The classification of $[p]$-nilpotent restricted Lie algebras of dimension 5

by

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Abstract

In this thesis, we will classify p-nilpotent restricted Lie algebras of dimension 5. Any finite dimensional p-nilpotent restricted Lie algebras is nilpotent by Engel's theorem. Therefore, we use as our starting point, the classification of nilpotent Lie algebras of dimension 5 and classify the possible equivalence classes of p-maps on these Lie algebras. We first explain the method that we used to classify p-nilpotent restricted Lie algebras of dimension 5 which is the analogue of Skjelbred-Sund method for classifying nilpotent Lie algebras. Then, we will give a complete classification of p-nilpotent restricted Lie algebras of dimension 5 over perfect fields of characteristic $p \geqslant 5$.

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Contents

Chapter 1

Introduction

Let L be a Lie algebra over a field $\mathbb{F}$ of positive characteristic p. Recall that L is called *restricted* if L affords a p-map that satisfies the following conditions for all $x, y \in L$ and $\lambda \in \mathbb{F}$

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $(\mathsf{ad}\, x)^p = \mathsf{ad}\,(x^{[p]})$;
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{j=1}^{p-1} s_j(x, y)$,
   where $js_j(x, y)$ is the coefficient of t^{j-1} in $(\mathsf{ad}\,(tx + y))^{p-1}(x)$, t an indeterminate.

Recall that a restricted Lie algebra L is called p-nilpotent, if there exists an integer n such that $x^{[p]^n} = 0$, for all $x \in L$. The purpose of this thesis is to classify all p-nilpotent restricted Lie algebras of dimension 5 over perfect fields of characteristic $p \geqslant 5$. It follows from the Engel Theorem that if L is finite-dimensional and p-nilpotent then L is nilpotent. Our work builds upon the recent work of Schneider and Usefi [12] on the classification of p-nilpotent restricted Lie algebras of dimension up to 4 over perfect fields of characteristic p. Our method is different than what is used in [12] as we describe below. The analogous classification for small dimensional nilpotent Lie algebras has a long history. The classification of all nilpotent Lie algebras of dimension up to five over any field has been known for a long time. However, in dimension 6, the characterization depends on the underlying field. In 1958 Morozov [8] gave a classification of nilpotent Lie algebras of dimension 6 over a field of characteristic zero, see also [1, 7, 9, 11] for a classification over other fields. These classifications, however, differ and it was not easy to compare them until recently that de Graaf [4] gave a complete classification over any field of characteristic other than 2. de Graaf's approach can be verified computationally and was later revised and extended to characteristic 2 in [2]. The classification in dimensions more than 6 is still in progress, see for example [13, 10].

We now describe the method used in [12] and explain why this method is not applicable in dimension 5. Note that in order to define a p-map on L, it is enough to define it on a basis of L and then extend it to linear combinations using properties (1) and (3). Let $\varphi_1,\ \varphi_2 : L \to L$ be two p-maps on L. Then the restricted Lie algebras (L, φ_1) and (L, φ_2) are isomorphic if and only if there exists $A \in \mathrm{Aut}(L)$ such that

$$A(\varphi_1(x)) = \varphi_2(A(x)) \quad \text{holds for all} \quad x \in L.$$

Hence, φ_1 and φ_2 define isomorphic restricted Lie algebras if and only if there exists $A \in \mathrm{Aut}(L)$ such that $A\varphi_1 A^{-1} = \varphi_2$; that is, they are conjugate under the automorphism group of L. In this case we say that the p-maps φ_1 and φ_2 are *equivalent*. This defines a left action of $\mathrm{Aut}(L)$ on the set of p-maps and the isomorphism classes of restricted Lie algebras correspond to the $\mathrm{Aut}(L)$-orbits under this action. The main task using this approach would be then to find the $\mathrm{Aut}(L)$-orbits. This is exactly what the authors did in [12] to determine all p-nilpotent restricted Lie algebras of dimension up to 4 over perfect fields. However, this task becomes computationally infeasible to carry out in dimension 5.

The method we use to classify p-nilpotent restricted Lie algebras of dimension 5 is the analogue of Skjelbred-Sund method for classifying nilpotent Lie algebras. We describe this method below.

Let L be a Lie algebra over $\mathbb{F}$ and M a vector space, a q-dimensional cochain of L with coefficients in M is a skew-symmetric, q-linear map on L taking values in M. We denote the space of q-dimensional cochains of a Lie algebra L with coefficients in M by $C^q_{\text{cl}}(L, M)$. The coboundary map $\delta^q : C^q_{\text{cl}}(L, M) \to C^{q+1}_{\text{cl}}(L, M)$ is defined by

$$(\delta^q\phi)(l_1, \ldots, l_{q+1}) \;=\; \sum_{1\leq s<t\leq q+1} (-1)^{s+t-1}\phi([l_s, l_t], l_1, \ldots, \widehat{l_s}, \ldots, \widehat{l_t}, \ldots, l_{q+1}),$$

where the symbol $\widehat{l_s}$ indicates that this term is to be omitted. Now, let L be a restricted Lie algebra and M a vector space. We view M as a trivial L-module. Let ϕ be a skew-symmetric, 2-linear map on L taking values in M and $\omega : L \to M$ a map. We say ω has $\star$-property with respect to ϕ, if for every $x, y \in L$ and $\lambda \in \mathbb{F}$, we have

1. $\omega(\lambda x) = \lambda^p\omega(x)$

2. $\omega(x + y) = \omega(x) + \omega(y) + \sum_{\substack{x_j=x \text{ or } y\\ x_1=x, x_2=y}} \frac{1}{\#x}\phi([x_1, x_2, \ldots, x_{p-1}], x_p)$, where $\#x$ is the number of x.

We define the space of 2-dimensional cochains of a restricted Lie algebra L with coefficients in M as the subspace spanned by all such (ϕ, ω). We denote this vector space by $C^2(L, M)$. Let $\psi : L \to M$ a map. It can be verified that $\tilde{\psi} : L \to M$ defined by $\tilde{\psi}(x) = \psi(x^{[p]})$ has the $\star$-property with to $\delta^1\psi$. We define

$$Z^2(L, M) = \{(\phi, \omega) \in C^2(L, M) \mid \delta^2\phi = 0, \phi(x, y^{[p]}) = \phi([x, \underbrace{y, \ldots, y}_{p-1}], y)\},$$

$$B^2(L, M) = \{(\phi, \omega) \in C^2(L, M) \mid \exists\psi \in C^1_{cl}(L, M),\ s.t.\ \delta^1\psi = \phi, \tilde{\psi} = \omega\}.$$

It can be seen that $B^2(L, M) \subseteq Z^2(L, M)$, so that, the quotient $Z^2(L, M)/B^2(L, M)$ is well-defined. The quotient $Z^2(L, M)/B^2(L, M)$ is called the second cohomology group of the restricted Lie algebra L with coefficients in M and we denote it by $H^2(L, M)$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, M)$. We set $L_\theta = L \oplus M$ as a vector space and define the Lie bracket and p-map on L_θ by:

$$[(x_1 + m_1), (x_2 + m_2)] = \phi(x_1, x_2) + [x_1, x_2], \quad (x + m)^{[p]} = \omega(x) + x^{[p]}.$$

Then L_θ with the given bracket and p-map is a restricted Lie algebra.

Now let K be a p-nilpotent restricted Lie algebra. Then its center $Z(K)$ is nonzero and there exists $x \in Z(K)$ such that $x^{[p]} = 0$. Let M be the one dimensional restricted ideal of K spanned by x, and set $L = K/M$. Let $\pi : K \to L$ be the projection map. We have the exact sequence of restricted Lie algebras:

$$0 \to M \to K \to L \to 0.$$

Choose an injective linear map $\sigma : L \to K$ such that $\pi\sigma = 1_L$. Define $\phi : L \times L \to M$ by $\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2])$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. It turns out that $[\theta] = [(\phi, \omega)] \in H^2(L, M)$ and $K \cong L_\theta$. Therefore, any p-nilpotent restricted Lie algebra K of dimension n can be constructed as a central extensions of a p-nilpotent restricted Lie algebras of dimension $n - 1$.

Now, the group of automorphisms $\mathrm{Aut}(L)$ acts on $H^2(L, M)$ in a natural way. Let $A \in \mathrm{Aut}(L)$ and $[\theta] = [(\phi, \omega)] \in H^2(L, M)$. We define $[A\theta] = [(A\phi, A\omega)]$, where $A\phi(x, y) = \phi(A(x), A(y))$ and $A\omega(x) = \omega(A(x))$. Let $[\theta_1], [\theta_2] \in H^2(L, M)$. It turns out that $[\theta_1]$ and $[\theta_2]$ are in the same $\mathrm{Aut}(L)$-orbit if and only if there exists an isomorphism $f : L_{\theta_1} \to L_{\theta_2}$ such that $f(M) = M$. Therefore, we use the action of $\mathrm{Aut}(L)$ to reduce the number of isomorphic restricted Lie algebras.

There are nine nilpotent Lie algebras of dimension 5 that we denote them by $L_{5,i}$, for $1 \leqslant i \leqslant 9$. Let i be in the range $1 \leqslant i \leqslant 9$ and set $K = L_{5,i}$. As we mentioned before, a

p-map is determined by its action on an $\mathbb{F}$-basis $x_1, \ldots, x_5$ of K. Since the p-maps are p-nilpotent, there exists a central element $x \in K$ such that $x^{[p]} = 0$. Then we let $L = K/\langle x \rangle$ and use the methods above to find all 1-dimensional central extensions of L that lead to K. That is we choose those $[\theta] = [(\phi, \omega)] \in H^2(L, \langle x \rangle)$ such that L_θ is isomorphic as a Lie algebra to K. Then we list all possible p-maps that are obtained via different choices of θ and x. We still need to detect and remove the isomorphic algebras from this list. Finally, we shall prove that the remaining algebras in the list are pairwise non-isomorphic. Please note that a copy of this thesis exists in Arxiv.

Chapter 2

Preliminaries

2.1 Restricted Lie algebras

We first recall some definitions and notations that are mostly adopted from [15].

Definition 2.1.1 *A restricted Lie algebra of characteristic $p > 0$ is a Lie algebra L of characteristic p together with a map $L \to L$, denoted by $x \mapsto x^{[p]}$, that satisfies*

- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$,
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{j=1}^{p-1} s_j(x,y)$
  where $js_j(x,y)$ is the coefficient of t^{j-1} in $(\mathsf{ad}\,(tx+y))^{p-1}(x)$, t an indeterminate,
- $[x, y^{[p]}] = [x, \underbrace{y, \ldots, y}_{p}]$,

for all $x, y \in L$ and all $\lambda \in \mathbb{F}$.

The map $x \mapsto x^{[p]}$ is referred to as the p-operator. Note that the second property is equivalent to

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{\substack{x_j = x \text{ or } y \\ x_1 = x, x_2 = y}} \frac{1}{\#x}[x_1, x_2, \ldots, x_p],$$

where $\#x$ denotes the number of x's among the x_j.

Note that long commutators are left-tapped, that is

$$[x_1, \dots, x_k, x_{k+1}] = [[x_1, \dots, x_k], x_{k+1}].$$

If L is a Lie algebra over $\mathbb{F}$ and M is vector space, a q-dimensional cochain of L with coefficients in M is a skew-symmetric, q-linear map on L taking values in M. We denote the space of q-dimensional cochains of a Lie algebra L with coefficients in M by $C^q_{\text{cl}}(L, M)$. So, we have

$$C^q_{\text{cl}}(L, M) = Hom_{\mathbb{F}}(\Lambda^q L, M).$$

The coboundary map $\delta^q : C^q_{\text{cl}}(L, M) \to C^{q+1}_{\text{cl}}(L, M)$ is defined by

$$(\delta^q \phi)(l_1, \dots, l_{q+1}) = \sum_{1 \le s < t \le q+1} (-1)^{s+t-1} \phi([l_s, l_t], l_1, \dots, \widehat{l_s}, \dots, \widehat{l_t}, \dots, l_{q+1}),$$

where the symbol $\widehat{l_s}$ indicates that this term is to be omitted.

Definition 2.1.2 *Let L be a restricted Lie algebra over $\mathbb{F}$ and M a vector space. If $\phi \in C^2_{cl}(L, M)$ and $\omega : L \to M$ a function, we say ω has the $\star$-property with respect to ϕ if for every $x, y \in L$ and $\lambda \in \mathbb{F}$, we have*

1. $\omega(\lambda x) = \lambda^p \omega(x)$

2. $\omega(x + y) = \omega(x) + \omega(y) + \sum_{\substack{x_j = x \text{ or } y \\ x_1 = x, x_2 = y}} \frac{1}{\#x} \phi([x_1, x_2, \dots, x_{p-1}], x_p)$*, where $\#x$ is the number of x.*

Now, we define the space of 2-dimensional cochains of a restricted Lie algbera L with coefficients in M as the subspace spanned by all (ϕ, ω) such that ϕ is skew-symmetric and $\omega : L \to M$ has the $\star$-property with respect to ϕ. We denote this vector space by $C^2(L, M)$.

Evidently if ω and ω' have the $\star$-property with respect to ϕ and ϕ' respectively, then $\omega+\omega'$ has the $\star$-property with respect to $\phi+\phi'$, and hence $C^2(L,M)$ is a vector space over $\mathbb{F}$ by point wise addition in each coordinate. We have adopted these definitions from [5, 6], however, the definition of $\star$-property in the whole generality given in [5, 6] is ambiguous.

Lemma 2.1.3 *Let M be a vector space and $\psi : L \to M$ a linear map. Then $\tilde{\psi} : L \to M$ defined by $\tilde{\psi}(x) = \psi(x^{[p]})$ has the $\star$-property with respect to $\delta^1\psi$.*

Proof. Clearly, $\tilde{\psi}(\lambda x) = \lambda^p \tilde{\psi}(x)$, for every $\lambda \in \mathbb{F}$. Since $\psi[x_1,\ldots,x_p] = (\delta^1\psi)([x_1,\ldots,x_{p-1}],x_p)$, we have

$$\tilde{\psi}(x+y) = \psi((x+y)^{[p]}) = \psi(x^{[p]}) + \psi(y^{[p]}) + \psi(\sum_{\substack{x_j=x \text{ or } y\\ x_1=x,x_2=y}} \frac{1}{\#x}[x_1,x_2,\ldots,x_p])$$

$$= \psi(x^{[p]}) + \psi(y^{[p]}) + \sum_{\substack{x_j=x \text{ or } y\\ x_1=x,x_2=y}} \frac{1}{\#(x)}(\delta^1\psi)([x_1,\ldots,x_{p-1}],x_p),$$

for every $x, y \in L$. ■

Definition 2.1.4 *We define*

$$Z^2(L,M) = \{(\phi,\omega) \in C^2(L,M) \mid \delta^2\phi = 0, \phi(x,y^{[p]}) = \phi([x,\underbrace{y,\ldots,y}_{p-1}],y)\},$$

$$B^2(L,M) = \{(\phi,\omega) \in C^2(L,M) \mid \exists\psi \in C^1_{cl}(L,M),\ s.t.\ \delta^1\psi = \phi, \tilde{\psi} = \omega\}.$$

Note that it is easy to verify that $Z^2(L,M)$ and $B^2(L,M)$ are subspaces of $C^2(L,M)$.

Theorem 2.1.5 $B^2(L,M) \subseteq Z^2(L,M)$, *so that, the quotient*

$$H^2(L,M) = Z^2(L,M)/B^2(L,M)$$

is well-defined.

Proof. Let $(\delta^1\psi, \tilde{\psi}) \in B^2(L, M)$. First, We claim that $\delta^2\delta^1\psi = 0$. Indeed, for all $x, y, z \in L$, we have

$$\begin{aligned}\delta^2\delta^1\psi(x,y,z) =&\delta^1\psi([x,y],z) + \delta^1\psi([y,z],x) + \delta^1\psi([z,x],y)\\ =&\psi([[x,y],z]]) + \psi([[y,z],x]]) + \psi([[z,x],y]])\\ =&\psi([[x,y],z]] + [[y,z],x]] + [[z,x],y]]),\end{aligned}$$

which is equal to zero by jacobi identity. Next, we claim that

$$(\delta^1\psi)(x, y^{[p]}) = (\delta^1\psi)([x, \underbrace{y, ..., y}_{p-1}], y),$$

for all $x, y \in L$. Indeed, for all $x, y \in L$, we have

$$\begin{aligned}(\delta^1\psi)(x, y^{[p]}) - (\delta^1\psi)([x, \underbrace{y, \ldots, y}_{p-1}], y) =&\psi[x, y^{[p]}] - \psi([x, \underbrace{y, \ldots, y}_{p}])\\ =&\psi([x, y^{[p]}] - [x, \underbrace{y, \ldots, y}_{p}])\\ =&0.\end{aligned}$$

The proof is complete. ■

We call $H^2(L, M)$ the second cohomology group of L with coefficients in M. Let $\theta = (\phi, \omega) \in Z^2(L, M)$. Then we denote by $[\theta]$ the image of θ in $H^2(L, M)$.

Definition 2.1.6 *A restricted Lie algebra M is called strongly abelian if, $[M, M] = 0$ and $M^{[p]} = 0$.*

2.2 Constructing p-nilpotent restricted Lie algebras

The analogue of the results of the remaining of this chapter for Lie algebras is known [14, 4]. However, we could not find a refrence for the corresponding results in the setting of restricted Lie algebras.

Let L be a restricted Lie algebra, M a vector space and $\theta = (\phi, \omega) \in Z^2(L, M)$. We construct a restricted extension of L by M as follows.

Lemma 2.2.1 *Let $L_\theta = L \oplus M$ as a vector space and define the Lie bracket and p-map on L_θ by:*

$$[(x_1 + m_1), (x_2 + m_2)] = [x_1, x_2] + \phi(x_1, x_2), \quad (x + m)^{[p]} = x^{[p]} + \omega(x).$$

Then L_θ with the given bracket and p-map is a resticted Lie algebra.

Proof. The bracket is clearly bilinear and skew symmetric and it is well known that the jacobi identity is equivalent to $\delta^2\phi = 0$. We claim that L is restricted with the given p-map. Let $x_1, \ldots, x_{k+1} \in L$, $m_1, \ldots m_{k+1} \in M$. Note that by induction we have

$$[x_1 + m_1, x_2 + m_2, \ldots, x_{k+1} + m_{k+1}] = [x_1, \ldots, x_{k+1}] + \phi([x_1, \ldots, x_k], x_{k+1}). \quad (2.1)$$

Now, we have

$$\begin{aligned}[x_1 + m_1, (x_2 + m_2)^{[p]}] =& [x_1 + m_1, x_2^{[p]} + \omega(x)] \\ =& [x_1, x_2^{[p]}] + \phi(x_1, x_2^{[p]}).\end{aligned}$$

On the other hand,

$$[x_1 + m_1, \underbrace{x_2 + m_2, \ldots, x_2 + m_2}_{p}] = [x_1, \underbrace{x_2, \ldots, x_2}_{p}] + \phi([x_1, \underbrace{x_2, \ldots, x_2}_{p-1}], x_2),$$

by equation (2.1). We have

$$\phi(x_1, x_2^{[p]}) = \phi([x_1, \underbrace{x_2, \ldots, x_2}_{p-1}], x_2),$$

also we have

$$[x_1, x_2^{[p]}] = [x_1, \underbrace{x_2, \ldots, x_2}_{p}].$$

Therefore,

$$[x_1+m_1,(x_2+m_2)^{[p]}]=[x_1+m_1,\underbrace{x_2+m_2,\ldots,x_2+m_2}_{p}].$$

Next, we have

$$\begin{aligned}(\lambda(x+m))^{[p]}=(\lambda x+\lambda m)^{[p]}=(\lambda x)^{[p]}+\omega(\lambda x)=&\lambda^p x^{[p]}+\lambda^p\omega(x)\\=&\lambda^p(x^{[p]}+\omega(x))\\=&\lambda^p(x+m)^{[p]}.\end{aligned}$$

Finally, we have

$$\begin{aligned}((x_1+m_1)+(x_2+m_2))^{[p]}=&((x_1+x_2)+(m_1+m_2))^{[p]}\\=&(x_1+x_2)^{[p]}+\omega(x_1+x_2)\\=&x_1^{[p]}+x_2^{[p]}+\sum_{\substack{x_{l_j}=x_1\text{ or }x_2\\x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}[x_{l_1},x_{l_2},\ldots,x_{l_p}]\\&+\omega(x_1)+\omega(x_2)+\sum_{\substack{x_{l_j}=x_1\text{ or }x_2\\x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}\phi([x_{l_1},x_{l_2},\ldots,x_{l_{p-1}}],x_{l_p}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(x_1+m_1)^{[p]}+(x_2+m_2)^{[p]}+\sum_{\substack{l_j=1\text{ or }2\\l_1=1,l_2=2}}\frac{1}{\#(x_1+m_1)}[x_{l_1}+m_{l_1},x_{l_2}+m_{l_2},\ldots,x_{l_p}+m_{l_p}]\\&=x_1^{[p]}+\omega(x_1)+x_2^{[p]}+\omega(x_2)+\sum_{\substack{x_{l_j}=x_1\text{ or }x_2\\x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}[x_{l_1},x_{l_2},\ldots,x_{l_p}]\\&+\sum_{\substack{x_{l_j}=x_1\text{ or }x_2\\x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}\phi([x_{l_1},x_{l_2},\ldots,x_{l_{p-1}}],x_{l_p}),\end{aligned}$$

by equation (2.1). Therefore,

$$((x_1+m_1)+(x_2+m_2))^{[p]} = (x_1+m_1)^{[p]} + (x_2+m_2)^{[p]}$$
$$+ \sum_{\substack{l_j=1 \text{ or } 2 \\ l_1=1, l_2=2}} \frac{1}{\# x_1 + m_1} [x_{l_1}+m_{l_1}, x_{l_2}+m_{l_2}, \ldots, x_{l_p}+m_{l_p}].$$

The proof is complete. ■

Now let K be a restricted Lie algebra, and suppose that its center $Z(K)$, is nonzero. Let $0 \neq M \subseteq Z(K)$ such that $M^{[p]} = 0$, and set $L = K/M$. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Choose an injective linear map $\sigma : L \to K$ such that $\pi\sigma = 1_L$. Note that we can easily show that $\pi([\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])) = 0$, for every $x_i, x_j \in L$ and $\pi(\sigma(x)^{[p]} - \sigma(x^{[p]})) = 0$, for every $x \in L$. Therefore, $[\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j]), \sigma(x)^{[p]} - \sigma(x^{[p]}) \in M$, for every $x_i, x_j, x \in L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. With these notations, we have:

Lemma 2.2.2 *Let $\theta = (\phi, \omega)$. Then $\theta \in Z^2(L, M)$ and $K \cong L_\theta$.*

Proof. It is easy to see that ϕ is a bilinear and skew-symmetric form on L. We claim that

ω has the $\star$-property with respect to ϕ. Indeed:

$$\begin{aligned}\omega(x_1+x_2) =&(\sigma(x_1+x_2))^{[p]}-\sigma((x_1+x_2)^{[p]})\\ =&(\sigma(x_1)+\sigma(x_2))^{[p]}-\sigma(x_1^{[p]}+x_2^{[p]}+\sum_{\substack{x_{l_j}=x_1 \text{ or } x_2\\ x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}[x_{l_1},x_{l_2},\dots,x_{l_p}])\\ =&\sigma(x_1)^{[p]}+\sigma(x_2)^{[p]}+\sum_{\substack{l_j=1 \text{ or } 2\\ l_1=1,l_2=2}}\frac{1}{\#\sigma(x_1)}[\sigma(x_{l_1}),\sigma(x_{l_2}),\dots,\sigma(x_{l_p)}])\\ &-\sigma(x_1^{[p]})-\sigma(x_2^{[p]})-\sum_{\substack{x_{l_j}=x_1 \text{ or } x_2\\ x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}\sigma([x_{l_1},x_{l_2},\dots,x_{l_p}])\\ =&\omega(x_1)+\omega(x_2)+\sum_{\substack{l_j=1 \text{ or } 2\\ l_1=1,l_2=2}}\frac{1}{\#x_1}[\sigma(x_{l_1}),\sigma(x_{l_2}),\dots,\sigma(x_{l_p)}])\\ &-\sum_{\substack{x_{l_j}=x_1 \text{ or } x_2\\ x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}\sigma([x_{l_1},x_{l_2},\dots,x_{l_p}]).\end{aligned}$$

Since $\phi(x,y)\in M\subseteq Z(L)$, we have

$$\begin{aligned}&\sigma([x_{l_1},x_{l_2},\dots,x_{l_p}])\\ &=[\sigma([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}]),\sigma(x_{l_p})]-\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}],x_{l_p})\\ &=[[\sigma([x_{l_1},x_{l_2},\dots,x_{l_{p-2}}]),\sigma(x_{l_{p-1}})]-\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-2}}],x_{l_{p-1}}),\sigma(x_{l_p})]\\ &-\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}],x_{l_p})\\ &=[[\sigma([x_{l_1},x_{l_2},\dots,x_{l_{p-2}}]),\sigma(x_{l_{p-1}})],\sigma(x_{l_p})]-\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}],x_{l_p}).\end{aligned}$$

If we repeat this procedure, we obtain that

$$\sigma([x_{l_1},x_{l_2},\dots,x_{l_p}])=[\sigma(x_{l_1}),\sigma(x_{l_2}),\dots,\sigma(x_{l_p)}])-\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}],x_{l_p}).$$

Therefore, we have

$$\omega(x_1+x_2)=\omega(x_1)+\omega(x_2)+\sum_{\substack{x_{l_j}=x_1 \text{ or } x_2\\ x_{l_1}=x_1,x_{l_2}=x_2}}\frac{1}{\#x_1}\phi([x_{l_1},x_{l_2},\dots,x_{l_{p-1}}],x_{l_p}).$$

Next, we claim that $\delta^2\phi = 0$. Indeed, for all $x, y, z \in L$, we have

$$\begin{aligned}(\delta^2\phi)(x,y,z) =& \phi([x,y],z) + \phi([y,z],x) + \phi([z,x],y)\\ =& [\sigma([x,y]),\sigma(z)] - \sigma([[x,y],z]) + [\sigma([y,z]),\sigma(x)] - \sigma([[y,z],x])\\ &+ [\sigma([z,x]),\sigma(y)] - \sigma([[z,x],y])\\ =& [\sigma([x,y]),\sigma(z)] + [\sigma([y,z]),\sigma(x)] + [\sigma([z,x]),\sigma(y)]\\ &- \sigma([[x,y],z] + [[y,z],x] + [[z,x],y]).\end{aligned}$$

By jacobi identity, $[x,y,z] + [y,z,x] + [z,x,y] = 0$. Therefore,

$$\begin{aligned}&\delta^2\phi(x,y,z)\\ &= [\sigma([x,y]),\sigma(z)] + [\sigma([y,z]),\sigma(x)] + [\sigma([z,x]),\sigma(y)]\\ &= [[\sigma(x),\sigma(y)] - \phi(x,y),\sigma(z)] + [[\sigma(y),\sigma(z)] - \phi(y,z),\sigma(x)]\\ &+ [[\sigma(z),\sigma(x)] - \phi(z,x),\sigma(y)]\\ &= [[\sigma(x),\sigma(y)],\sigma(z)] - [\phi(x,y),\sigma(z)] + [[\sigma(y),\sigma(z)],\sigma(x)] - [\phi(y,z),\sigma(x)]\\ &+ [[\sigma(z),\sigma(x)],\sigma(y)] - [\phi(z,x),\sigma(y)]\\ &= [[\sigma(x),\sigma(y)],\sigma(z)] + [[\sigma(y),\sigma(z)],\sigma(x)] + [[\sigma(z),\sigma(x)],\sigma(y)],\end{aligned}$$

which is equal to zero by jacobi identity.

Finally, we claim that $\phi(x, y^{[p]}) = \phi([x, \underbrace{y,\ldots,y}_{p-1}], y)$, for all $x, y \in L$. Indeed:

$$\begin{aligned}&\phi(x,y^{[p]}) - \phi([x,\underbrace{y,\ldots,y}_{p-1}],y)\\ &= [\sigma(x),\sigma(y^{[p]})] - \sigma([x,y^{[p]}]) - [\sigma([x,\underbrace{y,\ldots,y}_{p-1}]),\sigma(y)] + \sigma([x,\underbrace{y,\ldots,y}_{p}])\\ &= [\sigma(x),\sigma(y^{[p]})] - [\sigma([x,\underbrace{y,\ldots,y}_{p-1}]),\sigma(y)] + \sigma([x,\underbrace{y,\ldots,y}_{p}] - [x,y^{[p]}])\\ &= [\sigma(x),\sigma(y^{[p]})] - [\sigma([x,\underbrace{y,\ldots,y}_{p-1}]),\sigma(y)].\end{aligned}$$

We have

$$[\sigma([x,\underbrace{y,\dots,y}_{p-1}]),\sigma(y)] = [[\sigma([x,\underbrace{y,\dots,y}_{p-2}]),\sigma(y)] - \phi([x,\underbrace{y,\dots,y}_{p-2}],y),\sigma(y)]$$
$$= [[\sigma([x,\underbrace{y,\dots,y}_{p-2}]),\sigma(y)],\sigma(y)].$$

If we repeat this procedure, we obtain that

$$[\sigma([x,\underbrace{y,\dots,y}_{p-1}]),\sigma(y)] = [\sigma(x),\underbrace{\sigma(y),\dots,\sigma(y)}_{p}]$$
$$= [\sigma(x),(\sigma(y))^{[p]}].$$

Therefore,

$$\begin{aligned}
&\phi(x,y^{[p]}) - \phi([x,\underbrace{y,\dots,y}_{p-1}],y)\\
&= [\sigma(x),\sigma(y^{[p]})] - [\sigma(x),(\sigma(y))^{[p]}]\\
&= [\sigma(x),\sigma(y)^{[p]} - \omega(x)] - [\sigma(x),(\sigma(y))^{[p]}]\\
&= [\sigma(x),\sigma(y)^{[p]}] - [\sigma(x),(\sigma(y))^{[p]}]\\
&= 0.
\end{aligned}$$

Finally, we show that $K \cong L_\theta$. Let $x \in K$. Then there exist unique $y \in L$ and $z \in M$ such that $x = \sigma(y) + z$. Indeed, we can take $z = \sigma(\pi(x)) - x$ and $y = \pi(x)$. Since, $\pi(z) = 0$, we have $z \in M$. Define $f : K \to L_\theta$ by $f(x) = y + z$. Then f is an isomorphism. Indeed:

$$f(x^{[p]}) = f((\sigma(y)+z)^{[p]}) = f\Bigg(\sigma(y)^{[p]} + z^{[p]} + \sum_{\substack{g_j=\sigma(y) \text{ or } z\\ g_1=\sigma(y),\, g_2=z}} \frac{1}{\#(\sigma(y))}[g_1,\dots,g_p]\Bigg).$$

Since $z \in M$, we have

$$z^{[p]} = 0, \text{ and } \sum_{\substack{g_j=\sigma(y) \text{ or } z \\ g_1=\sigma(y),\, g_2=z}} \frac{1}{\#(\sigma(y))}[g_1, \ldots, g_p] = 0.$$

Thus,

$$f(x^{[p]}) = f(\sigma(y)^{[p]}) = f(\sigma(y^{[p]}) + \omega(y)) = y^{[p]} + \omega(y) = y^{[p]} + \sigma(y)^{[p]} - \sigma(y^{[p]}).$$

On the other hand, we have

$$f(x)^{[p]} = (y+z)^{[p]} = y^{[p]} + \omega(y) = y^{[p]} + \sigma(y)^{[p]} - \sigma(y^{[p]}).$$

Thus, $f(x^{[p]}) = f(x)^{[p]}$.

Similarly we can show that $f([x, y]) = [f(x), f(y)]$. ■

We conclude that any p-nilpotent restricted Lie algebra K of dimension n can be constructed from a restricted Lie algebra of lower dimension.

Lemma 2.2.3 *Let* $\theta = (\phi, \omega) \in Z^2(L, M)$ *and* $\eta = (\nu, \xi) \in B^2(L, M)$*. Then we have* $L_\theta \cong L_{\theta+\eta}$*.*

Proof. We have $\eta = (\nu, \xi) \in B^2(L, M)$. Therefore, there exits $\psi \in C^1_{\text{cl}}(L, M)$ such that $\delta^1(\psi) = \nu$ and $\tilde{\psi} = \xi$. Note that $f : L_\theta \to L_{\theta+\eta}$ such that $f(x) = x + \psi(x)$ for $x \in L$ and $f(m) = m$ for $m \in M$ is an isomorphism. Indeed, let $x_1, x_2 \in L$ and $m_1, m_2 \in M$. Then we have

$$f([x_1 + m_1, x_2 + m_2]) = f([x_1, x_2] + \phi(x_1, x_2)) = [x_1, x_2] + \psi([x_1, x_2]) + \phi(x_1, x_2).$$

On the other hand, we have

$$\begin{aligned} [f(x_1 + m_1), f(x_2 + m_2)] &= [x_1 + \psi(x_1) + m_1, x_2 + \psi(x_2) + m_2] \\ &= [x_1, x_2] + (\phi + \nu)(x_1, x_2) \\ &= [x_1, x_2] + \psi([x_1, x_2]) + \phi(x_1, x_2). \end{aligned}$$

Therefore, $f([x_1+m_1, x_2+m_2]) = [f(x_1+m_1), f(x_2+m_2)]$. Also, we have

$$f((x_1+m_1)^{[p]}) = f(x_1^{[p]} + \omega(x_1)) = x_1^{[p]} + \psi(x_1^{[p]}) + \omega(x_1).$$

On the other hand, we have

$$f(x_1+m_1)^{[p]} = (x_1 + \psi(x_1) + m_1)^{[p]} = x_1^{[p]} + (\omega+\xi)(x_1) = x_1^{[p]} + \psi(x_1^{[p]}) + \omega(x_1).$$

Therefore, $f((x_1+m_1)^{[p]}) = f(x_1+m_1)^{[p]}$. ■

2.3 Isomorphism

Let L to be a restricted Lie algebra and M a strongly abelian restricted Lie algebra which is considered as a trivial L-module. Let $e_1, \dots, e_s$ be a basis of M and set $\theta = (\phi, \omega) \in Z^2(L, M)$. Let

$$\phi(x,y) = \sum_{i=1}^{s} \phi_i(x,y)e_i, \text{ and } \omega(x) = \sum_{i=1}^{s} \omega_i(x)e_i.$$

Lemma 2.3.1 *For every i, we have $(\phi_i, \omega_i) \in Z^2(L, \mathbb{F})$.*

Proof. Since ϕ is skew symmetric and bilinear form, so is every ϕ_i. We claim that $\delta^2\phi_i = 0$. Indeed, $0 = \delta^2\phi(x,y) = \sum_{i=1}^{s} \delta^2\phi_i(x,y)e_i$, implying that $\delta^2\phi_i(x,y) = 0$.

Now we show that ω_i has $\star$-property with respect ϕ_i. First note that $\omega(\lambda x) = \lambda^p\omega(x)$. Thus, $\sum_{i=1}^{s} \omega_i(\lambda x)e_i = \lambda^p \sum_{i=1}^{s} \omega_i(x)e_i$ and so $\omega_i(\lambda x) = \lambda^p\omega_i(x)$. Furthermore, we have

$$\omega(x+y) = \omega(x) + \omega(y) + \sum_{\substack{x_j = x or y \\ x_1 = x, x_2 = y}} \frac{1}{\#(x)} \phi([x_1, \dots, x_{p-1}], x_p).$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{s} \omega_i(x+y)e_i \\ &= \sum_{i=1}^{s} \omega_i(x)e_i + \sum_{i=1}^{s} \omega_i(y)e_i + \sum_{\substack{x_j = x or y \\ x_1 = x, x_2 = y}} \frac{1}{\#(x)} \sum_{i=1}^{s} \phi_i([x_1, \dots, x_{p-1}], x_p)e_i, \end{aligned}$$

which implies that

$$\omega_i(x+y) = \omega_i(x) + \omega_i(y) + \sum_{\substack{x_j = x or y \\ x_1 = x, x_2 = y}} \frac{1}{\#(x)} \phi_i([x_1, \ldots, x_{p-1}], x_p).$$

Finally, we have

$$\begin{aligned} 0 &= \phi(x, y^{[p]}) - \phi([x, \underbrace{y, \ldots, y}_{\text{p-1}}], y) \\ &= \sum_{i=1}^{s} \phi_i(x, y^{[p]}) e_i - \sum_{i=1}^{s} \phi_i([x, \underbrace{y, \ldots, y}_{\text{p-1}}], y) e_i. \end{aligned}$$

Therefore

$$\phi_i(x, y^{[p]}) - \phi_i([x, \underbrace{y, ..., y}_{\text{p-1}}], y) = 0.$$

The proof is complete. ■

The group of automorphisms $\mathrm{Aut}(L)$ acts on $H^2(L, M)$ in a natural way. Let $A \in \mathrm{Aut}(L)$ and $\theta = (\phi, \omega) \in Z^2(L, M)$. We define $A\theta = (A\phi, A\omega)$, where $A\phi(x, y) = \phi(A(x), A(y))$ and $A\omega(x) = \omega(A(x))$. We claim that $A\theta \in Z^2(L, M)$. It is easy to see that $A\phi$ is skew symmetric and bilinear map. Since $\delta^2\phi = 0$, we have

$$\delta^2 A\phi(x, y) = \delta^2\phi(A(x), A(y)) = 0,$$

which implies that $\delta^2 A\phi = 0$. Also, $A\omega$ satisfies the $\star$-property with respect to $A\phi$. To see this, let $x, x_1, x_2 \in L$ and $\lambda \in \mathbb{F}$. Then we have,

$$A\omega(\lambda x) = \omega(A(\lambda x)) = \omega(\lambda A(x)) = \lambda^p \omega(A(x)) = \lambda^p A\omega(x),$$

and

$$
\begin{aligned}
A\omega(x_1+x_2) &= \omega(A(x_1+x_2)) = \omega(Ax_1+Ax_2)\\
&= \omega(Ax_1)+\omega(Ax_2)+\sum_{Ax_j=Ax_1 or Ax_2}\frac{1}{\#(Ax_1)}\phi([Ax_1,\dots,Ax_{p-1}],Ax_p)\\
&= \omega(Ax_1)+\omega(Ax_2)+\sum_{x_j=x_1 or x_2}\frac{1}{\#(x_1)}\phi(A[x_1,\dots,x_{p-1}],Ax_p)\\
&= \omega(Ax_1)+\omega(Ax_2)+\sum_{x_j=x_1 or x_2}\frac{1}{\#(x_1)}A\phi([x_1,\dots,x_{p-1}],x_p)\\
&= A\omega(x_1)+A\omega(x_2)+\sum_{x_j=x_1 or x_2}\frac{1}{\#(x_1)}A\phi([x_1,\dots,x_{p-1}],x_p).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&A\phi(x,y^{[p]})-A\phi([x,\underbrace{y,\dots,y}_{p-1}],y)\\
&=\phi(A(x),A(y^{[p]}))-\phi(A([x,\underbrace{y,\dots,y}_{p-1}]),A(y))\\
&=\phi(A(x),A(y)^{[p]})-\phi(([A(x),\underbrace{A(y),\dots,A(y)}_{p-1}]),A(y))\\
&=0.
\end{aligned}
$$

Now, we show that $\mathrm{Aut}(L)$ preserves $B^2(L,M)$. Let $(\delta\psi,\tilde{\psi})\in B^2(L,M)$, where $\psi: L\to M$ is linear. We have $A(\delta\psi,\tilde{\psi})=(A\delta\psi,A\tilde{\psi})$. Then

$A\delta\psi(x,y)=\delta\psi(Ax,Ay)=\psi([Ax,Ay])=\psi(A[x,y])=A\psi([x,y])=\delta(A\psi)(x,y),$ and $A\tilde{\psi}(x)=\tilde{\psi}(Ax)=\psi(A(x)^{[p]})=\psi(A(x^{[p]}))=A\psi(x^{[p]})=(\widetilde{A\psi})(x).$

So $A(\delta\psi,\tilde{\psi})=(\delta(A\psi),(\widetilde{A\psi}))$. Therefore, $\mathrm{Aut}(L)$ acts on $H^2(L,M)$.

Definition 2.3.2 *Let* $[\theta]\in H^2(L,M)$. *We define* $\bar{\theta}=\{A\theta\mid A\in\mathrm{Aut}(L)\}$. *We call* $\bar{\theta}$

an $\mathrm{Aut}(L)$*-orbit and* θ *an* $\mathrm{Aut}(L)$*-orbit representative. We say* θ_1 *and* θ_2 *are in the same* $\mathrm{Aut}(L)$*-orbit if there exists* $A \in \mathrm{Aut}(L)$ *such that* $A\theta_1 = \theta_2$.

Lemma 2.3.3 *Let* L *be a restricted Lie algebra,* M *a vector space and* $e_1, \ldots, e_s$ *be a basis of* M*. Let* $[\theta_1] = [(\phi_1, \omega_1)]$, $[\theta_2] = [(\phi_2, \omega_2)] \in H^2(L, M)$ *where* $\phi_1(x, y) = \sum_{i=1}^{s} \phi_{1i}(x, y)e_i$, $\phi_2(x, y) = \sum_{i=1}^{s} \phi_{2i}(x, y)e_i$, $\omega_1(x) = \sum_{i=1}^{s} \omega_{1i}(x)e_i$ *and* $\omega_2(x) = \sum_{i=1}^{s} \omega_{2i}(x)e_i$*. Then there exists an isomorphism* $f : L_{\theta_1} \to L_{\theta_2}$ *such that* $f(M) = M$ *if and only if there is an* $A \in Aut(L)$ *such that the images of* $(A\phi_{2i}, A\omega_{2i})$*'s span the same subspace of* $H^2(L, \mathbb{F})$ *as the images of* (ϕ_{1i}, ω_{1i})*'s.*

Proof. Let $f : L_{\theta_1} \to L_{\theta_2}$ be an isomorphism such that $f(M) = M$. Then f induces an isomorphism of $L_{\theta_1}/M \cong L$ to $L_{\theta_2}/M \cong L$, i.e., an automorphism of L. Denote this automorphism by A. Let L be spanned by $x_1, \ldots, x_n$. Then we write $f(x_i) = A(x_i) + v_i$, where $v_i \in M$, $f(e_i) = \sum_{j=1}^{s} a_{ji}e_j$, and $v_i = \sum_{l=1}^{s} \beta_{il}e_l$. Also write $[x_i, x_j]_L = \sum_{k=1}^{n} c_{ij}^k x_k$, and $x_i^{[p]} = \sum_{k=1}^{n} b_{ik}x_k$. We claim that

$$\phi_{2l}(A(x_i), A(x_j)) = \sum_{k=1}^{s} a_{lk}\phi_{1k}(x_i, x_j) + \sum_{k=1}^{n} c_{ij}^k \beta_{kl}. \tag{2.2}$$

To prove the claim, we note that $f([x_i, x_j]_{L_{\theta_1}}) = [f(x_i), f(x_j)]_{L_{\theta_2}}$. Then

$$f([x_i, x_j]_{L_{\theta_1}}) = f([x_i, x_j]_L + \phi_1(x_i, x_j)) = f([x_i, x_j]_L) + f(\phi_1(x_i, x_j)),$$

where

$$\begin{aligned} f([x_i, x_j]_L) = f(\sum_{k=1}^{n} c_{ij}^k x_k) = \sum_{k=1}^{n} c_{ij}^k f(x_k) &= \sum_{k=1}^{n} c_{ij}^k (A(x_k) + v_k) \\ &= \sum_{k=1}^{n} c_{ij}^k A(x_k) + \sum_{k=1}^{n} c_{ij}^k v_k, \end{aligned}$$

and

$$\begin{aligned} f(\phi_1(x_i,x_j)) = f(\sum_{k=1}^{s}\phi_{1k}(x_i,x_j)e_k) &= \sum_{k=1}^{s}\phi_{1k}(x_i,x_j)f(e_k) \\ &= \sum_{k=1}^{s}\phi_{1k}(x_i,x_j)\sum_{l=1}^{s}a_{lk}e_l \\ &= \sum_{l=1}^{s}\sum_{k=1}^{s}a_{lk}\phi_{1k}(x_i,x_j)e_l. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f([x_i,x_j]_{L_{\theta_1}}) &= \sum_{k=1}^{n}c_{ij}^{k}A(x_k) + \sum_{k=1}^{n}c_{ij}^{k}v_k + \sum_{l=1}^{s}\sum_{k=1}^{s}a_{lk}\phi_{1k}(x_i,x_j)e_l \\ &= \sum_{k=1}^{n}c_{ij}^{k}A(x_k) + \sum_{l=1}^{s}\sum_{k=1}^{n}c_{ij}^{k}\beta_{kl}e_l + \sum_{l=1}^{s}\sum_{k=1}^{s}a_{lk}\phi_{1k}(x_i,x_j)e_l. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [f(x_i),f(x_j)]_{L_{\theta_2}} &= [A(x_i)+v_i, A(x_j)+v_j]_{L_{\theta_2}} \\ &= [A(x_i),A(x_j)]_L + \phi_2(A(x_i),A(x_j)), \end{aligned}$$

where

$$[A(x_i),A(x_j)]_L = A([x_i,x_j]_L) = A(\sum_{k=1}^{n}c_{ij}^{k}x_k) = \sum_{k=1}^{n}c_{ij}^{k}A(x_k),$$

and

$$\phi_2(A(x_i),A(x_j)) = \sum_{l=1}^{s}\phi_{2l}(A(x_i),A(x_j))e_l.$$

Therefore, we have

$$[f(x_i),f(x_j)]_{L_{\theta_2}} = \sum_{k=1}^{n}c_{ij}^{k}A(x_k) + \sum_{l=1}^{s}\phi_{2l}(A(x_i),A(x_j))e_l.$$

As a result, we have

$$\sum_{l=1}^{s}\phi_{2l}(A(x_i),A(x_j))e_l = \sum_{l=1}^{s}\sum_{k=1}^{n}c_{ij}^{k}\beta_{kl}e_l + \sum_{l=1}^{s}\sum_{k=1}^{s}a_{lk}\phi_{1k}(x_i,x_j)e_l,$$

which implies that

$$\phi_{2l}(A(x_i), A(x_j)) = \sum_{k=1}^{n} c_{ij}^k \beta_{kl} + \sum_{k=1}^{s} a_{lk}\phi_{1k}(x_i, x_j),$$

for $1 \leqslant l \leqslant s$. This proves the claim. Next we claim that

$$\omega_{2l}(A(x_i)) = \sum_{k=1}^{n} b_{ik}\beta_{kl} + \sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i), \tag{2.3}$$

for every $1 \leqslant l \leqslant s$. To prove the claim, we note that $f(x_i^{[p]}) = f(x_i)^{[p]}$. So,

$$f(x_i^{[p]}) = f(\omega_1(x_i) + x_i^{[p]}) = f(\omega_1(x_i)) + f(x_i^{[p]}).$$

Now we have

$$\begin{aligned} f(\omega_1(x_i)) = f(\sum_{k=1}^{s}\omega_{1k}(x_i)e_k) = \sum_{k=1}^{s}\omega_{1k}(x_i)f(e_k) &= \sum_{k=1}^{s}\omega_{1k}(x_i)\sum_{l=1}^{s} a_{lk}e_l \\ &= \sum_{l=1}^{s}\sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i)e_l, \end{aligned}$$

and

$$f(x_i^{[p]}) = f(\sum_{k=1}^{n} b_{ik}x_k) = \sum_{k=1}^{n} b_{ik}f(x_k) = \sum_{k=1}^{n} b_{ik}(A(x_k) + v_k) = \sum_{k=1}^{n} b_{ik}A(x_k) + \sum_{k=1}^{n} b_{ik}v_k.$$

Therefore, we have

$$\begin{aligned} f(x_i^{[p]}) &= \sum_{k=1}^{n} b_{ik}A(x_k) + \sum_{k=1}^{n} b_{ik}v_k + \sum_{l=1}^{s}\sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i)e_l \\ &= \sum_{k=1}^{n} b_{ik}A(x_k) + \sum_{l=1}^{s}\sum_{k=1}^{n} b_{ik}\beta_{kl}e_l + \sum_{l=1}^{s}\sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i)e_l. \end{aligned}$$

On the other hand, we have

$$f(x_i)^{[p]} = (A(x_i) + v_i)^{[p]} = \omega_2(A(x_i)) + A(x_i)^{[p]},$$

where

$$\omega_2(A(x_i)) = \sum_{l=1}^{s}\omega_{2l}(A(x_i))e_l, \text{ and}$$

$$A(x_i)^{[p]} = A(x_i^{[p]}) = A(\sum_{k=1}^{n} b_{ik}x_k) = \sum_{k=1}^{n} b_{ik}A(x_k).$$

Therefore, we have

$$f(x_i)^{[p]} = \sum_{l=1}^{s} \omega_{2l}(A(x_i))e_l + \sum_{k=1}^{n} b_{ik}A(x_k).$$

As a result,

$$\sum_{l=1}^{s} \omega_{2l}(A(x_i))e_l = \sum_{l=1}^{s}\sum_{k=1}^{n} b_{ik}\beta_{kl}e_l + \sum_{l=1}^{s}\sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i)e_l,$$

which implies that

$$\omega_{2l}(A(x_i)) = \sum_{k=1}^{n} b_{ik}\beta_{kl} + \sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i),$$

for $1 \leqslant l \leqslant s$. This proves the second claim.

Now define the linear function $f_l : L \to \mathbb{F}$ by $f_l(x_k) = \beta_{kl}$. Then

$$(\delta^1 f_l)(x_i, x_j) = f_l([x_i, x_j]) = \sum_{k=1}^{n} c_{ij}^k \beta_{kl}, \text{ and}$$

$$\tilde{f}_l(x_i) = f_l(x_i^{[p]}) = f_l(\sum_{k=1}^{n} b_{ik}x_k) = \sum_{k=1}^{n} b_{ik}\beta_{kl}.$$

Hence, $(\delta^1 f_l, \tilde{f}_l) \in B^2(L, \mathbb{F})$. We deduce, by Equations (2.2) and (2.3), that the subspace spanned by all the $(A\phi_{2i}, A\omega_{2i})$'s is the same as the subspace spanned by all the (ϕ_{1i}, ω_{1i})'s modulo $B^2(L, \mathbb{F})$.

Conversely, suppose that the images of $(A\phi_{2i}, A\omega_{2i})$'s span the same subspace of $H^2(L, \mathbb{F})$ as the the images of (ϕ_{1i}, ω_{1i})'s. Then there are linear functions $f_l : L \to \mathbb{F}$ and $a_{lk} \in \mathbb{F}$ so that

$$A\phi_{2l}(x_i, x_j) = \sum_{k=1}^{s} a_{lk}\phi_{1k}(x_i, x_j) + f_l([x_i, x_j]), \text{ and}$$

$$A\omega_{2l}(x_i) = \sum_{k=1}^{s} a_{lk}\omega_{1k}(x_i) + \tilde{f}_l(x_i).$$

If we set $\beta_{kl} = f_l(x_k)$, then we see that Equations (2.2) and (2.3) hold. This means that, if we define $f : L_{\theta_1} \to L_{\theta_2}$ by $f(x_i) = A(x_i) + \sum_l \beta_{il}e_l$, $f(e_i) = \sum_j a_{ji}e_j$, then f is an isomorphism. ■

2.4 Finding a basis for $H^2(L,\mathbb{F})$

Remark 2.4.1 *Let L be a p-nilpotent restricted Lie algebra of dimension 5 and $(\phi,\omega)\in H^2(L,\mathbb{F})$. Then we have*

$$\phi(x,y^{[p]})=\phi([x,\underbrace{y,\dots,y}_{p-1}],y),$$

$$\omega(x+y)=\omega(x)+\omega(y)+\sum_{x_j=x\text{ or }y}\frac{1}{\#x}\phi([x,y,x_1,\dots,x_{p-3}],x_{p-2}),$$

for all $x,y\in L$. Since $p\geqslant 5$ and all of nilpotent Lie algebras of dimension 5 are nilpotent of class at most 4, we have $\phi([x,\underbrace{y,\dots,y}_{p-1}],y)=0$ and

$$\sum_{x_j=x\text{ or }y}\frac{1}{\#x}\phi([x,y,x_1,\dots,x_{p-3}],x_{p-2})=0,$$

for all $x,y\in L$. Therefore, $\phi(x,y^{[p]})=0$, for all $x,y\in L$ and ω is semilinear.

Let L be a restricted Lie algebra of dimension n. We know that $C^2(L,\mathbb{F})$ is the subspace spanned by all (ϕ,ω) such that $\phi : L\times L\to\mathbb{F}$ is a skew-symmetric and bilinear and $\omega : L\to\mathbb{F}$ has the $\star$-property with respect to ϕ.

Let $\{x_1,\dots,x_n\}$ be a basis for L and $(\phi,\omega)\in C^2(L,\mathbb{F})$. We represent ϕ by a skew symmetric matrix $B=\sum_{1\leqslant i<j\leqslant n}c_{ij}\Delta_{ij}$, where Δ_{ij} is the matrix with (i,j) element being 1, and (j,i) element being -1 and all the others 0, and ω by $\omega=\alpha_1 f_1+\cdots+\alpha_n f_n$, where $f_i(x_j)=\delta_{i,j}$.

Since we deal with nilpotent restricted Lie algebras of dimension 5 over a field of characteristic $p\geqslant 5$, we deduce by Remark 2.4.1 that ω is semilinear. Thus, the set

$$\{(\Delta_{12},0),\dots,(\Delta_{1n},0),\dots,(\Delta_{n1},0),\dots,(\Delta_{nn-1},0),(0,f_1),\dots,(0,f_n)\}\qquad(2.4)$$

forms a basis for $C^2(L,\mathbb{F})$. Now let $(\phi,\omega)\in C^2(L,\mathbb{F})$. If $(\phi,\omega)\in Z^2(L,\mathbb{F})$, then we must have $\delta^2\phi=0$, and $\phi(x,y^{[p]})=0$, for all $x,y\in L$, which will impose certain constraints on the c_{ij}'s. This way, we obtain a subset S of the set given in (2.4) that serves as a basis for $Z^2(L,\mathbb{F})$. For every $s \in S$, we denote by $[s]$ the image of s in $H^2(L,\mathbb{F}) = Z^2(L,\mathbb{F})/B^2(L,\mathbb{F})$.

Now if $(\phi,\omega) \in B^2(L,\mathbb{F})$, we must further have $\phi = \delta^1\psi$ and $\omega = \tilde{\psi}$, for some linear map $\psi : L \to \mathbb{F}$. These latter conditions impose further restrictions on the c_{ij}'s and α_k's. This way, we obtain a basis for $B^2(L,\mathbb{F})$ in which every basis element is expressed as a linear combination of elements of S. So every element of this basis of $B^2(L,\mathbb{F})$ serves as a dependence relation between the elements of S in $Z^2(L,\mathbb{F})/B^2(L,\mathbb{F})$. So, as a basis for $H^2(L,\mathbb{F})$ we take the elements $[s]$, with $s \in S$, modulo these dependence relations.

Recall that for a subset $X \subseteq L$, we denote by $\langle X\rangle_{\mathbb{F}}$ the subspace spanned by X. Note that we denote by T_2 the multiplicative group $\mathbb{F}^*/(\mathbb{F}^*)^2$ and by T_3 the multiplicative group $\mathbb{F}^*/(\mathbb{F}^*)^3$. Moreover, let G be the subgroup $G = \{\epsilon \in F^* \mid \epsilon^5 = 1\}$. We denote by $T_{2,5}$ and $T_{3,5}$ the multiplicative groups $\mathbb{F}^*/\big((\mathbb{F}^*)^2 \cap G\big)$ and $\mathbb{F}^*/\big((\mathbb{F}^*)^3 \cap G\big)$ respectively.

Lemma 2.4.2 *Let $f : L \to H$ be an isomorphism of restricted Lie algebras. Then f induces an isomorphism between $L/\langle L^{[p]}\rangle_p$ and $H/\langle H^{[p]}\rangle_p$.*

Proof. Let $\phi : H \to H/\langle H^{[p]}\rangle_p$ be the canonical homomorphism. Then $\psi = \phi o f : L \to H/\langle H^{[p]}\rangle_p$ is a surjective homomorphism of restricted Lie algeras with $\ker\psi = \langle L^{[p]}\rangle_p$. Therefore, $L/\langle L^{[p]}\rangle_p$ and $H/\langle H^{[p]}\rangle_p$ are isomorphic.

■

Chapter 3

Restriction maps on the abelian Lie algebra

Let $x_1, \ldots, x_5$ be a basis of the abelian Lie algebra $K_1 = L_{5,1}$ of dimension 5. Since we want to determine p-nilpotent maps on $L_{5,1}$, without loss of generality, we assume that $x_5^{[p]} = 0$. Let $L = L_{5,1}/\langle x_5 \rangle$. Since $\dim L = 4$, by [12], there are five restricted Lie algebra structures on L given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_2$;

I.3 $x_1^{[p]} = x_2, x_3^{[p]} = x_4$;

I.4 $x_1^{[p]} = x_2, x_2^{[p]} = x_3$;

I.5 $x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4$.

3.1 Extensions of ($L_{5,1}/\langle x_5\rangle$, trivial p-map)

We have $x_i^{[p]} \in \langle x_5\rangle$, for all $1 \leqslant i \leqslant 4$. Hence $x_i^{[p]} = \alpha_i x_5$, for some $\alpha_i \in \mathbb{F}$ and for all $1 \leqslant i \leqslant 4$. If $\alpha_i \neq 0$, then without loss of generality, we assume $\alpha_1 \neq 0$. Now, we rescale x_1 such that $x_1^{[p]} = x_5$. Now, if $\alpha_j \neq 0$ then we rescale x_j such that $x_j^{[p]} = x_5$. Finally, if $x_j^{[p]} = x_5$, then we replace x_j with $x_j - x_1$ such that $x_j^{[p]} = 0$.

Therefore, the possible p-maps are as follows:

$$K_1^1 = \langle x_1, \ldots, x_5\rangle;$$

$$K_1^2 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_5\rangle.$$

3.2 Extensions of ($L_{5,1}/\langle x_5\rangle, x_1^{[p]} = x_2$)

In this case we have $x_1^{[p]} - x_2 \in \langle x_5\rangle$, $x_2^{[p]}, x_3^{[p]}, x_4^{[p]} \in \langle x_5\rangle$. Hence, $x_1^{[p]} = x_2 + \alpha_1 x_5$, $x_2^{[p]} = \alpha_2 x_5$, $x_3^{[p]} = \alpha_3 x_5$, and $x_4^{[p]} = \alpha_4 x_5$, for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$. We replace x_2 with $x_2 + \alpha_1 x_5$ to obtain that $x_1^{[p]} = x_2$. We consider two cases:

Case 1. $\alpha_2 = 0$: First, if $\alpha_3 = \alpha_4 = 0$ then we have the following p-map:

$$K_1^3 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2\rangle.$$

Next, if one of α_3, α_4 is zero and one is nonzero, without loss of generality, we assume that $\alpha_4 = 0$. Now, we rescale x_3 so that $x_3^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^4 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_5\rangle.$$

Finally, if both α_3 and α_4 are non-zero then we rescale x_3 and x_4 so that $x_3^{[p]} = x_5$ and $x_4^{[p]} = x_5$. Now, we replace x_4 with $x_4 - x_3$ so that $x_4^{[p]} = 0$. Therefore, we obtain the same p-map as the previous one.

Case 2. $\alpha_2 \neq 0$: First, if $\alpha_3 = \alpha_4 = 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_2 x_5$ so that $x_2^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^5 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5 \rangle.$$

Next, if one of α_3, α_4 is zero and one is nonzero, without loss of generality, we assume that $\alpha_4 = 0$. Now, in $L_{5,1}$ we replace x_3 with $(\alpha_2/\alpha_3)^{1/p} x_3$ and x_5 with $\alpha_2 x_5$ so that $x_2^{[p]} = x_5$ and $x_3^{[p]} = x_5$. Then, we replace x_3 with $x_3 - x_2$ so that $x_3^{[p]} = 0$. Therefore, we obtain the same p-map as K_1^5.

Finally, if both α_3 and α_4 are non-zero then in $L_{5,1}$ we replace x_3 with $(\alpha_2/\alpha_3)^{1/p} x_3$, x_4 with $(\alpha_2/\alpha_4)^{1/p} x_4$ and x_5 with $\alpha_2 x_5$ so that $x_2^{[p]} = x_5$, $x_3^{[p]} = x_5$ and $x_4^{[p]} = x_5$. Now, we replace x_3 with $x_3 - x_2$ and x_4 with $x_4 - x_2$ so that $x_3^{[p]} = 0$ and $x_4^{[p]} = 0$. Therefore, we obtain the same p-map as K_1^5.

3.3 Extensions of ($L_{5,1}/\langle x_5 \rangle, x_1^{[p]} = x_2, x_3^{[p]} = x_4$)

In this case we have $x_1^{[p]} - x_2 \in \langle x_5 \rangle$, $x_3^{[p]} - x_4 \in \langle x_5 \rangle$, and $x_2^{[p]}, x_4^{[p]} \in \langle x_5 \rangle$. Hence, $x_1^{[p]} = x_2 + \alpha_1 x_5$, $x_3^{[p]} = x_4 + \alpha_3 x_5$, $x_2^{[p]} = \alpha_2 x_5$, and $x_4^{[p]} = \alpha_4 x_5$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$. We replace x_2 with $x_2 + \alpha_1 x_5$ so that $x_1^{[p]} = x_2$ and x_4 with $x_4 + \alpha_3 x_5$ so that $x_3^{[p]} = x_4$.

First, if $\alpha_2 = \alpha_4 = 0$ then we have the following p-map:

$$K_1^6 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4 \rangle.$$

Next, if $\alpha_2 \neq 0$ and $\alpha_4 = 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_2 x_5$ so that $x_2^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^7 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_2^{[p]} = x_5 \rangle.$$

Next, if $\alpha_2 = 0$ and $\alpha_4 \neq 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_4 x_5$ so that $x_4^{[p]} = x_5$.

Therefore, we have the following p-map:

$$K_1^8 = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

Finally, if $\alpha_2 \neq 0$ and $\alpha_4 \neq 0$ then in $L_{5,1}$ we replace x_3 with $((\alpha_2/\alpha_4)^{1/p})^{1/p}x_3$, x_4 with $(\alpha_2/\alpha_4)^{1/p}x_4$ and x_5 with $\alpha_2 x_5$ so that $x_2^{[p]} = x_5$ and $x_4^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^9 = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

3.4 Extensions of $(L_{5,1}/\langle x_5 \rangle, x_1^{[p]} = x_2, x_2^{[p]} = x_3)$

In this case we have $x_1^{[p]} - x_2 \in \langle x_5 \rangle$, $x_2^{[p]} - x_3 \in \langle x_5 \rangle$, and $x_3^{[p]}, x_4^{[p]} \in \langle x_5 \rangle$. Hence, $x_1^{[p]} = x_2 + \alpha_1 x_5$, $x_2^{[p]} = x_3 + \alpha_2 x_5$, $x_3^{[p]} = \alpha_3 x_5$, and $x_4^{[p]} = \alpha_4 x_5$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$. We replace x_2 with $x_2 + \alpha_1 x_5$ so that $x_1^{[p]} = x_2$ and x_3 with $x_3 + \alpha_2 x_5$ so that $x_2^{[p]} = x_3$.

First, if $\alpha_3 = \alpha_4 = 0$ then we have the following p-map:

$$K_1^{10} = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3 \rangle.$$

Next, if $\alpha_3 \neq 0$ and $\alpha_4 = 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_3 x_5$ so that $x_3^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^{11} = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle.$$

Next, if $\alpha_3 = 0$ and $\alpha_4 \neq 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_4 x_5$ so that $x_4^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^{12} = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_4^{[p]} = x_5 \rangle.$$

Finally, if $\alpha_3 \neq 0$ and $\alpha_4 \neq 0$ then in $L_{5,1}$ we replace x_4 with $(\alpha_3/\alpha_4)^{1/p}x_4$ and x_5 with $\alpha_3 x_5$ so that $x_3^{[p]} = x_5$ and $x_4^{[p]} = x_5$. Now, we replace x_4 with $x_4 - x_3$ so that $x_4^{[p]}$=0. Therefore, we have the following p-map:

$$K_1^{13} = \langle x_1, \dots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle.$$

3.5 Extensions of $(L_{5,1}/\langle x_5\rangle, x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4)$

In this case we have $x_1^{[p]} - x_2 \in \langle x_5\rangle$, $x_2^{[p]} - x_3 \in \langle x_5\rangle$, $x_3^{[p]} - x_4 \in \langle x_5\rangle$ and $x_4^{[p]} \in \langle x_5\rangle$. Hence, $x_1^{[p]} = x_2 + \alpha_1 x_5$, $x_2^{[p]} = x_3 + \alpha_2 x_5$, $x_3^{[p]} = x_4 + \alpha_3 x_5$, and $x_4^{[p]} = \alpha_4 x_5$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$. We replace x_2 with $x_2 + \alpha_1 x_5$ so that $x_1^{[p]} = x_2$, x_3 with $x_3 + \alpha_2 x_5$ so that $x_2^{[p]} = x_3$ and x_4 with $x_4 + \alpha_3 x_5$ so that $x_3^{[p]} = x_4$. First, if $\alpha_4 = 0$ then we have the following p-map:

$$K_1^{14} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4\rangle.$$

Next, if $\alpha_4 \neq 0$ then in $L_{5,1}$ we replace x_5 with $\alpha_4 x_5$ so that $x_4^{[p]} = x_5$. Therefore, we have the following p-map:

$$K_1^{15} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5\rangle.$$

Therefore, the list of all restricted Lie algebra structures on $L_{5,1}$ are as follows:

$$K_1^1 = \langle x_1, \ldots, x_5\rangle;$$

$$K_1^2 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_5\rangle;$$

$$K_1^3 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2\rangle;$$

$$K_1^4 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_5\rangle;$$

$$K_1^5 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5\rangle;$$

$$K_1^6 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_2^{[p]} = x_5\rangle;$$

$$K_1^7 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4\rangle;$$

$$K_1^8 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_4^{[p]} = x_5\rangle;$$

$$K_1^9 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5\rangle;$$

$$K_1^{10} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3\rangle;$$

$$K_1^{11} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_1^{12} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_4^{[p]} = x_5 \rangle;$$

$$K_1^{13} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_1^{14} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_1^{15} = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

Note that $K_1^2 \cong K_1^3$, $K_1^7 \cong K_1^4$, $K_1^{10} \cong K_1^5$, $K_1^{14} \cong K_1^{11}$, and $K_1^6 \cong K_1^8 \cong K_1^{12}$. Furthermore, K_1^{13} is identical to K_1^{11}.

Therefore, up to isomorphism the list of all restricted Lie algebra structures on $L_{5,1}$ are as follows:

L	$\dim L^{[p]}$	$\dim L^{[p]^2}$	$\dim L^{[p]^3}$
$L_{5,1}^1 = \langle x_1, \ldots, x_5 \rangle$	0	0	0
$L_{5,1}^2 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_5 \rangle$	1	0	0
$L_{5,1}^3 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_5 \rangle$	2	0	0
$L_{5,1}^4 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5 \rangle$	2	1	0
$L_{5,1}^5 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_2^{[p]} = x_5 \rangle$	3	1	0
$L_{5,1}^6 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$	3	2	0
$L_{5,1}^7 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle$	3	2	1
$L_{5,1}^8 = \langle x_1, \ldots, x_5 \mid x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$	4	3	2

Note that it is clear from the table that the above list of restricted Lie algebra structures on $L_{5,1}$ are pairwise non-isomorphic.

Chapter 4

Restriction maps on $L_{5,2}$

Let

$$K_2 = L_{5,2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle_{\mathbb{F}}.$$

We have $Z(L_{5,2}) = \langle x_3, x_4, x_5 \rangle_{\mathbb{F}}$. Note that there exists an element $\alpha x_3 + \beta x_4 + \gamma x_5 \in Z(L_{5,2})$ such that $(\alpha x_3 + \beta x_4 + \gamma x_5)^{[p]} = 0$, for some $\alpha, \beta, \gamma \in \mathbb{F}$. If $\gamma \neq 0$, then consider

$$H_1 = \langle x'_1, \ldots, x'_5 \mid [x'_1, x'_2] = x'_3 \rangle,$$

where $x'_1 = x_1, x'_2 = x_2, x'_3 = x_3, x'_4 = x_4, x'_5 = \alpha x_3 + \beta x_4 + \gamma x_5$. Let $\phi : K_2 \to H_1$ such that $x_1 \mapsto x'_1$, $x_2 \mapsto x'_2$, $x_3 \mapsto x'_3$, $x_4 \mapsto x'_4$, and $x_5 \mapsto x'_5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_5^{[p]} = 0$. Next, if $\gamma = 0$ and $\beta \neq 0$ then consider

$$H_2 = \langle x'_1, \ldots, x'_5 \mid [x'_1, x'_2] = x'_3 \rangle,$$

where $x'_1 = x_1, x'_2 = x_2, x'_3 = x_3, x'_4 = \alpha x_3 + \beta x_4, x'_5 = x_5$. Let $\phi : K_2 \to H_2$ such that $x_1 \mapsto x'_1$, $x_2 \mapsto x'_2$, $x_3 \mapsto x'_3$, $x_4 \mapsto x'_4$, and $x_5 \mapsto x'_5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_4^{[p]} = 0$. Finally, if $\gamma = \beta = 0$ and $\alpha \neq 0$, then consider

$$H_3 = \langle x'_1, \ldots, x'_5 \mid [x'_1, x'_2] = x'_3 \rangle,$$

where $x'_1 = x_1, x'_2 = x_2, x'_3 = \alpha x_3, x'_4 = x_4, x'_5 = x_5$. Let $\phi : K_2 \to H_3$ such that $x_1 \mapsto x'_1$, $x_2 \mapsto x'_2$, $x_3 \mapsto x'_3$, $x_4 \mapsto x'_4$, and $x_5 \mapsto x'_5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_3^{[p]} = 0$. Hence we have three cases:

I. $x_3^{[p]} = 0$;

II. $x_5^{[p]} = 0$;

III. $x_4^{[p]} = 0$.

Note that cases II and III yield the same restricted Lie algebras because given a restricted Lie algebra structure on $L_{5,2}$ for which $x_5^{[p]} = 0$, the automorphism of $L_{5,2}$ obtained by switching x_4 and x_5 gives rise to a restricted Lie algebra structure on $L_{5,2}$ for which $x_4^{[p]} = 0$.

4.1 Extensions of $L = \frac{L_{5,2}}{\langle x_3 \rangle}$

In this section we find all non-isomorphic p-maps on $L_{5,2}$ such that $x_3^{[p]} = 0$. We let

$$L = \frac{L_{5,2}}{\langle x_3 \rangle} \cong L_{4,1},$$

where $L_{4,1} = \langle x_1, x_2, x_4, x_5 \rangle$. Note that we denote the image of x_i in L by x_i again. We rename the x_i's so that $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_4$, $y_4 = x_5$, and $y_5 = x_3$ and at the end we will switch them. Therefore, we have $L = L_{4,1} = \langle y_1, y_2, y_3, y_4 \rangle$. The group $\mathrm{Aut}(L)$

consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Lemma 4.1.1 *Let $K = L_{5,2}$ and $[p] : K \to K$ be a p-map on K such that $x_3^{[p]} = 0$ and let $L = \frac{K}{M}$ where $M = \langle x_3 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{12}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ be a linear map given by $x_i \mapsto x_i$, for all i. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, for all i, j and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = [x_1, x_2] = x_3;$$

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = [x_1, x_3] = 0.$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{12}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{12}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K such that $x_3^{[p]} = 0$ can be obtained by an extension of L via $\theta = (\Delta_{12}, \omega)$, for some ω.

Note that by [12], there are five non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

I.1 Trivial p-map;

I.2 $y_1^{[p]} = y_2$;

I.3 $y_1^{[p]} = y_2, y_3^{[p]} = y_4$;

I.4 $y_1^{[p]} = y_2, y_2^{[p]} = y_3$;

I.5 $y_1^{[p]} = y_2, y_2^{[p]} = y_3, y_3^{[p]} = y_4$.

In the following subsections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 4.1.1, it suffices to assume $[\theta] = [(\Delta_{12}, \omega)]$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,2}$.

4.1.1 Extensions of (L, trivial p-map)

We have

$C^2(L, \mathbb{F}) = \langle (\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (\Delta_{34}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4) \rangle_{\mathbb{F}}$.

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$. Note that since L is an abelian Lie algebra and the p-map is trivial, $\delta^2\phi = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (\Delta_{34}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in B^2(L, \mathbb{F})$. Then there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$\phi(y_1, y_2) = \delta^1\psi(y_1, y_2) = \psi([y_1, y_2]) = 0.$$

But $\phi(y_1, y_2) = a$. So we deduce that $a = 0$. Similarly, we can show that $b = c = d = e = f = 0$. Also, we have

$$\omega(y_1) = \tilde{\psi}(y_1) = \psi(y_1^{[p]}) = 0.$$

Hence, $\alpha = 0$. Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = 0$ and hence $B^2(L, \mathbb{F}) = 0$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows

$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(\Delta_{34}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then, we have $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{12} + b'\Delta_{13} + c'\Delta_{14} + d'\Delta_{23} + e'\Delta_{24} + f'\Delta_{34}$, for some $a', b', c', d', e', f' \in \mathbb{F}$. Then

$$\begin{aligned}
A\phi(y_1, y_2) &= \phi(Ay_1, Ay_2) \\
&= \phi(a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4, a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4) \\
&= (a_{11}a_{22} - a_{12}a_{21})a + (a_{11}a_{32} - a_{31}a_{12})b + (a_{11}a_{42} - a_{41}a_{12})c \\
&+ (a_{21}a_{32} - a_{31}a_{22})d + (a_{21}a_{42} - a_{41}a_{22})e + (a_{31}a_{42} - a_{41}a_{32})f, \text{ and}
\end{aligned}$$

$$\begin{aligned}
A\phi(y_1, y_3) &= \phi(Ay_1, Ay_3) \\
&= \phi(a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4, a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4) \\
&= (a_{11}a_{23} - a_{21}a_{13})a + (a_{11}a_{33} - a_{31}a_{13})b + (a_{11}a_{43} - a_{13}a_{41})c \\
&+ (a_{21}a_{33} - a_{31}a_{23})d + (a_{21}a_{43} - a_{23}a_{41})e + (a_{31}a_{43} - a_{33}a_{41})f, \text{ and}
\end{aligned}$$

$$\begin{aligned}A\phi(y_1, y_4) &= \phi(Ay_1, Ay_4)\\ &= \phi(a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4, a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4)\\ &= (a_{11}a_{24} - a_{14}a_{21})a + (a_{11}a_{34} - a_{14}a_{31})b + (a_{11}a_{44} - a_{14}a_{41})c\\ &+ (a_{21}a_{34} - a_{24}a_{31})d + (a_{21}a_{44} - a_{24}a_{41})e + (a_{31}a_{44} - a_{34}a_{41})f, \text{ and}\end{aligned}$$

$$\begin{aligned}A\phi(y_2, y_3) &= \phi(Ay_2, Ay_3)\\ &= \phi(a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4, a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4)\\ &= (a_{12}a_{23} - a_{22}a_{13})a + (a_{12}a_{33} - a_{32}a_{13})b + (a_{12}a_{43} - a_{13}a_{42})c\\ &+ (a_{22}a_{33} - a_{32}a_{23})d + (a_{22}a_{43} - a_{23}a_{42})e + (a_{32}a_{43} - a_{33}a_{42})f, \text{ and}\end{aligned}$$

$$\begin{aligned}A\phi(y_2, y_4) &= \phi(Ay_2, Ay_4)\\ &= \phi(a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4, a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4)\\ &= (a_{12}a_{24} - a_{14}a_{22})a + (a_{12}a_{34} - a_{14}a_{32})b + (a_{12}a_{44} - a_{14}a_{42})c\\ &+ (a_{22}a_{34} - a_{24}a_{32})d + (a_{22}a_{44} - a_{24}a_{42})e + (a_{32}a_{44} - a_{34}a_{42})f, \text{ and}\end{aligned}$$

$$\begin{aligned}A\phi(y_3, y_4) &= \phi(Ay_3, Ay_4)\\ &= \phi(a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4, a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4)\\ &= (a_{13}a_{24} - a_{14}a_{23})a + (a_{13}a_{34} - a_{14}a_{33})b + (a_{13}a_{44} - a_{14}a_{43})c\\ &+ (a_{23}a_{34} - a_{24}a_{33})d + (a_{23}a_{44} - a_{24}a_{43})e + (a_{33}a_{44} - a_{34}a_{43})f.\end{aligned}$$

Therefore, the action of $\mathrm{Aut}(L)$ on the set of ϕ's in the matrix form is as follows:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{32}-a_{31}a_{12} & a_{11}a_{42}-a_{41}a_{12} & a_{21}a_{32}-a_{31}a_{22} & a_{21}a_{42}-a_{41}a_{22} & a_{31}a_{42}-a_{41}a_{32} \\ a_{11}a_{23}-a_{21}a_{13} & a_{11}a_{33}-a_{31}a_{13} & a_{11}a_{43}-a_{13}a_{41} & a_{21}a_{33}-a_{31}a_{23} & a_{21}a_{43}-a_{23}a_{41} & a_{31}a_{43}-a_{33}a_{41} \\ a_{11}a_{24}-a_{14}a_{21} & a_{11}a_{34}-a_{14}a_{31} & a_{11}a_{44}-a_{14}a_{41} & a_{21}a_{34}-a_{24}a_{31} & a_{21}a_{44}-a_{24}a_{41} & a_{31}a_{44}-a_{34}a_{41} \\ a_{12}a_{23}-a_{22}a_{13} & a_{12}a_{33}-a_{32}a_{13} & a_{12}a_{43}-a_{13}a_{42} & a_{22}a_{33}-a_{32}a_{23} & a_{22}a_{43}-a_{23}a_{42} & a_{32}a_{43}-a_{33}a_{42} \\ a_{12}a_{24}-a_{14}a_{22} & a_{12}a_{34}-a_{14}a_{32} & a_{12}a_{44}-a_{14}a_{42} & a_{22}a_{34}-a_{24}a_{32} & a_{22}a_{44}-a_{24}a_{42} & a_{32}a_{44}-a_{34}a_{42} \\ a_{13}a_{24}-a_{14}a_{23} & a_{13}a_{34}-a_{14}a_{33} & a_{13}a_{44}-a_{14}a_{43} & a_{23}a_{34}-a_{24}a_{33} & a_{23}a_{44}-a_{24}a_{43} & a_{33}a_{44}-a_{34}a_{43} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}.$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,2}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. Then we can verify that the action of $\mathrm{Aut}(L)$ on the set of ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ a_{13}^p & a_{23}^p & a_{33}^p & a_{43}^p \\ a_{14}^p & a_{24}^p & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's.

Let $\nu = \begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix}1 & 0 & (-\beta/\delta)^{1/p} & 0 & (\alpha/\delta)^{1/p} & 0\\ 0 & 1 & (-\gamma/\delta)^{1/p} & 0 & 0 & (\alpha/\delta)^{1/p}\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & (-\gamma/\delta)^{1/p} & (\beta/\delta)^{1/p}\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0 & -\alpha/\delta\\ 0 & 1 & 0 & -\beta/\delta\\ 0 & 0 & 1 & -\gamma/\delta\\ 0 & 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}\right]$$

$$= \left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\delta)^{1/p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1/\delta)^{1/p} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/\delta)^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\delta \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix} 1 & (-\beta/\gamma)^{1/p} & 0 & (\alpha/\gamma)^{1/p} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & (-\alpha/\gamma)^{1/p} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (-\beta/\gamma)^{1/p} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha/\gamma & 0 \\ 0 & 1 & -\beta/\gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1/\gamma)^{1/p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1/\gamma)^{1/p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/\gamma)^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

Next, if $\delta = \gamma = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & (-\alpha/\beta)^{1/p} & 0 & 0 \\ 0 & 0 & 1 & 0 & (-\alpha/\beta)^{1/p} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha/\beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix}1&0&0&0&0&0\\0&\beta^{1/p}&0&0&0&0\\0&0&\beta^{1/p}&0&0&0\\0&0&0&(1/\beta)^{1/p}&0&0\\0&0&0&0&(1/\beta)^{1/p}&0\\0&0&0&0&0&1\end{pmatrix},\begin{pmatrix}\beta&0&0&0\\0&1/\beta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right].$$

Finally, if $\delta=\gamma=\beta=0$, but $\alpha\neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0&0&0\\0&\alpha^{-1/p}&0&0&0&0\\0&0&\alpha^{-1/p}&0&0&0\\0&0&0&\alpha^{1/p}&0&0\\0&0&0&0&\alpha^{1/p}&0\\0&0&0&0&0&1\end{pmatrix},\begin{pmatrix}1/\alpha&0&0&0\\0&\alpha&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Now, we find the restricted Lie algebra structure corresponding to the orbit representative $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. We have $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=f_1$ and hence $\omega=f_1y_5$. First, by Lemma 2.2.1 we get

$$\begin{aligned}
y_1^{[p]}&=y_1^{[p]}+\omega(y_1)=y_5,\\
y_2^{[p]}&=y_2^{[p]}+\omega(y_2)=0,\\
y_3^{[p]}&=y_3^{[p]}+\omega(y_3)=0,\\
y_4^{[p]}&=y_4^{[p]}+\omega(y_4)=0.
\end{aligned}$$

Therefore, the corresponding restricted Lie algebra structure is as follows:

$$K_2^2 = \langle y_1, \ldots, y_5 \mid [y_1, y_2] = y_5, y_1^{[p]} = y_5 \rangle.$$

Next, we use the substitutions $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_4$, $y_4 = x_5$, and $y_5 = x_3$. Hence, we have

$$K_2^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle.$$

Similarly, we can obtain the restricted Lie algebra structures corresponding to the other orbit representatives. Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$
$$K_2^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$
$$K_2^3 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3 \rangle;$$
$$K_2^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$
$$K_2^5 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_5^{[p]} = x_3 \rangle.$$

4.1.2 Extensions of $(L, y_1^{[p]} = y_2)$

Note that $L^{[p]} = \langle y_2 \rangle$. Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$. Hence, $\phi(x, y_2) = 0$ for all $x \in L$. Therefore, $\phi(y_1, y_2) = 0$ which implies that $a = 0$. Since $\phi = \Delta_{12}$ gives us $L_{5,2}$, we deduce by Lemma 4.1.1 that $L_{5,2}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also get $a = 0$.

I.3 $y_1^{[p]} = y_2, y_3^{[p]} = y_4$;

I.4 $y_1^{[p]} = y_2, y_2^{[p]} = y_3$;

I.5 $y_1^{[p]} = y_2, y_2^{[p]} = y_3, y_3^{[p]} = y_4$.

4.2 Extensions of $L = \frac{L_{5,2}}{\langle x_5 \rangle}$

In this section we find all non-isomorphic p-maps on $L_{5,2}$ such that $x_5^{[p]} = 0$. We let

$$L = \frac{L_{5,2}}{\langle x_5 \rangle} \cong L_{4,2},$$

where $L_{4,2} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle$. The group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Lemma 4.2.1 *Let $K = L_{5,2}$ and $[p] : K \to K$ be a p-map on K such that $x_5^{[p]} = 0$ and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (0, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$,

$1 \leqslant i, j \leqslant 4$. and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = [x_1, x_2] - \sigma(x_3) = 0;$$

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = 0.$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = 0$. Now, by Lemma 2.2.2, we have $\theta = (0, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K such that $x_5^{[p]} = 0$ can be obtained by an extension of L via $\theta = (0, \omega)$, for some ω.

Note that by [12], there are eight non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

II.1 Trivial p-map;

II.2 $x_1^{[p]} = x_3$;

II.3 $x_1^{[p]} = x_4$;

II.4 $x_1^{[p]} = x_3, x_2^{[p]} = x_4$;

II.5 $x_3^{[p]} = x_4$;

II.6 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$;

II.7 $x_4^{[p]} = x_3$;

II.8 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$.

In the following subsections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 4.2.1, it suffices to assume $[\theta] = [(0, \omega)]$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,2}$.

4.2.1 Extensions of (L, trivial p-map)

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = d = e = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. Since we want L_θ and $L_{5,2}$ to be isomorphic as Lie algebras, we should have $\phi = 0$. Since 0 is preserved under $\mathrm{Aut}(L)$, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's.

Let $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Then, $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We can verify that the action of $\mathrm{Aut}(L)$ on the set of ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{4.1}$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & -\alpha/\gamma & 0 \\ 0 & 1 & -\beta/\gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta/\gamma & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1/\gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\begin{pmatrix} 1 & 0 & 0 & -\alpha/\delta \\ 0 & 1 & 0 & -\beta/\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1 & -\alpha/\beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\beta & 0 & 0 \\ 0 & 0 & 1/\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, if $\gamma = \delta = \beta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$
$$K_2^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5 \rangle;$$
$$K_2^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5 \rangle;$$
$$K_2^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5 \rangle;$$
$$K_2^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_5 \rangle.$$

4.2.2 Extensions of $(L, x_1^{[p]} = x_3)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and

$\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_3) = 0$, for all $x \in L$ and hence $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_4, x_3) = 0$ which implies that $b = d = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{14}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + c\Delta_{14} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$\begin{aligned} a &= \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \phi(x_1, x_2) = \psi(x_3), \text{ and} \\ c &= \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0. \end{aligned}$$

Similarly, we can show that $e = 0$. Also, we have

$$\begin{aligned} \alpha &= \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_3), \text{ and} \\ \beta &= \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0. \end{aligned}$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_3) = a = \alpha$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_1)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_1)\rangle_{\mathbb{F}}$. Note that

$$\left\{[(\Delta_{12}, 0)], [(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\right\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_1)] = [(\Delta_{12}, f_1)] = [0]$, then $[(0, f_1)]$ is an scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 6$. Therefore,

$$\left\{[(\Delta_{12}, 0)], [(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\right\}$$

forms a basis for $H^2(L,\mathbb{F})$.

Let $[\theta] = [(\phi,\omega)] \in H^2(L,\mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix}\beta'\\ \gamma'\\ \delta'\end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & r^p & 0\\ 0 & a_{34}^p & a_{44}^p\end{pmatrix}\begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}.$$

Note that $\omega = \beta f_2 + \gamma f_3 + \delta f_4$. Since $\omega(x_1) = 0$, we have $A\omega(x_1) = 0$ which implies that $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$.

Now we find the representatives of the orbits of this action. Note that we need to have $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$. Let $\nu = \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix} 1 & -\beta/\gamma & 0\\ 0 & 1 & 0\\ 0 & -\delta/\gamma & 1\end{pmatrix}\begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1/\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}.$$

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\begin{pmatrix} 1 & 0 & -\beta/\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1/\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$

$$K_2^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_4^{[p]} = x_5 \rangle.$$

4.2.3 Extensions of $(L, x_1^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and

$\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$\begin{aligned} a &= \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and} \\ b &= \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0. \end{aligned}$$

Similarly, we can show that $d = 0$. Also, we have

$$\begin{aligned} \alpha &= \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_4), \text{ and} \\ \beta &= \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0. \end{aligned}$$

Similarly, we can show that $\gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, \alpha f_1)$ and hence $B^2(L, \mathbb{F}) = \langle (\Delta_{12}, 0), (0, f_1) \rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & r^p & 0 \\ 0 & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Let $\nu = \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix} 1 & -\beta/\gamma & 0 \\ 0 & 1 & 0 \\ 0 & -\delta/\gamma & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\begin{pmatrix} 1 & 0 & -\beta/\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1/\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{15} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4 \rangle;$$

$$K_2^{16} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{17} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{18} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

4.2.4 Extensions of $(L, x_1^{[p]} = x_3, x_2^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_3) = 0$ and $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $b = c = d = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3).$$

Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_3), \text{ and}$$
$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = \psi(x_4), \text{ and}$$
$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = 0.$$

Similarly, we can show that $\delta = 0$. Note that $\psi(x_3) = a = \alpha$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_1 + \beta f_2)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_1), (0, f_2)\rangle_{\mathbb{F}}$. Note that

$$\Big\{[(\Delta_{12}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\Big\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_1)] = [(\Delta_{12}, f_1)] = [0]$, then $[(0, f_1)]$ is an scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 3$. Therefore,

$$\Big\{[(\Delta_{12}, 0)], [(0, f_3)], [(0, f_4)]\Big\}$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} r^p & 0 \\ a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Note that we need to have $A\omega(x_1) = 0$ which implies that $a_{31}^p\gamma + a_{41}^p\delta = 0$. Let $\nu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^2$. If $\nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix} 1 & 0 \\ -\delta/\gamma & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1/\gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{19} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4 \rangle;$$
$$K_2^{20} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$
$$K_2^{21} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

4.2.5 Extensions of $(L, x_3^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$
$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $d = 0$. Also, we have

$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$
$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, \gamma f_3)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (0, f_3)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)].$$

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If

$\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & -\beta/\delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1 & -\alpha/\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next, if $\delta = \beta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{22} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4 \rangle;$$
$$K_2^{23} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$
$$K_2^{24} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$
$$K_2^{25} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

4.2.6 Extensions of $(L, x_2^{[p]} = x_3, x_3^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_3) = 0$ and $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $b = c = d = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3).$$

Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0, \text{ and}$$
$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = \psi(x_3), \text{ and}$$
$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$
$$\delta = \omega(x_4) = \tilde{\psi}(x_4) = \psi(x_4^{[p]}) = 0.$$

Note that $\psi(x_3) = a = \beta$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_2 + \gamma f_3)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_2), (0, f_3)\rangle_{\mathbb{F}}$. Note that

$$\left\{[(\Delta_{12}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)]\right\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_2)] = [(\Delta_{12}, f_2)] = [0]$, then $[(0, f_2)]$ is an scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 3$. Therefore,

$$\left\{[(\Delta_{12}, 0)], [(0, f_1)], [(0, f_4)]\right\}$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{41}^p \\ 0 & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Note that we need to have $A\omega(x_2) = 0$ which implies that $a_{21}^p\alpha + a_{42}^p\delta = 0$.

Let $\nu = \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \in \mathbb{F}^2$. If $\nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & -\alpha/\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{26} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{27} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{28} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

4.2.7 Extensions of $(L, x_4^{[p]} = x_3)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_3) = 0$, for all $x \in L$ and hence $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_4, x_3) = 0$ which implies that $b = d = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{14}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + c\Delta_{14} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$\begin{aligned} a &= \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and} \\ c &= \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0. \end{aligned}$$

Similarly, we can show that $e = 0$. Also, we have

$$\begin{aligned} \delta &= \omega(x_4) = \tilde{\psi}(x_4) = \psi(x_4^{[p]}) = \psi(x_3), \text{ and} \\ \alpha &= \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0. \end{aligned}$$

Similarly, we can show that $\beta = \gamma = 0$. Note that $\psi(x_3) = a = \delta$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_4)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_4)\rangle_{\mathbb{F}}$. Note that

$$\left\{[(\Delta_{12}, 0)], [(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\right\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_4)] = [(\Delta_{12}, f_4)] = [0]$, then $[(0, f_4)]$ is an scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 6$. Therefore,

$$\left\{[(\Delta_{12}, 0)], [(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)]\right\}$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix}\alpha'\\ \beta'\\ \gamma'\end{pmatrix} = \begin{pmatrix}a_{11}^p & a_{21}^p & a_{31}^p\\ a_{12}^p & a_{22}^p & a_{32}^p\\ 0 & 0 & r^p\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Note that we need to have $A\omega(x_4) = 0$ which implies that $a_{34}^p\gamma = 0$.

Let $\nu = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix}1 & 0 & -\alpha/\gamma\\ 0 & 1 & -\beta/\gamma\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \gamma\end{pmatrix}, \text{ and}$$

$$\begin{pmatrix}1/\gamma & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/\gamma\end{pmatrix}\begin{pmatrix}0\\ 0\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Next, if $\gamma = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix}1 & -\alpha/\beta & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \beta\\ 0\end{pmatrix}, \text{ and}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1/\beta & 0\\ 0 & 0 & 1/\beta\end{pmatrix}\begin{pmatrix}0\\ \beta\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}.$$

Finally, if $\gamma = \beta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{29} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$
$$K_2^{30} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$
$$K_2^{31} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$
$$K_2^{32} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle.$$

4.2.8 Extensions of $(L, x_2^{[p]} = x_4, x_4^{[p]} = x_3)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_3) = 0$ and $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $b = c = d = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L,\mathbb{F})$. Let $(\phi,\omega) \in B^2(L,\mathbb{F})$. Since $B^2(L,\mathbb{F}) \subseteq Z^2(L,\mathbb{F})$, we have $(\phi,\omega) = (a\Delta_{12}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1,x_2) = \delta^1\psi(x_1,x_2) = \psi([x_1,x_2]) = \psi(x_3).$$

Also, we have

$$\begin{aligned}
\alpha &= \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0, \text{ and} \\
\beta &= \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = \psi(x_4), \text{ and} \\
\gamma &= \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = 0, \text{ and} \\
\delta &= \omega(x_4) = \tilde{\psi}(x_4) = \psi(x_4^{[p]}) = \psi(x_3).
\end{aligned}$$

Note that $\psi(x_3) = a = \delta$. Therefore, $(\phi,\omega) = (a\Delta_{12}, \beta f_2 + a f_4)$ and hence $B^2(L,\mathbb{F}) = \langle(\Delta_{12}, f_4), (0, f_2)\rangle_{\mathbb{F}}$. Note that

$$\Big\{[(\Delta_{12},0)], [(0,f_1)], [(0,f_3)], [(0,f_4)]\Big\}$$

spans $H^2(L,\mathbb{F})$. Since $[(\Delta_{12},0)] + [(0,f_4)] = [(\Delta_{12},f_4)] = [0]$, then $[(0,f_4)]$ is an scalar multiple of $[(\Delta_{12},0)]$ in $H^2(L,\mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 3$. Therefore,

$$\Big\{[(\Delta_{12},0)], [(0,f_1)], [(0,f_3)]\Big\}$$

forms a basis for $H^2(L,\mathbb{F})$.

Let $[\theta] = [(\phi,\omega)] \in H^2(L,\mathbb{F})$. As in Section 4.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. We can verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{31}^p \\ 0 & r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Note that we need to have $A\omega(x_4) = 0$ which implies that $a_{34}^p\gamma = 0$.

Let $\nu = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \in \mathbb{F}^2$. If $\nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\gamma \neq 0$. Then

$$\begin{pmatrix} 1 & -\alpha/\gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\gamma \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Next, if $\gamma = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_2^{33} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$
$$K_2^{34} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$
$$K_2^{35} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle.$$

4.2.9 A list of restricted Lie algebra structures on $L_{5,2}$

Therefore, the list of all (possibly redundant) restricted Lie algebra structures on $L_{5,2}$ is as follows:

$$K_2^1 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$

$$K_2^2 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$

$$K_2^3 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3 \rangle;$$

$$K_2^4 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$

$$K_2^5 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_5^{[p]} = x_3 \rangle;$$

$$K_2^6 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$

$$K_2^7 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5 \rangle;$$

$$K_2^8 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5 \rangle;$$

$$K_2^9 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{10} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{11} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$

$$K_2^{12} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{13} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{14} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{15} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4 \rangle;$$

$$K_2^{16} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{17} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{18} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{19} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4 \rangle;$$

$$K_2^{20} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{21} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{22} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{23} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{24} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{25} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{26} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{27} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{28} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{29} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{30} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{31} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{32} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{33} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{34} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{35} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle.$$

4.3 Detecting Isomorphisms

We can easily see that some of the algebras given above are identical. The following is the list of all irredundant restricted Lie algebra structures on $L_{5,2}$ and yet, as we shall see below, we prove that some of them are isomorphic.

$$K_2^{1} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$

$$K_2^{2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$

$$K_2^{3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3 \rangle;$$

$$K_2^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$

$$K_2^5 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_5^{[p]} = x_3 \rangle;$$

$$K_2^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5 \rangle;$$

$$K_2^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5 \rangle;$$

$$K_2^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{15} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4 \rangle;$$

$$K_2^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_2^{17} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{18} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{19} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4 \rangle;$$

$$K_2^{20} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_2^{21} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{22} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{23} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{24} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{25} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{26} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{27} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle;$$

$$K_2^{28} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_2^{30} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{31} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{32} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{33} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{34} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$K_2^{35} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle.$$

The group $\mathrm{Aut}(L_{5,2})$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} & a_{34} & a_{35} \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{pmatrix},$$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Note that the following automorphism of $L_{5,2}$ that maps $x_1 \mapsto -x_2$, $x_2 \mapsto -x_1$, $x_3 \mapsto -x_3$, $x_4 \mapsto -x_5$ and $x_5 \mapsto -x_4$

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

implies that

$$K_2^{24} \cong K_2^{17}, \quad K_2^{26} \cong K_2^{13}, \quad K_2^{27} \cong K_2^{20}.$$

Consider the following automorphism of $L_{5,2}$ that maps $x_1 \mapsto -x_2$, $x_2 \mapsto -x_1$, $x_3 \mapsto -x_3$,

$x_4 \mapsto -x_4$ and $x_5 \mapsto -x_5$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Therefore,

$$K_2^3 \cong K_2^2, \quad K_2^8 \cong K_2^7, \quad K_2^{31} \cong K_2^{30}.$$

Finally, using the following automorphism of $L_{5,2}$ that only switches x_4 and x_5

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

we get

$$K_2^5 \cong K_2^4, \quad K_2^{15} \cong K_2^7, \quad K_2^{19} \cong K_2^{12}, \quad K_2^{23} \cong K_2^{17}.$$

Theorem 4.3.1 *The list of all restricted Lie algebra structures on $L_{5,2}$, up to isomorphism,* *is as follows:*

$$L_{5,2}^{1} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle;$$

$$L_{5,2}^{2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{4} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{6} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{7} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{8} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{9} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{15} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,2}^{17} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{18} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{19} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{20} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_2^{[p]} = x_4, x_4^{[p]} = x_3 \rangle;$$

$$L_{5,2}^{21} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5, x_4^{[p]} = x_3 \rangle.$$

In the remaining of this section we establish that the algebras given in Theorem 4.3.1 are pairwise non-isomorphic, thereby completing the proof of Theorem 4.3.1.

It is clear that $L_{5,2}^1$ is not isomorphic to the other restricted Lie algebras.

We claim that $L_{5,2}^2$ and $L_{5,2}^3$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^3 \to L_{5,2}^2$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]}$$
$$(a_{11}a_{22} - a_{12}a_{21})x_3 = 0.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. Note that $L_{5,2}^2 \ncong L_{5,2}^4$ because

$$L_{5,2}^2/(L_{5,2}^2)^{[p]} \ncong L_{5,2}^4/(L_{5,2}^4)^{[p]}. \tag{4.2}$$

Similarly, $L_{5,2}^2 \ncong L_{5,2}^5$ and $L_{5,2}^2 \ncong L_{5,2}^6$. It is clear that $L_{5,2}^2$ is not isomorphic to the other restricted Lie algebras.

Similar argument as in (4.2) shows that $L_{5,2}^3 \ncong L_{5,2}^4$, $L_{5,2}^3 \ncong L_{5,2}^5$, and $L_{5,2}^3 \ncong L_{5,2}^6$. It is clear that $L_{5,2}^3$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,2}^4$ and $L_{5,2}^5$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^4 \to L_{5,2}^5$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}$$
$$0 = (a_{11}a_{22} - a_{12}a_{21})^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

Next, we claim that $L_{5,2}^4$ and $L_{5,2}^6$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^6 \to L_{5,2}^4$. Then we have

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$

$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$

$$0 = a_{11}^p x_5.$$

Therefore, $a_{11} = 0$. Also we have

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$

$$0 = (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$

$$0 = a_{12}^p x_5.$$

Therefore, $a_{12} = 0$. Hence, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

It is clear that $L_{5,2}^4$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,2}^5$ and $L_{5,2}^6$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^6 \to L_{5,2}^5$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$

$$0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}$$

$$0 = (a_{11}a_{22} - a_{12}a_{21})^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

It is clear that $L_{5,2}^5$ and $L_{5,2}^6$ are not isomorphic to the other restricted Lie algebras.

Note that $L_{5,2}^7$ and $L_{5,2}^8$ are not isomorphic because $(L_{5,2}^7)^{[p]^2} = 0$ but $(L_{5,2}^8)^{[p]^2} \neq 0$. Also, $L_{5,2}^7$ and $L_{5,2}^9$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^9 \to L_{5,2}^7$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$

$$0 = (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$

$$0 = a_{12}^p x_3 + a_{22}^p x_5.$$

Therefore, $a_{12} = 0$ and $a_{22} = 0$. Hence, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

Similar argument as in (4.2) shows that $L^7_{5,2} \ncong L^{10}_{5,2}$, $L^7_{5,2} \ncong L^{11}_{5,2}$, $L^7_{5,2} \ncong L^{12}_{5,2}$, and $L^7_{5,2} \ncong L^{15}_{5,2}$. Next, we claim that $L^7_{5,2}$ and $L^{17}_{5,2}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{17}_{5,2} \to L^7_{5,2}$. Then

$$
\begin{aligned}
&A(x_4^{[p]}) = A(x_4)^{[p]} \\
&A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]} \\
&(a_{11}a_{22} - a_{12}a_{21})x_3 = 0.
\end{aligned}
$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

Next, we claim that $L^7_{5,2}$ and $L^{18}_{5,2}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^7_{5,2} \to L^{18}_{5,2}$. Then

$$
\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]} \\
&0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]} \\
&0 = (a_{11}a_{22} - a_{12}a_{21})^p x_5.
\end{aligned}
$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. Note that $L^7_{5,2}$ and $L^{19}_{5,2}$ are not isomorphic because $(L^7_{5,2})^{[p]^2} = 0$ but $(L^1_{5,2}9)^{[p]^2} \neq 0$. It is clear that $L^7_{5,2}$ is not isomorphic to the remaining restricted Lie algebras.

Note that $L^8_{5,2}$ is not isomorphic to any of $L^9_{5,2}$, $L^{10}_{5,2}$, $L^{11}_{5,2}$, $L^{17}_{5,2}$, $L^{18}_{5,2}$ because $(L^9_{5,2})^{[p]^2} = (L^{10}_{5,2})^{[p]^2} = (L^{11}_{5,2})^{[p]^2} = (L^{17}_{5,2})^{[p]^2} = (L^{18}_{5,2})^{[p]^2} = 0$ but $(L^8_{5,2})^{[p]^2} \neq 0$.

Similar argument as in (4.2) shows that $L^8_{5,2}$ is not isomorphic to any of $L^{12}_{5,2}$ or $L^{15}_{5,2}$.

Next, we claim that $L^8_{5,2}$ and $L^{19}_{5,2}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{19}_{5,2} \to L^{8}_{5,2}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]}$$
$$(a_{11}a_{22} - a_{12}a_{21})x_3 = a_{34}^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. It is clear that $L^{8}_{5,2}$ is not isomorphic to the other restricted Lie algebras.

Similar argument as in (4.2) shows that $L^{9}_{5,2}$ is not isomorphic to any of $L^{10}_{5,2}$, $L^{11}_{5,2}$, $L^{12}_{5,2}$, or $L^{15}_{5,2}$.

Next, we claim that $L^{9}_{5,2}$ and $L^{17}_{5,2}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{17}_{5,2} \to L^{9}_{5,2}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]}$$
$$(a_{11}a_{22} - a_{12}a_{21})x_3 = a_{44}^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. Note that $L^{9}_{5,2}$ is not isomorphic to $L^{18}_{5,2}$ nor $L^{19}_{5,2}$ because $(L^{9}_{5,2})^{[p]^2} = 0$ but $(L^{18}_{5,2})^{[p]^2} \neq 0$ and $(L^{19}_{5,2})^{[p]^2} \neq 0$. It is clear that $L^{9}_{5,2}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^{10}_{5,2}$ and $L^{11}_{5,2}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{10}_{5,2} \to L^{11}_{5,2}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}$$
$$0 = (a_{11}a_{22} - a_{12}a_{21})^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. Note that $L^{10}_{5,2}$ is not isomorphic to $L^{12}_{5,2}$, $L^{15}_{5,2}$, $L^{18}_{5,2}$ nor $L^{19}_{5,2}$ because $(L^{10}_{5,2})^{[p]^2} = 0$ but $(L^{12}_{5,2})^{[p]^2} \neq 0$, $(L^{15}_{5,2})^{[p]^2} \neq 0$, $(L^{18}_{5,2})^{[p]^2} \neq 0$ and $(L^{19}_{5,2})^{[p]^2} \neq 0$.

Next, we claim that $L_{5,2}^{10}$ and $L_{5,2}^{17}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{17} \to L_{5,2}^{10}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]}$$
$$(a_{11}a_{22} - a_{12}a_{21})x_3 = 0.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. It is clear that $L_{5,2}^{10}$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,2}^{11}$ is not isomorphic to any of $L_{5,2}^{12}$, $L_{5,2}^{15}$, $L_{5,2}^{18}$, or $L_{5,2}^{19}$ because $(L_{5,2}^{11})^{[p]^2} = 0$ but $(L_{5,2}^{12})^{[p]^2} \neq 0$, $(L_{5,2}^{15})^{[p]^2} \neq 0$, $(L_{5,2}^{18})^{[p]^2} \neq 0$ and $(L_{5,2}^{19})^{[p]^2} \neq 0$. Next, we claim that $L_{5,2}^{11}$ and $L_{5,2}^{17}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{17} \to L_{5,2}^{11}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]}$$
$$(a_{11}a_{22} - a_{12}a_{21})x_3 = a_{34}^p x_5.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. It is clear that $L_{5,2}^{11}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,2}^{12}$ and $L_{5,2}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{12} \to L_{5,2}^{15}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}$$
$$0 = (a_{11}a_{22} - a_{12}a_{21})^p x_4.$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

Similar argument as in (4.2) shows that $L_{5,2}^{12}$ is not isomorphic to any of to any of $L_{5,2}^{17}$, $L_{5,2}^{18}$, or $L_{5,2}^{19}$. It is clear that $L_{5,2}^{12}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,2}^{13}$ and $L_{5,2}^{14}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{14} \to L_{5,2}^{13}$. Then

$$\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]}\\
&0 = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}\\
&0 = (a_{11}a_{22} - a_{12}a_{21})^p x_5.
\end{aligned}$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

Note that $L_{5,2}^{13}$ is not isomorphic to any of $L_{5,2}^{16}$ or $L_{5,2}^{21}$ because $(L_{5,2}^{13})^{[p]^3} = 0$ but $(L_{5,2}^{16})^{[p]^3} \neq 0$ and $(L_{5,2}^{21})^{[p]^3} \neq 0$. Next, we claim that $L_{5,2}^{13}$ and $L_{5,2}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{13} \to L_{5,2}^{20}$. Then

$$\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]}\\
&A(x_5) = ((a_{11}a_{22} - a_{12}a_{21})x_3)^{[p]}\\
&a_{35}x_3 + a_{45}x_4 + a_{55}x_5 = 0.
\end{aligned}$$

Therefore, $a_{35} = 0$, $a_{45} = 0$, and $a_{55} = 0$ which is a contradiction. It is clear that $L_{5,2}^{13}$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,2}^{14}$ is not isomorphic to any of $L_{5,2}^{16}$ or $L_{5,2}^{21}$ because $(L_{5,2}^{14})^{[p]^3} = 0$ but $(L_{5,2}^{16})^{[p]^3} \neq 0$ and $(L_{5,2}^{21})^{[p]^3} \neq 0$. Next, we claim that $L_{5,2}^{14}$ and $L_{5,2}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{14} \to L_{5,2}^{20}$. Then

$$\begin{aligned}
&A(x_1^{[p]}) = A(x_1)^{[p]}\\
&A(x_3) = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}\\
&(a_{11}a_{22} - a_{12}a_{21})x_3 = a_{11}^p x_5 + a_{21}^p x_4 + a_{41}^p x_3,
\end{aligned}$$

which implies that $a_{11} = 0$ and $a_{21} = 0$. Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. It is clear that $L_{5,2}^{14}$ is not isomorphic to the other restricted Lie algebras.

Similar argument as in (4.2) shows that $L_{5,2}^{15}$ is not isomorphic to any of to any of $L_{5,2}^{17}$, $L_{5,2}^{18}$, or $L_{5,2}^{19}$. It is clear that $L_{5,2}^{15}$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,2}^{16}$ and $L_{5,2}^{20}$ are not isomorphic because $(L_{5,2}^{16})^{[p]^3} \neq 0$ but $(L_{5,2}^{20})^{[p]^3} = 0$. Next, we claim that $L_{5,2}^{16}$ and $L_{5,2}^{21}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{21} \to L_{5,2}^{16}$. Then

$$
\begin{aligned}
&A(x_4^{[p]}) = A(x_4)^{[p]} \\
&A(x_3) = (a_{34}x_3 + a_{44}x_4 + a_{54}x_5)^{[p]} \\
&(a_{11}a_{22} - a_{12}a_{21})x_3 = a_{34}^p x_4 + a_{44}^p x_5.
\end{aligned}
$$

Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction. It is clear that $L_{5,2}^{16}$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,2}^{17}$ is not isomorphic to any of $L_{5,2}^{18}$ or $L_{5,2}^{19}$ because $(L_{5,2}^{17})^{[p]^2} = 0$ but $(L_{5,2}^{18})^{[p]^2} \neq 0$ and $(L_{5,2}^{19})^{[p]^2} \neq 0$. It is clear that $L_{5,2}^{17}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,2}^{18}$ and $L_{5,2}^{19}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,2}^{18} \to L_{5,2}^{19}$. Then

$$
\begin{aligned}
&A(x_1^{[p]}) = A(x_1)^{[p]} \\
&0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]} \\
&0 = a_{21}^p x_4 + a_{41}^p x_3,
\end{aligned}
$$

which implies that, $a_{21} = 0$. Also we have

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$

$$0 = (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$

$$0 = a_{22}^p x_4 + a_{42}^p x_3,$$

which implies that $a_{22} = 0$. Therefore, $a_{11}a_{22} - a_{12}a_{21} = 0$ which is a contradiction.

It is clear that $L_{5,2}^{18}$ and $L_{5,2}^{19}$ are not isomorphic to the other restricted Lie algebras.

Finally, $L_{5,2}^{20}$ and $L_{5,2}^{21}$ are not isomorphic because $(L_{5,2}^{20})^{[p]^3} = 0$ but $(L_{5,2}^{21})^{[p]^3} \neq 0$.

Chapter 5

Restriction maps on $L_{5,3}$

Let

$$K_3 = L_{5,3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle.$$

We have $Z(L_{5,3}) = \langle x_4, x_5 \rangle_{\mathbb{F}}$. Note that there exists an element $\alpha x_4 + \beta x_5 \in Z(L_{5,3})$ such that $(\alpha x_4 + \beta x_5)^{[p]} = 0$, for some $\alpha, \beta \in \mathbb{F}$. If $\beta \neq 0$ then consider

$$K = \langle x_1', \ldots, x_5' \mid [x_1', x_2'] = x_3', [x_1', x_3'] = x_4' \rangle,$$

where $x_1' = x_1, x_2' = x_2, x_3' = x_3, x_4' = x_4, x_5' = \alpha x_4 + \beta x_5$. Let $\phi : K_3 \to K$ given by $x_i \mapsto x_i'$, for $1 \leqslant i \leqslant 5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_5^{[p]} = 0$. If $\beta = 0$ then $\alpha \neq 0$ and we rescale x_4 so that $x_4^{[p]} = 0$. Hence we have two cases:

I. $x_4^{[p]} = 0$;

II. $x_5^{[p]} = 0$.

Note that the group $\mathrm{Aut}(L_{5,3})$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2a_{22} & a_{45} \\ a_{51} & a_{52} & 0 & 0 & a_{55} \end{pmatrix}.$$

5.1 Extensions of $L = \frac{L_{5,3}}{\langle x_4 \rangle}$

In this section we find all non-isomorphic p-maps on $L_{5,3}$ such that $x_4^{[p]} = 0$. We let

$$L = \frac{L_{5,3}}{\langle x_4 \rangle} \cong L_{4,2},$$

where $L_{4,2} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle$. Note that we denote the image of x_i in L by x_i again. We rename x_4 with x_5 and x_5 with x_4 and at the end we will switch them. The group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Lemma 5.1.1 *Let $K = L_{5,3}$ and $[p] : K \to K$ be a p-map on K such that $x_4^{[p]} = 0$ and let $L = \frac{K}{M}$, where $M = \langle x_4 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{13}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = [x_1, x_3] = x_4;$$

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0.$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{13}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{13}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K for which $x_4^{[p]} = 0$ can be obtained by an extension of L via $\theta = (\Delta_{13}, \omega)$. Note that by [12], there are eight non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_3$;

I.3 $x_1^{[p]} = x_4$;

I.4 $x_1^{[p]} = x_3, x_2^{[p]} = x_4$;

I.5 $x_3^{[p]} = x_4$;

I.6 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$;

I.7 $x_4^{[p]} = x_3$;

I.8 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$.

In the following subsections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that, by Lemma 5.1.1, we can assume that $\phi = \Delta_{13}$.

5.1.1 Extensions of (L, trivial p-map)

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = d = e = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then, we have $\phi = a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$, for some $a, b, c, d \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{14} + c'\Delta_{23} + d'\Delta_{24}$, for some $a', b', c', d' \in \mathbb{F}$. We can verify that the action of $\mathrm{Aut}(L)$ on the set of ϕ's in the matrix form is as follows:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{5.1}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,3}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. Then we have

$$\begin{aligned} A\omega(x_1) =& \omega(Ax_1) = \omega(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4) = a_{11}^p\alpha + a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta; \\ A\omega(x_2) =& a_{12}^p\alpha + a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta; \\ A\omega(x_3) =& r^p\gamma; \\ A\omega(x_4) =& a_{34}^p\gamma + a_{44}^p\delta. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{5.2}$$

Thus, we can write Equations (5.1) and (5.2) together as follows:

$$\left[\begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \right]$$
$$= \left[\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's.

Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\gamma \neq 0$. Then

$$\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&-\alpha/\gamma&0\\0&1&-\beta/\gamma&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right],\text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&\gamma^{1/p}&0&0\\0&0&\gamma^{-3/p}&0\\0&0&0&\gamma^{-2/p}\end{pmatrix},\begin{pmatrix}\gamma&0&0&0\\0&\gamma^{-2}&0&0\\0&0&1/\gamma&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\\delta\end{pmatrix}\right].$$

Next, if $\delta \neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&(1/\delta)^{1/p}&0&0\\0&0&1&0\\0&0&0&(1/\delta)^{1/p}\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1/\delta\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right].$$

If $\delta = 0$, then we have $\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$.

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&-\alpha/\delta\\0&1&0&-\beta/\delta\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right],\text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&(1/\delta)^{1/p}&0&0\\0&0&1&0\\0&0&0&(1/\delta)^{1/p}\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1/\delta\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right].$$

Next, if $\gamma=\delta=0$, but $\beta\neq 0$, then

$$\left[\begin{pmatrix}1&0&(-\alpha/\beta)^{1/p}&0\\0&1&0&(-\alpha/\beta)^{1/p}\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&-\alpha/\beta&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right].$$

Finally, if $\gamma=\delta=\beta=0$, but $\alpha\neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&\alpha^{-1/p}&0&0\\0&0&\alpha^{3/p}&0\\0&0&0&\alpha^{2/p}\end{pmatrix},\begin{pmatrix}1/\alpha&0&0&0\\0&\alpha^2&0&0\\0&0&\alpha&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Lemma 5.1.2 *The vectors* $\begin{pmatrix}0\\\beta_1\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\\beta_2\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1}\in(\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}\right].$$

From this we obtain that $ra_{11} = 1$, $a_{34} = 0$, $a_{12} = 0$, $a_{21} = 0$, and $\beta_2 = a_{22}^p\beta_1$ which gives that $\beta_2\beta_1^{-1} = a_{11}^{-2p} \in (\mathbb{F}^*)^2$. Assume next that $\beta_2\beta_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon^{-1/p} & 0 & 0 \\ 0 & 0 & \epsilon^{3/p} & 0 \\ 0 & 0 & 0 & \epsilon^{2/p} \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}\right],$$

as required. ■

Therefore, the corresponding restricted Lie algebra structures are as follows: (Note that we need to switch x_4 and x_5.)

$$K_3^1 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle;$$

$$K_3^2 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_5^{[p]} = x_4\rangle;$$

$$K_3^3 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4\rangle;$$

$$K_3^4(\beta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4\rangle;$$

$$K_3^5 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4\rangle;$$

$$K_3^6 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_5^{[p]} = x_4\rangle.$$

Lemma 5.1.3 *We have $K_3^4(\beta_1) \cong K_3^4(\beta_2)$ if and only if $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.*

Proof. Suppose that $f : K_3^4(\beta_1) \to K_3^4(\beta_2)$ is an isomorphism. Then, f is an automorphism of $L_{5,3}$ and $f(\langle x_4\rangle_{\mathbb{F}}) = \langle x_4\rangle_{\mathbb{F}}$. Thus, f induces an automorphism $A : K_3^4(\beta_1)/\langle x_4\rangle \to K_3^4(\beta_2)/\langle x_4\rangle$ and, by Lemma 2.3.3, we have that $A\theta_1 = c\theta_2$, for some $c \in \mathbb{F}^*$, where $\theta_1 = (\Delta_{13}, \beta_1 f_2)$ and $\theta_2 = (\Delta_{13}, \beta_2 f_2)$ which implies that $c^{-1}A\theta_1 = \theta_2$. Therefore, without loss of generality, we can suppose that $c = 1$. So, $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit and it follows by Lemma 5.1.2 that $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$. The converse is easy to see by Lemma 2.3.3. ■

5.1.2 Extensions of $(L, x_1^{[p]} = x_3)$

Note that $L^{[p]} = \langle x_3\rangle$. Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$, for some $a, b, c, d \in \mathbb{F}$. Hence, $\phi(x, x_3) = 0$, for all $x \in L$. Therefore, $\phi(x_1, x_3) = 0$ which implies that $a = 0$. Since

$\phi = \Delta_{13}$ gives us $L_{5,3}$, we deduce by Lemma 5.1.1 that $L_{5,3}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also get $a = 0$.

I.4 $x_1^{[p]} = x_3, x_2^{[p]} = x_4$;

I.6 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$;

I.7 $x_4^{[p]} = x_3$;

I.8 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$.

5.1.3 Extensions of $(L, x_1^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a

linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$
$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_4), \text{ and}$$
$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0.$$

Similarly, we can show that $\gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, \alpha f_1)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (0, f_1)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then, we have $\phi = a\Delta_{13} + c\Delta_{23}$, for some $a, c \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + c'\Delta_{23}$, for some $a', c' \in \mathbb{F}$. We determine a', c'. Note that

$$\begin{aligned} A\phi(x_1, x_3) = \phi(Ax_1, Ax_3) &= \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, rx_3) \\ &= a_{11}ra + a_{21}rc; \text{ and} \\ A\phi(x_2, x_3) = \phi(Ax_2, Ax_3) &= \phi(a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, rx_3) \\ &= a_{12}ra + a_{22}rc. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ c' \end{pmatrix} = \begin{pmatrix} a_{11}r & a_{21}r \\ a_{12}r & a_{22}r \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}. \tag{5.3}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,3}$.

Note that we need to have $A\phi(x_1, x_4) = A\phi(x_2, x_4) = A\phi(x_3, x_4) = 0$ which implies that

$$a_{11}a_{34}a + a_{21}a_{34}c = 0,$$
$$a_{12}a_{34}a + a_{22}a_{34}c = 0.$$

Also, we have $\omega = \beta f_2 + \gamma f_3 + \delta f_4$, for some $\beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\beta', \gamma', \delta' \in \mathbb{F}$. We have

$$\begin{aligned} A\omega(x_2) &= a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta; \\ A\omega(x_3) &= r^p\gamma; \\ A\omega(x_4) &= a_{34}^p\gamma + a_{44}^p\delta. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & r^p & 0 \\ 0 & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{5.4}$$

Thus, we can write Equations (5.3) and (5.4) together as follows:

$$\left[r \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & r^p & 0 \\ 0 & a_{34}^p & a_{44}^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ c' \end{pmatrix}, \begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's and

$$a_{11}a_{34}a + a_{21}a_{34}c = 0, \quad a_{12}a_{34}a + a_{22}a_{34}c = 0.$$

Let $\nu = \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix}0\\0\\0\end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0&-\beta/\delta\\0&1&0\\0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1/\delta\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 1\end{pmatrix}\right].$$

If $\delta \neq 0$ and $\gamma \neq 0$, then

$$\left[(1/\gamma)^{1/p}\begin{pmatrix}(1/\gamma)^{-1/p}&0\\0&(1/\gamma)^{2/p}\end{pmatrix}, \begin{pmatrix}(1/\gamma)^2&0&0\\0&1/\gamma&0\\0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 1\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right].$$

If $\delta \neq 0$ but $\gamma = 0$, then we have $\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right]$. Next, if $\delta = 0$ but $\gamma \neq 0$, then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&-\beta/\gamma&0\\0&1&0\\0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\beta\\ \gamma\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix}\right], \text{ and}$$

$$\left[(1/\gamma)^{1/p}\begin{pmatrix}(1/\gamma)^{-1/p}&0\\0&(1/\gamma)^{2/p}\end{pmatrix}, \begin{pmatrix}(1/\gamma)^2&0&0\\0&1/\gamma&0\\0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right].$$

Next, if $\delta = \gamma = 0$, then we have $\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\beta\\0\\0\end{pmatrix}\right]$.

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}\beta\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Lemma 5.1.4 *The vectors* $\begin{pmatrix}\beta_1\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}\beta_2\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}\beta_1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\beta_2\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix},\begin{pmatrix}a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & r^p & 0\\ 0 & a_{34}^p & a_{44}^p\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\beta_1\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\beta_2\\0\\0\end{pmatrix}\right].$$

From this we obtain that $ra_{11} = 1$, $a_{12} = 0$, and $\beta_2 = a_{22}^p\beta_1$ which gives that $\beta_2\beta_1^{-1} = a_{11}^{-2p} \in (\mathbb{F}^*)^2$. Conversely, suppose that $\beta_2\beta_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p}\begin{pmatrix}\epsilon^{-1/p} & 0\\ 0 & \epsilon^{2/p}\end{pmatrix},\begin{pmatrix}\epsilon^2 & 0 & 0\\ 0 & \epsilon & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\beta_1\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\beta_2\\0\\0\end{pmatrix}\right]$$

as required. ■

Therefore, the corresponding restricted Lie algebra structures are as follows: (Note that we need to switch x_4 and x_5.)

$$K_3^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5 \rangle;$$
$$K_3^8(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_2^{[p]} = \beta x_4 \rangle;$$
$$K_3^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$
$$K_3^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_5^{[p]} = x_4 \rangle;$$
$$K_3^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4, x_5^{[p]} = x_4 \rangle$$

where $\beta \in \mathbb{F}^*$. Similarly, as in Lemma 5.1.3, we can prove that $K_3^8(\beta_1) \cong K_3^8(\beta_2)$ if and only if $\beta_2 \beta_1^{-1} \in (\mathbb{F}^*)^2$.

5.1.4 Extensions of $(L, x_3^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4) = f.$$

Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$.

So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = 0$. Also, we have

$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, \gamma f_3)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (0, f_3)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)].$$

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then $\phi = a\Delta_{13} + c\Delta_{23}$. We can verify that the action of $\mathrm{Aut}(L)$ on the ϕ's in the matrix form is as follows:

$$\begin{pmatrix} a' \\ c' \end{pmatrix} = \begin{pmatrix} a_{11}r & a_{21}r \\ a_{12}r & a_{22}r \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}. \tag{5.5}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,3}$. Note that we need to have $A\phi(x_1, x_4) = A\phi(x_2, x_4) = A\phi(x_3, x_4) = 0$ which implies that

$$a_{11}a_{34}a + a_{21}a_{34}c = 0,$$

$$a_{12}a_{34}a + a_{22}a_{34}c = 0.$$

Let $\omega = \alpha f_1 + \beta f_2 + \delta f_4$, for some $\alpha, \beta, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \delta' f_4$, for some $\alpha', \beta', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p\alpha + a_{21}^p\beta + a_{41}^p\delta;$$

$$A\omega(x_2) = a_{12}^p\alpha + a_{22}^p\beta + a_{42}^p\delta;$$

$$A\omega(x_4) = a_{44}^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}. \tag{5.6}$$

Thus, we can write Equations (5.5) and (5.6) together as follows:

$$\left[r \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{44}^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ c' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we take $a_{34} = 0$. Then we have

$$a_{11}a_{34}a + a_{21}a_{34}c = 0,$$

$$a_{12}a_{34}a + a_{22}a_{34}c = 0.$$

Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & -\beta/\delta \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right].$$

Next, if $\delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix}1 & (-\alpha/\beta)^{1/p}\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -\alpha/\beta & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta\\ 0\end{pmatrix}\right].$$

Finally, if $\delta = \beta = 0$, but $\alpha \neq 0$, then

$$\left[\alpha^{1/p}\begin{pmatrix}\alpha^{-1/p} & 0\\ 0 & \alpha^{2/p}\end{pmatrix}, \begin{pmatrix}1/\alpha & 0 & 0\\ 0 & \alpha^2 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\alpha\\ 0\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\ \beta\\0\end{pmatrix}.$$

Lemma 5.1.5 *The vectors* $\begin{pmatrix}0\\ \beta_1\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\ \beta_2\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}0\\ \beta_1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\ \beta_2\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}, \begin{pmatrix}a_{11}^p & a_{21}^p & a_{41}^p\\ a_{12}^p & a_{22}^p & a_{42}^p\\ 0 & 0 & a_{44}^p\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_1\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_2\\ 0\end{pmatrix}\right].$$

From this we obtain that $ra_{11} = 1$, $a_{12} = 0$, $a_{21} = 0$, and $\beta_2 = a_{22}^p \beta_1$ which gives that $\beta_2\beta_1^{-1} = a_{11}^{-2p} \in (\mathbb{F}^*)^2$. Conversely, assume that $\beta_2\beta_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p}\begin{pmatrix}\epsilon^{-1/p} & 0\\ 0 & \epsilon^{2/p}\end{pmatrix}, \begin{pmatrix}\epsilon^{-1} & 0 & 0\\ 0 & \epsilon^2 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_1\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_2\\ 0\end{pmatrix}\right],$$

as required. ■

Therefore, the corresponding restricted Lie algebra structures are as follows: (Note that we need to switch x_4 and x_5.)

$$K_3^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5\rangle;$$
$$K_3^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_3^{[p]} = x_5\rangle;$$
$$K_3^{14}(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4, x_3^{[p]} = x_5\rangle;$$
$$K_3^{15} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5, x_5^{[p]} = x_4\rangle$$

where $\beta \in \mathbb{F}^*$. Similarly, as in Lemma 5.1.3, we can prove that $K_3^{14}(\beta_1) \cong K_3^{14}(\beta_2)$ if and only if $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.

5.2 Extensions of $L = \frac{L_{5,3}}{\langle x_5 \rangle}$

In this section we find all non-isomorphic p-maps on $L_{5,3}$ such that $x_5^{[p]} = 0$. We let

$$L = \frac{L_{5,3}}{\langle x_5 \rangle} \cong L_{4,3},$$

where $L_{4,3} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle$. The group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}r \end{pmatrix},$$

where $r = a_{11}a_{22} \neq 0$.

Lemma 5.2.1 *Let $K = L_{5,3}$ and $[p] : K \to K$ be a p-map on K such that $x_5^{[p]} = 0$ and let $L = \frac{K}{M}$ where $M = \langle x_5\rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (0, \omega) \in Z^2(L, M)$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0;$$

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = 0.$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = 0$. Now, by Lemma 2.2.2, we have $\theta = (0, \omega) \in Z^2(L, M)$ and $K \cong L_\theta$. ∎

We deduce that any p-map on K such that $x_5^{[p]} = 0$ can be obtained by an extension of L via $\theta = (0, \omega)$.

Note that by [12], there are four non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

II.1 Trivial p-map;

II.2 $x_1^{[p]} = x_4$;

II.3 $x_2^{[p]} = \xi x_4$;

II.4 $x_3^{[p]} = x_4$.

5.2.1 Extensions of (L, trivial p-map)

First, we find a basis for $Z^2(L,\mathbb{F})$. Let $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L,\mathbb{F})$. Then we must have $\delta^2\phi(x,y,z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x,y,z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1,x_2,x_4) = \phi([x_1,x_2],x_4) + \phi([x_2,x_4],x_1) + \phi([x_4,x_1],x_2) = \phi(x_3,x_4), \text{ and}$$
$$0 = (\delta^2\phi)(x_1,x_2,x_3) = \phi([x_1,x_2],x_3) + \phi([x_2,x_3],x_1) + \phi([x_3,x_1],x_2) = \phi(x_2,x_4).$$

Thus, we get $f = e = 0$. Since the p-map is trivial, $\phi(x,y^{[p]}) = \phi(x,0) = 0$, for all $x,y \in L$. Therefore, a basis for $Z^2(L,\mathbb{F})$ is as follows:

$$(\Delta_{12},0),(\Delta_{13},0),(\Delta_{14},0),(\Delta_{23},0),(0,f_1),(0,f_2),(0,f_3),(0,f_4).$$

Next, we find a basis for $B^2(L,\mathbb{F})$. Let $(\phi,\omega) \in B^2(L,\mathbb{F})$. Since $B^2(L,\mathbb{F}) \subseteq Z^2(L,\mathbb{F})$, we have $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x,y \in L$.

So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and}$$

$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0$$

Similarly, we can show that $d = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (\Delta_{13}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. Since we want L_θ and $L_{5,3}$ to be isomorphic as Lie algebras, we should have $\phi = 0$. Since 0 is preserved under $\mathrm{Aut}(L)$, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's.

Let $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We can easily verify that the action of $\mathrm{Aut}(L)$ on the ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If

$\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & 0 & -\alpha/\delta \\ 0 & 1 & 0 & -\beta/\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma/\delta & \gamma^2/\delta^2 \\ 0 & 0 & 1 & -\gamma/\delta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\delta & 0 & 0 \\ 0 & 0 & 1/\delta & 0 \\ 0 & 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\begin{pmatrix} 1 & 0 & -\alpha/\gamma & 0 \\ 0 & 1 & -\beta/\gamma & 0 \\ 0 & 0 & 1 & -\beta/\gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} (1/\gamma)^2 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & (1/\gamma)^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1 & -\alpha/\beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} (1/\beta)^{-2} & 0 & 0 & 0 \\ 0 & 1/\beta & 0 & 0 \\ 0 & 0 & (1/\beta)^{-1} & 0 \\ 0 & 0 & 0 & (1/\beta)^{-3} \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, if $\delta = \gamma = \beta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & (1/\alpha)^2 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_3^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle;$$
$$K_3^{17} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5 \rangle;$$
$$K_3^{18} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5 \rangle;$$
$$K_3^{19} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5 \rangle;$$
$$K_3^{20} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_4^{[p]} = x_5 \rangle.$$

5.2.2 Extensions of $(L, x_1^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4), \text{ and}$$
$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4).$$

Thus, we get $f = e = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$.

So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and}$$

$$c = \phi(x_2, x_3) = \delta^1\psi(x_2, x_3) = \psi([x_2, x_3]) = 0.$$

Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_4), \text{ and}$$

$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0.$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_4) = b = \alpha$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, bf_1)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{13}, f_1), (\Delta_{12}, 0)\rangle_{\mathbb{F}}$. Note that

$$\Big\{[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\Big\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{13}, 0)] + [(0, f_1)] = [(\Delta_{13}, f_1)] = [0]$, then $[(0, f_1)]$ is a scalar multiple of $[(\Delta_{13}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$\Big\{[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\Big\}$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 5.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's.

Let $\omega = \beta f_2 + \gamma f_3 + \delta f_4$, for some $\beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\beta', \gamma', \delta' \in \mathbb{F}$. We have

$$A\omega(x_2) = a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta;$$

$$A\omega(x_3) = r^p\gamma + a_{11}^p a_{32}^p\delta;$$

$$A\omega(x_4) = a_{11}^p r^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Note that we need to have $A\omega(x_1) = 0$ which implies that $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$. Now we find the representatives of the orbits of this action. Note that we take $a_{21} = a_{31} = a_{41} = 0$. Therefore, $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$. Let $\nu = \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & -\beta/\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & -\gamma/\delta & \gamma^2/\delta^2 \\ 0 & 1 & -\gamma/\delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1/\delta & 0 & 0 \\ 0 & 1/\delta & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\begin{pmatrix} 1 & -\beta/\gamma & 0 \\ 0 & 1 & -\beta/\gamma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} (1/\gamma)^{-1} & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & (1/\gamma)^3 \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} (1/\beta) & 0 & 0 \\ 0 & (1/\beta)^{-1} & 0 \\ 0 & 0 & (1/\beta)^{-3} \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_3^{21} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4 \rangle;$$
$$K_3^{22} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$
$$K_3^{23} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$
$$K_3^{24} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

5.2.3 Extensions of $(L, x_2^{[p]} = \xi x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4)$, and

$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4)$.

Thus, we get $f = e = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$\begin{aligned} a &= \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and} \\ b &= \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and} \\ c &= \phi(x_2, x_3) = \delta^1\psi(x_2, x_3) = \psi([x_2, x_3]) = 0. \end{aligned}$$

Also, we have

$$\begin{aligned} \beta &= \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = \xi\psi(x_4), \text{ and} \\ \alpha &= \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0. \end{aligned}$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_4) = b = \beta\xi^{-1}$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, b\xi f_2)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{13}, \xi f_2), (\Delta_{12}, 0)\rangle_{\mathbb{F}}$. Note that

$$\Big\{[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)]\Big\}$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{13}, 0)] + \xi[(0, f_2)] = [(\Delta_{13}, \xi f_2)] = [0]$, then $[(0, f_2)]$ is a scalar multiple of $[(\Delta_{13}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$\Big\{[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_3)], [(0, f_4)]\Big\}$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$. As in Section 5.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's.

Let $\omega = \alpha f_1 + \gamma f_3 + \delta f_4$, for some $\alpha, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \gamma', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p \alpha + a_{31}^p \gamma + a_{41}^p \delta;$$

$$A\omega(x_3) = r^p \gamma + a_{11}^p a_{32}^p \delta;$$

$$A\omega(x_4) = a_{11}^p r^p \delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{31}^p & a_{41}^p \\ 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of this action. Note that we need to have $A\omega(x_2) = 0$ which implies that $a_{32}^p \gamma + a_{42}^p \delta = 0$.

Let $\nu = \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & -\gamma/\delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\delta & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\begin{pmatrix} 1 & -\alpha/\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & 1/\gamma \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, if $\gamma = \delta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_3^{25}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4 \rangle;$$

$$K_3^{26}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_2^{[p]} = \xi x_4 \rangle;$$

$$K_3^{27}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_3^{28}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4, x_4^{[p]} = x_5 \rangle$$

where $\xi \in \mathbb{F}^*$. Note that $K_3^{25}(\xi)$ is identical to $K_3^4(\beta)$, $K_3^{26}(\xi)$ is identical to $K_3^8(\beta)$, and $K_3^{27}(\xi)$ is identical to $K_3^{14}(\beta)$.

Lemma 5.2.2 *We have* $K_3^{28}(\xi_1) \cong K_3^{28}(\xi_2)$ *if and only if* $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $K_3^{28}(\xi_1) \cong K_3^{28}(\xi_2)$. Then there exists an isomorphism $f : K_3^{28}(\xi_1) \to K_3^{28}(\xi_2)$. Therefore, we have $f(x_2^{[p]}) = f(x_2)^{[p]}$ which implies that

$$A(\xi_1 x_4) = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$

Hence, $\xi_1 a_{11}^2 a_{22} x_4 = a_{22}^p \xi_2 x_4 + a_{42}^p x_5$. We deduce that $\xi_2\xi_1^{-1} = a_{11}^2 a_{22}^{1-p} \in (\mathbb{F}^*)^2$. Conversely, assume that $\xi_2\xi_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then the following automorphism of $L_{5,3}$

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{2p} \end{pmatrix}$$

is an isomorphism between $K_3^{28}(\xi_1)$ and $K_3^{28}(\xi_2)$. ■

5.2.4 Extensions of $(L, x_3^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4)$, and

$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4)$.

Thus, we get $f = e = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L,\mathbb{F})$. Let $(\phi,\omega) \in B^2(L,\mathbb{F})$. Since $B^2(L,\mathbb{F}) \subseteq Z^2(L,\mathbb{F})$, we have $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1,x_2) = \delta^1\psi(x_1,x_2) = \psi([x_1,x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1,x_3) = \delta^1\psi(x_1,x_3) = \psi([x_1,x_3]) = \psi(x_4), \text{ and}$$

$$c = \phi(x_2,x_3) = \delta^1\psi(x_2,x_3) = \psi([x_2,x_3]) = 0.$$

Also, we have

$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \delta = 0$. Note that $\psi(x_4) = b = \gamma$. Therefore, $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13}, bf_3)$ and hence $B^2(L,\mathbb{F}) = \langle(\Delta_{13}, f_3), (\Delta_{12}, 0)\rangle_{\mathbb{F}}$. Note that

$$\left\{[(\Delta_{13},0)], [(\Delta_{23},0)], [(0,f_1)], [(0,f_2)], [(0,f_3)], [(0,f_4)]\right\}$$

spans $H^2(L,\mathbb{F})$. Since $[(\Delta_{13},0)] + [(0,f_3)] = [(\Delta_{13},f_3)] = [0]$, then $[(0,f_3)]$ is a scalar multiple of $[(\Delta_{13},0)]$ in $H^2(L,\mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$\left\{[(\Delta_{13},0)], [(\Delta_{23},0)], [(0,f_1)], [(0,f_2)], [(0,f_4)]\right\}$$

forms a basis for $H^2(L,\mathbb{F})$.

Let $[\theta] = [(\phi,\omega)] \in H^2(L,\mathbb{F})$. As in Section 5.2.1, it is enough to find $\mathrm{Aut}(L)$-representatives of the ω's. Let $\omega = \alpha f_1 + \beta f_2 + \delta f_4$, for some $\alpha, \beta, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \delta' f_4$, for some $\alpha', \beta', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p\alpha + a_{21}^p\beta + a_{41}^p\delta;$$

$$A\omega(x_2) = a_{22}^p\beta + a_{42}^p\delta;$$

$$A\omega(x_4) = a_{11}^p r^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ 0 & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}.$$

Note that we need to have $A\omega(x_3) = 0$ which implies that $a_{11}^p a_{32}^p \delta = 0$. Now we find the representatives of the orbits of this action. Note that we take $a_{32} = 0$. Therefore, $a_{11}^p a_{32}^p \delta = 0$. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & -\beta/\delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1/\delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, if $\delta = 0$, but $\beta \neq 0$, then

$$\begin{pmatrix} 1 & -\alpha/\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, if $\beta = \delta = 0$, but $\alpha \neq 0$, then

$$\begin{pmatrix} 1/\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1/\alpha)^2 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_3^{29} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4 \rangle;$$

$$K_3^{30} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_3^{31} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_3^{32} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

5.2.5 A list of restricted Lie algebra structures on $L_{5,3}$

Therefore, the list of all (possibly redundant) restricted Lie algebra structures on $L_{5,3}$ is as follows:

$$K_3^{1} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle;$$

$$K_3^{2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_5^{[p]} = x_4 \rangle;$$

$$K_3^{3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4 \rangle;$$

$$K_3^{4}(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4 \rangle;$$

$$K_3^{5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4 \rangle;$$

$K_3^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_5^{[p]} = x_4 \rangle;$

$K_3^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5 \rangle;$

$K_3^8(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_2^{[p]} = \beta x_4 \rangle;$

$K_3^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$

$K_3^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_5^{[p]} = x_4 \rangle;$

$K_3^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4, x_5^{[p]} = x_4 \rangle;$

$K_3^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{14}(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{15} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5, x_5^{[p]} = x_4 \rangle;$

$K_3^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle;$

$K_3^{17} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5 \rangle;$

$K_3^{18} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5 \rangle;$

$K_3^{19} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{20} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_4^{[p]} = x_5 \rangle;$

$K_3^{21} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4 \rangle;$

$K_3^{22} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$

$K_3^{23} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{24} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$

$K_3^{25}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4 \rangle;$

$K_3^{26}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_2^{[p]} = \xi x_4 \rangle;$

$K_3^{27}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4, x_3^{[p]} = x_5 \rangle;$

$K_3^{28}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4, x_4^{[p]} = x_5 \rangle;$

$K_3^{29} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4 \rangle;$

$K_3^{30} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$

$K_3^{31} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$

$K_3^{32} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$

5.3 Detecting isomorphisms

We can easily see that some of the algebras above are identical.

Theorem 5.3.1 *The list of all restricted Lie algebra structures on $L_{5,3}$, up to isomorphism, is as follows:*

$$L_{5,3}^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle;$$

$$L_{5,3}^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_5^{[p]} = x_4 \rangle;$$

$$L_{5,3}^3 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4 \rangle;$$

$$L_{5,3}^4(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4 \rangle;$$

$$L_{5,3}^5 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4 \rangle;$$

$$L_{5,3}^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_5^{[p]} = x_4 \rangle;$$

$$L_{5,3}^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5 \rangle;$$

$$L_{5,3}^8(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_2^{[p]} = \beta x_4 \rangle;$$

$$L_{5,3}^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$L_{5,3}^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_5^{[p]} = x_4 \rangle;$$

$$L_{5,3}^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{13}(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_5, x_5^{[p]} = x_4 \rangle;$$

$$L_{5,3}^{15} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{17} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{18} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{19}(\xi) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \xi x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{20} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$L_{5,3}^{21} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$L_{5,3}^{22} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_5, x_3^{[p]} = x_4, x_5^{[p]} = x_4 \rangle.$$

In the remaining of this section we establish that the algebras given in Theorem 5.3.1 are pairwise non-isomorphic, thereby completing the proof of Theorem 5.3.1.

It is clear that $L_{5,3}^1$ is not isomorphic to the other restricted Lie algebras. We claim that $L_{5,3}^2$ and $L_{5,3}^3$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^2 \to L_{5,3}^3$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^2$ and $L_{5,3}^4(\beta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^2 \to L_{5,3}^4(\beta)$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^2_{5,3}$ and $L^5_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,3} \to L^5_{5,3}$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^2_{5,3}$ and $L^6_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,3} \to L^6_{5,3}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$A(x_4) = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0,$$

which implies that $a_{11}^2 a_{22} = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^2_{5,3}$ and $L^7_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,3} \to L^7_{5,3}$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^2$ and $L_{5,3}^{11}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^2 \to L_{5,3}^{11}$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^2$ and $L_{5,3}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^2 \to L_{5,3}^{15}$. Then

$$A(x_5^{[p]}) = A(x_5)^{[p]}$$
$$A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^2$ and $L_{5,3}^{16}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^2 \to L_{5,3}^{16}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^2$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^3_{5,3}$ and $L^4_{5,3}(\beta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^4_{5,3}(\beta)$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p \beta x_4,$$

which implies that $a_{22}^p\beta = 0$. Since, $\beta \neq 0$, we have $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^5_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^5_{5,3}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_4.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^6_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^6_{5,3}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$A(x_4) = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = 0,$$

which implies that $a_{11}^2 a_{22} = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^7_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^7_{5,3}$. Then

$$\begin{aligned}
&A(x_1^{[p]}) = A(x_1)^{[p]} \\
&0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]} \\
&0 = a_{11}^p x_5.
\end{aligned}$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^{11}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^{11}_{5,3}$. Then

$$\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]} \\
&A(x_4) = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]} \\
&a_{11}^2 a_{22}x_4 = a_{11}^p a_{22}^p x_5,
\end{aligned}$$

which implies that $a_{11}^2 a_{22} = 0$ and $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^{15}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^{15}_{5,3}$. Then

$$\begin{aligned}
&A(x_2^{[p]}) = A(x_2)^{[p]} \\
&0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\
&0 = a_{22}^p x_5.
\end{aligned}$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,3}$ and $L^{16}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,3} \to L^{16}_{5,3}$. Then

$$\begin{aligned}
A(x_4^{[p]}) &= A(x_4)^{[p]} \\
0 &= (a_{11}^2 a_{22} x_4)^{[p]} \\
0 &= a_{11}^{2p} a_{22}^p x_5.
\end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L^3_{5,3}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^5_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,3}(\beta) \to L^5_{5,3}$. Then

$$\begin{aligned}
A(x_1^{[p]}) &= A(x_1)^{[p]} \\
0 &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]} \\
0 &= a_{11}^p x_4.
\end{aligned}$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^6_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^6_{5,3} \to L^4_{5,3}(\beta)$. Then

$$\begin{aligned}
A(x_5^{[p]}) &= A(x_5)^{[p]} \\
A(x_4) &= (a_{45}x_4 + a_{55}x_5)^{[p]} \\
a_{11}^2 a_{22} x_4 &= 0.
\end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^7_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,3}(\beta) \to L^7_{5,3}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^{11}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,3}(\beta) \to L^{11}_{5,3}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^{15}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,3}(\beta) \to L^{15}_{5,3}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$A(x_4) = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$a_{11}^2 a_{22}x_4 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^4_{5,3}(\beta)$ and $L^{16}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,3}(\beta) \to L^{16}_{5,3}$. Then

$$\begin{aligned} &A(x_4^{[p]}) = A(x_4)^{[p]} \\ &0 = (a_{11}^2 a_{22} x_4)^{[p]} \\ &0 = a_{11}^{2p} a_{22}^p x_5. \end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L^4_{5,3}(\beta)$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^5_{5,3}$ and $L^6_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^6_{5,3} \to L^5_{5,3}$. Then

$$\begin{aligned} &A(x_5^{[p]}) = A(x_5)^{[p]} \\ &A(x_4) = (a_{45}x_4 + a_{55}x_5)^{[p]} \\ &a_{11}^2 a_{22} x_4 = 0. \end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^5_{5,3}$ and $L^7_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^5_{5,3} \to L^7_{5,3}$. Then

$$\begin{aligned} &A(x_1^{[p]}) = A(x_1)^{[p]} \\ &A(x_4) = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]} \\ &a_{11}^2 a_{22} x_4 = a_{11}^p x_5. \end{aligned}$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^5$ and $L_{5,3}^{11}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^5 \to L_{5,3}^{11}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^5$ and $L_{5,3}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^5 \to L_{5,3}^{15}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^5$ and $L_{5,3}^{16}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^5 \to L_{5,3}^{16}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22}x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^5$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,3}^6$ and $L_{5,3}^7$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^6 \to L_{5,3}^7$. Then

$$\begin{aligned} A(x_1^{[p]}) &= A(x_1)^{[p]} \\ 0 &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]} \\ 0 &= a_{11}^p x_5. \end{aligned}$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^6$ and $L_{5,3}^{11}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^6 \to L_{5,3}^{11}$. Then

$$\begin{aligned} A(x_3^{[p]}) &= A(x_3)^{[p]} \\ 0 &= (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]} \\ 0 &= a_{11}^p a_{22}^p x_5, \end{aligned}$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^6$ and $L_{5,3}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^6 \to L_{5,3}^{15}$. Then

$$\begin{aligned} A(x_2^{[p]}) &= A(x_2)^{[p]} \\ 0 &= (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\ 0 &= a_{22}^p x_5. \end{aligned}$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^6$ and $L_{5,3}^{16}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^6 \to L_{5,3}^{16}$. Then

$$\begin{aligned} A(x_4^{[p]}) &= A(x_4)^{[p]} \\ 0 &= (a_{11}^2 a_{22} x_4)^{[p]} \\ 0 &= a_{11}^{2p} a_{22}^p x_5. \end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^6$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,3}^7$ and $L_{5,3}^{11}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^7 \to L_{5,3}^{11}$. Then

$$\begin{aligned} A(x_3^{[p]}) &= A(x_3)^{[p]} \\ 0 &= (a_{11} a_{22} x_3 + a_{11} a_{32} x_4)^{[p]} \\ 0 &= a_{11}^p a_{22}^p x_5, \end{aligned}$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^7$ and $L_{5,3}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^7 \to L_{5,3}^{15}$. Then

$$\begin{aligned} A(x_2^{[p]}) &= A(x_2)^{[p]} \\ 0 &= (a_{22} x_2 + a_{32} x_3 + a_{42} x_4 + a_{52} x_5)^{[p]} \\ 0 &= a_{22}^p x_5. \end{aligned}$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^7_{5,3}$ and $L^{16}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^7_{5,3} \to L^{16}_{5,3}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L^7_{5,3}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^8_{5,3}(\beta)$ and $L^9_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^8_{5,3}(\beta) \to L^9_{5,3}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Note that $L^8_{5,3}(\beta)$ is not isomorphic to any of $L^{10}_{5,3}$, $L^{14}_{5,3}$, $L^{18}_{5,3}$, $L^{19}_{5,3}(\xi)$, $L^{21}_{5,3}$, $L^{22}_{5,3}$ because $(L^8_{5,3})^{[p]^2} = 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L^8_{5,3}(\beta)$ and $L^{12}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^8_{5,3}(\beta) \to L^{12}_{5,3}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^8(\beta)$ and $L_{5,3}^{13}(\beta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^8(\beta) \to L_{5,3}^{13}(\beta)$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^8(\beta)$ and $L_{5,3}^{17}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^8(\beta) \to L_{5,3}^{17}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$A(\beta x_4) = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$\beta a_{11}^2 a_{22} x_4 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^8(\beta)$ and $L_{5,3}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^8(\beta) \to L_{5,3}^{20}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^8(\beta)$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,3}^9$ is not isomorphic to any of $L_{5,3}^{10}$, $L_{5,3}^{14}$, $L_{5,3}^{18}$, $L_{5,3}^{19}(\xi)$, $L_{5,3}^{21}$, $L_{5,3}^{22}$ because $(L_{5,3}^9)^{[p]^2} = 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L_{5,3}^9$ and $L_{5,3}^{12}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^9 \to L_{5,3}^{12}$. Then

$$
\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]} \\
&A(x_4) = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]} \\
&a_{11}^2 a_{22}x_4 = a_{11}^p a_{22}^p x_5,
\end{aligned}
$$

which implies that $a_{11}^2 a_{22} = 0$ and $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^9$ and $L_{5,3}^{13}(\beta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^9 \to L_{5,3}^{13}(\beta)$. Then

$$
\begin{aligned}
&A(x_2^{[p]}) = A(x_2)^{[p]} \\
&0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\
&0 = a_{22}^p \beta x_4 + a_{32}^p x_5,
\end{aligned}
$$

which implies that $a_{22}^p \beta = 0$. Since, $\beta \neq 0$ we have $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^9$ and $L_{5,3}^{17}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^9 \to L_{5,3}^{17}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^9$ and $L_{5,3}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^9 \to L_{5,3}^{20}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5 + a_{32}^p x_4.$$

Therefore, $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^9$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,3}^{10}$ is not isomorphic to any of $L_{5,3}^{12}$, $L_{5,3}^{13}(\beta)$, $L_{5,3}^{17}$, $L_{5,3}^{20}$ because $(L_{5,3}^{10})^{[p]^2} \neq 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L_{5,3}^{10}$ and $L_{5,3}^{14}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{10} \to L_{5,3}^{14}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{10}$ and $L_{5,3}^{18}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{10} \to L_{5,3}^{18}$. Then

$$\begin{aligned}
&A(x_4^{[p]}) = A(x_4)^{[p]} \\
&0 = (a_{11}^2 a_{22} x_4)^{[p]} \\
&0 = a_{11}^{2p} a_{22}^p x_5.
\end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{10}$ and $L_{5,3}^{19}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{10} \to L_{5,3}^{19}(\xi)$. Then

$$\begin{aligned}
&A(x_4^{[p]}) = A(x_4)^{[p]} \\
&0 = (a_{11}^2 a_{22} x_4)^{[p]} \\
&0 = a_{11}^{2p} a_{22}^p x_5.
\end{aligned}$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{10}$ and $L_{5,3}^{21}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{10} \to L_{5,3}^{21}$. Then

$$\begin{aligned}
&A(x_3^{[p]}) = A(x_3)^{[p]} \\
&0 = (a_{11} a_{22} x_3 + a_{11} a_{32} x_4)^{[p]} \\
&0 = a_{11}^p a_{22}^p x_4 + a_{11}^p a_{32}^p x_5,
\end{aligned}$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{10}$ and $L_{5,3}^{22}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{10} \to L_{5,3}^{22}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^{10}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,3}^{11}$ and $L_{5,3}^{15}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{11} \to L_{5,3}^{15}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{11}$ and $L_{5,3}^{16}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{11} \to L_{5,3}^{16}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22}x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^{11}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,3}^{12}$ and $L_{5,3}^{13}(\beta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{12} \to L_{5,3}^{13}(\beta)$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p \beta x_4 + a_{32}^p x_5,$$

which implies that $a_{22}^p \beta \neq 0$. Since, $\beta \neq 0$ we have $a_{22} = 0$ which is a contradiction.

Note that $L_{5,3}^{12}$ is not isomorphic to any of $L_{5,3}^{14}$, $L_{5,3}^{18}$, $L_{5,3}^{19}(\xi)$, $L_{5,3}^{21}$, $L_{5,3}^{22}$ because $(L_{5,3}^{12})^{[p]^2} = 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L_{5,3}^{12}$ and $L_{5,3}^{17}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{12} \to L_{5,3}^{17}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{12}$ and $L_{5,3}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{12} \to L_{5,3}^{20}$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_5 + a_{32}^p x_4.$$

Therefore, $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^{12}$ is not isomorphic to the other restricted Lie algebras.

Note that $L_{5,3}^{13}$ is not isomorphic to any of $L_{5,3}^{14}$, $L_{5,3}^{18}$, $L_{5,3}^{19}(\xi)$, $L_{5,3}^{21}$, $L_{5,3}^{22}$ because $(L_{5,3}^{13})^{[p]^2} = 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L^{13}_{5,3}(\beta)$ and $L^{17}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{13}_{5,3}(\beta) \to L^{17}_{5,3}$. Then

$$
\begin{aligned}
&A(x_1^{[p]}) = A(x_1)^{[p]}\\
&0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}\\
&0 = a_{11}^p x_4 + a_{21}^p x_5.
\end{aligned}
$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^{13}_{5,3}(\beta)$ and $L^{20}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{13}_{5,3}(\beta) \to L^{20}_{5,3}$. Then

$$
\begin{aligned}
&A(x_2^{[p]}) = A(x_2)^{[p]}\\
&A(\beta x_4) = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}\\
&\beta a_{11}^2 a_{22} x_4 = a_{22}^p x_5 + a_{32}^p x_4.
\end{aligned}
$$

Therefore, $a_{22} = 0$ which is a contradiction.

It is clear that $L^{13}_{5,3}(\beta)$ is not isomorphic to the other restricted Lie algebras.

Note that $L^{14}_{5,3}$ is not isomorphic to any of $L^{17}_{5,3}$, $L^{20}_{5,3}$ because $(L^{14}_{5,3})^{[p]^2} \neq 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L^{14}_{5,3}$ and $L^{18}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{14}_{5,3} \to L^{18}_{5,3}$. Then

$$
\begin{aligned}
&A(x_4^{[p]}) = A(x_4)^{[p]}\\
&0 = (a_{11}^2 a_{22} x_4)^{[p]}\\
&0 = a_{11}^{2p} a_{22}^p x_5.
\end{aligned}
$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{14}$ and $L_{5,3}^{19}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{14} \to L_{5,3}^{19}(\xi)$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{14}$ and $L_{5,3}^{21}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{14} \to L_{5,3}^{21}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{14}$ and $L_{5,3}^{22}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{22} \to L_{5,3}^{14}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$A(x_4) = (a_{11} a_{22} x_3 + a_{11} a_{32} x_4)^{[p]}$$
$$a_{11}^2 a_{22} x_4 = a_{11}^p a_{22}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^{14}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,3}^{15}$ and $L_{5,3}^{16}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{15} \to L_{5,3}^{16}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5.$$

Therefore, $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

It is clear that $L_{5,3}^{15}$ and $L_{5,3}^{16}$ are not isomorphic to the other restricted Lie algebras.

Note that $L_{5,3}^{17}$ is not isomorphic to any of $L_{5,3}^{18}$, $L_{5,3}^{19}(\xi)$, $L_{5,3}^{21}$, $L_{5,3}^{22}$ because $(L_{5,3}^{17})^{[p]^2} = 0$ but this is not true for those restricted Lie algebras.

Next, we claim that $L_{5,3}^{17}$ and $L_{5,3}^{20}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{17} \to L_{5,3}^{20}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{18}$ and $L_{5,3}^{19}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{18} \to L_{5,3}^{19}(\xi)$. Then

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$
$$0 = (a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$
$$0 = a_{22}^p x_4 + a_{42}^p x_5.$$

Therefore, $a_{22} = 0$ which is a contradiction.

Note that $L_{5,3}^{18}$ is not isomorphic to $L_{5,3}^{20}$ because $(L_{5,3}^{18})^{[p]^2} \neq 0$ but this is not true for $L_{5,3}^{20}$.

Next, we claim that $L_{5,3}^{18}$ and $L_{5,3}^{21}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{18} \to L_{5,3}^{21}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4 + a_{11}^p a_{32}^p x_5,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L_{5,3}^{18}$ and $L_{5,3}^{22}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{18} \to L_{5,3}^{22}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$
$$0 = (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]}$$
$$0 = a_{11}^p a_{22}^p x_4,$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Note that $L_{5,3}^{19}$ is not isomorphic to $L_{5,3}^{20}$ because $(L_{5,3}^{19})^{[p]^2} \neq 0$ but this is not true for $L_{5,3}^{20}$.

Next, we claim that $L^{19}_{5,3}(\xi)$ and $L^{21}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{19}_{5,3}(\xi) \to L^{21}_{5,3}$. Then

$$
\begin{aligned}
A(x_3^{[p]}) &= A(x_3)^{[p]} \\
0 &= (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]} \\
0 &= a_{11}^p a_{22}^p x_4 + a_{11}^p a_{32}^p x_5,
\end{aligned}
$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Next, we claim that $L^{19}_{5,3}$ and $L^{22}_{5,3}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{19}_{5,3} \to L^{22}_{5,3}$. Then

$$
\begin{aligned}
A(x_3^{[p]}) &= A(x_3)^{[p]} \\
0 &= (a_{11}a_{22}x_3 + a_{11}a_{32}x_4)^{[p]} \\
0 &= a_{11}^p a_{22}^p x_4,
\end{aligned}
$$

which implies that $a_{11}^p a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Note that $L^{20}_{5,3}$ is not isomorphic to $L^{21}_{5,3}$ and $L^{22}_{5,3}$ because $(L^{20}_{5,3})^{[p]^2} = 0$ but this is not true for $L^{21}_{5,3}$ and $L^{22}_{5,3}$.

Next, we claim that $L_{5,3}^{21}$ and $L_{5,3}^{22}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,3}^{22} \to L_{5,3}^{21}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$
$$0 = (a_{11}^2 a_{22} x_4)^{[p]}$$
$$0 = a_{11}^{2p} a_{22}^p x_5,$$

which implies that $a_{11}^{2p} a_{22}^p = 0$. Therefore, we have $a_{11} = 0$ or $a_{22} = 0$ which is a contradiction.

Chapter 6

Restriction maps on Lie algebras of 1-dimensional centre

6.1 Restriction maps on $L_{5,4}$

Let $L_{5,4} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_5, [x_3, x_4] = x_5 \rangle$. We have $Z(L_{5,4}) = \langle x_5 \rangle_{\mathbb{F}}$ and so $x_5^{[p]} = 0$. Let

$$L = \frac{L_{5,4}}{\langle x_5 \rangle_{\mathbb{F}}} \cong L_{4,1},$$

where $L_{4,1} = \langle x_1, x_2, x_3, x_4 \rangle$.

Lemma 6.1.1 *Let $K = L_{5,4}$ and $[p] : K \to K$ be a p-map on K and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{12} + \Delta_{34}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$,

$1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = [x_1, x_2] = x_5;$$

$$\phi(x_3, x_4) = [\sigma(x_3), \sigma(x_4)] - \sigma([x_3, x_4]) = x_5;$$

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = 0;$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = 0$. Therefore, $\phi = \Delta_{12} + \Delta_{34}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{12} + \Delta_{34}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{12} + \Delta_{34}, \omega)$, for some ω.

Note that by [12], there are five non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_2$;

I.3 $x_1^{[p]} = x_2, x_3^{[p]} = x_4$;

I.4 $x_1^{[p]} = x_2, x_2^{[p]} = x_3$;

I.5 $x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4$.

In the following sections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 6.1.1, it suffices to assume $[\theta] = [(\Delta_{12} + \Delta_{34}, \omega)] \in H^2(L, \mathbb{F})$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,4}$.

Note that the group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

6.1.1 Extension of $L_{5,4}/\langle x_5 \rangle$ via the trivial p-map

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Since L is abelian and the p-map is trivial, $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (\Delta_{34}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = 0.$$

Similarly, we can show that $b = c = d = e = f = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (0, 0)$ and hence $B^2(L, \mathbb{F}) = 0$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(\Delta_{34}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)],$
$[(0, f_4)]$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{12} + b'\Delta_{13} + c'\Delta_{14} + d'\Delta_{23} + e'\Delta_{24} + f'\Delta_{34}$, for some $a', b', c', d', e', f' \in \mathbb{F}$. Then

$$\begin{aligned} A\phi(x_1, x_2) &= \phi(Ax_1, Ax_2) \\ &= \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4) \\ &= (a_{11}a_{22} - a_{12}a_{21})a + (a_{11}a_{32} - a_{31}a_{12})b + (a_{11}a_{42} - a_{41}a_{12})c \\ &+ (a_{21}a_{32} - a_{31}a_{22})d + (a_{21}a_{42} - a_{41}a_{22})e + (a_{31}a_{42} - a_{41}a_{32})f, \text{ and} \end{aligned}$$

$$\begin{aligned} A\phi(x_1, x_3) &= \phi(Ax_1, Ax_3) \\ &= \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4) \\ &= (a_{11}a_{23} - a_{21}a_{13})a + (a_{11}a_{33} - a_{31}a_{13})b + (a_{11}a_{43} - a_{13}a_{41})c \\ &+ (a_{21}a_{33} - a_{31}a_{23})d + (a_{21}a_{43} - a_{23}a_{41})e + (a_{31}a_{43} - a_{33}a_{41})f, \text{ and} \end{aligned}$$

$$\begin{aligned} A\phi(x_1, x_4) &= \phi(Ax_1, Ax_4) \\ &= \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4) \\ &= (a_{11}a_{24} - a_{14}a_{21})a + (a_{11}a_{34} - a_{14}a_{31})b + (a_{11}a_{44} - a_{14}a_{41})c \\ &+ (a_{21}a_{34} - a_{24}a_{31})d + (a_{21}a_{44} - a_{24}a_{41})e + (a_{31}a_{44} - a_{34}a_{41})f, \text{ and} \end{aligned}$$

$$\begin{aligned}A\phi(x_2, x_3) &= \phi(Ax_2, Ax_3)\\ &= \phi(a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4)\\ &= (a_{12}a_{23} - a_{22}a_{13})a + (a_{12}a_{33} - a_{32}a_{13})b + (a_{12}a_{43} - a_{13}a_{42})c\\ &+ (a_{22}a_{33} - a_{32}a_{23})d + (a_{22}a_{43} - a_{23}a_{42})e + (a_{32}a_{43} - a_{33}a_{42})f, \text{ and}\end{aligned}$$

$$\begin{aligned}A\phi(x_2, x_4) &= \phi(Ax_2, Ax_4)\\ &= \phi(a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)\\ &= (a_{12}a_{24} - a_{14}a_{22})a + (a_{12}a_{34} - a_{14}a_{32})b + (a_{12}a_{44} - a_{14}a_{42})c\\ &+ (a_{22}a_{34} - a_{24}a_{32})d + (a_{22}a_{44} - a_{24}a_{42})e + (a_{32}a_{44} - a_{34}a_{42})f, \text{ and}\end{aligned}$$

$$\begin{aligned}A\phi(x_3, x_4) &= \phi(Ax_3, Ax_4)\\ &= \phi(a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4, a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)\\ &= (a_{13}a_{24} - a_{14}a_{23})a + (a_{13}a_{34} - a_{14}a_{33})b + (a_{13}a_{44} - a_{14}a_{43})c\\ &+ (a_{23}a_{34} - a_{24}a_{33})d + (a_{23}a_{44} - a_{24}a_{43})e + (a_{33}a_{44} - a_{34}a_{43})f.\end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{42} - a_{41}a_{12} & a_{21}a_{32} - a_{31}a_{22} & a_{21}a_{42} - a_{41}a_{22} & a_{31}a_{42} - a_{41}a_{32} \\ a_{11}a_{23} - a_{21}a_{13} & a_{11}a_{33} - a_{31}a_{13} & a_{11}a_{43} - a_{13}a_{41} & a_{21}a_{33} - a_{31}a_{23} & a_{21}a_{43} - a_{23}a_{41} & a_{31}a_{43} - a_{33}a_{41} \\ a_{11}a_{24} - a_{14}a_{21} & a_{11}a_{34} - a_{14}a_{31} & a_{11}a_{44} - a_{14}a_{41} & a_{21}a_{34} - a_{24}a_{31} & a_{21}a_{44} - a_{24}a_{41} & a_{31}a_{44} - a_{34}a_{41} \\ a_{12}a_{23} - a_{22}a_{13} & a_{12}a_{33} - a_{32}a_{13} & a_{12}a_{43} - a_{13}a_{42} & a_{22}a_{33} - a_{32}a_{23} & a_{22}a_{43} - a_{23}a_{42} & a_{32}a_{43} - a_{33}a_{42} \\ a_{12}a_{24} - a_{14}a_{22} & a_{12}a_{34} - a_{14}a_{32} & a_{12}a_{44} - a_{14}a_{42} & a_{22}a_{34} - a_{24}a_{32} & a_{22}a_{44} - a_{24}a_{42} & a_{32}a_{44} - a_{34}a_{42} \\ a_{13}a_{24} - a_{14}a_{23} & a_{13}a_{34} - a_{14}a_{33} & a_{13}a_{44} - a_{14}a_{43} & a_{23}a_{34} - a_{24}a_{33} & a_{23}a_{44} - a_{24}a_{43} & a_{33}a_{44} - a_{34}a_{43} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}.$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,4}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. Then we can verify that the action of $\mathrm{Aut}(L)$ on the set of ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ a_{13}^p & a_{23}^p & a_{33}^p & a_{43}^p \\ a_{14}^p & a_{24}^p & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\delta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & (-\beta/\delta)^{1/p} & 0 & (\alpha/\delta)^{1/p} & 0 \\ (-\alpha/\delta)^{1/p} & 1 & (\alpha\beta/\delta^2)^{1/p} & 0 & (-\alpha^2/\delta^2)^{1/p} & (\alpha/\delta)^{1/p} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ (-\beta/\delta)^{1/p} & 0 & (\beta^2/\delta^2)^{1/p} & 1 & (-\alpha\beta/\delta^2)^{1/p} & (\beta/\delta)^{1/p} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & (\beta/\delta)^{1/p} & 0 & (-\alpha/\delta)^{1/p} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -\alpha/\delta \\ 0 & 1 & 0 & -\beta/\delta \\ \beta/\delta & -\alpha/\delta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^{1/p} & (-\gamma)^{1/p} & 0 & 0 & 0 \\ 0 & 0 & (1/\delta)^{1/p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{1/p} & (-\gamma)^{1/p} & 0 \\ 0 & 0 & 0 & 0 & (1/\delta)^{1/p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta & -\gamma \\ 0 & 0 & 0 & 1/\delta \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right].$$

Next, suppose that $\delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & (\gamma/\beta)^{1/p} & 0 \\ (-\gamma/\beta)^{1/p} & 1 & 0 & (-\alpha/\beta)^{1/p} & (-\gamma^2/\beta^2)^{1/p} & (\gamma/\beta)^{1/p} \\ 0 & 0 & 1 & 0 & (-\alpha/\beta)^{1/p} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & (-\gamma/\beta)^{1/p} & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha/\beta & 0 & -\gamma/\beta \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma/\beta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta^{1/p} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^{1/p} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1/\beta)^{1/p} & 0 & 0 \\ 0 & 0 & 0 & 0 & (1/\beta)^{1/p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 1/\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right].$$

Next, if $\delta = \beta = 0$, but $\alpha \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & (\gamma/\alpha)^{1/p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ (\gamma/\alpha)^{1/p} & 0 & (\gamma^2/\alpha^2)^{1/p} & 1 & 0 & (-\gamma/\alpha)^{1/p} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & (-\gamma/\alpha)^{1/p} & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma/\alpha \\ -\gamma/\alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1/\alpha)^{1/p} & 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\alpha)^{1/p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{1/p} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^{1/p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/\alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right].$$

Next, if $\delta = \beta = \alpha = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1/\gamma)^{1/p} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma^{1/p} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1/\gamma)^{1/p} & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma^{1/p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

Now we claim that the following elements are in the same $\mathrm{Aut}(L)$-orbit representatives.

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right].$$

Indeed,

$$\left[\left[\begin{pmatrix}1&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right],\right.$$

$$\left[\begin{pmatrix}0&0&0&0&0&1\\0&-1&0&0&0&0\\0&0&0&-1&0&0\\0&0&-1&0&0&0\\0&0&0&0&-1&0\\1&0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right],$$

$$\left[\begin{pmatrix}0&0&0&0&0&1\\0&0&0&1&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&-1&0&0&0\\1&0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&-1&0&0\\1&0&0&0\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right].$$

Therefore, the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right],\left[\begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right].$$

6.1.2 Extensions of $(L_{5,4}/\langle x_5\rangle, x_1^{[p]}=x_2)$

Note that $L^{[p]}=\langle x_2\rangle_{\mathbb{F}}$. Let $[(\phi,\omega)]\in H^2(L,\mathbb{F})$. Then we must have $\phi(x,y^{[p]})=0$, for all $x,y\in L$, where $\phi=a\Delta_{12}+b\Delta_{13}+c\Delta_{14}+d\Delta_{23}+e\Delta_{24}+f\Delta_{34}$, for some $a,b,c,d,e,f\in\mathbb{F}$. Hence, $\phi(x,x_2)=0$, for all $x\in L$. Therefore,

$$\begin{aligned}\phi(x_1,x_2)&=a\Delta_{12}(x_1,x_2)+b\Delta_{13}(x_1,x_2)+c\Delta_{14}(x_1,x_2)\\&\quad+d\Delta_{23}(x_1,x_2)+a\Delta_{24}(x_1,x_2)+f\Delta_{34}(x_1,x_2)=0\end{aligned}$$

which implies that $a=0$. Since $\phi=\Delta_{12}+\Delta_{34}$ gives us $L_{5,4}$, we deduce by Lemma 6.1.1 that $L_{5,4}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also cannot construct $L_{5,4}$.

I.3 $x_1^{[p]}=x_2, x_3^{[p]}=x_4$;

I.4 $x_1^{[p]}=x_2, x_2^{[p]}=x_3$;

I.5 $x_1^{[p]}=x_2, x_2^{[p]}=x_3, x_3^{[p]}=x_4$.

Theorem 6.1.2 *The list of all restricted Lie algebra structures on $L_{5,4}$, up to isomorphism, is as follows:*

$$L_{5,4}^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_5, [x_3, x_4] = x_5 \rangle;$$
$$L_{5,4}^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_5, [x_3, x_4] = x_5, x_1^{[p]} = x_5 \rangle.$$

Note that by Lemma 2.3.3, the algebras in the Theorem above are pairwise non-isomorphic because these algebras correspond to different $\mathrm{Aut}(L)$-orbits.

6.2 Restriction maps on $L_{5,5}$

Let $L_{5,5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle$. We have $Z(L_{5,5}) = \langle x_5 \rangle_{\mathbb{F}}$. Let

$$L = \frac{L_{5,5}}{\langle x_5 \rangle_{\mathbb{F}}} \cong L_{4,2},$$

where $L_{4,2} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle$.

Lemma 6.2.1 *Let $K = L_{5,5}$ and $[p] : K \to K$ be a p-map on K and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{13} + \Delta_{24}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$,

$1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = [x_1, x_3] = x_5;$$

$$\phi(x_2, x_4) = [\sigma(x_2), \sigma(x_4)] - \sigma([x_2, x_4]) = x_5;$$

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0;$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{13} + \Delta_{24}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{13} + \Delta_{24}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{13} + \Delta_{24}, \omega)$, for some ω.

Note that by [12], there are eight non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

II.1 Trivial p-map;

II.2 $x_1^{[p]} = x_3$;

II.3 $x_1^{[p]} = x_4$;

II.4 $x_1^{[p]} = x_3, x_2^{[p]} = x_4$;

II.5 $x_3^{[p]} = x_4$;

II.6 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$;

II.7 $x_4^{[p]} = x_3$;

II.8 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$.

In the following sections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 6.2.1, it suffices to assume $[\theta] = [(\Delta_{13} + \Delta_{24}, \omega)] \in H^2(L, \mathbb{F})$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,5}$.

Note that the group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

6.2.1 Extension of $L_{5,5}/\langle x_5 \rangle$ via the trivial p-map

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L,\mathbb{F})$. Let $(\phi,\omega) \in B^2(L,\mathbb{F})$. Since $B^2(L,\mathbb{F}) \subseteq Z^2(L,\mathbb{F})$, we have $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$b = \phi(x_1,x_3) = \delta^1\psi(x_1,x_3) = \psi([x_1,x_3]) = 0.$$

Similarly, we can show that $c = d = e = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi,\omega) = (a\Delta_{12}, 0)$ and hence $B^2(L,\mathbb{F}) = \langle(\Delta_{12},0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L,\mathbb{F})$ is as follows:

$$[(\Delta_{13},0)], [(\Delta_{14},0)], [(\Delta_{23},0)], [(\Delta_{24},0)], [(0,f_1)], [(0,f_2)], [(0,f_3)], [(0,f_4)].$$

Let $[(\phi,\omega)] \in H^2(L,\mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$, for some $a,b,c,d \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{14} + c'\Delta_{23} + d'\Delta_{24}$, for some $a',b',c',d' \in \mathbb{F}$. We can verify that the action of $\mathrm{Aut}(L)$ on the set of ϕ's in the matrix form is as follows:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{6.1}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,5}$. Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha,\beta,\gamma,\delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We can verify that the action of $\mathrm{Aut}(L)$ on the set of ω's in the matrix form is as follows:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{6.2}$$

Thus, we can write Equations (6.1) and (6.2) together as follows:

$$\left[\begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \right]$$
$$= \left[\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\gamma \neq 0$. Then

$$\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&-\alpha/\gamma&0\\0&1&-\beta/\gamma&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&\gamma^{3/p}&0&0\\0&0&\gamma^{-3/p}&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}\gamma&0&0&0\\0&\gamma^{-2}&0&0\\0&0&\gamma^{-1}&0\\0&0&0&\gamma^2\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\\gamma^2\delta\end{pmatrix}\right].$$

Furthermore,

$$\left[\begin{pmatrix}1&0&\delta^{1/p}&0\\-\delta^{1/p}&1&-\delta^{2/p}&\delta^{1/p}\\0&0&1&0\\0&0&-\delta^{1/p}&1\end{pmatrix},\begin{pmatrix}1&\delta&0&0\\0&1&0&0\\0&0&1&0\\0&0&-\delta&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right].$$

Next, if $\gamma = 0$, but $\delta \neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&-\alpha/\delta\\0&1&0&-\beta/\delta\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right].$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & (-\alpha/\beta)^{1/p} & 0 \\ (\alpha/\beta)^{1/p} & 1 & (-\alpha/\beta)^{2/p} & (-\alpha/\beta)^{1/p} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (\alpha/\beta)^{1/p} & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha/\beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha/\beta & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right].$$

Finally, if $\gamma = \delta = \beta = 0$, but $\alpha \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^{-3/p} & 0 & 0 \\ 0 & 0 & \alpha^{3/p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^{-2} \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}.$$

Lemma 6.2.2 *The vectors* $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}\right].$$

From this we obtain that $ra_{11} = 1$ and $ra_{12} = 0$, and $\beta_2 = a_{22}^p\beta_1$ which gives that $r = a_{11}^{-1}$, $a_{12} = 0$. Therefore, $a_{22} = a_{11}^{-2}$. Hence, $\beta_2\beta_1^{-1} = a_{11}^{-2p} \in (\mathbb{F}^*)^2$. Conversely assume that $\beta_2\beta_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon^{-3/p} & 0 & 0 \\ 0 & 0 & \epsilon^{3/p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^{-2} \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}\right],$$

as required. ■

Lemma 6.2.3 *The vectors* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix} ra_{11} & 0 & ra_{21} & 0 \\ a_{11}a_{34} & a_{11}a_{44} & a_{21}a_{34} & a_{21}a_{44} \\ ra_{12} & 0 & ra_{22} & 0 \\ a_{12}a_{34} & a_{12}a_{44} & a_{22}a_{34} & a_{22}a_{44} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & 0 \\ 0 & 0 & a_{34}^p & a_{44}^p \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}\right].$$

From this we obtain that $ra_{11} = 1$ and $ra_{12} = 0$, $a_{12}a_{34} + a_{22}a_{44} = 1$, and $\delta_2 = a_{44}^p\delta_1$ which gives that $r = a_{11}^{-1}$, $a_{12} = 0$. Therefore, $a_{22} = a_{11}^{-2}$ and $a_{22}a_{44} = 1$. Hence, $\delta_2\delta_1^{-1} = a_{11}^{2p} \in (\mathbb{F}^*)^2$. Conversely assume that $\delta_2\delta_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon^{3/p} & 0 & 0 \\ 0 & 0 & \epsilon^{-3/p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon^{-2} & 0 & 0 \\ 0 & 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 0 & \epsilon^2 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}\right],$$

as required. ∎

6.2.2 Extensions of $(L_{5,5}/\langle x_5\rangle, x_1^{[p]} = x_3)$

Note that $L^{[p]} = \langle x_3\rangle_{\mathbb{F}}$. Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{12}+b\Delta_{13}+c\Delta_{14}+d\Delta_{23}+e\Delta_{24}+f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$.

Hence, $\phi(x_1, x_3) = 0$ which implies that $b = 0$. Since $\phi = \Delta_{13} + \Delta_{24}$ gives us $L_{5,5}$, we deduce by Lemma 6.2.1 that $L_{5,5}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also cannot construct $L_{5,5}$.

II.3 $x_1^{[p]} = x_4$;

II.4 $x_1^{[p]} = x_3, x_2^{[p]} = x_4$;

II.5 $x_3^{[p]} = x_4$;

II.6 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$;

II.7 $x_4^{[p]} = x_3$;

II.8 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$.

Theorem 6.2.4 *The list of all restricted Lie algebra structures on $L_{5,5}$, up to isomorphism, is as follows:*

$$L_{5,5}^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle;$$

$$L_{5,5}^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, x_1^{[p]} = x_5 \rangle;$$

$$L_{5,5}^3(\beta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, x_2^{[p]} = \beta x_5 \rangle;$$

$$L_{5,5}^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, x_3^{[p]} = x_5 \rangle;$$

$$L_{5,5}^5(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, x_4^{[p]} = \delta x_5 \rangle$$

where $\beta, \gamma \in T_2$.

Note that by Lemma 2.3.3, the algebras in the Theorem above are pairwise non-isomorphic because these algebras correspond to different $\mathrm{Aut}(L)$-orbits.

Lemma 6.2.5 *We have $L^3_{5,5}(\beta_1) \cong L^3_{5,5}(\beta_2)$ if and only if $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$.*

Proof. Suppose that $f : L^3_{5,5}(\beta_1) \to L^3_{5,5}(\beta_2)$ is an isomorphism. Then, f is an automorphism of $L_{5,5}$ and $f(\langle x_5\rangle_{\mathbb{F}}) = \langle x_5\rangle_{\mathbb{F}}$. Thus, f induces an automorphism $A : L^3_{5,5}(\beta_1)/\langle x_5\rangle \to L^3_{5,5}(\beta_2)/\langle x_5\rangle$ and, by Lemma 2.3.3, we have that $A\theta_1 = c\theta_2$, for some $c \in \mathbb{F}^*$, where $\theta_1 = (\Delta_{13} + \Delta_{24}, \beta_1 f_2)$ and $\theta_2 = (\Delta_{13} + \Delta_{24}, \beta_2 f_2)$ which implies that $c^{-1}A\theta_1 = \theta_2$. Therefore, without loss of generality, we can suppose that $c = 1$. So, $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit and it follows by Lemma 6.2.2 that $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^2$. The converse is easy to see by Lemma 2.3.3. ■

Similarly, as in Lemma 6.2.5, we can prove that $L^5_{5,5}(\delta_1) \cong L^5_{5,5}(\delta_2)$ if and only if $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^2$.

6.3 Restriction maps on $L_{5,6}$

Let $L_{5,6} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5\rangle$. We have $Z(L_{5,6}) = \langle x_5\rangle_{\mathbb{F}}$. Let

$$L = \frac{L_{5,6}}{\langle x_5\rangle_{\mathbb{F}}} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle \cong L_{4,3}.$$

Lemma 6.3.1 *Let $K = L_{5,6}$ and $[p] : K \to K$ be a p-map on K and let $L = \frac{K}{M}$ where $M = \langle x_5\rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{14} + \Delta_{23}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_4) = [\sigma(x_1), \sigma(x_4)] - \sigma([x_1, x_4]) = [x_1, x_4] = x_5;$$

$$\phi(x_2, x_3) = [\sigma(x_2), \sigma(x_3)] - \sigma([x_2, x_3]) = x_5;$$

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0;$$

Similarly, we can show that $\phi(x_1, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{14} + \Delta_{23}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{14} + \Delta_{23}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{14} + \Delta_{23}, \omega)$, for some ω.

Note that by [12], there are four non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_4$;

I.3 $x_2^{[p]} = \xi x_4$;

I.4 $x_3^{[p]} = x_4$.

In the following sections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 6.3.1, it suffices to assume $[\theta] = [(\Delta_{14} + \Delta_{23}, \omega)] \in H^2(L, \mathbb{F})$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,6}$.

Note that the group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}r \end{pmatrix},$$

where $r = a_{11}a_{22} \neq 0$.

6.3.1 Extensions of $L_{5,6}/\langle x_5 \rangle$ via the trivial p-map

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4), \text{ and}$$

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $e = f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0.$$

Similarly, we can show that $d = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (\Delta_{13}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{14} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{14} + b'\Delta_{23}$, for some $a', b' \in \mathbb{F}$. We determine a', b'. Note that

$$\begin{aligned} A\phi(x_1, x_4) =& \phi(Ax_1, Ax_4) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{11}rx_4) = a_{11}^2 ra; \\ A\phi(x_2, x_3) =& \phi(Ax_2, Ax_3) = \phi(a_{22}x_2 + a_{32}x_3 + a_{42}x_4, rx_3 + a_{11}a_{32}x_4) = a_{22}rb. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}^2 r & 0 \\ 0 & a_{22}r \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{6.3}$$

The orbit with representative $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,6}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We have

$$\begin{aligned} A\omega(x_1) =& \omega(Ax_1) = \omega(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4) = a_{11}^p\alpha + a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta; \\ A\omega(x_2) =& a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta; \\ A\omega(x_3) =& r^p\gamma + a_{11}^p a_{32}^p\delta; \\ A\omega(x_4) =& a_{11}^p r^p\delta. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{6.4}$$

Thus, we can write Equations (6.3) and (6.4) together as follows:

$$\left[r \begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's.

Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0&-\alpha/\delta\\0&1&0&-\beta/\delta\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right],\text{ and}$$

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&-\gamma/\delta&\gamma^2/\delta^2\\0&0&1&-\gamma/\delta\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&-\alpha/\gamma&0\\0&1&-\beta/\gamma&0\\0&0&1&-\beta/\gamma\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}\right].$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&-\alpha/\beta&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right].$$

Finally, if $\beta = \gamma = \delta = 0$, but $\alpha \neq 0$, then we have $\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}\right]$.

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}.$$

Lemma 6.3.2 *The vectors* $\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\alpha_2\alpha_1^{-1} = \epsilon^2$ *and* $\epsilon^5 = 1$*, for some* $\epsilon \in \mathbb{F}^*$.

Proof. First assume that $\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix}a_{11}^2 & 0\\ 0 & a_{22}\end{pmatrix}, \begin{pmatrix}a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p\\ 0 & a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & 0 & r^p & a_{11}^p a_{32}^p\\ 0 & 0 & 0 & a_{11}^p r^p\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}\right].$$

From this we obtain that $ra_{11}^2 = 1$, $ra_{22} = 1$, and $\alpha_2 = a_{11}^p\alpha_1$ which gives that $\alpha_2\alpha_1^{-1} = (a_{22}^p)^{-2} \in (\mathbb{F}^*)^2$ and $a_{22}^5 = 1$. Therefore, $\alpha_2\alpha_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$, where $\epsilon \in \mathbb{F}^*$. Conversely assume that $\alpha_2\alpha_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$, for some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p}\begin{pmatrix}\epsilon^{4/p} & 0\\ 0 & \epsilon^{-1/p}\end{pmatrix}, \begin{pmatrix}\epsilon^2 & 0 & 0 & 0\\ 0 & \epsilon^{-1} & 0 & 0\\ 0 & 0 & \epsilon & 0\\ 0 & 0 & 0 & \epsilon^3\end{pmatrix}\right]\left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}\right],$$

as required. ■

Lemma 6.3.3 *The vectors* $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1} = \epsilon^3$ *and* $\epsilon^5 = 1$*, for some* $\epsilon \in \mathbb{F}^*$*.*

Proof. First assume that $\begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r \begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix} \right].$$

From this we obtain that $ra_{11}^2 = 1$, $ra_{22} = 1$, and $\beta_2 = a_{22}^p\beta_1$ which gives that $\beta_2\beta_1^{-1} = (a_{11}^p)^{-3} \in (\mathbb{F}^*)^3$ and $a_{11}^5 = 1$. Therefore, $\beta_2\beta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$, where $\epsilon \in \mathbb{F}^*$. Conversely assume that $\beta_2\beta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$ with $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{2/p} \begin{pmatrix} \epsilon^{-2/p} & 0 \\ 0 & \epsilon^{3/p} \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & 0 & 0 & 0 \\ 0 & \epsilon^3 & 0 & 0 \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta_2 \\ 0 \\ 0 \end{pmatrix} \right],$$

as required. ■

Lemma 6.3.4 *The vectors* $\begin{pmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ \gamma_2 \\ 0 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\gamma_2\gamma_1^{-1} = \epsilon^2$ *and* $\epsilon^5 = 1$*, where* $\epsilon \in \mathbb{F}^*$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \gamma_2 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma_2 \\ 0 \end{pmatrix}\right].$$

From this we obtain that $ra_{11}^2 = 1$, $ra_{22} = 1$, and $\gamma_2 = r^p\gamma_1$ which gives that $\gamma_2\gamma_1^{-1} = (a_{11}^p)^{-2} \in (\mathbb{F}^*)^2$ and $a_{11}^5 = 1$. Therefore, $\gamma_2\gamma_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$, where $\epsilon \in \mathbb{F}^*$. Conversely assume that $\gamma_2\gamma_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{2/p}\begin{pmatrix} \epsilon^{-2/p} & 0 \\ 0 & \epsilon^{3/p} \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & 0 & 0 & 0 \\ 0 & \epsilon^3 & 0 & 0 \\ 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma_2 \\ 0 \end{pmatrix}\right],$$

as required. ■

Lemma 6.3.5 *The vectors* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if*

$\delta_2\delta_1^{-1} = \epsilon^2$ *and* $\epsilon^5 = 1$, *where* $\epsilon \in \mathbb{F}^*$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r \begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix} \right].$$

From this we obtain that $ra_{11}^2 = 1$, $ra_{22} = 1$, and $\delta_2 = a_{11}^p r^p \delta_1$ which gives that $\delta_2\delta_1^{-1} = (a_{22}^p)^{-3} \in (\mathbb{F}^*)^3$ and $a_{22}^5 = 1$. Therefore, $\delta_2\delta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$, where $\epsilon \in \mathbb{F}^*$. Conversely assume that $\delta_2\delta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$, where $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p} \begin{pmatrix} \epsilon^{4/p} & 0 \\ 0 & \epsilon^{-1/p} \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^3 \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix} \right],$$

as required. ■

6.3.2 Extensions of $(L_{5,6}/\langle x_5 \rangle, x_1^{[p]} = x_4)$

Note that $L^{[p]} = \langle x_4 \rangle_{\mathbb{F}}$. Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$. Hence, $\phi(x_1, x_4) = 0$ which implies that $c = 0$. Since $\phi = \Delta_{14} + \Delta_{23}$ gives us $L_{5,6}$, we deduce by Lemma 6.3.1 that $L_{5,6}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also get $c = 0$.

I.3 $x_2^{[p]} = \xi x_4$;

I.4 $x_3^{[p]} = x_4$.

Theorem 6.3.6 *The list of all restricted Lie algebra structures on $L_{5,6}$, up to isomorphism, is as follows:*

$L_{5,6}^1 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5\rangle$;

$L_{5,6}^2(\alpha) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5\rangle$;

$L_{5,6}^3(\beta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, x_2^{[p]} = \beta x_5\rangle$;

$L_{5,6}^4(\gamma) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, x_3^{[p]} = \gamma x_5\rangle$;

$L_{5,6}^5(\delta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, x_4^{[p]} = \delta x_5\rangle$

where $\alpha, \gamma, \delta \in T_{2,5}$ and $\beta \in T_{3,5}$.

Similarly, as in Lemma 6.2.5, we can prove that $L_{5,6}^2(\alpha_1) \cong L_{5,6}^2(\alpha_2)$ if and only if $\alpha_2\alpha_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$ where $\epsilon \in \mathbb{F}^*$, $L_{5,6}^3(\beta_1) \cong L_{5,6}^3(\beta_2)$ if and only if $\beta_2\beta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$ where $\epsilon \in \mathbb{F}^*$, $L_{5,6}^4(\gamma_1) \cong L_{5,6}^4(\gamma_2)$ if and only if $\gamma_2\gamma_1^{-1} = \epsilon^2$ and $\epsilon^5 = 1$ where $\epsilon \in \mathbb{F}^*$, and $L_{5,6}^5(\delta_1) \cong L_{5,6}^5(\delta_2)$ if and only if $\delta_2\delta_1^{-1} = \epsilon^3$ and $\epsilon^5 = 1$ where $\epsilon \in \mathbb{F}^*$. Furthermore, by Lemma 2.3.3, the algebras in the Theorem above are pairwise non-isomorphic because these algebras correspond to different $\mathrm{Aut}(L)$-orbits.

6.4 Restriction maps on $L_{5,7}$

Let $L_{5,7} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5\rangle$. We have $Z(L_{5,7}) = \langle x_5\rangle_{\mathbb{F}}$. Let

$$L = \frac{L_{5,7}}{\langle x_5\rangle} \cong L_{4,3},$$

where $L_{4,3} = \langle x_1, \dots, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle$.

Lemma 6.4.1 *Let $K = L_{5,7}$ and $[p] : K \to K$ be a p-map on K and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{14}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_4) = [\sigma(x_1), \sigma(x_4)] - \sigma([x_1, x_4]) = [x_1, x_4] = x_5;$$
$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0;$$

Similarly, we can show that $\phi(x_1, x_3) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{14}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{14}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$.
■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{14}, \omega)$, for some ω.

Note that by [12], there are four non-isomorphic restricted Lie algebra structures on $L_{4,3}$ given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_4$;

I.3 $x_2^{[p]} = \xi x_4$;

I.4 $x_3^{[p]} = x_4$.

In the following sections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 6.4.1, it suffices to assume $[\theta] = [(\Delta_{14}, \omega)] \in H^2(L, \mathbb{F})$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,7}$.

Note that the group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}r \end{pmatrix},$$

where $r = a_{11}a_{22} \neq 0$.

6.4.1 Extensions of $L_{5,7}/\langle x_5 \rangle$ via the trivial p-map

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4),$ and

$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$

Thus, we get $e = f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0.$$

Similarly, we can show that $d = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle (\Delta_{12}, 0), (\Delta_{13}, 0) \rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{14} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{14} + b'\Delta_{23}$, for some $a', b' \in \mathbb{F}$. We determine a', b'. Note that

$$\begin{aligned} A\phi(x_1, x_4) =& \phi(Ax_1, Ax_4) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{11}rx_4) = a_{11}^2 ra; \\ A\phi(x_2, x_3) =& \phi(Ax_2, Ax_3) = \phi(a_{22}x_2 + a_{32}x_3 + a_{42}x_4, rx_3 + a_{11}a_{32}x_4) = a_{22}rb. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}^2 r & 0 \\ 0 & a_{22}r \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{6.5}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,7}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega =$

$\alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We have

$$\begin{aligned} A\omega(x_1) =& \omega(Ax_1) = \omega(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4) = a_{11}^p\alpha + a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta; \\ A\omega(x_2) =& a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta; \\ A\omega(x_3) =& r^p\gamma + a_{11}^p a_{32}^p\delta; \\ A\omega(x_4) =& a_{11}^p r^p\delta. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{6.6}$$

Thus, we can write Equations (6.5) and (6.6) together as follows:

$$\left[r\begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & r^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p r^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$.

Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0&-\alpha/\delta\\0&1&0&-\beta/\delta\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&-\gamma/\delta&\gamma^2/\delta^2\\0&0&1&-\gamma/\delta\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right], \text{ and}$$

$$\left[\delta^{-2/p}\begin{pmatrix}\delta^{2/p}&0\\0&\delta^{-3/p}\end{pmatrix},\begin{pmatrix}\delta&0&0&0\\0&\delta^{-3}&0&\\0&0&\delta^{-2}&0\\0&0&0&\delta^{-1}\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&-\alpha/\gamma&0\\0&1&-\beta/\gamma&0\\0&0&1&-\beta/\gamma\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}\right].$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&-\alpha/\beta&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right].$$

Finally, if $\beta = \gamma = \delta = 0$, but $\alpha \neq 0$, then we have

$$\left[\alpha^{2/p}\begin{pmatrix}\alpha^{-2/p} & 0\\ 0 & \alpha^{3/p}\end{pmatrix}, \begin{pmatrix}\alpha^{-1} & 0 & 0 & 0\\ 0 & \alpha^3 & 0 & 0\\ 0 & 0 & \alpha^2 & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Lemma 6.4.2 *The vectors* $\begin{pmatrix}0\\\beta_1\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\\beta_2\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^3$.

Proof. First assume that $\begin{pmatrix}0\\\beta_1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\\beta_2\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix}a_{11}^2 & 0\\ 0 & a_{22}\end{pmatrix}, \begin{pmatrix}a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p\\ 0 & a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & 0 & r^p & a_{11}^p a_{32}^p\\ 0 & 0 & 0 & a_{11}^p r^p\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\beta_1\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\beta_2\\0\\0\end{pmatrix}\right].$$

From this we obtain that $ra_{11}^2 = 1$ and $\beta_2 = a_{22}^p\beta_1$ which gives that $\beta_2\beta_1^{-1} = a_{11}^{-3p} \in (\mathbb{F}^*)^3$. Conversely assume that $\beta_2\beta_1^{-1} = \epsilon^3$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{2/p}\begin{pmatrix}\epsilon^{-2/p} & 0\\ 0 & \epsilon^{3/p}\end{pmatrix}, \begin{pmatrix}\epsilon^{-1} & 0 & 0 & 0\\ 0 & \epsilon^3 & 0 & 0\\ 0 & 0 & \epsilon^2 & 0\\ 0 & 0 & 0 & \epsilon\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_1\\ 0\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ \beta_2\\ 0\\ 0\end{pmatrix}\right],$$

as required. ■

Lemma 6.4.3 *The vectors* $\begin{pmatrix}0\\0\\ \gamma_1\\ 0\end{pmatrix}$ *and* $\begin{pmatrix}0\\0\\ \gamma_2\\ 0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\gamma_2\gamma_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}0\\0\\ \gamma_1\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\ \gamma_2\\ 0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[r\begin{pmatrix}a_{11}^2 & 0\\ 0 & a_{22}\end{pmatrix}, \begin{pmatrix}a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p\\ 0 & a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & 0 & r^p & a_{11}^p a_{32}^p\\ 0 & 0 & 0 & a_{11}^p r^p\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \gamma_1\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \gamma_2\\ 0\end{pmatrix}\right].$$

From this we obtain that $ra_{11}^2 = 1$ and $\gamma_2 = r^p\gamma_1$ which gives that $\gamma_2\gamma_1^{-1} = a_{11}^{-2p} \in (\mathbb{F}^*)^2$.

Conversely assume that $\gamma_2\gamma_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{2/p}\begin{pmatrix}\epsilon^{-2/p} & 0\\ 0 & \epsilon^{3/p}\end{pmatrix}, \begin{pmatrix}\epsilon^{-1} & 0 & 0 & 0\\ 0 & \epsilon^3 & 0 & 0\\ 0 & 0 & \epsilon^2 & 0\\ 0 & 0 & 0 & \epsilon\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\gamma_1\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\gamma_2\\0\end{pmatrix}\right],$$

as required. ■

6.4.2 Extensions of $(L_{5,7}/\langle x_5\rangle, x_1^{[p]} = x_4)$

Note that $L^{[p]} = \langle x_4\rangle_{\mathbb{F}}$. Let $[(\phi,\omega)] \in H^2(L,\mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{12}+b\Delta_{13}+c\Delta_{14}+d\Delta_{23}+e\Delta_{24}+f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$. Hence, $\phi(x_1, x_4) = 0$ which implies that $c = 0$. Since $\phi = \Delta_{14}$ gives us $L_{5,7}$, we deduce by Lemma 6.4.1 that $L_{5,7}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also get $c = 0$.

I.3 $x_2^{[p]} = \xi x_4$;

I.4 $x_3^{[p]} = x_4$.

Theorem 6.4.4 *The list of all restricted Lie algebra structures on $L_{5,7}$, up to isomorphism, is as follows:*

$$L_{5,7}^1 = \langle x_1,\ldots,x_5 \mid [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5\rangle;$$

$$L_{5,7}^2 = \langle x_1,\ldots,x_5 \mid [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, x_1^{[p]} = x_5\rangle;$$

$$L_{5,7}^3(\beta) = \langle x_1,\ldots,x_5 \mid [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, x_2^{[p]} = \beta x_5\rangle;$$

$$L_{5,7}^4(\gamma) = \langle x_1,\ldots,x_5 \mid [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, x_3^{[p]} = \gamma x_5\rangle;$$

$$L_{5,7}^5 = \langle x_1,\ldots,x_5 \mid [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5, x_4^{[p]} = x_5\rangle$$

where $\beta \in T_3$ *and* $\gamma \in T_2$.

Note that, as in Lemma 6.2.5, we can show that $L^3_{5,7}(\beta_1) \cong L^3_{5,7}(\beta_2)$ if and only if $\beta_2\beta_1^{-1} \in (\mathbb{F}^*)^3$ and $L^4_{5,7}(\gamma_1) \cong L^4_{5,7}(\gamma_2)$ if and only if $\gamma_2\gamma_1^{-1} \in (\mathbb{F}^*)^2$. Furthermore, by Lemma 2.3.3, the algebras in the Theorem above are pairwise non-isomorphic because these algebras correspond to different $\mathrm{Aut}(L)$-orbits.

Chapter 7

Restriction maps on $L_{5,8}$

Let

$$K_8 = L_{5,8} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle.$$

We have $Z(L_{5,8}) = \langle x_4, x_5 \rangle_{\mathbb{F}}$. Note that there exists an element $\alpha x_4 + \beta x_5 \in Z(L_{5,8})$ such that $(\alpha x_4 + \beta x_5)^{[p]} = 0$, for some $\alpha, \beta \in \mathbb{F}$. If $\alpha \neq 0$ then consider

$$K = \langle x_1', \dots, x_5' \mid [x_1', x_2'] = x_4', [x_1', x_3'] = x_5' \rangle,$$

where $x_1' = x_1, x_2' = \alpha x_2 + \beta x_3, x_3' = x_3, x_4' = \alpha x_4 + \beta x_5, x_5' = x_5$. Let $\phi : K_8 \to K$ given by $x_i \mapsto x_i'$, for $1 \leqslant i \leqslant 5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_4^{[p]} = 0$. If $\alpha = 0$ then $\beta \neq 0$ and we rescale x_5 so that $x_5^{[p]} = 0$. Hence we can assume either $x_4^{[p]} = 0$ or $x_5^{[p]} = 0$. Consider the automorphism of K_8 given by $x_1 \mapsto x_1, x_2 \mapsto x_3, x_3 \mapsto x_2, x_4 \mapsto x_5$ and $x_5 \mapsto x_4$. Using this automorphism, we deduce that it is enough to determine all the p-maps on K_8 for which $x_5^{[p]} = 0$.

Lemma 7.0.5 *Let $K = L_{5,8}$ and $[p] : K \to K$ be a p-map on K such that $x_5^{[p]} = 0$ and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{13}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_1, x_3) = [\sigma(x_1), \sigma(x_3)] - \sigma([x_1, x_3]) = [x_1, x_3] = x_5;$$

$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0.$$

Similarly, we can show that $\phi(x_1, x_4) = \phi(x_2, x_3) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{13}$. Now, by Lemma, 2.2.2 we have $\theta = (\Delta_{13}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{13}, \omega)$, for some ω.

Let

$$L = \frac{L_{5,8}}{\langle x_5 \rangle} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_4 \rangle.$$

Then $L \cong L_{4,2}$. Indeed, the map $A : L_{4,2} \to L$ such that $x'_1 \mapsto x_1$, $x'_2 \mapsto x_2$, $x'_3 \mapsto x_4$, $x'_4 \mapsto x_3$ is an isomorphism, where $L_{4,2} = \langle x'_1, x'_2, x'_3, x'_4 \mid [x'_1, x'_2] = x'_3 \rangle$. Therefore, by [12], there are eight non-isomorphic restricted Lie algebra structures on L given by the following p-maps:

II.1 Trivial p-map;

II.2 $x_1^{[p]} = x_4$;

II.3 $x_1^{[p]} = x_3$;

II.4 $x_1^{[p]} = x_4, x_2^{[p]} = x_3$;

II.5 $x_4^{[p]} = x_3$;

II.6 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$;

II.7 $x_3^{[p]} = x_4$;

II.8 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$.

In the following sections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 7.0.5, it suffices to assume $[\theta] = [(\Delta_{13}, \omega)]$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,8}$.

Note that the group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & r \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

7.1 Extensions of (L, trivial p-map)

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and

$\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_4, x_3).$$

Thus, we get $f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (\Delta_{24}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = d = e = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$, for some $a, b, c, d \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{14} + c'\Delta_{23} + d'\Delta_{24}$, for some $a', b', c', d' \in \mathbb{F}$. We determine a', b', c', d'. Note that

$$A\phi(x_1, x_3) = a_{11}a_{33}a + a_{11}a_{43}b + a_{21}a_{33}c + a_{21}a_{43}d;$$

$$A\phi(x_1, x_4) = a_{11}rb + a_{21}rd;$$

$$A\phi(x_2, x_3) = a_{12}a_{33}a + a_{12}a_{43}b + a_{22}a_{33}c + a_{22}a_{43}d;$$

$$A\phi(x_2, x_4) = a_{12}rb + a_{22}rd.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} a_{11}a_{33} & a_{11}a_{43} & a_{21}a_{33} & a_{21}a_{43} \\ 0 & a_{11}r & 0 & a_{21}r \\ a_{12}a_{33} & a_{12}a_{43} & a_{22}a_{33} & a_{22}a_{43} \\ 0 & a_{12}r & 0 & a_{22}r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{7.1}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,8}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We determine $\alpha', \beta', \gamma', \delta'$. Note that

$$A\omega(x_1) = \omega(Ax_1) = \omega(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4) = a_{11}^p\alpha + a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta;$$

$$A\omega(x_2) = a_{12}^p\alpha + a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta;$$

$$A\omega(x_3) = a_{33}^p\gamma + a_{43}^p\delta;$$

$$A\omega(x_4) = r^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & a_{33}^p & a_{43}^p \\ 0 & 0 & 0 & r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{7.2}$$

Thus, we can write Equations (7.1) and (7.2) together as follows:

$$\left[\begin{pmatrix} a_{11}a_{33} & a_{11}a_{43} & a_{21}a_{33} & a_{21}a_{43} \\ 0 & a_{11}r & 0 & a_{21}r \\ a_{12}a_{33} & a_{12}a_{43} & a_{22}a_{33} & a_{22}a_{43} \\ 0 & a_{12}r & 0 & a_{22}r \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & a_{33}^p & a_{43}^p \\ 0 & 0 & 0 & r^p \end{pmatrix}\right]\left[\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix}\right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's.

Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & -\gamma/\delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\gamma/\delta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -\alpha/\delta \\ 0 & 1 & 0 & -\beta/\delta \\ 0 & 0 & 1 & -\gamma/\delta \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&\delta^{-2/p}&0&0\\0&0&\delta^{1/p}&0\\0&0&0&\delta^{-1/p}\end{pmatrix},\begin{pmatrix}1/\delta&0&0&0\\0&1&0&0\\0&0&\delta&0\\0&0&0&1/\delta\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&-\alpha/\gamma&0\\0&1&-\beta/\gamma&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\\gamma\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&\gamma^{2/p}&0&0\\0&0&\gamma^{-1/p}&0\\0&0&0&\gamma^{1/p}\end{pmatrix},\begin{pmatrix}\gamma&0&0&0\\0&1&0&0\\0&0&1/\gamma&0\\0&0&0&\gamma\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\gamma\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right].$$

Next, if $\delta = \gamma = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix}1&0&(-\alpha/\beta)^{1/p}&0\\0&1&0&(-\alpha/\beta)^{1/p}\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&-\alpha/\beta&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1&0&0&0\\0&(1/\beta)^{1/p}&0&0\\0&0&(1/\beta)^{1/p}&0\\0&0&0&(1/\beta)^{2/p}\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1/\beta&0&0\\0&0&1&0\\0&0&0&1/\beta\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\beta\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right].$$

Finally, if $\delta = \gamma = \beta = 0$, but $\alpha \neq 0$, then

$$\left[\begin{pmatrix}1&0&0&0\\0&\alpha^{-2/p}&0&0\\0&0&\alpha^{1/p}&0\\0&0&0&\alpha^{-1/p}\end{pmatrix},\begin{pmatrix}1/\alpha&0&0&0\\0&1&0&0\\0&0&\alpha&0\\0&0&0&1/\alpha\end{pmatrix}\right]\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_8^1 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle;$$

$$K_8^2 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_5 \rangle;$$

$$K_8^3 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5 \rangle;$$

$$K_8^4 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_5 \rangle;$$

$$K_8^5 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_4^{[p]} = x_5 \rangle.$$

7.2 Extensions of $(L, x_1^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_4, x_3).$$

Thus, we get $f = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a

linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_4), \text{ and}$$
$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_4), \text{ and}$$
$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0.$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_4) = a = \alpha$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_1)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_1)\rangle_{\mathbb{F}}$. Note that

$$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [0, f_1], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_1)] = [(\Delta_{12}, f_1)] = [0]$, then $[(0, f_1)]$ is an scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 6$. Therefore,

$$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{12} + c\Delta_{23}$, for some $a, b, c \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{12} + c'\Delta_{23}$, for some $a', b', c' \in \mathbb{F}$. We determine a', b', c'. Note that

$$A\phi(x_1, x_3) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{33}x_3 + a_{43}x_4) = a_{11}a_{33}a + a_{21}a_{33}c;$$
$$A\phi(x_1, x_2) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4)$$
$$= (a_{11}a_{32} - a_{31}a_{12})a + (a_{11}a_{22} - a_{12}a_{21})b + (a_{21}a_{32} - a_{31}a_{22})c;$$
$$A\phi(x_2, x_3) = \phi(a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, a_{33}x_3 + a_{43}x_4) = a_{12}a_{33}a + a_{22}a_{33}c.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a_{11}a_{33} & 0 & a_{21}a_{33} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{22} - a_{12}a_{21} & a_{21}a_{32} - a_{31}a_{22} \\ a_{12}a_{33} & 0 & a_{22}a_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{7.3}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,8}$.

Also, we have $\omega = \beta f_2 + \gamma f_3 + \delta f_4$, for some $\beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\beta', \gamma', \delta' \in \mathbb{F}$. We have

$$A\omega(x_2) = a_{22}^p \beta + a_{32}^p \gamma + a_{42}^p \delta;$$

$$A\omega(x_3) = a_{33}^p \gamma + a_{43}^p \delta;$$

$$A\omega(x_4) = r^p \delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & a_{33}^p & a_{43}^p \\ 0 & 0 & r^p \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{7.4}$$

Thus, we can write Equations (7.3) and (7.4) together as follows:

$$\left[\begin{pmatrix} a_{11}a_{33} & 0 & a_{21}a_{33} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{22} - a_{12}a_{21} & a_{21}a_{32} - a_{31}a_{22} \\ a_{12}a_{33} & 0 & a_{22}a_{33} \end{pmatrix}, \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & a_{33}^p & a_{43}^p \\ 0 & 0 & r^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}, \begin{pmatrix} \beta' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Note that $A\phi(x_1, x_4) = A\phi(x_2, x_4) = A\phi(x_3, x_4) = 0$ and $A\omega(x_1) = 0$ which imply that $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$. Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we need to have $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$.

Let $\nu = \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\beta/\delta \\ 0 & 1 & -\gamma/\delta \\ 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}\right], \text{ and}$$
$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\delta)^{1/p} & 0 \\ 0 & 0 & \delta^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1/\delta \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^{1/p} & 0 \\ 0 & 0 & (1/\gamma)^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ 1 \\ 0 \end{pmatrix}\right].$$

If $\beta = 0$, then we have $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\beta)^{1/p} & 0 \\ 0 & 0 & (1/\beta)^{1/p} \end{pmatrix}, \begin{pmatrix} 1/\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ 1 \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]$$

Finally, if $\delta = \gamma = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\beta)^{1/p} & 0 \\ 0 & 0 & (1/\beta)^{1/p} \end{pmatrix}, \begin{pmatrix} 1/\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_8^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4 \rangle;$$

$$K_8^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_8^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_8^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_8^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5, x_3^{[p]} = x_5 \rangle.$$

7.3 Extensions of $(L, x_1^{[p]} = x_3)$

Note that $L^{[p]} = \langle x_3 \rangle$. Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we must have $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$, where $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, for some $a, b, c, d, e, f \in \mathbb{F}$.

Hence, $\phi(x_1, x_3) = 0$ which implies that $b = 0$. Since $\phi = \Delta_{13}$ gives us $L_{5,8}$, we deduce by Lemma 7.0.5 that $L_{5,8}$ cannot be constructed in this case. Similarly, we can show that for the following p-maps we also get $b = 0$.

II.4 $x_1^{[p]} = x_4, x_2^{[p]} = x_3$;

II.5 $x_4^{[p]} = x_3$;

II.6 $x_4^{[p]} = x_3, x_2^{[p]} = x_4$;

II.8 $x_3^{[p]} = x_4, x_2^{[p]} = x_3$.

7.4 Extensions of $(L, x_3^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore, we have

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_4, x_3).$$

Thus, we get $f = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a

linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x,y) = \phi(x,y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_4), \text{ and}$$
$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = 0$. Also, we have

$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$
$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \delta = 0$. Note that $\psi(x_4) = a = \gamma$. Therefore, $(\phi, \omega) = (a\Delta_{12}, af_3)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, f_3)\rangle_{\mathbb{F}}$. Note that

$$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [0, f_1], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{12}, 0)] + [(0, f_3)] = [(\Delta_{12}, f_3)] = [0]$, then $[(0, f_3)]$ is a scalar multiple of $[(\Delta_{12}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 6$. Therefore,

$$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)].$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{12} + c\Delta_{23}$, for some $a, b, c \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{12} + c'\Delta_{23}$, for some $a', b', c' \in \mathbb{F}$. We determine a', b', c'. Note that

$$A\phi(x_1, x_3) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{33}x_3 + a_{43}x_4) = a_{11}a_{33}a + a_{21}a_{33}c;$$
$$A\phi(x_1, x_2) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4)$$
$$= (a_{11}a_{32} - a_{31}a_{12})a + (a_{11}a_{22} - a_{12}a_{21})b + (a_{21}a_{32} - a_{31}a_{22})c;$$
$$A\phi(x_2, x_3) = \phi(a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, a_{33}x_3 + a_{43}x_4) = a_{12}a_{33}a + a_{22}a_{33}c.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a_{11}a_{33} & 0 & a_{21}a_{33} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{22} - a_{12}a_{21} & a_{21}a_{32} - a_{31}a_{22} \\ a_{12}a_{33} & 0 & a_{22}a_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{7.5}$$

The orbit with representative $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of this action gives us $L_{5,8}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \delta f_4$, for some $\alpha, \beta, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \delta' f_4$, for some $\alpha', \beta', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p \alpha + a_{21}^p \beta + a_{41}^p \delta;$$

$$A\omega(x_2) = a_{12}^p \alpha + a_{22}^p \beta + a_{42}^p \delta;$$

$$A\omega(x_4) = r^p \delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{42}^p \\ 0 & 0 & r^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}. \tag{7.6}$$

Thus, we can write Equations (7.5) and (7.6) together as follows:

$$\left[\begin{pmatrix} a_{11}a_{33} & 0 & a_{21}a_{33} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{22} - a_{12}a_{21} & a_{21}a_{32} - a_{31}a_{22} \\ a_{12}a_{33} & 0 & a_{22}a_{33} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{42}^p \\ 0 & 0 & r^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} \right].$$

Note that $A\phi(x_1, x_4) = A\phi(x_2, x_4) = A\phi(x_3, x_4) = 0$ and $A\omega(x_3) = 0$ which imply that $a_{43}^p \delta = 0$. Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we take $a_{43} = 0$. Therefore, $a_{43}^p \delta = 0$. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Let $\nu \neq 0$. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & -\beta/\delta \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\delta)^{1/p} & 0 \\ 0 & 0 & (1/\delta)^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\delta & 0 \\ 0 & 0 & 1/\delta \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right].$$

Next, if $\delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & (-\alpha/\beta)^{1/p} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha/\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} \right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\beta)^{1/p} & 0 \\ 0 & 0 & (1/\beta)^{1/p} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right].$$

Finally, if $\delta = \beta = 0$, but $\alpha \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/\alpha)^{1/p} & 0 \\ 0 & 0 & \alpha^{1/p} \end{pmatrix}, \begin{pmatrix} 1/\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \end{pmatrix}\right]\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the corresponding restricted Lie algebra structures are as follows:

$$K_8^{11} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{12} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{13} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

7.5 Detecting isomorphisms

The following is the list of all restricted Lie algebra structures on $L_{5,8}$ and yet, as we shall see below, we prove that some of them are isomorphic.

$$K_8^{1} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle;$$

$$K_8^{2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_5 \rangle;$$

$$K_8^{3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5 \rangle;$$

$$K_8^{4} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_5 \rangle;$$

$$K_8^{5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_4^{[p]} = x_5 \rangle;$$

$$K_8^6 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4 \rangle;$$

$$K_8^7 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_8^8 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_8^9 = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$

$$K_8^{10} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5, x_3^{[p]} = x_5 \rangle;$$

$$K_8^{11} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{12} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{13} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$

$$K_8^{14} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

The group $\mathrm{Aut}(L_{5,8})$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} & a_{11}a_{23} \\ a_{51} & a_{52} & a_{53} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix}.$$

Note that the following automorphism of $L_{5,8}$ that maps $x_1 \mapsto x_1$, $x_2 \mapsto x_3$, $x_3 \mapsto x_2$, $x_4 \mapsto x_5$ and $x_5 \mapsto x_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

implies that

$$K_8^2 \cong K_8^6, \quad K_8^3 \cong K_8^{11}, \quad K_8^7 \cong K_8^{12}.$$

Moreover, $K_8^8 \cong K_8^{10}$ via the automorphism

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Theorem 7.5.1 *The list of all the restricted Lie algebra structures on $L_{5,8}$, up to isomorphism, is as follows:*

$$L_{5,8}^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle;$$
$$L_{5,8}^2 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_5 \rangle;$$
$$L_{5,8}^3 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5 \rangle;$$
$$L_{5,8}^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_5 \rangle;$$
$$L_{5,8}^5 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_4^{[p]} = x_5 \rangle;$$
$$L_{5,8}^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$
$$L_{5,8}^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$
$$L_{5,8}^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = x_5 \rangle;$$
$$L_{5,8}^9 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle;$$
$$L_{5,8}^{10} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle.$$

In the remaining of this section we establish that the algebras given in Theorem 7.5.1 are pairwise non-isomorphic, thereby completing the proof of Theorem 7.5.1.

It is clear that $L_{5,8}^1$ is not isomorphic to the other restricted Lie algebras.

We claim that $L_{5,8}^2$ and $L_{5,8}^3$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^3_{5,8} \to L^2_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^2_{5,8}$ and $L^4_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,8} \to L^2_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Next, we claim that $L^2_{5,8}$ and $L^5_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^5_{5,8} \to L^2_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

It is clear that $L^2_{5,8}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^3_{5,8}$ and $L^4_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,8} \to L^3_{5,8}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$

$$A(x_5) = (a_{23}x_2 + a_{33}x_3 + a_{43}x_4 + a_{53}x_5)^{[p]}$$

$$a_{11}a_{23}x_4 + a_{11}a_{33}x_5 = a_{23}^p x_5,$$

which implies that $a_{11}a_{23} = 0$ and $a_{11}a_{33} = a_{23}^p$. Therefore, $a_{11} = 0$ or $a_{23} = 0$. First, if $a_{11} = 0$, we have a contradiction. Next, if $a_{23} = 0$ then $a_{11}a_{33} = 0$ and $a_{11}a_{23} = 0$ which is a contradiction.

Next, we claim that $L^3_{5,8}$ and $L^5_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^5_{5,8} \to L^3_{5,8}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$

$$A(x_5) = (a_{11}a_{22}x_4 + a_{11}a_{32}x_5)^{[p]}$$

$$a_{11}a_{23}x_4 + a_{11}a_{33}x_5 = 0.$$

Therefore, $a_{11}a_{23} = 0$ and $a_{11}a_{33} = 0$ which is a contradiction.

It is clear that $L^3_{5,8}$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^4_{5,8}$ and $L^5_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^5_{5,8} \to L^4_{5,8}$. Then

$$A(x_4^{[p]}) = A(x_4)^{[p]}$$

$$A(x_5) = (a_{11}a_{22}x_4 + a_{11}a_{32}x_5)^{[p]}$$

$$a_{11}a_{23}x_4 + a_{11}a_{33}x_5 = 0.$$

Therefore, $a_{11}a_{23} = 0$ and $a_{11}a_{33} = 0$ which is a contradiction.

It is clear that $L^4_{5,8}$ and $L^5_{5,8}$ are not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^6_{5,8}$ and $L^7_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^7_{5,8} \to L^6_{5,8}$. Then

$$A(x_3^{[p]}) = A(x_3)^{[p]}$$

$$A(x_5) = (a_{23}x_2 + a_{33}x_3 + a_{43}x_4 + a_{53}x_5)^{[p]}$$

$$a_{11}a_{23}x_4 + a_{11}a_{33}x_5 = a_{23}^p x_5,$$

which implies that $a_{11}a_{23} = 0$ and $a_{11}a_{33} = a_{23}^p$. Therefore, $a_{11} = 0$ or $a_{23} = 0$. First, if $a_{11} = 0$, we have a contradiction. Next, if $a_{23} = 0$ then $a_{11}a_{33} = 0$ and $a_{11}a_{23} = 0$ which is a contradiction.

Note that $L^6_{5,8}$ is not isomorphic to any of $L^8_{5,8}$, $L^{10}_{5,8}$ because $(L^6_{5,8})^{[p]^2} = 0$ but $(L^8_{5,8})^{[p]^2} \neq 0$, $(L^{10}_{5,8})^{[p]^2} \neq 0$.

Next, we claim that $L^6_{5,8}$ and $L^9_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^9_{5,8} \to L^6_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$

$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$

$$0 = a_{11}^p x_4 + a_{21}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Note that $L^7_{5,8}$ is not isomorphic to any of $L^8_{5,8}$, $L^{10}_{5,8}$ because $(L^7_{5,8})^{[p]^2} = 0$ but $(L^8_{5,8})^{[p]^2} \neq 0$, $(L^{10}_{5,8})^{[p]^2} \neq 0$.

Next, we claim that $L^7_{5,8}$ and $L^9_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^9_{5,8} \to L^7_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_4 + a_{31}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Note that $L^8_{5,8}$ is not isomorphic to $L^9_{5,8}$, because $(L^9_{5,8})^{[p]^2} = 0$ but $(L^8_{5,8})^{[p]^2} \neq 0$.

Next, we claim that $L^8_{5,8}$ and $L^{10}_{5,8}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^{10}_{5,8} \to L^8_{5,8}$. Then

$$A(x_1^{[p]}) = A(x_1)^{[p]}$$
$$0 = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$
$$0 = a_{11}^p x_4 + a_{41}^p x_5.$$

Therefore, $a_{11} = 0$ which is a contradiction.

Note that $L^9_{5,8}$ is not isomorphic to $L^{10}_{5,8}$, because $(L^9_{5,8})^{[p]^2} = 0$ but $(L^{10}_{5,8})^{[p]^2} \neq 0$.

Chapter 8

Restriction maps on $L_{5,9}$

Let

$$K_9 = L_{5,9} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle.$$

Note that $Z(K_9) = \langle x_4, x_5 \rangle_{\mathbb{F}}$ and the group $\mathrm{Aut}(L_{5,9})$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & u & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} - a_{12}a_{31} & a_{11}u & a_{12}u \\ a_{51} & a_{52} & a_{21}a_{32} - a_{22}a_{31} & a_{21}u & a_{22}u \end{pmatrix},$$

where $u = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Note that there exists an element $\alpha x_4 + \beta x_5 \in Z(L_{5,9})$ such that $(\alpha x_4 + \beta x_5)^{[p]} = 0$, for some $\alpha, \beta \in \mathbb{F}$. If $\alpha \neq 0$ then consider

$$K = \langle x_1', \ldots, x_5' \mid [x_1', x_2'] = x_3', [x_1', x_3'] = x_4', [x_2', x_3'] = x_5' \rangle,$$

where $x_1' = \alpha x_1 + \beta x_2, x_2' = \alpha^{-1} x_2, x_3' = x_3, x_4' = \alpha x_4 + \beta x_5, x_5' = \alpha^{-1} x_5$. Let $\phi : K_9 \to K$ given by $x_i \mapsto x_i'$, for $1 \leqslant i \leqslant 5$. It is easy to see that ϕ is an isomorphism. Therefore, in this case we can suppose that $x_4^{[p]} = 0$. If $\alpha = 0$ then $\beta \neq 0$ and we rescale

x_5 so that $x_5^{[p]} = 0$. Hence we can assume either $x_4^{[p]} = 0$ or $x_5^{[p]} = 0$. Furthermore, we claim that it is enough to consider the case where $x_5^{[p]} = 0$. Indeed, suppose that there exists a p-map of $L_{5,9}$ such that $x_4^{[p]} = 0$. Then the image of the x_i's under the following automorphism of $L_{5,9}$ yields another basis y_i of $L_{5,9}$ such that $y_5^{[p]} = 0$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now we let

$$L = \frac{L_{5,9}}{\langle x_5 \rangle_{\mathbb{F}}} \cong L_{4,3},$$

where $L_{4,3} = \langle x_1, \ldots, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$. The group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & d & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}d \end{pmatrix},$$

where $d = a_{11}a_{22}$.

Lemma 8.0.2 *Let $K = L_{5,9}$ and $[p] : K \to K$ be a p-map on K such that $x_5^{[p]} = 0$ and let $L = \frac{K}{M}$ where $M = \langle x_5 \rangle_{\mathbb{F}}$. Then $K \cong L_\theta$ where $\theta = (\Delta_{23}, \omega) \in Z^2(L, \mathbb{F})$.*

Proof. Let $\pi : K \to L$ be the projection map. We have the exact sequence

$$0 \to M \to K \to L \to 0.$$

Let $\sigma : L \to K$ such that $x_i \mapsto x_i$, $1 \leqslant i \leqslant 4$. Then σ is an injective linear map and $\pi\sigma = 1_L$. Now, we define $\phi : L \times L \to M$ by $\phi(x_i, x_j) = [\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j])$, $1 \leqslant i, j \leqslant 4$ and $\omega : L \to M$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Note that

$$\phi(x_2, x_3) = [\sigma(x_2), \sigma(x_3)] - \sigma([x_2, x_3]) = [x_2, x_3] = x_5;$$
$$\phi(x_1, x_2) = [\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]) = 0.$$

Similarly, we can show that $\phi(x_1, x_3) = \phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$. Therefore, $\phi = \Delta_{23}$. Now, by Lemma 2.2.2, we have $\theta = (\Delta_{23}, \omega) \in Z^2(L, \mathbb{F})$ and $K \cong L_\theta$. ■

We deduce that any p-map on K can be obtained by an extension of L via $\theta = (\Delta_{23}, \omega)$, for some ω.

Note that by [12], there are four non-isomorphic restricted Lie algebra structures on $L_{4,3}$ given by the following p-maps:

I.1 Trivial p-map;

I.2 $x_1^{[p]} = x_4$;

I.3 $x_2^{[p]} = \xi x_4$;

I.4 $x_3^{[p]} = x_4$.

In the following subsections we consider each of the above cases and find all possible $[\theta] = [(\phi, \omega)] \in H^2(L, \mathbb{F})$ and construct L_θ. Note that by Lemma 8.0.2, it suffices to assume $[\theta] = [(\Delta_{23}, \omega)]$ and find all non-isomorphic restricted Lie algebra structures on $L_{5,9}$.

8.1 Extensions of (L, trivial p-map)

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4);$$

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $e = f = 0$. Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{14}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0.$$

Similarly, we can show that $d = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (\Delta_{13}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Let $[(\phi,\omega)] \in H^2(L,\mathbb{F})$. Then we have $\phi = a\Delta_{14} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{14} + b'\Delta_{23}$, for some $a', b' \in \mathbb{F}$. We determine a', b'. Note that

$$A\phi(x_1,x_4) = \phi(Ax_1,Ax_4) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, a_{11}dx_4) = a_{11}^2 da;$$

$$A\phi(x_2,x_3) = \phi(Ax_2,Ax_3) = \phi(a_{22}x_2 + a_{32}x_3 + a_{42}x_4, dx_3 + a_{11}a_{32}x_4) = a_{22}db.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}^2 d & 0 \\ 0 & a_{22}d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{8.1}$$

The orbit with representative $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,9}$. Also we have $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \beta', \gamma', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = \omega(Ax_1) = \omega(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4) = a_{11}^p\alpha + a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta;$$

$$A\omega(x_2) = a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta;$$

$$A\omega(x_3) = d^p\gamma + a_{11}^p a_{32}^p\delta;$$

$$A\omega(x_4) = a_{11}^p d^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p d^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{8.2}$$

Thus, we can write Equations (8.1) and (8.2) together as follows:

$$\left[d\begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p d^p \end{pmatrix}\right]\left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix}\right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^4$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -\alpha/\delta \\ 0 & 1 & 0 & -\beta/\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma/\delta & \gamma^2/\delta^2 \\ 0 & 0 & 1 & -\gamma/\delta \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}\right].$$

Hence the set of vectors $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$ with $\delta \neq 0$ form an single orbit with orbit representative

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta \end{pmatrix}$. Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha/\gamma & 0 \\ 0 & 1 & -\beta/\gamma & 0 \\ 0 & 0 & 1 & -\beta/\gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right], \text{ and}$$

$$\left[(1/\gamma)^{1/p}\begin{pmatrix} (1/\gamma)^{4/p} & 0 \\ 0 & (1/\gamma)^{-1/p} \end{pmatrix}, \begin{pmatrix} \gamma^{-2} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & \gamma^{-3} \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

Next, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha/\beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right], \text{ and}$$

$$\left[\beta^{1/p}\begin{pmatrix} \beta^{4/p} & 0 \\ 0 & \beta^{-1/p} \end{pmatrix}, \begin{pmatrix} \beta^2 & 0 & 0 & 0 \\ 0 & 1/\beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right].$$

Finally, if $\beta = \gamma = \delta = 0$, but $\alpha \neq 0$, then we have $\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$.

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\delta\end{pmatrix}.$$

Lemma 8.1.1 *The vectors* $\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbits. Then

$$\left[d\begin{pmatrix}a_{11}^2 & 0\\0 & a_{22}\end{pmatrix},\begin{pmatrix}a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p\\0 & a_{22}^p & a_{32}^p & a_{42}^p\\0 & 0 & d^p & a_{11}^p a_{32}^p\\0 & 0 & 0 & a_{11}^p d^p\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}\right].$$

From this we obtain that $da_{22} = 1$ and that $\alpha_2 = a_{11}^p\alpha_1$ which gives that $\alpha_2\alpha_1^{-1} = a_{22}^{-2p} \in (\mathbb{F}^*)^2$. Assume next that $\alpha_2\alpha_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p}\begin{pmatrix}(\epsilon)^{4/p} & 0\\0 & (\epsilon)^{-1/p}\end{pmatrix},\begin{pmatrix}\epsilon^2 & 0 & 0 & 0\\0 & \epsilon^{-1} & 0 & 0\\0 & 0 & \epsilon & 0\\0 & 0 & 0 & \epsilon^3\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_1\\0\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_2\\0\\0\\0\end{pmatrix}\right],$$

as required. ∎

Lemma 8.1.2 *The vectors* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[d\begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ 0 & a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & 0 & a_{11}^p d^p \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}\right].$$

From this we obtain that $\delta_2 = a_{11}^p d^p \delta_1$ and that $da_{22} = 1$ which gives that $\delta_2\delta_1^{-1} = a_{22}^{-3p} \in (\mathbb{F}^*)^3$. Assume next that $\delta_2\delta_1^{-1} = \epsilon^3$, with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\epsilon^{1/p}\begin{pmatrix} (\epsilon)^{4/p} & 0 \\ 0 & (\epsilon)^{-1/p} \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_2 \end{pmatrix}\right],$$

as required. ■

Hence, the corresponding restricted Lie algebra structures are as follows:

$$K_9^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle;$$

$$K_9^2(\alpha) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5 \rangle;$$

$$K_9^3 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = x_5 \rangle;$$

$$K_9^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_5 \rangle;$$

$$K_9^5(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_4^{[p]} = \delta x_5 \rangle$$

where $\alpha, \delta \in \mathbb{F}^*$.

Lemma 8.1.3 *We have $K_9^2(\alpha_1) \cong K_9^2(\alpha_2)$ if and only if $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$.*

Proof. Suppose that $K_9^2(\alpha_1) \cong K_9^2(\alpha_2)$. Then there exists an isomorphism $f : K_9^2(\alpha_1) \to K_9^2(\alpha_2)$. Therefore, we have

$$\begin{aligned} f(x_2^{[p]}) =& f(x_2)^{[p]} \\ 0 =& (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\ 0 =& a_{12}^p \alpha_2 x_5, \end{aligned}$$

which implies that $a_{12} = 0$. Hence, $f(\langle x_5 \rangle_{\mathbb{F}}) = \langle x_5 \rangle_{\mathbb{F}}$. Thus, f induces an automorphism $A : K_9^2(\alpha_1)/\langle x_5 \rangle \to K_9^2(\alpha_2)/\langle x_5 \rangle$ and, by Lemma 2.3.3, we have that $A\theta_1 = c\theta_2$, for some $c \in \mathbb{F}^*$, where $\theta_1 = (\Delta_{23}, \alpha_1 f_1)$ and $\theta_2 = (\Delta_{23}, \alpha_2 f_1)$ which implies that $c^{-1}A\theta_1 = \theta_2$. Therefore, without loss of generality, we can suppose that $c = 1$. So, $\begin{pmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \alpha_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit and it follows by Lemma 8.1.1 that $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$. The converse is easy to see by Lemma 2.3.3. ■

Similarly, as in Lemma 8.1.3, we can prove that $K_9^5(\delta_1) \cong K_9^5(\delta_2)$ if and only if $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

8.2 Extensions of $(L, x_1^{[p]} = x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4);$$

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $e = f = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and}$$

$$c = \phi(x_2, x_3) = \delta^1\psi(x_2, x_3) = \psi([x_2, x_3]) = 0.$$

Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = \psi(x_4), \text{ and}$$

$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = 0.$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_4) = b = \alpha$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, bf_1)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (\Delta_{13}, f_1)\rangle_{\mathbb{F}}$. Note that

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [0, f_1], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{13}, 0)] + [(0, f_1)] = [(\Delta_{13}, f_1)] = [0]$, then $[(0, f_1)]$ is a scalar multiple of $[(\Delta_{13}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{23}$ for some $a', b' \in \mathbb{F}$. We determine a, b'. Note that

$$\begin{aligned} A\phi(x_1, x_3) &= \phi(Ax_1, Ax_3) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, dx_3 + a_{11}a_{32}x_4) \\ &= a_{11}da + a_{21}db; \end{aligned}$$

$$A\phi(x_2, x_3) = \phi(Ax_2, Ax_3) = \phi(a_{22}x_2 + a_{32}x_3 + a_{42}x_4, dx_3 + a_{11}a_{32}x_4) = a_{22}db.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{8.3}$$

The orbit with representative $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,9}$.

Also, we have $\omega = \beta f_2 + \gamma f_3 + \delta f_4$, for some $\beta, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \beta' f_2 + \gamma' f_3 + \delta' f_4$, for some $\beta', \gamma', \delta' \in \mathbb{F}$. We have

$$\begin{aligned} A\omega(x_2) &= a_{22}^p\beta + a_{32}^p\gamma + a_{42}^p\delta; \\ A\omega(x_3) &= d^p\gamma + a_{11}^p a_{32}^p\delta; \\ A\omega(x_4) &= a_{11}^p d^p\delta. \end{aligned}$$

In the matrix form we can write this as

$$\begin{pmatrix}\beta'\\ \gamma'\\ \delta'\end{pmatrix} = \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & d^p & a_{11}^p a_{32}^p\\ 0 & 0 & a_{11}^p d^p\end{pmatrix}\begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}. \tag{8.4}$$

Thus, we can write Equations (8.3) and (8.4) together as follows:

$$\left[\begin{pmatrix} a_{11}d & a_{21}d\\ 0 & a_{22}d\end{pmatrix}, \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p\\ 0 & d^p & a_{11}^p a_{32}^p\\ 0 & 0 & a_{11}^p d^p\end{pmatrix}\right]\left[\begin{pmatrix} a\\ b\end{pmatrix}, \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix} a'\\ b'\end{pmatrix}, \begin{pmatrix}\beta'\\ \gamma'\\ \delta'\end{pmatrix}\right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix}0\\1\end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we need to have $A\omega(x_1) = 0$ which implies that $a_{21}^p\beta + a_{31}^p\gamma + a_{41}^p\delta = 0$.

Let $\nu = \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix}0\\0\\0\end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & -\beta/\delta\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix} 0\\ 1\end{pmatrix}, \begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix} 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right], \text{and}$$

$$\left[\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & -\gamma/\delta & \gamma^2/\delta^2\\ 0 & 1 & -\gamma/\delta\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix} 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix} 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \delta\end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\beta/\gamma & 0 \\ 0 & 1 & -\beta/\gamma \\ 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} (1/\gamma)^{3/p} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & (1/\gamma)^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

Finally, if $\gamma = \delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} (1/\beta)^{-3/p} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & (1/\beta)^{-3} \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right].$$

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}.$$

Lemma 8.2.1 *The vectors* $\begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix}, \begin{pmatrix} a_{22}^p & a_{32}^p & a_{42}^p \\ 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right].$$

From this we obtain that $\delta_2 = a_{11}^p d^p \delta_1$ and that $da_{22} = 1$ which gives that $\delta_2\delta_1^{-1} = a_{22}^{-3p} \in (\mathbb{F}^*)^3$. Assume next that $\delta_2\delta_1^{-1} = \epsilon^3$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix} \epsilon^{3/p} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon^{-1} & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right],$$

as required. ■

Hence, the corresponding restricted Lie algebra structurs are as follows:

$$K_9^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4 \rangle;$$

$$K_9^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$$

$$K_9^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$$

$$K_9^9(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = \delta x_5 \rangle$$

where $\delta \in \mathbb{F}^*$. Similarly, as in Lemma 8.1.3, we can prove that $K_9^9(\delta_1) \cong K_9^9(\delta_2)$ if and only if $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

8.3 Extensions of $(L, x_2^{[p]} = \xi x_4)$

First, we find a basis for $Z^2(L, \mathbb{F})$. Let $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4) \in Z^2(L, \mathbb{F})$. Then we must have $\delta^2\phi(x, y, z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4);$$

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $e = f = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$.

Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$
$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and}$$
$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0.$$

Also, we have

$$\beta = \omega(x_2) = \tilde{\psi}(x_2) = \psi(x_2^{[p]}) = \xi\psi(x_4), \text{ and}$$
$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\gamma = \delta = 0$. Note that $\psi(x_4) = b = \beta\xi^{-1}$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, b\xi f_2)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0), (\Delta_{13}, \xi f_2)\rangle_{\mathbb{F}}$. Note that

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [0, f_1], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{13}, 0)] + \xi[(0, f_2)] = [(\Delta_{13}, \xi f_2)] = [0]$, then $[(0, f_2)]$ is a scalar multiple of $[(\Delta_{13}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_3)], [(0, f_4)]$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{23}$ for some $a', b' \in \mathbb{F}$. Then we have

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{8.5}$$

The orbit with representative $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,9}$.

Also, we have $\omega = \alpha f_1 + \gamma f_3 + \delta f_4$, for some $\alpha, \gamma, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \gamma' f_3 + \delta' f_4$, for some $\alpha', \gamma', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p \alpha + a_{31}^p \gamma + a_{41}^p \delta;$$

$$A\omega(x_3) = d^p \gamma + a_{11}^p a_{32}^p \delta;$$

$$A\omega(x_4) = a_{11}^p d^p \delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{31}^p & a_{41}^p \\ 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix}. \tag{8.6}$$

Thus, we can write Equations (8.5) and (8.6) together as follows:

$$\left[\begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{31}^p & a_{41}^p \\ 0 & d^p & a_{11}^p a_{32}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \gamma' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we need to have $A\omega(x_2) = 0$ which implies that $a_{32}^p \gamma + a_{42}^p \delta = 0$.

Let $\nu = \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that

$\delta \neq 0$. Then

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & -\alpha/\delta\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}\alpha\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -\gamma/\delta\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \delta\end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\gamma \neq 0$, then

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -\alpha/\gamma & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}\alpha\\ \gamma\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix}(1/\gamma)^{3/p} & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}(1/\gamma)^2 & 0 & 0\\ 0 & 1/\gamma & 0\\ 0 & 0 & (1/\gamma)^3\end{pmatrix}\right]\left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ \gamma\\ 0\end{pmatrix}\right] = \left[\begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right].$$

Finally, if $\gamma = \delta = 0$, but $\alpha \neq 0$, then we get $\begin{pmatrix}\alpha\\ 0\\ 0\end{pmatrix}$.

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}\alpha\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \delta\end{pmatrix}.$$

Lemma 8.3.1 *The vectors* $\begin{pmatrix}\alpha_1\\ 0\\ 0\end{pmatrix}$ *and* $\begin{pmatrix}\alpha_2\\ 0\\ 0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix}a_{11}d & a_{21}d\\0 & a_{22}d\end{pmatrix},\begin{pmatrix}a_{11}^p & a_{31}^p & a_{41}^p\\0 & d^p & a_{11}^pa_{32}^p\\0 & 0 & a_{11}^pd^p\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}\right].$$

From this we obtain that $da_{22}=1$ and that $\alpha_2=a_{11}^p\alpha_1$ which gives that $\alpha_2\alpha_1^{-1}=a_{22}^{-2p}\in(\mathbb{F}^*)^2$. Assume next that $\alpha_2\alpha_1^{-1}=\epsilon^2$ with some $\epsilon\in\mathbb{F}^*$. Then

$$\left[\begin{pmatrix}\epsilon^{3/p} & 0\\0 & 1\end{pmatrix},\begin{pmatrix}\epsilon^2 & 0 & 0\\0 & \epsilon & 0\\0 & 0 & \epsilon^3\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}\right]=\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}\right],$$

as required. ■

Lemma 8.3.2 *The vectors* $\begin{pmatrix}0\\0\\\delta_1\end{pmatrix}$ *and* $\begin{pmatrix}0\\0\\\delta_2\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\delta_2\delta_1^{-1}\in(\mathbb{F}^*)^3$.

Proof. First assume that $\begin{pmatrix}0\\0\\\delta_1\end{pmatrix}$ and $\begin{pmatrix}0\\0\\\delta_2\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix}a_{11}d & a_{21}d\\0 & a_{22}d\end{pmatrix},\begin{pmatrix}a_{11}^p & a_{31}^p & a_{41}^p\\0 & d^p & a_{11}^pa_{32}^p\\0 & 0 & a_{11}^pd^p\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\delta_1\end{pmatrix}\right]=\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\delta_2\end{pmatrix}\right].$$

From this we obtain that $\delta_2 = a_{11}^p d^p \delta_1$ and that $da_{22} = 1$ which gives that $\delta_2\delta_1^{-1} = a_{22}^{-3p} \in (\mathbb{F}^*)^3$. Assume next that $\delta_2\delta_1^{-1} = \epsilon^3$ with some $\epsilon \in \mathbb{F}^*$.

$$\left[\begin{pmatrix} \epsilon^{3/p} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right],$$

as required. ■

Therefore, the corresponding restricted Lie algebra structures are as follows:

$K_9^{10}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4\rangle;$

$K_9^{11}(\xi, \alpha) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5, x_2^{[p]} = \xi x_4\rangle;$

$K_9^{12}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4, x_3^{[p]} = x_5\rangle;$

$K_9^{13}(\xi, \delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4, x_4^{[p]} = \delta x_5\rangle,$

where $\alpha, \delta, \xi \in \mathbb{F}^*$.

Lemma 8.3.3 *We have $K_9^{10}(\xi_1) \cong K_9^{10}(\xi_2)$ if and only if $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$.*

Proof. First assume that $f : K_9^{10}(\xi_1) \to K_9^{10}(\xi_2)$ is an isomorphism. Then

$$\begin{aligned} f(x_2^{[p]}) &= f(x_2)^{[p]} \\ f(\xi_1 x_4) &= (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\ \xi_1(a_{11}ux_4 + a_{21}ux_5) &= a_{22}^p \xi_2 x_4. \end{aligned}$$

Hence,

$$\xi_1 a_{11} u = a_{22}^p \xi_2, \quad \xi_2 a_{21} u = 0,$$

which implies that $\xi_2\xi_1^{-1} = a_{11}^2 a_{22}^{1-p} \in (\mathbb{F}^*)^2$.

Assume next that $\xi_2\xi_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then the following automorphism of $L_{5,9}$

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix},$$

is an isomorphism between $K_9^{10}(\xi_1)$ and $K_9^{10}(\xi_2)$. ■

Note that in Section 8.5.1 we shall prove that $K_9^{11}(\xi_1, \alpha_1) \cong K_9^{11}(\xi_2, \alpha_2)$ if and only if $\frac{\xi_1}{\alpha_1}\frac{\xi_2}{\alpha_2} \in (\mathbb{F}^*)^2$ and the equation

$$x^2 - \frac{\xi_1}{\alpha_1}y^2 = \frac{\alpha_2}{\alpha_1}$$

has a solution in $\mathbb{F}$.

Lemma 8.3.4 *We have $K_9^{12}(\xi_1) \cong K_9^{12}(\xi_2)$ if and only if $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$.*

Proof. Assume that $f : K_9^{12}(\xi_1) \to K_9^{12}(\xi_2)$ is an isomorphism. Therefore,

$$f(x_3^{[p]}) = f(x_3)^{[p]}$$

$$a_{12}ux_4 + a_{22}ux_5 = u^p x_5$$

which implies that $a_{12}u = 0$. Therefore, the isomorphism f takes x_5 to a multiple of x_5. Now, if $K_9^{12}(\xi_1) \cong K_9^{12}(\xi_2)$ then $K_9^{12}(\xi_1)/\langle x_5\rangle_{\mathbb{F}} \cong K_9^{12}(\xi_2)/\langle x_5\rangle_{\mathbb{F}}$ which implies that $L_{4,3}^3(\xi_1) \cong L_{4,3}^3(\xi_2)$. Note that by [12], $L_{4,3}^3(\xi_1) \cong L_{4,3}^3(\xi_2)$ if and only if $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$.

Conversely, suppose that $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$. Let $L_1 = K_9^{12}(\xi_1)/\langle x_5\rangle_{\mathbb{F}}$ and $L_2 = K_9^{12}(\xi_2)/\langle x_5\rangle_{\mathbb{F}}$. Since $\xi_2\xi_1^{-1} \in (\mathbb{F}^*)^2$, it follows from [12] that $L_1 \cong L_2$. Now we extend L_1 to obtain $K_9^{12}(\xi_1)$ and L_2 to obtain $K_9^{12}(\xi_2)$. Note that both $K_9^{12}(\xi_1)$ and $K_9^{12}(\xi_2)$ are constructed by the orbit (Δ_{23}, f_3). Therefore, by Lemma 2.3.3, $K_9^{12}(\xi_1) \cong K_9^{12}(\xi_2)$. ■

Lemma 8.3.5 *We have* $K_9^{13}(\xi_1,\delta_1)\cong K_9^{13}(\xi_2,\delta_2)$ *if and only if* $\xi_2\xi_1^{-1}\in(\mathbb{F}^*)^2$ *and* $\delta_2\delta_1^{-1}\in(\mathbb{F}^*)^3$.

Proof. Assume that $f: K_9^{13}(\xi_1,\delta_1)\to K_9^{13}(\xi_2,\delta_2)$ is an isomorphiism. Then

$$\begin{aligned} f(x_4^{[p]}) &= f(x_4)^{[p]} \\ f(\delta_1 x_5) &= (a_{11}ux_4+a_{21}ux_5)^{[p]} \\ \delta_1 a_{12}ux_4+\delta_1 a_{22}ux_5 &= a_{11}^p u^p \delta_2 x_5. \end{aligned}$$

Hence, $\delta_1 a_{12}u = 0$ which implies that $a_{12}=0$. So, f takes x_5 to a multiple of x_5 and induces an isomorphism $K_9^{13}(\xi_1,\delta_1)/\langle x_5\rangle_{\mathbb{F}}\cong K_9^{13}(\xi_2,\delta_2)/\langle x_5\rangle_{\mathbb{F}}$. Therefore, by [12], $\xi_2\xi_1^{-1}\in(\mathbb{F}^*)^2$.

Furthermore, the restricted Lie algebra $K_9^{13}(\xi,\delta)$ is constructed by the orbit $(\Delta_{23},\delta f_4)$. Since $K_9^{13}(\xi_1,\delta_1)\cong K_9^{13}(\xi_2,\delta_2)$, we deduce by Lemma 2.3.3 that two orbits $(\Delta_{23},\delta_1 f_4)$ and $(\Delta_{23},\delta_2 f_4)$ are in the same $\mathrm{Aut}(L)$-orbit. Therefore, $\delta_2\delta_1^{-1}\in(\mathbb{F}^*)^3$.

Conversely, suppose that $\xi_2\xi_1^{-1}\in(\mathbb{F}^*)^2$ and $\delta_2\delta_1^{-1}\in(\mathbb{F}^*)^3$. Let $L_1=K_9^{13}(\xi_1,\delta_1)/\langle x_5\rangle_{\mathbb{F}}$ and $L_2=K_9^{13}(\xi_2,\delta_2)/\langle x_5\rangle_{\mathbb{F}}$. Since $\xi_2\xi_1^{-1}\in(\mathbb{F}^*)^2$, it follows by [12], that $L_1\cong L_2$. Now we extend L_1 to obtain $K_9^{13}(\xi_1,\delta_1)$ and L_2 to obtain $K_9^{13}(\xi_2,\delta_2)$. Note that $K_9^{13}(\xi_1,\delta_1)$ is constructed by the orbit $(\Delta_{23},\delta_1 f_4)$ and $K_9^{13}(\xi_2,\delta_2)$ is construted by the orbit $(\Delta_{23},\delta_2 f_4)$. Since $\delta_2\delta_1^{-1}\in(\mathbb{F}^*)^3$, we deduce by Lemma 8.3.2 that $(\Delta_{23},\delta_1 f_4)$ and $(\Delta_{23},\delta_2 f_4)$ are in the same $\mathrm{Aut}(L)$-orbit. Therefore, by Lemma 2.3.3, $K_9^{13}(\xi_1,\delta_1)\cong K_9^{13}(\xi_2,\delta_2)$. ■

8.4 Extensions of $(L, x_3^{[p]}=x_4)$

First, we find a basis for $Z^2(L,\mathbb{F})$. Let $(\phi,\omega)=(a\Delta_{12}+b\Delta_{13}+c\Delta_{14}+d\Delta_{23}+e\Delta_{24}+f\Delta_{34},\alpha f_1+\beta f_2+\gamma f_3+\delta f_4)\in Z^2(L,\mathbb{F})$. Then we must have $\delta^2\phi(x,y,z)=0$ and

$\phi(x, y^{[p]}) = 0$, for all $x, y, z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1, x_2, x_3) = \phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2) = \phi(x_2, x_4);$$

$$0 = (\delta^2\phi)(x_1, x_2, x_4) = \phi([x_1, x_2], x_4) + \phi([x_2, x_4], x_1) + \phi([x_4, x_1], x_2) = \phi(x_3, x_4).$$

Thus, we get $e = f = 0$. Also, we have $\phi(x, y^{[p]}) = 0$. Therefore, $\phi(x, x_4) = 0$, for all $x \in L$ and hence $\phi(x_1, x_4) = \phi(x_2, x_4) = \phi(x_3, x_4) = 0$ which implies that $c = e = f = 0$. Therefore, $Z^2(L, \mathbb{F})$ has a basis consisting of:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3), (0, f_4).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Since $B^2(L, \mathbb{F}) \subseteq Z^2(L, \mathbb{F})$, we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4)$. Note that there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$a = \phi(x_1, x_2) = \delta^1\psi(x_1, x_2) = \psi([x_1, x_2]) = \psi(x_3), \text{ and}$$

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = \psi(x_4), \text{ and}$$

$$c = \phi(x_1, x_4) = \delta^1\psi(x_1, x_4) = \psi([x_1, x_4]) = 0.$$

Also, we have

$$\gamma = \omega(x_3) = \tilde{\psi}(x_3) = \psi(x_3^{[p]}) = \psi(x_4), \text{ and}$$

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \delta = 0$. Note that $\psi(x_4) = b = \gamma$. Therefore, $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13}, bf_3)$ and hence $B^2(L, \mathbb{F}) = \langle (\Delta_{12}, 0), (\Delta_{13}, f_3) \rangle_{\mathbb{F}}$. Note that

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [0, f_1], [(0, f_2)], [(0, f_3)], [(0, f_4)]$$

spans $H^2(L, \mathbb{F})$. Since $[(\Delta_{13}, 0)] + [(0, f_3)] = [(\Delta_{13}, f_3)] = [0]$, then $[(0, f_3)]$ is a scalar multiple of $[(\Delta_{13}, 0)]$ in $H^2(L, \mathbb{F})$. Note that $\dim H^2 = \dim Z^2 - \dim B^2 = 5$. Therefore,

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)]$$

forms a basis for $H^2(L, \mathbb{F})$.

Let $[(\phi, \omega)] \in H^2(L, \mathbb{F})$. Then we have $\phi = a\Delta_{13} + b\Delta_{23}$, for some $a, b \in \mathbb{F}$. Suppose that $A\phi = a'\Delta_{13} + b'\Delta_{23}$ for some $a', b' \in \mathbb{F}$. We determine a, b'. Note that

$$A\phi(x_1, x_3) = \phi(Ax_1, Ax_3) = \phi(a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, dx_3 + a_{11}a_{32}x_4)$$
$$= a_{11}da + a_{21}db; \text{ and}$$

$$A\phi(x_2, x_3) = \phi(Ax_2, Ax_3) = \phi(a_{22}x_2 + a_{32}x_3 + a_{42}x_4, dx_3 + a_{11}a_{32}x_4) = a_{22}db.$$

In the matrix form we can write this as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{8.7}$$

The orbit with representative $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of this action gives us $L_{5,9}$.

Also, we have $\omega = \alpha f_1 + \beta f_2 + \delta f_4$ for some $\alpha, \beta, \delta \in \mathbb{F}$. Suppose that $A\omega = \alpha' f_1 + \beta' f_2 + \delta' f_4$, for some $\alpha', \beta', \delta' \in \mathbb{F}$. We have

$$A\omega(x_1) = a_{11}^p\alpha + a_{21}^p\beta + a_{41}^p\delta;$$
$$A\omega(x_2) = a_{22}^p\beta + a_{42}^p\delta;$$
$$A\omega(x_4) = a_{11}^p d^p\delta.$$

In the matrix form we can write this as

$$\begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ 0 & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}. \tag{8.8}$$

Thus, we can write Equations (8.7) and (8.8) together as follows:

$$\left[\begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ 0 & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \right] = \left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \\ \delta' \end{pmatrix} \right].$$

Now we find the representatives of the orbits of the action of $\mathrm{Aut}(L)$ on the set of ω's such that the orbit represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is preserved under the action of $\mathrm{Aut}(L)$ on the set of ϕ's. Note that we need to have $A\omega(x_3) = 0$ which implies that $a_{11}^p a_{32}^p \delta = 0$.

Let $\nu = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} \in \mathbb{F}^3$. If $\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\{\nu\}$ is clearly an $\mathrm{Aut}(L)$-orbit. Suppose that $\delta \neq 0$. Then

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha/\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ \delta \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\beta/\delta \\ 0 & 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \\ \delta \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}\right].$$

Next, if $\delta = 0$, but $\beta \neq 0$, then

$$\left[\begin{pmatrix} (1/\beta)^{-3/p} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & \beta^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta^2\alpha \\ 1 \\ 0 \end{pmatrix}\right].$$

If $\alpha = 0$, we have $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and if $\alpha \neq 0$, we have $\begin{pmatrix} \beta^2\alpha \\ 1 \\ 0 \end{pmatrix}$.

Finally, if $\beta = \delta = 0$, but $\alpha \neq 0$, then we have $\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$.

Thus the following elements are $\mathrm{Aut}(L)$-orbit representatives:

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\delta\end{pmatrix}, \begin{pmatrix}\alpha\\1\\0\end{pmatrix}.$$

Lemma 8.4.1 *The vectors* $\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$.

Proof. First assume that $\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix}a_{11}d & a_{21}d\\0 & a_{22}d\end{pmatrix}, \begin{pmatrix}a_{11}^p & a_{21}^p & a_{41}^p\\0 & a_{22}^p & a_{42}^p\\0 & 0 & a_{11}^p d^p\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}\right].$$

From this we obtain that $da_{22} = 1$ and that $\alpha_2 = a_{11}^p\alpha_1$ which gives that $\alpha_2\alpha_1^{-1} = a_{22}^{-2p} \in (\mathbb{F}^*)^2$. Assume next that $\alpha_2\alpha_1^{-1} = \epsilon^2$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix}\epsilon^{3/p} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\epsilon^2 & 0 & 0\\0 & \epsilon^{-1} & 0\\0 & 0 & \epsilon^3\end{pmatrix}\right]\left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\alpha_1\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\alpha_2\\0\\0\end{pmatrix}\right],$$

as required. ■

Lemma 8.4.2 *The vectors* $\begin{pmatrix}0\\0\\\delta_1\end{pmatrix}$ *and* $\begin{pmatrix}0\\0\\\delta_2\end{pmatrix}$ *are in the same* $\mathrm{Aut}(L)$*-orbit if and only if* $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

Proof. First assume that $\begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}$ are in the same $\mathrm{Aut}(L)$-orbit. Then

$$\left[\begin{pmatrix} a_{11}d & a_{21}d \\ 0 & a_{22}d \end{pmatrix}, \begin{pmatrix} a_{11}^p & a_{21}^p & a_{41}^p \\ 0 & a_{22}^p & a_{42}^p \\ 0 & 0 & a_{11}^p d^p \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right].$$

From this we obtain that $\delta_2 = a_{11}^p d^p \delta_1$ and that $da_{22} = 1$ which gives that $\delta_2\delta_1^{-1} = a_{22}^{-3p} \in (\mathbb{F}^*)^3$. Assume next that $\delta_2\delta_1^{-1} = \epsilon^3$ with some $\epsilon \in \mathbb{F}^*$. Then

$$\left[\begin{pmatrix} \epsilon^{3/p} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 \\ 0 & 0 & \epsilon^3 \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right],$$

as required. ■

Therefore, the corresponding restricted Lie algebra structures are as follows:

$K_9^{14} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_4 \rangle$;

$K_9^{15}(\alpha) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5, x_3^{[p]} = x_4 \rangle$;

$K_9^{16} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_4, x_2^{[p]} = x_5 \rangle$;

$K_9^{17}(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = \delta x_5 \rangle$;

$K_9^{18}(\alpha) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle$

where $\alpha, \delta \in \mathbb{F}^*$. Similarly, as in Lemma 8.1.3, we can prove that $K_9^{15}(\alpha_1) \cong K_9^{15}(\alpha_2)$ if and only if $\alpha_2\alpha_1^{-1} \in (\mathbb{F}^*)^2$ and $K_9^{17}(\delta_1) \cong K_9^{17}(\delta_2)$ if and only if $\delta_2\delta_1^{-1} \in (\mathbb{F}^*)^3$.

Lemma 8.4.3 *We have* $K_9^{18}(\alpha) \cong K_9^{15}(\alpha)$.

Proof. Note that the following automorphism of $L_{5,9}$ is an isomorphim from $K_9^{15}(\alpha)$ to $K_9^{18}(\alpha)$.

$$\begin{pmatrix} 1 & -(1/\alpha)^{1/p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\alpha^{1/p}((1/\alpha)^{1/p})^{1/p} & 0 & 1 & 0 & 0 \\ 0 & 0 & -((1/\alpha)^{1/p})^{1/p} & 1 & -(1/\alpha)^{1/p} \\ 0 & 0 & \alpha^{1/p}((1/\alpha)^{1/p})^{1/p} & 0 & 1 \end{pmatrix}.$$

■

8.5 Detecting isomorphisms

The following is the list of all restricted Lie algebra structures on $L_{5,9}$ and yet as we shall see below we prove that some of them are isomorphic.

$K_9^1 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle;$

$K_9^2(\alpha) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5 \rangle;$

$K_9^3 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = x_5 \rangle;$

$K_9^4 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_5 \rangle;$

$K_9^5(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_4^{[p]} = \delta x_5 \rangle;$

$K_9^6 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4 \rangle;$

$K_9^7 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$

$K_9^8 = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$

$K_9^9(\delta) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = \delta x_5 \rangle;$

$K_9^{10}(\xi) = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4 \rangle;$

$K_9^{11}(\alpha,\xi) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_1^{[p]}=\alpha x_5, x_2^{[p]}=\xi x_4\rangle;$

$K_9^{12}(\xi) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_2^{[p]}=\xi x_4, x_3^{[p]}=x_5\rangle;$

$K_9^{13}(\xi,\delta) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_2^{[p]}=\xi x_4, x_4^{[p]}=\delta x_5\rangle;$

$K_9^{14} = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_3^{[p]}=x_4\rangle;$

$K_9^{15}(\alpha) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_1^{[p]}=\alpha x_5, x_3^{[p]}=x_4\rangle;$

$K_9^{16} = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_3^{[p]}=x_4, x_2^{[p]}=x_5\rangle;$

$K_9^{17}(\delta) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_3^{[p]}=x_4, x_4^{[p]}=\delta x_5\rangle;$

Note that the following automorphism of $L_{5,9}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

implies that

$$K_9^3 \cong K_9^6, \quad K_9^{10}(\xi) \cong K_9^2(\alpha), \quad K_9^4 \cong K_9^{14}, \quad K_9^{12}(\xi) \cong K_9^{15}(\alpha), \quad K_9^8 \cong K_9^{16}.$$

Theorem 8.5.1 *The list of all restricted Lie algebra structures on $L_{5,9}$, up to isomorphism, is as follows:*

$L_{5,9}^1 = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5\rangle;$

$L_{5,9}^2(\alpha) = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_1^{[p]}=\alpha x_5\rangle;$

$L_{5,9}^3 = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_2^{[p]}=x_5\rangle;$

$L_{5,9}^4 = \langle x_1,\ldots,x_5 \mid [x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5, x_3^{[p]}=x_5\rangle;$

$L^5_{5,9}(\delta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_4^{[p]} = \delta x_5 \rangle;$

$L^6_{5,9} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_2^{[p]} = x_5 \rangle;$

$L^7_{5,9} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_3^{[p]} = x_5 \rangle;$

$L^8_{5,9}(\delta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = x_4, x_4^{[p]} = \delta x_5 \rangle;$

$L^9_{5,9}(\xi, \alpha) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_1^{[p]} = \alpha x_5, x_2^{[p]} = \xi x_4 \rangle;$

$L^{10}_{5,9}(\xi) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4, x_3^{[p]} = x_5 \rangle;$

$L^{11}_{5,9}(\xi, \delta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_2^{[p]} = \xi x_4, x_4^{[p]} = \delta x_5 \rangle;$

$L^{12}_{5,9}(\delta) = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = \delta x_5 \rangle,$

where $\alpha, \xi \in T_2$ *and* $\delta \in T_3$.

In the remaining of this section we establish that the algebras given in Theorem 8.5.1 are pairwise non-isomorphic, thereby completing the proof of Theorem 8.5.1.

It is clear that $L^1_{5,9}$ is not isomorphic to the other restricted Lie algebras. We claim that $L^2_{5,9}(\alpha)$ and $L^3_{5,9}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,9}(\alpha) \to L^3_{5,9}$. Then $A(x_1^{[p]}) = A(x_1)^{[p]}$. So,

$$A(\alpha x_5) = (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5)^{[p]}$$

$$\alpha a_{12}ux_4 + \alpha a_{22}ux_5 = a_{11}^p x_1^{[p]} + a_{21}^p x_2^{[p]} + a_{31}^p x_3^{[p]} + a_{41}^p x_4^{[p]} + a_{51}^p x_5^{[p]}$$

$$\alpha a_{12}ux_4 + \alpha a_{22}ux_5 = a_{21}^p x_5,$$

which implies that $\alpha a_{12}u = 0$. Since $u \neq 0, \alpha \neq 0$, we have $a_{12} = 0$. Thus, $u = a_{11}a_{22}$. Also, we have

$$A(x_2^{[p]}) = A(x_2)^{[p]}$$

$$0 = (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]}$$

$$0 = a_{22}^p x_5,$$

which implies that $a_{22} = 0$. Therefore $u = 0$, which is a contradiction.

Next, we claim that $L^2_{5,9}(\alpha)$ and $L^4_{5,9}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,9}(\alpha) \to L^4_{5,9}$. Then

$$\begin{aligned} A(x_3^{[p]}) =& A(x_3)^{[p]} \\ 0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\ 0 =& u^p x_5. \end{aligned}$$

Therefore, $u = 0$, which is a contradiction.

Next, we claim that $L^2_{5,9}(\alpha)$ and $L^5_{5,9}(\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^2_{5,9}(\alpha) \to L^5_{5,9}(\delta)$. Then $A(x_4^{[p]}) = A(x_4)^{[p]}$. So,

$$\begin{aligned} 0 =& (a_{11}ux_4 + a_{21}ux_5)^{[p]} \\ 0 =& a_{11}^p u^p \delta x_5, \end{aligned}$$

which implies that $\alpha_{11}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{11} = 0$. Thus, $u = -a_{12}a_{21}$. Also, we have

$$\begin{aligned} A(x_5^{[p]}) =& A(x_5)^{[p]} \\ 0 =& (a_{12}ux_4 + a_{22}ux_5)^{[p]} \\ 0 =& a_{12}^p u^p \delta x_5, \end{aligned}$$

which implies that $a_{12}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{12} = 0$. Therefore $u = 0$, which is a contradiction.

It is clear that $L^2_{5,9}(\alpha)$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L_{5,9}^3$ and $L_{5,9}^4$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^3 \to L_{5,9}^4$. Then

$$\begin{aligned} A(x_3^{[p]}) =& A(x_3)^{[p]} \\ 0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\ 0 =& u^p x_5. \end{aligned}$$

Therefore, $u = 0$, which is a contradiction.

Next, we claim that $L_{5,9}^3$ and $L_{5,9}^5(\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^3 \to L_{5,9}^5(\delta)$. Then $A(x_4^{[p]}) = A(x_4)^{[p]}$. So,

$$\begin{aligned} 0 =& (a_{11}ux_4 + a_{21}ux_5)^{[p]} \\ 0 =& a_{11}^p u^p \delta x_5, \end{aligned}$$

which implies that $\alpha_{11}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{11} = 0$. Thus, $u = -a_{12}a_{21}$. Also, we have

$$\begin{aligned} A(x_5^{[p]}) =& A(x_5)^{[p]} \\ 0 =& (a_{12}ux_4 + a_{22}ux_5)^{[p]} \\ 0 =& a_{12}^p u^p \delta x_5, \end{aligned}$$

which implies that $a_{12}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{12} = 0$. Therefore $u = 0$, which is a contradiction.

It is clear that $L_{5,9}^3$ is not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^4_{5,9}$ and $L^5_{5,9}(\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^4_{5,9} \to L^5_{5,9}(\delta)$. Then $A(x_4^{[p]}) = A(x_4)^{[p]}$. So,

$$0 =(a_{11}ux_4 + a_{21}ux_5)^{[p]}$$
$$0 =a_{11}^p u^p \delta x_5,$$

which implies that $\alpha_{11}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{11} = 0$. Thus, $u = -a_{12}a_{21}$. Also, we have

$$A(x_5^{[p]}) =A(x_5)^{[p]}$$
$$0 =(a_{12}ux_4 + a_{22}ux_5)^{[p]}$$
$$0 =a_{12}^p u^p \delta x_5,$$

which implies that $a_{12}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{12} = 0$. Therefore $u = 0$, which is a contradiction.

It is clear that $L^4_{5,9}$ and $L^5_{5,9}(\delta)$ are not isomorphic to the other restricted Lie algebras.

Next, we claim that $L^6_{5,9}$ and $L^7_{5,9}$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L^6_{5,9} \to L^7_{5,9}$. Then

$$A(x_3^{[p]}) =A(x_3)^{[p]}$$
$$0 =(ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]}$$
$$0 =u^p x_5.$$

Therefore, $u = 0$, which is a contradiction.

Note that $L_{5,9}^6$ is not isomorphic to any of $L_{5,9}^8(\delta)$, $L_{5,9}^{11}(\xi,\delta)$, $L_{5,9}^{12}(\delta)$ because $(L_{5,9}^6)^{[p]^2} = 0$ but $(L_{5,9}^8(\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{11}(\xi,\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{12}(\delta))^{[p]^2} \neq 0$.

In Section 8.5.1, we shall show that $L_{5,9}^6$ and $L_{5,9}^9(\alpha,\xi)$ are not isomorphic.

Next, we claim that $L_{5,9}^6$ and $L_{5,9}^{10}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^6 \to L_{5,9}^{10}(\xi)$. Then

$$\begin{aligned} A(x_3^{[p]}) =& A(x_3)^{[p]} \\ 0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\ 0 =& u^p x_5. \end{aligned}$$

Therefore, $u = 0$, which is a contradiction.

Note that $L_{5,9}^7$ is not isomorphic to any of $L_{5,9}^8(\delta)$, $L_{5,9}^{11}(\xi,\delta)$, $L_{5,9}^{12}(\delta)$ because $(L_{5,9}^7)^{[p]^2} = 0$ but $(L_{5,9}^8(\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{11}(\xi,\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{12}(\delta))^{[p]^2} \neq 0$.

Next, we claim that $L_{5,9}^7$ and $L_{5,9}^9(\alpha,\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^9(\alpha,\xi) \to L_{5,9}^7$. Then

$$\begin{aligned} A(x_3^{[p]}) =& A(x_3)^{[p]} \\ 0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\ 0 =& u^p x_5. \end{aligned}$$

Therefore, $u = 0$, which is a contradiction.

Next, we claim that $L_{5,9}^7$ and $L_{5,9}^{10}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^7 \to L_{5,9}^{10}(\xi)$. Then

$$\begin{aligned} A(x_2^{[p]}) =& A(x_2)^{[p]} \\ 0 =& (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\ 0 =& a_{22}^p \xi x_4 + a_{32}^p x_5, \end{aligned}$$

which implies that $a_{22}^p \xi = 0$. Since $\xi \neq 0$, we have $a_{22} = 0$. Thus, $u = -a_{12}a_{21}$. Also, we have

$$\begin{aligned} A(x_3^{[p]}) =& A(x_3)^{[p]} \\ A(x_5) =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\ a_{12}ux_4 + a_{22}ux_5 =& u^p x_5, \end{aligned}$$

which imples that $a_{12}u = 0$. Since $u \neq 0$, we have $a_{12} = 0$. Therefore, $u = 0$, which is a contradiction.

Note that $L_{5,9}^8(\delta)$ is not isomorphic to any of $L_{5,9}^9(\alpha,\xi)$, $L_{5,9}^{10}(\xi)$ because $(L_{5,9}^9(\alpha,\xi))^{[p]^2} = (L_{5,9}^{10}(\xi))^{[p]^2} = 0$ but $(L_{5,9}^8(\delta))^{[p]^2} \neq 0$.

Next, we claim that $L_{5,9}^8(\delta)$ and $L_{5,9}^{11}(\xi,\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^8(\delta) \to L_{5,9}^{11}(\xi,\delta)$. Then

$$\begin{aligned} A(x_2^{[p]}) =& A(x_2)^{[p]} \\ 0 =& (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)^{[p]} \\ 0 =& a_{22}^p \xi x_4 + a_{42}^p \delta x_5, \end{aligned}$$

which implies that $a_{22}^p\xi = 0$. Since $\xi \neq 0$, we have $a_{22} = 0$. Thus, $u = -a_{12}a_{21}$. Also, we have

$$
\begin{aligned}
A(x_5^{[p]}) =& A(x_5)^{[p]} \\
0 =& (a_{12}ux_4 + a_{22}ux_5)^{[p]} \\
0 =& a_{12}^p u^p \delta x_5,
\end{aligned}
$$

which implies that $a_{12}^p u^p \delta = 0$. Since $u \neq 0, \delta \neq 0$, we have $a_{12} = 0$. Therefore, $u = 0$, which is a contradiction.

Next, we claim that $L_{5,9}^8(\delta)$ and $L_{5,9}^{12}(\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^8(\delta) \to L_{5,9}^{12}(\delta)$. Then

$$
\begin{aligned}
A(x_3^{[p]}) =& A(x_3)^{[p]} \\
0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\
0 =& u^p x_4 + (a_{11}a_{32} - a_{31}a_{12})^p \delta x_5.
\end{aligned}
$$

Therefore, $u = 0$, which is a contradiction.

Next, we claim that $L_{5,9}^9(\alpha, \xi)$ and $L_{5,9}^{10}(\xi)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^9(\alpha, \xi) \to L_{5,9}^{10}(\xi)$. Then

$$
\begin{aligned}
A(x_3^{[p]}) =& A(x_3)^{[p]} \\
0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\
0 =& u^p x_5.
\end{aligned}
$$

Therefore, $u = 0$, which is a contradiction.

Note that $L_{5,9}^{9}(\alpha,\xi)$ is not isomorphic to any of $L_{5,9}^{11}(\xi,\delta)$, $L_{5,9}^{12}(\delta)$ because $(L_{5,9}^{9}(\alpha,\xi))^{[p]^2} = 0$ but $(L_{5,9}^{11}(\xi,\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{12}(\delta))^{[p]^2} \neq 0$.

Note that $L_{5,9}^{10}(\xi)$ is not isomorphic to any of $L_{5,9}^{11}(\xi,\delta)$, $L_{5,9}^{12}(\delta)$ because $(L_{5,9}^{10}(\xi))^{[p]^2} = 0$ but $(L_{5,9}^{11}(\xi,\delta))^{[p]^2} \neq 0$, $(L_{5,9}^{12}(\delta))^{[p]^2} \neq 0$.

Finally, we claim that $L_{5,9}^{11}(\xi,\delta)$ and $L_{5,9}^{12}(\delta)$ are not isomorphic. Suppose to the contrary that there exists an isomorphism $A : L_{5,9}^{11}(\xi,\delta) \to L_{5,9}^{12}(\delta)$. Then

$$\begin{aligned}
A(x_3^{[p]}) =& A(x_3)^{[p]} \\
0 =& (ux_3 + (a_{11}a_{32} - a_{31}a_{12})x_4 + (a_{21}a_{32} - a_{31}a_{22}x_5)^{[p]} \\
0 =& u^p x_4 + (a_{11}a_{32} - a_{31}a_{12})^p \delta x_5.
\end{aligned}$$

Therefore, $u = 0$, which is a contradiction.

8.5.1 Detecting isomorphism for $L_{5,9}^{6}$ and $L_{5,9}^{9}(\xi,\alpha)$

In this section, we show that $L_{5,9}^{6} \ncong L_{5,9}^{9}(\xi,\alpha)$.

Let

$$L = \frac{L_{5,9}^{6}}{M} \cong \frac{L_{5,9}^{9}(\xi,\alpha)}{M} \cong \langle x_1, x_2, x_3 \mid [x_1,x_2] = x_3 \rangle \cong L_{3,2}.$$

First, we find a basis for $Z^2(L,\mathbb{F})$. Let $(\phi,\omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3) \in Z^2(L,\mathbb{F})$. Then we must have $\delta^2\phi(x,y,z) = 0$ and $\phi(x, y^{[p]}) = 0$, for all $x,y,z \in L$. Therefore,

$$0 = (\delta^2\phi)(x_1,x_2,x_3) = \phi([x_1,x_2],x_3) + \phi([x_2,x_3],x_1) + \phi([x_3,x_1],x_2) = 0.$$

Since the p-map is trivial, $\phi(x, y^{[p]}) = \phi(x, 0) = 0$, for all $x, y \in L$. Therefore, a basis for $Z^2(L, \mathbb{F})$ is as follows:

$$(\Delta_{12}, 0), (\Delta_{13}, 0), (\Delta_{23}, 0), (0, f_1), (0, f_2), (0, f_3).$$

Next, we find a basis for $B^2(L, \mathbb{F})$. Let $(\phi, \omega) \in B^2(L, \mathbb{F})$. Then we have $(\phi, \omega) = (a\Delta_{12} + b\Delta_{13} + c\Delta_{23}, \alpha f_1 + \beta f_2 + \gamma f_3)$. So, there exists a linear map $\psi : L \to \mathbb{F}$ such that $\delta^1\psi(x, y) = \phi(x, y)$ and $\tilde{\psi}(x) = \omega(x)$, for all $x, y \in L$. So, we have

$$b = \phi(x_1, x_3) = \delta^1\psi(x_1, x_3) = \psi([x_1, x_3]) = 0.$$

Similarly, we can show that $c = 0$. Also, we have

$$\alpha = \omega(x_1) = \tilde{\psi}(x_1) = \psi(x_1^{[p]}) = 0.$$

Similarly, we can show that $\beta = \gamma = \delta = 0$. Therefore, $(\phi, \omega) = (a\Delta_{12}, 0)$ and hence $B^2(L, \mathbb{F}) = \langle(\Delta_{12}, 0)\rangle_{\mathbb{F}}$. We deduce that a basis for $H^2(L, \mathbb{F})$ is as follows:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)].$$

The group $\mathrm{Aut}(L)$ consists of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & d \end{pmatrix},$$

where $d = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Let $(\phi, \omega) \in H^2(L, M)$ and $\phi = \Delta_{13}x_4 + \Delta_{23}x_5$. Let $A \in \mathrm{Aut}(L)$. Then $A\phi = A(\Delta_{13}x_4 + \Delta_{23}x_5) = A(\Delta_{13})x_4 + A(\Delta_{23})x_5$. Note that we have

$$A\Delta_{13}(x_1, x_3) = \Delta_{13}(Ax_1, Ax_3) = \Delta_{13}(a_{11}x_1 + a_{21}x_2 + a_{31}x_3, dx_3) = a_{11}d;$$

$$A\Delta_{13}(x_2, x_3) = \Delta_{13}(Ax_2, Ax_3) = \Delta_{13}(a_{12}x_1 + a_{22}x_2 + a_{32}x_3, dx_3) = a_{12}d;$$

Therefore, $A\Delta_{13} = a_{11}d\Delta_{13} + a_{12}d\Delta_{23}$. Similarly, we can show that $A\Delta_{23} = a_{21}d\Delta_{13} + a_{22}d\Delta_{23}$.

Also, we have

$$\begin{aligned} Af_1(x_1) &= f_1(Ax_1) = f_1(a_{11}x_1 + a_{21}x_2 + a_{31}x_3) = a_{11}; \\ Af_1(x_2) &= f_1(Ax_2) = f_1(a_{12}x_1 + a_{22}x_2 + a_{32}x_3) = a_{12}; \\ Af_1(x_3) &= 0. \end{aligned}$$

Therefore, $Af_1 = a_{11}f_1 + a_{12}f_2$. Similarly, we can show that $Af_2 = a_{21}f_1 + a_{22}f_2$.

Lemma 8.5.2 *The restricted Lie algebras* $L^6_{5,9}$ *and* $L^9_{5,9}(\alpha,\xi)$ *are not isomorphic for any values of* $\alpha, \xi \in T_2$.

Proof. Suppose that $L^6_{5,9} \cong L^9_{5,9}(\alpha,\xi)$. Then by Lemma 2.3.3, we have

$$\langle A(\Delta_{13}, \xi f_2), A(\Delta_{23}, \alpha f_1)\rangle_{\mathbb{F}} = \langle (\Delta_{13}, f_1), (\Delta_{23}, f_2)\rangle_{\mathbb{F}}.$$

Note that

$$A(\Delta_{13}, \xi f_2) = (a_{11}d\Delta_{13} + a_{12}d\Delta_{23}, \xi a_{21}f_1 + \xi a_{22}f_2).$$

Therefore, we need to have $a_{11}a_{32} - a_{31}a_{12} = 0$, $a_{11}d = \xi a_{21}$, and $a_{12}d = \xi a_{22}$. Hence, $a_{11}a_{22}d\xi = a_{21}a_{12}d\xi$ implying that $a_{11}a_{22} = a_{12}a_{21}$ which is a contradiction. Therefore, $L^6_{5,9} \not\cong L^9_{5,9}(\alpha,\xi)$. ■

Lemma 8.5.3 *The restricted Lie algebras* $K^{11}_9(\xi_1,\alpha_1)$ *and* $K^{11}_9(\xi_2,\alpha_2)$ *are isomorphic if and only if* $\frac{\xi_1}{\alpha_1}\frac{\xi_2}{\alpha_2} \in (\mathbb{F}^*)^2$ *and the equation*

$$x^2 - \frac{\xi_1}{\alpha_1}y^2 = \frac{\alpha_2}{\alpha_1}$$

has a solution in $\mathbb{F}$.

Proof. Note that, by Lemma 2.3.3, $L^9_{5,9}(\xi_1, \alpha_1)$ and $L^9_{5,9}(\xi_2, \alpha_2)$ are isomorphic if and only if there exists $A \in \mathrm{Aut}(L)$ such that

$$\langle(\Delta_{13}, \xi_1 f_2), (\Delta_{23}, \alpha_1 f_1)\rangle_{\mathbb{F}} = \langle A(\Delta_{13}, \xi_2 f_2), A(\Delta_{23}, \alpha_2 f_1)\,\rangle_{\mathbb{F}}$$
$$= \langle(a_{11}d\Delta_{13} + a_{12}d\Delta_{23}, \xi_2 a_{21} f_1 + \xi_2 a_{22} f_2), (a_{21}d\Delta_{13} + a_{22}d\Delta_{23}, \alpha_2 a_{11} f_1 + \alpha_2 a_{12} f_2)\rangle_{\mathbb{F}}.$$

We deduce that $L^9_{5,9}(\xi_1, \alpha_1)$ and $L^9_{5,9}(\xi_2, \alpha_2)$ are isomorphic if and only if there exist $a_{i,j}, 1 \leqslant i, j \leqslant 2$, that satisfy the equations

$$a_{11}d\xi_1 = \xi_2 a_{22}; \tag{8.9}$$

$$a_{12}d\alpha_1 = \xi_2 a_{21}; \tag{8.10}$$

$$a_{21}d\xi_1 = \alpha_2 a_{12}; \tag{8.11}$$

$$a_{22}d\alpha_1 = \alpha_2 a_{11}. \tag{8.12}$$

Let $\frac{\xi_1}{\alpha_1} = \epsilon_1$, $\frac{\xi_2}{\alpha_2} = \epsilon_2$, and $\frac{\alpha_2}{\alpha_1} = \epsilon_3$. First suppose that $K^{11}_9(\xi_1, \alpha_1) \cong K^{11}_9(\xi_2, \alpha_2)$. Note that if $a_{12} \neq 0$ then Equations (8.10) and (8.11) imply that

$$\frac{\xi_1}{\alpha_1}\frac{a_{21}}{a_{12}} = \frac{\alpha_2}{\xi_2}\frac{a_{12}}{a_{21}}. \tag{8.13}$$

Therefore, $\epsilon_1\epsilon_2 = (\frac{a_{12}}{a_{21}})^2 \in (\mathbb{F}^*)^2$. Also, if $a_{22} \neq 0$ then Equations (8.9) and (8.12) give us

$$\frac{\xi_1}{\alpha_1}\frac{a_{11}}{a_{22}} = \frac{\xi_2}{\alpha_2}\frac{a_{22}}{a_{11}}. \tag{8.14}$$

Therefore, $\epsilon_1\epsilon_2 = (\epsilon_2\frac{a_{22}}{a_{11}})^2 \in (\mathbb{F}^*)^2$. Now, if $a_{11} = 0$ then $a_{22} = 0$ and $d = a_{12}a_{21}$. Then Equation (8.11) implies that $-a_{21}^2\epsilon_1 = \epsilon_3$ and we can take $x = 0$ and $y = a_{21}$. If $a_{12} = 0$ then $a_{21} = 0$ and $d = a_{11}a_{22}$. Then Equation (8.12) implies that $a_{22}^2 = \epsilon_3$ and we can take $x = a_{22}$ and $y = 0$.

Suppose now that $a_{i,j} \neq 0$, for all $1 \leqslant i, j \leqslant 2$. Then we deduce from Equations (8.14) and (8.13) that

$$a_{22} = (\frac{\epsilon_1}{\epsilon_2})^{1/2}a_{11}, \quad a_{12} = (\epsilon_1\epsilon_2)^{1/2}a_{21}. \tag{8.15}$$

Thus, a_{22} and a_{12} are determined by a_{11} and a_{21}. Now, Equation (8.11) implies that

$$(\frac{\epsilon_1}{\epsilon_2})^{1/2}d = \epsilon_3. \tag{8.16}$$

Since, by Equation (8.15), $d = (\frac{\epsilon_1}{\epsilon_2})^{1/2}a_{11}^2 - (\epsilon_1\epsilon_2)^{1/2}a_{21}^2$, we deduce from Equation (8.16) that

$$\frac{\epsilon_1}{\epsilon_2}a_{11}^2 - \epsilon_1 a_{21}^2 = \epsilon_3$$

Since $\frac{\epsilon_1}{\epsilon_2} \in (\mathbb{F}^*)^2$, we deduce that the equation

$$x^2 - \epsilon_1 y^2 = \epsilon_3 \tag{8.17}$$

has a solution in $\mathbb{F}$. To prove the sufficiency, suppose that Equation (8.17) has a solution in $\mathbb{F}$ and $\epsilon_1\epsilon_2 \in (\mathbb{F}^*)^2$. It is easy to see that $\frac{\epsilon_1}{\epsilon_2} \in (\mathbb{F}^*)^2$. Then we take $a_{11} = (\frac{\epsilon_2}{\epsilon_1})^{1/2}x$, $a_{21} = y$, and find a_{12} and a_{22} from Equation (8.15). We can now verify that Equations (8.9), (8.10), (8.11), and (8.12) are satisfied. Thus, $K_9^{11}(\xi_1, \alpha_1) \cong K_9^{11}(\xi_2, \alpha_2)$. The proof is complete. ■

Lemma 8.5.4 *If $a \neq 0$ then the equation $x^2 - ay^2 = b$ has a solution in the prime field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, where p is a prime number.*

Proof. Consider the two sets $A = \{x^2 \mid x \in \mathbb{F}_p\}$ and $B = \{ay^2 + b \mid y \in \mathbb{F}_p\}$. Since $a \neq 0$, both sets have $\frac{p+1}{2}$ elements. Indeed, let $\{a_1, \ldots, a_{p-1}\}$ be the nonzero elements of F_p. Then $a_i^2 \equiv (p - a_i)^2 \mod p$, for every$1 \leqslant i \leqslant p-1$. Therefore, all quadratic residues must be among

$$a_1^2, \ldots, a_{\frac{p-1}{2}}^2.$$

Now, we prove that the above elements are distinct. For this, suppose that $a_i^2 \equiv a_j^2 \mod p$, for $1 \leqslant i, j \leqslant \frac{p-1}{2}$. Then, we have $(a_i - a_j)(a_i + a_j) \equiv 0 \mod p$, which implies that either $a_i - a_j \equiv 0 \mod p$, or $a_i + a_j \equiv 0 \mod p$. Note that if $a_i + a_j \equiv 0 \mod p$, then $p - a_i \equiv a_i \equiv -a_j \mod p$, and hence $a_i - a_j \equiv p \equiv 0 \mod p$, which lead us to $a_i \equiv a_j \equiv 0 \mod p$, a contradiction. Therefore, $a_i - a_j \equiv 0 \mod p$, and so $a_i \equiv a_j \mod p$. So, the number of elements of A is $\frac{p+1}{2}$. Note that we can easily define a bijection from set A to set B and hence the number of elements of B is also $\frac{p+1}{2}$. So, the sets A and B must overlap and hence a solution exists. ■

In fact, more can be said. We thank Behrang Noohi for the following lemma.

Lemma 8.5.5 *Let q be a power of a prime p. The number of solutions of the equation*

$$x^2 - ay^2 = b \tag{8.18}$$

over the finite field $\mathbb{F}_q$ is equal to $q + 1$, provided that $a \neq 0$ and $b \neq 0$.

Proof. First, if a is a square in $\mathbb{F}_q$, then the number of solutions is clearly $q - 1$ (factorize the left hand side). So assume that a is not a square in $\mathbb{F}_q$. Note that the number of solutions is at most $2q$ because for each choice of x there are at most 2 choices for y.

Consider the equation

$$x^2 - ay^2 - bz^2 = 0 \tag{8.19}$$

in $\mathbb{F}_q$. By the well known Ax-Katz theorem, the number of solutions to this equation is divisible by q. Therefore, the number of non-zero solutions to this equation is of the form $qk - 1$, for some positive integer k. Note that for all such solutions we must have $z \neq 0$ as a is not a square. Consider the map

$$(x, y, z) \mapsto (x/z, y/z).$$

It is a $(q-1)$-to-one surjective map from the set of solutions of (8.19) to (8.18). Therefore, the number of solutions of (8.18) is $\frac{qk-1}{q-1}$. The only chance for this to be a positive integer (of size at most $2q$) is when $k = 1$ or $k = q$. The case $k = 1$ would imply that (8.18) has a unique solution which is not possible (if x works, then so does $-x$). So we must have $k = q$, which means that (8.18) has exactly $q + 1$ solutions. ■

Corollary 8.5.6 *The restricted Lie algebras* $K_9^{11}(\xi_1, \alpha_1)$ *and* $K_9^{11}(\xi_2, \alpha_2)$ *are isomorphic if and only if* $\frac{\xi_1}{\alpha_1}\frac{\xi_2}{\alpha_2} \in (\mathbb{F}^*)^2$.

Proof. We have $\frac{\xi_1}{\alpha_1} \neq 0$. Then by Lemma 8.5.4, the equation

$$x^2 - \frac{\xi_1}{\alpha_1}y^2 = \frac{\alpha_2}{\alpha_1}$$

has a solution in the prime field $\mathbb{F}_p$. Note that since every field of characteristic p contains the prime field $\mathbb{F}_p$ as a subfield then the equation

$$x^2 - \frac{\xi_1}{\alpha_1}y^2 = \frac{\alpha_2}{\alpha_1}$$

has always a solution in $\mathbb{F}$. Therefore, by Lemma 8.5.3, the restricted Lie algebras $K_9^{11}(\xi_1, \alpha_1)$ and $K_9^{11}(\xi_2, \alpha_2)$ are isomorphic if and only if $\frac{\xi_1}{\alpha_1}\frac{\xi_2}{\alpha_2} \in (\mathbb{F}^*)^2$. ■

Chapter 9

Conclusion

In this thesis, we used the analogue of Skjelbred-Sund method to classify all $[p]$-nilpotent restricted Lie algebras of dimension 5 over perfect fields of characteristic $p \geqslant 5$. We obtained 8 equivalence classes of $[p]$-maps for $L_{5,1}$, 21 equivalence classes of $[p]$-maps for $L_{5,2}$, 22 equivalence classes of $[p]$-maps for $L_{5,3}$, 2 equivalence classes of $[p]$-maps for $L_{5,4}$, 5 equivalence classes of $[p]$-maps for $L_{5,5}$, 5 equivalence classes of $[p]$-maps for $L_{5,6}$, 5 equivalence classes of $[p]$-maps for $L_{5,7}$, 10 equivalence classes of $[p]$-maps for $L_{5,8}$, and 12 equivalence classes of $[p]$-maps for $L_{5,9}$. In this thesis, we have not included the equivalence classes of $[p]$-maps in characteristic 2 and 3. Let L be a restricted Lie algebra and M a vector space and $\theta = (\phi, \omega)$ be such that $[\theta] \in H^2(L, M)$. Then ω and the $[p]$-map are not semilinear in characteristic 2 and 3. One may use the analogue of Skjelbred-Sund method to classify all equivalence classes of $[p]$-maps in characteristic 2 and 3, but more computations must be performed. We wish to give a complete classification of p-nilpotent restricted Lie algebras of dimension 5 over perfect fields of characteristic 2 and 3 in the future.

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