\begin{aligned}
H_0(S^3\times S^1) &\cong H_0(S^3)\otimes H_0(S^1)\cong \mathbf{Z}\otimes\mathbf{Z}\cong\mathbf{Z}\\
H_1(S^3\times S^1) &\cong H_0(S^3)\otimes H_1(S^1)\cong \mathbf{Z}\otimes\mathbf{Z}\cong\mathbf{Z}\\
H_2(S^3\times S^1) &\cong 0\\
H_3(S^3\times S^1) &\cong H_3(S^3)\otimes H_0(S^1)\cong \mathbf{Z}\otimes\mathbf{Z}\cong\mathbf{Z}\\
H_4(S^3\times S^1) &\cong H_3(S^3)\otimes H_1(S^1)\cong \mathbf{Z}\otimes\mathbf{Z}\cong\mathbf{Z}
\end{aligned}$$

and $H_k(S^3\times S^1)\cong 0$ for all $k\geq 5$. Let a_0, a_1, b_0, and b_3 be the generators of $H_0(S^1)$, $H_1(S^1)$, $H_0(S^3)$, and $H_3(S^3)$ respectively. Notice that a_1 and b_3 are the fundamental classes of S^1 and S^3 respectively. Then, $b_0\times a_0$, $b_0\times a_1$, $b_3\times a_0$, and $b_3\times a_1$ are the generators of $H_0(S^3\times S^1)$, $H_1(S^3\times S^1)$, $H_3(S^3\times S^1)$, and $H_4(S^3\times S^1)$ respectively, where $\times$ here means the external product of the two homology classes. Moreover, up to sign, $b_3\times a_1$ is the fundamental class of $S^3\times S^1$. Thus, we write $(-1)^{\acute{t}}b_3\times a_1$, where $\acute{t}=0$ or $\acute{t}=1$, as the fundamental class of $S^3\times S^1$. On the other hand, let $D_k(S^1): H^k(S^1)\longrightarrow H_{1-k}(S^1)$, $D_s(S^3): H^s(S^3)\longrightarrow H_{3-s}(S^3)$, and $D_r(S^3\times S^1): H^r(S^3\times S^1)\longrightarrow H_{4-r}(S^3\times S^1)$ be the respective Poincare′

isomorphisms, where $0 \leq k \leq 1$, $0 \leq s \leq 3$, and $0 \leq r \leq 4$. In addition, let

a^0 be the generator of $H^0(S^1)$ such that $D_0(S^1)(a^0) = a^0 \cap a_1 = a_1$,

a^1 be the generator of $H^1(S^1)$ such that $D_1(S^1)(a^1) = a^1 \cap a_1 = a_0$,

b^0 be the generator of $H^0(S^3)$ such that $D_0(S^3)(b^0) = b^0 \cap b_3 = b_3$, and

b^3 be the generator of $H^3(S^3)$ such that $D_3(S^3)(b^3) = b^3 \cap b_3 = b_0$.

Thus, we have $D_k(S^1)(a^k) = a_{1-k}$ for $0 \leq k \leq 1$ and $D_s(S^3)(b^s) = b_{3-s}$ for $0 \leq s \leq 3$. Let us compute the image of $b^s \times a^k$, the external product of cohomology classes, under $D_{s+k}(S^3 \times S^1)$, where $0 \leq k \leq 1$, $0 \leq s \leq 3$, and $0 \leq s + k \leq 4$. In fact,

$$\begin{aligned}
D_{s+k}(S^3 \times S^1)(b^s \times a^k) &= (b^s \times a^k) \cap (-1)^{\acute{l}} (b_3 \times a_1) \\
&= (-1)^{\acute{l}} (-1)^{k(3-s)} \left((b^s \cap b_3) \times (a^k \cap a_1)\right) \\
&= (-1)^{\acute{l}+k(3-s)} \left(D_s(S^3)(b^s) \times D_k(S^1)(a^k)\right) \\
&= (-1)^{\acute{l}+k(3-s)} \left(b_{3-s} \times a_{1-k}\right) .
\end{aligned}$$

Put $\chi(s, k) = \acute{l} + k(3 - s)$. Hence,

$$D_{s+k}(S^3 \times S^1)(b^s \times a^k) = (-1)^{\chi(s,k)} \left(b_{3-s} \times a_{1-k}\right) . \tag{4.3.3}$$

Since D_{s+k} is an isomorphism, we get that

$$D^{-1}_{s+k}(S^3 \times S^1)(b_{3-s} \times a_{1-k}) = (-1)^{\chi(s,k)} \left(b^s \times a^k\right) , \tag{4.3.4}$$

or equivalently

$$D^{-1}_{4-s-k}(S^3 \times S^1)(b_s \times a_k) = (-1)^{\chi(3-s,1-k)} \left(b^{3-s} \times a^{1-k}\right) . \tag{4.3.5}$$

Now, we use Equations 4.3.3 and 4.3.4 (or 4.3.5) in order to compute the trace of the linear homomorphisms $\Theta_{\acute{l}} = D_{4-\acute{l}} \circ \widetilde{g}^* \circ D^{-1}_{4-\acute{l}} \circ \widetilde{f}_*$ for $0 \leq \acute{l} \leq 4$. We show the work for Θ_3.

$$\begin{aligned}
\Theta_3(b_3 \times a_0) &= D_1(S^3 \times S^1) \circ \widetilde{g}^* \circ D_1^{-1}(S^3 \times S^1) \circ \widetilde{f}_*(b_3 \times a_0) \\
&= D_1(S^3 \times S^1) \circ \widetilde{g}^* \circ D_1^{-1}(S^3 \times S^1)(6\, b_3 \times a_0) \\
&= 6\, D_1(S^3 \times S^1) \circ \widetilde{g}^* \circ D_1^{-1}(S^3 \times S^1)(b_3 \times a_0) \\
&= 6\, D_1(S^3 \times S^1) \circ \widetilde{g}^* \left((-1)^{\chi(0,1)}(b^0 \times a^1)\right) \\
&= 6\,(-1)^{\chi(0,1)}\, D_1(S^3 \times S^1) \circ \widetilde{g}^*(b^0 \times a^1) \\
&= 6\,(-1)^{\chi(0,1)}\, D_1(S^3 \times S^1)(b^0 \times 2\, a^1) \\
&= 12\,(-1)^{\chi(0,1)}\, D_1(S^3 \times S^1)(b^0 \times a^1) \\
&= 12\,(-1)^{\chi(0,1)}\,(-1)^{\chi(0,1)}\,(b_3 \times a_0) \\
&= 12\,(b_3 \times a_0).
\end{aligned}$$

Thus, $tr(\Theta_3) = 12$. Similarly, we get that $tr(\Theta_0) = 2$, $tr(\Theta_1) = 4$, $tr(\Theta_2) = 0$, and $tr(\Theta_4) = 24$. Therefore,

$$L(\widetilde{f}, \widetilde{g}) = tr(\Theta_0) - tr(\Theta_1) + tr(\Theta_2) - tr(\Theta_3) + tr(\Theta_4) = 2 - 4 + 0 - 12 + 24 = 10 \neq 0\,.$$

• It follows that $L\left(\widetilde{f}, (\omega^l, \lambda^t)\,\widetilde{g}\right) \neq 0$ for all t and l. Hence, $N_{ED}(f, g) = 0$ and $N(\widetilde{f}, \widetilde{g}) = |Coker(\widetilde{g}_\# - \widetilde{f}_\#)|$. We have $\pi_1(S^3 \times S^1, ((1, 0), 1)) \cong \pi_1(S^1, 1)$. If a_1 is the generator of $\pi_1(S^1, 1)$, by abuse of language we can write

$$(\widetilde{g}_\# - \widetilde{f}_\#)(a_1) = (\widetilde{g}_{2\,\#} - \widetilde{f}_{2\,\#})(a_1) = 2\, a_1 - 4\, a_1 = -2\, a_1\,.$$

That is $Im(\widetilde{g}_\# - \widetilde{f}_\#) = 2\mathbf{Z}$. Therefore, $|Coker(\widetilde{g}_\# - \widetilde{f}_\#)| = \left|\dfrac{\mathbf{Z}}{\mathbf{Z_2}}\right| = 2$ and $N(\widetilde{f}, \widetilde{g}) = 2$.

• Finally, by Equations 4.3.1 and 4.3.2

$$N(f, g) = 15 \times \frac{2}{2} = 15\,.$$

The next example is the main example in this chapter because the manifolds involved are non-orientable, and the covering transformations groups do not have a prime order. In addition, this example shows how our results are effective for non-orientable manifolds. On the other hand, this example also shows the way our result can be applied in the sense that we compute the Nielsen number of maps between non-orientable manifolds in terms of the Nielsen numbers of maps between orientable manifolds.

Example 4.3.18. Let $f_1, g_1 : \mathbf{RP^2} \longrightarrow \mathbf{RP^2}$ and $f_2, g_2 : S^1 \longrightarrow S^1$ be maps defined by $f_1[(x, y, z)] = [(-x, -y, z)]$, $g_1 = 1_{\mathbf{RP^2}}$, $f_2(e^{i\varphi}) = e^{i12\varphi}$, and $g_2(e^{i\varphi}) = e^{i3\varphi}$. Let $p, \acute{p} : S^1 \longrightarrow S^1$ be the covering maps defined by $p(z) = z^2$ and $\acute{p}(z) = z^3$ respectively, and $q : S^2 \longrightarrow \mathbf{RP^2}$ be the quotient map that defines the projective plane. Define $f, g : \mathbf{RP^2} \times S^1 \longrightarrow \mathbf{RP^2} \times S^1$ by $f = f_1 \times f_2$ and $g = g_1 \times g_2$. We have $q \times p$, $q \times \acute{p} : S^2 \times S^1 \longrightarrow \mathbf{RP^2} \times S^1$ are covering maps. Both f and g admit lifts $\widetilde{f} = \widetilde{f}_1 \times \widetilde{f}_2$ and $\widetilde{g} = \widetilde{g}_1 \times \widetilde{g}_2$, where $\widetilde{f}_1(x, y, z) = (-x, -y, z)$ for $(x, y, z) \in S^2$, $\widetilde{g}_1 = 1_{S^2}$, $\widetilde{f}_2(z) = z^8$, and $\widetilde{g}_2(z) = z^2$, for $z \in S^1$. Consider the commutative diagrams

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\widetilde{f}_1, \widetilde{g}_1} & S^2 \\
q \downarrow & & \downarrow q \\
\mathbf{RP^2} & \xrightarrow{f_1, g_1} & \mathbf{RP^2}
\end{array}
\qquad
\begin{array}{ccc}
S^1 & \xrightarrow{\widetilde{f}_2, \widetilde{g}_2} & S^1 \\
\downarrow p & & \acute{p} \downarrow \\
S^1 & \xrightarrow{f_2, g_2} & S^1
\end{array}
\qquad
\begin{array}{ccc}
S^2 \times S^1 & \xrightarrow{\widetilde{f}, \widetilde{g}} & S^2 \times S^1 \\
q \times p \downarrow & & \downarrow q \times \acute{p} \\
\mathbf{RP^2} \times S^1 & \xrightarrow{f, g} & \mathbf{RP^2} \times S^1
\end{array}
$$

where $\widetilde{f}_i$ and $\widetilde{g}_i$ are lifts of f_i and g_i respectively for $i = 1, 2$. Here the space $\mathbf{RP^2} \times S^1$ is a nonorientable connected closed smooth manifold and all maps considered are smooth. Moreover, the coverings are regular since the fundamental groups of $\mathbf{RP^2}$ and S^1 and $\mathbf{RP^2} \times S^1$ are abelian.

• By Lemma 4.3.13, $\mathcal{A}(S^2 \times S^1, q \times p) \cong \mathcal{A}(S^2, q) \oplus \mathcal{A}(S^1, p) \cong \mathbf{Z_2} \oplus \mathbf{Z_2}$. Similarly, $\mathcal{A}(S^2 \times S^1, q \times \acute{p}) \cong \mathcal{A}(S^2, q) \oplus \mathcal{A}(S^1, \acute{p}) \cong \mathbf{Z_2} \oplus \mathbf{Z_3} \cong \mathbf{Z_6}$. Let ω and λ be the square and the 3rd, primitive roots of unity, respectively. Then, we can write

$$\mathcal{A}(S^2 \times S^1, q \times p) = \left\{ (\omega^k, \omega^s) \mid k,\, s = 0, 1 \right\} ,$$

and

$$\mathcal{A}(S^2 \times S^1, q \times \acute{p}) = \langle (\omega, \lambda) \rangle = \left\{ (\omega^r, \lambda^t) \mid r = 0, 1 \text{ and } t = 0, 1, 2 \right\} .$$

We find $H-$Reidemeister representatives from among the members of the set $\left\{ \left(\widetilde{f}, (\omega^r, \lambda^t)\, \widetilde{g} \right) \mid r = 0, 1 \text{ and } t = 0, 1, 2 \right\}$. Of course, we can apply the same method as in the previous example to show that we have six Reidemeister representatives. We can however also use Corollary 4.1.21. Since the fundamental group of $\mathbf{RP^2} \times S^1$ is abelian, the number J depends only on the $H-$Nielsen class. Since $\mathcal{A}(\widetilde{M})$ is abelian, we have that $\mathrm{L}(\widetilde{f}, (\omega^r, \lambda^t)\, \widetilde{g})$ is normal for each $r = 0, 1$ and $t = 0, 1, 2$. By Corollary 4.1.21, the number of $H-$Reidemeister representatives Λ equals 6. Notice that $I = 4$ is the only feasible value which gives an integer result. This means that $(\widetilde{f}, (\omega^r, \lambda^t)\, \widetilde{g})$ is a distinct $H-$Reidemeister representative for each $r = 0, 1$ and $t = 0, 1, 2$. Now, we have

$$N_L(f, g) = \sum_{r=0}^{1} \sum_{t=0}^{2} \frac{N(\widetilde{f}, (\omega^r, \lambda^t)\, \widetilde{g})}{S(\widetilde{f}, (\omega^r, \lambda^t)\, \widetilde{g})} .$$

We have $(\widetilde{f}, (1_{S^2}, \lambda^t)\, \widetilde{g})$ is homotopic to $(\widetilde{f}, \widetilde{g})$ for each t, and $(\widetilde{f}, (-1_{S^2}, \lambda^t)\, \widetilde{g})$ is homotopic to $(\widetilde{f}, (-1_{S^2}, 1_{S^2})\,\widetilde{g})$ for each t. Thus,

$$N_L(f,g) = \sum_{t=0}^{2} \frac{N(\widetilde{f},(1_{S^2},\lambda^t)\,\widetilde{g})}{S(\widetilde{f},(1_{S^2},\lambda^t)\,\widetilde{g})} + \sum_{t=0}^{2} \frac{N(\widetilde{f},(-1_{S^2},\lambda^t)\,\widetilde{g})}{S(\widetilde{f},(-1_{S^2},\lambda^t)\,\widetilde{g})} \tag{4.3.6}$$

$$= 3\,\frac{N(\widetilde{f},\widetilde{g})}{S(\widetilde{f},\widetilde{g})} + 3\,\frac{N(\widetilde{f},(-1_{S^2},1_{S^2})\,\widetilde{g})}{S(\widetilde{f},(-1_{S^2},1_{S^2})\,\widetilde{g})}\,. \tag{4.3.7}$$

The next step is to compute $S(\widetilde{f},\widetilde{g})$ and $S(\widetilde{f},(-1_{S^2},1_{S^2})\,\widetilde{g})$. We start with $S(\widetilde{f},\widetilde{g})$.

- Since S^2 is simply connected, $\Phi(\widetilde{f}_1,\widetilde{g}_1)$ is the only Nielsen class for the pair $(\widetilde{f}_1,\widetilde{g}_1)$. Thus,

  $$I(\widetilde{f}_1,\widetilde{g}_1) = |\mathrm{L}(\widetilde{f}_1,\widetilde{g}_1)| = |\mathcal{A}(S^2)| = 2\,.$$

  Moreover, $I(\widetilde{f}_1,\widetilde{g}_1) = J(\widetilde{f}_1,\widetilde{g}_1)$.

- If u is the generator of $\pi_1(S^1)$, we get that $(g_{2\#} - f_{2\#})\,(u) = 3u - 12u = -9u$. This implies that $Ker\,(g_{2\#} - f_{2\#}) = 0$. Hence, $J(\widetilde{f}_2,\widetilde{g}_2) = |j\,(Ker\,(g_{2\#} - f_{2\#}))\,| = 1$. On the other hand,

  $$I(\widetilde{f}_2,\widetilde{g}_2) = |\mathrm{L}(\widetilde{f}_2,\widetilde{g}_2)| = |\mathcal{A}(S^1,p)| = 2\,.$$

- Parts (3) and (4) of Lemma 4.3.13 imply that

  $$J(\widetilde{f},\widetilde{g}) = J(\widetilde{f}_1,\widetilde{g}_1) \times J(\widetilde{f}_2,\widetilde{g}_2) = 2 \times 1 = 2\,,$$

  and

  $$I(\widetilde{f},\widetilde{g}) = I(\widetilde{f}_1,\widetilde{g}_1) \times I(\widetilde{f}_2,\widetilde{g}_2) = 2 \times 2 = 4\,.$$

Thus,

$$S(\widetilde{f},\widetilde{g}) = \frac{4}{2} = 2 \,. \tag{4.3.8}$$

We don't need to compute $S(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g})$, since, as we will soon see, $N(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g}) = 0$.

Next, we compute $N(\widetilde{f},\widetilde{g})$ and $N(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g})$. In order to give a variety of computation methods, we will not use the Lefschetz numbers of the $H-$Reidemeister representatives or Jiang space methods as in the previous examples. Instead, we will use the index formula for product maps.

Each Nielsen class of $(\widetilde{f},\widetilde{g})$ has the form

$$A_k = \{((0,0,1),\mu^k),((0,0,-1),\mu^k)\} \,.$$

where μ is the 6th primitive root of unity, and $k = 0,1,2,3,4,5$. Furthermore, since $index(\widetilde{f}_1,\widetilde{g}_1;\{(0,0,1),(0,0,-1)\}) = 2$ (See Example 3.1.14), and $\{\mu^k\}$ is essential for each k, we have

$$index(\widetilde{f},\widetilde{g};A_k) = index(\widetilde{f}_1,\widetilde{g}_1;\{(0,0,1),(0,0,-1)\}) \cdot index(\widetilde{f}_2,\widetilde{g}_2;\{\mu^k\}) \neq 0 \,.$$

Thus, we have 6 essential classes for $(\widetilde{f},\widetilde{g})$. Therefore, $N(\widetilde{f},\widetilde{g}) = 6$.

On the other hand, each Nielsen class of $(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g})$ has the form

$$B_k = \{((x,y,0),\mu^k) \mid x^2+y^2 = 1\} \,,$$

where $k = 0,1,2,3,4,5$. Since $index(\widetilde{f}_1,\widetilde{g}_1;\{(x,y,0) \mid x^2+y^2 = 1\}) = 0$ (see Example 3.3.2), the formula of index of product maps gives that

$$index(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g};B_k) = 0 \,.$$

Thus, there are no essential classes for $(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g})$. That is,

$$N(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g}) = 0\ .$$

Substituting $N(\widetilde{f}, \widetilde{g})$, $N(\widetilde{f}, (-1_{S^2}, 1_{S^2})\ \widetilde{g})$, and $S(\widetilde{f}, \widetilde{g})$ in Equation 4.3.6 gives that

$$N_L(f,g) = 3 \cdot \frac{6}{2} + 0 = 9\,. \tag{4.3.9}$$

The last step is to compute $N_{ED}(f,g)$. For this purpose we study the existence of the essential defective classes of (f,g) for which J is even. Since the pre-image of each such class by the covering map must be a union of inessential classes of the lifts of (f,g), we focus on the Nielsen classes of (f,g) that correspond to the $H-$Reidemeister representatives of the form $(\widetilde{f}, (-1_{S^2}, \lambda^t)\ \widetilde{g})$, where $t = 0,1,2$. Fixing t, we have shown in Example 3.3.2 that $q(\{(x^2,y^2,0)|\ x^2+y^2=1\})$ is inessential class. That is $|ind|(f_1,g_1;q(\{(x^2,y^2,0)|\ x^2+y^2=1\}))=0$. The semi-index formula for product maps implies that $|ind|(f,g;A)=0$ for any Nielsen class $A \subset q \times p\left(\Phi(\widetilde{f}, (-1_{S^2}, \lambda^t)\ \widetilde{g})\right)$. That is, there do not exist essential Nielsen classes which correspond to the $H-$ Reidemeister representatives $(\widetilde{f}, (-1_{S^2}, \lambda^t)\ \widetilde{g})$ where $t = 0,1,2$. Hence, $N_{ED}(f,g) = 0$.

Finally, we get that

$$N(f,g) = N_L(f,g) + N_{ED}(f,g) = 9 + 0 = 9\,.$$

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