Cyclic Block Designs from Skolem-type Sequences

by

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Abstract

M. Colbourn and R. Mathon [45] asked: "Can Skolem's partitioning problems be generalized to yield cyclic BIBD$(v, 4, 1)$?". Rosa [76] asked: "What is the format of Skolem-type sequences that leads to cyclic BIBD(v, k, λ) for $k \geq 4$?". In this thesis, we will address these two questions.

We introduce new Skolem-type sequences and then we use them to construct new cyclic BIBD(v, k, λ) for $k \geq 3$. Specifically, we use Skolem-type sequences to construct new cyclic BIBD$(v, 3, \lambda)$ for all admissible orders v and λ.

We use Skolem-type sequences to construct new cyclic BIBD(v, k, λ) for $k \geq 4$ and every v coprime with 6. We provide a complete set of examples of Skolem partitions that induce one cyclic BIBD$(v, 4, \lambda)$ for every admissible class.

We also use some known results and relative difference families to construct new cyclic BIBD$(v, 4, \lambda)$ for infinite values of v.

Moreover, we use Skolem-type sequences to construct cyclic, simple, and indecomposable BIBD$(v, 3, 3)$ for every v with some possible exceptions for $v = 9$ and $v = 24c + 9$, $c \geq 4$. We also construct infinitely many cyclically indecomposable but decomposable BIBD$(v, 3, 4)$ for some orders v.

Finally, we have many examples of simple and super-simple cyclic designs coming from Skolem-type sequences that produce optical orthogonal codes.

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Chapter 1

Introduction

Informally, a combinatorial design can be defined to be a way of selecting subsets from a finite or infinite set such that specific conditions are satisfied. As an example, suppose it is required to select 3-sets from the seven objects $\{a, b, c, d, e, f, g\}$, such that each object occurs in three of the 3-sets and every intersection of two 3-sets has precisely one member. The solution to such a problem is a combinatorial design. One possible example is $\{\{a, b, d\}, \{b, c, e\}, \{c, d, f\}, \{d, e, g\}, \{e, f, a\}, \{f, g, b\}, \{g, a, c\}\}$. This combinatorial design is also known as a balanced incomplete block design and denoted by BIBD$(7, 3, 1)$ (also known as a Steiner triple system of order 7 and denoted by STS(7)).

Combinatorial designs have numerous applications. For example, in the statistical design of experiments, Fisher [51] laid out mathematical principles of experimental designs and Yates [91] was the first to draw attention to the importance of block designs for the statistical design of experiments. Fisher and Yates used block designs to compare the effects of different treatments, the growth of different strains of an organism, and other similar problems. When the number of treatments was large, these designs helped to eliminate heterogeneity to a greater extent than was possible with randomized blocks and Latin squares. The precision of the estimate of a treatment effect depends on the number of replications of the treatment, i.e., the larger the number of replications, the greater is the precision. Similar is the case for the precision of estimates of the difference between two treatment effects. To ensure equal precision when comparing different pairs of treatment effects, the treatments are allocated to the experimental units in different blocks of equal sizes such that each treatment occurs at most once in a block, each treatment has an equal number of replications, and each pair of treatments has the same number of replications.

Various restrictions for block designs were considered in the hope that tools developed for restricted versions can be extended to the general cases. Restrictions are also motivated by a desire to consider interesting designs. Typical restrictions that were considered are those that constrain the automorphism group.

A design with v elements is *cyclic* when its automorphism group contains a v-cycle.

A cyclic block design can be represented as a design with elements $\{0, 1, \ldots, v-1\}$ where if $\{b_1, b_2, \ldots, b_k\}$ is a block, then $\{b_1+1, b_2+1, \ldots, b_k+1\}$ (addition performed mod v) is also a block. A cyclic design is always isomorphic to a design for which $V = \mathbb{Z}_v = \{0, 1, \ldots, v-1\}$ and the mapping $\alpha : i \to i+1 \pmod v$ is an automorphism. For example, our STS(7) is cyclic. One automorphism α, which is a 7-cycle, carries $g \to a \to b \to c \to d \to e \to f \to g \to a$. If we rename g as 0, and we map $f(g)$ to 1, $f(f(g))$ to 2, and so on, our STS(7) becomes:

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}, \{0, 1, 3\}.$$

This way we obtain a block from another block by adding 1 to each entry (mod v). This process provides a compact representation. Selecting one block under the cyclic automorphism α, we can easily reconstruct the system. This property makes cyclic designs attractive in applications and for testing purposes.

Moreover, cyclic designs with special properties (simple or super-simple) are equivalent to optimal optical orthogonal codes. The study of optical orthogonal codes (OOC for short) was motivated by an application in a fibre-optic code-division multiple access channel. Many users wish to transmit information over a common wide-band optical channel. The objective is to design a system that allows the users to share the common channel. For the construction of optimal optical orthogonal codes, cyclic block designs play an important role since a cyclic block design is equivalent to an optimal optical orthogonal code. As an example, $C = \{1100100000000, 1010000100000\}$ is a $(13, 3, 1)$ code with two codewords. In set theoretic notation $C = \{\{0, 1, 4\}, \{0, 2, 7\}\} \pmod{13}$, which gives a cyclic Steiner triple system of order 13. A survey about cyclic designs and their applications to optimal optical orthogonal codes is given in [11].

These applications, together with the obvious mathematical significance, illustrate the importance of the construction of block designs, particularly those which are cyclic.

1.1 Definitions and Known Results

In this section, we provide the basic definitions and known results necessary for the further proven results.

Definition 1.1.1 *A* balanced incomplete block design*, denoted by BIBD*(v, k, λ) *is a pair* $(V, \mathcal{B})$ *where* V *is a* v*-set of points and* $\mathcal{B}$ *is a set of* k*-subsets called* blocks *such that any 2-subset of the* v*-set appears in exactly* λ *of the* k*-subsets.*

Definition 1.1.2 *Two set systems* $(V, \mathcal{B})$ *and* $(W, \mathcal{D})$ *are* isomorphic *if there is a bijection (isomorphism)* Φ *from* V *to* W *so that the number of times* B *appears as a* *block in $\mathcal{B}$ is the same as the number of times $\Phi(B) = \{\Phi(x) : x \in B\}$ appears as a block in $\mathcal{D}$.*

Definition 1.1.3 *An isomorphism from a set system to itself is an* automorphism*.*

Definition 1.1.4 *A BIBD(v, k, λ) is* cyclic *if it admits an automorphism of order v.*

If (V, B) is a cyclic BIBD(v, k, λ), one may assume $V = \mathbb{Z}_v$, and $\alpha : i \rightarrow i + 1(\text{mod } v)$ is its cyclic automorphism. Let $B = \{b_1, b_2, \ldots, b_k\}$ be a block of a cyclic BIBD(v, k, λ). The block orbit containing B is defined by the set of distinct blocks

$$B + i = \{b_1 + i, \ldots, b_k + i\}(\text{mod } v)$$

for $i \in \mathbb{Z}_v$. If a block orbit has v blocks, then the block orbits is said to be *full*, otherwise it is said to be *short*. An arbitrary block from a block orbit is called a *base block*. A base block is also referred to as a *starter block* or an *initial block*. The block orbit that contains the block $\{0, \frac{v}{k}, \frac{2v}{k}, \ldots, \frac{(k-1)v}{k}\}$ is called a *regular short orbit*.

Definition 1.1.5 *If $\lambda = 1$ and $k = 3$ the design is called a* Steiner triple system*, denoted by STS(v). A cyclic STS(v) is denoted by CSTS(v).*

Definition 1.1.6 *A BIBD(v, k, λ) is* simple *if it contains no repeated blocks.*

Definition 1.1.7 *A BIBD(v, k, λ) is* super-simple *if the intersection of any two blocks has at most two elements.*

One way to build a block design with higher λ is to take the union of the sets of blocks of smaller designs based on the same set V. In particular, the union of a BIBD(v, k, λ_1) with a BIBD(v, k, λ_2) is a BIBD$(v, k, \lambda_1 + \lambda_2)$ design. Conversely, suppose that we can partition the blocks of a BIBD(v, k, λ) so that each part induces a design with a strictly smaller λ. Then, we say the design is *decomposable*.

Definition 1.1.8 *A BIBD(v, k, λ) is called* indecomposable *if its blocks set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1$, $\mathcal{B}_2$ of blocks of the form BIBD(v, k, λ_1) and BIBD(v, k, λ_2) respectively, where $\lambda_1 + \lambda_2 = \lambda$ with $\lambda_1, \lambda_2 \geq 1$.*

Definition 1.1.9 *A cyclic BIBD(v, k, λ) is called* cyclically indecomposable *if its block set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1$, $\mathcal{B}_2$ of blocks to form a cyclic BIBD(v, k, λ_1) and cyclic BIBD(v, k, λ_2) respectively, where $\lambda_1 + \lambda_2 = \lambda$ with $\lambda_1, \lambda_2 \geq 1$.*

Theorem 1.1.1 *[46] An STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$.*

Theorem 1.1.2 *[46] A CSTS(v) exists whenever $v \equiv 1, 3 \pmod 6$ and $v \neq 9$.*

When $v = nu$, $n > 1, u > 1$, then the group of integers modulo v will have a subgroup of order u and index n for each divisor u of v.

Definition 1.1.10 *A* cyclic subsystem *of a CSTS(v) having order u and index n, is a subsystem (H, B_u) where H is a subgroup of order u and index n which fixes B_u.*

Definition 1.1.11 *Let K and H be sets of positive integers and let λ be a positive integer. A* group divisible design *of index λ and order v, denoted by $(K, \lambda) - GDD$, is a triple $(\mathcal{V}, \mathcal{H}, \mathcal{B})$, where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{H}$ is a partition of $\mathcal{V}$ into parts called* groups *whose sizes lie in H, and $\mathcal{B}$ is a family of subsets called* blocks, *of $\mathcal{V}$ that satisfy:*

1. *if $B \in \mathcal{B}$ then $|B| \in K$;*
2. *every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks or one group, but not both;*
3. $|\mathcal{H}| > 1$.

If there are a_i groups of size g_i, $i = 1, \ldots, s$, then the $(K, \lambda) - GDD$ is of type $g_1^{a_1} g_2^{a_2} \ldots g_s^{a_s}$. If $K = \{k\}$, then the $(K, \lambda) - GDD$ is a $(k, \lambda) - GDD$.

Example 1.1.1 *A $(3,1)-GDD$ of type 2^4 has $\{\{1,5\},\{2,6\},\{3,7\},\{4,0\}\}$ as groups and $\{i, 1+i, 3+i\}$, $i = 0, \ldots, 7$ as blocks.*

Remark 1.1.1 *If a GD design is cyclic, then each group G_i must be the subgroup $m\mathbb{Z}_n = \{0, m, 2m, \ldots, (n-1)m\}$ of $\mathbb{Z}_{mn}$ or its coset.*

Let G be an additive group and let $B \subseteq G$. Denote by ΔB the list of differences from B, that is, the multiset of all differences between two distinct elements of B. In

ΔB an element $g \in G$ appears $\mu_g|G_B|$ times, where G_B is the stabilizer of B in G and μ_g is an integer. Denote by ∂B the list of partial differences from B, that is, the multiset of all differences between two distinct elements of B, with the property that an element $g \in G$ appears μ_g times in ∂B. If G_B is trivial, then $\Delta B = \partial B$.

Definition 1.1.12 *A* (G, k, λ)-difference family*, DF for short, is a family* $\mathcal{F}=\{B_1, \ldots, B_t\}$ *of* k*-subsets of* G *such that, in* $\partial\mathcal{F}= \partial B_1 \cup \ldots \cup \partial B_t$ *every element of* $G - \{0_G\}$ *appears exactly* λ *times. The elements of* $\mathcal{F}$ *are called* base blocks*. If* G *is the cyclic group* $\mathbb{Z}_v$*, then a* $(G, k, \lambda) - DF$ *is denoted by* $(v, k, \lambda) - DF$*.*

Let $\mathcal{F}=\{B_1, \ldots, B_t\}$ be a $(G, k, \lambda) - DF$. For $B_i \in \mathcal{F}$ denote by S_i a complete system of distinct representatives for the right cosets of G_{B_i} in G and by $O_i = \{B_i + s_i | s_i \in S_i\}$ the G-orbit of B_i; O_i is full or short according to whether G_{B_i} is trivial or not. The set $O_1 \cup \ldots \cup O_t$ is the set of blocks of a BIBD(v, k, λ) admitting G as an automorphism group acting sharply transitively on V. Conversely, a BIBD(v, k, λ) admitting a group G as an automorphism group acting sharply transitively on V is generated by a suitable $(G, k, \lambda) - DF$. Hence, a cyclic BIBD(v, k, λ) is generated by a suitable $(v, k, \lambda) - DF$.

Definition 1.1.13 *If* $v = k(k-1)t + 1$*, then* t *blocks* $B_i = \{b_{i,1}, b_{i,2}, \ldots, b_{i,k}\}$ *form a* $(v, k, 1)$ *perfect difference family over* $\mathbb{Z}_v$ *(*$(v, k, 1)$*-PDF for short) if the differences*

$b_{i,m} - b_{i,n}$, $i = 1, \ldots, t$, $1 \le m < n \le k$ *cover the set* $\{1, 2, \ldots, (v-1)/2\}$.

Definition 1.1.14 *A* (vg, g, k, λ) relative difference set *is a* k*-subset* B *of* $\mathbb{Z}_{gv}$ *with the property that its list of differences* $\Delta B = \{x - y | x, y \in B, x \neq y\}$ *has no element in* $v\mathbb{Z}_g$ *while it contains each element of* $\mathbb{Z}_{gv} - v\mathbb{Z}_g$ *exactly* λ *times.*

More generally, a (vg, g, k, λ) relative difference family ((vg, g, k, λ)-DF in short) is a family $\mathcal{F}$ of k subsets (base blocks) of $\mathbb{Z}_{gv}$ with the property that its list of differences $\Delta\mathcal{F} = \cup_{\mathcal{B}\in\mathcal{F}} \Delta B$ is λ times $\mathbb{Z}_{gv} - v\mathbb{Z}_g$.

Such a DF generates a cyclic (k, λ)-GDD of type g^v $(V, \mathcal{G}, \mathcal{B})$ with point-set $V = \mathbb{Z}_{gv}$, group set $\mathcal{G}$=$\{v\mathbb{Z}_g + i | 0 \le i < v\}$ and block multiset $\mathcal{B}$=$\{B + t | B \in \mathcal{F},\ t \in \mathbb{Z}_{gv}\}$.

Remark 1.1.2 *A* $(gv, g, k, 1)$*-DF is also called a* g-regular cyclic packing *CP*$(1, k; gv)$.

Definition 1.1.15 *Let* v, g, k, λ *be positive integers and* $\alpha \in [0, \lambda]$. *A* $(gv, \{g, k_\alpha\}, k, \lambda)$-difference family *in* $\mathbb{Z}_{gv}$ *(*$(gv, \{g, k_\alpha\}, k, \lambda)$*-DF in* $\mathbb{Z}_{gv}$ *in short) is a family of* k *subsets (base blocks) of* $\mathbb{Z}_{gv}$ *with the property that its list of differences* $\Delta\mathcal{F} = \cup_{\mathcal{B}\in\mathcal{F}} \Delta B$ *is* $\lambda\mathbb{Z}_{gv} \setminus (\lambda\{0, v, \ldots, (g-1)v\} \cup \alpha\{0, gv/k, \ldots, (k-1)gv/k\})$, *where* $\Delta B = \{b_i - b_j : 1 \le i, j \le k, i \neq j\}$ *if* $B = \{b_1, b_2, \ldots, b_k\}$, *such that* $\{0, v, \ldots, (g-1)v\} \cap \{0, gv/k, \ldots, (k-1)gv/k\} = \{0\}$.

Definition 1.1.16 *A $t \times \lambda u$ matrix $D = (d_{ij})$ with entries from $\mathbb{Z}_u$ is called a (u, t, λ)-*cyclic difference matrix *(denoted by (u, t, λ)-CDM) if every element of $\mathbb{Z}_u$ occurs exactly λ times among the differences $d_{ij} - d_{\ell j}, j = 1, \ldots, \lambda u$ for any $i \neq \ell$.*

Example 1.1.2

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \end{bmatrix}$$

is a $(5, 4, 1)$-CDM.

Definition 1.1.17 *Let D be a multiset of positive integers with $|D| = n$. A* Skolem-type sequence of order n *is a sequence $(s_1, \ldots, s_t), t \geq 2n$ of integers $i \in D$ such that for each $i \in D$ there is exactly one $j \in \{1, \ldots, t - i\}$ such that $s_j = s_{j+i} = i$. Positions in the sequence not occupied by integers $i \in D$ contain null elements. The null elements in the sequence are also called* hooks, zeros or holes.

Example 1.1.3 $(1, 1, 6, 2, 5, 2, 1, 1, 6, 5)$ *is a Skolem-type sequence of order* 5 *and* $(7, 5, 2, *, 2, *, 5, 7, 1, 1)$ *is a Skolem-type sequence of order* 4.

Definition 1.1.18 *A Skolem-type sequence is k-*extended *if it contains exactly one hook which is in position k.*

Now, we give the definitions of some special cases of Skolem-type sequences.

Definition 1.1.19 *A* Skolem sequence of order n *is a sequence* $S_n = (s_1, s_2, \ldots, s_{2n})$ *of* $2n$ *integers which satisfies the conditions:*

1. *for every* $k \in \{1, 2, \ldots, n\}$ *there are exactly two elements* $s_i, s_j \in S_n$ *such that* $s_i = s_j = k$, *and*

2. *if* $s_i = s_j = k,\ i < j$, *then* $j - i = k$.

Skolem sequences are also written as collections of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n,\ b_i - a_i = i\}$ *with* $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, \ldots, 2n\}$.

Example 1.1.4 $S_5 = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$ *or, equivalently, the collection* $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$ *is a Skolem sequence of order* 5 .

Definition 1.1.20 *Given a Skolem sequence* $S_n = (s_1, s_2, \ldots, s_{2n})$, *the* reverse $\overleftarrow{S}_n = (s_{2n}, \ldots, s_1)$ *is also a Skolem sequence.*

Definition 1.1.21 *A* hooked Skolem sequence of order n *is a sequence* $hS_n = (s_1, \ldots, s_{2n-1}, s_{2n+1})$ *of* $2n + 1$ *integers which satisfies Definition 1.1.19, as well as* $s_{2n} = 0$.

Example 1.1.5 $hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, *, 6)$ *or, equivalently, the collection* $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$ *is a hooked Skolem sequence of order* 6.

Theorem 1.1.3 *[84] A Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$.

Theorem 1.1.4 *[69] A hooked Skolem sequence of order n exists if and only if* $n \equiv 2, 3 \pmod 4$.

Definition 1.1.22 *A* k-extended Skolem sequence of order n *is sequence* $ES_n = (s_1, s_2, ..., s_{2n+1})$ *in which* $s_k = 0$ *and, for each* $j \in \{1, 2, ...n\}$*, there exists a unique* $i \in \{1, 2, ..., n\}$ *such that* $s_i = s_{i+j} = j$.

Example 1.1.6 3*-extended* $ES_4 = (4, 2, *, 2, 4, 3, 1, 1, 3)$ *is a* 3*-extended Skolem sequence of order* 4.

Theorem 1.1.5 *[7] The necessary and sufficient conditions for the existence of a* k*-extended Skolem sequence are* $n \equiv 0, 1 \pmod 4$ *for* k *odd, and* $n \equiv 2, 3 \pmod 4$ *for* k *even.*

Definition 1.1.23 *A Skolem sequence with a hook in the middle* ($s_{n+1} = 0$)*, is called a* Rosa sequence (or a split Skolem sequence) *and is denoted by* R_n. *A sequence with two hooks* ($s_{n+1} = 0$; $s_{2n+1} = 0$)*, is called a* hooked Rosa sequence (or a hooked split Skolem sequence) *and is denoted by* hR_n.

Example 1.1.7 $R_3 = (1, 1, 3, *, 2, 3, 2)$ *is a Rosa sequence of order* 3 *and* $hR_5 = (3, 1, 1, 3, 4, *, 5, 2, 4, 2, *, 5)$ *is a hooked Rosa sequence of order* 5.

Theorem 1.1.6 *[75]*

1. *A Rosa sequence of order n exists if and only if $n \equiv\ 0, 3 \pmod 4$.*

2. *A hooked Rosa sequence of order n exists if and only if $n \equiv\ 1, 2 \pmod 4$.*

Definition 1.1.24 *Let m, n be positive integers, with $m \leq n$. A* near-Skolem sequence of order n and defect m, *is a sequence $(s_1, s_2, \dots, s_{2n-2})$ of integers $s_i \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ which satisfies the following conditions:*

1. *for every $k \in \{1, 2, \dots, m-1, m+1, \dots, n\}$, there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and*

2. *if $s_i = s_j = k$, then $j - i = k$.*

Example 1.1.8 $(1, 1, 6, 3, 7, 5, 3, 2, 6, 2, 5, 7)$ *is a* 4*-near Skolem sequence of order* 7*.*

Definition 1.1.25 *A* hooked near-Skolem sequence of order n and defect m, *denoted m-near hS_n, is a sequence $(s_1, s_2, \dots, s_{2n-1})$ of integers $s_i \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ satisfying Definition 1.1.24 and $s_{2n-2} = 0$.*

Example 1.1.9 $(2, 5, 2, 4, 6, 7, 5, 4, 1, 1, 6, *, 7)$ *is a hooked* 3*-near Skolem sequence of order* 7*.*

Theorem 1.1.7 *[77] An m-near Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even.*

Theorem 1.1.8 *[77] A hooked m-near Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$ *and m is even, or* $n \equiv 2, 3 \pmod 4$ *and m is odd.*

Definition 1.1.26 *A* Langford sequence of order n and defect d, $n > d$ *(also called a perfect Langford) is a sequence* $L_d^n = (l_1, l_2, \ldots, l_{2n})$ *of* $2n$ *integers which satisfies both:*

1. *for every* $k \in \{d, d+1, \ldots, d+n-1\}$*, there exist exactly two elements* $l_i, l_j \in L$ *such that* $l_i = l_j = k$ *and*
2. *if* $l_i = l_j = k$ *with* $i < j$*, then* $j - i = k$*.*

Note that some authors use the term length instead of order for the Langford sequence. Also, the largest difference in a Langford sequence, $n + d - 1$, is sometimes called the order of the Langford sequence.

Example 1.1.10 $L_3^5 = (7, 5, 3, 6, 4, 3, 5, 7, 4, 6)$ *is a Langford sequence of order* 5 *and defect* 3*.*

Definition 1.1.27 *A* hooked Langford sequence of order n and defect d *is a sequence* $hL_d^m = (l_1, l_2, \ldots, l_{2n+1})$ *of* $2n+1$ *integers which satisfies Definition 1.1.26 and* $l_{2n} = 0$*.*

Example 1.1.11 $hL_2^5 = (4, 5, 6, 2, 4, 2, 5, 3, 6, *, 3)$ *is a hooked Langford sequence of order* 5 *and defect* 2*.*

Theorem 1.1.9 *[8, 66, 83] The necessary and sufficient conditions for the existence of a Langford sequence are:*

1. $n \geq 2d - 1$, *and*

2. $n \equiv 0,\ 1 \pmod 4$ *for* d *odd,* $n \equiv 0,\ 3 \pmod 4$ *for* d *even.*

Theorem 1.1.10 *[8, 66, 83] The necessary and sufficient conditions for the existence of a hooked Langford sequence* $(d, d+1, \ldots, d+n-1)$ *are:*

1. $n(n+1-2d) + 2 \geq 2$, *and*

2. $n \equiv 2,\ 3 \pmod 4$ *for* d *odd,* $n \equiv 1,\ 2 \pmod 4$ *for* d *even.*

Definition 1.1.28 *Let* G *be an additive abelian group of order* $v > 1$. *A* starter *in* G *is a set of unordered pairs* $S = \{\{x_i, y_i\} | 1 \leq i \leq (v-1)/2\}$ *which satisfies the following two properties:*

1. $\{x_i | 1 \leq i \leq (v-1)/2\} \cup \{y_i | 1 \leq i \leq (v-1)/2\} = G \setminus \{0\}$.

2. $\{\pm(x_i - y_i) | 1 \leq i \leq (v-1)/2\} = G \setminus \{0\}$.

Example 1.1.12 *A starter in* $\mathbb{Z}_7$ *is given by the pairs* $(3, 4), (2, 5), (1, 6)$. *This starter can also be written as a sequence* $(5, 3, 1, 1, 3, 5)$.

Definition 1.1.29 *Let G be an additive abelian group of order $v > 1$. A* skew starter *in G is a set of unordered pairs $S = \{\{x_i, y_i\} | 1 \leq i \leq (v-1)/2\}$ which satisfies the following three properties:*

1. $\{x_i | 1 \leq i \leq (v-1)/2\} \cup \{y_i | 1 \leq i \leq (v-1)/2\} = G \setminus \{0\}$.

2. $\{\pm(x_i - y_i) | 1 \leq i \leq (v-1)/2\} = G \setminus \{0\}$.

3. $\{\pm(x_i + y_i) | 1 \leq i \leq (v-1)/2\} = G \setminus \{0\}$.

Example 1.1.13 *A skew starter in $\mathbb{Z}_7$ is given by the pairs $(2,3),(4,6),(1,5)$. This skew starter can also be written as a sequence $(4,1,1,2,4,2)$.*

Definition 1.1.30 *A (v,k,λ)* optical orthogonal code *is a family of $(0,1)$-sequences of length v and weight k satisfying the following two properties:*

(a) The Auto-Correlation Property: $\sum_{t=0}^{v-1} x_t x_{t+i} \leq \lambda$ for any $x \in C$ and any integer $i \not\equiv 0 \pmod{v}$;

(b) The Cross-Correlation Property: $\sum_{t=0}^{v-1} x_t y_{t+i} \leq \lambda$ for any $x \neq y$ in C and any integer i.

Example 1.1.14 *$C = \{1100100000000, 1010000100000\}$ is a $(13,3,1)$ code with two codewords.*

A convenient way of viewing optical orthogonal codes, especially when k is much smaller than v, is from a set-theoretical perspective. A (v, k, λ)-OOC, $\mathcal{C}$, can be alternatively considered as a collection of k-sets of integers modulo v, in which each k-set corresponds to a codeword and the numbers in each k-set specify the nonzero bits of the codeword. The correlations properties can be reformulated in this set-theoretic perspective as follows:

1. The Auto-Correlation Property: $|(X + s_1) \cap (X + s_2)| \leq \lambda$ for any $X \in \mathcal{C}$ and any integers $s_1 \not\equiv s_2 \mod v$;

2. The Cross-Correlation Property: $|(X + s_1) \cap (Y + s_2)| \leq \lambda$ for any $X, Y \in \mathcal{C}$ with $X \neq Y$ and any integers s_1 and s_2;

Note that here $X + s = \{x + s \mod v : x \in X\}$ represents a cyclic shift of a codeword X of amount s.

Remark 1.1.3 *The size of a* (v, k, λ) *optical orthogonal code* C *is bounded by the Johnson bound:*

$$|C| \leq \frac{(v-1)(v-2)\cdots(v-\lambda)}{k(k-1)\ldots(k-\lambda)}.$$

Definition 1.1.31 *An optical orthogonal code is* optimal *if its size is the largest possible. It is* perfect *if its size meets the Johnson upper bound.*

1.2 Previous Work and Applications

The origin of cyclic designs lies in the late nineteenth century when Heffter [61] posed a problem equivalent to determining the spectrum of CSTS(v). In general, given k and λ, to establish the spectrum of values of v for which there exists a cyclic BIBD(v, k, λ) is a very difficult problem. The problem has been solved for $k = 3$ and $\lambda = 1$ by Peltesohn [70], and for $k = 3$ and $\lambda > 1$ by Colbourn and Colbourn [42].

In 1897, Heffter [61] stated two difference problems. The solution to these problems is equivalent to the existence of cyclic Steiner triple systems.

Heffter's first difference problem (denoted by $HDP_1(n)$) is: can a set $\{1, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 (\text{mod } 6n + 1)$? If such a partition is possible then $\{\{0, a_i, a_i + b_i\} | 1 \leq i \leq n\}$ will be the base blocks of a CSTS($6n + 1$).

Heffter's second difference problem (denoted by $HDP_2(n)$) is: can a set $\{1, \ldots, 3n+1\} \setminus \{2n+1\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 (\text{mod } 6n + 3)$? If such a partition is possible then $\{\{0, a_i, a_i + b_i\} | 1 \leq i \leq n\}$ with the addition of the base block $\{0, 2n+1, 4n+2\}$, having a short orbit of length $3n + 1$, will be the base blocks of a CSTS($6n + 3$).

In 1939, Peltesohn [70] solved both Heffter's difference problems, showing at least one solution exists for each case, and constructed cyclic Steiner triple systems of order v for $v \equiv 1, 3 \pmod 6), v \neq 9$.

In 1957, Skolem [84], studying Steiner triple systems, considered the existence of a partition of the set $\{1, 2, \ldots, 2n\}$ into n ordered pairs $\{(a_i, b_i) | 1 \leq i \leq n, b_i - a_i = i\}$. This began the study of Skolem sequences and their many generalizations. For any solution $\{(a_i, b_i) | 1 \leq i \leq n\}$ to Skolem's problem, the triples $\{(i, a_i + n, b_i + n) | 1 \leq i \leq n\}$ form a solution to Heffter's first difference problem. These triples yield the base blocks $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$ of a CSTS$(6n + 1)$. Also, $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a CSTS$(6n + 1)$.

Skolem [85] showed that such a distribution exists for $n \equiv 0$ or $1 \pmod 4$. He also conjectured that a similar partitioning of $\{1, 2, \ldots, 2n - 1, 2n + 1\}$ into n pairs $\{(a_i, b_i) | 1 \leq i \leq n, b_i - a_i = i\}$ is possible if and only if $n \equiv 2$ or $3 \pmod 4$. This conjecture was proved true by O'Keefe [69] in 1961. The combined results of Skolem and O'Keefe proved the sufficiency for the existence of a CSTS$(6n + 1)$.

In 1966, Rosa [75] showed that a partition of $\{1, 2, \ldots, n, n + 2, \ldots, 2n + 1\}$ into n pairs $\{(a_i, b_i) | 1 \leq i \leq n, b_i - a_i = i\}$ is possible if and only if $n \equiv 0$ or $3 \pmod 4$. For such a partition, the triples $\{(i, a_i + n, b_i + n) | 1 \leq i \leq n\}$ form a solution to Heffter's second difference problem. These triples yield the base blocks $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$. The base blocks $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, together with the base block $\{0, 2n+1, 4n+2\}$ having a short orbit of length $3n+1$, are the base blocks of a CSTS$(6n+3)$.

Rosa [75] also showed that a similar partition of $\{1, 2, \ldots, n, n+2, \ldots, 2n, 2n+2\}$ exists if and only if $n \equiv 1$ or $2 \pmod 4$, where $n \geq 2$. The existence of the two sequences above proved the sufficiency for the existence of a CSTS$(6n+3)$.

Skolem sequences were first studied for use in constructing cyclic Steiner triple systems. Later, these sequences were generalized in many ways and are applied in several areas such as: triple systems [47], factorization of complete graphs [72], balanced ternary designs [9, 10], and design of statistical models, such as a balanced sampling plan excluding contiguous units [89]. Some other papers in which these sequences have been very useful are [7, 14, 17, 25, 26].

The spectrum of cyclic BIBD$(v, 4, \lambda)$ is not determined yet, although it has been treated in many papers. The first one to consider cyclic designs with block size four was Bose [15]. He constructed an infinite family of cyclic BIBD$(v, 4, 1)$ for $v = 12n+1$ prime. His result was improved by Buratti [20] and by Chen and Zhu [35]. Bose [16] constructed another two infinite families of cyclic block designs for $k = 4$, $\lambda = 2$ or 3, and v a prime number. Wilson [90] also constructed infinite families of such designs. Particular attention was paid to cyclic BIBD$(v, 4, 1)$. Hanani [60] demonstrated that a cyclic BIBD$(v, 4, 1)$ can only exist when $v \equiv 1, 4 \pmod{12}$. It is reasonable to believe that a cyclic BIBD$(v, 4, 1)$ exists for any admissible orders $v \geq 37$, but the problem is far from being settled.

It was shown that there exists cyclic BIBD$(12t + 1, 4, 1)$ for $t \leq 1000$ with one exception of $t = 2$ [1,3,56] and there exist cyclic BIBD$(12t+4, 4, 1)$ for $3 \leq t \leq 50$ [36]. There is no cyclic BIBD$(12t + 4, 4, 1)$ for $t = 1, 2$ [36].

A few direct constructions for cyclic BIBD$(v, 4, 1)$ are known. They are given by Colbourn and Mathon [45], Mathon [67], and Rogers [74].

Furino [53] constructed new cyclic block designs for $v \equiv 1, 5, 7, 11 (\bmod\ 12)$ and $\lambda = 2, 3$ or 6. Also, a few recursive constructions for cyclic BIBD$(v, 4, \lambda)$ are known [21, 43, 62, 63].

The existence of cyclic BIBD$(v, 4, \lambda)$ for small values of v plays an important role in the recursive constructions for new cyclic designs. Some super-simple cyclic designs with small values of v were constructed [37]. Also some linear classes of super-simple cyclic designs are constructed in [38].

Although there exist direct and recursive constructions for cyclic BIBD$(v, 4, \lambda)$, the spectrum of the cyclic BIBD$(v, 4, \lambda)$ is still an open problem.

1.3 Outline of the Thesis

This thesis uses Skolem-type sequences to construct new cyclic BIBD(v, k, λ) for $k \geq 3$. We also use Skolem-type sequences to construct cyclic block designs with the following properties: simple, cyclic, indecomposable, and cyclically indecomposable but decomposable.

Chapter 1 gives the outline of the thesis. In this chapter, we give also some known definitions and known results which will be useful for the further proven results.

Chapter 2 introduces a new Skolem-type sequence with three hooks which will be used in Chapter 3 to construct cyclic BIBD$(v, 3, 12)$. This chapter also introduces new Skolem and Rosa arrays which can be used to construct cyclic BIBD$(v, 3, \lambda)$. These arrays will also be used in Chapter 4 to provide many examples of cyclic BIBD$(v, 4, \lambda)$.

Chapter 3 discusses the connection between Skolem-type sequences and cyclic block designs with block size 3. It was known that Skolem sequences give rise to cyclic block designs with block size 3 and $\lambda = 1$ and 2. In this chapter, we show that Skolem-type sequences can be used to generate cyclic designs with block size 3 and all admissible λs. These results were published in *Design, Codes and Cryptography* [82].

Chapter 4 discusses cyclic block designs with block size 4 constructed with the use of Skolem-type sequences. We give the necessary conditions for the existence of a cyclic block design with block size 4. We construct, using Skolem-type sequences, cyclic BIBD$(v, 4, 6)$ for all $v \equiv 1, 5, 7, 11 (\mathrm{mod}\, 12)$. Then, we generalize these constructions to construct cyclic BIBD(v, k, λ) for all $v \equiv 1, 5, 7, 11 (\mathrm{mod}\, 12)$. Furthermore, we provide a complete set of examples of Skolem partitions that induce cyclic BIBD$(v, 4, \lambda)$ for every admissible class. These results were submitted for publication [79].

Chapter 5 discusses cyclic block designs with block size four constructed with the use of other combinatorial structures. We construct several new linear classes of cyclic block designs with block size 4 and $\lambda > 1$. Specifically, we construct cyclic BIBD$(6t+1, 4, 2)$ for $t \leq 1000$ and cyclic BIBD$(v, 4, 2)$ for $v = 30t+7, 78t+7, 114t+25, 138t+31, 150t+31, 162t+31, 174t+37, 174t+43$ for every $t \leq 1000, t \neq 2, 3$. We also construct, using relative difference families, many new linear classes of cyclic BIBD$(v, 4, 2)$ for some values of $v \equiv 10 \,(\mathrm{mod}\, 12)$, new cyclic BIBD$(v, 4, 3)$ for some values of $v \equiv 0, 8, 9 \,(\mathrm{mod}\, 12)$, new cyclic BIBD$(v, 4, 6)$ for some values of $v \equiv 0, 2, 3, 6, 8 (\mathrm{mod}\, 12)$, and new cyclic BIBD$(v, 4, 4)$ for some $v \equiv 4 (\mathrm{mod}\ 12)$. For cyclic BIBD$(v, 4, 6)$ and $v \equiv 6 (\mathrm{mod}\ 12)$, our constructions cover all the values of v except for $v = 810, 30v', 810v'$ where v' is a product of primes greater than 5. The problem of constructing cyclic BIBD$(v, 4, \lambda)$ for all admissible orders v is still an open problem. This chapter is a considerable step forward to the solution of this problem.

The results of this chapter were submitted for publication [80].

Chapter 6 discusses block designs with block size three having different properties: cyclic, simple, indecomposable, and cyclically indecomposable but decomposable. We construct, using Skolem-type sequences, cyclic, simple, and indecomposable BIBD$(v, 3, 3)$ for all admissible orders v, with some possible exceptions for $v = 9$ and $v = 24c + 9$, $c \geq 4$. These results were accepted for publication in *Journal of Combinatorial Mathematics and Combinatorial Computing* [78]. We also construct cyclic block designs that are cyclically indecomposable but decomposable. We give examples of cyclically indecomposable but decomposable BIBD$(v, 3, 4)$, for $v \equiv 0, 1 (\text{mod } 3), v \leq 21$, and a few constructions which yield infinitely many such triple systems of order $v \equiv 0, 1(\text{mod } 3)$. The results were submitted for publication [59].

Chapter 7 describes the known constructions of optical orthogonal codes from Skolem-type sequences.

Chapter 8 concludes the thesis and provides several open questions.

Chapter 2

New Skolem-type Arrays

In this chapter, we introduce a new Skolem-type sequence with three hooks and new Skolem and Rosa-type arrays. Then, in Chapters 3 and 4, we are going to use these sequences to construct many new cyclic BIBD(v, k, λ), $k \geq 3$.

Besides the fact that these new Skolem-type sequences will produce new cyclic block designs, they can also be used to construct cyclic block designs with different properties, like cyclic and simple designs, cyclic and indecomposable designs, cyclic and decomposable designs, etc.

2.1 Skolem-type Sequences with Three Hooks

We introduce a new Skolem-type sequence. This is an m-near S_{2m-1} with three hooks in positions m, $2m$, $3m$ (i.e., a Skolem-type sequence with $t = 4m-1$, $D = \{1, \ldots, 2m-1\} \setminus \{m\}$ and $s_m = s_{2m} = s_{3m} = 0$).

Definition 2.1.1 *Let m be a positive integer. An m-*near Skolem sequence of order $2m-1$ with three hooks in positions m, $2m$, and $3m$ *is a sequence $N = (n_1, n_2, \ldots, n_{4m-1})$ of integers $n_i \in \{1, 2, \ldots, 2m-1\} \setminus \{m\}$ that satisfies the following conditions:*

1. *for every $k \in \{1, 2, \ldots, 2m-1\}, k \neq m$, there are exactly two elements $n_i, n_j \in N$ such that $n_i = n_j = k$;*

2. *if $n_i = n_j = k$ with $i < j$, then $j - i = k$;*

3. *$n_m = n_{2m} = n_{3m} = 0$.*

Theorem 2.1.1 *An m-near Skolem sequence of order $2m-1$ with three hooks in positions m, $2m$ and $3m$ exists for $m \equiv 2 \pmod 4, m \geq 6$.*

Proof A Skolem construction which yields m-near Skolem sequences of order $2m-1$ with hooks in positions m, $2m$ and $3m$ for $m \equiv 2 (\text{mod } 4), m \geq 6$ will be given. For $m = 6$ the sequence is $(9, 11, 5, 7, 4, *, 10, 5, 4, 9, 7, *, 11, 2, 8, 2, 10, *, 3, 1, 1, 3, 8)$.

Note that in the construction, a_i and b_i represent the two positions in the sequence of the element i, with $a_i < b_i$ and $1 \leq i \leq 2m-1, i \neq m$.

For $m \equiv 2 (\text{mod } 4)$, let $m = 4r+2, r \geq 2$. The required construction is given in Table 2.1.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$3m+2r-1$	$3m+2r$	1	-
(2)	$2m+2r$	$2m+2r+2$	2	-
(3)	$3m-2$	$3m+1$	3	-
(4)	$m-1$	$m+3$	4	-
(5)	$m+1$	$3m-1$	$2m-2$	-
(6)	$3m+2$	$4m-2$	$m-4$	-
(7)	$2m+3+2j$	$4m-1-2j$	$2m-4-4j$	r
(8)	$2m+2+2j$	$4m-4-2j$	$2m-6-4j$	$r-2$
(9)	$m-3-2j$	$m+2+2j$	$5+4j$	$2r-1$
(10)	$m-2-2j$	$m+5+2j$	$7+4j$	$2r-1$
(11)	$3m-3-j$	$3m+3+j$	$6+2j$	$2r-5$
Omit row (11) when $r=2$				

Table 2.1: m-near Skolem sequence of order $2m-1$, for $m \equiv 2 (\text{mod } 4)$

To verify that this construction forms an m-near Skolem sequence of order $2m-1$ with hooks in positions m, $2m$ and $3m$, it must be shown that each element of $\{1, 2, \ldots, m-1, m+1, \ldots, 2m-1, 2m+1, \ldots, 3m-1, 3m+1, \ldots 4m-1\} = \{1, \ldots, 4r+1, 4r+3, \ldots 8r+3, 8r+5, \ldots, 12r+5, 12r+7, \ldots, 16r+7\}$ appears in a pair (a_i, b_i) exactly once, and that the differences $b_i - a_i$ are exactly the elements $\{1, \ldots, m-1, m+1, \ldots, 2m-1\} = \{1, \ldots, 4r+1, 4r+3, \ldots, 8r+3\}$.

Consider the pairs (a_i, b_i). It is easy to check that there are $2m - 2 = 8r + 2$ such pairs, and so exactly $16r + 4$ elements a_i and b_i. Thus, if every element of $\{1, \ldots, 4r+1, 4r+3, \ldots 8r+3, 8r+5, \ldots, 12r+5, 12r+7, \ldots, 16r+7\}$ occurs in one of these pairs, each of these elements must occur exactly once.

Now, the elements $1, 3, \ldots, 4r-3, 4r-1$ occur in the pairs $(m-3-2j, m+2+2j)$ for $0 \le j \le 2r-1$, from row (9). The elements $2, 4, \ldots, 4r-2, 4r$ occur in the pairs $(m-2-2j, m+5+2j)$ for $0 \le j \le 2r-1$, from row (10). The element $4r+1$ is given by the pair $(m-1, m+3)$ from row (4). The element $4r+3$ is given by the pair $(m+1, 3m-1)$ from row (5). The elements $4r+4, 4r+6, \ldots, 8r+2$ occur in the pairs $(m-3-2j, m+2+2j)$ for $0 \le j \le 2r-1$, from row (9). The element $4r+5$ is given by the pair $(m-1, m+3)$, from row (4). The elements $4r+7, 4r+9, \ldots, 8r+5$ occur in the pairs $(m-2-2j, m+5+2j)$ for $0 \le j \le 2r-1$, from row (10). The elements $8r+6, 8r+8, \ldots, 10r+2$ occur in the pairs $(2m+2+2j, 4m-4-2j)$ for $0 \le j \le r-2$, from row (8). The elements $8r+7, 8r+9, \ldots, 10r+7$ occur in the pairs $(2m+3+2j, 4m-1-2j)$ for $0 \le j \le r$, from row (7).

Both the elements $10r+4$ and $10r+6$ appear in the pair $(2m+2r, 2m+2r+2)$, from row (2). The elements $10r+8, 10r+9, \ldots, 12r+3$, occur in the pairs $(3m-3-j, 3m+3+j)$ for $0 \le j \le 2r-5$, from row (11). The elements $12r+4$ and $12r+7$ appear both in the pair $(3m-2, 3m+1)$, from row (3). The element $12r+5$ occurs in the pair $(m+1, 3m-1)$ from row (5). The element $12r+8$ appears in the pair $(3m+2, 4m-2)$, from row (6). The elements $12r+9, 12r+10, \ldots, 14r+4$, occur in the pairs $(3m-3-j, 3m+3+j)$ for $0 \leq j \leq 2r-5$, from row (11).

Both the elements $14r+5$ and $14r+6$, occur in the pair $(3m+2r-1, 3m+2r)$, from row (1). The elements $14r+7, 14r+9, \ldots, 16r+7$, occur in the pairs $(2m+3+2j, 4m-1-2j)$ for $0 \leq j \leq r$, from row (7). The elements $14r+8, 14r+10, \ldots, 16r+4$, occur in the pairs $(2m+2+2j, 4m-4-2j)$ for $0 \leq j \leq r-2$, from row (8). And finally, the element $16r+6$ appears in the pair $(3m+2, 4m-2)$, from row (6).

Therefore, all the elements of $\{1, \ldots, 4r+1, 4r+3, \ldots 8r+3, 8r+5, \ldots, 12r+5, 12r+7, \ldots, 16r+7\}$, occur in the pairs (a_i, b_i). Hence, each such element occurs exactly once as either a_i or b_i for some i.

Secondly, it must be shown that the differences $b_i - a_i$ give all values $\{1, \ldots, 4r+1, 4r+3, \ldots, 8r+3\}$ exactly once. Again, there are $2m-2 = 8r+2$ such differences, so it must only be shown that each such element occurs at least once, which will then imply that each occurs exactly once.

Difference $1 = (3m+2r) - (3m+2r-1)$ can be found in row (1). Difference $2 = (2m+2r+2) - (2m+2r)$ appears in row (2). Difference $3 = (3m+1) - (3m-2)$ appears in row (3), and difference $4 = (m+3) - (m-1)$ is the difference of $b_i - a_i$, in row (4). The differences $(m+2+2j) - (m-3-2j) = 5+4j$ for $0 \leq j \leq 2r-1$, in row (9), give the numbers $5, 9, 13, \ldots, 8r+1$. The differences $(m+5+2j)-(m-2-2j) = 7+4j$ for $0 \leq j \leq 2r-1$, in row (10), give the numbers $7, 11, 15, \ldots, 8r+3$. So, all the odd differences are covered. The differences $(3m+3+j)-(3m-3-j) = 6+2j$ for $0 \leq j \leq 2r-5$ in row (11) give the numbers $6, 8, \ldots, 4r-4$. The difference $(4m-2)-(3m+2) = m-4 = 4r-2$ is the difference in row (6). The differences $(4m-1-2j)-(2m+3+2j) = 2m-4-4j$ for $0 \leq j \leq r$, in row (7), give the numbers $4r, 4r+4, 4r+8, \ldots, 8r$. The differences $(4m-4-2j)-(2m+2+2j) = 2m-6-4j$ for $0 \leq j \leq r-2$, in row (8), give the numbers $4r+6, 4r+10, \ldots, 8r-2$. To complete the even elements of $\{1, \ldots, 4r+1, 4r+3, \ldots, 8r+3\}$, $8r+2$ occurs as the difference $(3m-1)-(m+1)$, in row (5). Thus, the verification is complete. ■

2.2 *m*-fold Skolem Arrays

Definition 2.2.1 *A* 2-fold Skolem array of order n *is a* $2 \times 2n$ *array* $\begin{matrix} s_1^1 & s_2^1 & \ldots & s_{2n}^1 \\ s_1^2 & s_2^2 & \ldots & s_{2n}^2 \end{matrix}$ *such that for every integer* $i \in \{1, 2, \ldots, n\}$ *there are exactly two pairs* (j_1, j_1+i) *and* (j_2, j_2+i) *such that* $s_{j_1}^1 = s_{j_1+i}^1 = i$ *(or* $s_{j_1}^1 = s_{j_1+i}^2 = i$*) and* $s_{j_2}^2 = s_{j_2+i}^2 = i$ *(or* $s_{j_2}^2 = s_{j_2+i}^1 = i$*).*

Example 2.2.1 $\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{matrix}$ *and* $\begin{matrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{matrix}$ *are 2-fold Skolem arrays of order* 2.

Theorem 2.2.1 *There exists a 2-fold Skolem array of order n for every positive integer n.*

Proof To show that a 2-fold Skolem array of order n exists for every positive integer n, we consider three different cases.

Case 1: $n \equiv 0, 1 \pmod 4$

Take two Skolem sequences of order n.

Case 2: $n \equiv 2 \pmod 4$

A 2-fold Skolem array of order n is given in Table 2.2.

Let $n = 4r + 2$, $r \geq 0$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1 + j$	$n + 1 - j$	$n - 2j$	$2r$
(2)	$2r + 2$	$n + 2r + 2$	n	-
(3)	$4r + 4 + j$	$n + 4r + 2 - j$	$n - 2 - 2j$	$2r - 1$
(4)	$1 + j$	$n - j$	$n - 1 - 2j$	$2r$
(5)	$n + 1 + j$	$2n - j$	$n - 1 - 2j$	$2r$
Omit row (3) when $r = 0$				

Table 2.2: 2-fold Skolem arrays of order $n \equiv 2 \pmod 4$

Case 3: $n \equiv 3 \pmod 4$

A 2-fold Skolem array of order n is given in Table 2.3.

Let $n = 4r + 3$, $r \geq 0$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1+j$	$n+1-j$	$n-2j$	$2r+1$
(2)	$n+2+j$	$2n-j$	$n-2-2j$	$2r$
(3)	$2r+2$	$n+2r+2$	n	-
(4)	$1+j$	$n-j$	$n-1-2j$	$2r$
(5)	$4r+4+j$	$n+4r+3-j$	$n-1-2j$	$2r$

Table 2.3: 2-fold Skolem arrays of order $n \equiv 3 (\text{mod } 4)$

■

Definition 2.2.2 *An m-fold Skolem array of order n is an $m \times 2n$ array*

$$\begin{matrix} s_1^1 & s_2^1 & \ldots & s_{2n}^1 \\ \vdots & & & \\ s_1^m & s_2^m & \ldots & s_{2n}^m \end{matrix}$$

such that for every integer $i \in \{1, 2, \ldots, n\}$ there are exactly m pairs $(j_k, j_k + i)$, $k \in \{1, \ldots, m\}$ such that $s_{j_k}^x = s_{j_k+i}^y = i$ for $x, y \in \{1, 2, \ldots, m\}$.

Example 2.2.2

$$\begin{matrix} 4 & 2 & 3 & 2 & 4 & 3 & 1 & 1 \\ 1 & 1 & 4 & 2 & 3 & 2 & 4 & 3 \\ 1 & 1 & 4 & 2 & 3 & 2 & 4 & 3. \end{matrix}$$

is a 3-fold Skolem array of order 4.

Theorem 2.2.2 *An m-fold Skolem array of order n exists if and only if m is even or if m is odd and $n \equiv 0, 1 \, (\text{mod } 4)$.*

Proof We begin by providing the necessity of these conditions.

Let $(a_{1_i}, b_{1_i}), \ldots, (a_{m_i}, b_{m_i})$, $1 \leq i \leq n$ be the pairs of the positions of an integer i in the m-fold Skolem array of order n. Then we have:

$\sum_{i=1}^{n}(a_{1_i} + b_{1_i} \ldots + a_{m_i} + b_{m_i}) = m(1 + 2 + \ldots + 2n) = \dfrac{2mn(2n+1)}{2}$, and

$\sum_{i=1}^{n}(a_{1_i} - b_{1_i} + \ldots + a_{m_i} - b_{m_i}) = m(1 + 2 + \ldots + n) = \dfrac{mn(n+1)}{2}$.

Subtracting the above sums, we get $2\sum_{i=1}^{n}(b_{1_i} + \ldots + b_{m_i}) = \dfrac{mn(3n+1)}{2}$. So, $\sum_{i=1}^{n}(b_{1_i} + \ldots + b_{m_i}) = \dfrac{mn(3n+1)}{4}$. But $b_{1_i}, \ldots, b_{m_i}$ are integers which implies that $\dfrac{mn(3n+1)}{4}$ has to be an integer. This condition gives us m even or m odd and $n \equiv 0,\, 1 \pmod 4$.

To construct an m-fold Skolem array of order n for $n \equiv 0,\, 1 \pmod 4$ and every m, take m copies of a Skolem sequence of order n. To construct an m-fold Skolem array of order n for $n \equiv 2,\, 3 \pmod 4$ and m even, take $\dfrac{m}{2}$ copies of a 2-fold Skolem array of order n. ■

2.3 m-fold Hooked Skolem Arrays

Definition 2.3.1 *A* 2-fold hooked Skolem array of order n *is an* $2 \times (2n+1)$ *array* $\begin{matrix} s_1^1 & s_2^1 & \ldots & s_{2n+1}^1 \\ s_1^2 & s_2^2 & \ldots & s_{2n+1}^2 \end{matrix}$ *such that for every integer* $i \in \{1, 2, \ldots, n\}$ *there are exactly two pairs* $(j_1, j_1 + i)$ *and* $(j_2, j_2 + i)$ *such that* $s_{j_1}^1 = s_{j_1+i}^1 = i$ *(or* $s_{j_1}^1 = s_{j_1+i}^2 = i$*) and*

$s^2_{j_2} = s^2_{j_2+i} = i$ *(or* $s^2_{j_2} = s^1_{j_2+i} = i$*) and* $s^l_{2n} = 0$ *for* $l = 1, 2$.

Example 2.3.1

$$\begin{array}{ccccccccc} 3 & 1 & 4 & 2 & 4 & 2 & 4 & * & 4 \\ 1 & 1 & 1 & 3 & 2 & 3 & 2 & * & 3. \end{array}$$

is a 2-fold hooked Skolem array of order 4.

Theorem 2.3.1 *There exists a 2-fold hooked Skolem array of order* n *for every positive integer* n.

Proof To construct a 2-fold hooked Skolem array of order n for every positive integer n, we consider three different cases.

Case 1: $\boldsymbol{n \equiv 2, 3 \pmod 4}$

Take two hooked Skolem sequences of order n.

Case 2: $\boldsymbol{n \equiv 0 \pmod 4}$

A 2-fold hooked Skolem array of order 4 is given in Example 2.3.1.

A 2-fold hooked Skolem array of order $n \equiv 0 \pmod 4$, $n \geq 8$ is given in Table 2.4.

Let $n = 4r$, $r \geq 2$.

Case 3: $\boldsymbol{n \equiv 1 \pmod 4}$

Table 2.5 gives a 2-fold hooked Skolem array of order n.

Let $n = 4r + 1$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1+j$	$n-j$	$n-1-2j$	1
(2)	$3+j$	$n-1-j$	$n-4-2j$	$2r-3$
(3)	$2r+1$	$n+2r+1$	n	-
(4)	$4r+2j$	$n+4r-2-2j$	$n-2-4j$	$r-1$
(5)	$4r+3+2j$	$n+4r-1-2j$	$n-4-4j$	$r-2$
(6)	$n+1$	$2n+1$	n	-
(7)	$1+j$	$n-2-j$	$n-3-2j$	$2r-2$
(8)	$n+1$	$2n-1$	$n-2$	-
(9)	$n+2$	$2n+1$	$n-1$	-
(10)	$n+3+j$	$2n-2-j$	$n-5-2j$	$2r-3$
(11)	$2n$	$2n$	0	-

Table 2.4: 2-fold hooked Skolem array of order $n \equiv 0 (\text{mod } 4), n \geq 8$

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1+j$	$n+1-j$	$n-2j$	1
(2)	$3+j$	$n-j$	$n-3-2j$	$2r-2$
(3)	$2r+2$	$n+2r+2$	n	-
(4)	$4r+3$	$n+4r+2$	$n-1$	-
(5)	$4r+2+2j$	$n+4r-1-2j$	$n-3-4j$	$r-1$
(6)	$4r+5+2j$	$n+4r-2j$	$n-5-4j$	$r-2$
(7)	$1+j$	$n-1-j$	$n-2-2j$	$2r-1$
(8)	$4r+3$	$n+4r+2$	$n-1$	-
(9)	$4r+4+j$	$n+4r-j$	$n-4-2j$	$2r-2$
(11)	$2n$	$2n$	0	-
Omit row (6) when $r=1$				

Table 2.5: 2-fold hooked Skolem array of order $n \equiv 1 (\text{mod } 4), n \geq 5$

■

Definition 2.3.2 *An* m-fold hooked Skolem array of order n *is an* $m \times (2n+1)$ *array*

$$\begin{array}{cccc} s_1^1 & s_2^1 & \ldots & s_{2n+1}^1 \\ \vdots & & & \\ s_1^m & s_2^m & \ldots & s_{2n+1}^m \end{array}$$ *such that for every integer* $i \in \{1, 2, \ldots, n\}$ *there are exactly* m *pairs* $(j_k, j_k + i)$, $k \in \{1, \ldots, m\}$ *such that* $s^x_{j_k} = s^y_{j_k+i} = i$ *for* $x, y \in \{1, 2, \ldots, m\}$ *and* $s^\ell_{2n} = 0$ *for every* $\ell \in \{1, \ldots, m\}$.

Theorem 2.3.2 *An* m*-fold hooked Skolem array of order* n *exists if and only if* m *is even or if* m *is odd and* $n \equiv 2, 3 \pmod 4$.

The proof for the necessity of these conditions is similar to the proof of Theorem 2.2.2.

Proof To construct an m-fold hooked Skolem array of order n for $n \equiv 2, 3 \pmod 4$ and every m, take m copies of a hooked Skolem sequence of order n. To construct an m-fold Skolem array of order n for $n \equiv 0, 1 (\text{mod } 4)$ and m even, take $\dfrac{m}{2}$ copies of a 2-fold hooked Skolem array of order n. ■

2.4 m-fold Skolem Arrays with a hooked extension

Definition 2.4.1 *An* m-fold Skolem array of order n with a hooked extension *is defined similarly to* m*-fold Skolem arrays of order* n *with the exception that the last row has* $2n + 1$ *entries and* $s^m_{2n} = 0$.

Example 2.4.1

$$\begin{array}{ccccc} 2 & 2 & 2 & 2 & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 2 & * & 2. \end{array}$$

is a 3-fold Skolem array of order 2 with a hooked extension.

Theorem 2.4.1 *An m-fold Skolem array of order n with a hooked extension exists if and only if m is odd and $n \equiv 2, 3 \pmod 4$.*

The proof for the necessity of these conditions is similar to the proof of Theorem 2.2.2.

Proof To construct an m-fold Skolem array of order n with a hooked extension for $n \equiv 2, 3 \pmod 4$ and m odd, take $\frac{m-1}{2}$ copies of a 2-fold Skolem array of order n and for the last row take a hooked Skolem sequence of order n. ■

2.5 m-fold Rosa Arrays

Definition 2.5.1 *A* 2-fold Rosa array of order n *is a* $2 \times (2n+1)$ *array defined similarly to 2-fold Skolem arrays of order n with the exception that $s^{\ell}_{n+1} = 0$ for $\ell = 1, 2$.*

Example 2.5.1 $\begin{array}{ccccccccccc} 2 & 5 & 2 & 5 & 3 & * & 5 & 3 & 5 & 1 & 1 \\ 2 & 3 & 2 & 4 & 3 & * & 4 & 4 & 1 & 1 & 4 \end{array}$ *is a 2-fold Rosa array of order 5.*

Theorem 2.5.1 *There exists a 2-fold Rosa array of order* n *for every positive integer* n*,* $n > 2$*.*

Proof To construct a 2-fold Rosa array of order n for every positive integer n, we consider three different cases.

Case 1: $n \equiv 0,\ 3 \pmod 4$

Take two Rosa sequences of order n.

Case 2: $n \equiv 1 \pmod 4$

A 2-fold Rosa array of order n is given in Table 2.6. Let $n = 4r + 1$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1 + j$	$n - j$	$n - 1 - 2j$	$2r - 1$
(2)	$2r + 1$	$n + 2r + 1$	n	-
(3)	$n + 2 + j$	$2n - j$	$n - 2 - 2j$	$2r - 2$
(4)	$2r + 2$	$n + 2r + 2$	n	-
(5)	$n + 2$	$2n + 1$	$n - 1$	-
(6)	1	2	1	-
(7)	$3 + j$	$n - j$	$n - 3 - 2j$	$2r - 2$
(8)	$n + 3 + j$	$2n + 1 - j$	$n - 2 - 2j$	$2r - 1$
(9)	$n + 1$	$n + 1$	0	-

Table 2.6: 2-fold Rosa arrays of order $n \equiv 1 \pmod 4$

Case 3: $n \equiv 2 \pmod 4$

A 2-fold Rosa array of order n is given in Table 2.7. Let $n = 4r + 2$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1 + j$	$n - j$	$n - 1 - 2j$	$2r$
(2)	$n + 2 + j$	$2n - j$	$n - 2 - 2j$	$2r - 1$
(3)	$2r + 2$	$n + 2r + 2$	n	-
(4)	$2r + 3$	$n + 2r + 3$	n	-
(5)	$n + 2$	$2n + 1$	$n - 1$	-
(6)	1	2	1	-
(7)	$3 + j$	$n - j$	$n - 3 - 2j$	$2r - 2$
(8)	$n + 3 + j$	$2n + 1 - j$	$n - 2 - 2j$	$2r - 1$
(9)	$n + 1$	$n + 1$	0	-

Table 2.7: 2-fold Rosa arrays of order $n \equiv 2 (\mathrm{mod}\ 4)$

■

Definition 2.5.2 *An m-fold* Rosa array of order n *is an* $m \times (2n + 1)$ *array defined similarly to m-fold Skolem arrays of order n with the exception that* $s^{\ell}_{n+1} = 0$ *for* $\ell \in \{1, \ldots, m\}$.

Theorem 2.5.2 *An m-fold Rosa array of order n exists if and only if m is even or if m is odd and* $n \equiv 0, 3 \pmod 4$.

The proof for the necessity of these conditions is similar to the proof of Theorem 2.2.2.

Proof To construct an m-fold Rosa array of order n for $n \equiv 0,\, 3 \pmod 4$ and every m, take m copies of a Rosa sequence of order n. To construct an m-fold Rosa array of order n for $n \equiv 1,\, 2 (\text{mod } 4)$ and m even, take $\frac{m}{2}$ copies of a 2-fold Rosa array of order n. ■

2.6 m-fold Hooked Rosa Arrays

Definition 2.6.1 *A* 2-fold hooked Rosa array of order n *is a* $2 \times (2n+2)$ *array defined similarly to* 2*-fold Skolem arrays of order* n *with the exception that* $s^{\ell}_{n+1} = 0$, $s^{\ell}_{2n+1} = 0$ *for* $\ell = 1, 2$.

Example 2.6.1
$$\begin{array}{cccccccccc} 3 & 1 & 1 & 3 & * & 1 & 1 & 2 & * & 2 \\ 3 & 4 & 4 & 3 & * & 4 & 4 & 2 & * & 2 \end{array}$$
is a 2*-fold hooked Rosa array of order* 4.

Theorem 2.6.1 *There exists a* 2*-fold hooked Rosa array of order* n *for every positive integer* n.

Proof To construct a 2-fold hooked Rosa array of order n for every positive integer n we consider three different cases.

Case 1: $n \equiv 1,\, 2 \pmod 4$

Take two hooked Rosa sequences of order n.

Case 2: $n \equiv 0 \pmod 4$

A 2-fold hooked Rosa array of order 4 is given in Example 2.6.1.

A 2-fold hooked Rosa array of order $n \equiv 0 \pmod 8$ is given in Table 2.8.

Let $n = 8r$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1+j$	$n-j$	$n-1-2j$	$4r-1$
(2)	$n+2$	$2n+2$	n	-
(3)	$n+3+j$	$2n-j$	$n-3-2j$	$4r-2$
(4)	$1+j$	$n-1-j$	$n-2-2j$	$4r-2$
(5)	$4r$	$n+4r-1$	$n-1$	-
(6)	n	$2n-2$	$n-2$	-
(7)	$n+2$	$2n+2$	n	-
(8)	$n+3+2j$	$2n-1-2j$	$n-4-4j$	$2r-3$
(9)	$n+4+4j$	$2n-6-4j$	$n-10-8j$	$r-2$
(10)	$n+6+4j$	$2n-4j$	$n-6-8j$	$r-2$
(11)	$12r+j$	$12r+4-j$	$4-4j$	1
(12)	$n+1$	$n+1$	0	-
(13)	$2n+1$	$2n+1$	0	-
Omit rows (8), (9), (10) when $r=1$				

Table 2.8: 2-fold hooked Rosa arrays of order $n \equiv 0 (\mathrm{mod}\ 8)$

A 2-fold hooked Rosa array of order $n \equiv 4 \pmod 8$ is given in Table 2.9. Let $n = 8r+4$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1+j$	$n-j$	$n-1-2j$	$4r+1$
(2)	$n+2$	$2n+2$	n	-
(3)	$n+3+j$	$2n-j$	$n-3-2j$	$4r$
(4)	$1+j$	$n-1-j$	$n-2-2j$	$4r$
(5)	$4r+2$	$n+4r+1$	$n-1$	-
(6)	n	$2n-2$	$n-2$	-
(7)	$n+2$	$2n+2$	n	-
(8)	$n+3+2j$	$2n-1-2j$	$n-4-4j$	$2r-2$
(9)	$n+4+4j$	$2n-6-4j$	$n-10-8j$	$r-2$
(10)	$n+6+4j$	$2n-4j$	$n-6-8j$	$r-2$
(11)	$12r+6$	$12r+12$	6	-
(12)	$12r+4$	$12r+8$	4	-
(13)	$12r+7$	$12r+9$	2	-
(14)	$n+1$	$n+1$	0	-
(15)	$2n+1$	$2n+1$	0	-
Omit rows $(9),(10)$ when $r=1$				

Table 2.9: 2-fold hooked Rosa arrays of order $n \equiv 4 (\text{mod } 8)$

Case 3: $n \equiv 3\ (\text{mod } 4)$

A 2-fold hooked Rosa array of order 3 is: $\begin{array}{cccccccc} 1 & 1 & 3 & * & 3 & 3 & * & 3 \\ 1 & 1 & 2 & * & 2 & 2 & * & 2 \end{array}$.

A 2-fold hooked Rosa array of order $n \equiv 7\ (\text{mod } 8)$ is given in Table 2.10.

Let $n = 8r + 7$, $r \geq 0$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1 + j$	$n - 1 - j$	$n - 2 - 2j$	$4r + 2$
(2)	n	$2n$	n	-
(3)	$n + 2$	$2n + 2$	n	-
(4)	$n + 3 + j$	$2n - 1 - j$	$n - 4 - 2j$	$4r + 1$
(5)	$1 + j$	$n - j$	$n - 1 - 2j$	$4r + 2$
(6)	$4r + 4$	$n + 4r + 3$	$n - 1$	-
(7)	$n + 2 + 4j$	$2n - 3 - 4j$	$n - 5 - 8j$	$r - 1$
(8)	$n + 3 + 2j$	$2n - 2j$	$n - 3 - 4j$	$2r - 1$
(9)	$n + 4$	$2n + 2$	$n - 2$	-
(10)	$8r + 15 + 4j$	$n + 8r + 6 - 4j$	$n - 9 - 8j$	$r - 1$
(11)	$12r + 9$	$12r + 13$	4	-
(12)	$12r + 12$	$12r + 14$	2	-
(13)	$n + 1$	$n + 1$	0	-
(14)	$2n + 1$	$2n + 1$	0	-
Omit rows $(7), (8), (10)$ when $r = 0$				

Table 2.10: 2-fold hooked Rosa arrays of order $n \equiv 7 (\text{mod } 8)$

A 2-fold hooked Rosa array of order $n \equiv 3\,(\text{mod } 8), n \geq 11$ is given in Table 2.11.

Let $n = 8r + 3$, $r \geq 1$.

	a_i	b_i	i	$0 \leq j \leq$
(1)	$1 + j$	$n - 1 - j$	$n - 2 - 2j$	$4r$
(2)	n	$2n$	n	-
(3)	$n + 2$	$2n + 2$	n	-
(4)	$n + 3 + j$	$2n - 1 - j$	$n - 4 - 2j$	$4r - 1$
(5)	$1 + j$	$n - j$	$n - 1 - 2j$	$4r$
(6)	$4r + 2$	$n + 4r + 1$	$n - 1$	-
(7)	$n + 2 + 4j$	$2n - 3 - 4j$	$n - 5 - 8j$	$r - 1$
(8)	$n + 3 + 2j$	$2n - 2j$	$n - 3 - 4j$	$2r - 2$
(9)	$n + 4$	$2n + 2$	$n - 2$	-
(10)	$n + 8 + 4j$	$2n - 1 - 4j$	$n - 9 - 8j$	$r - 2$
(11)	$12r + 5 + j$	$12r + 9 - j$	$4 - 2j$	1
(12)	$n + 1$	$n + 1$	0	-
(13)	$2n + 1$	$2n + 1$	0	-
Omit row (10) when $r = 1$				

Table 2.11: 2-fold hooked Rosa arrays of order $n \equiv 11 (\text{mod } 8)$

■

Definition 2.6.2 *An* m-fold hooked Rosa array of order n *is an* $m \times (2n + 2)$ *array defined similarly to m-fold Skolem arrays of order n with the exception that* $s^{\ell}_{n+1} = 0$, $s^{\ell}_{2n+1} = 0$ *for* $\ell \in \{1, \ldots, m\}$.

Theorem 2.6.2 *An m-fold hooked Rosa array of order n exists if and only if m is even or if m is odd and* $n \equiv 1, 2 \pmod 4$.

The proof for the necessity of these conditions is similar to the proof of Theorem 2.2.2.

Proof To construct an m-fold Rosa array of order n for $n \equiv 1, 2 \pmod 4$ and every m, take m copies of a hooked Rosa sequence of order n. To construct an m-fold hooked Rosa array of order n for $n \equiv 0, 3 (\text{mod } 4)$ and m even, take $\frac{m}{2}$ copies of a 2-fold hooked Rosa array of order n. ■

2.7 m-fold Rosa Arrays with a hooked extension

Definition 2.7.1 *An* m-fold Rosa array of order n with a hooked extension *is defined similarly to m-fold Rosa arrays of order n with the exception that the last row has* $2n + 2$ *entries and* $s^m_{2n+1} = 0$.

Example 2.7.1
$$\begin{array}{cccccccccccc} 4 & 5 & 5 & 5 & 4 & * & 5 & 5 & 5 & 1 & 1 & \\ 4 & 2 & 4 & 2 & 4 & * & 4 & 2 & 2 & 2 & 2 & \\ 1 & 1 & 1 & 1 & 3 & * & 3 & 3 & 3 & 3 & * & 3 \end{array}$$
is a 3-fold Rosa array of order 5 *with a hooked extension.*

Theorem 2.7.1 *An m-fold Rosa array of order n with a hooked extension exists if and only if m is odd and* $n \equiv 1, 2 \pmod 4$.

The proof for the necessity of these conditions is similar to the proof of Theorem 2.2.2.

Proof To construct an m-fold Rosa array of order n with a hooked extension for $n \equiv 1, 2 \pmod 4$ and m odd, take $\dfrac{m-1}{2}$ copies of a 2-fold Rosa array of order n and for the last row take a hooked Rosa sequence of order n. ■

Chapter 3

Cyclic BIBD($v, 3, \lambda$) from Skolem-type Sequences

In 1939, Peltesohn [70] solved Heffter's two difference problems showing at least one solution exists and constructed cyclic Steiner triple systems of order v for $v \equiv 1, 3 \pmod 6$, $v \neq 9$. In 1981, Colbourn and Colbourn [42] solved the existence problem for cyclic block designs with block size 3 and $\lambda > 1$ using Peltesohn technique.

In 1957, Skolem [84] showed that Skolem sequences can be used to construct cyclic Steiner triple systems of order v for $v \equiv 1, 3 \pmod 6$, $v \neq 9$. With this solution, Skolem provided a second existence proof for the existence of cyclic Steiner triple systems. In 2000, Rees and Shalaby [73] showed that Skolem sequences can be used to construct cyclic BIBD$(v, 3, 2)$ for all admissible orders v.

Since Skolem sequences can be used to construct cyclic BIBD$(v, 3, \lambda)$ for $\lambda = 1, 2$, an obvious question is if these Skolem sequences can be used to construct cyclic BIBD$(v, 3, \lambda)$ for all admissible v and λ.

In this chapter, we use Skolem-type sequences to provide a complete proof for the existence of cyclic BIBD$(v, 3, \lambda)$ for all admissible orders v and λ.

First, we describe the known constructions that give cyclic BIBD$(v, 3, \lambda)$ for $\lambda = 1$ and $\lambda = 2$. Then we give six new constructions that will generate cyclic block designs for $\lambda = 3, 4, 6$ and 12.

Construction 3.0.1 *[84] From a Skolem sequence or a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$ for $1 \leq i \leq n$. The set of all triples $(i, a_i{+}n, b_i{+}n)$ for $1 \leq i \leq n$ is a solution to the first Heffter difference problem. These triples yield the base blocks for a CSTS$(6n+1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$. Also, $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a CSTS$(6n+1)$.*

Example 3.0.2 $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$, yields the pairs $\{(1, 2), (4, 6), (5, 8), (3, 7)\}$. These pairs yield in turn the triples $\{(1, 5, 6), (2, 8, 10), (3, 9, 12), (4, 7, 11)\}$ forming a solution to the first Heffter problem. These triples yield the base blocks for two CSTS(25)s*:*

1. $\{0,5,6\},\{0,8,10\},\{0,9,12\},\{0,7,11\} \pmod{25}$

2. $\{0,1,6\},\{0,2,10\},\{0,3,12\},\{0,4,11\} \pmod{25}$.

Construction 3.0.2 *[75] From a Rosa sequence or a hooked Rosa sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$ for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$ for $1 \le i \le n$ is a solution to the second Heffter difference problem. These triples yield the base blocks for a CSTS$(6n+3)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$ together with the short orbit $\{0, 2n+1, 4n+2\} \pmod{6n+3}$.*

Construction 3.0.3 *[73] Let $S_n = (s_1, s_2, \ldots, s_{2n})$ be a Skolem sequence of order n and let $\{(a_i, b_i) | 1 \le i \le n\}$ be the pairs of the positions in S_n for which $b_i - a_i = i$. Then the set $\{i, a_i + n, b_i + n\}$ partitions the set $\{1, \ldots, 3n\}$ into n triples (a, b, c) such that $a + b \equiv c \pmod{3n+1}$. Hence the set of triples $\{\{0, i, b_i + n\} | i = 1, \ldots, n\}$ forms the base blocks for a cyclic BIBD$(3n+1, 3, 2)$.*

Example 3.0.3 $S_5 = (5,1,1,3,4,5,3,2,4,2)$ gives the triples $\{(1,7,8), (2,13,15), (3,9,12), (4,10,14), (5,6,11)\}$. These triples give the base blocks of a cyclic BIBD$(16,3,2)$: $\{\{0,1,8\},\{0,2,15\},\{0,3,12\},\{0,4,14\},\{0,5,11\}\} \pmod{16}$.

Construction 3.0.4 *[73] Let $R_n = (r_1, r_2, \ldots, r_{2n+1})$ be a Rosa sequence of order n $(r_{n+1} = 0)$, and let $\{(a_i, b_i) | 1 \le i \le n\}$ be the pairs of positions in R_n for which*

$b_i - a_i = i$. *Then the set* $\{i, a_i + n + 1, b_i + n + 1\}$ *partitions the set* $\{1, \ldots, 3n + 2\} \backslash \{n+1, 2n+2\}$ *into* n *triples* (a, b, c) *such that* $a + b \equiv c \pmod{3n+3}$. *Hence the set of triples* $\{\{0, i, b_i + n + 1\} | i = 1, \ldots, n\}$ *forms the base blocks for a cyclic* $(3, 2)$*-GDD of type* 3^{n+1} *(whose groups are given by* $\{0, n+1, 2n+2\} \pmod{3n+3}$*) which in turn gives rise to a cyclic BIBD*$(3n + 3, 3, 2)$.

Example 3.0.4 A Rosa sequence of order 3, $R_3 = (1, 1, 3, *, 2, 3, 2)$ gives the triples $\{(1, 5, 6), (2, 9, 11), (3, 7, 10)\}$. These triples give the base blocks of a cyclic BIBD$(12, 3, 2)$: $\{\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\}\}$ (with two copies of $\{0, 4, 8\}$)(mod 12).

Next, we provide six new constructions that will give cyclic BIBD$(v, 3, \lambda)$ for $\lambda \geq 3$.

Construction 3.0.5 *Let* $S_n = (s_1, s_2, \ldots, s_{2n})$ *be a Skolem sequence of order* n *and let* $\{(a_i, b_i) | 1 \leq i \leq n\}$ *be the pairs of positions in* S_n *for which* $b_i - a_i = i$. *Then the set* $\mathcal{F}= \{\{0, i, b_i\} | 1 \leq i \leq n\} \pmod{2n+1}$ *is a* $(2n + 1, 3, 3)$*-DF. Hence, the set of triples in* $\mathcal{F}$ *forms the base blocks of a cyclic BIBD*$(2n + 1, 3, 3)$.

Proof Set $B_i = \{0, i, b_i\}$ and consider the elements of B_i mod $2n + 1$. We have $\partial B = \Delta B = \pm\{i, b_i, a_i\}$ whence $\partial\mathcal{F}=\pm\{1, \ldots, n\} \cup (\cup_{i=1}^{n} \pm \{a_i, b_i\}) = \pm\{1, \ldots, n\} \cup \pm\{1, \ldots, 2n\}$, as $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, \ldots, 2n\}$. Since we consider the elements of $\partial\mathcal{F}$ mod $2n + 1$, we have $\pm\{1, \ldots, n\} = \{1, \ldots, 2n\}$ and $\pm\{1, \ldots, 2n\} = \{1, \ldots, 2n\} \cup$

$\{1, \ldots, 2n\}$. One can see that, in $\partial\mathcal{F}$, every integer mod $2n+1$ appears exactly 3 times; hence $\mathcal{F}$ is a $(2n+1, 3, 3)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(2n+1, 3, 3)$. ■

Example 3.0.5 $S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ gives the base blocks for a cyclic BIBD$(9, 3, 3)$: $\{\{0, 1, 2\}, \{0, 2, 7\}, \{0, 3, 6\}, \{0, 4, 8\}\}(\text{mod } 9)$.

The proof of Construction 3.0.6 is similar to that of Construction 3.0.5, and is thus omitted.

Construction 3.0.6 *Let $hS_n = (s_1, s_2, \ldots, s_{2n-1}, s_{2n+1})$ be a hooked Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in hS_n for which $b_i - a_i = i$. Then the set $\mathcal{F} = \{\{0, i, b_i + 1\} | 1 \leq i \leq n\}(\text{mod } 2n+1)$ is a $(2n+1, 3, 3)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(2n+1, 3, 3)$.*

Example 3.0.6 $hS_3 = (1, 1, 2, 3, 2, *, 3)$ gives the base blocks for a cyclic BIBD$(7, 3, 3)$: $\{0, 2, 6\}$ and two copies of $\{0, 1, 3\}$ (mod 7).

Construction 3.0.7 *Let n be even. Let $ES_n = (s_1, s_2, \ldots, s_{\frac{3n}{2}}, s_{\frac{3n}{2}+2}, \ldots, s_{2n+1})$ be an $(\frac{3n}{2}+1)$-extended Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in ES_n for which $b_i - a_i = i$. Then the set $\mathcal{F} = \{\{0, i, b_i\} | 1 \leq i \leq n\}(\text{mod } \frac{3n}{2}+1)$ is a $(\frac{3n}{2}+1, 3, 4)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(\frac{3n}{2}+1, 3, 4)$.*

Proof Note that $n \equiv 0, 2 \pmod 4$ whence $\frac{3n}{2} + 1 \equiv 0, 1 (\text{mod } 2)$, respectively; hence Theorem 1.1.5 holds. Set $B_i = \{0, i, b_i\}$ and consider the elements of B_i mod $(\frac{3n}{2}+1)$. We have $\partial B_i = \Delta B_i = \pm\{i, b_i, a_i\}$ whence $\partial\mathcal{F} = \pm\{1, \ldots, n\} \cup \pm\{1, \ldots, \frac{3n}{2}\} \cup \pm\{\frac{3n}{2} + 2, \ldots, 2n + 1\}$. Since we consider the elements of $\partial\mathcal{F}$ mod $(\frac{3n}{2} + 1)$, we have $\pm\{1, \ldots, n\} = \{1, \ldots, n\} \cup \{\frac{n}{2}+1, \ldots, \frac{3n}{2}\}$ and $\pm\{\frac{3n}{2}+2, \ldots, 2n+1\} = \{1, \ldots, \frac{n}{2}\} \cup \{n+1, \ldots, \frac{3n}{2}\}$. Whence $\partial\mathcal{F} = \{1, \ldots, \frac{3n}{2}\} \cup \{1, \ldots, \frac{3n}{2}\} \cup \pm\{1, \ldots, \frac{3n}{2}\}$, that is a $(\frac{3n}{2}+1, 3, 4)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(\frac{3n}{2}+1, 3, 4)$.

■

Example 3.0.7 The 7-extended Skolem sequence of order 4, $ES_4 = (3, 1, 1, 3, 4, 2, *, 2, 4)$ gives the base blocks for a cyclic BIBD$(7, 3, 4)$: $\{\{0, 1, 3\}, \{0, 2, 1\}, \{0, 3, 4\}, \{0, 4, 2\}\} (\text{mod } 7)$.

Construction 3.0.8 *Let n be odd. Let $ES_n = (s_1, s_2, \ldots, s_{2n-\lfloor \frac{n}{2} \rfloor}, s_{2n-\lfloor \frac{n}{2} \rfloor+2}, \ldots, s_{2n+1})$ be an $(2n - \lfloor \frac{n}{2} \rfloor + 1)$-extended Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in ES_n for which $b_i - a_i = i$. Then the set $\mathcal{F} = \{\{0, i, b_i\} | 1 \leq i \leq n\} (\text{mod } 2n - \lfloor \frac{n}{2} \rfloor + 1)$ together with the block $\{0, \lfloor \frac{n}{2} \rfloor + 1, n + 1\} (\text{mod } 2n - \lfloor \frac{n}{2} \rfloor + 1)$ having a short orbit of length $\frac{2n - \lfloor \frac{n}{2} \rfloor + 1}{3}$ is a $(2n - \lfloor \frac{n}{2} \rfloor + 1, 3, 4) - DF$. Hence, the set of triples in $\mathcal{F}$ together with the block $\{0, \lfloor \frac{n}{2} \rfloor + 1, n + 1\} (\text{mod } 2n - \lfloor \frac{n}{2} \rfloor + 1)$ forms the base blocks of a cyclic BIBD$(2n - \lfloor \frac{n}{2} \rfloor + 1, 3, 4)$.*

Proof Note that $n \equiv 1, 3 \pmod 4$, that is $\lfloor \frac{n}{2} \rfloor \equiv 0, 1 \pmod 2$, respectively, whence $2n - \lfloor \frac{n}{2} \rfloor + 1 \equiv 1, 0 \pmod 2$ respectively. Hence Theorem 1.1.5 holds. Set $B_i = \{0, i, b_i\}$ and consider the elements of B_i modulo $v = 2n - \lfloor \frac{n}{2} \rfloor + 1$. We have $\partial B_i = \Delta B_i = \pm\{i, b_i, a_i\}$ whence $\partial\mathcal{F} = \pm\{1, \ldots, n\} \cup (\cup_{i=1}^{n}\{a_i, b_i\}) = \pm\{1, \ldots, n\} \cup \pm\{1, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\} \cup \pm\{2n - \lfloor \frac{n}{2} \rfloor + 2, \ldots, 2n+1\}$. Since we consider the elements of $\partial\mathcal{F}$ mod $(2n - \lfloor \frac{n}{2} \rfloor + 1)$, we have $\pm\{1, \ldots, n\} = \{1, \ldots, n\} \cup \{\lfloor \frac{n}{2} \rfloor + 2, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\}$ and $\pm\{2n - \lfloor \frac{n}{2} \rfloor + 2, \ldots, 2n+1\} = \pm\{1, \ldots, \lfloor \frac{n}{2} \rfloor\} = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\} \cup \{n+2, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\}$. Whence $\partial\mathcal{F} = \{1, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\} \cup \{1, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\} \cup (\{1, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\} - \{n+1\}) \cup (\{1, \ldots, 2n - \lfloor \frac{n}{2} \rfloor\} - \{\lfloor \frac{n}{2} \rfloor + 1\})$. One can see that in $\partial\mathcal{F}$ every integer mod $2n - \lfloor \frac{n}{2} \rfloor + 1$ different from $\pm(n+1)$ appears exactly 4 times and the integers $\pm(n+1)$ appear 3 times. Set $B = \{0, \lfloor \frac{n}{2} \rfloor + 1, n+1\}$ and consider the elements of B mod $v = 2n - \lfloor \frac{n}{2} \rfloor + 1$. We have $\partial B = \pm\{\lfloor \frac{n}{2} \rfloor + 1, n - \lfloor \frac{n}{2} \rfloor, n+1\} = \pm\{n+1, n+1, n+1\}$ whence $\partial B = \pm\{n+1\}$, since G_B is the subgroup of order 3 of $\mathbb{Z}_v$.

It follows that in $\partial\mathcal{F}$ every integer mod $2n - \lfloor \frac{n}{2} \rfloor + 1$ appears exactly 4 times, hence $\mathcal{F}$ is a $(2n - \lfloor \frac{n}{2} \rfloor + 1, 3, 4) - DF$. Hence, the set of triples in $\mathcal{F}$ together with the block $\{0, \lfloor \frac{n}{2} \rfloor + 1, n+1\} \pmod{2n - \lfloor \frac{n}{2} \rfloor + 1}$ forms the base blocks of a cyclic BIBD$(2n - \lfloor \frac{n}{2} \rfloor + 1, 3, 4)$. ■

Example 3.0.8 A 9-extended Skolem sequence of order 9, $ES_5 = (5, 3, 1, 1, 3, 5, 4, 2, *, 2, 4)$ gives the base blocks for a cyclic BIBD$(9, 3, 4)$: $\{\{0, 1, 4\},$

$\{0,2,1\},\{0,3,5\},\{0,4,2\},\{0,5,6\}\}(\text{mod } 9)$ together with the block $\{0,3,6\}(\text{mod } 9)$ having a short orbit of length 3.

The proof of Construction 3.0.9 is similar to that of Construction 3.0.7, and is thus omitted.

Construction 3.0.9 *Let $R_n = (r_1, r_2, \ldots, r_{2n+1})$ be a Rosa sequence of order n, ($r_{n+1} = 0$), and let $\{(a_i, b_i)|1 \leq i \leq n\}$ be the pairs of positions in R_n for which $b_i - a_i = i$. Then the set $\mathcal{F}= \{\{0, i, b_i\}|1 \leq i \leq n\}(\text{mod } n+1)$ is a $(n+1, 3, 6)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(n+1, 3, 6)$.*

Example 3.0.9 $R_4 = (2, 4, 2, 3, *, 4, 3, 1, 1)$ gives the base blocks of a cyclic BIBD(5, 3, 6): $\{\{0,1,4\},\{0,2,3\},\{0,3,2\},\{0,4,1\}\}(\text{mod } 5)$.

To construct cyclic BIBD$(v, 3, 12)$, we use the near Skolem sequences with three hooks introduced in Chapter 2.

Construction 3.0.10 *Let $m \equiv 2(\text{mod } 4), m \geq 6$ and let $N = (n_1, \ldots, n_{4m-1})$ be an m-near Skolem sequence of order $2m-1$ with $n_m = 0, n_{2m} = 0$, and $n_{3m} = 0$. Let $\{(a_i, b_i)|1 \leq i \leq 2m-1, i \neq m\}$ be the pairs of positions in N for which $b_i - a_i = i$. Then the set $\mathcal{F}$=$\{\{0, i, b_i\}|1 \leq i \leq 2m-1, i \neq m\}(\text{mod } m)$ is a $(m, 3, 12)$-DF. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(m, 3, 12)$.*

Proof Set $B_i = \{0, i, b_i\}$ and consider the elements of B_i mod m. We have $\partial B_i = \Delta B_i = \pm\{i, b_i, a_i\}$ whence $\partial\mathcal{F}=\pm(\{1,\ldots,2m-1\}-\{m\}) \cup \pm(\{1,\ldots,4m-1\}-\{m,2m,3m\})$. Since we consider the elements of $\partial\mathcal{F}$ mod m, we have $\pm(\{1,\ldots,2m-1\}-\{m\}) = \pm\{1,\ldots,m-1\}\cup\pm\{1,\ldots,m-1\}$ and $\pm(\{1,\ldots,4m-1\}-\{m,2m,3m\}) = \pm\{1,\ldots,m-1\}\cup\pm\{1,\ldots,m-1\}\cup\pm\{1,\ldots,m-1\}\cup\pm\{1,\ldots,m-1\}$. One can see that in $\partial\mathcal{F}$ every integer mod m appears exactly 12 times, hence $\mathcal{F}$ is a $(m,3,12)-DF$. Hence, the set of triples in $\mathcal{F}$ forms the base blocks of a cyclic BIBD$(m,3,12)$. ■

Example 3.0.10 $N = (17,19,13,15,9,11,5,7,4,*,18,5,4,9,7,13,11,17,15,*,19,14,16,2,12,2,8,3,18,*,3,6,1,1,8,14,12,6,16)$ is a 10-near S_{19} with hooks in positions $10,20$ and 30. This sequence gives the base blocks for a cyclic BIBD$(10,3,12)$: $\{\{0,1,4\},\{0,2,6\},\{0,3,1\},\{0,4,3\},\{0,5,2\},\{0,6,8\},\{0,7,5\},\{0,8,5\},\{0,9,4\},$ $\{0,1,7\},\{0,2,7\},\{0,3,6\},\{0,4,6\},\{0,5,9\},\{0,6,9\},\{0,7,8\},\{0,8,9\},\{0,9,1\}\}$ (mod 10).

Now, we are ready to prove the sufficiency of Theorem 3.0.4 using Skolem-type sequences. We will construct cyclic block designs, using Skolem-type sequences, for all admissible v and λ. For the necessary conditions we outline the proof of Colbourn and Colbourn [42]. Then, we prove the sufficiency of the Main Theorem 3.0.4 using Skolem-type sequences.

In 1961, Hanani [60] solved the existence problem of BIBD$(v, 3, \lambda)$.

Theorem 3.0.2 *[60] Necessary and sufficient conditions for the existence of BIBD$(v, 3, \lambda)$ are:*

1. $\lambda \equiv 1, 5 (\mathrm{mod}\ 6)$ *and* $v \equiv 1, 3 (\mathrm{mod}\ 6)$ *or*

2. $\lambda \equiv 2, 4 (\mathrm{mod}\ 6)$ *and* $v \equiv 0, 1 (\mathrm{mod}\ 3)$ *or*

3. $\lambda \equiv 3 (\mathrm{mod}\ 6)$ *and* $v \equiv 1 (\mathrm{mod}\ 2)$ *or*

4. $\lambda \equiv 0 (\mathrm{mod}\ 6)$ *and* $v \geq 3$.

In 1981, Colbourn and Colbourn [42] solved the existence problem of cyclic BIBD$(v, 3, \lambda)$. They defined $D(v, \lambda)$ the multiset containing each i for $0 < i < \frac{v}{2}$, λ times when v is odd. When v is even, $D(v, \lambda)$ contains in addition the difference $\frac{v}{2}$, $\frac{\lambda}{2}$ times. When $v \equiv 0 (\mathrm{mod}\ 3)$, they also defined $D_0(v, \lambda) = D(v, \lambda)$ and $D_m(v, \lambda) = D_{m-1}(v, \lambda) - \{\frac{v}{3}\}$.

Colbourn and Colbourn [42], posed the following generalized versions of Heffter's problems for arbitrary λ:

- If $v \not\equiv 0$ (mod 3), can $D(v, \lambda)$ be partitioned into difference triples?

- If $v \equiv 0$ (mod 3), is there an m for which $D_m(v, \lambda)$ can be partitioned into difference triples?

The resolution of these two difference problems is equivalent to a complete determination of cyclic BIBD$(v, 3, \lambda)$.

Lemma 3.0.3 *[42] If $v \equiv 2 \pmod 4$ and $\lambda \equiv 2 \pmod 4$, there is no cyclic BIBD$(v, 3, \lambda)$.*

The following theorem was proved by Colbourn and Colbourn [42] using Peltesohn's technique. We give a new proof using Skolem-type sequences. While Colbourn and Colbourn's construction [42] provides only one cyclic design for each order v, our constructions will provide many nonisomorphic cyclic designs for each order v.

Theorem 3.0.4 *(Main Theorem) [42]*

Necessary and sufficient conditions for the existence of a cyclic BIBD$(v, 3, \lambda)$ are:

1. $\lambda \equiv 1, 5, 7, 11 \pmod{12}$ *and* $v \equiv 1, 3 \pmod 6$ *or*

2. $\lambda \equiv 2, 10 \pmod{12}$ *and* $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ *or*

3. $\lambda \equiv 3, 9 \pmod{12}$ *and* $v \equiv 1 \pmod 2$ *or*

4. $\lambda \equiv 4, 8 \pmod{12}$ *and* $v \equiv 0, 1 \pmod 3$ *or*

5. $\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0, 1, 3 \pmod 4$ *or*

6. $\lambda \equiv 0 \pmod{12}$ *and* $v \geq 3$,

with only two exceptions: CSTS(9) and cyclic BIBD$(9,3,2)$ do not exist.

Proof *Necessity:* Theorem 3.0.2 and Lemma 3.0.3 give the necessary conditions. For $v \equiv 0 \pmod 3$, the designs may have short orbits. To see the designs that have short orbits, we arrange the necessary conditions of the above theorem in Table 3.1. We denote with $-$ the designs with no short orbits, and with $+$ the designs that have short orbits having the length equal to $1/3$ of the full orbit. An empty cell in the table means that such a design does not exist.

$v/\lambda \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11
0	-		+		+		-		+		+	
1	-	-	-	-	-	-	-	-	-	-	-	-
2	-											
3	-	+	+	-	+	+	-	+	+	-	+	+
4	-		-		-		-		-		-	
5	-			-			-			-		
6	-				+				+			
7	-	-	-	-	-	-	-	-	-	-	-	-
8	-						-					
9	-	+	+	-	+	+	-	+	+	-	+	+
10	-				-				-			
11	-			-			-			-		

Table 3.1: Necessary conditions for the existence of a cyclic BIBD$(v,3,\lambda)$

Sufficiency: We use the fact that we can construct a cyclic BIBD$(v,3,n\lambda)$ from a cyclic BIBD$(v,3,\lambda)$ by simply taking each block of the cyclic BIBD$(v,3,\lambda)$ n times. Using this observation, it will suffice to consider $\lambda = 1, 2, 3, 4, 6$, and 12.

Case 1: $\lambda = 1$ and $v \equiv 1, 3 \pmod 6, v \neq 9$

For $v \equiv 1 (\text{mod } 6)$, see Construction 3.0.1.

For $v \equiv 3 (\text{mod } 6), v \neq 9$, see Construction 3.0.2.

Case 2: $\lambda = 2$ and $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}, v \neq 9$

For $v \equiv 0, 3 (\text{mod } 12)$, see Construction 3.0.4.

For $v \equiv 1, 4 (\text{mod } 12)$, see Construction 3.0.3.

For $v \equiv 7 (\text{mod } 12)$, take two copies of a CSTS$(6n + 1)$ from Construction 3.0.1.

For $v \equiv 9 (\text{mod } 12), v \neq 9$, take two copies of a CSTS$(6n + 3)$ from Construction 3.0.2.

It is known that a cyclic BIBD$(9, 3, 2)$ does not exist [42].

Case 3: $\lambda = 3$ and $v \equiv 1 \pmod 2$

For $v \equiv 1, 3 (\text{mod } 8)$, see Construction 3.0.5.

For $v \equiv 5, 7 (\text{mod } 8)$, see Construction 3.0.6.

Case 4: $\lambda = 4$ and $v \equiv 0, 1 \pmod 3$

For $v \equiv 0 (\text{mod } 3)$, see Construction 3.0.8.

For $v \equiv 1 (\text{mod } 3)$, see Construction 3.0.7.

Case 5: $\lambda = 6$ and $v \equiv 0, 1, 3 \pmod 4$

For $v \equiv 0, 1 (\text{mod } 4)$, see Construction 3.0.9.

For $v \equiv 3 (\text{mod } 8)$, take two copies of a cyclic BIBD$(v, 3, 3)$ from Construction

3.0.5.

For $v \equiv 7 \pmod 8$, take two copies of a cyclic BIBD$(v, 3, 3)$ from Construction 3.0.6.

Case 6: $\lambda = 12$ and $v \geq 3$

For $v \equiv 0 \pmod 3$, take three copies of a cyclic BIBD$(v, 3, 4)$ from Construction 3.0.8.

For $v \equiv 1 \pmod 3$, take three copies of a cyclic BIBD$(v, 3, 4)$ from Construction 3.0.7.

For $v \equiv 2 \pmod{12}$, see Construction 3.0.10.

For $v \equiv 5, 8 \pmod{12}$, take two copies of a cyclic BIBD$(v, 3, 6)$ from Construction 3.0.9.

For $v \equiv 3 \pmod 8$, take four copies of a cyclic BIBD$(v, 3, 3)$ from Construction 3.0.5.

For $v \equiv 7 \pmod 8$, take four copies of a cyclic BIBD$(v, 3, 3)$ from Construction 3.0.6. ■

Remark 3.0.1 *Some of the above cyclic BIBD$(v, 3, \lambda)$ can be also constructed using m-fold Skolem and Rosa arrays.*

Chapter 4

Cyclic BIBD($v, 4, \lambda$) from Skolem-type Sequences

M. Colbourn and Mathon [45] asked: "Can Skolem's partitioning problems be generalized to yield cyclic BIBD($v, 4, 1$)?". Rosa [76] asked: "What is the format of Skolem-type sequences that leads to cyclic BIBD(v, k, λ) for $k \geq 4$?".

In this chapter, we will answer these two questions. We give the necessary conditions for the existence of a cyclic BIBD($v, 4, \lambda$) and we construct several new cyclic designs using Skolem-type sequences. We also show that there exists an example of Skolem partitions that induces cyclic BIBD($v, 4, \lambda$) for every admissible class.

4.1 Necessary Conditions for Cyclic BIBD$(v, 4, \lambda)$

There are many papers dealing with cyclic block designs with block size four, but to the best of our knowledge, the necessary conditions for the existence of cyclic BIBD$(v, 4, \lambda)$ are not stated. Here is our successful effort with this problem.

Hanani [60] proved that the necessary conditions are also sufficient for the existence of a BIBD$(v, 4, \lambda)$.

Theorem 4.1.1 *[60] The necessary conditions for the existence of a BIBD$(v, 4, \lambda)$ are:*

1. $v \equiv 1,\ 4 \pmod{12}$ *and all* λ*s;*

2. $v \equiv 1 \pmod 3$ *and* $\lambda \equiv 2,\ 4 (\text{mod } 6)$*;*

3. $v \equiv 0,\ 1 \pmod 4$ *and* $\lambda \equiv 3 (\text{mod } 6)$*;*

4. *all* v *and* $\lambda \equiv 0 (\text{mod } 6)$.

It is evident that cyclic BIBD$(v, 4, \lambda)$ is a subset of BIBD$(v, 4, \lambda)$. For a given v, the necessary conditions $\lambda(v-1) \equiv 0 (\text{mod } 3)$ and $\lambda v(v-1) \equiv 0 (\text{mod } 12)$ for the existence of a cyclic BIBD$(v, 4, \lambda)$ determine a minimum value of λ. It is enough to consider these parameters since a cyclic BIBD$(v, 4, n\lambda)$ can be constructed from a cyclic BIBD$(v, 4, \lambda)$ by taking n copies.

Theorem 4.1.1 and the fact that cyclic BIBD$(v, 4, \lambda)$ with small values of v exists for every admissible pair (v, λ) (see Appendix B), gives us the following necessary conditions.

Theorem 4.1.2 *(***Necessary conditions***)* *The necessary conditions for the existence of a cyclic BIBD*$(v, 4, \lambda)$ *are:*

1. $v \equiv 1,\ 4 \pmod{12}$ *and all* λ*s;*

2. $v \equiv 1 \pmod 3$ *and* $\lambda \equiv 2,\ 4,\ 8,\ 10 \pmod{12}$*;*

3. $v \equiv 0,\ 1 \pmod 4$ *and* $\lambda \equiv 3,\ 9 \pmod{12}$*;*

4. *all* v *and* $\lambda \equiv 0,\ 6 \pmod{12}$*;*

with the exceptions of: cyclic BIBD$(v, 4, 1)$ *for* $v = 16, 25, 28$ *do not exist, cyclic BIBD*$(8, 4, 3)$ *and cyclic BIBD*$(10, 4, 2)$ *do not exist.*

It is known that cyclic BIBD$(v, 4, 1)$ for $v = 16, 25, 28$ do not exist [41].

For a cyclic BIBD$(8, 4, 3)$ to exist, the design needs to have one full orbit and a regular orbit repeated three times, or one full orbit, one regular orbit and one short orbit. Assume the cyclic BIBD$(8, 4, 3)$ has the regular orbit $\{0, 2, 4, 6\}$ repeated three times. This orbit covers the differences $2, 4, 6$ exactly three times and it is impossible to find a full orbit that will cover the remaining differences exactly three times. Assume now that the cyclic BIBD$(8,4,3)$ has the regular orbit $\{0,2,4,6\}$ and one short orbit, say $\{0,1,4,5\}$. These orbits will cover difference 4 exactly three times and will cover the differences $1,2,3$, each exactly once. Then it is impossible to find a full orbit that will cover the differences $1,2,3$ or their inverses exactly twice. A similar argument can be used if the short orbit is $\{0,3,4,7\}$. Therefore, a cyclic BIBD$(8,4,3)$ does not exist.

For a cyclic BIBD$(10,4,2)$ to exist, the design must have 15 blocks. So 15 can be expressed as the sum of the lengths of the orbits of the blocks. The stabilizer of a block has order 1 or 2, therefore the orbit of a block has length either 10 or 5. In the first case a set of base blocks has the form $\{0,a,5,a+5\}$, $\{0,b,5,b+5\}$, $\{0,c,5,c+5\}$ which is obviously impossible since we would have three blocks through 0 and 5. Thus, we necessarily are in the second case and a set of base blocks has the form $\{0,a,b,c\}$, $\{0,d,5,d+5\}$. One the other hand one can see, even by hand, that no quadruple (a,b,c,d) can realize such a set of base blocks. Therefore, a cyclic BIBD$(10,4,2)$ does not exist.

For $v \equiv 0 \pmod 2$, the designs may have short orbits. There are two types of short orbits:

- the regular short orbit $\{0, \frac{v}{4}, \frac{2v}{4}, \frac{3v}{4}\}$ which has length equal to $\frac{1}{4}$ of the full orbit;

- the short orbit $\{0, i, \frac{v}{2}, i+\frac{v}{2}\}, 1 \leq i < \frac{v}{2}$ which has length equal to a half of the full orbit.

To see the designs that have short orbits, we exhibit the necessary conditions of the above theorem in Table 4.1.

$v/\lambda \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11
0	-			$+^{rs}$			$+^{s}$			$+^{r}$		
1	-	-	-	-	-	-	-	-	-	-	-	-
2	-						$+^{s}$					
3	-						-					
4	-	$+^{r}$	$+^{s}$	$+^{rs}$	-	$+^{r}$	$+^{s}$	$+^{rs}$	-	$+^{r}$	$+^{s}$	$+^{rs}$
5	-			-			-			-		
6	-						$+^{s}$					
7	-		-		-		-		-		-	
8	-			$+^{rs}$			$+^{s}$			$+^{r}$		
9	-			-			-			-		
10	-		$+^{s}$		-		$+^{s}$		-		$+^{s}$	
11	-						-					

Table 4.1: Necessary conditions for the existence of a cyclic BIBD$(v, 4, \lambda)$

We denote with $-$ the designs with no short orbits, with $+^{r}$ the designs that contain the regular short orbit, with $+^{s}$ the designs that contain the short orbit or the regular short orbit twice, with $+^{rs}$ the designs that contain the regular short orbit and the short orbit or the regular short orbit three times. An empty cell in the table shows that no such design exists.

To prove that the necessary conditions are also sufficient for cyclic block designs with block size 4 is a difficult task. We will prove some of the necessary conditions by constructing new linear classes of such designs.

4.2 Cyclic BIBD$(v, 4, 1)$

4.2.1 Heffter's Problem for $k = 4$

In her doctoral thesis, M.J Colbourn [41] tried to construct cyclic block designs with block size four using Peltesohn's proof technique. We present her idea here: the construction of a cyclic BIBD$(v, 4, 1)$, $v = 12n + 1$, is equivalent to partitioning the integers $\{1, 2, \ldots, 12n\}$ into n 12-subsets $\{a, v-a, b, v-b, c, v-c, a+b, v-a-b, b+c, v-b-c, a+b+c, v-a-b-c\}$. Then the base blocks $\{0, a, a+b, a+b+c\}$ will be the base blocks of a cyclic BIBD$(12n + 1, 4, 1)$.

In subsection 4.2.2, we reduce this problem in half.

Example 4.2.1 *For $v = 49$ and $k = 4$, a partition of the numbers $\{1, \ldots, 48\}$ into four 12-subsets is:*

1. $\{1, 48, 11, 37, 6, 43, 12, 37, 17, 32, 18, 31\}$;

2. $\{2, 47, 5, 44, 15, 34, 7, 42, 20, 29, 22, 27\}$;

3. $\{3, 46, 13, 36, 8, 41, 16, 33, 21, 28, 24, 25\}$;

4. $\{4, 45, 10, 39, 9, 40, 14, 35, 19, 30, 23, 26\}$.

and the base blocks of the cyclic BIBD$(49, 4, 1)$ *are:* $\{\{0, 1, 12, 18\}, \{0, 2, 7, 22\}, \{0, 3, 16, 24\}, \{0, 4, 14, 23\}\}$.

When $v = 12n+4$, the designs will contain the extra starter block $\{0, 3n+1, 6n+2, 9n+3\}$. Hence, the construction of a cyclic BIBD$(12n+4, 4, 1)$, is equivalent to partitioning the set of integers $\{1, \ldots, 3n, 3n+2, \ldots, 6n+1, 6n+3, \ldots, 9n+2, 9n+4, \ldots, 12n+3\}$ into n 12-subsets $\{a, v-a, b, v-b, c, v-c, a+b, v-a-b, b+c, v-b-c, a+b+c, v-a-b-c\}$ [41].

4.2.2 Skolem Partitioning Problem for $\boldsymbol{k = 4}$ and $\boldsymbol{\lambda = 1}$

The problem can be reduced in half as follows:

Remark 4.2.1 *If there is a partition of the numbers* $\{1, \ldots, 6n\}$ *into* n *six-subsets* $\{a, b, c, a+b, b+c, a+b+c\}$*, then* $\{0, a, a+b, a+b+c\}$ *will be the base blocks of a cyclic BIBD*$(12n+1, 4, 1)$*. These* n *six-subsets can be arranged into triangles of the form:*

$$\begin{array}{ccccc} & & a+b+c & & \\ & a+b & & b+c & \\ a & & b & & c. \end{array}$$

Example 4.2.2 *For $v = 49$ and $k = 4$, a partition of the numbers $\{1, \dots, 24\}$ into four six-subsets is:*

1. $\{1, 11, 6, 12, 17, 18\}$;
2. $\{2, 5, 15, 7, 20, 22\}$;
3. $\{3, 13, 8, 16, 21, 24\}$;
4. $\{4, 10, 9, 14, 19, 23\}$.

The triangle representation of these sets is:

$$\begin{array}{ccccc|ccccc|ccccc|ccccc}
 & & 18 & & & & & 22 & & & & & 24 & & & & & 23 & & \\
 & 12 & & 17 & & & 7 & & 20 & & & 16 & & 21 & & & 14 & & 19 & \\
1 & & 11 & & 6 & 2 & & 5 & & 15 & 3 & & 13 & & 8 & 4 & & 10 & & 9
\end{array}$$

and the base blocks of the cyclic BIBD$(49, 4, 1)$ are: $\{\{0, 1, 12, 18\}, \{0, 2, 7, 22\}, \{0, 3, 16, 24\}, \{0, 4, 14, 23\}\}$.

The problem is similar for the case $v = 12n + 4$.

Remark 4.2.2 *If there is a partition of the numbers $\{1, \dots, 3n, 3n + 2, \dots 6n + 1\}$ into n six-subsets $\{a, b, c, a + b, b + c, a + b + c\}$, then $\{0, a, a + b, a + b + c\}$ together with the extra short orbit $\{0, 3n + 1, 6n + 2, 9n + 3\}$ will be the base blocks of a cyclic BIBD$(12n + 4, 4, 1)$.*

4.2.3 Cyclic BIBD$(12n+1,4,1)$

In this subsection, we give several examples that use Skolem-type sequences to construct cyclic BIBD$(v,4,1)$, and some constructions that reduce the problem to a problem similar to Heffter's difference problem.

Remark 4.2.3 *Let* $(a_i,b_i), 1 \le i \le n$, *be the pairs of a (hooked) Skolem sequence of order* n, *and let* x_i, $1 \le x_i \le n$, *be positive integers. If there exists a partition of the numbers* $\{n+1,\dots,4n\}$ *into triples of the form* $(x_i - i, x_i, b_i + 4n - x_i)$, $\forall 1 \le i \le n$, *then* $\{0,i,x_i,b_i+4n\}$, $1 \le i \le n$, *will be the base blocks of a cyclic BIBD*$(12n+1,4,1)$.

Proof By the definition of a (hooked) Skolem sequence, $b_i - a_i = i, \forall 1 \le i \le n \Rightarrow b_i - i = a_i, \forall 1 \le i \le n$. Taking the differences $\pm i, \pm(b_i+4n), \pm(b_i+4n-i)$ from $\{0,i,x_i,b_i+4n\}$ we see that:

- $\pm i$ gives all the differences $\{1,\dots,n\}$, each exactly once,

- for $n \equiv 0,1 \pmod 4$, $\pm(b_i+4n)$ and $\pm(b_i+4n-i) = (a_i+4n)$ give all the differences $\{4n+1,\dots,6n\}$, each exactly once, since (a_i,b_i) is a partition of $\{1,\dots,2n\}$,

- for $n \equiv 2,3 \pmod 4$, $\pm(b_i+4n)$ and $\pm(b_i+4n-i) = (a_i+4n)$ give all the differences $\{4n+1,\dots,6n-1,6n+1\}$, each exactly once, since (a_i,b_i) is a partition of $\{1,\dots,2n-1,2n+1\}$.

So, the differences $\{1, \ldots, n\}$ and $\{4n+1, \ldots, 6n\}$ (respectively $\{4n+1, \ldots, 6n-1, 6n+1\}$) are covered exactly once.

Suppose we can partition the numbers $\{n+1, \ldots, 4n\}$ into triples of the form $(x_i - i, x_i, b_i + 4n - x_i)$, $1 \leq i \leq n$. Then $\{0, i, x_i, b_i + 4n\}$, $1 \leq i \leq n$, covers all the differences $\{1, \ldots, 6n\}$, each exactly once. Therefore, $\{0, i, x_i, b_i + 4n\}$, $1 \leq i \leq n$, will be the base blocks of a cyclic BIBD$(12n+1, 4, 1)$. ■

Example 4.2.3 *Let* $v = 49$ *and* $(6,7), (1,3), (2,5), (4,8)$ *be the pairs of a Skolem sequence of order* 4. *Then the base blocks* $\{0, 1, x_1, 23\}$, $\{0, 2, x_2, 19\}$, $\{0, 3, x_3, 21\}$, $\{0, 4, x_4, 24\}$ *cover the differences* $\{1, \ldots, 4\}$ *and* $\{17, \ldots, 24\}$. *In order to complete the quadruples we need to partition the numbers* $\{5, \ldots, 16\}$ *(or* $\{1, \ldots, 12\}$*) into triples* $(x_i - i, x_i, b_i + 4n - x_i)$, $1 \leq i \leq 4$.

A possible solution is: $(6, 7, 16)$, $(12, 14, 5)$, $(10, 13, 8)$, $(11, 15, 9)$.

If we subtract 4 *from each of the numbers above, we get the triples* $(2, 3, 12)$, $(8, 10, 1)$, $(6, 9, 4)$, $(7, 11, 5)$ *which can be represented as a Skolem sequence with* 4 *hooks as follows: from the first triple* $(2, 3, 12)$ *we place a* 1 *in positions* 2 *and* 3 *and a hook in position* 12, *from the second triple* $(8, 10, 1)$ *we place a* 2 *in positions* 8 *and* 10 *and a hook in position* 1 *and so on. The sequence is:*

$\star$	*1*	*1*	$\star$	$\star$	*3*	*4*	*2*	*3*	*2*	*4*	$\star$
-	-	-	-	-	-	-	-	-	-	-	-
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*

So the base blocks of the cyclic $BIBD(49, 4, 1)$ *are:* $\{\{0, 1, 7, 23\}, \{0, 2, 14, 19\}, \{0, 3, 13, 21\}, \{0, 4, 15, 24\}\}$.

One possible way to find the integers x_i is to find a Skolem sequence with n hooks that gives the required partition. Unfortunately, not every Skolem sequence with n hooks will solve the problem.

Problem 4.2.1 *What is the format of the Skolem-type sequence with* n *hooks that gives the required partition?*

There are many solutions for different orders. For example, there are three solutions for $v = 49$, three solutions for $v = 61$, thirty-six solutions for $v = 73$ and several solutions for $v = 85$ and $v = 97$. It is known that the number of Skolem sequences of order n grows exponentially as the order of the sequence grows. It seems that the number of solutions we get using our approach also grows exponentially. A few more examples of such designs are given in Appendix A.

4.2.4 Cyclic BIBD$(12n + 4, 4, 1)$

In this case, the designs will have the short orbit $\{0, 3n+1, 6n+2, 9n+3\}$. We split this problem in two cases.

Case 1: $n \equiv 0$, 1(mod 4)

Remark 4.2.4 *Let $(a_i, b_i), 1 \le i \le n$, be a Skolem sequence of order n, and let x_i, $1 \le x_i \le n$, be positive integers. If there exists a partition of the numbers $\{n+1, \dots, 4n+1\} \setminus \{3n+1\}$ into triples of the form $(x_i - i, x_i, b_i + 4n + 1 - x_i)$, $\forall 1 \le i \le n$, then $\{0, i, x_i, b_i+4n+1\}$, $1 \le i \le n$, together with the short orbit $\{0, 3n+1, 6n+2, 9n+3\}$, will be the base blocks of a cyclic BIBD$(12n + 4, 4, 1)$.*

In this case the base blocks $\{0, i, x_i, b_i+4n+1\}$, $1 \le i \le n$, cover all the differences $\{1, \dots, n\}$ and $\{4n + 2, \dots, 6n + 1\}$ exactly once. The short orbit $\{0, 3n + 1, 6n + 2, 9n + 3\}$ covers the differences $3n + 1$ and $6n + 2$.

Using the above approach, the problem is reduced to partitioning the numbers $\{n + 1, \dots, 4n + 1\} \setminus \{3n + 1\}$ into triples of the form $(x_i - i, x_i, b_i + 4n + 1 - x_i)$, $\forall 1 \le i \le n$. Subtracting n from each of the numbers above, the problem is equivalent to partitioning the numbers $\{1, \dots, 3n\} \setminus \{2n + 1\}$ into triples of the form $\{x_i - i - n, x_i - n, b_i + 3n + 1 - x_i\}$, $1 \le i \le n$.

Example 4.2.4 *Let $v = 64$ and $(2, 3), (6, 8), (7, 10), (1, 5), (4, 9)$ be the pairs of* *a Skolem sequence of order* 5. *Then the base blocks* $\{0,1,x_1,24\}$, $\{0,2,x_2,29\}$, $\{0,3,x_3,31\}$, $\{0,4,x_4,26\}$, $\{0,5,x_5,30\}$, $\{0,16,32,48\}$ *cover the differences* $\{1,\ldots,5\}$ *and* $\{22,\ldots,31\}$ *and their inverses. The last block covers the differences* 16 *and* 32 *and their inverses. In order to complete the quadruples, we need to partition the numbers* $\{6,\ldots,21\}\setminus\{16\}$ *(or* $\{1,\ldots,16\}\setminus\{11\}$*) into triples* $(x_i - i, x_i, b_i + 4n + 1 - x_i)$, $1 \le i \le 5$.

A possible solution is: $(9,10,14)$, $(6,8,21)$, $(17,20,11)$, $(15,19,7)$, $(13,18,12)$.

If we subtract 5 *from each of the numbers above, we get* $(4,5,9)$, $(1,3,16)$, $(12,15,6)$, $(10,14,2)$, $(8,13,7)$, *which can be represented as a Skolem sequence with* 6 *hooks (the extra hook is in position* 11*) as follows:*

2	⋆	*2*	*1*	*1*	⋆	⋆	*5*	⋆	*4*	⋆	*3*	*5*	*4*	*3*	⋆
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*

So the base blocks of the cyclic BIBD$(64,4,1)$ *are:* $\{\{0,1,10,24\}, \{0,2,8,29\}, \{0,3,20,31\}, \{0,4,19,26\}, \{0,5,18,30\}, \{0,16,32,48\}\}$.

More examples of such designs are given in Appendix A.

Case 2: $\boldsymbol{n \equiv 2,\ 3 (\mathrm{mod}\ 4)}$

Remark 4.2.5 *Let* $(a_i, b_i), 1 \le i \le n$, *be a reversed hooked Skolem sequence of order* n, *and let* x_i, $1 \le x_i \le n$, *be positive integers. If there exists a partition of the* *numbers* $\{n+1,\ldots,4n+2\}\setminus\{3n+1,4n+1\}$ *into triples of the form* $(x_i - i, x_i, b_i + 4n - x_i)$, $1 \le i \le n$, *then* $\{0, i, x_i, b_i + 4n\}$, $1 \le i \le n$, *together with the short orbit* $\{0, 3n+1, 6n+2, 9n+3\}$, *will be the base blocks of a cyclic BIBD*$(12n+4,4,1)$.

Using the above approach our problem is reduced to partitioning the numbers $\{n+1,\ldots,4n+2\}\setminus\{3n+1,4n+1\}$ into triples of the form $(x_i - i, x_i, b_i + 4n - x_i)$, $\forall 1 \le i \le n$. Subtracting n from each of the numbers above, the problem is equivalent to partitioning the numbers $\{1,\ldots,3n+2\}\setminus\{2n+1,3n+1\}$ into triples of the form $\{x_i - i - n, x_i - n, b_i + 3n - x_i\}$, $\forall 1 \le i \le n$.

Example 4.2.5 *Let* $v = 76$ *and* $(6,7), (1,3), (10,13), (8,12), (4,9), (5,11)$ *be the pairs of a reversed hooked Skolem sequence of order* 5. *Then the base blocks* $\{0,1,x_1,31\}$, $\{0,2,x_2,27\}$, $\{0,3,x_3,37\}$, $\{0,4,x_4,36\}$, $\{0,5,x_5,33\}$, $\{0,6,x_6,35\}$, $\{0,19,38,57\}$ *cover the differences* $\{1,\ldots,6\}$ *and* $\{25,\ldots,37\}\setminus\{26\}$ *and their inverses. The last block covers the differences* 19 *and* 38 *and their inverses. In order to complete the quadruples we need to partition the numbers* $\{7,\ldots,26\}\setminus\{19,25\}$ *(or* $\{1,\ldots,20\}\setminus\{13,19\}$*) into triples* $(x_i - i, x_i, b_i + 4n - x_i)$, $1 \le i \le 6$.

A possible solution is: $(7,8,23)$, $(10,12,15)$, $(21,24,13)$, $(18,22,14)$, $(11,16,17)$, $(20,26,9)$.

If we subtract 6 *from each of the numbers above, we get:* $(1,2,17)$, $(4,6,9)$,

$(15, 18, 7)$, $(12, 16, 8)$, $(5, 10, 11)$, $(14, 20, 3)$, *which can be represented as a Skolem sequence with* 8 *hooks (the extra hooks are in positions* 13, 19*) as follows:*

1	1	⋆	2	5	2	⋆	⋆	⋆	5	⋆	4	⋆	6	3	4	⋆	3	⋆	6
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

So the base blocks of the cyclic BIBD$(76, 4, 1)$ *are:* $\{\{0, 1, 8, 31\}, \{0, 2, 12, 27\}, \{0, 3, 24, 37\}, \{0, 4, 22, 36\}, \{0, 5, 16, 33\}, \{0, 6, 26, 35\}, \{0, 19, 38, 57\}\}$.

Finding x_i for all $1 \leq i \leq n$ will solve the problem. There are fifteen solutions for $v = 76$ and many solutions for $v = 88$. In Appendix A we list a few more examples of such designs.

4.3 Cyclic BIBD$(v, 4, 2)$

4.3.1 Cyclic BIBD$(12n + 1, 4, 2)$

We use the 2-fold Skolem arrays of order n to construct cyclic BIBD$(12n + 1, 4, 2)$. We also give a construction that reduces the problem to a problem similar to Heffter's difference problem.

Remark 4.3.1 *Let* (a_i, b_i) *and* $(c_i, d_i), 1 \leq i \leq n$, *be a 2-fold Skolem array of order* n, *and let* x_i, x_i' *be positive integers. If there exists a partition of the numbers* $\{n +$

$1, \ldots, 4n\} \cup \{n+1, \ldots, 4n\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 4n - x_i)$ and $(x'_i - i, x'_i, d_i + 4n - x'_i)$, $1 \leq i \leq n$, then $\{0, i, x_i, b_i + 4n\}$ and $\{0, i, x'_i, d_i + 4n\}$, $1 \leq i \leq n$, will be the base blocks of a cyclic BIBD$(12n+1, 4, 2)$.

Using the above approach our problem is reduced to partitioning the numbers $\{n+1, \ldots, 4n\} \cup \{n+1, \ldots, 4n\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 4n - x_i)$ and $(x'_i - i, x'_i, d_i + 4n - x'_i)$, $1 \leq i \leq n$. Finding x_i and x'_i for all $1 \leq i \leq n$ will solve the problem.

Example 4.3.1 *Let $v = 37$ and $(1,2), (5,6), (2,4), (3,5), (1,4), (3,6)$ be the pairs of a 2-fold Skolem array of order 3. Then, the base blocks $\{0, 1, x_1, 14\}$, $\{0, 1, x'_1, 18\}$, $\{0, 2, x_2, 17\}$, $\{0, 2, x'_2, 16\}$, $\{0, 3, x_3, 16\}$, $\{0, 3, x'_3, 18\}$, cover the differences $\{1, \ldots, 3\}$ and $\{13, \ldots, 18\}$, each exactly twice. In order to complete the quadruples we need to partition the numbers $\{4, \ldots, 12\}$ (or $\{1, \ldots, 9\}$) into triples $(x_i - i, x_i, b_i + 4n - x_i)$, $(x'_i - i, x'_i, d_i + 4n - x'_i)$, $1 \leq i \leq 3$.*

A possible solution is: $(4,5,9)$, $(11,12,6)$, $(10,12,5)$, $(8,10,6)$, $(4,7,9)$, $(8,11,7)$.

If we subtract 3 from each of the numbers above we get: $(1,2,6)$, $(8,9,3)$, $(7,9,2)$, $(5,7,3)$, $(1,4,6)$, $(5,8,4)$, which can be represented as a 2-fold Skolem array with 6 hooks as follows:

3	$\star$	$\star$	3	3	$\star$	2	3	2
1	1	$\star$	$\star$	2	$\star$	2	1	1
-	-	-	-	-	-	-	-	-
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*

So the base blocks of the cyclic BIBD$(37,4,2)$ *are:* $\{\{0,1,5,14\}$, $\{0,1,12,18\},\{0,2,12,17\},\{0,2,10,16\},\{0,3,7,16\},\{0,3,11,18\}\}$.

In Appendix A, we list a few more examples of such designs.

4.3.2 Cyclic BIBD$(12n+4,4,2)$

Similar to Section 4.3.1 with the only exception being that the cyclic BIBD$(12n+4,4,2)$ will have the extra short orbit $\{0,3n+1,6n+2,9n+3\}$.

4.3.3 Cyclic BIBD$(12n+7,4,2)$

We write $12n+7=6(2n+1)+1$.

Remark 4.3.2 *Let* $(a_i,b_i), 1\le i\le 2n+1$ *be a (hooked) Skolem sequence of order* $2n+1$*. If there exists a partition of the numbers* $\{1,\ldots,6n+3\}$ *or the numbers* $\{1,\ldots,6n+2,6n+4\}$ *(or their inverses) into triples of the form* $(x_i-i, x_i, b_i+2n+$

$1 - x_i)$, $1 \leq i \leq 2n+1$, *then* $\{0, i, x_i, b_i + 2n + 1\}$, $1 \leq i \leq 2n+1$, *will be the base blocks of a cyclic BIBD*$(12n+1, 4, 2)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n+3\}$ or the numbers $\{1, \ldots, 6n+2, 6n+4\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 2n + 1 - x_i)$ $\forall 1 \leq i \leq 2n+1$. Finding x_i will solve the problem.

Example 4.3.2 *Let* $v = 19$ *and* $\{(1,2), (3,5), (4,7)\}$ *be the pairs of a hooked Skolem sequence of order* 3. *We need to partition* $\{1, \ldots, 8, 10\}$ *into triples* $(x_i - i, x_i, b_i + 3 - x_i)$, $1 \leq i \leq 3$. *The partition is* $(5, 6, -1), (8, 10, -2), (4, 7, 3)$. *Then* $\{0, 1, 6, 5\}$, $\{0, 2, 10, 8\}$, $\{0, 3, 7, 10\}$ *are the base blocks of a cyclic BIBD*$(19, 4, 2)$.

4.3.4 Cyclic BIBD$(12n + 10, 4, 2)$

We write $12n + 10 = 6(2n+1) + 4$.

Case 1: $\boldsymbol{n \equiv 0 \pmod 2}$

Remark 4.3.3 *Let* $(a_i, b_i), 1 \leq i \leq 2n+1$ *be a Skolem sequence of order* $2n+1$, *and let* x_i *be positive integers. If there exists a partition of the numbers* $\{2, \ldots, 6n+4\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i + 2n + 1 - x_i)$, $1 \leq i \leq$

$2n+1$, *then* $\{0, i, x_i, b_i + 2n + 1\}$, $1 \leq i \leq 2n+1$, *together with the extra short orbit* $\{0, 1, 6n+4, 6n+5\}$, *will be the base blocks of a cyclic BIBD*$(12n+10, 4, 2)$.

Using the above approach our problem is reduced to partitioning the numbers $\{2, \ldots, 6n+4\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 2n + 1 - x_i)$ $\forall 1 \leq i \leq 2n+1$.

Case 2: $\boldsymbol{n \equiv 1 \pmod 2}$

Remark 4.3.4 *Let* $(a_i, b_i), 1 \leq i \leq 2n+1$, *be a hooked Skolem sequence of order* $2n+1$, *and let* x_i *be positive integers. If there exists a partition of the numbers* $\{1, 3, \ldots, 6n+4\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i + 2n + 1 - x_i)$, $1 \leq i \leq 2n+1$, *then* $\{0, i, x_i, b_i + 2n + 1\}$, $1 \leq i \leq 2n+1$, *together with the extra short orbit* $\{0, 6n+3, 6n+5, 12n+8\}$, *will be the base blocks of a cyclic BIBD*$(12n+10, 4, 2)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, 3, \ldots, 6n+4\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 2n + 1 - x_i)$ $\forall 1 \leq i \leq 2n+1$.

Example 4.3.3 *Let* $v = 34$ *and* $\{(2,3), (6,8), (7,10), (1,5), (4,9)\}$ *be the pairs of a Skolem sequence of order* 5. *Then* $\{0,1,31,8\}$, $\{0,2,7,13\}$, $\{0,3,24,15\}$, $\{0,4,12,10\}$, $\{0,5,20,14\}$ *together with the short orbit* $\{0,1,16,17\}$ *are the base blocks of a cyclic BIBD*$(34,4,2)$.

4.4 Cyclic BIBD$(v, 4, 3)$

In this section we give examples of Skolem partitioning problem for each admissible class of cyclic designs with $\lambda = 3$.

4.4.1 Cyclic BIBD$(12n, 4, 3)$

We write $12n = 4(3n-1)+4$.

Case 1: $n \equiv 0,\ 3 \pmod 4$

Remark 4.4.1 *Let $(a_i, b_i), 1 \le i \le 3n-1$, be a $(3n)$-extended Skolem sequence of order $3n-1$, and let x_i, $1 \le i \le 3n-1$, be positive integers. If there exists a partition of the numbers $\{1,\dots,3n-1\} \cup \{1,\dots,3n-1\} \cup \{3n+1,\dots,6n-1\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 3n - x_i)$, $1 \le i \le 3n-1$, then $\{0, i, x_i, b_i + 3n\}$, $1 \le i \le 3n-1$ together with the regular short orbit $\{0, 3n, 6n, 9n\}$ taken three times, will be the base blocks of a cyclic BIBD$(12n, 4, 3)$.*

Using the above approach our problem is reduced to partitioning the numbers $\{1,\dots,3n-1\} \cup \{1,\dots,3n-1\} \cup \{3n+1,\dots,6n-1\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 3n - x_i)$ $\forall 1 \le i \le 3n-1$.

Case 2: $n \equiv 1,\ 2 \pmod 4$

Remark 4.4.2 *Let* $(a_i, b_i), 1 \leq i \leq 3n-1$, *be a* $(3n+1)$*-extended Skolem sequence of order* $3n-1$, *and let* x_i, $1 \leq i \leq 3n-1$ *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 3n-2\} \cup \{1, \ldots, 6n-1\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i + 3n - 1 - x_i)$, $1 \leq i \leq 3n-1$, *then* $\{0, i, x_i, b_i + 3n - 1\}$, $1 \leq i \leq 3n-1$, *together with the regular short orbit* $\{0, 3n, 6n, 9n\}$ *and together with the short orbit* $\{0, 3n-1, 6n, 9n-1\}$, *will be the base blocks of a cyclic BIBD*$(12n, 4, 3)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 3n-2\} \cup \{1, \ldots, 6n-1\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 3n - 1 - x_i)$, $1 \leq i \leq 3n-1$.

Example 4.4.1 *Let* $v = 12$ *and* $\{(1, 2), (3, 5)\}$ *be the pairs of a* 4*-extended Skolem sequence of order* 2*. Then* $\{0, 1, 11, 4\}$, $\{0, 2, 11, 7\}$ *together with the regular short orbit* $\{0, 3, 6, 9\}$ *and together with the short orbit* $\{0, 2, 6, 8\}$ *are the base blocks of a cyclic BIBD*$(12, 4, 3)$.

4.4.2 Cyclic BIBD$(12n + 1, 4, 3)$

We use a 3-fold (hooked) Skolem array of order n to construct cyclic BIBD$(12n + 1, 4, 3)$.

Remark 4.4.3 *Let* (a_i, b_i), (c_i, d_i) *and* $(e_i, f_i), 1 \leq i \leq n$, *be a* 3*-fold (hooked) Skolem* *array of order* n*, and let* x_i, x'_i, x''_i *be positive integers. If there exists a partition of the numbers* $\{n{+}1,\ldots,4n\}\cup\{n{+}1,\ldots,4n\}\cup\{n{+}1,\ldots,4n\}$ *(or their inverses) into triples of the form* (x_i-i, x_i, b_i+4n-x_i)*,* $(x'_i-i, x'_i, b_i+4n-x'_i)$ *and* $(x''_i-i, x''_i, f_i+4n-x''_i)$*,* $1\le i\le n$*, then* $\{0, i, x_i, b_i+4n\}$*,* $\{0, i, x'_i, d_i+4n\}$*, and* $\{0, i, x''_i, f_i+4n\}$ $1\le i\le n$*, will be the base blocks of a cyclic BIBD*$(12n+1, 4, 3)$*.*

Using the above approach our problem is reduced to partitioning the numbers $\{n+1,\ldots,4n\}\cup\{n+1,\ldots,4n\}\cup\{n+1,\ldots,4n\}$ (or their inverses) into triples of the form (x_i-i, x_i, b_i+4n-x_i), $(x'_i-i, x'_i, b_i+4n-x'_i)$ and $(x''_i-i, x''_i, f_i+4n-x''_i)$, $1\le i\le n$.

Example 4.4.2 *Let* $v=13$ *and* $\{(1,2),(1,2),(1,2)\}$ *be the pairs of a* 3*-fold Skolem sequence of order* 1*. Then* $\{0,1,4,6\}$*,* $\{0,1,4,6\}$*,* $\{0,1,4,6\}$ *are the base blocks of a cyclic BIBD*$(13,4,3)$*.*

4.4.3 Cyclic BIBD$(12n+5, 4, 3)$

We write $12n+5 = 4(3n+1)+1$.

Remark 4.4.4 *Let* $(a_i, b_i), 1\le i\le 3n+1$*, be a (hooked) Skolem sequence of order* $3n+1$*, and let* x_i*,* $1\le i\le 3n+1$*, be positive integers. If there exists a partition of the numbers* $\{1,\ldots,6n+2\}\cup\{3n+2,\ldots,6n+2\}$ *(or their inverses) into triples of* *the form* $(x_i - i, x_i, b_i - x_i)$, $1 \le i \le 3n+1$, *then* $\{0, i, x_i, b_i\}$, $1 \le i \le 3n+1$, *will be the base blocks of a cyclic BIBD*$(12n+5, 4, 3)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n+2\} \cup \{3n+2, \ldots, 6n+2\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$, $1 \le i \le 3n+1$.

Example 4.4.3 *Let* $v = 17$ *and* $\{(1,2), (4,6), (5,8), (3,7)\}$ *be the pairs of a Skolem sequence of order* 4. *Then* $\{0,1,11,2\}$, $\{0,2,13,6\}$, $\{0,3,15,8\}$, $\{0,4,16,7\}$ *are the base blocks of a cyclic BIBD*$(17, 4, 3)$.

4.4.4 Cyclic BIBD$(12n+8, 4, 3)$

We write $12n + 8 = 4(3n+1) + 4$.

Case 1: $n \equiv 0, 3 \pmod 4$

Remark 4.4.5 *Let* $(a_i, b_i), 1 \le i \le 3n+1$, *be a* $(3n+3)$*-extended Skolem sequence of order* $3n+1$, *and let* x_i, $1 \le i \le 3n+1$, *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 3n\} \cup \{1, \ldots, 6n+3\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i + 3n + 1 - x_i)$, $1 \le i \le 3n+1$, *then* $\{0, i, x_i, b_i + 3n + 1\}$, $1 \le i \le 3n+1$ *together with the regular short orbit* $\{0, 3n+2, 6n+4, 9n+6\}$ *and*

and together with the short orbit $\{0, 3n+1, 6n+4, 9n+5\}$*, will be the base blocks of a cyclic BIBD*$(12n+8, 4, 3)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 3n\} \cup \{1, \ldots, 6n+3\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 3n + 1 - x_i)$ $\forall 1 \leq i \leq 3n+1$.

Case 2: $\boldsymbol{n \equiv 1,\ 2 \pmod 4}$

Remark 4.4.6 *Let* $(a_i, b_i), 1 \leq i \leq 3n+1$*, be a* $(3n+2)$*-extended Skolem sequence of order* $3n+1$*, and let* x_i*,* $1 \leq i \leq 3n+1$*, be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 3n+1\} \cup \{1, \ldots, 3n+1\} \cup \{3n+3, \ldots, 6n+3\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i + 3n + 2 - x_i)$*,* $1 \leq i \leq 3n+1$*, then* $\{0, i, x_i, b_i + 3n + 2\}$*,* $1 \leq i \leq 3n+1$*, together with the regular short orbit* $\{0, 3n+2, 6n+4, 9n+6\}$ *taken three times, will be the base blocks of a cyclic BIBD*$(12n+8, 4, 3)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 3n+1\} \cup \{1, \ldots, 3n+1\} \cup \{3n+3, \ldots, 6n+3\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 3n + 2 - x_i)$, $1 \leq i \leq 3n+1$.

Example 4.4.4 *Let* $v = 20$ *and* $\{(8, 9), (1, 3), (4, 7), (2, 6)\}$ *be the pairs of a 5-ext Skolem sequence of order 4. Then* $\{0, 1, 18, 14\}$*,* $\{0, 2, 19, 8\}$*,* $\{0, 3, 4, 12\}$*,*

$\{0,4,18,11\}$ *together with the regular short orbit* $\{0,5,10,15\}$ *taken three times are the base blocks of a cyclic BIBD*$(20,4,3)$.

4.4.5 Cyclic BIBD$(12n+9,4,3)$

We write $12n+9=4(3n+2)+1$.

Remark 4.4.7 *Let* $(a_i,b_i), 1\leq i\leq 3n+2$, *be a (hooked) Skolem sequence of order* $3n+2$, *and let* x_i, $1\leq i\leq 3n+2$ *be positive integers. If there exists a partition of the numbers* $\{1,\ldots,6n+4\}\cup\{3n+3,\ldots,6n+4\}$ *(or their inverses) into triples of the form* (x_i-i,x_i,b_i-x_i), $1\leq i\leq 3n+2$, *then* $\{0,i,x_i,b_i\}$, $1\leq i\leq 3n+2$, *will be the base blocks of a cyclic BIBD*$(12n+9,4,3)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1,\ldots,6n+4\}\cup\{3n+3,\ldots,6n+4\}$ (or their inverses) into triples of the form (x_i-i,x_i,b_i-x_i), $1\leq i\leq 3n+2$.

Example 4.4.5 *Let* $v=9$ *and* $\{(1,2),(3,5)\}$ *be the pairs of a hooked Skolem sequence of order* 2. *Then* $\{0,1,2,4\}$, $\{0,2,5,6\}$ *are the base blocks of a cyclic BIBD*$(9,4,3)$.

4.5 Cyclic BIBD$(v, 4, 6)$

4.5.1 Cyclic BIBD$(v, 4, 6), v \equiv 1, 5, 7, 11 (\text{mod } 12)$

For $\lambda = 6$ and $v \equiv 1, 5, 7, 11 (\text{mod } 12)$ we give a few constructions of cyclic block designs using Skolem-type sequences. Cyclic designs for $\lambda = 6$ and $v \equiv 1, 5, 7, 11 (\text{mod } 12)$ were also constructed by Furino [53] using an algebraic proof. We give a very simple combinatorial proof of the same problem.

Theorem 4.5.1 *Let $v \equiv 1, 5, 7, 11 (\text{mod } 12)$. Let $S = (v-4, v-6, \ldots, 1, 1, \ldots, v-6, v-4, 2, 0, 2)$ be a Skolem-type sequence where $D = \{1, 2, 3, 5, \ldots, v-4\}$ and let $\{(a_i, b_i) | i \in D\}$ be the pairs of positions in S_n for which $b_i - a_i = i$. Then the set $\mathcal{F}$ $= \{\{0, i, a_i + 1, b_i + 1\} | i \in D\}$ (mod v) is a $(v, 4, 6)$-DF. Hence, the set of quadruples in $\mathcal{F}$ form the base blocks of a cyclic BIBD$(v, 4, 6)$.*

Proof Set $B_i = \{\{0, i, a_i + 1, b_i + 1\} | i \in D\}$ and consider the elements of B_i mod v. We have $\partial B_i = \Delta B_i = \pm\{i, a_i + 1, b_i + 1, a_i + 1 - i, a_i + 1, i\}$ whence $\partial\mathcal{F} = \pm\{1, 2, 3, 5, \ldots, (v-4)\} \cup \pm\{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\} \cup \pm\{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\} \cup \pm\{2, 5, 8, \ldots, (v-3), 1, 4, \ldots, (\lfloor \frac{v}{2} \rfloor - 1)\} \cup \pm\{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\} \cup \pm\{1, 2, 3, 5, \ldots, (v-4)\}$.

Since we consider the elements of $\partial\mathcal{F}$ mod v, we have $\pm\{1, 2, 3, 5, \ldots, (v-4)\} = \{1, 2, \ldots, v-1\}$, $\pm\{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\} = \{1, 2, \ldots, v-1\}$ and $\pm\{2, 5, 8, \ldots, (v-$

3), 1, 4, ..., $(\lfloor \frac{v}{2} \rfloor - 1)\} = \{1, 2, \ldots, v-1\}$. Whence, $\partial\mathcal{F}$ is a $(v, 4, 6)$-DF. Hence, the set of quadruples in $\mathcal{F}$ form the base blocks of a cyclic BIBD$(v, 4, 6)$. ■

Example 4.5.1 *For $v = 11$, the Skolem-type sequence is* $(7, 5, 3, 1, 1, 3, 5, 7, 2, 0, 2)$ *with the pairs* $(4, 5), (9, 11), (3, 6), (2, 7), (1, 8)$. *These pairs yield the base blocks of a cyclic BIBD*$(11, 4, 6)$*:* $\{0, 1, 5, 6\}, \{0, 2, 10, 1\}, \{0, 3, 4, 7\}, \{0, 5, 3, 8\}$ *and* $\{0, 7, 2, 9\}$ (mod 11).

Theorem 4.5.1 can be generalized for all $k \geq 5$ as in Theorem 4.5.2. The proof is similar to Theorem 4.5.1 and is thus omitted.

Theorem 4.5.2 *Let $v \equiv 1, 5, 7, 11 \pmod{12}$, $k \geq 5$ and $v \geq 2k - 5$. Let $S = (v-4, v-6, \ldots, 1, 1, \ldots, v-6, v-4, 2, 0, 2)$ be a Skolem-type sequence where $D = \{1, 2, 3, 5, \ldots, v-4\}$ and let $\{(a_i, b_i) | i \in D\}$ be the pairs of positions in S_n for which $b_i - a_i = i$. Then the set $\mathcal{F}$=$\{\{0, i, a_i+1, b_i+1, i+b_i+1, 2i+b_i+1, \ldots, (k-4)i+b_i+1\} | i \in D\}$ (mod v) is a $(v, k, \frac{k(k-1)}{2})$-DF. Hence, the set of quadruples in $\mathcal{F}$ form the base blocks of a cyclic BIBD$(v, k, \frac{k(k-1)}{2})$.*

Example 4.5.2 *For $v = 11$, the Skolem-type sequence is* $(7, 5, 3, 1, 1, 3, 5, 7, 2, 0, 2)$ *with the pairs* $(4, 5), (9, 11), (3, 6), (2, 7), (1, 8)$. *These pairs yield the base blocks of a cyclic BIBD*$(11, 6, 15)$*:* $\{0, 1, 5, 6, 7, 8\}, \{0, 2, 10, 1, 3, 5\}, \{0, 3, 4, 7, 10, 2\}, \{0, 5, 3, 8, 2, 7\}$ *and* $\{0, 7, 2, 9, 5, 1\}$ (mod 11).

We can also get some of the above cyclic designs using skew starters.

Theorem 4.5.3 *If there exists a skew starter in* $\mathbb{Z}_v$ *then there exists a cyclic BIBD*$(v, 4, 6)$.

Proof Let S= $\{\{x_i, y_i\} | 1 \leq i \leq v - 1/2\}$ be a skew starter in $\mathbb{Z}_v$ and let B be the set of subsets $\{0, x_i, y_i, x_i + y_i\} | 1 \leq i \leq v - 1/2\}$. It is easy to see that these are the base blocks of a cyclic BIBD$(v, 4, 6)$. ■

It is known that skew starters exist for all v such that $\gcd(v, 150) = 1$ or 25 and that there do not exist skew starters for $v \equiv 0 (\text{mod } 3)$ or v even [34, 50]. It is conjectured that skew starters exist for all v such that $\gcd(v, 6) = 1$ [50] but the conjecture is still open.

4.5.2 Cyclic BIBD$(12n + 2, 4, 6)$

We write $12n + 2 = 2(6n) + 2$.

Case 1: $n \equiv 0, 2 \pmod 4$

Remark 4.5.1 *Let* $(a_i, b_i), 1 \leq i \leq 6n$, *be a Skolem sequence of order* $6n$, *and let* x_i, $1 \leq i \leq 6n$ *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 6n - 1\} \cup \{6n + 1\} \cup \{1, \ldots, 6n\} \cup \{1, \ldots, 6n\}$ *(or their inverses) into triples of the form*

$(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n$, *then* $\{0, i, x_i, b_i\}$, $1 \leq i \leq 6n$ *together with the short orbit* $\{0, 1, 6n+1, 6n+2\}$, *will be the base blocks of a cyclic BIBD*$(12n+2, 4, 6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n-1\} \cup \{6n+1\} \cup \{1, \ldots, 6n\} \cup \{1, \ldots, 6n\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$ $\forall 1 \leq i \leq 6n$.

Case 2: $\boldsymbol{n \equiv 1, 3 \pmod 4}$

Remark 4.5.2 *Let* $(a_i, b_i), 1 \leq i \leq 6n$, *be a hooked Skolem sequence of order* $6n$, *and let* x_i, $1 \leq i \leq 6n$ *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 6n-2\} \cup \{6n, 6n+1\} \cup \{1, \ldots, 6n\} \cup \{1, \ldots, 6n\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n$, *then* $\{0, i, x_i, b_i\}$, $1 \leq i \leq 6n$, *together with the short orbit* $\{0, 2, 6n+1, 6n+3\}$, *will be the base blocks of a cyclic BIBD*$(12n+2, 4, 6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n-2\} \cup \{6n, 6n+1\} \cup \{1, \ldots, 6n\} \cup \{1, \ldots, 6n\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n$.

Example 4.5.3 *Let* $v = 14$ *and* $\{(1,2), (9,11), (3,6), (4,8), (5,10), (7,13)\}$ *be the pairs of a hooked Skolem sequence of order* 6. *Then* $\{0, 1, 11, 2\}$, $\{0, 2, 10, 11\}$,

$\{0,3,4,6\}$, $\{0,4,7,8\}$, $\{0,5,8,10\}$, $\{0,6,8,13\}$ *together with the short orbit* $\{0,2,7,9\}$ *are the base blocks of a cyclic BIBD*$(14,4,6)$.

4.5.3 Cyclic BIBD$(12n+3,4,6)$

We write $12n+3 = 2(6n+1)+1$.

Case 1: $n \equiv 0 \pmod 2$

Remark 4.5.3 *Let* $(a_i, b_i), 1 \le i \le 6n+1$, *be a Skolem sequence of order* $6n+1$ *and let* x_i, $1 \le i \le 6n+1$, *be positive integers. If there exists a partition of the numbers* $\{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i - x_i)$, $1 \le i \le 6n+1$, *then* $\{0, i, x_i, b_i\}$, $1 \le i \le 6n+1$, *will be the base blocks of a cyclic BIBD*$(12n+3,4,6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$ $\forall 1 \le i \le 6n+1$.

Case 2: $n \equiv 1 \pmod 2$

Remark 4.5.4 *Let* $(a_i, b_i), 1 \le i \le 6n+1$, *be a Rosa sequence of order* $6n+1$, *and let* x_i, $1 \le i \le 6n+1$, *be positive integers. If there exists a partition of the numbers* $\{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\} \cup \{1,\ldots,6n+1\}$ *(or their inverses) into triples of* *the form* $(x_i - i, x_i, b_i + 6n + 1 - x_i)$, $1 \leq i \leq 6n + 1$, *then* $\{0, i, x_i, b_i + 6n + 1\}$, $1 \leq i \leq 6n + 1$ *will be the base blocks of a cyclic BIBD*$(12n + 3, 4, 6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n + 1\} \cup \{1, \ldots, 6n + 1\} \cup \{1, \ldots, 6n + 1\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i + 6n + 1 - x_i)$, $1 \leq i \leq 6n + 1$.

Example 4.5.4 *Let* $v = 15$ *and* $\{(12, 13), (4, 6), (11, 14), (1, 5), (2, 7), (9, 15), (3, 10)\}$ *be the pairs of a Rosa sequence of order* 7. *Then* $\{0, 1, 2, 5\}$, $\{0, 2, 9, 13\}$, $\{0, 3, 13, 6\}$, $\{0, 4, 13, 12\}$, $\{0, 5, 9, 14\}$, $\{0, 6, 10, 7\}$, $\{0, 7, 14, 2\}$ *are the base blocks of a cyclic BIBD*$(15, 4, 6)$.

4.5.4 Cyclic BIBD$(12n + 6, 4, 6)$

We write $12n + 6 = 2(6n + 2) + 2$.

Case 1: $n \equiv 0,\ 2 \pmod 4$

Remark 4.5.5 *Let* $(a_i, b_i), 1 \leq i \leq 6n + 2$, *be a hooked Skolem sequence of order* $6n+2$, *and let* x_i, $1 \leq i \leq 6n+2$ *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 6n\} \cup \{6n + 2, 6n + 4\} \cup \{1, \ldots, 6n + 2\} \cup \{1, \ldots, 6n + 2\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n+2$, *then* $\{0, i, x_i, b_i\}$,

$1 \leq i \leq 6n+2$, *together with the short orbit* $\{0, 2, 6n+3, 6n+5\}$, *will be the base blocks of a cyclic BIBD*$(12n+6, 4, 6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n\} \cup \{6n+2, 6n+4\} \cup \{1, \ldots, 6n+2\} \cup \{1, \ldots, 6n+2\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n+2$.

Case 2: $n \equiv 1, 3 \pmod 4$

Remark 4.5.6 *Let* $(a_i, b_i), 1 \leq i \leq 6n+2$, *be a Skolem sequence of order* $6n+2$, *and let* x_i, $1 \leq i \leq 6n+2$, *be positive integers. If there exists a partition of the numbers* $\{1, \ldots, 6n+1\} \cup \{6n+3\} \cup \{1, \ldots, 6n+2\} \cup \{1, \ldots, 6n+2\}$ *(or their inverses) into triples of the form* $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n+2$, *then* $\{0, i, x_i, b_i\}$, $1 \leq i \leq 6n+2$, *together with the short orbit* $\{0, 1, 6n+3, 6n+4\}$, *will be the base blocks of a cyclic BIBD*$(12n+6, 4, 6)$.

Using the above approach our problem is reduced to partitioning the numbers $\{1, \ldots, 6n+1\} \cup \{6n+3\} \cup \{1, \ldots, 6n+2\} \cup \{1, \ldots, 6n+2\}$ (or their inverses) into triples of the form $(x_i - i, x_i, b_i - x_i)$, $1 \leq i \leq 6n+2$.

Example 4.5.5 *Let* $v = 18$ *and* $\{(6,7), (1,3), (13,16), (11,15), (4,9), (8,14), (5,12), (2,10)\}$ *be the pairs of a Skolem sequence of order* 8. *Then* $\{0, 1, 17, 7\}$, $\{0, 2, 13, 3\}$,

$\{0, 3, 14, 16\}$, $\{0, 4, 13, 1\}$, $\{0, 5, 6, 9\}$, $\{0, 6, 3, 14\}$, $\{0, 7, 13, 12\}$, $\{08, 14, 10\}$ *together with the short orbit* $\{0, 1, 9, 10\}$ *are the base blocks of a cyclic BIBD*$(18, 4, 6)$.

Remark 4.5.7 *We have shown that there exists an example of Skolem partitions that induces a cyclic BIBD*$(v, 4, \lambda)$ *for every admissible class in Table 4.1.*

It is known that cyclic BIBD$(v, 4, 1)$ for $v = 16, 25, 28$ do not exist, and we showed at the beginning of this chapter that cyclic BIBD$(8, 4, 3)$ and cyclic BIBD$(10, 4, 2)$ do not exist. These mat be the only exceptions or there may be more exceptions for some small orders.

Conjecture 4.5.1 *For all admissible orders v and λ, with some possible exceptions, there exists a Skolem partition that induces a cyclic BIBD*$(v, 4, \lambda)$.

Chapter 5

Cyclic BIBD($v, 4, \lambda$) from Other Structures

In this chapter we use some existing results and relative difference families to construct new cyclic BIBD($v, 4, \lambda$) for $\lambda > 1$. Then, we provide a summary of what we have done in this thesis and what is known about cyclic BIBD($v, 4, \lambda$).

We provide many new linear classes of cyclic BIBD($v, 4, \lambda$) for some orders v, in particular for $v \equiv 6 \pmod{12}$ and $\lambda = 6$, our constructions cover all the values of v except for $v = 810, 30v', 810v'$ where v' is a product of primes greater than 5. The problem of constructing cyclic BIBD($v, 4, \lambda$) for all admissible orders v is still an open problem. This chapter is a considerable step forward to the solution of this problem.

5.1 Cyclic BIBD$(v, 4, 2)$

Using some known existing results, we construct cyclic BIBD$(6t + 1, 4, 2)$, for every $t \leq 1000$, and cyclic BIBD$(v, 4, 2)$, for $v = 30t+7, 78t+7, 114t+25, 138t+31, 150t+31, 162t+31, 174t+37, 174t+43$, for every $t \leq 1000, t \neq 2, 3$.

Lemma 5.1.1 *If there exists a* $(12t + 1, 4, 1)$*-PDF, then there exists a cyclic BIBD*$(6t + 1, 4, 2)$*.*

Proof Let $\mathcal{B}$ be the set of base blocks of a $(12t + 1, 4, 1)$-PDF. These base blocks cover all the differences $\{1, 2, \ldots, 6t\}$ each exactly once. Then the set of base blocks $\mathcal{B}$ are the base blocks of a cyclic BIBD$(6t + 1, 4, 2)$. ■

Theorem 5.1.2 *There exists cyclic BIBD*$(6t + 1, 4, 2)$*, for every* $t \leq 1000$*.*

Proof There exists a $(12t + 1, 4, 1)$-PDF for any $t \leq 1000$ except for $t = 2, 3$ [56]. By Lemma 5.1.1, there exists a cyclic BIBD$(6t + 1, 4, 2)$ for any $t \leq 1000$ except for $t = 2, 3$. A cyclic BIBD$(13, 4, 2)$ is given by the base blocks: $\{0, 1, 4, 6\}, \{0, 1, 3, 9\}$ and a cyclic BIBD$(19, 4, 2)$ is given by the base blocks: $\{0, 1, 2, 6\}, \{0, 2, 8, 11\}$, $\{0, 3, 7, 12\}$. ■

Example 5.1.1 *Let* $\{0, 1, 4, 6\}$ *be the base block of a* $(13, 4, 1)$*-PDF. Then* $\{0, 1, 4, 6\}$ *is the base block of a cyclic BIBD*$(7, 4, 2)$*.*

Theorem 5.1.3 *There exists cyclic BIBD*$(v,4,2)$ *for* $v = 30t+7, 78t+7, 114t+25, 138t+31, 150t+31, 162t+31, 174t+37, 174t+43$ *for every* $t \leq 1000$*,* $t \neq 2,3$.

Proof In [67] it is proved that if a $(12t+1,4,1)$-PDF exists, then there exists $(60t+13,4,1)$-PDF, $(156t+13)$-PDF, $(228t+49)$-PDF, $(276t+61)$-PDF, $(300t+61)$-PDF, $(324t+61)$-PDF, $(348t+73,4,1)$-PDF and $(348t+85,4,1)$-PDF. We apply Lemma 5.1.1 to the above perfect difference families. ■

5.2 Cyclic BIBD$(v,4,\lambda)$ from Relative Difference Families

Skew starters are special Skolem-type sequences. Direct and recursive constructions are used to construct many relative difference families from skew starters; see for example [31, 32, 55]. As an example, to construct an $(126,18,4,1)$-DF [55], from the skew starter $S = \{(2,3),(4,6),(1,5)\}$ in $\mathbb{Z}_7$, take the base blocks $\{(x,0),(y,0),(x+y,7),(0,8)\}$, $\{(0,0),(x+y,1),(x,13),(y,15)\}$, $\{(0,0),(-x-y,9),(-x,3),(-y,5)\}$ where $\{x,y\}$ run over all pairs in S and $-x,-y$ are the inverses of x,y in $\mathbb{Z}_7$.

So, the base blocks in our case are: $\{(2,0),(3,0),(5,7),(0,8)\}$, $\{(4,0),(6,0),(3,7),(0,8)\}$, $\{(1,0),(5,0),(6,7),(0,8)\}$, $\{(0,0),(5,1),(2,13),(3,15)\}$, $\{(0,0),(3,1),(4,13),(6,15)\}$, $\{(0,0),(6,1),(1,13),(5,15)\}$, $\{(0,0),(2,9),(5,3),$

$(4,5)\}$, $\{(0,0),(4,9),(3,3),(1,5)\}$,$\{(0,0),(1,9),(6,3),(2,5)\}$.

Using the ring isomorphism $\phi : (a,b) \in \mathbb{Z}_7 \oplus \mathbb{Z}_{18} \rightarrow 36a + 91b \in \mathbb{Z}_{126}$, one may see that the $(126,18,4,1)$-DF has the following base blocks: $\{72,108,61,98\}$, $\{18,90,115,98\}$, $\{36,54,97,98\}$, $\{0,19,121,87\}$, $\{0,73,67,69\}$, $\{0,55,85,33\}$, $\{0,9,75,95\}$, $\{0,81,3,113\}$, $\{0,99,111,23\}$.

We are going to use relative difference families and known recursive constructions to construct new cyclic block designs with block size 4 and $\lambda > 1$.

We construct new linear classes of cyclic BIBD$(v,4,2)$ for some values of $v \equiv 10 \pmod{12}$, new cyclic BIBD$(v,4,3)$ for some values of $v \equiv 0,8,9 \pmod{12}$, new cyclic BIBD$(v,4,6)$ for some values of $v \equiv 0,2,3,6,8 \pmod{12}$, and new cyclic BIBD$(v,4,4)$ for some values of $v \equiv 4 \pmod{12}$.

First, we outline what is known about relative difference families, cyclic difference matrices (CDM) and the known recursive constructions.

It is known that no $(9,4,1)$-CDM and no $(g,4,1)$-CDM exist for $g \equiv 0 \pmod 2$ [54]. It is also known that a $(27,4,1)$-CDM and a $(81,4,1)$- CDM exist [55].

In the case of $k = 4$, it is well known that the $4 \times v$ matrix with rows $R = (0,1,\ldots,v-1)$, $-R$, $2R$ and $-2R$ is a $(v,4,1)$ cyclic difference matrix provided that $\gcd(v,6) = 1$.

To construct new cyclic designs it will be necessary to build families of $(v,g,4,1)$-

DFs. The following constructions provide recursive methods to obtain an infinite number of such families.

Theorem 5.2.1 *[92] Suppose that both a* $(v, g, 4, 1)$*-DF and a* $(g, r, 4, 1)$*-DF exist. Then a* $(v, r, 4, 1)$*-DF also exists.*

Theorem 5.2.2 *[92] Suppose that both a* $(v, g, 4, 1)$*-DF and a* $(m, 4, 1)$*-CDM exist. Then there exists a* $(mv, gm, 4, 1)$*-DF.*

Theorem 5.2.3 *[92] Suppose that there exists:*

1. *a* $(v, g, 4, 1)$*-DF,*

2. *a* $(m, 4, 1)$*-CDM, and*

3. *a* $(gm, r, 4, 1)$*-DF,*

Then there exists an $(mv, r, 4, 1)$*-DF.*

Theorem 5.2.4 *[21] If there exists an* $(nv, n, 4, 1)$*-DF, an* $(nw, n, 4, 1)$*-DF, and a* $(w, 4, 1)$ *difference matrix, then there exists an* $(nvw, n, 4, 1)$*-DF.*

We are going to use the following lemma, to construct new cyclic BIBD$(v, 4, \lambda)$.

Lemma 5.2.5 *If there exists a cyclic* $(v, g, 4, 1)$*-DF and a cyclic BIBD*$(g, 4, \lambda)$*, then there exists a cyclic BIBD*$(v, 4, \lambda)$*.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, g, 4, 1)$-DF and let $\mathcal{E}$ be the base blocks of a cyclic BIBD$(g, 4, \lambda)$. Denote $u = v/g$. For every $B \in \mathcal{E}$, we construct new base blocks uB as follows: $uB = \{ub (\text{mod } v) | b \in B\}$.

Let $u\mathcal{E} = \{uB | B \in \mathcal{E}\}$ and let $\mathcal{H}$ be the family of starter blocks in which each starter block in $\mathcal{F}$ is taken λ times. Then, one can easily check that $u\ \mathcal{E} \cup \mathcal{H}$ is the required family of base blocks of a cyclic BIBD$(v, 4, \lambda)$. ■

The above lemma can be used even more generally as follows:

Lemma 5.2.6 *If there exist a cyclic $(v, g, 4, \lambda_1)$-DF and a cyclic BIBD$(g, 4, \lambda_2)$, then there exists a cyclic BIBD$(v, 4, \lambda)$ with λ=LCM(λ_1, λ_2).*

5.2.1 Cyclic BIBD$(v, 4, 2)$ for some values $v \equiv 10(\text{mod } 12)$

We construct infinitely many new cyclic BIBD$(v, 4, 2)$ for some values of $v \equiv 10(\text{mod } 12)$. Specifically, we construct cyclic BIBD$(22p, 4, 2)$ for every prime $p \equiv 1(\text{mod } 6)$.

Theorem 5.2.7 *Let v be a non-negative integer. If there exists a $(v, 22, 4, 1)$-DF, then there exists a cyclic BIBD$(v, 4, 2)$.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 22, 4, 1)$-DF and let $\mathcal{B}$

be the base blocks of a cyclic BIBD$(22, 4, 2)$ (see Appendix B). Now, apply Lemma 5.2.5. ■

Theorem 5.2.8 *There exists a cyclic BIBD*$(22p, 4, 2)$ *for every prime* $p \equiv 1 (\text{mod } 6)$.

Proof There exists a $(2p, 2, 4, 1)$-DF for every prime $p \equiv 1 (\text{mod } 6)$ [19]. Apply Theorem 5.2.2 for $v = 2p$, $g = 2$ and $m = 11$ where $p \equiv 1 (\text{mod } 6)$ a prime to get a $(22p, 22, 4, 1)$-DF for every prime $p \equiv 1 (\text{mod } 6)$. Apply Theorem 5.2.7 to the $(22p, 22, 4, 1)$-DF given above for every prime $p \equiv 1 (\text{mod } 6)$. ■

5.2.2 Cyclic BIBD$(v, 4, 3)$ for some values of $v \equiv 0 (\text{mod } 12)$

The following theorems give many new cyclic block designs with block size 4 and $v \equiv 0 \,(\text{mod}\, 12)$. Specifically, we construct cyclic BIBD$(v, 4, 3)$ for $v = 120, 144, 192, 216, 240, 288$. We construct cyclic BIBD$(24v, 4, 3)$ for all integers v such that $\gcd(v, 6) = 1$. We construct cyclic BIBD$(24v \cdot 3^{4t}, 4, 3)$ and cyclic BIBD$(24v \cdot 3^{4t+2}, 4, 3)$ for all integers $t \geq 0$ and all integers v such that $\gcd(v, 6) = 1$. Finally, we also construct cyclic BIBD$(24 \cdot u \cdot 5^s, 4, 3)$ for every integer $s \geq 1$ and $u = 5, 6, 8, 9, 10, 12$.

Theorem 5.2.9 *Let* v *be a non-negative integer. If there exists a* $(v, 24, 4, 1)$*-DF, then there exists a cyclic BIBD*$(v, 4, 3)$.

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 24, 4, 1)$-DF and let $\{\{0,1,2,4\}, \{0,1,5,10\}, \{0,2,9,17\}, \{0,3,10,14\}, \{0,3,11,16\}\}$ together with the regular orbit $\{0,6,12,18\}$ taken three times be the base blocks of a cyclic BIBD$(24, 4, 3)$. Now, apply Lemma 5.2.5. ■

Theorem 5.2.10 *There exists a cyclic BIBD*$(24v \cdot 3^{4t}, 4, 3)$ *and a cyclic BIBD*$(24v \cdot 3^{4t+2}, 4, 3)$*, for all integers* $t \geq 0$ *and all integers* v *such that gcd*$(v, 6) = 1$*.*

Proof There exists a $(24v \cdot 3^{4t}, 24, 4, 1)$-DF for all integers $t \geq 0$ and all integers v such that $\gcd(v, 6) = 1$ [55]. There exists a $(24v \cdot 3^{4t+2}, 24, 4, 1)$-DF for all integers $t \geq 0$ and all integers v such that $\gcd(v, 6) = 1$ [55]. Apply Theorem 5.2.9 to the $(24v \cdot 3^{4t}, 24, 4, 1)$-DF and the $(24v \cdot 3^{4t+2}, 24, 4, 1)$-DF given above to get a cyclic BIBD$(24 \cdot v \cdot 3^{4t}, 4, 3)$ and a cyclic BIBD$(24v \cdot 3^{4t+2}, 4, 3)$ for all integers $t \geq 0$ and all integers v such that $\gcd(v, 6) = 1$. ■

Theorem 5.2.11 *There exists cyclic BIBD*$(v, 4, 3)$ *for* $v = 120, 144, 192, 216, 240, 288$*.*

Proof There exist an $(120, 24, 4, 1)$-DF and a $(216, 24, 4, 1)$-DF [92]. There exist an $(144, 24, 4, 1)$-DF and a $(288, 24, 4, 1)$-DF [30]. There exist also an $(192, 24, 4, 1)$-DF and a $(240, 24, 4, 1)$-DF [31]. Apply Theorem 5.2.9 to the $(v, 24, 4, 1)$-DF for $v = 120, 144, 192, 216, 240, 288$. ■

Theorem 5.2.12 *There exists a cyclic BIBD*$(24 \cdot u \cdot 5^s, 4, 3)$ *for every integer* $s \geq 1$ *and* $u = 5, 6, 8, 9, 10, 12$.

Proof Apply Theorem 5.2.4 for $n = 24$, $w = 5$ and $v = u \cdot 5^s$, $s \geq 0$ to get a $(24 \cdot u \cdot 5^s, 24, 4, 1)$-DF for every $s \geq 1$. Then apply Theorem 5.2.9. ■

5.2.3 Cyclic BIBD$(v, 4, 3)$ for some values of $v \equiv 8 \pmod{12}$

We construct many new cyclic BIBD$(v, 4, 3)$ for some values of $v \equiv 8 \pmod{12}$. Specifically, we construct cyclic BIBD$(20p, 4, 3)$ for every prime $p \equiv 1 \pmod{12}$.

Theorem 5.2.13 *Let* v *be a non-negative integer. If there exist a* $(v, 20, 4, 1)$*-DF, then there exists a cyclic BIBD*$(v, 4, 3)$.

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 20, 4, 1)$-DF and let $\mathcal{B}$ be the base blocks of a cyclic BIBD$(20, 4, 3)$ (see Appendix B). Now, apply Lemma 5.2.5. ■

Theorem 5.2.14 *There exists a cyclic BIBD*$(20p, 4, 3)$ *for every prime* $p \equiv 1 \pmod{12}$.

Proof There exists a $(4p, 4, 4, 1)$-DF for every prime $p \equiv 1 \pmod{12}$ [22, 33]. Apply Theorem 5.2.2 for $v = 4p$, $g = 4$ and $m = 5$ where $p \equiv 1 \pmod{12}$ a prime to get a $(20p, 20, 4, 1)$-DF for every prime $p \equiv 1 \pmod{12}$. Apply Theorem 5.2.13 to the $(20p, 20, 4, 1)$-DF given above for every prime $p \equiv 1 \pmod{12}$. ■

5.2.4 Cyclic BIBD$(v, 4, 3)$ for some values of $v \equiv 9 \pmod{12}$

In this subsection, we construct cyclic BIBD$(v, 4, 3)$ for some $v \equiv 9 \pmod{12}$, using Lemma 5.2.5 and the known recursive constructions for difference families. Specifically, we construct a cyclic BIBD$(81, 4, 3)$ and infinite families of cyclic BIBD$(9v, 4, 3)$ and cyclic BIBD$(9v^2, 4, 3)$ for any positive integer v whose prime factors are all congruent to $1 \pmod 4$ and greater than 5.

Theorem 5.2.15 *Let v be a non-negative integer. If there exists a $(9v, 9, 4, 1)$-DF, then there exists a cyclic BIBD$(9v, 4, 3)$.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(9v, 9, 4, 1)$-DF and let $\{\{0, 1, 2, 4\}, \{0, 1, 4, 6\}\}$ be the base blocks of a cyclic BIBD$(9, 4, 3)$. Now, apply Lemma 5.2.5 to get the required design. ■

Example 5.2.1 *[56] The following are two examples of base blocks for a $(9v, 9, 4, 1)$-DF with $v = 9, 13$.*

1. *An* $(81, 9, 4, 1)$-*DF:* $\{\{0, 6, 28, 40\}, \{0, 3, 16, 26\}, \{0, 2, 31, 35\}, \{0, 5, 19, 30\}, \{0, 1, 21, 38\}, \{0, 7, 15, 39\}\}$.

2. *An* $(117,9,4,1)$-*DF:* $\{\{0,19,27,50\},\{0,6,47,54\},\{0,9,29,34\},\{0,4,42,53\},$
$\{0,1,18,58\},\{0,2,30,46\},\{0,32,35,56\},\{0,12,45,55\},\{0,14,36,51\}\}$.

It is known that no $(45,9,4,1)$-DF exists [52].

Example 5.2.2 *From Example 5.2.1, we know that an* $(81,9,4,1)$-*DF and an* $(117,9,4,1)$-*DF exist. Applying Theorem 5.2.15 we get the new cyclic BIBD*$(81,4,3)$ *and BIBD*$(117,4,3)$.

Note that the new cyclic BIBD$(81,4,3)$ cannot be obtained by any known direct or recursive construction.

Theorem 5.2.16 *There exists a cyclic BIBD*$(9v,4,3)$ *for any positive integer* v *whose prime factors are all congruent to* $1(\text{mod } 4)$ *and greater than* 5.

Proof There exists a $(9v,9,4,1)$-DF for any positive integer v whose prime factors are all congruent to $1(\text{mod } 4)$ and greater than 5 [52]. Apply Theorem 5.2.15 to the $(9v,9,4,1)$-DF given above. ■

Theorem 5.2.17 *There exists a cyclic BIBD*$(9^2\cdot v',4,3)$ *for any positive integer* $v'=p_1p_2\cdots p_r$ *whose prime factors* $p_1,p_2,\ldots,p_r$ *are all congruent to* $1(\text{mod } 4)$ *and greater than* 5.

Proof Apply Theorem 5.2.4 for $n = 9$, $w = p_i$ and $v = 9p_j$ for any primes $p_i, p_j \equiv 1 (\mathrm{mod}\ 4), p_i, p_j > 5$ to get a $(9^2 \cdot p_i \cdot p_j, 9, 4, 1)$-DF. Then apply again Theorem 5.2.4 for $n = 9$, $w = p_k$ and $v = 9p_ip_j$ for any primes $p_i, p_j, p_k \equiv 1(\mathrm{mod}\ 4), p_i, p_j > 5$ to get a $(9^2 \cdot p_i \cdot p_j \cdot p_k, 9, 4, 1)$-DF. Continuing in this way, we get a $(9^2 \cdot v', 9, 4, 1)$-DF for any positive integer $v' = p_1p_2 \cdots p_r$ whose prime factors $p_1, p_2, \ldots, p_r$ are all congruent to $1(\mathrm{mod}\ 4)$ and greater than 5. Now, we apply Theorem 5.2.15 to get a cyclic BIBD$(9^2 \cdot v', 4, 3)$ for any positive integer v' whose prime factors are all congruent to $1(\mathrm{mod}\ 4)$ and greater than 5. ■

5.2.5 Cyclic BIBD$(v, 4, 4)$ for some values of $v \equiv 4(\mathrm{mod}\ 12)$

The theorems in this section give many cyclic block designs with block size 4 and $v \equiv 4 \,(\mathrm{mod}\, 12)$. Specifically, we construct cyclic BIBD$(16p, 4, 4)$ for every prime $p \equiv 1(\mathrm{mod}\ 6)$. We construct cyclic BIBD$(16^2 \cdot v', 4, 4)$ for all v' whose prime factors are congruent to $1(\mathrm{mod}\ 6)$.

Theorem 5.2.18 *Let v be a non-negative integer. If there exists a $(v, 16, 4, 1)$-DF, then there exists a cyclic BIBD$(v, 4, 4)$.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 16, 4, 1)$-DF and let $\{\{0, 1, 2, 3\}, \{0, 2, 7, 13\}, \{0, 3, 9, 10\}, \{0, 5, 7, 10\}\}$ together with the short orbit

$\{0,4,8,12\}$ taken four times, be the base blocks of a cyclic BIBD$(16,4,4)$. Now, apply Lemma 5.2.5. ■

Theorem 5.2.19 *There exists a cyclic BIBD*$(16v,4,4)$ *for all v whose prime factors are congruent to* $1(\text{mod } 6)$.

Proof Apply Theorem 5.2.4 for $n = 16$, $v = p_1p_2\cdots p_r$, $w = p_j$ for every prime $p_i \equiv 1(\text{mod } 6)$. ■

Theorem 5.2.20 *There exists a cyclic BIBD*$(16^2 \cdot v', 4, 4)$ *for all v' whose prime factors are congruent to* $1(\text{mod } 6)$.

Proof There exists a $(256,16,4,1)$-DF [30] and a $(112,16,4,1)$-DF [32]. Apply Theorem 5.2.4 for $n = 16$, $w = v'$ and $v = 16$ to get a $(16^2 \cdot v', 16, 4, 1)$-DF. Then apply Theorem 5.2.18. ■

5.2.6 Cyclic BIBD$(v,4,6)$ for some values of $v \equiv 2(\text{mod } 12)$

We construct many new cyclic BIBD$(v,4,6)$ for some values of $v \equiv 2(\text{mod } 12)$. Specifically, we construct cyclic BIBD$(14p,4,6)$ for every prime $p \equiv 1(\text{mod } 6)$.

Theorem 5.2.21 *Let v be a non-negative integer. If there exists a* $(v,14,4,1)$*-DF, then there exists a cyclic BIBD*$(v,4,6)$.

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 14, 4, 1)$-DF and let $\mathcal{B}$ be the base blocks of a cyclic BIBD$(14, 4, 6)$ (see Appendix B). Now, apply Lemma 5.2.5. ■

Theorem 5.2.22 *There exists a cyclic BIBD*$(14p, 4, 6)$ *for every prime* $p \equiv 1 \pmod 6$.

Proof There exists a $(2p, 2, 4, 1)$-DF for every prime $p \equiv 1 \pmod 6$ [19]. Apply Theorem 5.2.2 for $v = 2p$, $g = 2$ and $m = 7$ where $p \equiv 1 \pmod 6$ a prime to get a $(14p, 14, 4, 1)$-DF for every prime $p \equiv 1 \pmod 6$. Apply Theorem 5.2.21 to the $(14p, 14, 4, 1)$-DF given above for every prime $p \equiv 1 \pmod 6$. ■

5.2.7 Cyclic BIBD$(v, 4, 6)$ for some values of $v \equiv 3 \pmod{12}$

We construct infinite families of cyclic BIBD$(v, 4, 6)$ for some values of $v \equiv 3 \pmod{12}$. Specifically, we construct infinite families of cyclic BIBD$(27v, 4, 6)$ for any v coprime with 6. We also construct infinite families of cyclic BIBD$(v, 4, 6)$ for $v = 3^5 \cdot v', 3^7 \cdot v'$ for any positive integer v' whose prime factors are all congruent to $1 \pmod 4$ and greater than 5.

Theorem 5.2.23 *Let v be a non-negative integer. If there exists a* $(v, 27, 4, 1)$*-DF, then there exists a cyclic BIBD*$(v, 4, 6)$.

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 27, 4, 1)$-DF. A cyclic BIBD$(27, 4, 6)$ exists [13]. Now, apply Lemma 5.2.5. ■

Theorem 5.2.24 *There exists a cyclic BIBD*$(27v, 4, 6)$ *for any* v *coprime with* 6*.*

Proof A cyclic BIBD$(27, 4, 6)$ exists [13] and the existence of a cyclic $(v, 4, 6)$-DF for any v coprime with 6 is given in Theorem 4.5.1. Then, using [64], there exists a $(27v, 4, 6)$-DF in $\mathbb{Z}_{27} \times \mathbb{Z}_v$ which is isomorphic to $\mathbb{Z}_{27v}$, by the Chinese remainder theorem. ■

Theorem 5.2.25 *There exists a cyclic BIBD*$(3^{2s+3}, 4, 6)$ *for any positive integer* $s \geq 1$*.*

Proof There exists a $(3^{2s+3}, 27, 4, 1)$-DF for any positive integer $s \geq 1$ [32]. Apply Theorem 5.2.23 to the $(3^{2s+3}, 27, 4, 1)$-DF given above for any positive integer $s \geq 1$. ■

Theorem 5.2.26 *There exist a cyclic BIBD*$(3^5 \cdot v, 4, 6)$ *and a cyclic BIBD*$(3^7 \cdot v, 4, 6)$ *for any positive integer* v *whose prime factors are all congruent to* $1 (\mathrm{mod}\ 4)$ *and greater than* 5*.*

Proof There exist a $(9 \cdot v', 9, 4, 1)$-DF [52] and a $(9^2 \cdot v', 9, 4, 1)$-DF for any positive integer v' whose prime factors are all congruent to $1 (\mathrm{mod}\ 4)$ and greater than 5 (see the proof of Theorem 5.2.17). There exists a $(27 \cdot 9, 27, 4, 1)$-DF [32]. Apply Theorem 5.2.3 for $v = 9 \cdot v'$, $m = 27$, $g = 9$ and $r = 27$ to get a $(3^5 \cdot v', 27, 4, 1)$-DF for any positive integer v' whose prime factors are all congruent to 1(mod 4) and greater than 5. Apply Theorem 5.2.3 for $v = 9^2 \cdot v'$, $m = 27$, $g = 9$ and $r = 27$ to get a $(3^7 \cdot v', 27, 4, 1)$-DF for any positive integer v' whose prime factors are all congruent to 1(mod 4) and greater than 5. Apply Theorem 5.2.23 to the $(3^5 \cdot v, 27, 4, 1)$-DF and to the $(3^7 \cdot v, 27, 4, 1)$-DF given above. ■

5.2.8 Cyclic BIBD$(v, 4, 6)$ for some values of $v \equiv 6 (\text{mod } 12)$

We construct infinite families of cyclic BIBD$(6v, 4, 6)$ for $v \equiv 6\,(\text{mod } 12)$. Specifically, we construct cyclic BIBD$(v, 4, 6)$ for $v = 9, 15, 27, 45, 81$. We construct cyclic BIBD$(6v, 4, 6)$ for all values of v whose prime factors are greater than 5. We also construct cyclic BIBD$(6vv', 4, 6)$ for $v = 3^{3s}, 3^{3s+1}, 3^{3s+2}, 3^{3s+1} \cdot 5, 3^{3s+2} \cdot 5, 3^{4s+1} \cdot 5, 3^{4s+2} \cdot 5$ for $s \geq 1$ and all v' whose prime factors are greater than 5. The only values of v not covered by our constructions are $v = 810, 30v', 810v'$ where v' has all the prime factors greater than 5.

Theorem 5.2.27 *Let v be a non-negative integer. If there exists a $(v, 6, 4, 1)$-DF, then there exists a cyclic BIBD$(v, 4, 6)$.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 6, 4, 1)$-DF, and let

$\{0,1,2,3\}$, $\{0,2,3,4\}$ together with the short orbit $\{0,1,3,4\}$ be the base blocks of a cyclic BIBD$(6,4,6)$. Now, apply Lemma 5.2.5. ■

Theorem 5.2.28 *There exists a cyclic BIBD*$(6v,4,6)$ *for all primes* $v > 5$*.*

Proof There exists a $(6v,6,4,1)$-DF for all primes $v > 5$ [23, 34]. Apply Theorem 5.2.27 to the $(6v,6,4,1)$-DF given above. ■

Theorem 5.2.29 *There exist a cyclic BIBD*$(6v,4,6)$ *for* $v = 9,15,27,45,81$ *and a cyclic BIBD*$(6v',4,6)$ *for all integers* v' *whose prime factors are greater than* 5*.*

Proof Apply Theorem 5.2.27 for $(6v,6,4,1)$-DF, $v = 9,15,27,45,81$ which can be found in [12, 55]. Apply Theorem 5.2.4 for $n = 6$, $w = p_i$ and $v = p_j$ for any primes $p_i, p_j > 5$ to get a $(6 \cdot p_i p_j, 6, 4, 1)$-DF, where p_i, p_j are prime factors greater than 5. Then apply again Theorem 5.2.4 for $n = 6$, $w = p_k$ and $v = p_i p_j$ to get a $(6 \cdot p_i p_j p_k, 6, 4, 1)$-DF, where p_i, p_j, p_k are prime factors greater than 5. Continuing this procedure we get a $(6 \cdot v', 6, 4, 1)$-DF, where v' has all prime factors greater than 5. Then apply Theorem 5.2.27 to get a cyclic BIBD$(6 \cdot v', 4, 6)$ for any positive integer v' whose prime factors are all greater than 5. ■

Theorem 5.2.30 *There exists a cyclic BIBD*$(6 \cdot u \cdot v', 4, 6)$ *, for any positive integer* v' *whose prime factors are all greater than* 5*, and* $u = 9,15,27,45,81$*.*

Proof Apply Theorem 5.2.4 for $n = 6$, $w = v'$ and $v = 9, 15, 27, 45, 81$ to get a $(6 \cdot u \cdot v', 6, 4, 1)$-DF for any positive integer v' whose prime factors are all greater than 5. Apply Theorem 5.2.27 to this relative difference family. ■

Theorem 5.2.31 *There exists a cyclic BIBD*$(6 \cdot 3^{3s} \cdot v', 4, 6)$ *for any positive v' whose prime factors are all greater than* 5 *and for any integer $s \geq 1$.*

Proof Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = v'$ to get a $(6 \cdot 3^{3s} \cdot v', 6, 4, 1)$-DF for every integer $s \geq 1$. Apply Theorem 5.2.27 to this relative difference family. ■

Theorem 5.2.32 *There exists a cyclic BIBD*$(v, 4, 6)$ *for $v = 6 \cdot 3^{3s}$, $6 \cdot 3^{3s+2}$, $6 \cdot 3^{3s+1}$ and every integer $s \geq 1$.*

Proof Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = 3^{3s}$ for every integer $s \geq 1$ to get a $(6 \cdot 3^{3s}, 6, 4, 1)$-DF for every integer $s \geq 1$. Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = 3^{3s-1}$ for every integer $s \geq 1$ to get a $(6 \cdot 3^{3s+2}, 6, 4, 1)$-DF. Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = 3^{3s+1}$ for every integer $s \geq 1$ to get a $(6 \cdot 3^{3s+1}, 6, 4, 1)$-DF.

Apply Theorem 5.2.27 to the above relative difference families. ■

Theorem 5.2.33 *There exists cyclic BIBD*$(6vv', 4, 6)$ *for $v = 3^{3s}, 3^{3s+1}, 3^{3s+2}$, $s \geq 1$ and all v' whose prime factors are all greater than* 5.

Proof Apply Theorem 5.2.4 for $n = 6$, $v = 3^{3s}, 3^{3s+1}, 3^{3s+2}$, $s \geq 1$ and $w = v'$. ■

Theorem 5.2.34 *There exists a cyclic BIBD*$(v, 4, 6)$ *for* $v = 6 \cdot 3^{3s+1} \cdot 5$, $6 \cdot 3^{3s+2} \cdot 5$, $6 \cdot 3^{4s+1} \cdot 5$, $6 \cdot 3^{4s+2} \cdot 5$ *and for every integer* $s \geq 1$.

Proof Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = 3^{3s-2} \cdot 5$ and for $n = 6$, $w = 81$ and $v = 3^{4s+1} \cdot 5$, for every integer $s \geq 1$. Apply Theorem 5.2.4 for $n = 6$, $w = 27$ and $v = 3^{3s-1} \cdot 5$ for every integer $s \geq 1$. Apply Theorem 5.2.4 for every integer $s \geq 1$ to get a $(6 \cdot 3^{4s+1} \cdot 5, 6, 4, 1)$-DF. Apply Theorem 5.2.4 for $n = 6$, $w = 81$ and $v = 3^{4s+2} \cdot 5$ for every integer $s \geq 0$ to get a $(6 \cdot 3^{4s+2} \cdot 5, 6, 4, 1)$-DF. Apply Theorem 5.2.27 to the above relative difference families for every integer $s \geq 1$. ■

Theorem 5.2.35 *There exists cyclic BIBD*$(6vv', 4, 6)$ *for* $v = 3^{3s+2} \cdot 5, 3^{3s+1} \cdot 5, 3^{4s+1} \cdot 5, 3^{4s+2} \cdot 5$, $s \geq 1$ *and all* v' *whose prime factors are all greater than* 5.

Proof Apply Theorem 5.2.4 for $n = 6$, $v = 3^{3s+2} \cdot 5, 3^{3s+1} \cdot 5, 3^{4s+1} \cdot 5, 3^{4s+2} \cdot 5$, $s \geq 1$ and $w = v'$. ■

Theorem 5.2.36 *There exists a cyclic BIBD*$(6 \cdot u \cdot v', 4, 6)$ *for* $u = 9, 15, 45$ *and every* v' *whose prime factors are greater than* 5.

Proof Apply Theorem 5.2.4 for $n = 6$, $w = v'$ and $v = u$ to get a $(6 \cdot u \cdot v', 6, 4, 1)$-DF. A $(6u, 6, 4, 1)$-DF for $u = 9, 15, 45$ exists, see [12, 55]. Apply Theorem 5.2.27 to the $(6 \cdot u \cdot v', 6, 4, 1)$-DF. ■

Theorem 5.2.37 *There exists a cyclic BIBD*$(6 \cdot 3^{4s}, 4, 6)$ *for every integer* $s \geq 1$*.*

Proof Apply Theorem 5.2.4 for $n = 6$, $w = 3^4$ and $v = 3^{4s}$ to get a $(6 \cdot 3^{4s+4}, 6, 4, 1)$-DF. Apply Theorem 5.2.27 to the $(6 \cdot 3^{4s}, 6, 4, 1)$-DF for every $s \geq 1$. ■

Theorem 5.2.38 *There exists a cyclic BIBD*$(6 \cdot 3^{4s} \cdot v', 4, 6)$ *for every integer* $s \geq 1$ *and every* v' *whose prime factors are greater than* 5*.*

Proof Apply Theorem 5.2.4 for $n = 6$, $w = v'$ and $v = 3^{4s}$ to get a $(6 \cdot 3^{4s} \cdot v', 6, 4, 1)$-DF. Apply Theorem 5.2.27 to the $(6 \cdot 3^{4s} \cdot v', 6, 4, 1)$-DF for every $s \geq 1$. ■

Theorem 5.2.39 *Let* v *be a non-negative integer. If there exists a* $(v, 18, 4, 1)$*-DF, then there exists a cyclic BIBD*$(v, 4, 6)$*.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 18, 4, 1)$-DF, and let $\{0, 2, 5, 7\}, \{0, 1, 5, 15\}, \{0, 1, 12, 13\}, \{0, 1, 7, 9\}, \{0, 6, 8, 10\}, \{0, 4, 8, 11\}, \{0, 5, 9, 12\}, \{0, 1, 3, 6\}$ together with the short orbit $\{0, 1, 9, 10\}$ be the base blocks of a cyclic BIBD$(18, 4, 6)$. Now, apply Lemma 5.2.5. ■

Theorem 5.2.40 *There exists a cyclic BIBD*$(6 \cdot 3 \cdot v, 4, 6)$ *for every* v *such that* *gcd*$(v, 150) = 1$ *or* 25*.*

Proof There exists a $(18v, 18, 4, 1)$-DF for every v such that $\gcd(v, 150) = 1$ or 25 [55]. Apply Theorem 5.2.39 to the $(18v, 18, 4, 1)$-DF given above. ■

There exists a cyclic BIBD$(30, 4, 6)$ (see Appendix B).

Summarizing all the above results, we have the following theorem:

Theorem 5.2.41 *There exists a cyclic BIBD*$(v, 4, 6)$ *for every positive* v *except* $810, 30v', 810v'$*, where* v' *has all the prime factors greater than* 5*.*

5.2.9 Cyclic BIBD$(v, 4, 6)$ for some values of $v \equiv 8(\text{mod } 12)$

The theorems in this section give many new cyclic block designs with block size 4 and $v \equiv 8 \,(\text{mod}\, 12)$. Specifically, we construct cyclic BIBD$(8v, 4, 6)$ for all orders v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime $\equiv 1(\text{mod } 6)$. We also construct cyclic BIBD$(8 \cdot 16 \cdot v, 4, 6)$ for all orders v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime congruent to $1(\text{mod } 6)$.

Theorem 5.2.42 *Let* v *be a non-negative integer. If there exist a* $(v, 8, 4, 1)$*-DF, then there exists a cyclic BIBD*$(v, 4, 6)$*.*

Proof Let $\mathcal{F}$ be the family of starter blocks of the given $(v, 8, 4, 1)$-DF and let $\{\{0, 1, 4, 5\}, \{0, 1, 4, 5\}, \{0, 1, 6, 7\}, \{0, 2, 5, 7\}, \{0, 2, 4, 6\}, \{0, 2, 4, 6\}\}$, be the base blocks of a cyclic BIBD$(8, 4, 6)$. Apply Lemma 5.2.5. ■

Theorem 5.2.43 *There exists a cyclic BIBD*$(8v, 4, 6)$ *for all* v *of the form* $v = p_1 p_2 \cdots p_r$ *such that each* p_i *is a prime congruent to* $1(\text{mod } 6)$*.*

Proof There exists a $(8v, 8, 4, 1)$-DF for all v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime congruent to $1 (\text{mod } 6)$ [23]. Apply Theorem 5.2.42 for the $(8v, 8, 4, 1)$-DF for all v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime congruent to $1 (\text{mod } 6)$ given above. ■

Theorem 5.2.44 *There exists a cyclic BIBD*$(8 \cdot 16 \cdot v', 4, 6)$ *for all* v'*s of the form* $v' = p_1 p_2 \cdots p_r$ *such that each* p_i *is a prime congruent to* $1 (\text{mod } 6)$*.*

Proof There exists an $(128, 8, 4, 1)$-DF [30]. Apply Theorem 5.2.4 for $n = 8$, $w = v'$ and $v = 16$ to get a $(8 \cdot 16 \cdot v', 8, 4, 1)$-DF for all v' of the form $v' = p_1 p_2 \cdots p_r$ such that each p_i is a prime congruent to $1 (\text{mod } 6)$. Now apply Theorem 5.2.42 to get the desired result. ■

5.3 Summary of Known Results for Cyclic BIBD$(v, 4, \lambda)$

We have constructed many linear classes of new cyclic BIBD$(v, 4, \lambda)$. We summarize the results obtained in this thesis, as well as the known results for cyclic BIBD$(v, 4, \lambda)$:

$\lambda = 1$ and $v \equiv 1, 4$ (mod 12)

For $v \equiv 1 (\text{mod } 12)$, $v \neq 25$, there exists cyclic BIBD$(12t + 1, 4, 1)$ for every

$t \leq 1000, t \neq 2$ [2,3,56]. For $v \equiv 4 (\mathrm{mod}\ 12)$ $v \neq 16, 28$, there exists cyclic BIBD$(12t + 4, 4, 1)$ for every $t \leq 50, t \neq 1, 2$ [36].

Some linear classes of cyclic BIBD$(v, 4, 1)$ are known: there exists a cyclic BIBD$(p, 4, 1)$ for any prime $p \equiv 1$ (mod 12) [20,35]; there exist a cyclic BIBD$(v, 4, 1)$ and a cyclic BIBD$(4v, 4, 1)$, where v is a product of primes congruent to 1 modulo 12 [24, 40]; there exists a cyclic BIBD$(4p, 4, 1)$ for any prime $p \equiv 1 (\mathrm{mod}\ 6)$ such that $\frac{p-1}{6}$ has a prime factor q not greater than 19 [23]; there exists a cyclic BIBD$(pq, 4, 1)$ for $p \leq q < 1000$ primes with $p \equiv q \equiv 7$ (mod 12) [27]; there exists a cyclic BIBD$(4^n u, 4, 1)$ with u a product of primes congruent to 1 modulo 6 for any integer $n \geq 3$. There exists a cyclic BIBD$(16u, 4, 1)$ where u is a product of primes congruent to 1 modulo 6 and $\gcd(u, 7 \cdot 13 \cdot 19) \neq 1$ [29]. If a cyclic BIBD$(12t + 1, 4, 1)$ exists, then there exists cyclic BIBD$(60t + 13, 4, 1)$, cyclic BIBD$(84t + 13, 4, 1)$, cyclic BIBD$(156t + 13, 4, 1)$, cyclic BIBD$(228t + 49, 4, 1)$, cyclic BIBD$(276t + 61, 4, 1)$, cyclic BIBD$(300t + 61, 4, 1)$, and cyclic BIBD$(300t + 79, 4, 1)$ designs [67, 74].

$\lambda = 2$ and $v \equiv 1, 4, 7, 10$ (mod 12)

For $v \equiv 1, 4$ (mod 12), take two copies of a cyclic BIBD$(v, 4, 1)$.

For $v \equiv 1, 7$ (mod 12), there exists cyclic BIBD$(6t + 1, 4, 2)$ for every $t \leq 1000$ (see Theorem 5.1.2).

There exists cyclic BIBD$(v, 4, 2)$ for $v = 30t + 7, 78t + 7, 114t + 25, 138t + 31, 150t +$

$31, 162t+31, 174t+37, 174t+43$ for every $t \leq 1000$, $t \neq 2, 3$ (see Theorem 5.1.3).

For $v \equiv 1 \pmod 6$ a prime power, there exists cyclic BIBD$(v, 4, 2)$ [16].

There exists a cyclic BIBD$(v, 4, 2)$ design for any v that is square-free and $v = \prod p_i^{n_i} \prod q_j^{2m_j} \equiv 7 \pmod{12}$ where $p_i \equiv 1 \pmod 6$, $q_j \equiv 5 \pmod 6$ and $n_i, m_j \in \mathbb{N}$ [53].

For $v \equiv 10 \pmod{12}$, there exists cyclic BIBD$(v, 4, 2)$ for every $v = 22p$ with $p \equiv 1 \pmod 6$ a prime (see Section 5.2.1). There exists cyclic BIBD$(v, 4, 2)$ for $v = 22, 34, 46$ [2].

$\boldsymbol{\lambda = 3}$ and $\boldsymbol{v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}}$

For $v \equiv 1, 4 \pmod{12}$, take three copies of a cyclic BIBD$(v, 4, 1)$.

For $v \equiv 0 \pmod{12}$, there exists cyclic BIBD$(v, 4, 3)$ for $v = 120, 144, 192, 216, 240, 288$; there exists cyclic BIBD$(24v, 4, 3)$ for all integers v such that $\gcd(v, 6) = 1$; there exists cyclic BIBD$(24v \cdot 3^{4t}, 4, 3)$ and cyclic BIBD$(24v \cdot 3^{4t+2}, 4, 3)$ for all integers $t \geq 0$ and all integers v such that $\gcd(v, 6) = 1$; there exists cyclic BIBD$(24 \cdot u \cdot 5^s, 4, 3)$ for every integer $s \geq 1$ and $u = 5, 6, 8, 9, 10, 12$ (see Section 5.2.2).

For $v \equiv 9 \pmod{12}$, there exist cyclic BIBD$(81, 4, 3)$ and infinite families of cyclic BIBD$(9v, 4, 3)$ and cyclic BIBD$(9v^2, 4, 3)$ for any positive integer v whose prime factors are all congruent to $1 \pmod 4$ and greater than 5 (see Section 5.2.4).

For $v = 4n+1$ a prime power, there exists a cyclic BIBD$(v, 4, 3)$ [16]. There is a cyclic BIBD$(v, 4, 3)$ when v has one of the following forms: $v \leq 149$ and $v \equiv 1 \pmod 4$ [4]; $v = (2w+1)^2$ with $w < 50$ [4]; $v = 2^{2n} + 2^n + 1$ with $n \geq 2$ [28]; $v = 9^n w$ with $n \geq 1$ and every prime factor of w congruent to $1 \pmod 4$ [5].

For $v \equiv 8 (\text{mod } 12)$, there exists cyclic BIBD$(v, 4, 3)$ for $v = 20p$ where $p \equiv 1 (\text{mod } 12)$ is a prime (see Section 5.2.3). There exists a cyclic BIBD$(20, 4, 3)$ (see Appendix B).

$\boldsymbol{\lambda = 4}$ and $\boldsymbol{v \equiv 1, 4, 7, 10}$ (mod 12)

For $v \equiv 1, 4 \pmod{12}$, take four copies of a cyclic BIBD$(v, 4, 1)$.

For $v \equiv 7, 10 \pmod{12}$, take two copies of a cyclic BIBD$(v, 4, 2)$.

For $v \equiv 4 \pmod{12}$, there exists cyclic BIBD$(16p, 4, 4)$ for every prime $p \equiv 1 (\text{mod } 6)$; there exists cyclic BIBD$(16^2 \cdot v', 4, 4)$ for all v' whose prime factors are congruent to $1 (\text{mod } 6)$ (see Section 5.2.5).

$\boldsymbol{\lambda = 5}$ and $\boldsymbol{v \equiv 1, 4}$ (mod 12)

For $v \equiv 1, 4 \pmod{12}$, take five copies of a cyclic BIBD$(v, 4, 1)$.

$\boldsymbol{\lambda = 6}$ and all $\boldsymbol{v}$

For $v \equiv 0 \pmod{12}$, take two copies of a cyclic BIBD$(v, 4, 3)$.

For $v \equiv 1, 5, 7, 11 \pmod{12}$, there exists cyclic BIBD$(v, 4, 6)$ (see Theorem 4.5.1).

For $v \equiv 2 \pmod{12}$, there exists cyclic BIBD$(v, 4, 6)$ for $v = 14p$ where $p \equiv 1 (\text{mod } 6)$ a prime (see Section 5.2.6). There exists a cyclic BIBD$(14, 4, 6)$ (see

Appendix B).

For $v \equiv 3 \pmod{12}$, there exists cyclic BIBD$(27v, 4, 6)$ for any v coprime with 6; there exist infinite families of cyclic BIBD$(v, 4, 6)$ for $v = 3^5 \cdot v', 3^7 \cdot v'$ for any positive integer v' whose prime factors are all congruent to $\equiv 1 \pmod 4$ and greater than 5 (see Section 5.2.7).

For $v \equiv 4 \pmod{12}$, take six copies of a cyclic BIBD$(v, 4, 1)$.

For $v \equiv 6 \pmod{12}$, there exists cyclic BIBD$(v, 4, 6)$ for every admissible v except for $v = 810, 30v', 810v'$, where v' is a product of primes each greater than 5 (see Section 5.2.8).

For $v \equiv 8 \pmod{12}$, there exists cyclic BIBD$(8v, 4, 6)$ for all v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime $\equiv 1 \pmod 6$; there exists cyclic BIBD$(8 \cdot 16 \cdot v', 4, 6)$ for all v of the form $v = p_1 p_2 \cdots p_r$ such that each p_i is a prime $\equiv 1 \pmod 6$ (see Section 5.2.9).

For $v \equiv 9 \pmod{12}$, take two copies of a cyclic BIBD$(v, 4, 3)$.

For $v \equiv 10 \pmod{12}$, take three copies of a cyclic BIBD$(v, 4, 2)$.

Chapter 6

Simple Cyclically Indecomposable BIBD $(v, 3, \lambda)$

The constructions of BIBDs$(v, 3, \lambda)$ with the properties cyclic, simple, and indecomposable, have been studied by many researchers one property at a time; for example, cyclic BIBD$(v, 3, \lambda)$ for all λs were constructed in [40, 82], simple for $\lambda = 2$ in [86] and simple for every v and λ satisfying the necessary conditions in [48].

Also some of the properties were combined in studies. In [87], cyclic and simple BIBD$(v, 3, 2)$ for all admissible orders were constructed, while in [6, 47, 49, 65, 68, 94], simple and indecomposable designs for $\lambda = 2, 3, 4, 5, 6$ and all admissible v were constructed. In [93], simple and indecomposable designs were constructed for all $v \geq$

$24\lambda - 5$ satisfying the necessary conditions. For the general case of $\lambda > 6$, Colbourn and Colbourn [44] constructed a single indecomposable BIBD$(v, 3, \lambda)$ for each odd λ. Shen [81] used the Colbourn and Colbourn result and some recursive constructions to prove the necessary conditions are asymptotically sufficient. Specifically, if λ is odd, then there exists a constant v_0 depending on λ with an indecomposable simple BIBD$(v, 3, \lambda)$ for all $v \geq v_0$ satisfying the necessary conditions.

In [73], the authors constructed BIBD$(v, 3, 2)$ having all three properties of being cyclic, simple and indecomposable for all admissible orders v. They acknowledged that the analogous problem for $\lambda = 3$ is more difficult.

In this chapter we construct cyclic, simple and indecomposable BIBD$(v, 3, 3)$ for all admissible orders v with some possible exceptions for $v = 9$ and $v = 24c+9$, $c \geq 4$.

Rees and Shalaby [73] introduced the notion of cyclically indecomposable triple systems. Grüttmüller, Rees, and Shalaby [57], constructed cyclically indecomposable BIBD$(v, 3, 2)$ for all admissible orders. The authors also checked exhaustively the cyclic BIBD$(v, 3, \lambda)$ for $\lambda = 2,\ v \leq 33$ and $\lambda = 3,\ v \leq 21$ that are cyclically indecomposable and determined if they are decomposable (to non cyclic) or not. Grüttmüller, Rees, Shalaby [57] found that there are exactly 3 inequivalent cyclically indecomposable but decomposable cyclic BIBD$(9, 3, 3)$, exactly 45 cyclically indecomposable but decomposable cyclic BIBD$(15, 3, 3)$, and exactly 7247 cyclically indecomposable but decomposable cyclic BIBD$(21,3,3)$. They pointed out that many of the sub-STSs are generated $+3 (\mathrm{mod}\ v)$; that is, the automorphism group contains no cycle of length v but a permutation which consists of three disjoint cycles of length $v/3$ each.

Grüttmüller [58] constructed cyclically indecomposable but decomposable cyclic BIBD$(v,3,3)$ for $v \equiv 3 (\mathrm{mod}\ 6)$. Cyclically indecomposable but decomposable cyclic BIBD$(v,3,3)$ for $v \equiv 1 (\mathrm{mod}\ 6)$ are more difficult to find. It is known that there is no such system for $v = 7$ or 13, but there is such a system for $v = 19$ [57]. Grüttmüller [58] constructed cyclically indecomposable but decomposable cyclic BIBD$(19v,3,3)$ for $v \equiv 1 (\mathrm{mod}\, 6)$.

We construct in this chapter, many new linear classes of cyclically indecomposable but decomposable BIBD$(v,3,4)$.

6.1 Cyclic, Simple, and Indecomposable BIBD$(v,3,3)$

In this section, we prove that there exist cyclic, simple, and indecomposable BIBD$(v,3,3)$ for all admissible orders v with some possible exceptions for $v = 9$ and $v = 24c + 9$, $c \geq 4$.

In 1974, Kramer [65] constructed all indecomposable BIBD$(v,3,3)$. We noticed that Kramer's construction for $v \equiv 1$ or $5 \pmod 6$ also gives cyclic and simple designs. We also noticed that this construction can be obtained using the canonical starter $v-2, v-4, \ldots, 3, 1, 1, 3, \ldots, v-4, v-2$ and taking the base blocks $\{\{0, i, b_i\} (\text{mod } v) | i = 1, 2, \ldots, \frac{1}{2}(v-1)\}$. So, Kramer's construction can be obtained using Skolem-type sequences.

We prove next that Kramer's construction for indecomposable BIBD$(v, 3, 3)$ produces simple designs.

Theorem 6.1.1 *The blocks* $\{0, \alpha, -\alpha\} (\text{mod } v) | \alpha = 0, 1, \ldots, \frac{1}{2}(v-1)$ *for* $v \equiv 1$ or $5 \pmod 6$ *form a cyclic, simple, and indecomposable BIBD*$(v, 3, 3)$.

Proof Let $v = 6n+1$. The design is cyclic and indecomposable [65]. We prove that the cyclic BIBD$(v, 3, 3)$ produced by $\{\{0, \alpha, -\alpha\} (\text{mod } v) | \alpha = 1, \ldots, \frac{1}{2}(v-1)\}$ is also simple.

Suppose that the construction above produces $\{x, y, z\}$ as a repeated block. Any block $\{x, y, z\}$ is of the form $\{0, i, 6n+1-i\} + k$ for some $i = 1, 2, \ldots, \frac{1}{2}(v-1)$ and $k \in \mathbb{Z}_{6n+1}$. Hence, if $\{x, y, z\}$ is a repeated block we have:

$$\{0, i_1, 6n+1-i_1\} + k_1 = \{0, i_2, 6n+1-i_2\} + k_2$$

whence

$$\{0, i_2, 6n+1-i_2\} = \{0, i_1, 6n+1-i_1\} + k$$

, for some $i_1, i_2 \in \{1, 2, \ldots, \frac{1}{2}(v-1)\}$ and some $k \in \mathbb{Z}_{6n+1}$.

If $k = 0$, we have $i_2 = 6n+1-i_1$ and $i_1 = 6n+1-i_2$ which is impossible since $6n+1-i_1 > i_2$ and $6n+1-i_2 > i_1$ by definition (i.e., $i_1, i_2 \in \{1, 2, \ldots, 3n\}$ while $6n+1-i_1$, $6n+1-i_2 \in \{3n+1, \ldots, 6n\}$.

If $k = i_2$, we have $\begin{cases} i_1 + i_2 = 6n+1 \\ 6n+1-i_1+i_2 = 6n+1-i_2 \end{cases}$ or $\begin{cases} i_1 + i_2 = 6n+1-i_2 \\ 6n+1-i_1+i_2 = 6n+1. \end{cases}$

Since both i_1 and i_2 are at most $3n$, it is impossible to have $i_1+i_2 = 6n+1$. Also $i_1 \neq i_2$.

If $k = 6n+1-i_2$, we have $\begin{cases} i_1 + 6n+1-i_2 = 6n+1 \\ 6n+1-i_1+6n+1-i_2 = i_2+6n+1 \end{cases}$ $\Leftrightarrow$ $\begin{cases} i_1 + i_2 = 6n+1-i_1 \\ 6n+1-i_2+i_1 = 6n+1 \end{cases}$ or $\begin{cases} i_1 + 6n+1-i_2 = i_2 \\ 6n+1-i_1+6n+1-i_2 = 6n+1. \end{cases}$

Since $6n+1-i_2 > i_2$, it is impossible to have $i_1+6n+1-i_2 = i_2$. It follows that our design is simple.

The case for $v = 6n + 5$ is similar. ■

In order to complete the case for $\lambda = 3$, we need to find cyclic, simple, and indecomposable BIBD$(v, 3, 3)$ for $v \equiv 3 \pmod 6$.

6.1.1 Simple BIBD $(v, 3, 3)$

We use Constructions 3.0.5 and 3.0.6 to construct simple BIBD$(v, 3, 3)$ for $v \equiv 3 \pmod 6$, $v \geq 15$ with a possible exception for $v = 24c + 57$, $c \geq 2$.

Lemma 6.1.2 *For every $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$, there is a Skolem sequence of order n starting with a 1 and ending with a 2.*

Proof To get a Skolem sequence of order n for $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$, take $(1, 1, hL_3^{n-2})$, replace the hook with a 2 and add the other 2 at the end of the sequence.

For $n = 8$, take $hL_3^6 = (8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, *, 6)$, for $n = 12$ take $hL_3^{10} = (9, 11, 3, 12, 4, 3, 7, 10, 4, 9, 8, 5, 11, 7, 6, 12, 5, 10, 8, *, 6)$ and for the remaining hL_3^{n-2}, hook a hL_4^{n-3} (see [83], Theorem 2, Case 1) to $(3, *, *, 3)$.

For $n \equiv 1 \pmod 4$, $n \geq 9$, take hL_3^{n-2} (see [83], Theorem 2, Case 1). ■

Example 6.1.1 *From Lemma 6.1.2, we have* S_8=$(1, 1, 8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 2, 6, 2)$, S_{12}=$(1, 1, 9, 11, 3, 12, 4, 3, 7, 10, 4, 9, 8, 5, 11, 7, 6, 12, 5, 10, 8, 2, 6, 2)$

and $S_{16} = (1,1,9,6,4,14,15,11,4,6,13,9,16,7,12,10,8,5,11,14,7,15,5,13,8,10,12,$ $3,16,2,3,2)$.

Lemma 6.1.3 *For every* $n \equiv 2$ or $3 \pmod 4$*,* $n \geq 7$*, there is a hooked Skolem sequence of order* n *starting with a* 1 *and ending with a* 2*.*

Proof For $n \equiv 2$ or $3 \pmod 4$, $n \geq 7$, take $\mathrm{hS}_n = (1,1,L_3^{n-2},2,*,2)$.

When $n \equiv 2(\text{mod } 4)$, take L_3^{n-2} (see [83], Theorem 1, Case 3).

When $n \equiv 3(\text{mod } 4)$, take $L_3^5 = (6,7,3,4,5,3,6,4,7,5)$ and for $n \geq 11$ take L_3^{n-2} (see [8], Theorem 2). ■

Theorem 6.1.4 *There exists simple BIBD*$(6n+3,3,3)$*,* $n \geq 2$*,* $n \equiv 0$ or $1 \pmod 4$ *except for* $6n+3 = 24c+9$*,* $c \geq 4$*.*

Proof First, let $v = 2n+1$, $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$. Apply Construction 3.0.5 to a Skolem sequence of order n starting with a 1 and ending with a 2 given by Lemma 6.1.2. We prove now that the cyclic BIBD$(v,3,3)$, $v = 6n+3$, $n \geq 2$, $v \neq 24c+57$, $c \geq 2$ produced by this construction is simple.

Suppose that the construction above produces $\{x,y,z\}$ as a repeated block. With regards to Construction 3.0.5, any block $\{x,y,z\}$ is of the form $\{0,i,b_i\}+k$ for some

$i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}_{2n+1}$. Hence if $\{x, y, z\}$ is a repeated block we have:

$$\{0, i_1, b_{i_1}\} + k_1 = \{0, i_2, b_{i_2}\} + k_2$$

whence,

$$\{0, i_2, b_{i_2}\} = \{0, i_1, b_{i_1}\} + k$$

for some $i_1, i_2 \in \{1, 2, \ldots, n\}$ and some $k \in \mathbb{Z}_{2n+1}$.

If $k = 0$, we have $i_2 = b_{i_1}$ and $i_1 = b_{i_2}$ which is impossible since $b_{i_1} \geq i_1 + 1$ and $b_{i_2} \geq i_2 + 1$ from the definition of a Skolem sequence.

If $k = i_2$, we have $\begin{cases} i_1 + i_2 = 2n+1 \\ b_{i_1} + i_2 = b_{i_2} \end{cases}$ or $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n+1. \end{cases}$

Since both i_1 and i_2 are at most n, it is impossible to have $i_1 + i_2 = 2n + 1$.

If $k = b_{i_2}$, we have $\begin{cases} i_1 + b_{i_2} = 2n+1 \\ b_{i_1} + b_{i_2} = i_2 + 2n + 1 \end{cases} \Leftrightarrow \begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n+1 \end{cases}$ or

$\begin{cases} i_1 + b_{i_2} = i_2 \\ b_{i_1} + b_{i_2} = 2n+1. \end{cases}$

Since $b_{i_2} > i_2$, it is impossible to have $i_1 + b_{i_2} = i_2$.

So, to prove that a system has no repeated blocks is enough to show that:

$$\begin{cases} i_1 + i_2 = b_{i_2} \\ \\ b_{i_1} + i_2 = 2n+1 \end{cases} \text{or} \begin{cases} i_1 + i_2 = b_{i_1} \\ \\ b_{i_2} + i_1 = 2n+1 \end{cases}$$ are not satisfied. We will prove also that $i = \frac{v}{3}$ and $b_i = \frac{2v}{3}$ are not allowed.

For $n = 8$ and $n = 12$, it is easy to see that the Skolem sequences of order n given by Lemma 6.1.2 produce simple designs.

For $n \equiv 0 (\text{mod } 4),\ n \geq 16$, let S_n be the Skolem sequence given by Lemma 6.1.2. This Skolem sequence is constructed using the hooked Langford sequence hL_4^{n-3} (see [83], Theorem 2, Case 1). Since $d = 4$, we will use only lines $(1)-(7), (14), (8*), (10*)$ and $(11*)$ in Simpson's Table [83]. Note that $n - 3 = 9 + 4r$ in Simpson's Table, so $n = 12 + 4r$ and $v = 25 + 8r$ in this case. Because we add the pair $(1, 1)$ at the beginning of the Langford sequence hL_4^{n-3}, a_i and b_i will be shifted to the right by two positions.

To make it easier for the reader we give, in Table 6.1, the hL_4^{n-3} taken from Simpson's Table and adapted for our case.

So the base blocks of the cyclic designs produced by Construction 3.0.5 are $\{0, 1, 2\}, \{0, 2, v-1\}, \{0, 3, v-2\}$ and $\{0, i, b_i + 2\}$ for $i = 4, \ldots, n$.

First, we show that the above designs have no short orbits, i.e., $i = \frac{v}{3}$ and $b_i + 2 = \frac{2v}{3}$ are not allowed. In the first three base blocks, it is obvious that $i \neq \frac{v}{3}$. For the remaining base blocks, we check lines $(1)-(7), (14), (8*), (10*)$ and $(11*)$ in Simpson's

	a_i+2	b_i+2	$i=b_i-a_i$	$0 \leq j \leq$
(1)	$2r+3-j$	$2r+7+j$	$4+2j$	r
(2)	$r+2-j$	$3r+9+j$	$2r+7+2j$	$r-1$
(3)	$6r+12-j$	$6r+17+j$	$5+2j$	$r-1$
(4)	$5r+12-j$	$7r+18+j$	$2r+6+2j$	r
(5)	$3r+8$	$7r+17$	$4r+9$	-
(6)	$4r+9$	$8r+21$	$4r+12$	-
(7)	$2r+6$	$6r+13$	$4r+7$	-
(14)	$2r+5$	$6r+16$	$4r+11$	-
(8$*$)	$4r+11$	$8r+19$	$4r+8$	-
(10$*$)	$4r+10$	$6r+15$	$2r+5$	-
(11$*$)	$2r+4$	$6r+14$	$4r+10$	-

Table 6.1: hL_4^{n-3}

Table.

Line (1): $\begin{cases} 4+2j = \frac{25+8r}{3} \\ 2r+7+j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{45}{12}$ which is impossible since $r \geq 1$.

Line (2): $\begin{cases} 2r+7+2j = \frac{25+8r}{3} \\ 3r+9+j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{7}{2}$ which is impossible since $r \geq 1$.

Line (3): $\begin{cases} 5+2j = \frac{25+8r}{3} \\ 6r+17+j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow r = -1$ which is impossible since $r \geq 1$

and also integer.

Line (4): $\begin{cases} 2r+6+2j = \frac{25+8r}{3} \\ 7r+18+j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{5}{4}$ which is impossible since $r \geq 1$.

Line (5): $\begin{cases} 4r+9 = \frac{25+8r}{3} \\ 7r+17 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (6) $\begin{cases} 4r+12 = \frac{25+8r}{3} \\ 8r+21 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (7): $\begin{cases} 4r+7 = \frac{25+8r}{3} \\ 6r+13 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (14) $\begin{cases} 4r+11 = \frac{25+8r}{3} \\ 6r+16 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (8∗): $\begin{cases} 4r+8 = \frac{25+8r}{3} \\ 8r+19 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (10∗) $\begin{cases} 2r+5 = \frac{25+8r}{3} \\ 6r+15 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (11∗): $\begin{cases} 4r+10 = \frac{25+8r}{3} \\ 6r+14 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset$.

Next, we have to check that neither $\begin{cases} i_1+i_2 = b_{i_2} \\ b_{i_1}+i_2 = 2n+1 \end{cases}$ nor $\begin{cases} i_1+i_2 = b_{i_1} \\ b_{i_2}+i_1 = 2n+1 \end{cases}$ are satisfied.

Lines (1) − (1): $\begin{cases} 4+2j_1+4+2j_2 = 2r+7+j_2 \\ 4+2j_2+2r+7+j_1 = 25+8r \end{cases} \Leftrightarrow \begin{cases} j_1 = \frac{-2r-15}{3} \\ j_2 = \frac{10r+27}{3} \end{cases}$ which is impossible since $j_1 \geq 0$ and also integer.

Lines $(1)-(2)$: $\begin{cases} 4+2j_1+2r+7+2j_2 = 3r+9+j_2 \\ \\ 2r+7+2j_2+2r+7+j_1 = 25+8r \end{cases} \Leftrightarrow \begin{cases} j_1 = \frac{-2r-15}{3} \\ \\ j_2 = r-2-2j_1 \end{cases}$

which is impossible since $j_1 \geq 0$ and also integer.

Lines $(1)-(3)$: $\begin{cases} 4+2j_1+5+2j_2 = 6r+17+_2 \\ \\ 5+2j_2+2r+7+j_1 = 25+8r \end{cases} \Leftrightarrow \begin{cases} j_1 = j_2 - 5 \\ \\ j_2 = 2r+9 \end{cases}$ which is impossible since $j_2 \leq r-1$.

Lines $(1)-(4)$: $\begin{cases} 4+2j_1+2r+6+2j_2 = 7r+18+j_2 \\ \\ 2r+6+2j_2+2r+7+j_1 = 25+8r \end{cases} \Leftrightarrow \begin{cases} j_1 = j_2 + r - 4 \\ \\ j_2 = \frac{3r+16}{3} \end{cases}$

which is impossible since $j_2 \leq r$.

Lines $(1)-(5)$: $\begin{cases} 4r+2j+13 = 7r+17 \\ \\ j+6r+16 = 25+8r \end{cases} \Leftrightarrow r = -14$ which is impossible since $r \geq 1$.

Lines $(1)-(6)$: $\begin{cases} 4r+2j+16 = 8r+21 \\ \\ j+6r+19 = 25+8r \end{cases} \Leftrightarrow \emptyset.$

Lines $(1)-(7)$: $\begin{cases} 4r+2j+11 = 6r+13 \\ \\ j+6r+14 = 25+8r \end{cases} \Leftrightarrow r = -12$ which is impossible since $r \geq 1$.

Lines $(1)-(14)$: $\begin{cases} 4r+2j+15 = 6r+16 \\ \\ j+6r+18 = 25+8r \end{cases} \Leftrightarrow j = -6$ which is impossible since

$j \geq 0$.

Lines (1) − (8∗): $\begin{cases} 4r + 2j + 15 = 8r + 19 \\ j + 6r + 15 = 25 + 8r \end{cases} \Leftrightarrow \emptyset.$

Lines (1) − (10∗): $\begin{cases} 2r + 2j + 9 = 6r + 15 \\ j + 4r + 12 = 25 + 8r \end{cases} \Leftrightarrow r = -5$ which is impossible since $r \geq 1$.

Lines (1) − (11∗): $\begin{cases} 4r + 2j + 14 = 6r + 14 \\ j + 6r + 17 = 25 + 8r \end{cases} \Leftrightarrow r = -8$ which is impossible since $r \geq 1$.

Similarly, we can check any combinations of two lines in Simpson's Table. This can be done easily using a program in Mathematica that checks all the pairs of rows in Simpson's Table using the above approach. The code for the program and the results can be found in [78]. From the results, we can easily see that if we check any combination of two lines in Simpson's Table, the conditions are not satisfied in almost all of the cases. There are two cases where these conditions are satisfied. The first case is when we check line (3) with line (1), and we get that, for $r = 4 + 3c$, $j_1 = 2c$, and $j_2 = 6 + 2c$, $c \geq 2$, the system is not simple. This implies that our system is not simple when $v = 24c + 9$, $c \geq 4$. The second case is when we check line (3) with line (2). Here, we get $r = 5$ and therefore $v = 59$. But $v = 59$ is not congruent to 3 (mod

6). A cyclic BIBD$(59, 3, 3)$ is simple, cyclic, and indecomposable by Theorem 6.1.1.

For $n \equiv 1 (\text{mod } 4)$, let S_n be the Skolem sequence given by Lemma 6.1.2. This Skolem sequence is constructed using hL_3^{n-2} from [83], Theorem 2, Case 1. Since $d = 3$, will use only lines $(1) - (6), (14), (7'), (8')$ and $(10')$ in Simpson's Table. Note that $n - 2 = 7 + 4r$ in Simpson's Table, so $n = 9 + 4r$ and $v = 19 + 8r$ in this case. Because we add the pair $(1, 1)$ at the beginning of the Langford sequence hL_3^{n-2}, a_i and b_i will be shifted to the right by two positions.

Table 6.2 gives the hL_3^{n-2} from Simpson's Table adapted to our case.

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r + 3 - j$	$2r + 6 + j$	$3 + 2j$	r
(2)	$r + 2 - j$	$3r + 8 + j$	$2r + 6 + 2j$	$r - 1$
(3)	$6r + 10 - j$	$6r + 14 + j$	$4 + 2j$	$r - 1$
(4)	$5r + 10 - j$	$7r + 15 + j$	$2r + 5 + 2j$	r
(5)	$3r + 7$	$7r + 14$	$4r + 7$	-
(6)	$4r + 8$	$8r + 17$	$4r + 9$	-
(14)	$2r + 4$	$6r + 12$	$4r + 8$	-
(7′)	$2r + 5$	$6r + 11$	$4r + 6$	-
(10′)	$4r + 9$	$6r + 13$	$2r + 4$	-

Table 6.2: hL_3^{n-2}

So, the base blocks of the cyclic designs produced by Construction 3.0.5 are $\{0, 1, 2\}, \{0, 2, v - 1\}$ and $\{0, i, b_i + 2\}$ for $i = 3, \ldots, n$. Using the same argument we show that these designs have no short orbits.

We show that $i = \frac{v}{3}$ and $b_i + 2 = \frac{2v}{3}$ are not allowed in the above system. In the first two base blocks it is obvious that $i \neq \frac{v}{3}$. For the remaining base blocks we check lines $(1)-(6),(14),(7')$ and $(10')$ in Table 6.2.

Line (1): $\begin{cases} 3+2j = \frac{19+8r}{3} \\ 2r+6+j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{5}{2}$ which is impossible since $r \geq 0$ and also integer.

Line (2): $\begin{cases} 2r+6+2j = \frac{19+8r}{3} \\ 3r+8+j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{9}{4}$ which is impossible since $r \geq 0$ and also integer.

Line (3): $\begin{cases} 4+2j = \frac{19+8r}{3} \\ 6r+14+j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow j = -\frac{1}{2}$ which is impossible since $j \geq 0$ and also integer.

Line (4): $\begin{cases} 2r+5+2j = \frac{19+8r}{3} \\ 7r+15+j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow r = -\frac{3}{2}$ which is impossible since $r \geq 0$ and also integer.

Line (5): $\begin{cases} 4r+7 = \frac{19+8r}{3} \\ 7r+14 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (6) $\begin{cases} 4r+9 = \frac{19+8r}{3} \\ 8r+17 = 2\frac{19+8r}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (14): $\begin{cases} 4r+8 = \frac{19+8r}{3} \\ 6r+12 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (7′) $\begin{cases} 4r+6 = \frac{19+8r}{3} \\ 6r+11 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line $(10')$: $\begin{cases} 2r+4 = \frac{19+8r}{3} \\ \\ 6r+13 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$

Next, we have to check that $\begin{cases} i_1+i_2 = b_{i_2} \\ \\ b_{i_1}+i_2 = 2n+1 \end{cases}$ or $\begin{cases} i_1+i_2 = b_{i_1} \\ \\ b_{i_2}+i_1 = 2n+1 \end{cases}$ are not satisfied. As with the previous case, the results can be found in [78]. When we check line (3) and line (1) the conditions are satisfied. But, in this case $v = 24c+35$, $c \geq 1$ which is not congruent to 3 (mod 6). So, a BIBD$(24c+35, 3, 3)$ for $c \geq 1$ is cyclic, simple, and indecomposable by Theorem 6.1.1. ■

Theorem 6.1.5 *There exists simple BIBD*$(6n+3, 3, 3)$ *for all* $n \equiv 2 \text{ or } 3 \pmod 4, n \geq 2$.

Proof Let $v = 2n+1$, $n \equiv 2$ or $3 \pmod 4$, $n \geq 7$. Apply Construction 3.0.6 to a hooked Skolem sequence of order n starting with a 1 and ending with a 2 given by Lemma 6.1.3. The proof is similar to Theorem 6.1.4.

Let $v = 2n+1$, $n \equiv 2$ or $3 \pmod 4$, $n \geq 10$.

For $n \equiv 2 (\text{mod } 4)$, $n \geq 10$, let hS_n be the hooked Skolem sequence given by Lemma 6.1.3. This hooked Skolem sequence is constructed using the Langford sequence L_3^{n-2} from [83], Theorem 1, Case 3.

Since $d = 3$, will use only lines $(1)-(4), (6), (9), (11)$ and (13) in Simpson's Table.

Note that $m = n - 2 = 4r$ in Simpson's Table, so $n = 4r + 2$ and $v = 8r + 5$, $r \geq 2$, $d = 3$, $s = 1$ in this case. Because we add the pair $(1, 1)$ at the beginning of the hooked Langford sequence hL_3^{n-2}, a_i and b_i will be shifted to the right by two positions. To make it easier for the reader we give in Table 6.3 the L_3^{n-2} taken from Simpson's Table and adapted for our case.

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r - j$	$2r + 4 + j$	$4 + 2j$	$r - 3$
(2)	$r + 2 - j$	$3r + 3 + j$	$2r + 1 + 2j$	$r - 1$
(3)	$6r + 1 - j$	$6r + 4 + j$	$3 + 2j$	$r - 2$
(4)	$5r + 2 - j$	$7r + 4 + j$	$2r + 2 + 2j$	$r - 2$
(6)	$2r + 3$	$4r + 3$	$2r$	-
(9)	$3r + 2$	$7r + 3$	$4r + 1$	-
(11)	$2r + 1$	$6r + 3$	$4r + 2$	-
(13)	$2r + 2$	$6r + 2$	$4r$	-
Omit row (1) when $r = 2$				

Table 6.3: L_3^{n-2}

So, the base blocks of the cyclic designs produced by Construction 3.0.6 are $\{0, 1, 3\}, \{0, 2, 1\}$ and $\{0, i, b_i + 2 + 1\}$ for $i = 3, \ldots, n$ and $i = b_i - a_i$.

First, we show that $i = \frac{v}{3}$ and $b_i + 2 + 1 = \frac{2v}{3}$ are not allowed in the above system. In the first two base blocks, it is obvious that $i \neq \frac{v}{3}$. For the remaining base blocks we check lines $(1) - (4), (6), (9), (11)$ and (13) in Table 6.3.

Line (1): $\begin{cases} 4+2j = \frac{8r+5}{3} \\ 2r+5+j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = \frac{1}{4}$ which is impossible since $r \geq 2$ and also integer.

Line (2): $\begin{cases} 2r+1+2j = \frac{8r+5}{3} \\ 3r+4+j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = \frac{1}{2}$ which is impossible since $r \geq 2$ and also integer.

Line (3): $\begin{cases} 3+2j = \frac{8r+5}{3} \\ 6r+5+j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = -\frac{1}{2}$ which is impossible since $r \geq 2$ and also integer.

Line (4): $\begin{cases} 2r+2+2j = \frac{8r+5}{3} \\ 7r+5+j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = -\frac{3}{4}$ which is impossible since $r \geq 2$ and also integer.

Line (6): $\begin{cases} 2r = \frac{8r+5}{3} \\ 4r+4 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (9): $\begin{cases} 4r+1 = \frac{8r+5}{3} \\ 7r+4 = 2\frac{8r+5}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (11): $\begin{cases} 4r+2 = \frac{8r+5}{3} \\ 6r+4 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (13): $\begin{cases} 4r = \frac{8r+5}{3} \\ 6r+3 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$.

Next, we have to show that $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n+1 \end{cases}$ or $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n+1 \end{cases}$ are not satisfied. The results for this can be found in [78]. As before, when we check line (3) with line (1), the conditions are satisfied. But $v = 24c+5$, $c \geq 2$ in this case which is not congruent to 3 (mod 6). So, by Theorem 6.1.1, there exists a cyclic, simple, and indecomposable BIBD$(24c+5, 3, 3)$ for $c \geq 2$.

For $n \equiv 3 (\mathrm{mod}\, 4), n \geq 11$, let hS_n be the hooked Skolem sequence given by Lemma 6.1.3. This hooked Skolem sequence is constructed using a L_3^{n-2} ([8], Theorem 2). Since $d = 3$ will use only lines $(1) - (4), (6) - (10)$ in [8]. Note that $m = n - 2 = 4r + 1, r \geq 2$, $e = 4$ in [8], so $n = 4r + 3$ and $v = 8r + 7$ in this case. Because we add the pair $(1, 1)$ at the beginning of the Langford sequence L_3^{n-2}, a_i and b_i will be shifted to the right by two positions.

Table 6.4 gives the L_3^{n-2} from [8] adapted to our case.

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r + 2 - j$	$2r + 6 + j$	$4 + 2j$	$r - 2$
(2)	$r + 2 - j$	$3r + 5 + j$	$2r + 3 + 2j$	$r - 2$
(3)	3	$4r + 4$	$4r + 1$	-
(4)	$2r + 4$	$4r + 5$	$2r + 1$	-
(6)	$r + 3$	$5r + 5$	$4r + 2$	-
(7)	$2r + 5$	$6r + 5$	$4r$	-
(8	$2r + 3$	$6r + 6$	$4r + 3$	-
(9)	$6r + 4 - j$	$6r + 7 + j$	$3 + 2j$	$r - 2$
(10)	$5r + 4 - j$	$7r + 6 + j$	$2r + 2 + 2j$	$r - 2$

Table 6.4: L_3^{n-2}

So, the base blocks of the cyclic designs produced by Construction 3.0.6 are

$\{0,1,3\},\{0,2,1\}$ and $\{0,i,b_i+2+1\}$ for $i=3,\ldots,n$. Using the same argument as before, we show that these designs are simple.

First, we show that $i=\frac{v}{3}$ and $b_i+2+1=\frac{2v}{3}$ are not allowed in the above system. In the first two base blocks is obvious that $i\neq\frac{v}{3}$. For the remaining base blocks we check lines $(1)-(4),(6)-(10)$ in Table 6.4.

Line (1): $\begin{cases} 4+2j=\frac{8r+7}{3} \\ 2r+7+j=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r=\frac{3}{4}$ which is impossible since $r\geq 2$ and also integer.

Line (2): $\begin{cases} 2r+3+2j=\frac{8r+7}{3} \\ 3r+6+j=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r=\frac{1}{2}$ which is impossible since $r\geq 2$ and also integer.

Line (3): $\begin{cases} 4r+1=\frac{8r+7}{3} \\ 4r+5=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (4) $\begin{cases} 2r+1=\frac{8r+7}{3} \\ 4r+6=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (6): $\begin{cases} 4r+2=\frac{8r+7}{3} \\ 5r+6=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$. Line (7) $\begin{cases} 4r=\frac{8r+7}{3} \\ 6r+6=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (8): $\begin{cases} 4r+3=\frac{8r+7}{3} \\ 6r+7=\frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$.

Line (9): $\begin{cases} 3+2j = \frac{8r+7}{3} \\ \\ 6r+8+j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r = -\frac{3}{2}$ which is impossible since $r \geq 2$.

Line (10): $\begin{cases} 2r+2+2j = \frac{8r+7}{3} \\ \\ 7r+7+j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r = -\frac{5}{4}$ which is impossible since $r \geq 2$.

Next, we have to check that $\begin{cases} i_1 + i_2 = b_{i_2} \\ \\ b_{i_1} + i_2 = 2n+1 \end{cases}$ or $\begin{cases} i_1 + i_2 = b_{i_1} \\ \\ b_{i_2} + i_1 = 2n+1 \end{cases}$ are not satisfied. The results for this can be found in [78]. Here, for $v = 3c-1$, $c \geq 4$ and for $v = 55$ the conditions are satisfied but these orders are not congruent to 3 (mod 6). Therefore, by Theorem 6.1.1, there exists cyclic, simple, and indecomposable BIBD$(3c-1, 3, 3)$ for $c \geq 4$ and cyclic, simple, and indecomposable BIBD$(55, 3, 3)$.

■

6.1.2 Indecomposable BIBD $(v, 3, 3)$

We use Constructions 3.0.5 and 3.0.6 to construct indecomposable BIBD$(v, 3, 3)$ for $v \equiv 3 \pmod 6$, $v \geq 15$.

Theorem 6.1.6 *There exists an indecomposable BIBD*$(v, 3, 3)$, *for every* $v \equiv 3 \pmod 6$, $v \geq 15$.

Proof Let $v \equiv 3 \pmod{6}$, $v \geq 15$. Suppose that $v = 2n + 1$, $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$.

Apply Construction 3.0.5 to a Skolem sequence of order n starting with a 1 and ending with a 2. By Lemma 6.1.2, such a Skolem sequence of order n exists for every $n \geq 8$.

Now, for a BIBD$(2n+1, 3, 3)$ to be decomposable, there must be an STS$(2n+1)$ inside the BIBD$(2n+1, 3, 3)$.

If $2n + 1 \equiv 3 \pmod{6}$, let $\{x_i, x_j, x_k\}$ be a triple using symbols from $\mathrm{N}_{2n+1} = \{0, 1, \ldots, 2n\}$. Let $d_{ij} = \min\{|x_i - x_j|, 2n + 1 - |x_i - x_j|\}$ be the difference between x_i and x_j. An STS$(2n + 1)$ on N_{2n+1} must have a set of triples with the property that each difference d, $1 \leq d \leq n$, occurs exactly $2n + 1$ times. Assume there is an STS$(2n + 1)$ inside our BIBD$(2n + 1, 3, 3)$ and let f_α be the number of triples inside the STS$(2n + 1)$ which are a cyclic shift of $\{0, \alpha, b_\alpha\}$.

We look at the first two base blocks of our BIBD$(2n + 1, 3, 3)$. These are $\{0, 1, 2\} \pmod{2n+1}$ and $\{0, 2, 2n\} \pmod{2n+1}$. Then the existence of an STS$(2n+1)$ inside our BIBD$(2n + 1, 3, 3)$ requires that the equation $2f_1 + f_2 = 2n + 1$ must have a solution in nonnegative integers (we need the difference 1 to occur exactly $2n + 1$ times).

Case 1: $f_1 = 1$

Suppose we choose one block from the orbit $\{0,1,2\}(\text{mod } 2n+1)$. Since this orbit uses the difference 1 twice and the difference 2, and the orbit $\{0,2,2n\}(\text{mod } 2n+1)$ uses the differences 1, 2 and 3, whenever we pick one block from the first orbit we cannot choose three blocks from the second orbit (i.e., those blocks where the pairs $(0,1)$, $(0,2)$ and $(1,2)$ are included). So, we just have $2n-2$ blocks in the second orbit to choose from. But we need $2n-1$ blocks from the second orbit in order to cover difference 1 exactly $2n+1$ times.

Therefore, we have no solution in this case.

Case 2: $f_1 = 2$

Since $f_2 = \frac{2n-3}{2}$ is not an integer, we have no solution in this case.

Case 3: $f_1 = 3, 5, \ldots, n$ (or $n-1$)

Similar to Case 1. So, there is no solution in this case.

Case 4: $f_1 = 4, 6, \ldots, n$ (or $n-1$)

Similar to Case 2. So, there is no solution in this case.

Case 5: $f_1 = 0$

Note that our cyclic BIBD$(v,3,3)$ has no short orbits (see Theorem 6.1.4) while a cyclic STS(v) will have a short orbit. Therefore, if a design exists inside our BIBD$(v,3,3)$, that design is not cyclic.

Now, we choose no block from the first orbit and all the blocks in the second orbit (i.e., $f_1 = 0$, $f_2 = 2n+1$). Therefore differences 1, 2 and 3 are all covered each exactly $2n+1$ times in the STS(v). From the remaining $n-2$ orbits $\{0, i, b_i\}$, $i \geq 3$ there will be two or three orbits which will use differences 2 and 3. Since differences 1, 2 and 3 are already covered, we cannot choose any block from those orbits that uses these three differences. So, we are left with $n-4$ or $n-5$ orbits to choose from. We need to cover differences $4, 5, \ldots, n$ ($n-3$ differences) each exactly $v = 2n+1$ times.

We form a system of $n-3$ equations with $n-4$ or $n-5$ unknowns in the following way: when a difference appears in different orbits, the sum of the blocks that we choose from each orbit has to equal v, i.e., if difference 4 appears in f_5, f_7 and f_{10} we have $f_5 + f_7 + f_{10} = v$ or if difference 4 appears in f_7 twice and in f_9 once we have $2f_7 + f_9 = v$. The system that we form is non-singular and it has the unique solution $f_{i_1} = f_{i_2} = \ldots = f_{i_k} = v$ for some $4 \leq i_1, i_2, \ldots, i_k \leq n$ and $f_{j_1} = f_{j_2} = \ldots = f_{j_k} = 0$ for some $4 \leq j_1, j_2, \ldots, j_k \leq n$. But this implies that the STS(v) inside our BIBD($v, 3, 3$) is cyclic which is impossible.

Therefore, we have no solution in this case. It follows that our BIBD($2n+1, 3, 3$) is indecomposable.

Now, suppose that $v = 2n+1$, $n \equiv 2$ or $3 \pmod 4$, $n \geq 7$. Apply Construction 3.0.6 to a hooked Skolem sequence of order n starting with a 1 and ending with a 2. By Lemma 6.1.3, such a hooked Skolem sequence of order n exists for every $n \geq 7$. Let f_α be the number of triples inside the STS$(2n+1)$ which are a cyclic shift of $\{0, \alpha, b_\alpha + 1\}$. Using the same argument as before it can be shown that the BIBD$(2n+1, 3, 3)$ is indecomposable. ■

6.1.3 Cyclic, Simple, and Indecomposable BIBD $(v, 3, 3)$

Now we are ready to prove our main result.

Theorem 6.1.7 *There exist cyclic, simple, and indecomposable BIBD$(v, 3, 3)$, for every $v \equiv 1 (\text{mod } 2)$, $v \geq 5, v \neq 9$ and $v \neq 24c + 9$, $c \geq 4$.*

Proof Let $v \equiv 1$ or 5 (mod 6) and take the base blocks $\{0, \alpha, -\alpha\} (\text{mod } v) | \alpha = 0, 1, \ldots, \frac{1}{2}(v-1)$. By Theorem 6.1.1, these will be the base blocks of a cyclic, simple and indecomposable BIBD$(v, 3, 3)$.

Let $v \equiv 3$ (mod 6), $v = 2n+1$, $n \equiv 0$ or $1 \,(\text{mod}\, 4)$, $n \geq 8$. Apply Construction 3.0.5 to the Skolem sequence of order n given by Lemma 6.1.2. These designs are cyclic by Construction 3.0.5, simple except for $v = 24c + 9$, $c \geq 4$ by Theorem 6.1.4 and indecomposable by Theorem 6.1.6.

Let $v \equiv 3$ (mod 6), $v = 2n+1$, $n \equiv 2$ or $3 \,(\text{mod}\, 4)$, $n \geq 7$. Apply Construction 3.0.6 to the hooked Skolem sequence of order n given by Lemma 6.1.3. These designs are cyclic by Construction 3.0.6, simple by Theorem 6.1.5 and indecomposable by Theorem 6.1.6. ■

6.2 Cyclically Indecomposable but Decomposable BIBD$(v, 3, 4)$

In this section we give examples of cyclically indecomposable but decomposable BIBD$(v, 3, 4)$ for $v \leq 21$. We obtained these designs using Constructions 3.0.7 and 3.0.8 from Chapter 2. Then, we use these examples and some new constructions to generate infinitely many new cyclically indecomposable but decomposable BIBD$(v, 3, 4)$.

6.2.1 Examples of Cyclically Indecomposable but Decomposable BIBD$(v, 3, 4)$ for $v \leq 21$

Lemma 6.2.1 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(9, 3, 4)$.

Proof From the 9-extended Skolem sequence of order 5, $(5, 3, 1, 1, 3, 5, 4, 2, *, 2, 4)$, take the base block of the form $\{0, i, b_i\}$ (mod 9). So, the canonical base blocks of the cyclic BIBD$(9,3,4)$ are: $\{0,1,4\}$, $\{0,2,1\}$, $\{0,3,5\}$, $\{0,4,2\}$, $\{0,5,6\}$, and the short orbit $\{0,6,3\}$.

This system is cyclically indecomposable since there exists no CSTS(9) nor cyclic BIBD$(9,3,2)$. This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form an STS(9): $\{0,1,2\}$, $\{0,3,8\}$, $\{0,4,7\}$, $\{0,5,6\}$, $\{1,3,7\}$, $\{1,4,6\}$, $\{1,5,8\}$, $\{2,3,6\}$, $\{2,4,8\}$, $\{2,5,7\}$,$\{3,4,5\}$,$\{6,7,8\}$. The base blocks of the BIBD$(9,3,3)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(9,3,4)$. ■

Lemma 6.2.2 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(10,3,4)$.

Proof From the 10-extended Skolem sequence of order 6, $(5,3,1,1,3,5,6,4,2,*,2,4,6)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 10). So, the canonical base blocks of the cyclic BIBD$(10,3,4)$ are: $\{0,1,4\}$, $\{0,2,1\}$, $\{0,3,5\}$, $\{0,4,2\}$, $\{0,5,6\}$, $\{0,6,3\}$. This system is cyclically indecomposable since there exists no CSTS(10) nor cyclic BIBD$(10,3,2)$. On the other hand, this design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(10,3,2)$: $\{0,1,2\}$, $\{0,1,4\}$, $\{0,2,4\}$, $\{0,3,7\}$, $\{0,3,9\}$, $\{0,5,6\}$, $\{0,5,8\}$, $\{0,6,7\}$, $\{0,8,9\}$, $\{1,2,6\}$,$\{1,3,5\}$,$\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,8\}$, $\{1,6,9\}$, $\{1,7,9\}$,$\{2,3,4\}$, $\{2,3,6\}$, $\{2,5,7\}$, $\{2,5,9\}$, $\{2,7,8\}$,$\{2,8,9\}$, $\{3,4,8\}$,

$\{3,5,7\}$, $\{3,6,9\}$, $\{4,5,6\}$, $\{4,5,9\}$, $\{4,6,8\}$, $\{4,7,9\}$, $\{6,7,8\}$. The base blocks of the second BIBD$(10,3,2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(10,3,4)$. ■

Lemma 6.2.3 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(12,3,4)$.

Proof From the 12-extended Skolem sequence of order 7, $(1,1,4,2,6,2,4,7,5,3,6,*,3,5,7)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 12) together with the short base block $\{0,4,8\}$. So, the canonical base blocks of the cyclic BIBD$(12,3,4)$ are: $\{0,1,2\}$, $\{0,2,6\}$, $\{0,3,1\}$, $\{0,4,7\}$, $\{0,5,2\}$, $\{0,6,11\}$, $\{0,7,3\}$, and the short base block $\{0,4,8\}$. There exists no CSTS(12) and therefore no cyclic decomposition into a CSTS(12) and a cyclic BIBD$(12,3,3)$. A decomposition into two cyclic BIBD$(12,3,2)$ would require 2×2 short base blocks but there is only one. Therefore the design is cyclically indecomposable.

This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(12,3,2)$: $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,5\}$, $\{0,3,7\}$, $\{0,4,7\}$, $\{0,4,8\}$, $\{0,5,9\}$, $\{0,6,8\}$, $\{0,6,11\}$, $\{0,9,10\}$,$\{0,10,11\}$,$\{1,2,4\}$, $\{1,3,6\}$, $\{1,4,8\}$, $\{1,5,9\}$, $\{1,5,11\}$,$\{1,6,7\}$, $\{1,7,9\}$, $\{1,8,10\}$, $\{1,10,11\}$, $\{2,3,4\}$,$\{2,3,9\}$, $\{2,5,10\}$, $\{2,6,9\}$, $\{2,6,11\}$, $\{2,7,8\}$, $\{2,7,11\}$, $\{2,8,10\}$, $\{3,4,5\}$, $\{3,5,8\}$, $\{3,6,10\}$, $\{3,7,10\}$,$\{3,8,11\}$, $\{3,9,11\}$, $\{4,5,11\}$, $\{4,6,9\}$,

$\{4,6,10\}$, $\{4,7,11\}$, $\{4,9,10\}$, $\{5,6,7\}$, $\{5,6,8\}$, $\{5,7,10\}$, $\{7,8,9\}$,$\{8,9,11\}$. The base blocks of the second BIBD$(12,3,2)$ are just the complement of the blocks above with respect to the blocks of the cyclic BIBD$(12,3,4)$. ■

Lemma 6.2.4 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(13,3,4)$.

Proof From the 13-extended Skolem sequence of order 8, $(1,1,6,4,2,8,2,4,6,7,5,3,*,8,3,5,7)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 13). So, the canonical base blocks of the cyclic BIBD$(13,3,4)$ are: $\{0,1,2\}$, $\{0,2,7\}$, $\{0,3,2\}$, $\{0,4,8\}$, $\{0,5,3\}$, $\{0,6,9\}$, $\{0,7,4\}$, $\{0,8,1\}$. Here we have to show that neither a decomposition into a CSTS(13) and a cyclic BIBD$(13,3,3)$, nor into two cyclic BIBD$(13,3,2)$ is possible. For a CSTS(13) we need two base blocks whose differences give each of $\{1,2,\ldots,6\}$ exactly once. But these do not exist. Now, suppose there exists a decomposition into two cyclic BIBD$(13,3,2)$, denoted C_1 and C_2 respectively. Assume that the base block $\{0,4,8\}$ providing the repeated difference 4 belongs to C_1. Then, the two base blocks $\{0,6,9\}$, $\{0,7,4\}$ which cover difference 4 as well need to occur in C_2. Both blocks cover differences 3 and 6 each. Therefore, base blocks $\{0,2,7\}$ and $\{0,8,1\}$ are forced to be contained in C_1. But then difference 5 occurs three times among the differences provided by base blocks of C_1, a contradiction. Hence, the cyclic BIBD$(13,3,4)$ above is cyclically indecomposable.

The design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(13,3,2)$: $\{0,1,2\}$, $\{0,1,8\}$, $\{0,2,3\}$, $\{0,3,7\}$, $\{0,4,7\}$, $\{0,4,8\}$, $\{0,5,9\}$, $\{0,5,11\}$, $\{0,6,9\}$, $\{0,6,10\}$,$\{0,10,12\}$,$\{0,11,12\}$, $\{1,2,9\}$, $\{1,3,4\}$, $\{1,3,11\}$, $\{1,4,6\}$,$\{1,5,8\}$, $\{1,5,10\}$, $\{1,6,12\}$, $\{1,7,10\}$, $\{1,7,11\}$,$\{1,9,12\}$, $\{2,3,10\}$, $\{2,4,5\}$, $\{2,4,12\}$, $\{2,5,9\}$, $\{2,6,11\}$, $\{2,6,12\}$, $\{2,7,8\}$, $\{2,7,11\}$, $\{2,8,10\}$, $\{3,4,5\}$,$\{3,5,6\}$, $\{3,6,10\}$, $\{3,7,12\}$, $\{3,8,9\}$, $\{3,8,12\}$, $\{3,9,11\}$, $\{4,6,11\}$, $\{4,7,9\}$, $\{4,8,11\}$, $\{4,9,10\}$, $\{4,10,12\}$,$\{5,6,7\}$, $\{5,7,12\}$, $\{5,8,12\}$, $\{5,10,11\}$, $\{6,7,8\}$, $\{6,8,9\}$, $\{7,9,10\}$, $\{8,10,11\}$, $\{9,11,12\}$. The base blocks of the second BIBD$(13,3,2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(13,3,4)$. ■

Lemma 6.2.5 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(15,3,4)$.

Proof From the 15-extended Skolem sequence of order 9, $(1,1,6,4,2,8,2,4,6,9,7,5,3,8,*,3,5,7,9)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 15). So, the canonical base blocks of the cyclic BIBD$(15,3,4)$ are:

$\{0,1,2\}$, $\{0,2,7\}$, $\{0,3,1\}$, $\{0,4,8\}$, $\{0,5,2\}$, $\{0,6,9\}$, $\{0,7,3\}$, $\{0,8,14\}$, $\{0,9,4\}$, together with the short orbit $\{0,5,10\}$.

With regard to cyclic decomposability it is easily seen that a decomposition into two cyclic BIBD$(15,3,2)$ would require four short base blocks, but there is only one. Now, suppose there exists a decomposition into a CSTS(15) and a cyclic BIBD$(15,3,3)$, denoted C_1 and C_2 respectively. The short base block has to occur in C_1 and therefore covers all pairs of points with difference 5. Consequently, the remaining differences $\{1,2,3,4,6,7\}$ have to be covered exactly once by two base blocks. But there are no such two base blocks. Hence, the cyclic BIBD$(15,3,4)$ above is cyclically indecomposable.

This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form an STS(15): $\{0,1,3\}$, $\{0,2,5\}$, $\{0,4,9\}$, $\{0,6,10\}$, $\{0,7,11\}$, $\{0,8,14\}$, $\{0,12,13\}$, $\{1,2,10\}$, $\{1,4,14\}$, $\{1,5,9\}$,$\{1,6,12\}$,$\{1,7,8\}$, $\{1,11,13\}$, $\{2,3,11\}$, $\{2,4,7\}$, $\{2,6,13\}$,$\{2,8,9\}$, $\{2,12,14\}$, $\{3,4,6\}$, $\{3,5,8\}$, $\{3,7,12\}$,$\{3,9,13\}$, $\{3,10,14\}$, $\{4,5,13\}$, $\{4,8,12\}$, $\{4,10,11\}$, $\{5,6,14\}$, $\{5,7,10\}$, $\{5,11,12\}$, $\{6,7,9\}$, $\{6,8,11\}$, $\{7,13,14\}$,$\{8,10,13\}$, $\{9,10,12\}$, $\{9,11,14\}$. The base blocks of the second BIBD$(15,3,3)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(15,3,4)$.

■

Lemma 6.2.6 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(16, 3, 4)$.

Proof From the 16-extended Skolem sequence of order 10, $(1, 1, 8, 6, 4, 2, 10, 2, 4, 6, 8, 9, 7, 5, 3, *, 10, 3, 5, 7, 9)$, take the base blocks of the form $\{0, i, b_i\}$ (mod 16). So, the canonical base blocks of the cyclic BIBD$(16, 3, 4)$ are: $\{0, 1, 2\}$, $\{0, 2, 8\}$, $\{0, 3, 2\}$, $\{0, 4, 9\}$, $\{0, 5, 3\}$, $\{0, 6, 10\}$, $\{0, 7, 4\}$, $\{0, 8, 11\}$, $\{0, 9, 5\}$, $\{0, 10, 1\}$. A CSTS(16) does not exist. So, we only have to show that a decomposition into two cyclic BIBD$(16, 3, 2)$ is not possible. Suppose to the contrary that there exists a decomposition into two cyclic BIBD$(16, 3, 2)$, denoted C_1 and C_2 respectively. Assume that the base block $\{0, 1, 2\}$ providing the repeated difference 1 belongs to C_1. Then the two base blocks $\{0, 3, 2\}$, $\{0, 10, 1\}$ which cover difference 1 as well need to occur in C_2. The second of these blocks covers difference 6. Therefore, base block $\{0, 6, 10\}$ with the repeated difference 6 has to be in C_1 and base block $\{0, 2, 8\}$ has to be in C_2. Moreover, the latter base block provides difference $8 = v/2$ in C_2 and thus base block $\{0, 8, 11\}$, with the second difference 8, is forced to be contained in C_1. We observe that difference 2 now already occurs twice in base blocks of C_2 and, therefore, base block $\{0, 5, 3\}$ belongs to C_1, covers a second difference 3 there, and forces the seventh base block $\{0, 7, 4\}$ to be in C_2. This in turn provides a second difference 7 in C_2 such that base blocks $\{0, 4, 9\}$ and $\{0, 9, 5\}$ need to be in

C_1. But then difference 5 occurs three times among the differences provided by base blocks of C_1, a contradiction. Hence, the cyclic BIBD$(16, 3, 4)$ above is cyclically indecomposable.

Moreover, this design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(16, 3, 2)$: $\{0,1,2\}$, $\{0,1,10\}$, $\{0,2,3\}$, $\{0,3,5\}$, $\{0,4,7\}$, $\{0,4,9\}$, $\{0,5,12\}$, $\{0,6,12\}$, $\{0,6,14\}$, $\{0,7,11\}$,$\{0,8,10\}$,$\{0,8,11\}$, $\{0,9,13\}$, $\{0,13,15\}$, $\{0,14,15\}$, $\{1,2,11\}$,$\{1,3,9\}$, $\{1,3,14\}$, $\{1,4,9\}$, $\{1,4,13\}$, $\{1,5,11\}$,$\{1,5,12\}$, $\{1,6,10\}$, $\{1,6,14\}$, $\{1,7,8\}$, $\{1,7,15\}$, $\{1,8,13\}$, $\{1,12,15\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,5,10\}$, $\{2,6,9\}$,$\{2,6,13\}$, $\{2,7,14\}$, $\{2,7,15\}$, $\{2,8,12\}$, $\{2,8,14\}$, $\{2,9,13\}$, $\{2,10,13\}$, $\{2,11,15\}$, $\{3,4,13\}$, $\{3,5,11\}$, $\{3,6,8\}$,$\{3,6,11\}$, $\{3,7,12\}$, $\{3,7,13\}$, $\{3,8,12\}$, $\{3,9,15\}$, $\{3,10,14\}$, $\{3,10,15\}$, $\{4,5,14\}$, $\{4,6,7\}$,$\{4,6,12\}$, $\{4,8,11\}$, $\{4,8,15\}$, $\{4,10,11\}$, $\{4,10,14\}$, $\{4,12,15\}$, $\{5,6,7\}$, $\{5,6,15\}$, $\{5,7,13\}$, $\{5,8,10\}$, $\{5,8,13\}$,$\{5,9,14\}$, $\{5,9,15\}$, $\{6,8,9\}$, $\{6,10,13\}$, $\{6,11,15\}$, $\{7,8,9\}$, $\{7,9,10\}$, $\{7,10,12\}$, $\{7,11,14\}$,$\{8,14,15\}$, $\{9,10,11\}$, $\{9,11,12\}$, $\{9,12,14\}$, $\{10,13,15\}$, $\{11,12,13\}$, $\{11,13,14\}$, $\{12,13,14\}$. The base blocks of the second BIBD$(16, 3, 2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(16, 3, 4)$. ■

Lemma 6.2.7 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(18, 3, 4)$.

Proof From the 18-extended Skolem sequence of order 11, $(11, 9, 7, 5, 3, 1, 1, 3, 5, 7, 9, 11, 10, 8, 6, 4, 2, *, 2, 4, 6, 8, 10)$, take the base blocks of the form $\{0, i, b_i\}$ (mod 18). So, the canonical base blocks of the cyclic BIBD$(18, 3, 4)$ are: $\{0, 1, 7\}$, $\{0, 2, 1\}$, $\{0, 3, 8\}$, $\{0, 4, 2\}$, $\{0, 5, 9\}$, $\{0, 6, 3\}$, $\{0, 7, 10\}$, $\{0, 8, 4\}$, $\{0, 9, 11\}$, $\{0, 10, 5\}$, $\{0, 11, 12\}$, and the short orbit $\{0, 6, 12\}$. This system is cyclically indecomposable since there exists no CSTS(18) nor a cyclic BIBD$(18, 3, 2)$. This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(18, 3, 2)$: $\{0, 1, 2\}$, $\{0, 1, 7\}$, $\{0, 2, 4\}$, $\{0, 3, 6\}$, $\{0, 3, 8\}$, $\{0, 4, 8\}$, $\{0, 5, 13\}$, $\{0, 5, 15\}$, $\{0, 6, 17\}$, $\{0, 7, 10\}$,$\{0, 9, 11\}$,$\{0, 9, 14\}$, $\{0, 10, 13\}$, $\{0, 11, 12\}$, $\{0, 12, 15\}$, $\{0, 14, 16\}$,$\{0, 16, 17\}$, $\{1, 2, 8\}$, $\{1, 3, 10\}$, $\{1, 3, 17\}$, $\{1, 4, 7\}$,$\{1, 4, 12\}$, $\{1, 5, 9\}$, $\{1, 5, 14\}$, $\{1, 6, 11\}$, $\{1, 6, 16\}$, $\{1, 8, 11\}$, $\{1, 9, 14\}$, $\{1, 10, 15\}$, $\{1, 12, 13\}$, $\{1, 13, 16\}$, $\{1, 15, 17\}$,$\{2, 3, 4\}$, $\{2, 3, 9\}$, $\{2, 5, 8\}$, $\{2, 5, 10\}$, $\{2, 6, 10\}$, $\{2, 6, 16\}$, $\{2, 7, 15\}$, $\{2, 7, 17\}$, $\{2, 9, 12\}$, $\{2, 11, 13\}$, $\{2, 11, 16\}$, $\{2, 12, 15\}$,$\{2, 13, 14\}$, $\{2, 14, 17\}$, $\{3, 4, 10\}$, $\{3, 5, 7\}$, $\{3, 5, 12\}$, $\{3, 6, 14\}$, $\{3, 7, 11\}$, $\{3, 8, 13\}$, $\{3, 9, 15\}$,$\{3, 11, 16\}$, $\{3, 12, 17\}$, $\{3, 13, 16\}$, $\{3, 14, 15\}$, $\{4, 5, 6\}$, $\{4, 5, 11\}$, $\{4, 6, 13\}$, $\{4, 7, 15\}$, $\{4, 8, 12\}$, $\{4, 9, 13\}$, $\{4, 9, 17\}$,$\{4, 10, 16\}$, $\{4, 11, 14\}$, $\{4, 14, 17\}$, $\{4, 15, 16\}$, $\{5, 6, 12\}$, $\{5, 7, 14\}$, $\{5, 8, 16\}$, $\{5, 9, 13\}$, $\{5, 10, 15\}$,$\{5, 11, 17\}$, $\{5, 16, 17\}$, $\{6, 7, 8\}$, $\{6, 7, 13\}$, $\{6, 8, 15\}$, $\{6, 9, 12\}$, $\{6, 9, 17\}$, $\{6, 10, 14\}$, $\{6, 11, 15\}$, $\{7, 8, 14\}$,$\{7, 9, 11\}$, $\{7, 9, 16\}$, $\{7, 10, 13\}$, $\{7, 12, 16\}$, $\{7, 12, 17\}$, $\{8, 9, 10\}$, $\{8, 9, 15\}$,

$\{8, 10, 17\}$, $\{8, 10, 14\}$,$\{8, 12, 16\}$, $\{8, 13, 17\}$, $\{9, 10, 16\}$, $\{10, 11, 12\}$, $\{10, 11, 17\}$, $\{10, 12, 14\}$, $\{11, 13, 15\}$, $\{12, 13, 14\}$, $\{13, 15, 17\}$, $\{14, 15, 16\}$. The base blocks of the second BIBD$(18, 3, 2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(18, 3, 4)$. ■

Lemma 6.2.8 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(19, 3, 4)$.

Proof From the 19-extended Skolem sequence of order 12, $(11, 9, 7, 5, 3, 1, 1, 3, 5, 7, 9, 11, 12, 10, 8, 6, 4, 2, *, 2, 4, 6, 8, 10, 12)$, take the base blocks of the form $\{0, i, b_i\}$ (mod 19). So, the canonical base blocks of the cyclic BIBD$(19, 3, 4)$ are: $\{0, 1, 7\}$, $\{0, 2, 1\}$, $\{0, 3, 8\}$, $\{0, 4, 2\}$, $\{0, 5, 9\}$, $\{0, 6, 3\}$, $\{0, 7, 10\}$, $\{0, 8, 4\}$, $\{0, 9, 11\}$, $\{0, 10, 5\}$, $\{0, 11, 12\}$, $\{0, 12, 6\}$. Again, we have to show that neither a decomposition into a CSTS(19) and a cyclic BIBD$(19, 3, 3)$, nor into two cyclic BIBD$(19, 3, 2)$ is possible. First, suppose there exists a decomposition into a CSTS(19) and a cyclic BIBD$(19, 3, 3)$, denoted C_1 and C_2 respectively. Then all base blocks with repeated differences must belong to C_2. These cover differences 4 and 2 three times such that the remaining base blocks with differences 4 and 2, that is $\{0, 5, 9\}$ and $\{0, 9, 11\}$, need to occur in C_1. But then the difference 9 occurs twice in C_1, a contradiction. Now, suppose there exists a decomposition into two BIBD$(19, 3, 2)$, denoted C_1 and C_2 respectively. Assume that the base block

$\{0,2,1\}$ providing the repeated difference 1 belongs to C_1. Then, the two base blocks $\{0,1,7\}$, $\{0,11,12\}$ which cover difference 1 as well need to occur in C_2. Both blocks cover difference 7 each. Therefore, the seventh and twelveth base blocks $\{0,7,10\}$ and $\{0,12,6\}$ are forced to be contained in C_1. The last base block provides the repeated difference 6, hence the 6th base block $\{0,6,3\}$ must be in C_2 providing a repeated difference 3. This in turn forces the third base block $\{0,3,8\}$ to occur in C_1. Continuing in this way difference 5 now forces the tenth base block $\{0,10,5\}$ to be in C_2 and the fifth base block to be in C_1. Finally, difference 4 forces the eighth base block $\{0,8,4\}$ to be part of C_2 and the fourth base block $\{0,4,2\}$ to belong to C_1. The latter gives a repeated difference 2 such that, together with the very first base block, this difference occurs three times among the differences provided by base blocks of C_1, a contradiction. Hence, the cyclic BIBD$(19,3,4)$ above is cyclically indecomposable.

The design above is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(19,3,2)$: $\{0,1,2\}$, $\{0,1,7\}$, $\{0,2,10\}$, $\{0,3,6\}$, $\{0,3,8\}$, $\{0,4,8\}$, $\{0,4,14\}$, $\{0,5,9\}$, $\{0,5,16\}$, $\{0,6,18\}$,$\{0,7,13\}$,$\{0,9,16\}$, $\{0,10,15\}$, $\{0,11,12\}$, $\{0,11,14\}$, $\{0,12,13\}$,$\{0,15,17\}$, $\{0,17,18\}$, $\{1,2,3\}$, $\{1,3,18\}$, $\{1,4,9\}$,$\{1,4,17\}$, $\{1,5,15\}$, $\{1,5,16\}$, $\{1,6,10\}$, $\{1,6,17\}$, $\{1,7,13\}$, $\{1,8,11\}$, $\{1,8,14\}$, $\{1,9,18\}$, $\{1,10,12\}$,

$\{1,11,16\}$,$\{1,12,15\}$, $\{1,13,14\}$, $\{2,3,9\}$, $\{2,4,6\}$, $\{2,4,12\}$, $\{2,5,10\}$, $\{2,5,14\}$, $\{2,6,16\}$, $\{2,7,11\}$, $\{2,7,18\}$, $\{2,8,14\}$,$\{2,8,15\}$, $\{2,9,15\}$, $\{2,11,18\}$, $\{2,12,17\}$, $\{2,13,16\}$, $\{2,13,17\}$, $\{3,4,5\}$, $\{3,4,11\}$, $\{3,5,7\}$,$\{3,6,11\}$, $\{3,7,17\}$, $\{3,8,12\}$, $\{3,9,15\}$, $\{3,10,13\}$, $\{3,10,16\}$, $\{3,12,14\}$, $\{3,13,18\}$, $\{3,14,17\}$, $\{3,15,16\}$, $\{4,5,6\}$,$\{4,7,10\}$, $\{4,7,12\}$, $\{4,8,18\}$, $\{4,9,13\}$, $\{4,10,16\}$, $\{4,11,14\}$, $\{4,13,15\}$, $\{4,15,18\}$, $\{4,16,17\}$,$\{5,6,13\}$, $\{5,7,9\}$, $\{5,8,13\}$, $\{5,8,17\}$, $\{5,10,14\}$, $\{5,11,17\}$, $\{5,11,18\}$, $\{5,12,15\}$, $\{5,12,18\}$, $\{6,7,8\}$,$\{6,7,14\}$, $\{6,8,10\}$, $\{6,9,12\}$, $\{6,9,14\}$, $\{6,11,15\}$, $\{6,12,18\}$, $\{6,13,16\}$, $\{6,15,17\}$, $\{7,8,9\}$,$\{7,10,15\}$, $\{7,11,15\}$, $\{7,12,16\}$, $\{7,14,17\}$, $\{7,16,18\}$, $\{8,9,16\}$, $\{8,10,12\}$, $\{8,11,16\}$, $\{8,13,17\}$, $\{8,15,18\}$,$\{9,10,11\}$, $\{9,10,17\}$, $\{9,11,13\}$, $\{9,12,17\}$, $\{9,14,18\}$, $\{10,11,17\}$, $\{10,13,18\}$, $\{10,14,18\}$, $\{11,12,13\}$,$\{12,14,16\}$, $\{13,14,15\}$, $\{14,15,16\}$, $\{16,17,18\}$. The base blocks of the second BIBD$(19,3,2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(19,3,4)$.

■

Lemma 6.2.9 *There exists a cyclically indecomposable but decomposable cyclic BIBD*$(21,3,4)$.

Proof From the 21-extended Skolem sequence of order 13, $(4,2,8,2,4,6,3,12,10,3,8,6,13,11,9,7,1,1,10,12,*,5,7,9,11,13,5)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 21) together with the short orbit $\{0,7,14\}$.

So, the canonical base blocks of the cyclic BIBD$(21,3,4)$ are: $\{0,1,18\}$, $\{0,2,4\}$, $\{0,3,10\}$, $\{0,4,5\}$, $\{0,5,6\}$, $\{0,6,12\}$, $\{0,7,2\}$, $\{0,8,11\}$, $\{0,9,3\}$, $\{0,10,19\}$, $\{0,11,4\}$, $\{0,12,20\}$, $\{0,13,5\}$. The existence of four short base blocks is a necessary condition for the existence of a decomposition into two BIBD$(v,3,2)$ which is clearly not satisfied. Therefore, if the design is cyclically decomposable, it must be into a CSTS(21) and a cyclic BIBD$(21,3,3)$, denoted C_1 and C_2. The short block with difference 7 belongs to C_1. Therefore, all other base blocks with difference 7 as well as all base blocks with repeated differences belong to C_2. These are the second, third, sixth, seventh, eleventh and thirteenth base blocks. Among the differences provided by these blocks, 2 occurs three times. Hence, the tenth base block $\{0,10,19\}$ with difference 2 must occur in C_1 providing differences 9 and 10. This forces the eightth, nineth and twelveth base blocks to be in C_2 with the consequence that difference 8 occurs now four times in C_2, a contradiction.

The design is clearly decomposable since the following blocks which are chosen from the orbits of the base blocks above form a BIBD$(21,3,2)$: $\{0,1,13\}$, $\{0,1,16\}$, $\{0,2,4\}$, $\{0,2,7\}$, $\{0,3,9\}$, $\{0,3,10\}$, $\{0,4,11\}$, $\{0,5,6\}$, $\{0,5,19\}$, $\{0,6,12\}$,$\{0,7,18\}$,$\{0,8,9\}$, $\{0,8,13\}$, $\{0,10,18\}$, $\{0,11,14\}$, $\{0,12,15\}$,$\{0,14,16\}$, $\{0,15,20\}$, $\{0,17,19\}$, $\{0,17,20\}$, $\{1,2,14\}$,$\{1,2,18\}$, $\{1,3,8\}$, $\{1,3,13\}$, $\{1,4,5\}$, $\{1,4,10\}$, $\{1,5,12\}$, $\{1,6,7\}$, $\{1,6,14\}$, $\{1,7,19\}$, $\{1,8,15\}$,

$\{1,9,12\}$,$\{1,9,17\}$, $\{1,10,16\}$, $\{1,11,19\}$, $\{1,11,20\}$, $\{1,15,17\}$, $\{1,18,20\}$, $\{2,3,19\}$, $\{2,3,20\}$, $\{2,4,9\}$, $\{2,5,6\}$, $\{2,5,15\}$,$\{2,6,13\}$, $\{2,7,8\}$, $\{2,8,17\}$, $\{2,9,19\}$, $\{2,10,11\}$, $\{2,10,15\}$, $\{2,11,13\}$, $\{2,12,16\}$, $\{2,12,20\}$,$\{2,14,17\}$, $\{2,16,18\}$, $\{3,4,16\}$, $\{3,4,20\}$, $\{3,5,7\}$, $\{3,5,10\}$, $\{3,6,7\}$, $\{3,6,13\}$, $\{3,8,16\}$, $\{3,9,15\}$, $\{3,11,12\}$,$\{3,11,14\}$, $\{3,12,18\}$, $\{3,14,17\}$, $\{3,15,18\}$, $\{3,17,19\}$, $\{4,5,17\}$, $\{4,6,8\}$, $\{4,6,16\}$, $\{4,7,14\}$,$\{4,7,17\}$, $\{4,8,9\}$, $\{4,10,19\}$, $\{4,11,18\}$, $\{4,12,15\}$, $\{4,12,20\}$, $\{4,13,15\}$, $\{4,13,19\}$, $\{4,14,18\}$, $\{5,7,12\}$,$\{5,8,14\}$, $\{5,8,18\}$, $\{5,9,10\}$, $\{5,9,16\}$, $\{5,11,17\}$, $\{5,11,20\}$, $\{5,13,16\}$, $\{5,13,18\}$, $\{5,14,20\}$,$\{5,15,19\}$, $\{6,8,18\}$, $\{6,9,10\}$, $\{6,9,19\}$, $\{6,10,11\}$, $\{6,11,19\}$, $\{6,12,18\}$, $\{6,14,15\}$, $\{6,15,17\}$, $\{6,16,20\}$,$\{6,17,20\}$, $\{7,8,20\}$, $\{7,9,11\}$, $\{7,9,14\}$, $\{7,10,16\}$, $\{7,10,17\}$, $\{7,11,18\}$, $\{7,12,13\}$, $\{7,13,19\}$,$\{7,15,16\}$, $\{7,15,20\}$, $\{8,10,15\}$, $\{8,10,20\}$, $\{8,11,12\}$, $\{8,11,17\}$, $\{8,12,19\}$, $\{8,13,14\}$, $\{8,16,19\}$, $\{9,11,16\}$, $\{9,12,13\}$, $\{9,13,20\}$, $\{9,14,15\}$, $\{9,17,18\}$, $\{9,18,20\}$, $\{10,12,14\}$, $\{10,12,17\}$, $\{10,13,14\}$, $\{10,13,20\}$, $\{10,18,19\}$, $\{11,13,15\}$, $\{11,15,16\}$, $\{12,14,19\}$, $\{12,16,17\}$, $\{13,16,17\}$, $\{13,17,18\}$, $\{14,16,18\}$, $\{14,19,20\}$, $\{15,18,19\}$, $\{16,19,20\}$. The base blocks of the second BIBD$(21,3,2)$ are just the complement of the blocks above with respect to the blocks in the cyclic BIBD$(21,3,4)$. ■

6.2.2 Cyclically Indecomposable BIBD$(v, 3, 4)$ for infinite values of v

In this subsection, we are going to construct some linear classes of cyclically indecomposable but decomposable BIBD$(v, 3, 4)$. To construct these designs we are using cyclic systems of order nv with cyclic sub-systems of order v and index n and relative difference families $(nv, n, 3, 1)$-DF.

By Theorem 2.1 in [71] there exists a cyclic triple system of order pn containing a cyclic sub-system of order p on the point set $0, n, 2n, \ldots, (p-1)n$.

Construction 6.2.1 *Let $p \equiv 1 \,\text{or}\, 3 \pmod 6, p \neq 9$ and $n \equiv 1 \pmod 6$ and suppose that there exists a cyclically indecomposable but decomposable cyclic BIBD$(p, 3, 4)$. Then there exists a cyclically indecomposable but decomposable cyclic BIBD$(pn, 3, 4)$.*

Proof Let the set $\mathcal{B}$ contain all the blocks of the cyclic sub-system. From the cyclic system of order pn, remove all blocks which are in the orbit of $\mathcal{B}$. Let $\mathcal{B}'$ contain all remaining blocks. Taking each of these blocks four times yields an incomplete cyclic BIBD$(pn, 3, 4)$ with n holes, each hole containing p points. Let $\mathcal{R}$ be the set of canonical base blocks of the given cyclic BIBD$(p, 3, 4)$. Now insert the base block $n \cdot R$ for each $R \in \mathcal{R}$ and develop these blocks mod pn to fill all holes. This gives a cyclic BIBD$(pn, 3, 4)$, say C.

It remains to show that this design C is cyclically indecomposable and to present a decomposition into a non-cyclic STS(pn) and a BIBD($pn, 3, 3$), or a decomposition into two non-cyclic BIBD($pn, 3, 2$).

Suppose first that there exists a decomposition of C into a CSTS(pn) and a cyclic BIBD($pn, 3, 3$), with canonical base block lists $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Let $\mathcal{R}_1'$ contain those base blocks from $\mathcal{R}_1$ which cover a difference of the form $\pm dn$ where $d \in \mathbb{Z}_p \setminus \{0\}$. By construction, each base block from $\mathcal{R}_1'$ is a translate of a base block $n \cdot R$ with $R \in \mathcal{R}$. So define $\mathcal{R}'$ to contain all base blocks $R \in \mathcal{R}$ with $n \cdot R + i \in \mathcal{R}_1'$ for some $i \in \mathbb{Z}_{pn}$. The base blocks in $\mathcal{R}_1'$ cover all differences of the form $\pm dn$ exactly once and, therefore, the base blocks in $\mathcal{R}'$ cover all differences of the form $\pm d$ ($d \in \mathbb{Z}_p \setminus \{0\}$) exactly once. This implies that the base blocks in $\mathcal{R}'$ generate a cyclic STS(p) and the base blocks in $\mathcal{R} \setminus \mathcal{R}'$ generate a cyclic BIBD($p, 3, 3$) which form together a cyclic decomposition of the cyclic BIBD($p, 3, 4$) which is cyclically indecomposable by hypothesis, a contradiction.

Suppose now that there exists a decomposition of C into two cyclic BIBD($pn, 3, 2$), with canonical base block lists $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Following the same argument as before, it is easy to show that such a decomposition is impossible.

Assume that the cyclically indecomposable but decomposable BIBD($p, 3, 4$) has a decomposition into a non-cyclic STS(p) and a BIBD($pn, 3, 3$). A decomposition into a non-cyclic STS(pn) and a BIBD($pn, 3, 3$) can be obtained as follows. Take all blocks $n \cdot B + i$, where B is a block from the STS(p) and $i \in \mathbb{Z}_n$, and adjoin the blocks in $\mathcal{B}'$ (which remained after the first step of the construction). This yields an STS(pn), say S, whose blocks are clearly contained in the block set of C. The blocks of the BIBD($pn, 3, 3$) are just the complement of the blocks of S with respect to the blocks in C.

If the cyclically indecomposable but decomposable BIBD($p, 3, 4$) has a decomposition into two non-cyclic BIBD($p, 3, 2$), follow the same argument as before to get a decomposition of the BIBD($pn, 3, 4$) into two non-cyclic BIBD($pn, 3, 2$). ■

Corollary 6.2.10 *Let $n \equiv 1 \pmod 6$. Then there exists cyclically indecomposable but decomposable cyclic BIBD($v, 3, 4$) for $v = 13n$, $v = 15n$, $v = 19n$ and $v = 21n$.*

Proof There exists a cyclically indecomposable but decomposable cyclic BIBD($v, 3, 4$) for $v = 13$ (Lemma 6.2.4), $v = 15$ (Lemma 6.2.5), $v = 19$ (Lemma 6.2.8) and $v = 21$ (Lemma 6.2.9). Apply Construction 6.2.1 for $p = 13, 15, 19$ and 21. ■

Construction 6.2.2 *Let $p \equiv 3 \pmod 6, p \neq 9$ and $n \equiv 3 \,\text{or}\, 5 \pmod 6, n \neq 3$ and suppose that there exists a cyclically indecomposable but decomposable cyclic BIBD($p, 3, 4$). Then there exists a cyclically indecomposable but decomposable cyclic*

BIBD$(pn, 3, 4)$.

Proof By Theorem 2.1 in [71] there exists a cyclic triple system of order pn containing a cyclic sub-system of order p on the point set $0, n, 2n, \ldots, (p-1)n$. The proof is then similar to Construction 6.2.1. ■

Corollary 6.2.11 *Let* $n \equiv 3 \,\text{or}\, 5 \pmod 6, n \neq 3$. *Then there exists cyclically indecomposable but decomposable cyclic BIBD*$(15n, 3, 4)$ *and cyclic BIBD*$(21n, 3, 4)$.

Proof There exists a cyclically indecomposable but decomposable BIBD$(v, 3, 4)$ for $v = 15$ (Lemma 6.2.5) and for $v = 21$ (Lemma 6.2.9). Apply Construction 6.2.2 for $p = 15$ and $p = 21$. ■

Construction 6.2.3 *Let* $p \equiv 1 \pmod 6, p \geq 7$ *and suppose that there exists a cyclically indecomposable but decomposable cyclic BIBD*$(p, 3, 4)$. *Then there exists a cyclically indecomposable but decomposable cyclic BIBD*$(9p, 3, 4)$.

Proof By Theorem 2.1 in [71] there exists a cyclic STS$(9p)$ containing a cyclic subsystem of order p on the point set $0, 9, 18, \ldots, 9(p-1)$. The proof is then similar to Construction 6.2.1. ■

Corollary 6.2.12 *There exists cyclically indecomposable but decomposable cyclic BIBD*$(117, 3, 4)$ *and cyclic BIBD*$(171, 3, 4)$.

Proof There exists a cyclically indecomposable but decomposable cyclic BIBD$(v, 3, 4)$ for $v = 13$ (Lemma 6.2.4) and $v = 19$ (Lemma 6.2.8). Apply Construction 6.2.3 for $p = 13$ and 19. ■

Construction 6.2.4 *There exists a cyclically indecomposable but decomposable BIBD*$(16v, 3, 4)$ *for every* $v \equiv 1(\text{mod } 3)$.

Proof By Lemma 3.8 in [88] there exists a $(16v, 16, 3, 1)$-DF for every $v \equiv 1(\text{mod } 3)$. By Lemma 6.2.6 there exists a cyclically indecomposable but decomposable BIBD$(16, 3, 4)$. The rest of the proof is similar to the proof of Construction 6.2.1. ■

Remark 6.2.1 *Construction 6.2.1 can be generalized.*

Theorem 6.2.13 *If there exists a* $(pn, p, k, 1)$*-DF and a cyclically indecomposable but decomposable BIBD*(p, k, λ)*, then there exists a cyclically indecomposable but decomposable BIBD*(pn, k, λ).

Theorem 6.2.14 *If there exists a* $(gv, \{g, 3\}, k, \lambda)$*-DF and a cyclically indecomposable but decomposable BIBD*(g, k, λ)*, then there exists a cyclically indecomposable but decomposable BIBD*(gv, k, λ).

We now summarize the results of this section. There exists cyclically indecomposable but decomposable BIBD$(v, 3, 4)$ for $v \equiv 1 (\text{mod } 3)$ if $v = 10, 13, 16, 19, 78k + 13, 114k + 19, 48k + 16$ where k is a positive integer. There exists cyclically indecomposable but decomposable BIBD$(v, 3, 4)$ for $v \equiv 0 (\text{mod } 3)$ if $v = 9, 12, 15, 18, 21, 117, 171, 90k + 75, 90k' + 45, 126k' + 63, 126k + 105, 90k + 15, 126k + 21$, where k and k' are integers, $k \geq 0, k' \geq 1$.

Chapter 7

Applications to Optical Orthogonal Codes

In this chapter, we want to discuss some attempts to use direct constructions of cyclic designs from Skolem-type sequences to obtain optical orthogonal codes with $\lambda = 1$ and $\lambda = 2$.

Cyclic designs with special properties (simple or super-simple) are equivalent to optimal optical orthogonal codes. The study of optical orthogonal codes (OOC for short) was motivated by an application in a fibre-optic code-division multiple access channel. Many users wish to transmit information over a common wide-band optical channel. The objective is to design a system that allows the users to share the common channel. For the construction of optimal optical orthogonal codes, cyclic block designs play an important role since a cyclic block design is equivalent to an optimal optical orthogonal code. As an example, $C = \{1100100000000, 1010000100000\}$ is a $(13, 3, 1)$ code with two codewords. In set theoretic notation $C = \{\{0, 1, 4\}, \{0, 2, 7\}\}$ (mod 13), which gives a cyclic Steiner triple system of order 13. A survey about cyclic designs and their applications to optimal optical orthogonal codes is given in [11].

Determining the parameters v, k and λ for which an optimal (v, k, λ)-OOC exists is a difficult task. For $k = 3$, optimal optical orthogonal codes always exist except when $v = 6n + 2$ and $n \equiv 2$ or 3 (mod 4) [39].

7.1 Construction of $(v, 3, 1)$-OOC

A direct construction for optimal orthogonal codes for every order v except when $v = 6n + 2$ and $n \equiv 2$ or 3 (mod 4) was given by Skolem and O'Keefe [69, 84] using Skolem-type sequences.

Brickell and Wei [18] gave a direct construction for near optimal optical orthogonal codes (missing one codeword) with weight 3 using a shell-like structure. A shell S_j is a set of codewords of the form $S_j(i) = \{0, a_j m + d_j + i, b_j m + e_j + 2i\}$ for $L_j \leq i \leq U_j$, where $U_j - L_j = m$. Brickell and Wei's shell structures can be written as a Skolem sequence with two hooks for $v = 6n + x$, $0 \le x \le 5$, n even, and it can be written as a Skolem sequence with one hook for $v = 6n + x$, $0 \le x \le 5$, n odd. When n is even, the Skolem-type sequence is $n-2, n-4, \ldots, 2, *, 2, 4, \ldots, n-2, *, n-1, n-3, \ldots, 1, 1, 3, \ldots, n-1$. For n odd, the Skolem-type sequence is $n-1, n-3, \ldots, 2, *, 2, 4, \ldots, n-3, n-1, n-2, n-4, \ldots, 1, 1, 3, \ldots, n-4, n-2$. In both cases, the pairs (a_i, b_i), $1 \le i \le n-1$ given by the Skolem-type sequence give the codewords $\{0, a_i + n, b_i + n\}$, $1 \le i \le n-1$ for a near optimal optical orthogonal code of order v, except when $v = 6n+2$ and $n \equiv 2$ or $3 \pmod 4$. Then, Brickell and Wei refined their construction to obtain perfect codes for every $v \equiv 1 \pmod 6$. Their construction for perfect codes is the same as that found by Skolem and O'Keefe [69, 84].

7.2 Construction of $(v, 4, 1)$-OOC

7.2.1 Shell construction

Brickell and Wei [18] attempted to extend their use of shells to obtain optimal optical orthogonal codes with weight 4 and $\lambda = 1$. They needed to construct six shells for this case and they had to enlist the help of a computer. The problem proved more difficult than they had expected and they were fully successful only in the case when $v = 72n + 1$.

For this value of v, they constructed optical orthogonal codes with two codewords less than optimal. In fact, values of v of this form satisfy the congruence $v \equiv 1(\mathrm{mod}\ k(k-1))$ and so cyclic designs will provide perfect optical orthogonal codes for those values of v for which they exist. However, Brickell and Wei pointed out that, with the help of an even longer computer search, their method should also be succsessful for other values of v of the form $72n + h$, for $h \neq 1$.

They looked for shells of the form $\{0, a_j k + d_j + i, b_j k + e_j + 2i, c_j k + f_j + 3i\}$ with $L_j \leq i \leq U_j$ and $1 \leq j \leq 6$, and where $0 \leq a_j, b_j, c_j \leq 71, d_j = 0, 1, e_j = 0, 1, 2, f_j = 0, 1, 2, 3, L_j = 0, 1$, and $U_j = n-2, n-1$, such that no repeated differences exist.

After an extensive computer search, they found four sets of values for the parameters a_j, b_j, c_j, which would give a complete shell structure of six shells. A further computer search was used to find optimal values for the parameters d_j, e_j, f_j, L_j and U_j.

As an example, the following $(145, 4, 1)$-OOC is obtained using their method: $\{0, 31, 13, 1\}, \{0, 32, 15, 4\}, \{0, 77, 44, 3\}, \{0, 78, 58, 2\}, \{0, 79, 60, 5\}, \{0, 112, 14, 37\}$, $\{0, 119, 27, 35\}, \{0, 120, 29, 38\}, \{0, 95, 43, 36\}, \{0, 96, 45, 39\}$.

Brickell and Wei pointed out that there might be a different choice of the parameters that could be extended to a cyclic design.

7.2.2 Examples from Skolem-type sequences

It is known that super-simple cyclic designs are equivalent to optimal optical orthogonal codes. All the cyclic BIBD$(v, 4, 1)$ constructed in this thesis are super-simple and thus they give $(v, 4, 1)$ optimal optical orthogonal codes. See Chapter 4 and Appendix A for examples of super-simple cyclic BIBD$(v, 4, 1)$.

7.3 Construction of $(v, 3, 2)$-OOC

It is well known that simple cyclic BIBD$(v, 3, \lambda)$ are equivalent to $(v, 3, 2)$ optical orthogonal codes. So, the cyclic, simple, and indecomposable BIBD$(v, 3, 3)$ we found in Chapter 3 give $(v, 3, 2)$ optical orthogonal codes for every v except for $v = 24c+57$, $c \geq 2$. Note that these codes are not optimal as they are missing many codewords.

A direct construction for $(v, 3, 2)$ optimal optical orthogonal codes for each $v \not\equiv 0 \pmod 3$, $v \geq 4$ was given by Chen and Wei [38].

7.4 Construction of $(v, 4, 2)$-OOC

Many of the cyclic BIBD$(v, 4, \lambda)$ for $\lambda > 1$ constructed in this thesis are super-simple and so they give rise to $(v, 4, 2)$ optical orthogonal codes.

Below we list just a few examples of such designs.

Example 7.4.1 *Cyclic BIBD*$(25,4,2)$*: 2-fold Skolem array* $S_2 = \{(1,2),(3,4),(1,3),(2,4)\} \Rightarrow \{0,1,6,10\},\ \{0,1,8,12\},\ \{0,2,8,11\},\ \{0,2,5,12\}$

Example 7.4.2 *Cyclic BIBD*$(37,4,2)$*: 2-fold Skolem array* $S_3 = \{(1,2),(5,6),(2,4),(3,5),(1,4),(3,6)\} \Rightarrow \{0,1,5,14\},\ \{0,1,12,18\},\ \{0,2,12,17\},\ \{0,2,10,16\},\ \{0,3,7,16\},\ \{0,3,11,18\}$.

Example 7.4.3 *Cyclic BIBD*$(49,4,2)$*: 2-fold Skolem array* $S_4 = \{(1,2),(6,7),(3,5),(6,8),(1,4),(2,5),(3,7),(4,8)\} \Rightarrow \{0,1,6,18\},\ \{0,1,16,23\},\ \{0,2,15,21\},\ \{0,2,16,24\},\ \{0,3,8,20\},\ \{0,3,10,21\}, \{0,4,14,23\},\ \{0,4,13,24\}$.

Example 7.4.4 *Cyclic BIBD*$(17,4,3)$*: Skolem sequence of order* 4 $\{(1,2),(4,6),(5,8),(3,7)\} \Rightarrow \{0,1,11,2\},\ \{0,2,13,6\},\ \{0,3,15,8\},\ \{0,4,16,7\}$.

Chapter 8

Conclusions and Future Research

8.1 Conclusions

The problem of constructing cyclic BIBD$(v, 4, 1)$ by Peltesohn's method has been open for over 35 years. We introduced in this thesis a Skolem partitioning problem which seems significantly easier. We showed that there exists an example of Skolem partitions that induces a cyclic BIBD$(v, 4, \lambda)$ for every admissible class in Table 4.1.

It was known that Skolem sequences can be used to construct cyclic BIBD$(v, 3, \lambda)$ for $\lambda = 1, 2$. We showed in this thesis that Skolem sequences can be used to construct cyclic BIBD$(v, 3, \lambda)$ for all admissible orders v and λ. Moreover, we showed that Skolem sequences can be used to construct cyclic BIBD(v, k, λ) for $k \geq 4$.

Furthermore, we used Skolem-type sequences to construct cyclic designs with the following properties: simple, indecomposable, and cyclically indecomposable but decomposable.

Simple designs with $k = 3$ and super-simple designs with $k = 4$ are equivalent to optical othogonal codes. All the cyclic BIBD$(v, 4, 1)$ constructed in this thesis are super-simple, and thus they give $(v, 4, 1)$ optimal optical orthogonal codes. We believe that many of the cyclic BIBD$(v, 4, \lambda)$ constructed in this thesis are super-simple and thus many applications of these designs may be found in the future.

8.2 Future Research

We will outline below some open questions related to our work.

8.2.1 New Skolem-type sequences

In Chapter 2, we introduced new Skolem-type sequences. We showed that there exists m-near Skolem sequences of order $2m - 1$ with three hooks in positions m, $2m$, and $3m$ for every $m \equiv 2 \pmod 4$, $m \geq 6$.

Problem 8.2.1 *Can m-near Skolem sequences of order $2m - 1$ with three hooks in positions m, $2m$, and $3m$ be constructed for every m?*

In Chapter 3, we used the m-near Skolem sequences of order $2m-1$ with three hooks in positions m, $2m$, and $3m$ to construct cyclic BIBD$(12n+2, 3, 12)$.

Problem 8.2.2 *Are there any other applications of these sequences?*

In Chapter 2, we also introduced m-fold Skolem-type arrays. The obvious applications of these sequences is that they give cyclic BIBD$(v, 3, \lambda)$ with $\lambda = m$.

Problem 8.2.3 *Are there any other applications of these Skolem-type arrays?*

8.2.2 Cyclic BIBD$(v, 3, \lambda)$

In Chapter 3, we constructed cyclic block designs with block size 3 for all admissible λ using Skolem-type sequences. One important thing about our constructions is that they provide lots of solutions. For example, to obtain a cyclic BIBD$(2 \times 4 + 1, 3, 3)$, we use a Skolem sequence of order 4. But there are six Skolem sequences of order 4 and that means we can get six different cyclic BIBD$(9, 3, 3)$. As the order of our Skolem sequences gets larger, the number of Skolem sequences that we obtain grows exponentially. For example, there are 504 Skolem sequences of order 8 and 2656 Skolem-sequences of order 9. An obvious questions is:

Problem 8.2.4 *Are all the designs of a certain order v isomorphic?*

Problem 8.2.5 *How many of the cyclic designs constructed in Chapter 3 are simple? How many are indecomposable? What other properties do these designs have?*

Problem 8.2.6 *Construct the cyclic packings and the cyclic coverings for BIBD*$(v, 3, \lambda)$.

8.2.3 Cyclic BIBD$(v, 4, \lambda)$

In Chapter 4 we outlined the necessary conditions for the existence of a cyclic BIBD$(v, 4, \lambda)$.

Problem 8.2.7 *Are the necessary conditions also sufficient for the existence of cyclic BIBD*$(v, 4, \lambda)$*?*

In her doctoral thesis, Marlene Colbourn [41] tried to construct cyclic block designs with block size four using Peltesohn's proof technique. We used Skolem-type sequences to reduce the partitioning problem in half.

Problem 8.2.8 *Is there a partition of the numbers* $\{1, \ldots, 6n\}$ *into* n *six-subsets* $\{a, b, c, a+b, b+c, a+b+c\}$*?*

Using Skolem-type sequences we reduced the problem but we are still far from solving it.

Problem 8.2.9 *Is there a partition of the numbers* $\{n+1,\ldots,4n\}$ *into triples* (x_i, y_i, z_i) *such that* $x_i = y_i - i$, $z_i = b_i + 4n - y_i, \forall 1 \leq i \leq n$*?*

Subtracting n from each of the numbers above, the problem is equivalent to:

Problem 8.2.10 *Is there a partition of the numbers* $\{1,\ldots,3n\}$ *into triples* (x_i, y_i, z_i) *such that* $x_i = y_i - i$, $z_i = b_i + 4n - y_i, \forall 1 \leq i \leq n$*?*

In Chapter 4, we used also Skolem-type sequences to construct cyclic block designs with block size 4 and $\lambda = 6$.

Problem 8.2.11 *Can Skolem-type sequences be used to construct cyclic block designs with block size* 4*, for all admissible* v *and* λ*?*

In Chapter 5, we used relative difference families to get many new infinite classes of cyclic block design.

Problem 8.2.12 *Can relative difference families be used to construct cyclic block designs with block size* 4*, for all admissible* v *and* λ*?*

Problem 8.2.13 *Construct the cyclic packings and the cyclic coverings for* *BIBD*$(v, 4, \lambda)$*.*

8.2.4 Cyclically Indecomposable BIBD$(v, 3, \lambda)$

In Chapter 6, we showed that there exists cyclic, simple, and indecomposable BIBD$(v, 3, 3)$ for all admissible $v \geq 15$ and a possible exception for $v = 9$. The six cyclic BIBD$(9, 3, 3)$ obtained from the six Skolem sequences of order 4 are all decomposable.

Problem 8.2.14 *Are there cyclic, simple and indecomposable BIBD$(9, 3, 3)$?*

Our efforts to solve the problem for $\lambda = 4$ were not successful. We have no example of a cyclic, simple, and indecomposable BIBD$(v, 3, 4)$.

Problem 8.2.15 *Are there cyclic, simple, and indecomposable BIBD$(v, 3, \lambda)$, for $\lambda \geq 4$?*

In Chapter 6, we also found cyclically indecomposable but decomposable designs for $\lambda = 4$.

Problem 8.2.16 *Are there cyclically indecomposable but decomposable BIBD$(v, 3, \lambda)$ for $\lambda \geq 5$?*

8.2.5 Optical Orthogonal Codes

It is known that perfect optical orthogonal codes are equivalent to cyclic designs. Brickel and Wei [18] developed shell-like structures for $k = 3$ that give optimal orthogonal codes for every v except when $v = 6n + 2$ and $n \equiv 2$ or 3 (mod 4). They also developed shell-like structures for $k = 4$ that give optimal orthogonal codes for $v = 72n + 1$.

Problem 8.2.17 *Can the shell-like structure for $k = 4$, developed by Brickel and Wei, be refined to give perfect optical orthogonal codes?*

Problem 8.2.18 *Are there shell-structures, for $k \geq 5$?*

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Appendix A

Examples of Cyclic Designs from Skolem-type Sequences

Here are the cyclic designs we found using Skolem-type sequences.

For a cyclic BIBD$(49, 4, 1)$ we found three designs. Here are the base blocks of these cyclic designs:

1. $S_4 = \{(1,2),(4,6),(5,8),(3,7)\} \Rightarrow$

   $\{0,1,12,18\},\ \{0,2,7,22\},\ \{0,3,16,24\},\ \{0,4,14,23\}$.

2. $S_4 = \{(6,7),(1,3),(2,5),(4,8)\} \Rightarrow$

   $\{0,1,7,23\},\ \{0,2,14,19\},\ \{0,3,13,21\},\ \{0,4,15,24\}$.

3. $S_4 = \{(1,2),(4,6),(5,8),(3,7)\} \Rightarrow$

$\{0,1,12,18\},\ \{0,2,15,22\},\ \{0,3,8,24\},\ \{0,4,14,23\}.$

For a cyclic BIBD$(61,4,1)$, we found three cyclic designs.

1. $S_5 = \{(2,3),(6,8),(7,10),(1,5),(4,9)\} \Rightarrow$

   $\{0,1,10,23\},\ \{0,2,20,28\},\ \{0,3,14,30\},\ \{0,4,19,25\},\ \{0,5,12,29\}.$

2. $S_5 = \{(2,3),(6,8),(7,10),(1,5),(4,9)\} \Rightarrow$

   $\{0,1,14,23\},\ \{0,2,20,28\},\ \{0,3,19,30\},\ \{0,4,10,25\},\ \{0,5,12,29\}.$

3. $S_5 = \{(8,9),(3,5),(1,4),(6,10),(2,7)\} \Rightarrow$

   $\{0,1,20,29\},\ \{0,2,15,25\},\ \{0,3,17,24\},\ \{0,4,12,30\},\ \{0,5,11,27\}.$

For a cyclic BIBD$(64,4,1)$ we found two cyclic designs. The first one was given in Chapter 4, and the second one is given below.

1. $S_5 = \{(9,10),(5,7),(1,4),(2,6),(3,8)\} \Rightarrow$

   $\{0,1,13,31\},\ \{0,2,21,28\},\ \{0,3,11,25\},\ \{0,4,10,27\},\ \{0,5,20,29\},$

   $\{0,16,32,48\}.$

For a cyclic BIBD$(73,4,1)$ we found seven cyclic designs.

1. $S_6 = \{(10,11),(5,7),(1,4),(2,6),(8,13),(3,9)\} \Rightarrow$

   $\{0,1,24,35\}$, $\{0,2,10,31\}$, $\{0,3,19,28\}$, $\{0,4,18,30\}$, $\{0,5,22,37\}$, $\{0,6,13,33\}$.

2. $S_6 = \{(10,11),(2,4),(6,9),(1,5),(3,8),(7,13)\} \Rightarrow$

   $\{0,1,24,35\}$, $\{0,2,21,28\}$, $\{0,3,13,33\}$, $\{0,4,12,29\}$, $\{0,5,14,32\}$, $\{0,6,22,37\}$.

3. $S_6 = \{(10,11),(2,4),(3,6),(5,9),(8,13),(1,7)\} \Rightarrow$

   $\{0,1,24,35\}$, $\{0,2,21,28\}$, $\{0,3,12,30\}$, $\{0,4,20,33\}$, $\{0,5,15,37\}$, $\{0,6,14,31\}$.

4. $S_6 = \{(2,3),(11,13),(5,8),(6,10),(4,9),(1,7)\} \Rightarrow$

   $\{0,1,17,27\}$, $\{0,2,22,37\}$, $\{0,3,24,32\}$, $\{0,4,11,34\}$, $\{0,5,14,33\}$, $\{0,6,18,31\}$.

5. $S_6 = \{(2,3),(8,10),(4,7),(9,13),(1,6),(5,11)\} \Rightarrow$

   $\{0,1,16,27\}$, $\{0,2,24,34\}$, $\{0,3,12,31\}$, $\{0,4,17,37\}$, $\{0,5,23,30\}$, $\{0,6,14,35\}$.

6. $S_6 = \{(1,2),(9,11),(3,6),(4,8),(5,10),(7,13)\} \Rightarrow$

   $\{0,1,9,26\}$, $\{0,2,23,35\}$, $\{0,3,19,30\}$, $\{0,4,22,32\}$, $\{0,5,20,34\}$, $\{0,6,13,37\}$.

7. $S_6 = \{(10,11),(4,6),(2,5),(9,13),(3,8),(1,7)\} \Rightarrow$

   $\{0,1,11,35\}$, $\{0,2,17,30\}$, $\{0,3,21,29\}$, $\{0,4,23,37\}$, $\{0,5,12,32\}$, $\{0,6,22,31\}$.

For a cyclic BIBD$(76,4,1)$ we found fifteen cyclic BIBD$(76,4,1)$. The base blocks of some of these cyclic BIBD$(76,4,1)$ are:

1. $S_6 = \{(6,7),(1,3),(10,13),(8,12),(4,9),(5,11)\} \Rightarrow$

   $\{0,1,8,31\}$, $\{0,2,12,27\}$, $\{0,3,24,37\}$, $\{0,4,22,36\}$, $\{0,5,16,33\}$, $\{0,6,26,35\}$.

2. $S_6 = \{(3,4),(11,13),(6,9),(8,12),(5,10),(1,7)\} \Rightarrow$

   $\{0,1,10,28\}$, $\{0,2,22,37\}$, $\{0,3,26,33\}$, $\{0,4,12,36\}$, $\{0,5,21,34\}$, $\{0,6,17,31\}$.

For a cyclic BIBD$(88,4,1)$ we found many cyclic BIBD$(88,4,1)$. The base blocks of some of these cyclic BIBD$(88,4,1)$ are:

1. $S_7 = \{(4,5),(9,11),(12,15),(10,14),(3,8),(1,7),(6,13)\} \Rightarrow$

   $\{0,1,9,33\}$, $\{0,2,18,39\}$, $\{0,3,30,43\}$, $\{0,4,14,42\}$, $\{0,5,25,36\}$, $\{0,6,23,35\}$, $\{0,7,26,41\}$.

2. $S_7 = \{(4,5),(9,11),(12,15),(10,14),(3,8),(1,7),(6,13)\} \Rightarrow$

   $\{0,1,9,33\}$, $\{0,2,23,39\}$, $\{0,3,30,43\}$, $\{0,4,14,42\}$, $\{0,5,25,36\}$, $\{0,6,18,35\}$, $\{0,7,26,41\}$.

For a cyclic BIBD$(97,4,1)$ we found two cyclic block designs.

1. $S_8 = \{(1,2),(13,15),(5,8),(6,10),(11,16),(3,9),(7,14),(4,12)\} \Rightarrow$

   $\{0,1,13,34\}$, $\{0,2,32,47\}$, $\{0,3,26,40\}$, $\{0,4,20,42\}$, $\{0,5,29,48\}$,

   $\{0,6,31,41\}$, $\{0,7,18,46\}$, $\{0,8,17,44\}$.

2. $S_8 = \{(1,2),(13,15),(4,7),(10,14),(6,11),(3,9),(5,12),(8,16)\} \Rightarrow$

   $\{0,1,13,34\}$, $\{0,2,31,47\}$, $\{0,3,14,39\}$, $\{0,4,27,46\}$, $\{0,5,15,43\}$,

   $\{0,6,32,41\}$, $\{0,7,24,44\}$, $\{0,8,30,48\}$.

For a cyclic BIBD$(13,4,2)$ we found one cyclic design. The base blocks are:

1. 2-fold Skolem array $S_1 = \{(1,2),(1,2)\} \Rightarrow$

   $\{0,1,4,6\}$, $\{0,1,4,6\}$

For a cyclic BIBD$(25,4,2)$ we found two cyclic design. The base blocks are:

1. 2-fold Skolem array $S_2 = \{(1,2),(3,4),(1,3),(2,4)\} \Rightarrow$

   $\{0,1,6,10\}$, $\{0,1,8,12\}$, $\{0,2,8,11\}$, $\{0,2,5,12\}$.

2. 2-fold Skolem array $S_2 = \{(1,2),(3,4),(1,3),(2,4)\} \Rightarrow$

   $\{0,1,4,10\}$, $\{0,1,8,12\}$, $\{0,2,8,11\}$, $\{0,2,7,12\}$.

For a cyclic BIBD$(37,4,2)$ we found four cyclic design. The base blocks are:

1. 2-fold Skolem array $S_3 = \{(1,2),(5,6),(2,4),(3,5),(1,4),(3,6)\} \Rightarrow$

   $\{0,1,5,14\}$, $\{0,1,12,18\}$, $\{0,2,12,17\}$, $\{0,2,10,16\}$, $\{0,3,7,16\}$, $\{0,3,11,18\}$.

2. 2-fold Skolem array $S_3 = \{(1,2),(5,6),(2,4),(3,5),(1,4),(3,6)\} \Rightarrow$

   $\{0,1,5,14\}$, $\{0,1,12,18\}$, $\{0,2,7,17\}$, $\{0,2,10,16\}$, $\{0,3,12,16\}$, $\{0,3,11,18\}$.

3. 2-fold Skolem array $S_3 = \{(1,2),(5,6),(2,4),(3,5),(1,4),(3,6)\} \Rightarrow$

   $\{0,1,5,14\}$, $\{0,1,7,18\}$, $\{0,2,12,17\}$, $\{0,2,10,16\}$, $\{0,3,12,16\}$, $\{0,3,11,18\}$.

4. 2-fold Skolem array $S_3 = \{(1,2),(5,6),(2,4),(3,5),(1,4),(3,6)\} \Rightarrow$

   $\{0,1,5,14\}$, $\{0,1,7,18\}$, $\{0,2,10,17\}$, $\{0,2,12,16\}$, $\{0,3,11,16\}$, $\{0,3,12,18\}$.

For a cyclic BIBD$(49,4,2)$ we found many cyclic design. The base blocks of some of them are:

1. 2-fold Skolem array $S_4 = \{(1,2),(6,7),(3,5),(6,8),(1,4),(2,5),(3,7),(4,8)\} \Rightarrow$

   $\{0,1,6,18\}$, $\{0,1,16,23\}$, $\{0,2,15,21\}$, $\{0,2,16,24\}$, $\{0,3,8,20\}$, $\{0,3,10,21\}$, $\{0,4,14,23\}$, $\{0,4,13,24\}$.

2. 2-fold Skolem array $S_4 = \{(1,2),(7,8),(3,5),(5,7),(1,4),(3,6),(2,6),(4,8)\} \Rightarrow$

   $\{0,1,6,18\}$, $\{0,1,15,24\}$, $\{0,2,15,21\}$, $\{0,2,16,23\}$, $\{0,3,10,20\}$, $\{0,3,11,22\}$, $\{0,4,16,22\}$, $\{0,4,9,24\}$.

Appendix B

Examples of Cyclic BIBD$(v, 4, \lambda)$ for some values of v

We give a few cyclic BIBD$(v, 4, \lambda)$ for some small values of v. The short orbits are shown *slanted*. We obtained the following cyclic designs using a hill-climbing algorithm. The algorithm removes the differences that appear in the short orbit and hill climbs on the remaining differences.

BIBD$(7, 4, 2)$: $\{0, 1, 2, 4\}$;

BIBD$(22, 4, 2)$: $\{0, 4, 16, 17\}, \{0, 12, 14, 21\}, \{0, 14, 16, 19\}, \{\mathit{0}, \mathit{4}, \mathit{11}, \mathit{15}\}$;

BIBD$(20, 4, 3)$: $\{0, 4, 11, 12\}, \{0, 3, 7, 9\}, \{0, 6, 7, 18\}, \{0, 2, 3, 6\}, \{\mathit{0}, \mathit{5}, \mathit{10}, \mathit{15}\}, \{\mathit{0}, \mathit{5}, \mathit{10}, \mathit{15}\}, \{\mathit{0}, \mathit{5}, \mathit{10}, \mathit{15}\}$;

BIBD(9, 4, 3): $\{0, 1, 3, 5\}, \{0, 1, 3, 4\}$;

BIBD(6, 4, 6): $\{0, 1, 2, 3\}, \{0, 2, 3, 4\}, \{\mathit{0}, \mathit{1}, \mathit{3}, \mathit{4}\}$;

BIBD(8, 4, 6): $\{0, 1, 4, 5\}, \{0, 1, 4, 5\}, \{0, 1, 6, 7\}, \{0, 2, 5, 7\}, \{\mathit{0}, \mathit{2}, \mathit{4}, \mathit{6}\}, \{\mathit{0}, \mathit{2}, \mathit{4}, \mathit{6}\}$;

BIBD(11, 4, 6): $\{0, 1, 2, 4\}, \{0, 2, 4, 8\}, \{0, 4, 5, 8\}, \{0, 5, 8, 10\}, \{0, 5, 9, 10\}$;

BIBD(14, 4, 6): $\{0, 4, 5, 10\}, \{0, 2, 4, 7\}, \{0, 3, 4, 5\}, \{0, 3, 6, 8\}, \{0, 8, 10, 11\}, \{0, 2, 6, 7\}, \{\mathit{0}, \mathit{1}, \mathit{7}, \mathit{8}\}$;

BIBD(15, 4, 6): $\{0, 1, 2, 3\}, \{0, 2, 4, 6\}, \{0, 4, 8, 12\}, \{0, 1, 8, 9\}, \{0, 6, 9, 12\}, \{0, 1, 5, 10\}, \{0, 2, 5, 10\}$;

BIBD(8, 4, 15): $\{0, 1, 4, 6\}, \{0, 1, 4, 6\}, \{0, 1, 2, 6\}, \{0, 1, 2, 5\}, \{0, 3, 4, 5\}, \{0, 2, 3, 5\}, \{0, 1, 2, 7\}, \{0, 1, 2, 4\}, \{\mathit{0}, \mathit{1}, \mathit{4}, \mathit{5}\}, \{\mathit{0}, \mathit{2}, \mathit{4}, \mathit{6}\}$;