MEMORIAL UNIVERSITY OF NEWFOUNDLAND

MASTER'S THESIS

Relative Growth Rate of Subgroups of Free Groups

Author:
Valentin GRUZDEV

Supervisor:
Dr. Yuri BAHTURIN

A thesis submitted in fulfilment of the requirements for Master's Degree

in the

Faculty of Science
Department of Mathematics and Statistics

August 2014

Declaration of Authorship

I, Valentin Gruzdev, declare that this thesis titled, 'Relative Growth Rate of Subgroups of Free Groups' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a Master's degree at Memorial University of Newfoundland.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at Memorial University of Newfoundland or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

15 August 2014

"The theory of groups is a branch of mathematics in which one does something to something and then compares the results with the result of doing the same thing to something else, or something else to the same thing. "

James Newman

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

Abstract

Faculty of Science

Department of Mathematics and Statistics

Master's Degree

Relative Growth Rate of Subgroups of Free Groups

by Valentin Gruzdev

In this thesis we investigate the following problem: given a free group of a finite rank greater or equal to 2, and its non-cyclic finitely generated subgroup of infinite index, is it true that the relative growth rate of the subgroup is strictly less than the growth rate of the free group itself?

Acknowledgements

The author would like to thank his supervisor Yuri Bahturin for his invaluable help in writing this thesis and guidance throughout the programme, and Aleksandr Olshanskii for suggestion of the proof of Theorem 5.3.

Contents

Symbols

$\blacksquare$	indicates the end of a proof;
$\blacktriangle$	indicates the end of an example or a definition;
❏	means either no further proof will be given or it is postponed.

Set Theory:

$\varnothing$	is the empty set;
$\#X$	denotes the cardinal of the set X;
$\bigcup$	denotes union of sets;
$\bigsqcup$	denotes disjoint union of sets;
$\bigcap$	denotes intersection of sets;
$\subseteq$	denotes inclusion;
$\subset$	denotes proper inclusion;
$\in$	denotes set membership.

Chapter 0

Introduction

Walther von Dyck was the first to introduce free groups in 1882, but the interest in them did not arise until 1924 when Jakob Nielsen coined the term and initiated the first deep study of their properties. Five years later John von Neumann introduced the notion of amenable groups. He observed that if a countable discrete group contains a free subgroup on two generators, then it is not amenable. The converse to this statement was known as the von Neumann conjecture, which was refuted by Rostislav Grigorchuk in 1978, see [13]. In his work Grigorchuk used the notion of relative growth rate of a subgroup of a free group. In this thesis we are going to show that given a finitely generated free group of rank greater or equal to 2, and its non-cyclic finitely generated subgroup of infinite index, the relative growth rate of the subgroup is strictly less than the growth rate of the free group itself.

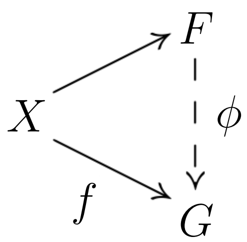
This thesis contains seven chapters. Chapter 1 is auxiliary, it contains mainly definitions and, most important, it introduces the notation that will be used in the thesis. Chapter 2 introduces the notion of the relative growth rate of a subgroup of a free group. It also contains some examples, the most important of which is Example 2.2 that shows that the relative growth rate of a subgroup of a free group depends on the choice of generators of the free group. The method to compute the relative growth rate in case when the generating set of the subgroup is Nielsen reduced, is described in Chapter 3. The last four chapters are dedicated directly to the answer to the question posed. Chapter 4 covers the case with no cancellations between generators of the subgroup, we derive yet another method to calculate the relative growth rate in this case and compare this result with the result obtained in Chapter 3. We use a classical result of B.Neumann to answer the question in the general case in Chapter 5. Finally in Chapter 6 we outline the proof of an even more general result, in the case of hyperbolic groups, which uses the notion of amenability.

Chapter 1

Free groups and their growth

First we give a formal definition of free groups and then indicate how these groups can be constructed.

Definition 1.1. If X is a subset of a group F, then F is a *free group* with basis X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\phi : F \to G$ making the following diagram commute:



▲

In this event we say that X is a *free basis* of F and that F is *freely generated* by X, we write $F = \langle X \rangle$. Occasionally, for a set $X = \{a_1, a_2, \dots, a_r\}$ we simply write

$F = \langle a_1, a_2, \dots, a_r \rangle$. The cardinality of free basis of F is the *rank* of the free group F, denoted rank(F).

Let X, the set of generators, be given. Consider $X^{-1} = \{a^{-1}, a \in X\}$, where a^{-1} is the *inverse* of element a. We define $X^{\pm 1} = X \cup X^{-1}$. The elements of $X^{\pm 1}$ are called *letters*. By a *word* in X we mean a finite, possibly empty, sequence of letters, $w = a_1 \dots a_n$ with $n \geq 0$ and all $a_i \in X^{\pm 1}$. The *inverse* of a word w is $w^{-1} = a_n^{-1} \dots a_1^{-1}$.

A word w that does not contain a subword of the type aa^{-1}, that is $w = a_1 \dots a_n$ ($a_i \in X^{\pm 1}$) where for any $i = 1, \dots, n-1$ condition $a_i \neq a_{i+1}^{-1}$ holds, is said to be *reduced*. The number n is called the *length* of the word w, we denote it by $|w| = n$. The word of length zero is the *empty* word, that is $w = 1$ if $|w| = 0$. Clearly, the empty word is reduced.

A group F is a free group if there exists a generating set X of F such that every non-empty reduced word in X defines a non-trivial element of F.

The *product* of two words is formed by juxtaposition with the convention that $w1 = w = 1w$. The set $W = W(X)$ of all words in X becomes a unital semigroup under juxtaposition.

We define an equivalence relation on W as follows. Two words w_1 and w_2 are *equivalent*, $w_1 \sim w_2$, if it is possible to obtain one word from the other by inserting or deleting a finite number of subwords of the type aa^{-1}. Clearly, $\sim$ defines an equivalence relation on the set W. It also preserves the structure of W as unital semigroup since $u_1 \sim u_2$ and $v_1 \sim v_2$ implies $u_1v_1 \sim u_2v_2$, and $u_1 \sim u_2$ implies $u_1^{-1} \sim u_2^{-1}$. Therefore we can define the quotient semigroup $F = W/\sim$, which clearly is a group. In fact, as we will see in Theorem 1.3, it is a free group with basis the images of the $a \in X$.

Theorem 1.2 ([12], p. 3)**.** *Each equivalence class of words in X contains a unique reduced word.*

Proof. Since successive deletion of subwords of the type aa^{-1} from any word w must lead to a reduced word, each equivalence class must contain at least one reduced word. We have to show that distinct reduced words u and v are not equivalent. Suppose on the contrary that there is a chain of words $u = w_1, \ldots, w_n = v$ with each w_{i+1} obtained from w_i $(1 \leq i < n)$ either by insertion or deletion of subwords of the type aa^{-1}, and with $N = \sum |w_i|$ a minimum. Since $u \neq v$ and both words are reduced, we have $n > 1$, $|w_2| > |w_1|$ and $|w_{n-1}| > |w_n|$. It follows that for some i $(1 < i < n)$ we must have $|w_i| > |w_{i-1}|$, and $|w_i| > |w_{i+1}|$. Suppose w_{i-1} and w_{i+1} are obtained from w_i by deletion of subwords aa^{-1} and bb^{-1} respectively. If these two subwords coincide, then $w_{i-1} = w_{i+1}$ contrary to the minimality of N. If these two subwords overlap without coinciding, then w_i has a subword $aa^{-1}a$, and w_{i-1} and w_{i+1} are both obtained by replacing this subword by a, hence again $w_{i-1} = w_{i+1}$. Finally if the two subwords are disjoint, w can be replaced by the result w' of deleting both parts to obtain a new chain with $N' = N - 4$, contrary to the minimality of N. ∎

Theorem 1.3 ([12], p. 3)**.** *F is a free group with basis the set $[X]$ of equivalence classes of elements from X, and $|[X]| = |X|$.*

Proof. Let G be a group and f a map from $[X]$ into G. To prove that $|[X]| = |X|$, we observe that if $a_1, a_2 \in X$ and $a_1 \neq a_2$ then $[a_1] \neq [a_2]$ since the two one-letter words a_1 and a_2 are reduced. Then f determines a map $f_1 : X \to G$ with $f([a]) = f_1(a)$. Define an extension ϕ_1 of f_1 from W into G by setting $\phi_1(w) = \phi_1(a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}) = (f_1(a_1))^{\varepsilon_1} \dots (f_1(a_n))^{\varepsilon_n}$, where $a_i \in X$ and $\varepsilon_i = \pm 1$. If $w_1 \sim w_2$ then $\phi_1(w_1) = \phi_1(w_2)$, whence ϕ_1 maps equivalent words onto the same element of G, thereby inducing a map $\phi : F \to G$ that is a homomorphism and an extension of f. ■

Corollary 1.4 ([12], p. 3)**.** *Given a set X, there exists a free group F with basis X.* ❑

Corollary 1.5 ([12], p. 87)**.** *Every group G is a quotient of a free group.*

Proof. Define a set $X = \{a_g : g \in G\}$ so that $f : a_g \mapsto g$ is a bijection from X to G. If F is free with basis X, then there is a homomorphism $\phi : F \to G$ extending f, and ϕ is a surjection because f is. Hence $G \simeq F/ker\phi$. ■

Some properties of free groups readily follow from the construction. For instance, if rank$(F) > 1$ then F is not abelian, and, moreover, its center is trivial. Also, two free groups are isomorphic if and only if their free bases have the same cardinality. Thus for every cardinal number, there is, up to isomorphism, exactly one free group of the rank of that number.

Let X be a set and let Δ be a family of words on X. We say that a group G has *generators* X and *relations* Δ if $G \simeq F/R$, where F is the free group with basis X and R is the normal subgroup of F generated by Δ. The ordered pair (X, Δ) is called a *presentation* of G.

Informally, no other relation may exist between the generators of a free group apart from the existence of inverses, and different reduced words in X define different elements in F.

More properties of free groups can be found in [12]. The one that is of importance to us is the following.

Theorem 1.6 (Nielsen-Schreier Theorem). *Every subgroup of a free group is free.*

❑

A concept that we will be dealing with in this work is the concept of *growth* of a group. Suppose we have a finitely generated group G which does not have to be free. Each element of G has a unique expression as a reduced word in the generators. Let $f(n) = \#G_n$ and $h(n) = \#\bigcup_{i=0}^{n} G_i$ be the numbers of elements represented by reduced words of length n and at most n respectively, here G_i is the set of elements of G which can be written as reduced words of length i. It is obvious that $h(n) = \sum_{i=0}^{n} f(i)$. We call $f(n)$ the *strict growth function*, whereas $h(n)$ is termed the *cumulative growth function* of F.

In the case of a finitely generated free group F of a rank r, a reduced word w of length $n+1$ with $n \geq 1$, can be obtained by writing a letter after a reduced word

$\hat{w} = a_1 \dots a_n$ of length n. We have $2r-1$ letters to choose from as it can be any letter $a \in X^{\pm 1}$, except a_n^{-1}. Hence we have

$$f(n+1) = f(n)(2r-1), \quad n \geq 1.$$

Using initial conditions $f(0) = 1$ and $f(1) = 2r$ we obtain

$$f(n) = 2r(2r-1)^{n-1}, \text{ for } n \geq 1. \tag{1.1}$$

Also,

$$h(n) = 1 + \sum_{i=1}^{n} 2r(2r-1)^{i-1}, \text{ for } n \geq 1. \tag{1.2}$$

Upon taking the n^{th} root and then the upper limit of equation (1.1), we obtain what we call the *growth rate* of free group of rank r

$$\alpha = \limsup_{n\to\infty} (f(n))^{\frac{1}{n}} = 2r - 1. \tag{1.3}$$

Note that instead of taking the upper limit we could take simply a limit and of course the growth rate could also be defined as $\lim_{n\to\infty} (h(n))^{\frac{1}{n}}$. However, as we will see in the next subsection, the definition of the growth rate of the form (1.3) is preferred.

Next we turn our attention to relative growth rate of subgroups of free groups.

Chapter 2

Relative growth rate of subgroups of free groups

We have seen the growth functions and the growth rate of free groups in the previous chapter. For subgroups of free groups we can define relative growth function and relative growth rate in a similar manner, but first let us recall what a subgroup generated by a subset of a group is.

Let Y be a subset of a group G (not necessarily free), then, the subgroup generated by Y is the smallest subgroup of G containing every element of Y, or in other words, it is the intersection of all subgroups containing all the elements of Y. Equivalently, the subgroup generated by Y is the subgroup consisting of all elements of G that can be expressed as a finite product of elements in Y and their inverses. We usually denote a subgroup of a group by H and, in the case when H is generated by a set Y, we write $H = \langle Y \rangle$. Sometimes for a set $Y = \{b_1, b_2, \ldots, b_s\}$ we simply write $H = \langle b_1, b_2, \ldots, b_s \rangle$.

We define the *strict relative growth function* of H as $d(n) = \#H_n$ and the *cumulative relative growth function* of H as $g(n) = \# \bigcup_{i=0}^{n} H_i$, where $H_i = H \cap F_i$.

The relative growth rate of a subgroup is defined similar to the growth rate of the group.

Definition 2.1. The *relative growth rate* of a subgroup H is $\alpha_H = \limsup_{n\to\infty} (d(n))^{\frac{1}{n}}$.

▲

Note that we cannot replace the upper limit with simply a limit in the Definition 2.1. For instance, consider the subgroup of a free group that consists of all elements of even length then $d(n) = 0$ for all odd n. Of course, the relative growth rate of H could also be defined as $\limsup_{n\to\infty} (g(n))^{\frac{1}{n}}$ (or as $\lim_{n\to\infty} (g(n))^{\frac{1}{n}}$), but the former definition is preferred as it allows us to find the relative growth rate using Hilbert series for the subgroup H.

The relative growth rate can be computed as the inverse of the radius of convergence of the series $\mathcal{S}(H, t) = \sum_{n=0}^{\infty} (\#H_n) t^n$. In the examples to follow we compute the radius of convergence using the following theorem.

Theorem 2.2 ([9], p. 499)**.** *Suppose $p(t)/q(t)$ is a quotient of two polynomials p and q. Suppose $q(t_0) \neq 0$. Then the power series expansion for p/q about t_0 has radius of convergence equal to the closest distance from t_0 to the roots (including complex roots) of q.* ❑

In the following example we compute the relative growth rate of a subgroup of a group freely generated by a set consisting of just two elements.

Example 2.1. Consider the free group $F = \langle a, b \rangle$ and its subgroup $H = \langle a^2, ab, ab^{-1} \rangle$.

First we show that H is a subgroup of F that consists of all elements of even length. Let $\widehat{H}$ be the subgroup of words of even length of F, then we have to show that $H = \widehat{H}$.

A reduced word of even length is a product of reduced words of length 2. There are 12 such words, namely: aa, ab, ab^{-1}, $a^{-1}a^{-1}$, $a^{-1}b$, $a^{-1}b^{-1}$, ba, ba^{-1}, bb, $b^{-1}a$, $b^{-1}a^{-1}$, $b^{-1}b^{-1}$. So $\widehat{H}$, the subgroup of words of even length of F, is generated by these twelve words. Removing inverses we have only a^2, ab, $a^{-1}b$, ab^{-1}, $a^{-1}b^{-1}$, and b^2 left. Now $a^{-1}b = (a^2)^{-1}(ab)$, $a^{-1}b^{-1} = (a^2)^{-1}(ab^-1)$ and $b^2 = (ab)^{-1}(ab)(ab^{-1})^{-1}(ab)$. Hence $\widehat{H} \subseteq H$. On the other hand every $h \in H$ also belongs to $\widehat{H}$, that is, $H \subseteq \widehat{H}$. Hence the subgroup containing all words of even length of F is generated by $\{a^2, ab, ab^{-1}\}$.

Now the Hilbert series for H is

$$\mathcal{S}(H, t) = \sum_{n=1}^{\infty} (\#F_{2n}) t^{2n},$$

where F_{2n} is the set of all words of length $2n$ belonging to F. We have

$$\mathcal{S}(H, t) = 1 + \frac{4}{3} \sum_{n=1}^{\infty} (3t)^{2n} = \frac{1 + 3t^2}{1 - (3t)^2}.$$

According to Theorem 2.2, the radius of convergence of $\mathcal{S}(H,t)$ is $\frac{1}{3}$. Hence the relative growth rate of H is $\alpha_H = 3$. ▲

Example 2.1 demonstrates the fact that a proper finitely generated subgroup of finite index of a free group may have relative growth rate equal to the growth rate of the group itself. Whether it is always true for subgroups of finite index, we will determine in subsequent chapters.

The following example shows that the relative growth rate of a subgroup of a free group depends on the choice of generators of the free group.

Example 2.2. Let the free group $F = \langle a_1, a_2, a_3\rangle$ and $H = \langle a_1, a_2\rangle$ be a subgroup of F. Clearly the relative growth rate of H is equal to the growth rate of a group freely generated by a set consisting of two elements, that is $\alpha_H = 3$. Since a free basis of a free group is not unique we can choose another basis for F, say $\{a_1a_3, a_2, a_3\}$. Let $b_1 = a_1a_3$, $b_2 = a_2$ and $b_3 = a_3$. In this case the subgroup H is generated by $\{b_1b_3^{-1}, b_2\}$. We will show that for this choice of generators of the free group the subgroup H has relative growth rate not equal to 3.

Let $H_n^{b_1b_3}$ be the set of reduced words of length n in H ending either with $b_1b_3^{-1}$ or $b_3b_1^{-1}$ and $H_n^{b_2}$ be the set of reduced words of length n ending either with b_2 or b_2^{-1}. We have:

$$\#H_n^{b_1b_3} = \#H_{n-2}^{b_1b_3} + 2\#H_{n-2}^{b_2},$$

$$\#H_n^{b_2} = 2\#H_{n-1}^{b_1b_3} + \#H_{n-1}^{b_2}.$$

Hence we obtain the following system of equations in terms of generating functions $\mathcal{S}(H^{b_1b_3}, t)$ and $\mathcal{S}(H^{b_2}, t)$:

$$\begin{cases} \mathcal{S}(H^{b_1b_3}, t) = 2t^2 + t^2\mathcal{S}(H^{b_1b_3}, t) + 2t^2\mathcal{S}(H^{b_2}, t) \\ \mathcal{S}(H^{b_2}, t) = 2t + 2t\mathcal{S}(H^{b_1b_3}, t) + t\mathcal{S}(H^{b_2}, t) \end{cases}$$

Now the Hilbert series for H is

$$\mathcal{S}(H, t) = 1 + \mathcal{S}(H^{b_1b_3}, t) + \mathcal{S}(H^{b_2}, t) = \frac{t^3 + t^2 + t + 1}{-3t^3 - t^2 - t + 1}.$$

According to Theorem 2.2, the radius of convergence of $\mathcal{S}(H, t)$ is equal to $\frac{1}{9}\Big(-1 - \frac{4\times 2^{2/3}}{\sqrt[3]{67+9\sqrt{57}}} + \sqrt[3]{2(67 + 9\sqrt{57})}\Big) \approx 0.46940$. So $\alpha_H \approx 2.13038$. ▲

Even though the example above demonstrates the dependence of the relative growth rate on the choice of generators of the free group, we will not reflect this fact in our notation, and simply write α_H (instead of writing, say, α_H^X), assuming that the generating set of the free group is clear from the context.

In contrast to the fact that the relative growth rate depends on the choice of generators of the free group, it clearly does not depend on the choice of generators of the subgroup itself. For instance, in Example 2.1 we could take the set $\{a^2, b^2, ab\}$ as the generating set for H. The result, however, would still be the same: $\alpha_H = 3$.

A natural question to ask is whether the relative growth rate of a subgroup of a finitely generated free group always exists. The affirmative answer for normal subgroups was given [13].

Theorem 2.3 ([13], p. 44)**.** *Let N be a non-trivial normal subgroup of a free group F of rank r. If all elements are of even length then there exists an upper limit $\alpha_N = \limsup_{n\to\infty}(\#N_{2n})^{\frac{1}{2n}}$, otherwise there exists an upper limit $\alpha_N = \limsup_{n\to\infty}(\#N_n)^{\frac{1}{n}}$.* ❏

A more general result stating that the relative growth rate of a subgroup of a finitely generated free group is a real number, can be found in [3].

Theorem 2.4 ([3], p. 2)**.** *For any subgroup H of a finitely generated free group F there exists the relative growth rate.* ❏

Chapter 3

Properties of the relative growth rate

In this chapter, which is mostly based on the results obtained in [13], we will see some of the properties of the relative growth rate of subgroups of free groups. We start with the following.

Theorem 3.1 ([13], p. 42)**.** *If N is a non-trivial normal subgroup of a free group F of rank r then the relative growth rate $\alpha_N \geq \sqrt{2r-1}$.*

Proof. Let $w \in N$ be a reduced word in letters a_ν^ε, where $\nu = 1, \ldots, r$ and $\varepsilon = \pm 1$. Consider set of words $a_\nu^\varepsilon w a_\nu^{-\varepsilon}$. This set contains at least $2r-2$ reduced words of length $|w|+2$. Let w_1 be one of those words. Then set of words $a_\nu^\varepsilon w_1 a_\nu^{-\varepsilon}$ contains $2r-1$ reduced words of length $|w|+4$. Choosing a word w_2 from the latter set and applying the same argument, we obtain $2r-1$ reduced words of length $|w|+6$.

As a result we obtain set $\Omega(w) \subset N$ which contains at least $(2r-2)(2r-1)^{n-1}$ words of length $|w|+2n$. Hence $\alpha_N \geq \limsup_{n\to\infty}((2r-2)(2r-1)^{n-1})^{\frac{1}{|w|+2n}} = \sqrt{2r-1}$.

■

Note that, in view of the Example 2.2, the relative growth rate of a non-trivial normal subgroup of F can be different depending on the choice of generators of F, but it is always greater or equal to $\sqrt{2r-1}$.

The following theorem deals with a sequence of normal subgroups.

Theorem 3.2 ([13], p. 46)**.** *Let $N_1 \subset N_2 \subset \ldots$ be a sequence of normal subgroups of a finitely generated free group F then for $N = \cup_{n=1}^{\infty} N_n$ we have*

$\alpha_N = \limsup_{n\to\infty} \alpha_{N_n}$. ❏

We need the notion of *Nielsen reduced set* before we can proceed. Let F be a free group on the free generators $\{a_\nu\}$ and let $\{w_i(a_\nu)\}$ be reduced words which generate H, words $w_i(a_\nu)$ and $w_i^{-1}(a_\nu)$ are called *w-symbols*. Every $v \in H$ can be expressed as a word $v(w_i)$ in symbols w_i and as a word $v(w_i(a_\nu))$ in a_ν. So we have two notions of length, namely $|v|$ length of v in a_ν's and $|v|_w$ length of v in w_i's. According to Theorem 1.6, any subgroup of a free group is free. Its generating set $\{w_i(a_\nu)\}$ can be chosen in such a way that it is *Nielsen reduced*, see [14].

Definition 3.3. A set of words is *Nielsen reduced* if

(i) For every w-reduced word $v(w_i) = w_{i_1}^{\varepsilon_1} \ldots w_{i_r}^{\varepsilon_r}$, $\varepsilon = \pm 1$, a-reduction of $v(w_i(a_\nu))$ leaves at least one letter $a_{\nu_j}^{\mu_j}$, $\mu_j = \pm 1$ from $w_{i_j}^{\varepsilon_j}$ for each j;

(ii) The a-length of $v(w_i(a_\nu))$ is at least as large as the a-length of any w-symbol occurring in $v(w_i)$. ▲

Let us have a look at a familiar example.

Example 3.1. Consider $F = \langle a, b \rangle$ and its subgroup $H = \langle a^2, ab, ba \rangle$. Take $v(w_i) = (ab)^{-1}a^2(ba)^{-1}$. Upon applying a-reduction we obtain $v(w_i(a_\nu)) = b^{-2}$, which violates (i) of Definition 3.3 of Nielsen reduced set.

On the other hand H is also generated by $\{a^2, ab, ba^{-1}\}$ and for $v(w_i) = (ab)^{-1}a^2(ba^{-1})^{-1}$ a-reduction produces $v(w_i(a_\nu)) = b^{-1}a^2b^{-1}$. In fact Lemma 3.5 shows that the generating set of H is Nielsen reduced in this case. ▲

To formulate Lemma 3.5 which gives a characterization of Nielsen reduced set, we inroduce some terminology.

Definition 3.4. An initial segment of w-symbol is called *isolated* if it does not occur as an initial segment of any other w-symbol. Similarly, a terminal segment is *isolated* if it is a terminal segment of a unique w-symbol.

Let $w(a_\nu)$ be an a-reduced word. The initial segment s of $w(a_\nu)$ such that $\frac{1}{2}|w| < |s| \leq \frac{1}{2}|w| + 1$, is called *the major initial segment* of w. The *major terminal segment* is defined similarly.

If the a-length of an a-reduced word $w(a_\nu)$ is even we define *left half* and *right half* in the obvious way. ▲

By $\beta(w_i^\mu, w_k^\varepsilon)$ we will understand the number of elements $a_\nu^{\pm 1}$ that can be removed from the word $w_i^\mu w_k^\varepsilon$ by cancellation, $\mu, \varepsilon = \pm 1$.

Below is a characterization of a Nielsen reduced set. Note that we are interested in Corollary 3.6 which provides us with some useful information about words in a Nielsen reduced set.

Lemma 3.5 ([14], p. 123)**.** *Let $\{w_i(a_\nu)\}$ be a set of a-reduced words. Then $\{w_i(a_\nu)\}$ is Nielsen reduced if and only if the following conditions are satisfied:*

(i) Both the major initial and major terminal segments of each w_i are isolated;

(ii) For each w_i of even length, either its left or its right half is isolated. ❏

Corollary 3.6 ([14], p. 124)**.** *For a Nielsen reduced set $\{w_i(a_\nu)\}$ the following inequalities hold:*

$$|w_k^\varepsilon| \geq \beta(w_i^\mu, w_k^\varepsilon),$$

$$|w_k^\varepsilon| \geq \beta(w_k^\varepsilon, w_i^\mu),$$

$i, k = 1, 2 \ldots$ and $\varepsilon, \mu = \pm 1$ unless $i = k$ and $\mu = -\varepsilon$ are satisfied simultaneously.

❏

In what follows we will consider subgroups H with a Nielsen reduced set of generators. Our aim is to derive an expression for finding relative growth rate of a subgroup in this case.

We say that a w-reduced word u *ends* with the word w_k^ε if $u = w_{i_1}^{\varepsilon_1} \ldots w_{i_n}^{\varepsilon_n} w_k^\varepsilon$. By $H_n^{k,\varepsilon}$ we understand the set of a-reduced words belonging to H that have a-length

n and end with the word w_k^ε and we define $\mathcal{S}(H^{k,\varepsilon},t) = \sum_{n=1}^{\infty}(\#H_n^{k,\varepsilon})t^n$. We have

$$\mathcal{S}(H,t) = 1 + \sum_{k=1}\sum_{\varepsilon=\pm 1}\mathcal{S}(H^{k,\varepsilon},t). \tag{3.1}$$

Note that w-words ending with different w_k^ε cannot have the same a-reduction as H is freely generated by $\{w_i(a_\nu)\}$ and, therefore, there are no no-trivial relations between the generators.

Theorem 3.7 ([13], p. 50)**.** *The functions $\mathcal{S}(H^{k,\varepsilon},t)$ for $k = 1, 2\ldots$ and $\varepsilon = \pm 1$ satisfy the following linear system of equations*

$$\mathcal{S}(H^{k,\varepsilon},t) = t^{|w_k^\varepsilon|} + \sum_{m=1}\sum_{\mu=\pm 1} t^{|w_k^\varepsilon| - \beta(w_m^\mu, w_k^\varepsilon)}\mathcal{S}(H^{m,\mu},t), \tag{3.2}$$

with $\mu = -\varepsilon$ for $m = k$ being excluded from the second sum.

Proof. Consider a w-reduced word u belonging to $H_n^{k,\varepsilon}$, that is, $u = w_{i_1}^{\varepsilon_1}\ldots w_m^\mu w_k^\varepsilon$. The a-length of the word $v = w_{i_1}^{\varepsilon_1}\ldots w_m^\mu$ in this case is given by $n - |w_k^\varepsilon| + \beta(w_m^\mu, w_k^\varepsilon)$, where w_m^μ is the word that v ends with.

The w-reduced words that end with w_k^ε are obtained from the words of a-length $n - |w_k^\varepsilon| + \beta(w_m^\mu, w_k^\varepsilon)$ ending with w_m^μ (all possible) by writing the word w_k^ε after them. Hence,

$$\#H_n^{k,\varepsilon} = \sum_{m=1}^{\infty}\sum_{\mu=\pm 1}\#H^{m,\mu}_{n-|w_k^\varepsilon|+\beta(w_m^\mu,w_k^\varepsilon)}. \tag{3.3}$$

Upon multiplying equations (3.3) by t^n and adding them, we obtain (3.2). ■

Corollary 3.8 ([13], p. 51)**.** *If H is a finitely generated subgroup of a free group then its relative growth rate is an inverse of the smallest positive root of the equation* $\det M = 0$*, where M is the matrix obtained from the matrix multiplication form of the system of equations* (3.2)*.*

Proof. Let H be generated by a set $\{w_1, \ldots, w_p\}$. The linear system of equations (3.2) becomes

$$\begin{cases} \mathcal{S}(H^{1,1},t) = t^{|w_1|} + t^{|w_1|-\beta(w_1,w_1)}\mathcal{S}(H^{1,1},t) + 0 + t^{|w_1|-\beta(w_2,w_1)}\mathcal{S}(H^{2,1},t) + \ldots + t^{|w_1|-\beta(w_p^{-1},w_1)}\mathcal{S}(H^{p,-1},t) \\ \mathcal{S}(H^{1,-1},t) = t^{|w_1|} + 0 + t^{|w_1|-\beta(w_1^{-1},w_1^{-1})}\mathcal{S}(H^{1,-1},t) + t^{|w_1|-\beta(w_2,w_1^{-1})}\mathcal{S}(H^{2,1},t) + \ldots + t^{|w_1|-\beta(w_p^{-1},w_1^{-1})}\mathcal{S}(H^{p,-1},t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathcal{S}(H^{p,-1},t) = t^{|w_p|} + t^{|w_p|-\beta(w_p^{-1},w_1)}\mathcal{S}(H^{1,1},t) + t^{|w_p|-\beta(w_p^{-1},w_1^{-1})}\mathcal{S}(H^{1,-1},t) + \ldots + t^{|w_p|-\beta(w_p^{-1},w_p^{-1})}\mathcal{S}(H^{p,-1},t), \end{cases}$$

which is equivalent to

$$\begin{cases} (t^{|w_1|-\beta(w_1,w_1)} - 1)\mathcal{S}(H^{1,1},t) + 0 + t^{|w_1|-\beta(w_2,w_1)}\mathcal{S}(H^{2,1},t) + \ldots + t^{|w_1|-\beta(w_p^{-1},w_1)}\mathcal{S}(H^{p,-1},t) = -t^{|w_1|} \\ 0 + (t^{|w_1|-\beta(w_1^{-1},w_1^{-1})} - 1)\mathcal{S}(H^{1,-1},t) + t^{|w_1|-\beta(w_2,w_1^{-1})}\mathcal{S}(H^{2,1},t) + \ldots + t^{|w_1|-\beta(w_p^{-1},w_1^{-1})}\mathcal{S}(H^{p,-1},t) = -t^{|w_1|} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t^{|w_p|-\beta(w_p^{-1},w_1)}\mathcal{S}(H^{1,1},t) + t^{|w_p|-\beta(w_p^{-1},w_1^{-1})}\mathcal{S}(H^{1,-1},t) + \ldots + (t^{|w_p|-\beta(w_p^{-1},w_p^{-1})} - 1)\mathcal{S}(H^{p,-1},t) = -t^{|w_p|}. \end{cases}$$

Instead of solving the aforementioned system of equations for $\mathcal{S}(H^{k,\varepsilon},t)$ ($k = 1, 2\ldots$ and $\varepsilon = \pm 1$), and then plugging the obtained results into (3.1), we notice that the radius of convergence of $\mathcal{S}(H,t)$ is equal to the least absolute value of poles of the function $\mathcal{S}(H,t)$ (see Theorem 2.2), which, according to Cramer's rule, can be found as roots of $\det M = 0$, where M is the matrix associated with the system. The result follows from the fact that the relative growth rate of H is equal to the inverse of the radius of convergence of $\mathcal{S}(H,t)$. ■

Example 3.2. Let $H = \langle a^k, b^l, c^p, d^q \rangle$ be a subgroup of $F = \langle a, b, c, d \rangle$, with $k, l, p, q \geq 1$. The relative growth rate α_H can be found as the inverse of the smallest positive root of $\det M = 0$, where M is the following 8×8 matrix

$$\begin{pmatrix} t^k-1 & 0 & t^k & t^k & t^k & t^k & t^k & t^k \\ 0 & t^k-1 & t^k & t^k & t^k & t^k & t^k & t^k \\ t^l & t^l & t^l-1 & 0 & t^l & t^l & t^l & t^l \\ t^l & t^l & 0 & t^l-1 & t^l & t^l & t^l & t^l \\ t^p & t^p & t^p & t^p & t^p-1 & 0 & t^p & t^p \\ t^p & t^p & t^p & t^p & 0 & t^p-1 & t^p & t^p \\ t^q & t^q & t^q & t^q & t^q & t^q & t^q-1 & 0 \\ t^q & t^q & t^q & t^q & t^q & t^q & 0 & t^q-1 \end{pmatrix}$$

Table A.1 in the Appendix contains the least absolute values of roots of the equation $\det M = 0$ for some values of k, l, p, q, and the corresponding values of the relative growth rates.

When $k = l = p = q$, for some $k \geq 1$, we have $\det(M) = -(t^k-1)^4(7t^k-1)(t^k+1)^3$. The least positive root of $\det M = 0$ in this case is $7^{-\frac{1}{k}}$, which implies that $\alpha_H = 7^{\frac{1}{k}}$ for such subgroups.

From the table we can see that for the values of k, l, p, q that we chose, the smallest value of the least positive root of the equation $\det M = 0$ corresponds to the case when the subgroup H is the entire group itself. The relative growth rate in this case is equal to 7. ▲

Example 3.2 is an example of no cancellations between the generators of the subgroup. In Chapter 4 we will give an alternative formula which allows us to find the relative growth rate of such subgroups of free groups. We conclude this chapter with one more result.

Theorem 3.9 ([13], p. 51)**.** *If $\{w_i\}$ is a Nielsen reduced generating set of $H \subseteq F_r$ then $\alpha_H \geq t_0^{-1}$, where t_0 is the smallest positive root of*

$$1 - \sum_{i=1} \sum_{\varepsilon=\pm 1} t^{|w_i^{\varepsilon}|} = 0.$$

Proof. If we repeatedly substitute (3.2) into (3.1) we obtain:

$$\begin{aligned} \mathcal{S}(H,t) &= 1 + \sum_{k=1} \sum_{\varepsilon=\pm 1} \Big(t^{|w_k^{\varepsilon}|} + \sum_{m=1} \sum_{\mu=\pm 1} t^{|w_k^{\varepsilon}| - \beta(w_m^{\mu}, w_k^{\varepsilon})} \mathcal{S}(H^{m,\mu}, t) \Big) = \\ &= 1 + \sum_{k=1} \sum_{\varepsilon=\pm 1} t^{|w_k^{\varepsilon}|} + \sum_{k=1} \sum_{\varepsilon=\pm 1} \sum_{m=1} \sum_{\mu=\pm 1} t^{|w_k^{\varepsilon}| - \beta(w_m^{\mu}, w_k^{\varepsilon})} \mathcal{S}(H^{m,\mu}, t) = \\ &= 1 + \sum_{k=1} \sum_{\varepsilon=\pm 1} t^{|w_k^{\varepsilon}|} + \sum_{k=1} \sum_{\varepsilon=\pm 1} \sum_{m=1} \sum_{\mu=\pm 1} t^{|w_k^{\varepsilon}| - \beta(w_m^{\mu}, w_k^{\varepsilon})} + \cdots \end{aligned}$$

Since

$$\frac{1}{1-t} \mathcal{S}(H,t) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \# H_k \Big) t^n,$$

the radius of convergence of $\mathcal{S}(H,t)$ is equal to the radius of convergence of $\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\#H_k\Big)t^n$ (note that $t=1$ cannot be the smallest root of the denominator in the LHS of the equation above as the existence of α_H is guaranteed by Theorem 2.4 and, hence, the radius of convergence of $\mathcal{S}(H,t)$ is strictly less than 1), which, in its turn, is not greater than the radius of convergence of the series

$$1+\sum_{k=1}\sum_{\varepsilon=\pm 1}t^{|w_k^\varepsilon|}+\sum_{k=1}\sum_{\varepsilon=\pm 1}\sum_{m=1}\sum_{\mu=\pm 1}t^{|w_k^\varepsilon|+|w_m^\mu|}+\cdots=\frac{1}{1-\sum\limits_{k=1}\sum\limits_{\varepsilon=\pm 1}t^{|w_k^\varepsilon|}}.$$

■

In this chapter we have introduced notation that we are going to use in the subsequent chapters, made ourselves familiar with the relative growth rate and described a method to compute it. In the next chapter we will describe another method to find the relative growth rate.

Chapter 4

The case with restriction on generators of H

In the chapters to follow we give some proofs in several special cases that given a free group $F = \langle a_1, \dots, a_r \rangle$ with $r \geq 2$ and its finitely generated subgroup H of infinite index, the relative growth rate of the subgroup is strictly less than the growth rate of the free group itself. This chapter is dedicated to the case where there are no cancellations between generators of the subgroup. Apart from proving the fact stated above we will also derive an alternative way to compute the relative growth rate in this case and compare the result with the result obtained in Chapter 3 in the case of the free group of rank 2.

In this chapter we impose the following restriction on generators of $H = \langle g_1, g_2, \dots, g_m \rangle$ with length $|g_i| \geq 1$ for $i = 1, \dots, m$: $|g_i^\varepsilon g_j^\mu| = |g_i^\varepsilon| + |g_j^\mu|$ for $i \neq j$ and $\varepsilon \neq -\mu$

($\varepsilon, \mu = \pm 1$), that is, we only consider the case when there are no cancellations between the generators of the subgroup and their inverses apart from the cancellations of the type $g_i g_i^{-1}$ or $g_i^{-1} g_i$.

Let $d_{i,\varepsilon}(n)$ denote the number of words of length n in H ending with g_i^{ε}. Then the number of words of length n belonging to H is given by

$$d(n) = \sum_{i=1}^{m} \sum_{\varepsilon=\pm 1} d_{i,\varepsilon}(n). \tag{4.1}$$

Now since $|g_i^{\varepsilon}| = |g_i^{-\varepsilon}|$ we have

$$\begin{aligned} d_{i,\varepsilon}(n) &= d(n - |g_i^{\varepsilon}|) - d_{i,-\varepsilon}(n - |g_i^{\varepsilon}|) = \\ &= d(n - |g_i^{\varepsilon}|) - d(n - 2|g_i^{\varepsilon}|) + d_{i,\varepsilon}(n - 3|g_i^{\varepsilon}|) = \cdots \\ &= \sum_{j=0}^{\infty} \Big(d(n - |g_i^{\varepsilon}|(1+2j)) - d(n - 2|g_i^{\varepsilon}|(1+j)) \Big). \end{aligned} \tag{4.2}$$

All terms in expression (4.2) turn into zero after some j is reached, this j is determined by n and $|g_i^{\varepsilon}|$. Nevertheless, we shall leave infinity at the upper limit of the sum in this equation for our convenience.

If we substitute (4.2) into (4.1) and let $s_i := |g_i^{\varepsilon}|$, we obtain:

$$d(n) = 2 \sum_{i=1}^{m} \sum_{j=0}^{\infty} \Big(d(n - s_i(1+2j)) - d(n - s_i(1+j)) \Big). \tag{4.3}$$

Let us introduce some notation.

Denote by K_t the set of all t-element subsets of $\{1, 2, \ldots, m\}$. For any $\sigma \in K_t$ we write $\sigma = \{k_1, k_2, \ldots, k_t\}$. Also let $L_\sigma = \{l \in \{1, 2, \ldots, m\} : l \notin \sigma\}$.

Lemma 4.1. *The number of words of length $n \geq \sum_{i=1}^{m} s_i + \min\{s_1, s_2, \ldots s_m\}$ in subgroup H is given by:*

$$d(n) = \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1)\Big(d(n - \sum_{i=1}^{t} s_{k_i})\Big).$$

Proof. From (4.3) we deduce that for a fixed $t \leq m$:

$$\sum_{\sigma \in K_t} d(n - s_{k_1} - \ldots - s_{k_t}) =$$

$$= 2\sum_{\sigma \in K_t} \sum_{i=1}^{m} \sum_{j=0}^{\infty} \Big(d(n - s_{k_1} - \ldots - s_{k_t} - s_i(1+2j)) - d(n - s_{k_1} - \ldots - s_{k_t} - 2s_i(1+j))\Big) =$$

$$= 2\sum_{\sigma \in K_t} \sum_{l \in L_\sigma} \sum_{j=0}^{\infty} \Big(d(n - s_{k_1} - \ldots - s_{k_t} - s_l(1+2j)) - d(n - s_{k_1} - \ldots - s_{k_t} - 2s_l(1+j))\Big) +$$

$$+2\sum_{\sigma \in K_{t-1}} \sum_{l \in L_\sigma} \sum_{j=0}^{\infty} \Big(d(n - s_{k_1} - \ldots - s_{k_{t-1}} - 2s_l(1+j)) - d(n - s_{k_1} - \ldots - s_{k_{t-1}} - s_l(3+2j))\Big). \tag{4.4}$$

Upon adding equations (4.4) for all $t = 1, \ldots, m$ and $d(n)$ from (4.3) we obtain:

$$d(n) + \sum_{t=1}^{m} \sum_{\sigma \in K_t} d(n - s_{k_1} - \ldots - s_{k_t}) = \sum_{t=1}^{m} \sum_{\sigma \in K_t} 2t d(n - s_{k_1} - \ldots - s_{k_t}),$$

that is, $d(n) = \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1)\Big(d(n - \sum_{i=1}^{t} s_{k_i})\Big)$. ∎

Adopting the same technique as in [2] we define a function of real variable ζ

$$F(\zeta) = \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1)(z-\zeta)^{-\sum_{i=1}^{t} s_{k_i}},$$

where z is a positive parameter.

Note that

$$F(0) \leq 1 \Leftrightarrow z^{\sum_{i=1}^{m} s_i} - \sum_{t=1}^{m-1} \sum_{\sigma \in K_{m-t}} (2t-1) z^{\sum_{i=1}^{m-t} s_{k_i}} - (2m-1) \geq 0. \qquad (4.5)$$

Replacing K_{m-t} by K_t, we find that $F(0) \leq 1$ if $z \geq z_0$, where z_0 is the positive root of the equation (the uniqueness of this positive root is guaranteed by Descartes' rule of signs)

$$z^{\sum_{i=1}^{m} s_i} - \sum_{t=1}^{m-1} \sum_{\sigma \in K_t} (2t-1) z^{\sum_{i=1}^{t} s_{k_i}} - (2m-1) = 0. \qquad (4.6)$$

On the other hand if $z < z_0$ then $F(0) > 1$.

Recall that Descartes' rule of signs states that if the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. In our case the number of sign differences between consecutive nonzero coefficients is 1.

Before we find the relative growth rate α_H, we need to do some preliminary work.

Lemma 4.2. *If $z \geq z_0$ then there exists a positive constant c such that $d(n) \leq cz^n$ for every $n \geq \sum_{i=1}^{m} s_i + \min\{s_1, s_2, \ldots s_m\}$.*

Proof. Proof by induction on n.

Let us fix parameter $z \geq z_0$. There exists a word of length $\sum_{i=1}^{m} s_i + \min\{s_1, s_2, \dots s_m\}$. We can always find a positive constant c such that the length of this word will be less than or equal to cz^n.

Suppose the statement holds for some $n \geq \sum_{i=1}^{m} s_i + \min\{s_1, s_2, \dots s_m\}$. Then, according to Lemma 4.1,

$$\begin{aligned} d(n+1) &= \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1)\Big(d(n+1-\textstyle\sum_{i=1}^{t} s_{k_i})\Big) \leq \\ \leq \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1) c z^{n+1-\sum_{i=1}^{t} s_{k_i}} &= c z^{n+1} \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1) z^{-\sum_{i=1}^{t} s_{k_i}} = \\ = c z^{n+1} F(0) &\leq c z^{n+1} . \end{aligned}$$ ■

Lemma 4.3. *If $z < z_0$ then there exists a positive constant c such that $d(n) > cz^n$ for every $n \geq \sum_{i=1}^{m} s_i + \min\{s_1, s_2, \dots s_m\}$.*

Proof. As before proof by induction on n.

Let us fix parameter $z < z_0$. There exists a word of length $\sum_{i=1}^{m} s_i + \min\{s_1, s_2, \dots s_m\}$. We can always find a positive constant c such that the length of this word will be greater than cz^n.

Suppose the statement holds for some $n \geq \sum_{i=1}^{m} s_i + \min\{s_1, s_2, \dots s_m\}$. Then, according to Lemma 4.1,

$$\begin{aligned} d(n+1) &= \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1)\Big(d(n+1-\textstyle\sum_{i=1}^{t} s_{k_i})\Big) > \\ > \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1) c z^{n+1-\sum_{i=1}^{t} s_{k_i}} &= c z^{n+1} \sum_{t=1}^{m} \sum_{\sigma \in K_t} (2t-1) z^{-\sum_{i=1}^{t} s_{k_i}} = \\ = c z^{n+1} F(0) &> c z^{n+1}. \end{aligned}$$ ■

The following theorem is the main result of this chapter.

Theorem 4.4. *Assume a finitely generated subgroup H of a free group F of finite rank r has a generating set with no cancellations and s_i are the lengths of the generators of H , $i = 1, \dots, m$. Then the relative growth rate of H is given by:*

$$\alpha_H = \lim_{n\to\infty} (g_H(n))^{\frac{1}{n}} = z_0,$$

where z_0 is the positive root of the equation

$$z^{\sum_{i=1}^{m} s_i} - \sum_{t=1}^{m-1} \sum_{\sigma \in K_t} (2t-1) z^{\sum_{i=1}^{t} s_{k_i}} - (2m-1) = 0.$$

Proof. According to Lemma 4.2, for $z \geq z_0$ we have $d(n) \leq cz^n$ for sufficiently large n. Therefore $g_H(n) \leq c_1 z^n$ for a positive constant c_1. By Lemma 4.3, on the other hand, for $z' < z_0$ we have $d(n) > c'(z')^n$ for sufficiently large n, which yields $g_H(n) > c_2(z')^n$ for a positive c_2. Upon taking the n^{th} root of the inequalities for $g_H(n)$ and passing to the limit as $n \to \infty$, we obtain $\limsup (g_H(n))^{\frac{1}{n}} \leq z$ and $\liminf (g_H(n))^{\frac{1}{n}} \geq z'$, that is $\lim_{n\to\infty} (g_H(n))^{\frac{1}{n}} = z_0$. ∎

Example 4.1. Consider $F = \langle x, y\rangle$ and its subgroup $H = \langle x^k, y^l\rangle$ with $k, l \geq 1$. Then, by Theorem 4.4, $\alpha_H = z_0$ is the positive root of

$$z^{k+l} - z^k - z^l - 3 = 0. \tag{4.7}$$

Let us compare this result with the one obtained by the method described in the previous chapter.

According to Corollary 3.8, α_H can also be computed as the inverse to the smallest positive root of the equation:

$$\det M = 0, \tag{4.8}$$

where M is a matrix given by

$$\begin{pmatrix} 1-t^k & 0 & t^k & t^k \\ 0 & 1-t^k & t^k & t^k \\ t^l & t^l & 1-t^l & 0 \\ t^l & t^l & 0 & 1-t^l \end{pmatrix}$$

Now

$$\det M = 0 \Leftrightarrow 2t^{2k+l} + 2t^{k+2l} - 3t^{2k+2l} + t^{2k} - 2t^k + t^{2l} - 2t^l + 1 = 0 \Leftrightarrow$$
$$\Leftrightarrow (t^k - 1)(t^l - 1)(3t^{k+l} + t^k + t^l - 1) = 0.$$

The positive root of $3t^{k+l} + t^k + t^l - 1 = 0$ lies somewhere in the interval $(0;1)$ and hence is the smallest positive root of (4.8). Not surprisingly, for $t = z_0^{-1}$ we have

$$\frac{3}{z_0^{k+l}} + \frac{1}{z_0^k} + \frac{1}{z_0^l} - 1 = \frac{-z_0^{k+l} + z_0^k + z_0^l + 3}{z_0^{k+l}},$$

that is z_0^{-1} is the required root of (4.8).

Let us evaluate z_0. Many methods of finding the upper bound of the zeros of an equation are known, we use the one described in [11].

If $k = l = 1$ then (4.7) becomes a quadratic equation with unique positive solution $z_0 = 3$.

Now suppose the condition $k = l = 1$ does not hold. Without loss of generality we may assume $k \geq l$.

Claim: $z_0 \leq \max\{r_0^{\frac{1}{l}}, r_0^{\frac{2}{k}}, (3r_0^3)^{\frac{1}{k+l}}\}$, where r_0 is a real solution of $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} = 1$.

Proof of the claim: Suppose that $z_0 > \max\{r_0^{\frac{1}{l}}, r_0^{\frac{2}{k}}, (3r_0^3)^{\frac{1}{k+l}}\}$ then

$$z_0 > r_0^{\frac{1}{l}}, \qquad z_0 > r_0^{\frac{2}{k}}, \qquad z_0 > (3r_0^3)^{\frac{1}{k+l}}.$$

But then we have

$$\frac{1}{r_0} > \frac{1}{z_0^l}, \qquad \frac{1}{r_0^2} > \frac{1}{z_0^k}, \qquad \frac{1}{r_0^3} > \frac{3}{z_0^{k+l}}.$$

Adding the last three inequalities together gives us: $\frac{1}{r_0}+\frac{1}{r_0^2}+\frac{1}{r_0^3} > \frac{1}{z_0^l}+\frac{1}{z_0^k}+\frac{3}{z_0^{k+l}} = 1$.

Contradiction.

Now since $l \geq 1$ and $k \geq 2$ we have

$$r_0^{\frac{1}{l}} \leq r_0, \qquad r_0^{\frac{2}{k}} \leq r_0, \qquad (3r_0^3)^{\frac{1}{k+l}} \leq 3^{\frac{1}{k+l}} r_0.$$

Hence $z_0 \leq 3^{\frac{1}{k+l}} r_0$. Now $r_0 = \frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}) \approx 1.8393$, which implies that for $k+l \geq 3$ the relative growth rate of H is less than 2.6527, whereas the growth rate of F is 3. ■

Now we are in position to show that the relative growth rate of a proper finitely generated subgroup of a free group is less than growth rate of the entire group.

As before let the free group be $F = \langle a_1, a_2, \ldots, a_r \rangle$ and its subgroup $H = \langle g_1, g_2, \ldots, g_m \rangle$ with length $|g_i| \geq 1$ for $i = 1, \ldots, m$ and no cancellations between generators and their inverses apart from cancellation of the type $g_i g_i^{-1}$.

Since cancellations are not allowed m cannot be greater than r. If $m = r$ and $s_i := |g_i| = 1$ for $i = 1, \ldots, m$ then, as Theorem 4.4 tells us, $2r - 1$ is the positive root of

$$z^r - \sum_{t=1}^{r} (2t-1) \binom{r}{t} z^{r-t} = 0,$$

where $\binom{r}{t}$ denotes the binomial coefficient. That is, it is the positive root of $F(0) = 1$ in which all $s_{k_i} = 1$. Let us introduce a new function

$$I(z) := z^{\sum_{i=1}^{m} s_i} - \sum_{t=1}^{m-1} \sum_{\sigma \in K_{m-t}} (2t-1) z^{\sum_{i=1}^{m-t} s_{k_i}} - (2m-1).$$

From (4.5) we see that $I(z) > 0 \Leftrightarrow F(0) < 1$. Clearly $I(0) < 0$. Now if $z = 2r - 1$ and if we increase any of s_{k_i} then $F(0)$ becomes less than 1, which means that $I(2r-1) > 0$. We have $I(0) < 0$ and $I(2r-1) > 0$. From (4.6) we conclude that $z_0 < 2r - 1$. Hence, if $m = r$ the relative growth rate of finitely generated proper subgroup of a free group is less than the growth rate of the the entire group.

A similar argument applies if $m < r$. In this case $F(0) < 1$ as well and, therefore, z_0 is strictly less then $2r - 1$.

Chapter 5

Cosets of subgroups conjugate to H

In this chapter we consider the case when the condition of non-cancellation between generators of the subgroup H is removed. The idea of the proof of Theorem 5.3 belongs to Aleksandr Olshanskii.

We start with folklore property of subsets of a finitely generated free group. The case of free monoids is covered in [1].

Theorem 5.1 ([4], p. 478)**.** *Let X , a set of cardinality $r \geq 2$, be a generating set of a free group F; w a nonempty reduced word in $X^{\pm 1}$ and $A(w)$ the set of reduced words in F which do not contain w as a subword. Then there are positive numbers c and ε, such that $d_A(n) := \#(A(w) \cap F(n)) \leq c(2r - 1 - \varepsilon)^n$ for all $n \in \mathbb{N}$, where $F(n)$ is the set of all reduced words of length n in F.*

Proof. Suppose w has length $|w| = m$ then $d_A(m) = 2r(2r-1)^{m-1} - 1$. The number of reduced words of length mq, with $q \in \mathbb{N}$, which do not contain w as subword in this case is at most $(2r(2r-1)^{m-1}-1)((2r-1)^m-1)^{q-1}$, since each word of this length can be written as a product of q words of length m. According to Quotient-Remainder Theorem for arbitrary n we have $n = mq+p$ with $0 \leq p < m$. Since there are $(2r-1)^p$ ways to extend a word of length mq to a word of length $mq+p$ we conclude that:

$$
\begin{aligned}
d_A(n) &\leq (2r(2r-1)^{m-1} - 1)((2r-1)^m - 1)^{q-1}(2r-1)^p \\
&= (2r(2r-1)^{m-1} - 1)((2r-1)^m - 1)^{\lfloor \frac{n}{m} \rfloor - 1}(2r-1)^p \\
&\leq \Big(\frac{2r}{2r-1}(2r-1)^m - 1\Big)((2r-1)^m - 1)^{\lfloor \frac{n}{m} \rfloor - 1}(2r-1)^{m-1} \\
&< \Big(\frac{2r}{2r-1}(2r-1)^m\Big)((2r-1)^m - 1)^{\lfloor \frac{n}{m} \rfloor - 1}(2r-1)^{m-1} \\
&= \Big(\frac{2r}{(2r-1)^2}(2r-1)^m\Big)((2r-1)^m - 1)^{\lfloor \frac{n}{m} \rfloor - 1}(2r-1)^m \\
&< (2r-1)^m((2r-1)^m - 1)^{\lfloor \frac{n}{m} \rfloor - 1}(2r-1)^m \\
&\leq (2r-1)^{2m}((2r-1)^m - 1)^{\frac{n}{m} - 1} \\
&< (2r-1)^{2m}((2r-1)^m - 1)^{\frac{n}{m}}.
\end{aligned}
$$

By taking $c = (2r-1)^{2m}$ and considering the fact that $((2r-1)^m-1)^{\frac{1}{m}} < (2r-1)$ we prove the lemma. ∎

The following result is due to B.H.Neumann.

Lemma 5.2 ([6], p. 239)**.** *Let a group G be the union of finitely many, say n, cosets of subgroups $H_1, H_2, \ldots, H_n$:*

$$G = \bigcup_{i=1}^{n} H_i g_i.$$

Then the index of (at least) one of H_i does not exceed n. In particular at least one subgroup H_i has finite index in G.

Proof. We prove the lemma by induction on the number of distinct subgroups among $H_1, \ldots, H_n$.

If all H_i coincide then G is the union of n cosets of one subgroup, that is, the subgroup has index at most n.

Now assume that the claim holds for at most $k-1$ distinct subgroups H_i. We have to show that the statement holds for k distinct subgroups among $H_1, H_2, \ldots, H_n$. Take one of these subgroups, say H_n. The union

$$G = \bigcup_{i=1}^{n} H_i g_i$$

can be arranged in such a way that $H_1, \ldots, H_m$ are different from H_n and $H_{m+1} = H_{m+2} = \cdots = H_n$. In this case either

$$G = \bigcup_{i=m+1}^{n} H_n g_i,$$

that is H_n has index at most n in G; or else there is an element h such that

$$h \notin \bigcup_{i=m+1}^{n} H_n g_i.$$

Since any two cosets of a subgroup either coincide or are disjoint we have:

$$H_n h \cap \bigcup_{i=m+1}^{n} H_n g_i = \varnothing,$$

and hence

$$H_n h \subseteq \bigcup_{i=1}^{m} H_i g_i$$

or

$$H_n g \subseteq \bigcup_{i=1}^{m} H_i g_i h^{-1} g,$$

for any $g \in G$. Hence we have that every right coset of H_n is contained in a finite union of right cosets of the other $k-1$ subgroups H_i. But then G can also be covered by a union of finitely many right cosets of these $k-1$ subgroups, and by the induction hypothesis one of them has finite index in G. ■

In what follows we will utilize the idea of *Schreier coset graph* and its *core*. The Schreier coset graph $\Gamma(G, H, X)$ is a labeled graph associated with a group G generated by the set $X = \{a_1, \dots a_r\}$, and its subgroup H. The set of vertices of the graph is the set of right cosets of H, $\mathcal{H} = \{Hw : w \in G\}$. Given $Hw \in \mathcal{H}$ and $a \in X$, there is unique edge e with initial vertex $e_- = Hw$, terminal vertex $e_+ = Hwa$ and label $\mathrm{Lab}(e) = a$. A path $p = e_1 \dots e_n$ is *reduced* if and only if its label $\text{Lab}(p) := \text{Lab}(e_1)\dots\text{Lab}(e_n)$ is a reduced word in the group G. A reduced path $p = e_1 \dots e_n$ is a *loop* if $(e_1)_- = (e_n)_+$.

The minimal subgraph $\mathcal{C}(\Gamma)$ of $\Gamma(G, H, X)$ which contains H and all reduced loops originating in H, is called the *core* of Schreier coset graph of G. Let $p_1, \dots, p_s, \dots$ be the reduced loops corresponding to the reduced forms of some generators $h_1, \dots, h_s, \dots$ of H. Any element of H can be written as a loop p obtained after all possible cancellations from a product $p(1)\dots p(t)$ where $p(i) \in \{p_1^{\pm 1}, \dots, p_s^{\pm 1}, \dots\}$, that is, $\mathcal{C}(\Gamma)$ contains only edges from the paths $p_1, \dots, p_s, \dots$. Hence for a finitely generated H the core $\mathcal{C}(\Gamma)$ is finite.

Theorem 5.3 (Olshanskii)**.** *The relative growth rate of a finitely generated subgroup of infinite index of a free group $F = \langle a_1, \dots, a_r \rangle$ is strictly less then the growth rate of the free group F itself:*

$$\alpha_H < 2r - 1.$$

Proof. Proof by contradiction. Suppose the statement of the theorem is false. Since the relative growth rate cannot be greater than $2r - 1$ we can assume that there exists a finitely generated subgroup H of infinite index of the free group F such that $\alpha_H = 2r - 1$. If this is the case then every word in F must be a subword of a word in H, since otherwise we obtain a contradiction to Theorem 5.1.

Consider the core $\mathcal{C}(\Gamma)$ of Schreier coset graph Γ associated with F, H and the generating set of F. Every word w in H can be associated with a loop l in $\mathcal{C}(\Gamma)$ originating in H, and every subword s of this word, that is, $w = u_1 s u_2$, can be associated with a path p on this loop such that $l = p_1 p p_2$, where p_1 is the reduced path on l connecting H to Hu_1 and p_2 is the reduced path on l connecting $Hu_1 s$ to H. Since $\mathcal{C}(\Gamma)$ is finite, the set of reduced paths connecting H and Hu_1, and also $Hu_1 s$ and H for all possible subwords s of all words w, is finite. It follows that $V = \{v_1, \ldots, v_n\}$, the set of reduced words corresponding to reduced paths between H and Hu_1, and also $Hu_1 s$ and H for all possible subwords s of all words w, is finite. If any word in F is a subword of a word in H then for any $s \in F$ we can find v_1 and v_2 from the finite set V such that $v_1 s v_2 \in H$, that is, $s \in v_1^{-1} H v_2^{-1} = v_1^{-1} H v_1 (v_1^{-1} v_2^{-1})$.

Since V is finite it follows that F is the union of finitely many cosets of a finite number of subgroups conjugate to H:

$$F = \bigcup_{i,j=1}^{n} (v_i^{-1} H v_i) g_{ij},$$

where $g_{ij} = v_i^{-1} v_j^{-1}$, with $v_i, v_j \in V$. By Lemma 5.2, at least one of subgroups conjugate to H must have finite index. Clearly then H itself has finite index, a contradiction. ■

So we have seen in this chapter that for any finitely generated subgroup H of infinite index, we always have $\alpha_H < 2r - 1$. In the next chapter we will show that an even more general fact can be proven using the notion of *amenability*.

Chapter 6

Non-amenability of Schreier graphs

In this and subsequent chapters we outline the proof, due to Ilya Kapovich [10], of even more general result, namely, Theorem 6.8, which states that the Schreier coset graph associated with a finitely generated hyperbolic group and a quasiconvex subgroup of infinite index, is non-amenable. As we will see, free groups are hyperbolic and their finitely generated subgroups are quasiconvex, hence the associated Schreier graphs are non-amenable, which implies that the relative growth rate of a subgroup is strictly less than the growth rate of the free group.

In the previous chapter we encountered the notion of Schreier coset graph which sometimes is also called *relative Cayley graph*. The Cayley graph $\Gamma(G, A)$ of a group G generated by set A is, of course, the Schreier coset graph for the trivial subgroup $H = \{1\}$. The Schreier coset graph $\Gamma(G, H, A)$ is connected and becomes equipped with metric m by identifying each edge with the unit interval in $\mathbb{R}$ and defining $m(x, y)$ to be the length of the shortest path joining x to y. Thus $(\Gamma(G, H, A), m)$ and in particular $(\Gamma(G, A), m)$ become metric spaces.

We can also define *word metric* m_A on a group generated by set A as follows. If $w_1, w_2 \in G$ then we define $m_A(w_1, w_2)$ to be the length of reduced word $w_1^{-1}w_2$. It is not hard to see that that the underlying set of G together with word metric forms a metric space (G, m_A). Note that in the metric space $(\Gamma(G, A), m)$ we have $m(x, y)$ equals to $m_A(x, y)$ if x and y are vertices of $\Gamma(G, A)$.

A path in $(\Gamma(G, A), m)$ of minimal length that connects points x and y is called a *geodesic segment*, denoted $[x, y]$. A geodesic triangle with vertices x, y and z in $\Gamma(G, A)$ is the union of three geodesic segments $[x, y]$, $[y, z]$, $[z, x]$.

In an arbitrary metric space there may not exist a geodesic segment from one point to another, and if geodesic segments exist, they are not necessarily unique. If for any pair of points in a metric space there exists geodesic segment between these points then the metric space is said to be *geodesic*. If for a geodesic metric space there exists a global constant δ such that each edge of each geodesic triangle in this metric space is contained in the δ-neighbourhood of the union of the other two sides, then this metric space is called *hyperbolic*. Finally, a finitely generated group is *hyperbolic* if for some, and hence for any, of its finite generating sets the Cayley graph of the group is hyperbolic.

Every tree, in graph-theoretical meaning of this word, is a hyperbolic metric space. Since any two points are connected by a unique shortest path, this metric space is geodesic and any side of geodesic triangle is contained in the union of the other two sides. Since Cayley graphs of free groups are trees, every free group is hyperbolic.

A subset Y of a geodesic metric space is *quasiconvex* if there is a constant ε such that for any $x, y \in Y$ any geodesic $[x, y]$ in Y is completely within an ε-neighbourhood of Y. A subgroup of a group is *quasiconvex* if the vertices in the subgroup form a quasiconvex set in the Cayley graph.

Lemma 6.1 ([5], p. 77)**.** *A subgroup of a finitely generated free group is quasiconvex if and only if it is finitely generated.* ❏

Before turning our attention to amenability we need to introduce the *Gromov product.* Roughly speaking, Gromov product measures how long geodesics remain close together.

Definition 6.2. Let (Z, m) be a metric space and suppose $x, y, z \in Z$. The Gromov product, denoted by $(x, y)_z$ is defined as

$$(x, y)_z = \frac{1}{2}(m(z, x) + m(z, y) - m(x, y)).$$

▲

It is straightforward to verify that the Gromov product is symmetric:

$$(x, y)_z = (y, x)_z, \tag{6.1}$$

zero at the endpoints:

$$(x, y)_x = (x, y)_y = 0, \tag{6.2}$$

and, finally, for any $p, q, x, y, z \in Z$,

$$m(x, y) = (x, z)_y + (y, z)_x, \tag{6.3}$$

$$0 \leq (x, y)_z \leq \min\{m(x, z), m(y, z)\}, \tag{6.4}$$

$$\left|(y, z)_p - (y, z)_q\right| \leq m(p, q), \tag{6.5}$$

$$\left|(x, y)_p - (x, z)_p\right| \leq m(y, z). \tag{6.6}$$

In fact we could define hyperbolic spaces as geodesic metric spaces for which there exists a constant $\delta \geq 0$ such that for all p, q, x, y, z in this space

$$(x, z)_p \geq \min\{(x, y)_p, (y, z)_p\} - \delta.$$

Definition 6.3. Let (Z, m) be a metric space and let $x \in Z$ and $P, Q \subseteq Z$. The *Gromov product* for sets is defined as

$$(P, Q)_x = \sup\{(p, q)_x : p \in P, q \in Q\}. \qquad \blacktriangle$$

Next we introduce notion of *amenability* of a group action.

Definition 6.4. Given a group G acting on a set S, *an invariant mean* is a G-invariant map μ from the collection of all subsets of S to $[0, 1]$ such that:

(i) $\mu(B \cup C) = \mu(B) + \mu(C)$ when $B \cap C = \varnothing$;

(ii) $\mu(S) = 1$. If such a mean exists, the action is called *amenable*.

A group is said to be *amenable* if its action on itself is amenable, and *non-amenable* otherwise.

A Schreier coset graph $\Gamma(G, H, A)$ is called *amenable* if the corresponding action of G on G/H is amenable. ▲

Theorem 6.5 ([13], p. 55). *Let F be a free group generated by X with $\#X = r$, H be its subgroup. Then the Schreier coset graph $\Gamma(F, H, X)$ is amenable if and only if relative growth rate α_H equals $2r - 1$.* ❑

Let Γ be a connected graph of bounded degree with associated metric m, and S be a finite nonempty subset of the vertex set of Γ. Then for S and an integer $k \geq 1$ we define by $\mathcal{N}_k(S)$ the set of all vertices v in Γ such that $m(v, S) \leq k$. The following is a particular case of a result proved in [8], Theorem 32.

Lemma 6.6 ([8], p. 19). *Let Γ be a Schreier graph with metric m.*

The following are equivalent:

(i) Γ *is non-amenable;*

(ii) *There is some $k \geq 1$ such that for any finite nonempty subset S of vertices of Γ the following inequality holds:*

$$\#\mathcal{N}_k(S) \geq 2(\#S);$$

(iii) *For any integer q there exists $k \geq 1$ such that for any finite nonempty subset S of vertex set of Γ the following inequality holds:*

$$\#\mathcal{N}_k(S) \geq q(\#S).$$

❏

The original definition of amenability of groups, in terms of a finitely additive invariant measure on the subsets of G, was introduced by John von Neumann in 1929. Von Neumann observed that the class of amenable groups is closed under the operation of taking subgoups and that the free group of rank two is non-amenable. It follows that a group which contains a subgroup isomorphic to the free group on two generators, is non-amenable. The next lemma which can be found in [7], Theorem 37, implies that hyperbolic groups are non-amenable unless they are virtually cyclic, that is unless they contain a cyclic subgroup of finite index.

Lemma 6.7 ([7], p. 157)**.** *Let G be a hyperbolic group with a finite generating set A. Then a subgroup $H \leq G$ is either virtually cyclic or H contains a free group of rank two which is quasiconvex in G.* ❏

A hyperbolic group that contains an infinite cyclic subgroup of a finite index is called *elementary*. From now on we will be talking about non-elementary hyperbolic groups.

Theorem 6.8 ([10], p. 2)**.** *Let G be a finitely generated non-elementary hyperbolic group generated by a set A and let H be a quasiconvex subgroup of infinite index in G. Then the Schreier coset graph $\Gamma(G, H, A)$ is non-amenable.* ❑

We postpone the proof of Theorem 6.8 for now. Let us first have a look at the corollaries of this theorem.

Corollary 6.9. *Let F be a free group generated by X with $\#X = r \geq 2$ and let H be a finitely generated subgroup of F of infinite index. Then $\alpha_H < 2r - 1$.*

Proof. This claim is obvious when H is cyclic and follows from Lemma 6.1, Theorem 6.5, Theorem 6.8 otherwise. ∎

Corollary 6.10. *Let F be a free group generated by X with $\#X = r \geq 2$ and let H be a finitely generated subgroup of F. Then H has finite index in F if and only if its relative growth rate $\alpha_H = 2r - 1$.*

Proof. Corollary 6.9 provides us with the implication $\alpha_H = 2r - 1 \Rightarrow [F : H]$ is finite.

For converse, let $\mathcal{H} = \{Hx : x \in G\}$ be the set of right cosets of H. Consider the action of the free group F on $\mathcal{H}$. Let $\mathcal{S}$ be a collection of subsets of $\mathcal{H}$. We can define an F-invariant map

$$\begin{aligned} \mu : \mathcal{S} \to \quad & [0, 1] \\ A \mapsto \quad & \#A/\#\mathcal{H}. \end{aligned}$$

Hence, if H has finite index then the F-action on F/H is amenable. Now the action of a free group F on the set of right cosets of H is amenable if and only if $\Gamma(F, H, X)$ is amenable if and only if $\alpha_H = 2r - 1$, by Theorem 6.5. This proves the other implication. ■

Chapter 7

Proof of Theorem 6.8

In this chapter we will prove Theorem 6.8. Recall that we consider only non-elementary hyperbolic groups. Let G be such a group generated by a set A, let $\Gamma(G, A)$ be the Cayley graph of G with respect to A. Let $\delta \geq 1$ be the parameter involved in the definition of the hyperbolic space $(\Gamma(G, A), m)$. Consider a quasiconvex subgroup $H \leq G$ of infinite index. We set $|g|_A = m_A(1, g)$ for any $g \in G$.

Lemma 7.1 ([10], p. 8)**.** *There exists an integer non-negative constant $K = K(G, H, A)$ such that if $g \in G$ is the closest to 1 element with respect to m_A in the coset class Hg then $(g, h)_1 \leq K$ for all $h \in H$.* ❑

Note that from Lemma 7.1 it follows that $(g, H)_1 \leq K$.

Lemma 7.2 ([10], p. 8)**.** *Let $g \in G$ be such that $(g, H)_1 \leq P_1$ and $|g|_A > P_1 + P_2 + \delta$ with $P_1, P_2 > 0$, and $f \in G$ be such that $|f|_A \leq P_2$. Then $(gf, H)_1 \leq P_1 + \delta$.*

Proof. First we note that $|g|_A = m(1, g) = (g, gf)_1 + (1, gf)_g$, as the special case of (6.3).

Now due to (6.4), $(1, gf)_g \leq m(g, gf) = |f|_A \leq P_2$, which implies that

$$(g, gf)_1 = |g|_A - (1, gf)_g > P_1 + P_2 + \delta - P_2 = P_1 + \delta. \tag{7.1}$$

Employing the lemma's condition $(g, H)_1 \leq P_1$, for any $h \in H$ we have

$$P_1 + \delta \geq (g, h)_1 + \delta \geq \min\{(g, gf)_1, (gf, h)_1\},$$

since G is hyperbolic. But this implies that $(gf, h)_1 \leq P_1 + \delta$, since (7.1) tells us that $(g, gf)_1 > P_1 + \delta$. Since $h \in H$ was arbitrary, the proof is complete. ■

Lemma 7.3 ([10], p. 9)**.** *If $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$ then there exists $h \in H$ such that $hg_1 = g_2$ and $|h|_A \leq (g_1, H)_1 + (g_2, H)_1$.*

Proof. Clearly $Hg_1 = Hg_2$ implies existence of $h \in H$ such that $hg_1 = g_2$. According to (6.3), we have:

$$|h|_A = m(1, h) = (h, g_2)_1 + (1, g_2)_h = (h, g_2)_1 + (1, hg_1)_h =$$
$$= (h, g_2)_1 + (h^{-1}, g_1)_1 \leq (g_1, H)_1 + (g_2, H)_1.$$

■

Now we are in position to prove the main result of this chapter.

Proof of Theorem 6.8. Let the Schreier coset graph $\Gamma(G, H, A)$ have metric m. First of all we note that Lemma 7.1 provides us with a non-negative constant $K = K(G, H, A)$ such that if $g \in G$ is the closest to 1 element with respect to m_A in the coset class Hg then $(g, H)_1 \leq K$. The Schreier coset graph $\Gamma(G, H, A)$ is infinite since index of H in G is infinite, it is also connected, as action of G on H is transitive, and $2r$-regular, r here is the number of elements in A.

Denote N_1 the number of elements belonging to G such that $|g|_A \leq 2(K + \delta)$. Then $\Gamma(G, H, A)$ has at most N_1 vertices within the distance $2(K + \delta)$ from H. As Lemma 6.7 tells us, non-elementary hyperbolic groups are non-amenable and, hence, the Cayley graph $\Gamma(G, A)$ is non-amenable. The fact of non-amenability of $\Gamma(G, A)$ implies existence of a constant $k_1 > 0$ such that for any finite non-empty subset S of vertex set $\Gamma(G, A)$

$$\#\mathcal{N}_{k_1}(S) \geq 4N_1(\#S), \tag{7.2}$$

by Lemma 6.6.

Consider the vertices of $\Gamma(G, H, A)$ within the distance $K + \delta + k_1$ from H, that is, the vertices Hg such that $m(H, Hg) \leq K + \delta + k_1$. Let N_2 be the number of elements of G with length at most $K + \delta + k_1$. Then we can always find a constant $k_2 > 1$ such that for any vertex Hg with $m(H, Hg) \leq K + \delta + k_1$ the k_2-neighbourhood of Hg has at least $4N_2$ vertices. Note that existence of k_2 is guaranteed by the fact that $\Gamma(G, H, A)$ is infinite. Take $k = \max\{k_1, k_2\}$.

Let T be a nonempty finite subset of the vertex set of $\Gamma(G, H, A)$. We can write

$T = T_1 \sqcup T_2$ with T_1 being the intersection of T with the closed ball in $\Gamma(G, H, A)$ of radius $K + \delta + k_1$ centered at H. We have two possibilities: either $\#T_1 \geq \frac{1}{2}\#T$ or $\#T_1 < \frac{1}{2}\#T$. We will show that in both cases $\#\mathcal{N}_k(T) \geq 2\#T$ which, by Lemma 6.6, would imply non-amenability of $\Gamma(G, H, A)$.

Case 1: Suppose $\#T_1 \geq \frac{1}{2}\#T$. We have

$$\#T_1 \geq \frac{1}{2}\#T \Rightarrow \#T \leq 2\#T_1 \leq 2N_2 \Rightarrow 2\#T \leq 4N_2.$$

Since $k \geq k_2$ and $T_1 \subseteq T$ we have

$$\#\mathcal{N}_k(T) \geq \#\mathcal{N}_k(T_1) \geq \#\mathcal{N}_{k_2}(T_1) \geq 4N_2.$$

Case 2: Suppose $\#T_1 < \frac{1}{2}\#T$, that is, $\#T_2 \geq \frac{1}{2}\#T$. We can write the elements of T_2 as Hg_i, $i = 1, \ldots, t$, where the g_i's are the shortest with respect to m_A in Hg_i, that is, $T_2 = \{Hg_1, \ldots, Hg_t\}$ where $\#T_2 = t$ of course, and $|g_i|_A > K + \delta + k_1$, by the nature of T_2. By the choice of K, we have $(g_i, H)_1 \leq K$ for all i. Also for any $f \in G$ with $|f|_A < k_1$ and for each i we have $(g_i f, H)_1 \leq K + \delta$, by Lemma 7.2. Let $S = \{g_1, \ldots, g_t\}$ and S' be the set of all vertices in $\Gamma(G, A)$ contained in the k_1-neighbourhood of S. Now

$$\#S' \geq 4N_1(\#S) = 4N_1(\#T_2). \tag{7.3}$$

Employing Lemma 7.3, we conclude that if $g, g' \in S'$ are such that $Hg = Hg'$ then there exists $h \in H$ such that $hg = g'$ with $|h|_A \leq 2(K + \delta)$.

Let $F' = \{Hg, g \in S'\}$ and recall that N_1 is the number of elements of G such that $|g|_A \leq 2(K+\delta)$. But then F' has at least $\frac{\#S'}{N_1}$ distinct elements. Using equation (7.3) we conclude that $\#F' \geq 4\#T_2 \geq 2\#T$. But, by its choice, F' is contained in the k-neighbourhood of T in $\Gamma(G, H, A)$, in and $k \geq k_1$. Therefore,

$$\#\mathcal{N}_k(T) \geq \#F' \geq 2\#T$$

We proved that $\#\mathcal{N}_k(T) 2\#T$ for any finite nonempty subset T of vertex set of $\Gamma(G, H, A)$ and, hence, by Lemma 6.6, $\Gamma(G, H, A)$ is nonamenable. ■

Appendix

k	l	p	q	$\det M$	The least positive root	α_H
1	1	1	1	$-(t-1)^4(t+1)^3(7t-1)$	$\frac{1}{7}$	7
1	1	1	2	$-(t-1)^4(t+1)^3(7t^3+t^2+5t-1)$	$\approx$0.18442	$\approx$5.42241
1	1	1	3	$-(t-1)^4(t+1)^3(t^2+t+1)^2(7t^3-6t^2+6t+1)$	$\approx$0.19640	$\approx$ 5.09165
1	1	2	2	$-(t-1)^4(t+1)^3(7t^3+3t^2+3t-1)$	$\approx$0.24184	$\approx$4.1350
1	1	2	3	$-(t-1)^4(t+1)^3(t^2+t+1)^2(7t^5-4t^4+9t^3-3t^2+4t-1)$	$\approx$0.26351	$\approx$3.794922
1	1	3	3	$-(t-1)^4(t^2+t+1)^2(t+1)^3(t^2-t+1)(7t^3-4t^2+4t-1)$	$\approx$0.29167	$\approx$3.42853
1	2	2	2	$-(t-1)^4(t+1)^3(7t^3+5t^2+t-1)(t^2+1)^2$	$\approx$ 0.31016	$\approx$3.22414
1	2	3	3	$-(t-1)^4(t^2+t+1)^2(t+1)^3(t^2-t+1)(7t^5-2t^4+7t^3-t^2+2t-1)$	$\approx$0.37703	$\approx$2.65230
2	2	2	2	$-(t-1)^4(t+1)^4(7t^2-1)(t^2+1)^3$	$\frac{1}{\sqrt{7}}$	$\sqrt{7}$
2	2	2	3	$-(t-1)^4(t+1)^3(t^2+t+1)^2(t^2+1)^2(7t^5+t^3+5t^2-1)$	$\approx$0.41170	$\approx$2.42895
1	3	3	3	$-(t-1)^4(t^2+t+1)^3(t+1)^3(t^2-t+1)(7t^3-2t^2+2t-1)$	$\approx$0.41855	$\approx$2.38920
1	2	2	3	$-(t-1)^4(t+1)^3(t^2+t+1)^2(t^2+1)(7t^5-2t^4+5t^3+t^2+t-1)$	$\approx$0.42078	$\approx$2.37653
2	3	3	3	$-(t-1)^4(t^2+t+1)^3(t+1)^3(t^2-t+1)^2(7t^5+5t^3+t^2-1)$	$\approx$0.48601	$\approx$2.05757
3	3	3	3	$-(t-1)^4(t^2+t+1)^4(t+1)^3(t^2-t+1)^3(7t^3-1)$	$\frac{1}{\sqrt[3]{7}}$	$\sqrt[3]{7}$

Table A.1: The relative growth rate for subgroups of the free group of rank 4.

Bibliography

[1] Bahturin Y., Olshanskii A. Actions of maximal growth. *Proc. London Math. Soc.*, 1(101):27–72, January 2010. URL http://plms.oxfordjournals.org/content/101/1/27.abstract.

[2] Bahturin Y., Olshanskii A. Growth of subalgebras and subideals in free Lie algebras. *Preprint*, pages 1–31, 2013. URL http://arxiv.org/abs/1307.1438.

[3] Olshanskii A. Subnormal subgroups in free groups, their growth and cogrowth. *Preprint*, pages 1–45, 2013. URL http://arxiv.org/abs/1312.0129.

[4] Sambusetti A. Growth tightness of free and amalgamated products. *Ann. Scient. Ec. Norm. Sup.*, 35(4):477–488, July-August 2002. URL http://www.sciencedirect.com/science/article/pii/S0012959302011011.

[5] Rosenberger G., Spellman D. Bencsath K., Gaglione A., Fine B., Myasnikov A., Roehl F.,. Aspects of the theory of free groups. *Algorithmic problems in groups and semigroups*, ISBN 3-7643-4130-0, 2000.

[6] Neumann B.H. Groups covered by permutable subsets. *J. London Math. Soc.*, 2(s1-29):236–248, January 1954. URL http://jlms.oxfordjournals.org/content/s1-29/2/236.extract.

[7] de la Harpe P., Ghys E. Sur les groupes hyperboliques d'après Mikhael Gromov (French). *Progress in Mathematics series*, ISBN 0-8176-3508-4, 1990.

[8] Ceccherini-Silberstein T., de la Harpe P., Grigorchuk R. Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces. *Proc. of the Steklov Inst. of Math.*, 224:57–97, 1998. URL http://www.unige.ch/math/folks/delaharpe/articles/18-CGH.pdf.

[9] Adkins W. A., Davidson M. G. Ordinary differential equations. *Springer New York Heidelberg Dordrecht London*, ISBN 978-1-4614-3617-1, 2010.

[10] Kapovich I. The non-amenability of Schreier graphs for infinite index quasi-convex subgroups of hyperbolic groups. *Ensign. Math.*, 3-4(48):359–375, January 2002. URL http://www.math.uiuc.edu/~kapovich/PAPERS/am4.pdf.

[11] Kioustelidis J.B. Bounds for positive roots of polynomials. *Journal of Computational and Applied Mathematics*, 2(16):241–244, May 1986.

[12] Lyndon R., Schupp P. Combinatorial group theory. *Springer-Verlag Berlin Heidelberg*, ISBN 3-540-07642-5, 1977.

[13] Grigorchuk R. Banach invariant means on homogeneous spaces and random walks. *Thesis*, (Russian):1–132, 1978. URL http://www.math.tamu.edu/~grigorch/publications/Phd/Ph.D.%20Thesis.html.

[14] Karass A., Solitar D., Magnus W. Combinatorial group theory. *Dover Publications INC.*, ISBN 0-486-63281-4, 1976.