ON THE ORDER STATISTICS FROM CORRELATED NORMAL DISTRIBUTION

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By

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Abstract

The inferences for the order statistics for normal random variables with a general correlation structure, where correlations can be unequal or equal, positive or negative are discussed in this thesis. Specifically, based on a small correlations approach, we, first, develop the joint density function of the order statistics under the general correlation set-up. We, then, provide an approximation for the distribution of a single order statistic under the same correlation set-up. Special attention is given to the derivations for the distributions of the maxima and minima. The computational aspects of the distribution of the maxima, for example, are discussed in details for the homoscedastic equi-correlation, homoscedastic unequal correlations, and heteroscedastic unequal correlations cases. The applications of the proposed small correlations approach to compute the percentile points of the maxima are shown for the homoscedastic correlated normal variables following a stationary auto-regressive process of order one, and for the heteroscedastic correlated normal variables following a nonstationary antedependence model. Furthermore, the small correlations approach for the maxima is compared with the Bonferroni bounds approximation for unequally homoscedastic and heteroscedastic correlated normal variables.

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Contents

List of Tables

Chapter 1

Introduction

1.1 Motivation of the problem

Over the last few decades, order statistics inferences for the independent normal variables have been widely discussed in the literature. For example, we refer to David (1981), Galambos (1987), Balakrishnan & Cohen (1991), Reiss (1989), Gumbel (1958), Barnett & lewis (1978) for such inferences. In practice, order statistics problems arise in many practical situations, for example, in the context of flood control in a given place. In a flood control problem, let $Y_1, Y_2, \ldots, Y_j, \ldots, Y_n$ be the yearly water levels of a river for n years, where Y_j denotes the water level of the river for the jth year. For this kind of yearly data, it is reasonable to assume that they are mutually independent. In order to take any remedial measure in preventing the future flood spread, it is important to study the pattern of the maximum water level of the river. On the other hand, if we want to use the river as a source of electrical energy in order to maintain a minimum level, then our interest would be to study the pattern of minimum water level of the river. In notations, the former problem requires to find $Pr(Y_{(n)} \leq a)$, and the latter problem requires to find $Pr(Y_{(1)} \geq b)$, where $Y_{(n)} = max(Y_j, 1 \leq j \leq n)$, $Y_{(1)} = min(Y_j, 1 \leq j \leq n)$ and a & b are the specified water levels.

But, in practice, there are also many situations where the observations may be correlated but still follow the Gaussian distribution. For example, consider the widespread production of rice in Hungary (say), for which a minimum amount of rainfall per week is needed. Similarly, consider the production of potatoes in West Africa (say) where the production may be affected if the daily high temperatures exceed a maximum amount of daily temperature. Here, let Y_j, $j = 1, 2, \ldots, n$ be the amount of rainfall in Hungary or the maximum temperature in West Africa during the $jth, j = 1, 2, \ldots, n$ week or day. It then follows that in the rice production problem one would study the pattern of $Y_{(1)} = min(Y_{(j)}, 1 \leq j \leq n)$ and in the potato production problem one would be interested to study the pattern of $Y_{(n)} = max(Y_{(j)}, 1 \leq j \leq n)$. Note, however, that in both of these problems, as the observations are collected successively over time, week or day, it is reasonable to assume that the original observations $Y_1, Y_2, \ldots, Y_n$ are correlated random variables. . Consequently, these problems reduce to the order statistics problems where observations are correlated.

Futhermore, under the cases when variables are correlated, most of the order statistics problems discussed so far in the literature belong to equi-correlated case. For such analyses, we refer to Gupta, Pillai & Steck (1964), Gupta, Nagel & Panchapakesan (1973), Owen & Steck (1962), Hoffman & Saw (1975), Rawlings (1976), among others. Gupta, Pillai and Steck (1964) considered the distributions of a linear function and ratio of two linear functions of order statistics from an equally correlated set of normal random variables. Later on, Gupta, Nagel & Panchapakesan (1973) have studied the distribution theory of the maxima which arises in the context of ranking and multiple comparison problems. More specifically, these authors have discussed some general distribution theory for certain order statistics from correlated normal random variables with a special correlation structure $\rho_{ij} = \rho \geq 0$, where ρ_{ij} is the correlation coefficient between Y_i and Y_j, for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.

The order statistics inferences for the equi-correlated normal random variables, was also studied by Owen and Steck (1962). In their study, they have shown how the marginal moments and product moments of the order statistics may be obtained from the corresponding moments and product moments for the independence case, $\rho = 0$. Rawlings (1976) studied the distribution of the maxima for such equi-correlated random variables. More specifically, Rawlings extended the distribution of the maxima of one group of k- dimensional equi-correlated variables studied by Gupta (1973) to the case of m independent groups of k-dimentional equicorrelated random variables.

Unlike the above authors, Hoffman and Saw (1975) attempted to include the negative equicorrelated case in finding the distribution function of the maxima that requires certain integrations in the complex domain, which may not be easy in general.

There has also been a few studies on the order statistics inferences, where correlations may be unequal. But, they were done for very special correlations structure. For example, by expressing a multivariate probability integral as a power series of the univariate probability integral, Greig (1967) has provided an approximation to the distribution of extreme values in corr lated normal population, for a very special situation when the correlation matrix has dominant elements adjacent to the leading diagonal with $\rho_{i,i+1} \neq 0$ but $\rho_{ij} = 0$ for all $i \neq j$ & $j \neq i+1$.

With regard to the detection of outliers in a simple regression set up, Ellenberg (1973, 1976) has used Bonferroni inequlities approach to compute the limits for the probability of maxima of standerized least squares residuals. Here in this problem, the residuals have unequal correlations based on the structure of the design matrix involved in the linear model.

Observe that all the studies mentioned above deal with either equicorrelated or special types of unequal correlated structures. But, as in reality, the normal variables can be unequally positively or negatively correlated, in this thesis, we deal with such general correlation structures and study the order statistics inferences for such cases. The specific plan of the thesis is as follows :

1.2 Objective of the Thesis

1. In chapter two, we provide detailed background of order statistics problems for correlated normal data.

2. Chapter 3 concerns the inferences for the order statistics obtained from unequal positively or negatively correlated normal variables. In Section 3.1, based on a small correlations approach, we develop the joint density function for the order statistics for correlated normal variables with general correlation structure. An approximate marginal distribution of a single order statistic is simplified in Section 3.2. In Section 3.3, we provide the distributions of maxima and minima as two special cases.

3. The computational aspects for the percentile points of the maxima for normal

variables with general correlation structures are given, in chapter 4, for three special situations. First, in Section 4.2, we discuss the computation of the percentile points of the maxima for homoscedastic equi-correlated (positive or negative) normal variables. As our results are obtained based on small correlations approach, we compare them with the existing results due to Gupta (1973) for positive equi-correlated cases. Our results appear to agree quite well with those in Gupta (1973) for small correlations. In Section 4.3, the percentile points of the maxima are computed for the homoscedastic but unequally correlated (positive or negative) normal variables case. This we have done in the context of auto-regressive process of order one, where variances (of the variables) are equal and correlations are unequal following a decaying pattern for increasing lags. Next, in Section 4.4, we discuss the computation of the percentile points of the maxima, for the heteroscedastic but unequally positively or negatively correlated normal variables case. Unlike the last case, we have done this in the context of antedependence (nonstationary) models. To assess the adequacy of our small correlations approach for the maxima in this case, a limited simulation study is also carried out.

Furthermore, we compare our approach with the well-known Bonferroni bounds approximation for both homoscedastic and heteroscedastic cases.

4. Capter 5 contains the summary of the present work and provides some suggested topics for further research.

Chapter 2

Background of Order Statistics Problems For Correlated Normal Data

In order to make inferences for order statistics from an equally correlated set of normal random variables. Owen and Steck (1962) showed that the moments and product moments of the order statistics for normal variables for any ρ, ρ being the equi correlation coefficient between any two variables, can be obtained from the corresponding moments and product moments of the order statistics for independent ($\rho = 0$) normal variables. Suppose that $X_1, X_2, \ldots, X_n$ are independently and normally distributed random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$, for all $i = 1, 2, \ldots, n$. Also suppose that X_0 is another standardized normal variable but with $E(X_0X_i) = 0$, for $\rho > 0$ and $E(X_0X_i) = -(-\rho)^{\frac{1}{2}}/(1-\rho)^{\frac{1}{2}}$, for $\rho < 0$. Let $Y_1, \ldots, Y_i, \ldots, Y_n$ be the n correlated random variables such that $E(Y_i) = 0$, $E(Y_i^2) = 1$ and $E(Y_iY_j) = \rho$, for all $i \neq j$. Now to obtain the marginal as well as product moments of these correlated random variables Y_i $(i = 1, 2, \ldots, n)$, Owen & Steck expressed Y_i $(i = 1, 2, \ldots, n)$ as a function of the standard normal random variables $X_0, X_1, \ldots, X_n$, given by

$$Y_{(i)} = \rho^{\frac{1}{2}} X_0 + (1-\rho)^{\frac{1}{2}} X_{(i)} \qquad (2.1)$$

where $X_{(i)}$ and $Y_{(i)}$ be, respectively, the ith order statistic of the sample $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$. Based on this transformation, they obtained the characteristic function of Y_i as well as the joint characteristic function of Y_i & Y_j, $i \neq j$, which were then exploited to compute

$$\begin{aligned}
E(Y_{(i)}) &= (1-\rho)^{\frac{1}{2}} E(X_{(i)}), \\
E(Y_{(i)}^2) &= \rho + (1-\rho) E(X_{(i)}^2), \\
E(Y_{(i)}^3) &= 3\rho(1-\rho)^{\frac{1}{2}} E(X_{(i)}) + (1-\rho)^{\frac{3}{2}} E(X_{(i)}^3), \\
E(Y_{(i)}^4) &= 3\rho^2 + 6\rho(1-\rho) E(X_{(i)}^2) + (1-\rho)^2 E(X_{(i)}^4), \\
\text{and} \qquad E(Y_{(i)}Y_{(j)}) &= \rho + (1-\rho) E(X_{(i)}X_{(j)}).
\end{aligned}$$

These authors provided the means, standard deviations, and the third and the fourth central moments of $Y_{(i)}$, in tabular form, for selected values of n and ρ.

Using the above transformation (2.1), Gupta, Pillai & Steck (1964) have derived the distributions of the linear function $Z = \sum_{i=1}^{n} a_i Y_{(i)}$. More specifically, they obtained the distribution function of Z in terms of the distribution function of $\sum_{i=1}^{n} a_i X_{(i)}$ given by

$$Pr\{Z \leq z\} = Pr\{\sum_{i=1}^{n} a_i X_{(i)} \leq -[\rho/(1-\rho)]^{\frac{1}{2}} (\sum_{i=1}^{n} a_i) X_0 + z/(1-\rho)^{\frac{1}{2}}\}. \qquad (2.2)$$

for $\rho \geq 0$, and

$$Pr\{Z \le z\} = Pr\{\sum_{i=1}^{n} a_i X_{(i)} \le -[-\rho/(1-\rho)]^{\frac{1}{2}}(\sum_{i=1}^{n} a_i)X_0 + z/(1-\rho)^{\frac{1}{2}}\}, \quad (2.3)$$

for $\rho < 0$.

There are, however, some practical situations where correlations may be different. For example, in familial analysis, it may be necessary to study the order pattern among n family members, where the variable under consideration for these family members may be unequally correlated. Let ρ_{ij}, for all $i \neq j$ be the correlation coefficient between ith & jth members of the family. For such a general situation with correlation structure $E(Y_iY_j) = \rho_{ij}$ $(i \neq j)$, Gupta et al. (1961) also obtained the distribution function of the range $W = maxY_i - minY_i$ of correlated normal random variables for $n = 3, 4$ based on the V-function described by Nicholson (1943). For the trivariate case, the distribution function for W was given as

$$\begin{aligned} Pr(W \le w) = & -2[V(\frac{w}{a_{12}}, \frac{w\theta_{12}}{a_{12}\sqrt{(1-\theta_{12}^2)}}) + V(\frac{w}{a_{12}}, \frac{w\theta_{13}}{a_{12}\sqrt{(1-\theta_{13}^2)}}) \\ & + V(\frac{w}{a_{13}}, \frac{w\theta_{13}}{a_{13}\sqrt{(1-\theta_{13}^2)}}) + V(\frac{w}{a_{13}}, \frac{w\theta_{23}}{a_{13}\sqrt{(1-\theta_{23}^2)}}) \\ & + V(\frac{w}{a_{23}}, \frac{w\theta_{12}}{a_{23}\sqrt{(1-\theta_{12}^2)}}) + V(\frac{w}{a_{23}}, \frac{w\theta_{23}}{a_{23}\sqrt{(1-\theta_{23}^2)}})] \end{aligned} \quad (2.4)$$

where $V(l, m)$ is the V-function described by Nicholson (1943), and

$$a_{ij} = \sqrt{2}(1-\rho_{ij})^{\frac{1}{2}}$$

$$\theta_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23}$$

$$\theta_{13} = -(1 + \rho_{23} - \rho_{12} - \rho_{13})/a_{12}a_{13}$$

and $$\theta_{23} = -(1 + \rho_{12} - \rho_{13} - \rho_{23})/a_{13}a_{23}.$$

Later on, Gupta, Nagel & Panchapakesan (1973) presented the cumulative distribution function of maxima, but for equal correlations $\rho \geq 0$. Based on the transformation $Y_{(i)} = \rho^{\frac{1}{2}} X_0 + (1-\rho)^{\frac{1}{2}} X_{(i)}$ as discussed above, they derived the distribution function of $Y_{(n)} = max(Y_i, 1 \leq i \leq n)$ as

$$F_n(H;\rho) \equiv Pr\{Y_{(n)} \leq H\} = \int_{-\infty}^{\infty} \Phi^n\{(y\rho^{\frac{1}{2}} + H)/(1-\rho)^{\frac{1}{2}}\}\phi(y)dy \tag{2.5}$$

where $\Phi(y)$ and $\phi(y)$ are, respectively, the cumulative distribution function and the density function of a standardized normal random variable Y. Further, they provided the percentage points of $Y_{(n)}$, namely, the values of H satisfying $F_n(H;\rho) = 1-\alpha$ for selected values of α and ρ in the form of tables. Note, however, that this approach does not permit one to compute the distribution function of $Y_{(n)}$ for $\rho < 0$ as well as for unequal ρ's.

In a n variate equi-correlated situation, Rawlings (1976) considered the probability integral for each of the $s \leq n$ variates to have magnitude less than h and the remaining $n-s$ variates to have their magnitudes more than h. Following Gupta (1963) and Curnow & Dunnett (1962), Rawlings (1976) computed the probability that Y_1 to Y_s fall below h and Y_{s+1} to Y_n fall above h, given by

$$\begin{aligned} L_n(h;s,n-s,\rho) &= \int_h^{\infty}\cdots\int_h^{\infty}\left[\int_{-\infty}^{h}\cdots\int_{-\infty}^{h}\phi_n(y_1,\ldots,y_n;\rho)dy_1,\ldots,dy_s\right] \\ &\qquad\qquad \cdot dy_{s+1}, dy_{s+2},\ldots,dy_n \\ &= \int_{-\infty}^{\infty}\phi(x)[\Phi(w)]^s[1-\Phi(w)]^{n-s}dx \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \Phi(w) &= \int_{-\infty}^{w}\phi(v)dv, \\ \text{with} \qquad w &= \frac{h+\rho^{\frac{1}{2}}x}{(1-\rho)^{\frac{1}{2}}} \end{aligned}$$

Suppose there are m independent groups or clusters with n equicorrelated normal variables in each. For example, in a familial data, there may be m independent full-sib families each with n sibs having equal correlation ρ. In such situations, Rawlings (1976) has then used the above probability $L_n(h: s, n-s, \rho)$ to compute the probability density function for the $(n_0 - i)th$ order statistic, $Y_{(n_0 - i:n_0,m,\rho)}$, given by

$$f(Y_{(n_0-i:n_0,m,\rho)}) = \sum_{j=1}^{2i} N_{(p,u_m,\{u_j\})} P_{(y;p,u_m,\{u_j\})} \tag{2.7}$$

where $n_0 = nm$, and

$$N_{(p,u_m,\{u_j\})} = \frac{\prod_{j=1}^{p} \binom{n}{u_j}}{B(p+1, m-p)\, B(u_m+1, n-u_m)}$$

$$P_{(y;p,u_m,\{u_j\})} = \left[L_n(y; n, 0, \rho)\right]^{m-p-1} \left[\prod_{j=1}^{p} L_n(y; n-u_j, u_j, \rho)\right] \times \left[L_{n-1}(w; n-u_m-1, u_m, \rho^*)\right]$$

with $B(.)$ as the usual beta function,

$$w = y\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}},$$

and

$$\rho^* = \rho/(1+\rho)$$

u_j be the number of variates in subset j which are greater than y and p is the number of subsets having at least one variable greater than y. In his paper, he tabulated the expectations of order statistics in correlated samples of m independent sets of k equicorrelated multinormal variates for selected $i+1$, n, m and ρ.

For handling negative equicorrelations, Hoffman and Saw (1975) provided a computationally feasible method for finding the cumulative distribution of the rth ranked of a set of equicorrelated normal variables. More specifically, they computed $P(Y_{(r)} \leq h)$, $r = 1, 2, \ldots, n$ based on Tchebychev-Hermite and Legendre polynomial. The probability was given by

$$P_{r,n}(h, \rho) = P(Y_{(r)} \leq h) \quad = \quad \sum_{k=0}^{m} d_k(r, n) J_k(h, \rho) \tag{2.8}$$

where

$$d_k(r, n) \quad = \quad \frac{n!(2k+1)!}{(k!)^2(n+k+1)!}\phi_k(r, n)$$

$$\text{and} \quad J_k(h, \rho) \quad = \quad \int_{-\infty}^{\infty} L_k[2P(\alpha + \theta u) - 1]z(u)du$$

with $\alpha = h/(1-\rho)^{\frac{1}{2}}$, $\theta = \rho^{\frac{1}{2}}$, $z(u)$ being the standard normal ordinate, $\phi_k(r, n)$ being the kth order Tchebychev-Hermite polynomial on $r \in \{0, 1, \ldots, m\}$ scaled, $L_k(v)$ being the kth order Legendre polynomial in v, and $P(\alpha + \theta u)$ being a suitable complex function where θ is complex. Note that the probability obtained as above by Hoffman & Saw (1975) is also valid for equal positive correlations. For $0 \leq k \leq 10$ and $r = m = 10$, they tabulated the values of $d_k(r, n)$ and $J_k(h, \rho)$ for selected h and ρ. Using these values, they obtained $P_{10,10}(0, -1/9) = 0.000000$; $P_{10,10}(0, 1/8) = 0.008939$; $P_{10,10}(2, 1/8) = 0.809761$; $P_{10,10}(1, 1/2) = 0.460560$; $P_{10,10}(2, 1/2) = 0.866909$.

For a very special case of general (equal or unequal) correlation structures, Greig (1967) developed an approximate formula for the moments of the smallest values in a correlated normal random variables with $E(Y_i Y_{i+1}) = \rho_{i,i+1} \neq 0$, and $E(Y_i Y_j) = 0$, for all

$i \neq j$ & $j \neq i+1$. Since the application of the normal multivariate integral to calculate the exact moments of the minima is cumbersome, Greig provided first an approximate expression for multivariate probability integral Φ_n in terms of Φ_1 as

$$\begin{aligned} \Phi_n(y, \ldots, y; \rho_{ij}) &= \int_y^\infty \cdots \int_y^\infty \phi_n(u_1, \ldots, u_n; \rho_{ij}) du_1 \ldots du_n \\ &\simeq \Phi_1(y) \prod_{i=1}^{n-1} [1 - (1 - \rho_{i,i+1})^{\frac{1}{2}} (1 - \Phi_1(y))] \end{aligned} \tag{2.9}$$

where $\phi_n(u_1, \ldots, u_n; \rho_{ij})$ is the multivariate normal density, and

$$\Phi_1(y) = \int_y^\infty \phi_1(u) du \tag{2.10}$$

with $\phi_1(u)$ as the density of a standard normal random variable. Utilizing this expression, the author, then, obtained an approximate result for the moments for the minima as

$$\mu_{ns} \simeq \prod_{i=1}^{n-1} [1 - (1 - \rho_{i,i+1})^{\frac{1}{2}} (1 - \mu'_s)] \tag{2.11}$$

where μ_{ns} is the sth moment of the smallest and $\mu'_s = n \int_{-\infty}^{\infty} y^s \phi_1(y) \{\Phi_1(y)\}^s dy$. The above approach taken by Greig (1967) does not appear to be realistic. This is because, in reality, the other off-diagonal elements may not be negligible, although they were neglected in this approach.

Note that order statistics inference is also essential in the linear regression analysis, mainly, for the detection of outliers or influential observations. For example, consider a simple linear regression model

$$Y = X\beta + \epsilon \tag{2.12}$$

where $Y = (Y_1, Y_2, \ldots, Y_n)$ is a $n \times 1$ response variable, X is a known design matrix of order $n \times k$, β is a $k \times 1$ vector of unknown parameters and ϵ is a $n \times 1$ error variable distributed as $\epsilon \sim N(0, \sigma^2 I_n)$, I_n being the $n \times n$ identity matrix. To test for the presence of a single outlier in a linear regression model, the maximum studentized residual test statistic defined as $R_n = max \mid e_i/s_i \mid$ is widely used. Here, $e_i = y_i - X_i^T\hat{\beta}$, X_i^T is the ith $(i = 1, 2, \ldots, n)$ row of the design matrix X, $\hat{\beta} = (X^TX)^{-1}X^Ty$ is the least square estimate of β , and s_i^2 is the ith diagonal element of $\hat{V}(e) = (I_n - V)\hat{\sigma}^2$, the estimated variance-covariance matrix of the residuals, with $V = X(X^TX)^{-1}X^T$ and $\hat{\sigma}^2 = e^Te/(n-k) = y^T(I_n - V)y/(n-k)$, where $e = (e_1, e_2, \ldots, e_n)^T$ is the $n \times 1$ residual vector. This R_n statistic is recommended for use in situations where the variances of the individual residuals are expected to vary a great deal among themselves. There is another statistic, namely, the maximum normed residual test statistic defined as $R_n^* = \{n/(n-k)\}^{\frac{1}{2}} max \mid e_i/\hat{\sigma} \mid$ which is also frequently used but for the situations when all residuals have a common variance. Note, however, that the exact critical values for these two statistics are not available.

Ellenberg (1973, 1976) has used the first Bonferroni bounds in approximating the critical values of the maximum studentized residual test statistic $R_n = max \mid \xi_i \mid$, where $\xi_i = e_i/s_i$ $(i = 1, 2, \ldots, n)$ is the standardized least squares residuals. These Bonferroni bounds may be simplified as

$$nPr(\mid \xi_i \mid > C_\alpha) - \sum_{i \leq j} Pr(\mid \xi_i \mid > C_\alpha, \mid \xi_j \mid > C_\alpha) \\ \leq Pr(max \mid \xi_i \mid > C_\alpha) \leq nPr(\mid \xi_i \mid > C_\alpha) \qquad (2.13)$$

with C_α as any critical constant. For the maximum normed residual test statistic R_n^*, Stefansky (1971, 1972) devoloped the bounds for the percentage points and it was shown how successive improvements could be made to the initial upper and lower bounds. It was also shown that in many situations the first upper (or lower) bound is either equal to or extremely close to the exact percentage point, the first upper bound for the $100(1-\alpha)$

percentage point of R_n^* is being given by $[(n-k)F/(n-k-1+F)]^{\frac{1}{2}}$, where F is the $100(1-\alpha/n)$ percentage point of the F distribution with 1 and $n-k-1$ degrees of freedom.

In small sample cases, these R_n and R_n^* tests are not equivalent. It has been shown in Sutradhar (1996) [see also Sutradhar & Chu (1995)] through a simulation experiment that the maximum normed residual test is more powerful than the maximum studentized residual test, irrespective of the situations whether outlier arises due to slippage of the mean or inflation of the variance of the data. Consequently, between the two tests, it was recommended in Sutradhar (1996) to use the maximum normed residual test statistic for detecting a single mean-shifted or variance-inflated outlier in the linear models with fixed designs. Note, however, that as discussed in Sutradhar and Chu (1995), the first upper (or lower) bound for the critical value of R_n^* found in Stefansky (1971, 1972) may be quite liberal in approximating the exact critical value of this statistic itself, especially when the residuals are heteroscadastic for certain choices of the design matrix.

It then follows from the above findings that in certain situations when variances of the random variables under consideration are unequal and when one is interested to find the p-value of a test statistic similar to R_n^*, the application of the first Bonferroni bounds may not be a good approximation to the exact p-value. Consequently, it seems quite appropriate to seek for alternative ways to calculate the critical value for the test statistics such as maxima, minima or general order statistics for normal correlated variables with unequal (or equal) variances. Motivated by this, we, in the present thesis, generalize the distributions of order statistics for the equi-correlated (as well as certain special case of unequally correlated) normal variables [cf. Gupta, Pillai & Steck (1964), Greig (1967), Gupta, Nagel & Panchapakesan (1973), Rawlings (1976), among others] to the case where heteroscadastic normal variables have positive or negative unequal correlations. We adopt a small correlations approach to achieve this goal.

Chapter 3

Order Statistics From Unequal Correlated Normal Variables : A Small Correlation Approach

3.1 Joint Density Function

Let $Y_1, Y_2, \ldots, Y_n$ be normally distributed with mean zero and variance covariance matrix Σ, where $\Sigma = D^{\frac{1}{2}} R D^{\frac{1}{2}}$

$$\text{with} \quad R = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \dots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \dots & \rho_{2n} \\ \rho_{31} & \rho_{32} & 1 & \dots & \rho_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \dots & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}$$

where ρ_{ij} in the R matrix is the correlation coefficients between Y_i & Y_j, and σ_i^2 in the diagonal matrix D is the variance of Y_i. Now we assume that ρ_{ij}'s are small in magnitude. This assumption about the small correlations is reasonable for many practical situations, for example, in clusterd regression problems, where the within cluster correlations are usually small. In such cases (for example in familial data), the sample size n is usually small too.

Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ be the order statistics of the random sample $Y_1, Y_2, \dots, Y_n$. The primary goal of this chapter is to derive the general marginal distribution of the rth order statistic, $Y_{(r)}$ $(r = 1, 2, \dots, n)$. In order to do this, we require the joint probability density function of the order statistics $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ which we will derive directly from the joint p.d.f. of the original variables $Y_1, Y_2, \dots, Y_n$ under the assumption that ρ_{ij}'s are small for all $i \neq j$. For the derivation, we, first, approximate the joint probability density function (p.d.f.) of the original random variables $Y_1, Y_2, \dots, Y_n$

$$f(y_1, y_2, \dots, y_n; \Sigma) = \frac{1}{[2\pi]^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}Y'\Sigma^{-1}Y} \tag{3.1}$$

where $Y = (y_1, y_2, \dots, y_n)'$ is a $n \times 1$ random vector, by its Taylor's series expansion about $\rho_{ij} = 0$. More specifically, for small ρ_{ij}'s, the Taylor's series expansion of the joint p.d.f. of the original variables, $f(y_1, y_2, \dots, y_n; \Sigma)$, when evaluated at $\rho_{ij} = 0$, is given by

$$
\begin{aligned}
f(y_1, y_2, \ldots, y_n; \Sigma) \simeq & \Big[U_n + \sum_{i<j}^{n} V_{ij} y_i y_j - \frac{1}{2}\sum_{i=1}^{n} V_i y_i^2 + \frac{1}{2}\sum_{i<j}^{n} W_{ij} y_i^2 y_j^2 \\
& + \sum_{i \neq j \neq k}^{n} W_{iijk} y_i^2 y_j y_k + \sum_{i \neq j \neq k \neq l}^{n} W^{*}_{ijkl} y_i y_j y_k y_l\Big] \\
& \times f(y_1, y_2, \ldots, y_n; D) \qquad (3.2)
\end{aligned}
$$

where

$$
\begin{aligned}
f(y_1, y_2, \ldots, y_n; D) &= \frac{1}{(2\pi)^{n/2} \mid D \mid^{1/2}} e^{-\frac{1}{2}Y' D^{-1} Y} \\
U_n &= 1 + \frac{1}{2}\sum_{i<j}^{n} \rho_{ij}^2 \\
V_{ij} &= \frac{1}{\sigma_i \sigma_j}\left(\rho_{ij} - \sum_{k \neq i \neq j}^{n} \rho_{ik}\rho_{jk}\right) \\
V_i &= \frac{1}{\sigma_i^2}\left(\sum_{j \neq i}^{n} \rho_{ij}^2\right) \\
W_{ij} &= \frac{\rho_{ij}^2}{\sigma_i^2 \sigma_j^2} \\
W_{iijk} &= \frac{\rho_{ij}\rho_{ik}}{\sigma_i^2 \sigma_j \sigma_k} \\
\text{and} \quad W^{*}_{ijkl} &= \frac{\rho_{ij}\rho_{kl} + \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk}}{\sigma_i \sigma_j \sigma_k \sigma_l}
\end{aligned}
$$

with $D = diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$.

Let $Y_{(1)} \leq \ldots \leq Y_{(n)}$ be the correlated order statistics of $Y_{(1)} \ldots Y_{(n)}$. Then, given the realizations of the order statistics to be $y_{(1)} \leq \ldots \leq y_{(n)}$, the original variables Y_i $(i = 1, 2, \ldots, n)$ are constrained to take on the values $y_{(i_k)}$ which yields the same expression for the similar terms in equation (3.2) for all $n!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. Consequently, we may obtain the joint probability density function $g^{*}(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; \Sigma)$ of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ as given in Theorem 3.1.

THEOREM 3.1. Let the approximate joint density function $f(y_1, y_2, \ldots, y_n)$ of $Y_1, Y_2, \ldots, Y_n$ be given by (3.2). Then the joint p.d.f. $g^*(y_{(1)}, y_{(2)}, \ldots, y_{(n)})$ of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$\begin{aligned} g^*\left(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; \Sigma\right) \simeq\ & \Big[U_n^* + D_n^* \sum_{i<j}^{n} y_{(i)}y_{(j)} - Q_n^* \sum_{i=1}^{n} y_{(i)}^2 + S_n^* \sum_{i<j}^{n} y_{(i)}^2 y_{(j)}^2 \\ & + T_n^* \sum_{i \neq j \neq k}^{n} y_{(i)}^2 y_{(j)} y_{(k)} + M_n^* \sum_{i \neq j \neq k \neq l}^{n} y_{(i)} y_{(j)} y_{(k)} y_{(l)}\Big] \\ & \times f\left(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; D\right) \end{aligned} \qquad (3.3)$$

where

$$\begin{aligned} U_n^* &= n! U_n \\ D_n^* &= 2(n-2)! \sum_{i<j}^{n} V_{ij} \\ Q_n^* &= \frac{1}{2}(n-1)! \sum_{i=1}^{n} V_i \\ S_n^* &= (n-2)! \sum_{i<j}^{n} W_{ij} \\ T_n^* &= 2(n-3)! \sum_{i \neq j \neq k}^{n} W_{iijk} \\ \text{and} \quad M_n^* &= 4!(n-4)! \sum_{i \neq j \neq k \neq l}^{n} W_{ijkl}^*, \end{aligned}$$

$U_n, V_{ij}, V_i, W_{ij}, W_{iijk}$ and W_{ijkl}^* being given by (3.2).

Derivation of Theorem 3.1 : For simplicity, we start with $n = 2$. It follows by (3.2) that for $n = 2$, the joint p.d.f. for the original variables Y_1, Y_2 is given by

$$\begin{aligned} f(y_1, y_2; \Sigma) \simeq\ & \left[U_2 + V_{12} y_1 y_2 + \frac{1}{2} W_{12} y_1^2 y_2^2 - \frac{1}{2} V_1 y_1^2 - \frac{1}{2} V_2 y_2^2\right] \\ & \times f(y_1, y_2; D) \end{aligned} \qquad (3.4)$$

where

$$
\begin{aligned}
f(y_1, y_2; D) &= \frac{1}{(2\pi)\mid D\mid^{1/2}} e^{-\frac{1}{2}y' D^{-1} y} \\
U_2 &= 1 + \frac{\rho_{12}^2}{2} \\
V_{12} &= \frac{\rho_{12}}{\sigma_1\sigma_2} \\
V_1 &= \frac{\rho_{12}^2}{\sigma_1^2} \\
V_2 &= \frac{\rho_{12}^2}{\sigma_2^2}, \\
\text{and} \qquad W_{12} &= \frac{\rho_{12}^2}{\sigma_1^2\sigma_2^2}.
\end{aligned}
$$

Now, consider the set A which is the union of the two mutually disjoint sets A_1 $\{(y_1, y_2): y_1 < y_2\}$ and $A_2 = \{(y_1, y_2): y_2 < y_1\}$. There are two of these sets because we can arrange y_1, y_2 in precisely $2! = 2$ ways. Let B be the set of the order statistics defined as $B = \{(y_{(1)}, y_{(2)}): y_{(1)} < y_{(2)}\}$. It follows that there exists one-to-one transfomations that map each of A_1, A_2 onto the set B. Inversely, in set B, $y_{(1)} = y_1, y_{(2)} = y_2$ for the points in A_1, and $y_{(1)} = y_2, y_{(2)} = y_1$ for the points in A_2. The absolute value of the jacobian of the undertaking transformation for each set of A is 1. Thus the joint p.d.f. of order statistics $Y_{(1)}, Y_{(2)}$ may be written as

$$
\begin{aligned}
g_2^* \left(y_{(1)}, y_{(2)}; \Sigma\right) &\simeq \left[U_2 + V_{12}y_{(1)}y_{(2)} + \frac{1}{2}W_{12}y_{(1)}^2y_{(2)}^2 - \frac{1}{2}V_1y_{(1)}^2 - \frac{1}{2}V_2y_{(2)}^2\right] \\
&\quad \times f\left(y_{(1)}, y_{(2)}; D\right) + [U_2 + V_{12}y_{(2)}y_{(1)} + \frac{1}{2}W_{12}y_{(2)}^2y_{(1)}^2 \\
&\quad -\frac{1}{2}V_1y_{(2)}^2 - \frac{1}{2}V_2y_{(1)}^2]f\left(y_{(2)}, y_{(1)}; D\right) \\
&= [2!U_2 + 2(2-2)!V_{12}y_{(1)}y_{(2)} - \frac{(2-1)!}{2}\left\{\sum_{i=2}^{2} V_i\right\}\sum_{i=2}^{2} y_{(i)} \\
&\quad +(2-2)!W_{12}y_{(1)}^2y_{(2)}^2]f\left(y_{(1)}, y_{(2)}; D\right)
\end{aligned}
$$

$$= \left[U_2^* + D_2^* \sum_{i<j}^{2} y_{(i)} y_{(j)} - Q_2^* \sum_{i=1}^{2} y_{(i)} + S_2^* \sum_{i<j}^{2} y_{(i)}^2 y_{(j)}^2\right] \times f\left(y_{(1)}, y_{(2)}; D\right) \tag{3.5}$$

where

$$\begin{aligned}
U_2^* &= 2! U_2 \\
D_2^* &= 2(2-2)! V_{12} = 2(2-2)! \sum_{i<j}^{2} V_{ij} \\
Q_2^* &= \frac{(2-1)!}{2} \sum_{i=1}^{2} V_i, \\
\text{and} \quad S_2^* &= (2-2)! W_{12} = (2-2)! \sum_{i<j}^{2} W_{ij}
\end{aligned}$$

Similarly, for $n = 3$, the joint p.d.f. of Y_1, Y_2, Y_3 obtained from equation (3.2) is given by

$$\begin{aligned}
f(y_1, y_2, y_3; \Sigma) \simeq \; & [U_3 + \sum_{i<j}^{3} V_{ij} y_i y_j - \frac{1}{2} \sum_{i=1}^{3} V_i y_i^2 + \frac{1}{2} \sum_{i<j}^{3} W_{ij} y_i^2 y_j^2 \\
& + \sum_{i \neq j \neq k}^{3} W_{iijk} y_i^2 y_j y_k] f(y_1, y_2, y_3; D)
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
f(y_1, y_2, y_3; D) &= \frac{1}{(2\pi)^{3/2} \mid D \mid^{1/2}} e^{-\frac{1}{2} Y' D^{-1} Y} \\
U_3 &= 1 + \frac{1}{2} \sum_{i<j}^{3} \rho_{ij}^2 \\
V_{ij} &= \frac{1}{\sigma_i \sigma_j} \left(\rho_{ij} - \sum_{k \neq i \neq j}^{3} \rho_{ik} \rho_{jk} \right) \\
V_i &= \frac{1}{\sigma_i^2} \left(\sum_{j \neq i}^{3} \rho_{ij}^2 \right)
\end{aligned}$$

$$W_{ij} = \frac{\rho_{ij}^2}{\sigma_i^2\sigma_j^2},$$

and
$$W_{ijk} = \frac{\rho_{ij}\rho_{ik}}{\sigma_i^2\sigma_j\sigma_k}$$

In this case, the realizations y_1, y_2, y_3 of the original random variables are rearranged in ascending order of magnitude in $3! = 6$ ways. Now, let E and F denote two sets, one for the original variables and one for order statistics respectively, where the set E is the union of the six mutually disjoint sets as $E_1 = \{(y_1, y_2, y_3); y_1 < y_2 < y_3\}$, $E_2 = \{(y_1, y_2, y_3); y_2 < y_1 < y_3\}$, $E_3 = \{(y_1, y_2, y_3); y_1 < y_3 < y_2\}$, $E_4 = \{(y_1, y_2, y_3); y_3 < y_1 < y_2\}$, $E_5 = \{(y_1, y_2, y_3); y_2 < y_3 < y_1\}$ & $E_6 = \{(y_1, y_2, y_3); y_3 < y_2 < y_1\}$, and $F = \{(y_{(1)}, y_{(2)}, y_{(3)}); y_{(1)} < y_{(2)} < y_{(3)}\}$. Similar to the case $n = 2$, we can make an one-to-one transformation from each disjoint set of E onto the set F and $|J| = 1$ for each set of E. For simplicity, we, first, compute the ordered function for each term in equation (3.6) considering the above transformations for each disjoint set of E. So, for $U_3 f(y_1, y_2, y_3; D)$ in equation (3.6), we obtain the ordered function as

$$\begin{aligned} m(y_{(1)}, y_{(2)}, y_{(3)}; D) &= U_3[f(y_{(1)}, y_{(2)}, y_{(3)}; D) + f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\ &\quad + f(y_{(1)}, y_{(3)}, y_{(2)}; D) + f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\ &\quad + f(y_{(3)}, y_{(1)}, y_{(2)}; D) + f(y_{(3)}, y_{(2)}, y_{(1)}; D)] \\ &= 3! U_3 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \end{aligned} \tag{3.7}$$

The ordered functions for the first term of $\sum_{i<j}^{3} V_{ij} y_i y_j f(y_1, y_2, y_3; D) = [V_{12}y_1y_2 + V_{13}y_1y_3 + V_{23}y_2y_3] f(y_1, y_2, y_3; D)$ in (3.6) is given by

$$\begin{aligned} h_1(y_{(1)}, y_{(2)}, y_{(3)}; D) &= V_{12}[y_{(1)}y_{(2)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) + y_{(2)}y_{(1)} f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\ &\quad + y_{(1)}y_{(3)} f(y_{(1)}, y_{(3)}, y_{(2)}; D) + y_{(2)}y_{(3)} f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\ &\quad + y_{(3)}y_{(1)} f(y_{(3)}, y_{(1)}, y_{(2)}; D) + y_{(3)}y_{(2)} f(y_{(3)}, y_{(2)}, y_{(1)}; D)] \end{aligned}$$

$$= \quad 2V_{12} \sum_{i<j}^{3} y_{(i)} y_{(j)} f(y_{(1)}, y_{(2)}, y_{(3)}; D)$$

Similarly, we may derive the second and third terms as follows

$$\begin{aligned} h_2(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_{13} \sum_{i<j}^{3} y_{(i)} y_{(j)} f(y_{(1)}, y_{(2)}, y_{(3)}; D), \\ \text{and} \quad h_3(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_{23} \sum_{i<j}^{3} y_{(i)} y_{(j)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \end{aligned}$$

Combining the last three expressions, we obtain the ordered function of $\sum_{i<j}^{3} V_{ij} y_i y_j \times f(y_1, y_2, y_3; D)$ as

$$\begin{aligned} h(y_{(1)}, y_{(2)}, y_{(3)}; D) &= h_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + h_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \\ &\quad + h_3(y_{(1)}, y_{(2)}, y_{(3)}) \\ &= 2\left\{\sum_{i<j}^{3} V_{ij}\right\} \sum_{i<j}^{3} y_{(i)} y_{(j)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \qquad (3.8) \end{aligned}$$

Similarly, we obtain the ordered function for each term of $\sum_{i<j}^{3} W_{ij} y_i^2 y_j^2 f(y_1, y_2, y_3; D) = [W_{12} y_1^2 y_2^2 + W_{13} y_1^2 y_3^2 + W_{23} y_2^2 y_3^2] f(y_1, y_2, y_3; D)$ in (3.6) as

$$\begin{aligned} i_1(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2W_{12} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\ i_2(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2W_{13} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\ \text{and} \quad i_3(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2W_{23} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \end{aligned}$$

respectively, yielding the ordered function of $\sum_{i<j}^{3} W_{ij} y_i^2 y_j^2 f(y_1, y_2, y_3; D)$ as

$$\begin{aligned} i(y_{(1)}, y_{(2)}, y_{(3)}; D) &= i_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + i_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \\ &\quad + i_3(y_{(1)}, y_{(2)}, y_{(3)}; D) \\ &= 2\left\{\sum_{i<j}^{3} W_{ij}\right\} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \qquad (3.9) \end{aligned}$$

In the manner similar to the derivation of (3.8) & (3.9), the ordered functions for each term of $\sum_{i<j}^{3} V_i y_i^2 f(y_1, y_2, y_3; D) = [V_1 y_1^2 + V_2 y_2^2 + V_3 y_3^2] f(y_1, y_2, y_3; D)$ in (3.6) are obtained, respectively, as

$$\begin{aligned}
j_1(y_{(1)}, y_{(2)}, y_{(3)}; D) &= V_1[y_{(1)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) + y_{(2)}^2 f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\
&\quad + y_{(1)}^2 f(y_{(1)}, y_{(3)}, y_{(2)}; D) + y_{(2)}^2 f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\
&\quad + y_{(3)}^2 f(y_{(3)}, y_{(1)}, y_{(2)}; D) + y_{(3)}^2 f(y_{(3)}, y_{(2)}, y_{(1)}; D)] \\
&= 2V_1 \sum_{i<j}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
j_2(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_2 \sum_{i<j}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D), \\
\text{and} \quad j_3(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_3 \sum_{i<j}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D),
\end{aligned}$$

which are exploited to compute the combined ordered function of $\sum_{i<j}^{3} V_i y_i^2 f(y_1, y_2, y_3; D)$ as

$$\begin{aligned}
j(y_{(1)}, y_{(2)}, y_{(3)}; D) &= j_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + j_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
&\quad + j_3(y_{(1)}, y_{(2)}, y_{(3)}) \\
&= 2\left\{\sum_{i=1}^{3} V_i\right\} \sum_{i=1}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \qquad (3.10)
\end{aligned}$$

In the similar way, we also obtain the ordered function for each term of $\sum_{i<j}^{3} W_{iijk} y_i^2 y_j y_k f(y_1, y_2, y_3; D) = [W_{1123} y_1^2 y_2 y_3 + W_{1223} y_1 y_2^2 y_3 + W_{1233} y_1 y_2 y_3^2] f(y_1, y_2, y_3; D)$ in (3.6), respectively, as

$$k_1(y_{(1)}, y_{(2)}, y_{(3)}; D) = W_{1123}[y_{(1)}^2 y_{(2)} y_{(3)} f(y_{(1)}, y_{(2)}, y_{(3)}; D)$$

$$
\begin{aligned}
& +y_{(2)}^2 y_{(1)} y_3 f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\
& +y_{(1)}^2 y_{(3)} y_{(2)} f(y_{(1)}, y_{(3)}, y_{(2)}; D) \\
& +y_{(2)}^2 y_{(3)} y_{(1)} f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\
& +y_{(3)}^2 y_{(1)} y_{(2)} f(y_{(3)}, y_{(1)}, y_{(2)}; D) \\
& +y_{(3)}^2 y_{(2)} y_{(1)} f(y_{(3)}, y_{(2)}, y_{(1)}; D)] \\
= {} & 2W_{1123} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)} y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D),
\end{aligned}
$$

$$
k_2(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{1223} \sum_{<j}^{3} y_{(i)}^2 y_{(j)} y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D),
$$

$$
\text{and} \quad k_3(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{1233} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)} y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D)
$$

By using the above expressions, we obtain the ordered function of $\sum_{i<j}^{3} W_{ii,jk} y_i^2 y_j y_k \times f(y_1, y_2, y_3; D)$ as

$$
\begin{aligned}
k(y_{(1)}, y_{(2)}, y_{(3)}; D) &= k_1(y_{(1)}, k_{(2)}, y_{(3)}; D) + k_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
&\quad + k_3(y_{(1)}, y_{(2)}, y_{(3)}) \\
&= 2\left\{\sum_{i<j}^{3} W_{iijk}\right\} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)} y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \qquad (3.11)
\end{aligned}
$$

Next, combining the results of equations (3.7)-(3.11) for the five terms in equation (3.6), we obtain the joint p.d.f. of $Y_{(1)}, Y_{(2)}, Y_{(3)}$ as

$$
\begin{aligned}
g_3^*(y_{(1)}, y_{(2)}, y_{(3)}; \Sigma) &\simeq [6U_3 + 2\left\{\sum_{i<j}^{3} V_{ij}\right\} \sum_{i<j}^{3} y_{(i)} y_{(j)} - \frac{1}{2} 2\left\{\sum_{i=1}^{3} V_i\right\} \sum_{i=1}^{3} y_{(i)}^2 \\
&\quad + \frac{1}{2} 2\left\{\sum_{i<j}^{3} W_{ij}\right\} \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2 + 2\left\{\sum_{i\neq j\neq k}^{3} W_{iijk}\right\} \\
&\quad \times \sum_{i\neq j\neq k}^{3} y_{(i)}^2 y_{(j)} y_{(k)}] f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
&= [U_3^* + D_3^* \sum_{i<j}^{3} y_{(i)} y_{(j)} - Q_3^* \sum_{i=1}^{3} y_{(i)}^2 + S_3^* \sum_{i<j}^{3} y_{(i)}^2 y_{(j)}^2
\end{aligned}
$$

$$+ T_3^* \sum_{i \neq j \neq k}^{3} y_{(i)}^2 y_{(j)} y_{(k)}] f(y_{(1)}, y_{(2)}, y_{(3)}; D) \qquad (3.12)$$

where

$$U_3^* = 3! U_3$$

$$D_3^* = 2\sum_{i<j}^{3} V_{ij} = 2(3-2)! \sum_{i<j}^{3} V_{ij}$$

$$Q_3^* = \frac{1}{2} 2 \sum_{i=1}^{3} V_i = \frac{1}{2}(3-1)! \sum_{i=1}^{3} V_i$$

$$S_3^* = \frac{1}{2} 2 \sum_{i<j}^{3} W_{ij} = (3-2)! \sum_{i<j}^{3} W_{ij},$$

and

$$T_3^* = 2 \sum_{i \neq j \neq k}^{3} W_{iijk} = 2(3-3)! \sum_{i \neq j \neq k}^{3} W_{iijk}$$

As done in the last case, we now start with $n = 1$. For $n = 1$, the joint p.d.f. of the original variables Y_1, Y_2, Y_3, Y_4 by (3.2) is

$$f(y_1, y_2, y_3, y_4; \Sigma) \simeq [U_4 + \sum_{i<j}^{4} V_{ij} y_i y_j - \frac{1}{2} \sum_{i=1}^{4} V_i y_i^2 + \frac{1}{2} \sum_{i<j}^{4} W_{ij} y_i^2 y_j^2$$
$$+ \sum_{i \neq j \neq k}^{4} W_{iijk} y_i^2 y_j y_k + W_{1234}^* y_1 y_2 y_3 y_4]$$
$$\times f(y_1, y_2, y_3, y_4; D) \qquad (3.13)$$

where

$$f(y_1, y_2, y_3, y_4; D) = \frac{1}{(2\pi)^2 \mid D \mid^{1/2}} e^{-\frac{1}{2} Y' D^{-1} Y}$$

$$U_4 = 1 + \frac{1}{2} \sum_{i<j}^{4} \rho_{ij}^2$$

$$V_{ij} = \frac{1}{\sigma_i \sigma_j} \left(\rho_{ij} - \sum_{k \neq i \neq j}^{4} \rho_{ik} \rho_{jk} \right)$$

$$V_i = \frac{1}{\sigma_i^2} \left(\sum_{j \neq i}^{4} \rho_{ij}^2 \right)$$

$$W_{ij} = \frac{\rho_{ij}^2}{\sigma_i^2\sigma_j^2}$$
$$W_{iijk} = \frac{\rho_{ij}\rho_{ik}}{\sigma_i^2\sigma_j\sigma_k}$$
and
$$W_{1234}^* = \frac{\rho_{12}\rho_{34}+\rho_{13}\rho_{24}+\rho_{14}\rho_{23}}{\sigma_1\sigma_2\sigma_3\sigma_4}$$

Now, similar to the cases for $n = 2, 3$, we consider Y_1, Y_2, Y_3, Y_4 and $Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)}$ as the sets of original and ordered variables respectively. Here the set of original variables is the union of the 24 mutually disjoint sets because of $4! = 24$ permutations of $(1, 2, 3, 4)$. By making one-to-one transformation from each disjoint set of original variables to the set of order statistics, we can obtain the ordered function for each term of (3.13) in the way similar to that for the case $n = 3$. After a straightforward algebra, the ordered functions for $U_4 f(y_1, y_2, y_3, y_4; D)$, $\sum_{i<j}^4 V_{ij} y_i y_j f(y_1, y_2, y_3, y_4; D)$, $\sum_{i=1}^4 V_i y_i^2 f(y_1, y_2, y_3, y_4; D)$, $\sum_{i<j}^4 W_{ij} y_i^2 y_j^2 f(y_1, y_2, y_3, y_4; D)$ and $\sum_{i\neq j\neq k}^4 W_{iijk} y_i^2 y_j y_k f(y_1, y_2, y_3, y_4; D)$ are, respectively, simplified as

$$m^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4! U_4 f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \tag{3.14}$$
$$h^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4\left\{\sum_{i<j}^4 V_{ij}\right\}\sum_{i<j}^4 y_{(i)}y_{(j)} \times f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \tag{3.15}$$
$$j^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 6\left\{\sum_{i=1}^4 V_i\right\}\sum_{i=1}^4 y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \tag{3.16}$$
$$i^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4\left\{\sum_{i<j}^4 W_{ij}\right\}\sum_{i<j}^4 y_{(i)}^2 y_{(j)}^2 \times f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \tag{3.17}$$
and
$$k^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 2\left\{\sum_{i\neq j\neq k}^4 W_{iijk}\right\}\sum_{i\neq j\neq k}^4 y_{(i)}^2 y_{(j)} y_{(k)} \times f(y_{(1)}, y_{(2)}, y_{(3)} y_{(4)}; D) \tag{3.18}$$

For the last term $W_{1234}^* y_1 y_2 y_3 y_4 f(y_1, y_2, y_3 y_4; D)$ in equation (3.13), we get the same product function of the set of order statistics $Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)}$ for each disjoint set of original variables Y_1, Y_2, Y_3, Y_4 which are exploited to compute the ordered function of this last term as

$$v^*(y_{(1)}, y_{(2)}, y_{(3)} y_{(4)}; D) = 24 y_{(1)} y_{(2)} y_{(3)} y_{(4)} f(y_{(1)}, y_{(2)}, y_{(3)} y_{(4)}; D) \qquad (3.19)$$

Based on the equations (3.14)-(3.19) for the six terms in equation (3.13), we then obtain the joint p.d.f. of $Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)}$ given by

$$\begin{aligned} g_4^*\left(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; \Sigma\right) &\simeq [24U_4 + 4\left\{\sum_{i<j}^{4} V_{ij}\right\} \sum_{i<j}^{4} y_{(i)} y_{(j)} - \frac{1}{2} 6 \left\{\sum_{i=1}^{4} V_i\right\} \sum_{i=1}^{4} y_{(i)}^2 \\ &\quad + \frac{1}{2} 4 \left\{\sum_{i<j}^{4} W_{ij}\right\} \sum_{i<j}^{4} y_{(i)}^2 y_{(j)}^2 + 2\left\{\sum_{i\neq j\neq k}^{4} W_{ijk}\right\} \\ &\quad \times \sum_{i\neq j\neq k}^{4} y_{(i)}^2 y_{(j)} y_{(k)} + 24 W_{1234}^* y_{(1)} y_{(2)} y_{(3)} y_{(4)}] \\ &\quad \times f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \\ &= [U_4^* + D_4^* \sum_{i<j}^{4} y_{(i)} y_{(j)} - Q_4^* \sum_{i=1}^{4} y_{(i)}^2 + S_4^* \sum_{i<j}^{n} y_{(i)}^2 y_{(j)}^2 \\ &\quad + T_4^* \sum_{i\neq j\neq k}^{4} y_{(i)}^2 y_{(j)} y_{(k)} + M_4^* \sum_{i\neq j\neq k\neq l}^{4} y_{(i)} y_{(j)} y_{(k)} y_{(l)}] \\ &\quad \times f\left(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D\right) \qquad (3.20) \end{aligned}$$

where

$$\begin{aligned} U_4^* &= 24U_4 = 4! U_4 \\ D_4^* &= 4\sum_{i<j}^{4} V_{ij} = 2(4-2)! \sum_{i<j}^{4} V_{ij} \\ Q_4^* &= \frac{1}{2} 6 \sum_{i=1}^{4} V_i = \frac{1}{2}(4-1)! \sum_{i=1}^{4} V_i \\ S_4^* &= \frac{1}{2} 4 \sum_{i<j}^{4} W_{ij} = (4-2)! \sum_{i<j}^{4} W_{ij} \end{aligned}$$

$$T_4^* = 2\sum_{i\neq j\neq k}^{4} W_{iijk} = 2(4-3)!\sum_{i\neq j\neq k}^{4} W_{iijk},$$

$$\text{and} \quad M_4^* = 24W_{1234}^* = 4!(4-4)!\sum_{i\neq j\neq k\neq l}^{4} W_{ijkl}^*$$

Finally, following the patterns for the joint probability density functions in the equations (3.5), (3.12) & (3.20) for $n = 2,3$ & 4 respectively, one may easily obtain, in general, the joint p.d.f. of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ as given in Theorem 3.1.

3.2 Approximation to the Distribution of a Single Order Statistic

We now turn to the distribution theory of a single order statistic under the assumption that ρ_{ij}'s are small. An approximation to the distribution of $Y_{(r)}$ $(1 \le r \le n)$, the rth order statistic, is provided in Theorem 3.2.

THEOREM 3.2. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the order statistics with joint p.d.f. as given in Theorem 3.1. Then the marginal density function of the rth order statistic, $Y_{(r)}$ is given by

$$\begin{aligned} g_r^*(y_{(r)}) \simeq\ & U_n^*\phi(y_{(r)}) + D_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{1100}(y_{(r)}) - Q_n^*\sum_{i=1}^{n}\lambda_{i(jkl)}^{2000}(y_{(r)}) + S_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{2200}(y_{(r)}) \\ & + T_n^*\sum_{i\neq j\neq k}^{n}\lambda_{ijk(l)}^{2110}(y_{(r)}) + M_n^*\sum_{i\neq j\neq k\neq l}^{n}\lambda_{ijkl}^{1111}(y_{(r)}), \quad -\infty \le y_{(r)} \le \infty \end{aligned} \tag{3.21}$$

where $U_n^*, D_n^*, Q_n^*, S_n^*, T_n^*$ & M_n^* are defined as in Theorem 3.1. Further in (3.21),

$$\phi(y_{(r)}) = \frac{1}{(r-1)!(n-r)!}[F(y_{(r)})]^{r-1}[1-F(y_{(r)})]^{n-r}f(y_{(r)})$$

$$\text{with} \quad F(y_{(r)}) = \int_{-\infty}^{y_{(r)}} f(x)dx,\ f(x) \text{ being the p.d.f. of normal variable.}$$

and for example, for $i < r$ & $j, k > r$

$$\lambda^{t_i t_j 00}_{ij(kl)}(y_{(r)}) = f(y_{(r)})\lambda^{t_i 000U}_{i(jkl)}(y_{(r)})\lambda^{t_j 000}_{j(ikl)L}(y_{(r)}) , \quad t_i, t_j = 1,2$$

$$\lambda^{t_i t_j t_k 0}_{ijk(l)}(y_{(r)}) = f(y_{(r)})\lambda^{t_i 000U}_{i(jkl)}(y_{(r)})\lambda^{t_j t_k 00}_{jk(il)L}(y_{(r)}) , \quad t_i, t_j, t_k = 1,2$$

and for $i < r$, $j = r$ & $k > r$

$$\lambda^{t_i t_j 00}_{ij(kl)}(y_{(r)}) = y^{t_j}_{(r)} f(y_{(r)})\lambda^{t_i 000U}_{i(jkl)}(y_{(r)})\lambda^{0000}_{(ijkl)L}(y_{(r)}) , \quad t_i, t_j = 1,2$$

$$\lambda^{t_i t_j t_k 0}_{ijk(l)}(y_{(r)}) = y^{t_j}_{(r)} f(y_{(r)})\lambda^{t_i 000U}_{i(jkl)}(y_{(r)})\lambda^{t_k 000}_{k(ijl)L}(y_{(r)}) , \quad t_i, t_j, t_k = 1,2$$

with

$$\lambda^{t_i t_j 00U}_{ij(kl)}(y_{(r)}) = \int^{2,r} \prod_{a=1(1)\overline{r-1}} y^{t_a}_{(a)} f(y_{[1,r-1]})\,dy_{[1,r-1]}, \quad t_a = 1,2 \text{ for } a = i,j \qquad (3.22)$$

$$\& \quad t_a = 0 \quad \text{for } a \neq i,j$$

$$\lambda^{t_i t_j t_k 0}_{ijk(l)L}(y_{(r)}) = \int_{r,n-1} \prod_{a=n(-1)\overline{r+1}} y^{t_a}_{(a)} f(y_{[r+1,n]})\,dy_{[r+1,n]}, \quad t_a = 1,2 \text{ for } a = i,j,k \qquad (3.23)$$

$$\& \quad t_a = 0 \quad \text{for } a \neq i,j,k$$

where $\int^{2,r}$ and $\int_{r,n-1}$ represent the multiple integrals $\int_{-\infty}^{y_{(r)}} \cdots \int_{-\infty}^{y_{(2)}}$ and $\int_{y_{(r)}}^{\infty} \cdots \int_{y_{(n-1)}}^{\infty}$ respectively and

$$f(y_{[1,r-1]}) \equiv f(y_{(1)})f(y_{(2)})\cdots f(y_{(r-1)})$$

$$f(y_{[r+1,n]}) \equiv f(y_{(n)})f(y_{(n-1)})\cdots f(y_{(r+1)})$$

$$dy_{[1,r-1]} \equiv dy_{(1)}dy_{(2)}\cdots dy_{(r-1)}$$

$$dy_{[r+1,n]} \equiv dy_{(n)}dy_{(n-1)}\cdots dy_{(r+1)}$$

Derivation of Theorem 3.2 : We start with the approximate joint probability density function of all n order statistics in (3.3) and then integrate out the variables

$Y_{(1)},\ldots,Y_{(r-1)}, Y_{(r+1)},\ldots,Y_{(n)}$ in order to derive the marginal density function of the rth order statistic, $Y_{(r)}$ $(1\le r\le n)$ as

$$\begin{aligned} g_r^*(y_{(r)}) &\simeq \int^{2,r}\int_{r,n-1} g^*(y_{(1)},\ldots,y_{(n)};\Sigma)dy_{[r+1,n]}dy_{[1,r-1]} \\ &= \int^{2,r}\int_{r,n-1}[U_n^* + D_n^*\sum_{i<j}^{n} y_{(i)}y_{(j)} - Q_n^*\sum_{i=1}^{n} y_{(i)}^2 + S_n^*\sum_{i<j}^{n} y_{(i)}^2y_{(j)}^2 \\ &\quad +T_n^*\sum_{i\neq j\neq k}^{n} y_{(i)}^2y_{(j)}y_{(k)} + M_n^*\sum_{i\neq j\neq k\neq l}^{n} y_{(i)}y_{(j)}y_{(k)}y_{(l)}] \\ &\quad \times f(y_{(1)},y_{(2)},\ldots,y_{(n)};D)dy_{[r+1,n]}dy_{[1,r-1]} \end{aligned} \tag{3.24}$$

Now, by computing the integration for each term in equation (3.24), we obtain the probability density function as in the theorem. The steps for the integrations are given below. For the first term, we simplify the integral as

$$\begin{aligned} &\int^{2,r}\int_{r,n-1} f(y_{(1)},\ldots,y_{(n)};D)dy_{[r+1,n]}dy_{[1,r-1]} \\ &= \left\{\int^{2,r} f(y_{[1,r-1]})dy_{[1,r-1]}\right\}\left\{\int_{r,n-1} f(y_{[r+1,n]})dy_{[r+1,n]}\right\} f(y_{(r)}) \\ &= \frac{1}{(r-1)!}[F(y_{(r)})]^{r-1}\frac{1}{(n-r)!}[1-F(y_{(r)})]^{n-r}f(y_{(r)}) \end{aligned} \tag{3.25}$$

which is $\phi(y_{(r)})$, as defined in (3.21).

Next, for $i<r$ and $j>r$, the integral in the second term may be expressed as

$$\begin{aligned} &\int^{2,r}\int_{r,n-1} y_{(i)}y_{(j)}f(y_{(1)},\ldots,y_{(n)};D)dy_{[r+1,n]}dy_{[1,r-1]} \\ &= f(y_{(r)})\left\{\int^{2,r} y_{(i)}f(y_{[1,r-1]})dy_{[1,r-1]}\right\}\left\{\int_{r,n-1} f(y_{[r+1,n]})dy_{[r+1,n]}\right\} \end{aligned} \tag{3.26}$$

Now, by using $t_i=1$, $t_j=1$ and $t_a=0$ for $a\neq i,j$, the integral in (3.26) may further be expressed as

$$\int^{2,r}\int_{r,n-1} y_{(i)}y_{(j)}f(y_{(1)},\ldots,y_{(n)};D)dy_{[r+1,n]}dy_{[1,r-1]}$$

$$
\begin{aligned}
&= f(y_{(r)}) \left\{ \int^{2,r} \prod_{a=1(1)\overline{r-1}} y_{(a)}^{l_a} f(y_{[1,r-1]}) dy_{[1,r-1]} \right\} \\
&\qquad \times \left\{ \int_{r,n-1} \prod_{a=n(-1)\overline{r+1}} y_{(a)}^{l_a} f(y_{[r+1,n]}) dy_{[r+1,n]} \right\} \\
&= f(y_{(r)}) \lambda^{1000l'}_{i(jkl)}(y_{(r)}) \lambda^{1000}_{j(ikl)L}(y_{(r)})
\end{aligned}
$$

which is $\lambda^{1100}_{i,j(kl)}(y_{(r)})$.

Similarly, for $i, j < r$, we obtain

$$
\begin{aligned}
\int^{2,r} \int_{r,n-1} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D) dy_{[r+1,n]} dy_{[1,r-1]} &= f(y_{(r)}) \lambda^{1100l'}_{i,j(kl)}(y_{(r)}) \lambda^{0000}_{(i,jkl)L}(y_{(r)}) \\
&= \lambda^{1100}_{i,j(kl)}(y_{(r)})
\end{aligned}
$$

and for $i < r$ and $j = r$, we obtain

$$
\begin{aligned}
\int^{2,r} \int_{r,n-1} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D) dy_{[r+1,n]} dy_{[1,r-1]} &= y^{1}_{(r)} f(y_{(r)}) \lambda^{1000l'}_{i(jkl)}(y_{(r)}) \\
&\quad \cdot \lambda^{0000}_{(ijkl)L}(y_{(r)}) \\
&= \lambda^{1100}_{i,j(kl)}(y_{(r)})
\end{aligned}
$$

In the manner similar to the computations of the above integral, we may obtain the expressions for the remaining integrals in (3.24).

3.3 Special Cases : Distributions of Maxima and Minima

The maximum and the minimum order statistics in samples of size n are of special interest in numerous practical applications. An approximation to the distributions of the maximum and minimum order statistics follows from equation (3.21). The probability density functions of the maxima and minima are, respectively, given by

$$g_n^*(y_{(n)}) \simeq U_n^*\phi(y_{(n)}) + D_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{1100}(y_{(n)}) - Q_n^*\sum_{i=1}^{n}\lambda_{i(jkl)}^{2000}(y_{(n)}) + S_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{2200}(y_{(n)})$$
$$+ T_n^*\sum_{i\neq j\neq k}^{n}\lambda_{ijk(l)}^{2110}(y_{(n)}) + M_n^*\sum_{i\neq j\neq k\neq l}^{n}\lambda_{ijkl}^{1111}(y_{(n)}), \quad -\infty \leq y_{(n)} \leq \infty \qquad (3.27)$$

and

$$g_1^*(y_{(1)}) \sim U_n^*\phi(y_{(1)}) + D_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{1100}(y_{(1)}) - Q_n^*\sum_{i=1}^{n}\lambda_{i(jkl)}^{2000}(y_{(1)}) + S_n^*\sum_{i<j}^{n}\lambda_{ij(kl)}^{2200}(y_{(1)})$$
$$+ T_n^*\sum_{i\neq j\neq k}^{n}\lambda_{ijk(l)}^{2110}(y_{(1)}) + M_n^*\sum_{i\neq j\neq k\neq l}^{n}\lambda_{ijkl}^{1111}(y_{(1)}), \quad -\infty \leq y_{(1)} \leq \infty \qquad (3.28)$$

In equation (3.27),

$$\phi(y_{(n)}) = \frac{1}{(n-1)!}[F(y_{(n)})]^{n-1}f(y_{(n)}),$$

and for example, for $i, j, k < n$,

$$\lambda_{ij(kl)}^{t_it_j00}(y_{(n)}) = f(y_{(n)})\lambda_{ij(kl)}^{t_it_j00F}(y_{(n)}), \quad t_i, t_j = 1, 2$$
$$\lambda_{ijk(l)}^{t_it_jt_k0}(y_{(n)}) = f(y_{(n)})\lambda_{ijk(l)}^{t_it_jt_k0F}(y_{(n)}), \quad t_i, t_j, t_k = 1, 2$$

and for $i, k < n$ & $j = n$,

$$\lambda_{ij(kl)}^{t_it_j00}(y_{(n)}) = y_{(r)}^{t_j}f(y_{(n)})\lambda_{i(jkl)}^{t_i000F}(y_{(n)}), \quad t_i, t_j = 1, 2$$
$$\lambda_{ijk(l)}^{t_it_jt_k0}(y_{(n)}) = y_{(r)}^{t_j}f(y_{(n)})\lambda_{ik(j)}^{t_it_k00F}(y_{(n)}), \quad t_i, t_j, t_k = 1, 2$$

with

$$\lambda_{ij(kl)}^{t_it_j00F}(y_{(n)}) = \int^{2,n}\prod_{a=1(1)\overline{n-1}} y_{(a)}^{t_a}f(y_{[1,n-1]})dy_{[1,n-1]}, \quad t_a = 1, 2 \text{ for } a = i, j \qquad (3.29)$$
$$\& \quad t_a = 0 \quad \text{for} \quad a \neq i, j$$

for example.

Similarly, in equation (3.28), we have

$$\phi(y_{(1)}) = \frac{1}{(n-1)!}[1 - F(y_{(1)})]^{n-1} f(y_{(1)}).$$

and for example, for $i, j, k > 1$,

$$\lambda^{t_i t_j 00}_{ij(kl)}(y_{(1)}) = f(y_{(1)})\lambda^{t_i t_j 00}_{ij(kl)L}(y_{(1)}), \quad t_i, t_j = 1, 2$$

$$\lambda^{t_i t_j t_k 0}_{ijk(l)}(y_{(1)}) = f(y_{(1)})\lambda^{t_i t_j t_k 0}_{ijk(l)L}(y_{(1)}), \quad t_i, t_j, t_k = 1, 2$$

and for $j = 1$ & $i, k > 1$,

$$\lambda^{t_i t_j 00}_{ij(kl)}(y_{(1)}) = y^{t_j}_{(1)} f(y_{(1)})\lambda^{t_i 000}_{i(jkl)L}(y_{(1)}), \quad t_i, t_j = 1, 2$$

$$\lambda^{t_i t_j t_k 0}_{ijk(l)}(y_{(1)}) = y^{t_j}_{(1)} f(y_{(1)})\lambda^{t_i t_k 00}_{ik(jl)L}(y_{(1)}), \quad t_i, t_j, t_k = 1, 2$$

with

$$\lambda^{t_i t_j t_k 0}_{ijk(l)L}(y_{(1)}) = \int_{1,n-1} \prod_{a=n(-1)2} y^{t_a}_{(a)} f(y_{[2,n]}) dy_{[2,n]}, \quad t_a = 1, 2 \text{ for } a = i, j, k \quad (3.30)$$

$$\& \quad t_a = 0 \text{ for } a \neq i, j, k$$

for example.

Note that in the manner similar to that of (3.29) and (3.30), one may write the integrals for the remaining λ's in (3.27) and (3.28). Further note that in (3.29) and (3.30), we just have expressed the specific λ's in terms of integrals which we now simplify in the following section.

3.3.1 Computation of the Integral in (3.29) for general t_a

In practice, one requires the exact expressions for the above probability density functions of maxima and minima in equations (3.27) and (3.28). In order to do this, one needs to compute the integrations for $\lambda_{ij(kl)}^{t_it_j00t_L}(y_{(n)})$ and $\lambda_{ijk(l)L}^{t_it_jt_k0}(y_{(n)})$ in equations (3.29) & (3.30) respectively for general t_a. For the sake of simplicity, we show the integration technique below to obtain the result for $\lambda_{ij(kl)}^{t_it_j00t_L}(y_{(n)})$ only for (3.29). More specifically, in this subsection, we compute the integral of the form

$$I(y_{(n)}) \;=\; \int^{2,n} \prod_{a=1(1)\overline{n-1}} y_{(a)}^{t_a} f(y_{[1,n-1]})dy_{[1,n-1]}. \tag{3.31}$$

for general t_a.

3.3.1.1 Aids to Compute (3.31)

As shown in the following section, we need to evaluate two integrals

$$\eta(x) \;=\; \int_{-\infty}^{x} u^{t} e^{-\frac{u^2}{2\sigma^2}} du \tag{3.32}$$

and

$$\xi(z) \;=\; \int_{-\infty}^{z} w^{2d+T} e^{-\frac{w^2}{2}V^*} dw \tag{3.33}$$

for $x, z \in \Re$ to solve the integral in (3.31). In (3.32) and (3.33), t can take values 0 or 1 or 2, $0 < d < \infty$, and T is the sum of suitable number of t. We, now, first solve these two integrals as in the following for positive x and z.

It is well known that the incomplete gamma function $I_\mu = \int_0^\mu s^{a-1}e^{-a}ds$ may be expressed in the form of a partial sum as

$$I_\mu = \int_0^\mu s^{a-1}e^{-a}ds = \Gamma(a)\sum_{r=a}^{\infty} \frac{\mu^r e^{-\mu}}{r!}$$

[cf. Gupta (1960, 1962) and Prescott (1971)], which may be rewritten as

$$I_\mu = \Gamma(a)\sum_{r=0}^{\infty} \frac{\mu^{r+a} e^{-\mu}}{(r+a)!}$$

Now, $x > 0$, direct exploitation of this result yields

$$
\begin{aligned}
\eta(x) &= (-1)^t 2^{\frac{t-1}{2}} \sigma^{t+1} \Gamma\left(\frac{t+1}{2}\right) + \sum_{r=0}^{\infty} \frac{\Gamma(\frac{t+1}{2})}{2^{r+1}\sigma^{2r}(r+\frac{t+1}{2})!} x^{2r+t+1} e^{-\frac{x^2}{2\sigma^2}} \\
&= Q(t,\sigma^2) + \sum_{r=0}^{\infty} G(t,\sigma^2;r) x^{2r+t+1} e^{-\frac{x^2}{2\sigma^2}} \qquad (3.34)
\end{aligned}
$$

where

$$
\begin{aligned}
Q(t,\sigma^2) &= (-1)^t 2^{\frac{t-1}{2}} \sigma^{t+1} \Gamma\left(\frac{t+1}{2}\right) \\
G(t,\sigma^2;r) &= \frac{\Gamma(\frac{t+1}{2})}{2^{r+1}\sigma^{2r}(r+\frac{t+1}{2})!}
\end{aligned}
$$

and for $z > 0$,

$$
\begin{aligned}
\xi(z) &= (-1)^T \frac{2^{d+\frac{T-1}{2}} \Gamma(d+\frac{T+1}{2})}{V^{*d+\frac{T+1}{2}}} + \sum_{r=0}^{\infty} \frac{V^{*r}\Gamma(d+\frac{T+1}{2})}{2^{r+1}(d+\frac{T+1}{2}+r)!} z^{2(d+r)+T+1} e^{-\frac{z^2}{2}V^*} \\
&= Q(d,T,V^*) + \sum_{r=0}^{\infty} G(d,T,V^*;r) z^{2(d+r)+T+1} e^{-\frac{z^2}{2}V^*} \qquad (3.35)
\end{aligned}
$$

where

$$
\begin{aligned}
Q(d,T,V^*) &= (-1)^T \frac{2^{d+\frac{T-1}{2}} \Gamma(d+\frac{T+1}{2})}{V^{*d+\frac{T+1}{2}}} \\
G(d,T,V^*;r) &= \frac{V^{*r}\Gamma(d+\frac{T+1}{2})}{2^{r+1}(d+\frac{T+1}{2}+r)!}
\end{aligned}
$$

Next, for $x < 0$ and $z < 0$, by similar operation, we have

$$
\eta(x) = Q(t,\sigma^2) + \sum_{r=0}^{\infty} G^*(t,\sigma^2;r) x^{2r+t+1} e^{-\frac{x^2}{2\sigma^2}} \qquad (3.36)
$$

where $\qquad G^*(t,\sigma^2;r) = (-1)^{t+1} G(t,\sigma^2;r)$

$$
\xi(z) = Q(d,T,V^*) + \sum_{r=0}^{\infty} G^*(d,T,V^*;r) z^{2(d+r)+T+1} e^{-\frac{z^2}{2}V^*} \qquad (3.37)
$$

with $$G^*(d, T, V^*; r) = (-1)^{T+1} G(d, T, V^*; r)$$

The above integrations for $\eta(x)$ and $\xi(z)$ are done for general t, σ^2, d, T and V^*. But, the integration in (3.31) requires the integration results for these functions for all possible $t_i, \sigma_i^2, d_i, T_{ji}$ and V_{ji}^*. To accomodate all these cases, we define

$$\begin{aligned}
Q_{i0} &= Q(t_i, \sigma_i^2), \quad i = 1, 2, \ldots, n-1 \\
G_{i0r_k} &= G(t_i, \sigma_i^2; r_k), \quad i, k = 1, 2, \ldots, n-1 \\
G^*_{i0r_k} &= (-1)^{t_i+1} G_{i0r_k}, \quad i, k = 1, 2, \ldots, n-1
\end{aligned}$$

where

$$\begin{aligned}
Q(t_i, \sigma_i^2) &= (-1)^{t_i} 2^{\frac{t_i-1}{2}} \sigma^{t_i+1} \Gamma\left(\frac{t_i+1}{2}\right) \\
G(t_i, \sigma_i^2; r_k) &= \frac{\Gamma(\frac{t_i+1}{2})}{2^{r_k+1} \sigma^{2r_k} (r_k + \frac{t_i+1}{2})!}
\end{aligned}$$

as in (3.34).

Also define

$$\begin{aligned}
Q_{jd_i} &= Q(d_i, T_{ji}, V_{ji}^*), \quad j = 1, 2, \ldots, n-1, \ i = 1, 2, \ldots, n-2 \\
G_{jd_ir_h} &= G(d_i, T_{ji}, V_{ji}^*; r_h), \quad j, h = 1, 2, \ldots, n-1, \ i = 1, 2, \ldots, n-2 \\
G^*_{jr_ir_h} &= (-1)^{T_{ji}+1} G_{jd_ir_h}, \quad j, h = 1, 2, \ldots, n-1, \ i = 1, 2, \ldots, n-2 \\
T_{ji} &= \sum_{k=j-i}^{j-1} (t_k + 1) + t_j, \quad j = 1, 2, \ldots, n-1, \ i = 1, 2, \ldots, n-2 \\
V_{ji}^* &= + \sum_{k=j-i}^{j-1} \frac{1}{\sigma_k^2} + \frac{1}{\sigma_i^2}, \quad j = 1, 2, \ldots, n-1, \ i = 1, 2, \ldots, n-2 \\
\text{and} \qquad C_n &= \frac{1}{(2\pi)^{n/2} \mid D \mid^{\frac{1}{2}}}
\end{aligned}$$

where, as in (3.35),

$$Q(d_i, T_{ji}, V_{ji}^{*}) = (-1)^{T_{ji}} \frac{2^{d_i + \frac{T_{ji}-1}{2}} \Gamma(d_i + \frac{T_{ji}+1}{2})}{V_{ji}^{*\, d_i + \frac{T_{ji}+1}{2}}}$$

$$G(d_i, T_{ji}, V_{ji}^{*}; r_h) = \frac{V_{ji}^{*\, r_h} \Gamma(d_i + \frac{T_{ji}+1}{2})}{2^{r_h+1}(d_i + \frac{T_{ji}+1}{2} + r_h)!}$$

with

$$d_i = \sum_{h=1}^{i} r_h^{*}, \quad i = 1, 2, \ldots, n-2$$

where r_h^{*} $(h = 1, 2, \ldots, i)$ is the index value of i number of r_h's used in the preceding consequtive G function in a particular product. For example, for $G_{10r_1} Q_{2d_1} G_{30r_2} Q_{4d_1}$, d_1 of Q_{2d_1} and Q_{4d_1} functions are r_1 and r_2 respectively, and for $G_{10r_1} G_{2d_1 r_2} Q_{3d_2}$, $d_1 = r_1$ and $d_2 = r_1 + r_2$

Further, for positive Y_i $(i = 1, \ldots, n)$ (implying $x > 0$ in (3.31), and $z > 0$ in (3.35)) and for $n_2 > n_1$, let ${}^{n_1}H^{**(n_2)}(Q, G)$ denote a single combination of the product of n_1 'G' functions and $n_2 - n_1$ 'Q' functions. As G and Q functions can be arranged in ${}^{n_1}C_{n_2} = q_{n_1,n_2}$ (say) possible ways to make such a product, for convenience of summation of all these product combinations, we label them as ${}^{n_1}H_1^{**(n_2)}(Q, G), \ldots, {}^{n_1}H_i^{**(n_2)}(Q, G), \ldots, {}^{n_1}H_{q_{n_1,n_2}}^{**(n_2)}$ (Q, G). For example, for $n_1 = 3$ and $n_2 = 2$, all possible combinations are

$$\begin{aligned}
{}^{3}H_1^{**(2)}(Q, G) &= Q_{10} G_{20r_1} G_{3d_1 r_2} \\
{}^{3}H_2^{**(2)}(Q, G) &= G_{10r_1} Q_{2d_1} G_{30r_2} \\
\text{and} \quad {}^{3}H_3^{**(2)}(Q, G) &= G_{10r_1} G_{2d_1 r_2} Q_{3d_2}
\end{aligned}$$

Note that without any lose of generality, one may label the second product $G_{10r_1} Q_{2d_1} G_{30r_2}$ by ${}^{3}H_1^{**(2)}(Q, G)$ or ${}^{3}H_3^{**(2)}(Q, G)$.

In this case, $d_1 = r_1$, $d_2 = r_1 + r_2$ and in ${}^3H_1^{**(2)}(Q,G)$, $G_{3d_1r_2} = G(d_1, T_{31}, V_{31}^*; r_2)$, and for $n_1 = 4$ and $n_2 = 2$, all possible combinations are

$$\begin{aligned}
{}^4H_1^{**(2)}(Q,G) &= Q_{10}Q_{20}G_{30r_1}G_{4d_1r_2} \\
{}^4H_2^{**(2)}(Q,G) &= Q_{10}G_{20r_1}Q_{3d_1}G_{40r_2} \\
{}^4H_3^{**(2)}(Q,G) &= G_{10r_1}Q_{2d_1}G_{30r_2}Q_{4d_1} \\
{}^4H_4^{**(2)}(Q,G) &= Q_{10}G_{20r_1}G_{3d_1r_2}Q_{4d_2} \\
{}^4H_5^{**(2)}(Q,G) &= G_{10r_1}G_{2d_1r_2}Q_{3d_2}Q_{40} \\
\text{and} \quad {}^4H_6^{**(2)}(Q,G) &= G_{10r_1}Q_{2d_1}Q_{30}G_{40r_2}
\end{aligned}$$

Here d_1 may be r_1 or r_2, $d_2 = r_1 + r_2$ and in ${}^4H_1^{**2}(Q,G)$, $G_{4d_1r_2} = G(d_1, T_{41}, V_{41}^*; r_2)$. Note that any $G_{jd_ir_h}$ or Q_{jd_i} function will appear in the product combination only if it is preceded by a G function. More specifically, in any product combination, $G_{jd_ir_h}$ function will be preceded by $(j-1)$ 'Q' or 'G' functions and i number of G functions. Similarly, in any product combination, Q_{jd_i} function will also be preceded by $(j-1)$ 'Q' or 'G' functions and i number of G functions. Further, note that, for the case when smaller order statistics take negative values, that is, $Y_i < 0$ (implying $x < 0$ in (3.36), and $z < 0$ in (3.37)), the G functions in each term will be replaced by corresponding G^* functions.

3.3.1.2 Expressions for (3.31) for $n = 2, 3$ and 4

In this subsection, we discuss the integration technique in details for special cases with $n = 2, 3$ and 4. By using the notations in (3.22), for $n = 2$, it follows from (3.31) that

$$I(y_{(2)}) = \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} f(y_{(1)}) dy_{(1)}$$

$$= C_1 \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} e^{-\frac{y_{(1)}^2}{2\sigma_1^2}} dy_{(1)} \tag{3.38}$$

where

$$C_1 = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_1}$$

Notice that in (3.38), the integral is of the form of $\eta(x)$ given in (3.32). Now, by using the result for $\eta(x)$ from (3.31), we obtain

$$\begin{aligned} I(y_{(2)}) &= C_1 Q_{10} + \sum_{r_1=0}^{\infty} G_{10r_1} y_{(2)}^{2r_1+t_1+1} e^{-\frac{y_{(2)}^2}{2\sigma_1^2}} \\ &= \Delta_1^{(1)} + \sum_{r_1=0} \psi_1^{(1)} y_{(2)}^{2r_1+t_1+1} e^{-\frac{y_{(2)}^2}{2\sigma_1^2}} \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} \Delta_1^{(1)} &= C_1 Q_{10} \\ \text{and} \quad \psi_1^{(1)} &= C_1 G_{10r_1} \\ \text{with} \quad Q_{10} &= Q(t_1, \sigma_1^2) \\ \text{and} \quad G_{10r_1} &= G(t_1, \sigma_1^2; r_1) \end{aligned}$$

Similarly, for $n = 3$, by using (3.22), the equation (3.31) yields

$$\begin{aligned} I(y_{(3)}) &= \int_{-\infty}^{y_{(3)}} \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} y_{(2)}^{t_2} f(y_{(1)}) f(y_{(2)}) dy_{(1)} dy_{(2)} \\ &= C_2 \int_{-\infty}^{y_{(3)}} \left\{ \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} e^{-\frac{y_{(1)}^2}{2\sigma_1^2}} dy_{(1)} \right\} y_{(2)}^{t_2} e^{-\frac{y_{(2)}^2}{2\sigma_2^2}} dy_{(2)} \end{aligned}$$

where

$$C_2 = \frac{1}{(2\pi)\sigma_1\sigma_2}$$

Now, by (3.38), this integral reduces to

$$\begin{aligned} I(y_{(3)}) &= C_2 \int_{-\infty}^{y_{(3)}} \left\{ Q_{10} + \sum_{r_1=0}^{\infty} G_{10r_1} y_{(2)}^{2r_1+t_1+1} e^{-\frac{y_{(2)}^2}{2\sigma_1^2}} \right\} y_{(2)}^{t_2} e^{-\frac{y_{(2)}^2}{2\sigma_2^2}} \, dy_{(2)} \\ &= C_2 Q_{10} \int_{-\infty}^{y_{(3)}} y_{(2)}^{t_2} e^{-\frac{y_{(2)}^2}{2\sigma_2^2}} \, dy_{(2)} \\ &\quad + C_2 \sum_{r_1=0}^{\infty} G_{10r_1} \int_{-\infty}^{y_{(3)}} y_{(2)}^{2r_1+T_{21}} e^{-\frac{y_{(2)}^2}{2} V_{21}^{*2}} \, dy_{(2)} \end{aligned} \tag{3.40}$$

Note that in (3.40), the first integral (as in the case with $n = 2$) is of the form $\eta(x)$ in (3.32) and the second integral is of the form $\xi(z)$ defined in (3.33). Consequently, by using (3.34) and (3.35), we obtain

$$\begin{aligned} I(y_{(3)}) &= C_2 Q_{10} \left\{ Q_{20} + \sum_{r_1=0}^{\infty} G_{20r_1} y_{(3)}^{2r_1+t_2+1} e^{-\frac{y_{(3)}^2}{2\sigma_2^2}} \right\} + C_2 \sum_{r_2=0}^{\infty} G_{10r_1} [Q_{2d_1} \\ &\quad + \sum_{r_2=0}^{\infty} G_{2d_1 r_2} y_{(3)}^{2(d_1+r_2)+T_{21}+1} e^{-\frac{y_3^2}{2} V_{21}^{*}}] \\ &= C_2 Q_{10} Q_{2d_1} + C_2 \sum_{r_1=0}^{\infty} \left\{ Q_{10} G_{20r_1} y_{(3)}^{2r_1+t_2+1} e^{-\frac{y_{(3)}^2}{2\sigma_2^2}} + G_{10r_1} Q_{2d_1} \right\} \\ &\quad + C_2 \sum_{r_1=0}^{\infty} G_{10r_1} G_{2d_1 r_2} y_{(3)}^{2(d_1+r_2)+T_{21}+1} e^{-\frac{y_3^2}{2} V_{21}^{*}} \end{aligned} \tag{3.41}$$

$$\begin{aligned} &= \Delta_1^{(2)} + \sum_{r_1=0}^{\infty} \left\{ \Theta_1^{*(2)} y_{(3)}^{2r_1+t_2+1} e^{-\frac{y_{(3)}^2}{2\sigma_2^2}} + \Theta_2^{*(2)} \right\} \\ &\quad + \sum_{r_1=0} \sum_{r_2=0} \Psi_1^{(2)} y_{(3)}^{2(d_1+r_2)+T_{21}+1} e^{-\frac{y_3^2}{2} V_{21}^{*}} \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} V_{21}^{*} &= \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} = \frac{1}{\sigma_2^2} + \sum_{k=2-1}^{2-1} \frac{1}{\sigma_k^2} \\ T_{21} &= t_1 + t_2 + 1 = t_2 + \sum_{k=2-1}^{2-1} (t_k + 1) \end{aligned}$$

$$\begin{aligned}
\Delta_1^{(2)} &= C_2 Q_{10} Q_{20} = C_2 \prod_{i=1}^{2} Q_{i0} \\
\Theta_1^{*(2)} &= C_2 Q_{10} G_{20r_1} = C_2 \Delta_1^1 G_{20r_1} \\
\Theta_2^{*(2)} &= C_2 G_{10r_1} Q_{2d_1} \\
\Psi_1^{(2)} &= C_2 G_{10r_1} G_{2d_1 r_2} = C_2 \prod_{j=1}^{2} G_{jd_{j-1}r_j}, \quad d_0 = 0 \;\&\; d_1 = r_1
\end{aligned}$$

$$\begin{aligned}
\text{with} \quad Q_{20} &= Q(t_2, \sigma_2^2) \\
Q_{2d_1} &= Q(d_1, T_{21}, V_{21}^*) \\
\text{and} \quad G_{2d_1 r_2} &= G(d_1, T_{21}, V_{21}^*; r_2)
\end{aligned}$$

Now, for the case $n = 4$, by (3.22), we write the integral from (3.31)

$$\begin{aligned}
I(y_{(4)}) &= \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(3)}} \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} y_{(2)}^{t_2} y_{(3)}^{t_3} f(y_{(1)}) f(y_{(2)}) f(y_{(3)}) dy_{(1)} dy_{(2)} dy_{(3)} \\
&= C_3 \int_{-\infty}^{y_{(4)}} \left\{ \int_{-\infty}^{y_{(3)}} \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} e^{-\frac{y_{(1)}^2}{2\sigma_1^2}} dy_{(1)} y_{(2)}^{t_2} e^{-\frac{y_{(2)}^2}{2\sigma_2^2}} dy_{(2)} \right\} y_{(3)}^{t_3} e^{-\frac{y_{(3)}^2}{2\sigma_3^2}} dy_{(3)}
\end{aligned}$$

where

$$C_3 = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_1 \sigma_2 \sigma_3}$$

Now, in the manner similar to that of $n = 2$ and 3 and by using the results of $\eta(x)$ and $\xi(z)$ in different steps, we obtain

$$\begin{aligned}
I(y_{(4)}) &= C_3 Q_{10} Q_{20} Q_{30} + C_3 \sum_{r_1=0}^{\infty} \{ Q_{10} Q_{20} G_{30r_1} y_{(4)}^{2r_1+t_3+1} e^{-\frac{y_{(4)}^2}{2\sigma_3^2}} + Q_{10} G_{20r_1} Q_{3d_1} \\
&+ G_{10r_1} Q_{2d_1} Q_{30} \} + C_3 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} [G_{10r_1} Q_{2d_1} G_{30r_2} y_{(4)}^{2r_2+t_3+1} e^{-\frac{y_{(4)}^2}{2\sigma_3^2}} \\
&+ Q_{10} G_{20r_1} G_{3d_1 r_2} y_4^{2(d_1+r_2)+T_{31}+1} e^{-\frac{y_4^2}{2} V_{31}^*} + G_{10r_1} G_{2d_1 r_2} Q_{3d_2}] \\
&+ C_3 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} G_{10r_1} G_{2d_1 r_2} G_{3d_2 r_3} y_4^{2(d_2+r_3)+T_{32}+1} e^{-\frac{y_{(4)}^2}{2} V_{32}^*}
\end{aligned} \tag{3.43}$$

$$= \Delta_1^{(3)} + \sum_{r_1=0}^{\infty}\left\{\Theta_1^{*(3)} y_{(4)}^{2r_1+t_3+1} e^{-\frac{y_{(4)}^2}{2\sigma_3^2}} + \sum_{l=2}^{3}\Theta_l^{*(3)}\right\} + \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} {}^3B^{*(2)}(r_1,r_2)$$

$$+ \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty} \Psi_1^{(3)} y_4^{2(d_2+r_3)+T_{32}+1} e^{-\frac{y_{(4)}^2}{2}V_{32}^*} \tag{3.44}$$

where

$$V_{31}^* = \frac{1}{\sigma_3^2} + \frac{1}{\sigma_2^2} = \frac{1}{\sigma_3^2} + \sum_{k=3-1}^{3-1}\frac{1}{\sigma_k^2}$$

$$V_{32}^* = \frac{1}{\sigma_3^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2} = \frac{1}{\sigma_3^2} + \sum_{k=3-2}^{3-1}\frac{1}{\sigma_k^2}$$

$$T_{31} = t_3 + t_2 + 1 = t_3 + \sum_{k=3-1}^{3-1}(t_k+1)$$

$$T_{32} = t_3 + t_2 + t_1 + 1 = t_3 + \sum_{k=3-2}^{3-1}(t_k+1)$$

$$\Delta_1^{(3)} = C_3 Q_{10} Q_{20} Q_{30} = C_3 \prod_{i=1}^{3} Q_{i0}$$

$$\Theta_1^{*(3)} = C_3 Q_{10} Q_{20} G_{30r_1} = C_3 \Delta_1^{(2)} G_{30r_1}$$

$$\Theta_1^{*(3)} = C_3 Q_{10} G_{20r_1} Q_{3d_1} = C_3 \Delta_1^{(1)} G_{20r_1} Q_{3d_1}$$

$$\Theta_2^{*(3)} = C_2 G_{10r_1} Q_{2d_1} Q_{30} = C_3 G_{10r_1} Q_{2d_1} \Delta_1^{(1)}$$

$$\Psi_1^{(3)} = C_3 G_{10r_1} G_{2d_1 r_2} G_{3d_2 r_3} = C_3 \prod_{j=1}^{3} G_{jd_{j-1}r_j}, \quad d_0 = 0$$

and $\quad {}^3B^{*(2)}(r_1,r_2) = C_3 \sum_{i=1}^{q_3} {}^3H_i^{*(2)}(Q,G)$

with $\quad q_3 = {}^3C_2 = 3$

and
$$ {}^3H_1^{*(2)}(Q,G) = {}^3H_1^{**(2)}(Q,G) y_{(4)}^{2r_2+t_3+1} e^{-\frac{y_{(1)}^2}{2\sigma_3^2}}$$

$${}^3H_2^{*(2)}(Q,G) = {}^3H_{(2)}^{**(2)}(Q,G) y_{(4)}^{2(d_1+r_2)+T_{31}+1} e^{-\frac{y_{(4)}^2}{2}V_{31}^*}$$

$${}^3H_3^{*(2)}(Q,G) = {}^3H_3^{**(2)}(Q,G)$$

where

$$
\begin{aligned}
{}^3H_1^{**(2)}(Q,G) &= G_{10r_1}Q_{2d_1}G_{30r_2} \\
{}^3H_2^{**(2)}(Q,G) &= Q_{10}G_{20r_1}G_{3d_1r_2} \\
{}^3H_3^{**(2)}(Q,G) &= G_{10r_1}G_{2d_1r_2}Q_{3d_2} \\
\text{with} \quad Q_{30} &= Q(t_3,\sigma_3^2) \\
Q_{3d_1} &= Q(d_1,T_{31},V_{31}^*) \\
Q_{3d_2} &= Q(d_2,T_{32},V_{32}^*) \\
G_{3d_1r_2} &= G(d_1,T_{31},V_{31}^*;r_2) \\
\text{and} \quad G_{3d_2r_3} &= G(d_2,T_{32},V_{32}^*;r_3)
\end{aligned}
$$

Here d_1 and d_2 will be r_1 and $r_1 + r_2$ respectively.

3.3.1.3 Expression for (3.31) for general n

It, now, follows from subsection 3.3.1.2 that for general n, one obtains

$$
\begin{aligned}
I(y_{(n)}) &= \int^{2,n} \prod_{a=1(1)n-1} y_{(a)}^{t_a} f(y_{[1,n-1]})dy_{[1,n-1]} \\
&= C_{n-1}\int_{-\infty}^{y_{(n)}} \cdots \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} e^{-\frac{y_{(1)}^2}{2\sigma_1^2}} \cdots y_{(n-1)}^{t_{n-1}} e^{-\frac{y_{(n-1)}^2}{2\sigma_{n-1}^2}} dy_{(1)}\cdots dy_{(n-1)} \\
&= C_{n-1}\Delta_1^{(n-1)} + \sum_{r_1=0}^{\infty}\left\{\Theta_1^{*(n-1)}(r_1)y_{(n)}^{2r_1+t_{n-1}}e^{-\frac{y_{(n)}^2}{2\sigma_{n-1}^2}} + \sum_{l=2}^{n-1}\Theta_l^{*(n-1)}(r_1)\right\} \\
&\quad + \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} {}^{n-1}B^{*(2)}(r_1,r_2) + \ldots + \sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty} {}^{n-1}B^{*(m)}(r_1,\ldots,r_m) + \ldots \\
&\quad + \sum_{r_1=0}^{\infty}\cdots\sum_{r_{n-2}=0}^{\infty} {}^{n-1}B^{*(n-2)}(r_1,\ldots,r_{n-2}) + \sum_{r_1=0}^{\infty}\cdots\sum_{r_{n-1}=0}^{\infty} \psi_1^{(n-1)}(r_1,\ldots,r_{n-1}) \\
&\quad \times y_{(n)}^{2(d_{n-2}+r_{n-1})+T_{(n-1)(n-2)}+1}e^{-\frac{y_{(n)}^2}{2}V_{(n-1)(n-2)}^*}
\end{aligned}
\qquad (3.45)
$$

where

$$C_{n-1} = \frac{1}{(2\pi)^{\frac{n-1}{2}} \mid D \mid^{\frac{1}{2}}}$$

$$\Delta_1^{(n-1)} = C_{n-1} \prod_{i=1}^{n-1} Q_{i0}$$

$$\psi_1^{(n-1)}(r_1, \ldots, r_{n-1}) = C_{n-1} \prod_{j=1}^{n-1} G_{jd_{j-1}r_j} \quad \text{with} \quad d_0 = 0$$

$$\Theta_l^{*(n-1)}(r_1) = C_{n-1}\Delta_1^{((n-1)-l)} G_{[(n-1)-l+1]0r_1} \left[Q_{[(n-1)-l+2]d_1}\right]^{I_{s_1}} \left[\Delta_{(n-1)-l+3}^{(n-1)}\right]^{I_{s_2}}$$

with $I_{s_1} = 0$ if $(n-1) - l + 2 > n - 1$ and $I_{s_1} = 1$, otherwise. Similarly $I_{s_2} = 0$ if $(n-1) - l + 3 > n - 1$ and $I_{s_2} = 1$, otherwise. Furthermore, in (3.45), for $m = 2, 3, \ldots, n-2$, we have

$${}^{n-1}B^{*(m)}(r_1, \ldots, r_m) = C_{n-1} \sum_{i=1}^{q_{n-1}} {}^{n-1}H_i^{*(m)}(Q, G)$$

with

$$q_{n-1,m} = {}^{n-1}C_m$$

and $${}^{n-1}H_i^{*(m)}(Q, G) = {}^{n-1}H_i^{**(m)}(Q, G) \left[y_{(n)}^{2(d^*+r^*)+T^{**}+1} e^{-\frac{y^2_{(n)}}{2}V^{**}} \right]^{I_G}$$

where $I_G = 1$ if the product function ${}^{n-1}H^{**(m)}$ is ended by a G function (cf. section 3.3.1.1) and $I_G = 0$, otherwise. For the case when $I_G = 1$, d^*, r^*, T^{**} and V^{**} in the square bracket [ ] are replaced by d, r, T and V, respectively, where the latter functions are taken from the last G function in ${}^{n-1}H_i^{**(m)}(Q, G)$.

Note that one may exploit the integration result in (3.45) to obtain any λ's necessary in (3.27) by putting appropriate values of t_a. For example, for $\lambda_{ij(kl)}^{t_i t_j 00 U'}(y_{(n)})$ in (3.29), we require to put $t_a = 1, 2$ for $a = i, j$ and $t_a = 0$ for all $a \neq i, j$ in equation (3.45).

Chapter 4

Computational Aspects and Applications

4.1 Computation of Percentile Points of $Y_{(n)}$

In this chapter, we provide several exact percentile points of the distribution of the maxima for the correlated normal variables under three different situations. In the first situation, we compute percentile points of the maxima for the equi-correlated (positive or negative) normal variables with zero mean and equal variance σ^2. Our results for positive correlations are verified with the results provided by Gupta (1973) among others. Also our results for negative correlation will supplement some of the results provided by Hoffman and Saw (1975). In the second situation, we provide the percentile points of the maxima for the homoscedastic but unequally (positively or negatively) correlated normal variables. These distributional results for the homoscedastic unequal correlated normal variables case may be applied to certain repeated measures data, for example, to any data sets, where the correlations of the data follow the correlation structure of a stationary Gaussian process, say, AR(1). In the last situation, we compute the percentile points of the maxima for the unequal positively or negatively correlated normal variables with heteroscedastic variances. In this chapter, we also compare the performance of the Bonferroni bounds approximations with our small correlations approach in computing the percentile points of the maxima for both homoscedastic and heteroscedistic cases.

4.2 First case : Homoscedastic equi-correlated (positive and negative) normal variables

Based on the transformation as in the equation (2.1), Gupta (1973) provided tables for the $(1-\alpha)$ percentile points of $Y_{(n)}$ for selected values of n, α and positive ρ, where $Y_{(n)}$ is the maxima of the n standardized normal random variables $Y_1, Y_2, \ldots, Y_n$ having correlation matrix $\rho J_n - \rho I_n$ with J_n as the $n \times n$ unit matrix and I_n as the $n \times n$ identity matrix. This type of transformations is not suitable to handle the negatively equi-correlated normal variables cases. Further more, this approach requires a difficult integration (cf. section 2.1) to compute the percentile points of the maxima. Instead of solving this integral, Gupta (1973) has used a numerical approximation to resolve this problem. To examine the performance of our approach, we exploit our method discussed in the last chapter and compare the percentile points of $Y_{(n)}$ for $\alpha = 0.05, 0.025$ & 0.01 and $\rho = 0.100, 0.125, 0.200$ & 0.250 with those given in Gupta (1973).

For the equi-correlated normal variables $Y_1, Y_2, \ldots, Y_n$ with $E(Y_i) = 0$, $E(Y_i^2) = \sigma^2$, for all $i = 1, 2, \ldots, n$ and $E(Y_i Y_j) = \rho$, for all $i \neq j$, we obtain the probability density function of the maxima by putting $\rho_{ij} = \rho$ for all $i \neq j$ and $\sigma_i^2 = \sigma^2$ for $i = 1, 2, \ldots, n$, in (3.27). The density is given by

$$\begin{aligned} g_n^*(y_{(n)}, \rho, \sigma^2) \;\simeq\; & U_n^* \phi(y_{(n)}, \sigma^2) + D_n^* \sum_{i<j}^{n} \lambda_{ij(kl)}^{1100}(y_{(n)}, \sigma^2) - Q_n^* \sum_{i=1}^{n} \lambda_{i(kl)}^{2000}(y_{(n)}, \sigma^2) \\ & + S_n^* \sum_{i<j}^{n} \lambda_{ij(kl)}^{2200}(y_{(n)}, \sigma^2) + T_n^* \sum_{i \neq j \neq k}^{n} \lambda_{ijk(l)}^{2110}(y_{(n)}, \sigma^2) \\ & + M_n^* \sum_{i \neq j \neq k \neq l}^{n} \lambda_{ijkl}^{1111}(y_{(n)}, \sigma^2) \end{aligned} \qquad (4.1)$$

where

$$\begin{aligned} U_n^* &= n!\{1 + \frac{1}{2} n(n-1)\rho^2\} \\ D_n^* &= n!\{\rho - (n-2)\rho\}\frac{1}{\sigma^2} \\ Q_n^* &= n!(n-1)\rho^2 \frac{1}{\sigma^2} \\ S_n^* &= n!\rho^2 \frac{1}{2\sigma^4} \\ T_n^* &= n!\rho^2 \frac{1}{\sigma^4} \\ M_n^* &= 3n!\rho^2 \frac{1}{\sigma^4} \\ \phi(y_{(n)}, \sigma^2) &= \phi(y_{(n)}) \mid_{\sigma_1^2 = \ldots = \sigma_n^2 = \sigma^2}, \end{aligned}$$

and for example,

$$\lambda_{ij(kl)}^{l_i l_j 00}(y_{(n)}, \sigma^2) = \lambda_{ij(kl)}^{l_i l_j 00}(y_{(n)}) \mid_{\sigma_1^2 = \ldots = \sigma_n^2 = \sigma^2} \quad , \quad l_i, l_j = 1, 2$$

with $\phi(y_{(n)})$ and $\lambda_{ij(kl)}^{l_i l_j 00}(y_{(n)})$'s as in equation (3.27). Note that the coefficients $U_n^*, D_n^*, Q_n^*, S_n^*, T_n^*$, and M_n^* in (4.1) are the special cases of the coefficients defined in (3.27).

Next, to compute the percentile points of $Y_{(n)}$, namely, the values of h, we may easily compute the distribution function of maxima from (4.1) as given by

$$
\begin{aligned}
G_n^{**}(h,\rho,\sigma^2) &= pr\{Y_{(n)} \leq h\} \\
&= \int_{-\infty}^{h} g_n^*(y_{(n)},\sigma^2,\rho)dy_{(n)} \\
&\simeq C_n^*\Phi(h,\sigma^2) + D_n^*\sum_{i<j}^{n}\Lambda_{ij(kl)}^{1100}(h,\sigma^2) - Q_n^*\sum_{i=1}^{n}\Lambda_{i(jkl)}^{2000}(h,\sigma^2) \\
&+ S_n^*\sum_{i<j}^{n}\Lambda_{ij(kl)}^{2200}(h,\sigma^2) + T_n^*\sum_{i\neq j\neq k}^{n}\Lambda_{ijk(l)}^{2110}(h,\sigma^2) \\
&+ M_n^*\sum_{i\neq j\neq k\neq l}^{n}\Lambda_{ijkl}^{1111}(h,\sigma^2)
\end{aligned} \tag{4.2}
$$

where

$$
\begin{aligned}
\Phi(h,\sigma^2) &= \int_{-\infty}^{h}\phi(y_{(n)},\sigma^2)dy_{(n)} \\
&= \left[\int_{-\infty}^{h}\phi(y_{(n)})dy_{(n)}\right]_{\sigma_1^2=\ldots=\sigma_n^2=\sigma^2} \\
&= \Phi(h)\mid_{\sigma_1^2=\ldots=\sigma_n^2=\sigma^2} \\
\text{with}\quad \Phi(h) &= \int_{-\infty}^{h}\phi(y_{(n)})dy_{(n)} \\
&= \frac{1}{n!}[F(h)]^n
\end{aligned} \tag{4.3}
$$

And for example,

$$
\begin{aligned}
\Lambda_{ij(kl)}^{t_it_j00}(h,\sigma^2) &= \int_{-\infty}^{h}\lambda_{ij(kl)}^{t_it_j00}(y_{(n)},\sigma^2)dy_{(n)} \quad , \quad t_i,t_j = 1,2 \\
&= \left[\int_{-\infty}^{h}\lambda_{ij(kl)}^{t_it_j00}(y_{(n)})dy_{(n)}\right]_{\sigma_1^2=\ldots=\sigma_n^2=\sigma^2} \\
&= \Lambda_{ij(kl)}^{t_it_j00}(h)\mid_{\sigma_1^2=\ldots=\sigma_n^2=\sigma^2} \\
\text{with}\quad \Lambda_{ij(kl)}^{t_it_j00}(h) &= \int_{-\infty}^{h}\lambda_{ij(kl)}^{t_it_j00}(y_{(n)})dy_{(n)} \quad , \quad t_i,t_j = 1,2
\end{aligned} \tag{4.4}
$$

In equation (4.3), for $i,j < n$, and $t_a = 1,2$ for $a = i,j$ & $t_a = 0$ for $a \neq i,j$, $\Lambda_{ij(kl)}^{t_it_j00}(h)$

is evaluated by using (3.27) & (3.29) given by

$$\Lambda_{ij(kl)}^{t_it_j00}(h) = \int_{-\infty}^{h} f(y_{(n)})\Lambda_{ij(kl)}^{t_it_j00l^*}(y_{(n)})dy_{(n)}$$
$$= \int_{-\infty}^{h} y_r^{l_n} f(y_{(n)})\left\{\int^{2,a} \prod_{a=1(1)\overline{n-1}} y_a^{l_a} f(y_{[1,n-1]})dy_{[1,n-1]}\right\} dy_{(n)} \quad (4.5)$$

where $l_n = 0$

Next, by using the results of $\eta(x)$, $\xi(z)$ and $I(y_{(n)})$ in the equations (3.31), (3.35) and (3.43), respectively, we obtain

$$\begin{aligned}\Lambda_{ij(kl)}^{t_it_j00}(h) &= C_n\Delta_1^{(n)} + \sum_{r_1=0}^{\infty}\left\{\Theta_1^{*(n)}(r_1)h^{2r_1+l_n}e^{-\frac{h^2}{2\sigma_n^2}} + \sum_{l=2}^{n}\Theta_l^{*(n)}\right\}\\ &+ \sum_{r_1}^{\infty}\sum_{r_2}^{\infty} {}^nB^{*(2)}(r_1,r_2;h) + \ldots + \sum_{r_1}^{\infty}\cdots\sum_{r_m}^{\infty} {}^nB^{*(m)}(r_1,\ldots,r_m;h) + \ldots\\ &+ \sum_{r_1}^{\infty}\cdots\sum_{r_{n-1}}^{\infty} {}^nB^{*(n-1)}(r_1,\ldots,r_{n-1};h) + \sum_{r_1}^{\infty}\cdots\sum_{r_n}^{\infty}\Psi_1^{(n)}(r_1,\ldots,r_n)\\ &\times\ h^{2(d_{n-1}+r_{n-1})+T_{n(n-1)}+1}e^{-\frac{h^2}{2}}V_{n(n-1)}^{**}\end{aligned} \quad (4.6)$$

where Δ, Θ^*, B^* and Ψ are defined as in equation (3.43) and $l_a = 1, 2$ for $a = i, j$, $l_a = 0$ for $a \neq i, j$ & $a = n$, in this case. Note that one may exploit the integration result in (4.4) to obtain any Λ's necessary in (4.2) by putting appropriate values of l_a.

In order to examine the performance of the proposed procedure, we now compute the probability $G_n^{**}(h, \rho, \sigma^2)$ given by (4.2) for selected values of $\alpha = 0.010, 0.025$ and 0.050, $\rho = 0.100, 0.125, 0.200$ and 0.250 as in Gupta (1973). Without loss of generality, we consider $\sigma^2 = 1$, and $n = 2, 3$ & 4 and compute $G_n^{**}(h, \rho, \sigma^2)$ by borrowing the values of h from Gupta (1973). These probabilities along with the cumulative probabilites computed by Gupta (1973) are shown in the Table 4.1. More specifically, for different

Table 4.1: The Actual Probabilities for the maxima of positive equi-correlated normal variables based on CTM and SCA for selected α, ρ and n with $\sigma^2 = 1$, corresponding to the Nominal $100(1-\alpha)\%$ probabilities with $\alpha = 0.01, 0.025$ and 0.05.

		n = 2		
$\rho \downarrow$	$\alpha =$	0.010	0.025	0.050
0.100	h	2.5739	2.2368	1.9508
	CTM	0.989799	0.974801	0.949806
	SCA	0.989832	0.974685	0.950282
0.125	h	2.5736	2.2361	1.9497
	CTM	0.989801	0.974801	0.949812
	SCA	0.989842	0.974728	0.949266
0.200	h	2.5722	2.2336	1.9456
	CTM	0.989802	0.974807	0.949809
	SCA	0.989877	0.974877	0.949708
0.250	h	2.5709	2.2314	1.9423
	CTM	0.989802	0.974807	0.949809
	SCA	0.989906	0.974993	0.950045

		n = 3		
$\rho \downarrow$	$\alpha =$	0.010	0.025	0.050
0.100	h	2.7105	2.3878	2.1158
	CTM	0.989801	0.974802	0.949803
	SCA	0.988399	0.974687	0.948522
0.125	h	2.7099	2.3829	2.1111
	CTM	0.989799	0.974805	0.949801
	SCA	0.988554	0.974806	0.949234
0.200	h	2.7078	2.3829	2.1080
	CTM	0.989801	0.974807	0.949813
	SCA	0.991875	0.975631	0.952218
0.250	h	2.7058	2.3795	2.1029
	CTM	0.989802	0.974807	0.949811
	SCA	0.991132	0.976198	0.954731
		n=4		
0.100	h	2.8041	2.4907	2.2276
	CTM	0.989801	0.974804	0.949806
	SCA	0.990062	0.975166	0.950091
9.125	h	2.8034	2.4894	2.2255
	CTM	0.989799	0.974801	0.949801
	SCA	0.990824	0.975913	0.950789
0.200	h	2.7078	2.3829	2.1080
	CTM	0.989802	0.974801	0.949814
	SCA	0.994899	0.975946	0.954433
0.250	h	2.7083	2.4804	2.2116
	CTM	0.989802	0.974807	0.949801
	SCA	0.994904	0.975926	0.957025

h, Gupta's and our results are shown under the heading CTM (correlation transformation method) and SCA (small correlations approach) respectively.

It is clear from Table 4.1 that the SCA based actual probabilities are pretty close to the CTM based actual probabilities of Gupta's (1973), for the equi-correlated standard normal variables case for the equi-correlation coefficients $\rho \leq 0.250$. However, for $\rho > 0.250$, our SCA based results will not be the same as Gupta's (1973) CTM based results, which is expected. This is because, our approximation is devoloped based on small values of correlations. We, however, computed some of the probabilities for $\rho \geq 0.300$ and find that for $n = 3$ and $\rho = 0.300$, the SCA yields the actual probabilities 0.996865, 0.980302 and 0.961371 corresponding to 99%, 97.5% and 95% nominal probabilities respectively. Similarly, for $n = 4$, these actual probabilites were founded to be 0.997539, 0.979913 and 0.961389 respectively, showing the departure from the nominal values.

For negative equi-correlations case, Gupta's (1973) method given in (2.4) is not suitable to compute the percentile points of the maxima. For this case, however, Hoffman and Saw (1975) proposed an alternative method to compute the percentile points of the maxima but they did not provide detailed numerical results in the paper. While our main interest is to obtain the percentage points of the maxima for unequally (positively or negatively) correlated normal random variables, we still computed the percentile points of the maxima based on our method for several negative small equi-correlations $\rho = -0.100, -0.125, -0.200$ and -0.250. The results are shown in Table 4.2, which may be verified with the results of Hoffman and Saw (1975).

Note that the percentile points shown in table 4.2 are computed by considering the results for the positive equi-correlation provided by Gupta (1973) as the initial values. Moreover, it was found that the results in the above table do not remain the same

Table 4.2: The SCA based 100(1 − α)% percentile points of the maxima for negative equi-correlations and selected α and n with $\sigma^2 = 1$

n	ρ	α =	0.010	0.025	0.050
2	-0.100		2.5123	2.2519	1.9878
	-0.125		2.5067	2.2408	1.9707
	-0.200		2.4956	2.2397	1.9667
	-0.250		2.4811	2.2251	1.9513
3	-0.100		2.5405	2.3065	2.0907
	-0.125		2.4799	2.2799	2.0509
	-0.200		2.3778	2.1829	2.0218
	-0.250		2.2858	2.1295	1.9837
4	-0.100		2.7279	2.4567	2.2156
	-0.125		2.6801	2.3912	2.1967
	-0.200		2.5478	2.3369	2.1613
	-0.250		2.4765	2.2801	2.1409

respective to the results given by Gupta (1973) for the positive equi-correlations case.

4.3 Second Case : Homoscedastic but unequally (positively or negatively) correlated normal variables

There also exists a few methods to study certain specific inference problems for the order statistics of the unequally positively or negatively correlated normal variables. Based on the V-function described by Nicholson (1943), Gupta et al. (1964) have studied the distribution of the range, $Y_{(n)} - Y_{(1)}$, of unequally correlated normal variables for $n = 3$ and 4. For a very special case of unequal correlation structures such that $E(Y_iY_j) = 0$, for all $i \neq j$ & $j \neq i + 1$ and non-zero $E(Y_iY_{i+1}) = \rho_{i,i+1}$, Greig (1967) studied the distributional properties of the order statistics of the correlated normal random variables.

As discussed in chapter 3, we have studied the distributions of the order statistics of unequally (positively or negatively) correlated normal random variables. But this was done for small correlations. The small correlations among normal variables arise in many practical situations, in particular, in the context of cluster regression analysis. In cluster analysis, the cluster sizes are usually small. But for generality, we have, however, provided the theory in the last chapter for general n. For cluster regression analysis with small ρ's among the observations within the cluster, we refer to Rao, Sutradhar and Yue (1993), and Wu, Holt and Holmes (1988), among others.

4.3.1 Application to AR(1) Models

We now consider a non-regression set-up where cluster observations may be generated by repeatations of measurments of single individual over a period of time. In such cases, it is likely that there will be a decay in the correlation with increasing time lags and the data may behave like an autoregressive normal process of order one given by

$$y_t = \phi_1 y_{t-1} + \epsilon_t \qquad (4.7)$$

where ϵ_t is independently and normally distributed random variables with mean zero and constant variance, σ^{*2}, and ϕ_1 is the parameter of the autoregressive process of order one. Here $\mathbf{Y} = [Y_1, \ldots, Y_t, \ldots, Y_n]'$ is jointly normal with $E(\mathbf{Y}) = 0$ and $Disp(\mathbf{Y}) = \sigma^{*2}\Upsilon$, for all $i = 1, 2, \ldots, n$, Υ being the $n \times n$ non-singular matrix given by $\Upsilon = (\phi_1^{|t-t'|}/(1-\phi_1^2))$, $t, t' = 1, 2, \ldots, n$. Also we have the correlation matrix of $\mathbf{Y}$ as $corr(\mathbf{Y}) = \phi_1^{|t-t'|}$, $t, t' = 1, 2, \ldots, n$.

Now, in order to compute the percentile points of $Y_{(n)}$ for unequally correlated (positive or negative) normal variables with constant variances σ^2, we obtain the distribution function of the maxima from equation (3.27) by putting $\sigma_i^2 = \sigma^2$ in the manner similar to that of (4.2) and after some adjustments in the coefficients, we write the distribution function as

$$\begin{aligned} Pr(Y_{(n)} \leq h) &= G_n^{**}(h, \rho_{ij}, \sigma^2) \\ &\simeq U_n^* \Phi(h, \sigma^2) + D_n^* \sum_{i<j}^{n} \Lambda_{ij(kl)}^{1100}(h, \sigma^2) - Q_n^* \sum_{i=1}^{n} \Lambda_{i(ikl)}^{2000}(h, \sigma^2) \\ &+ S_n^* \sum_{i<j}^{n} \Lambda_{ij(kl)}^{2200}(h, \sigma^2) + T_n^* \sum_{i \neq j \neq k}^{n} \Lambda_{ijk(l)}^{2110}(h, \sigma^2) \\ &+ M_n^* \sum_{i \neq j \neq k \neq l}^{n} \Lambda_{ijkl}^{1111}(h, \sigma^2) \end{aligned} \qquad (4.8)$$

where

$$
\begin{aligned}
U_n^* &= n!\left\{1+\frac{1}{2}\sum_{i<j}^{n}\rho_{ij}^2\right\} \\
D_n^* &= \frac{2(n-2)!}{\sigma^2}\sum_{i<j}^{n}\left\{\rho_{ij}-\sum_{i\neq j\neq k}^{n}\rho_{ik}\rho_{jk}\right\} \\
Q_n^* &= \frac{(n-1)!}{2\sigma^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\rho_{ij}^2 \\
S_n^* &= \frac{(n-2)!}{\sigma^4}\sum_{i<j}^{n}\rho_{ij}^2 \\
T_n^* &= \frac{2(n-3)!}{\sigma^4}\sum_{i\neq j\neq k}^{n}\rho_{ij}\rho_{ik} \\
M_n^* &= \frac{4!(n-4)!}{\sigma^4}\sum_{i\neq j\neq k\neq l}^{n}(\rho_{ij}\rho_{kl}+\rho_{ik}\rho_{jl}+\rho_{il}\rho_{jk}).
\end{aligned}
$$

and $\Phi(h,\sigma^2)$ and $\Lambda(h,\sigma^2)$'s are defined as in equations (4.3) and (4.4) respectively.

Next, we compute the $100(1-\alpha)\%$ percentile points of maxima for unequal positive or negative correlations for selected values of α = 0.010, 0.025 and 0.050 based on $G_n^{**}(h,\rho_{ij},\sigma^2)$ given in (4.8). In our numerical computations, we actually considered a homoscedastic normal variable case with $\text{var}(y_t) = 1$ for all t but the correlation structures similar to AR(1) with $\phi_1 = \pm 0.100, \pm 0.125, \pm 0.200$ and ± 0.250, and $n = 3$ and 4. For specific value of ϕ_1, we may easily obtain the correlation coefficients $\rho_{ij} = \phi_1^{|i-j|}$ between two repeated variables with lag $|i-j|$ according to the correlation matrix of AR(1) model. For example, for $\phi_1 = 0.250$ and $n = 4$, the correlation coefficients are : $\rho_{12} = 0.250$, $\rho_{13} = 0.0625$, $\rho_{14} = 0.0156$; $\rho_{23} = 0.250$, $\rho_{24} = 0.0625$; and $\rho_{34} = 0.250$. The percentile points h for this situation along with the correlations are shown in Table 4.3. To compute the percentile points of the maxima for the selected values of $\phi_1 = \pm 0.100, \pm 0.125, \pm 0.200$ and ± 0.250, the trial and error method was applied where the initial values of h were chosen from Gupta (1973) for the positive values of

Table 4.3: The SCA based $100(1-\alpha)\%$ percentile points of the maxima for AR(1) data with unit variance and selected ϕ_1, α and n, ϕ_1 being the parameter of the AR(1) process.

n	ϕ_1	ρ_{ij}	$i<j$	$i,j=1,\ldots,n$	$\alpha=$	0.010	0.025	0.050
3	0.100	0.1000	0.0100	0.1000		3.1151	2.5078	2.1858
	0.150	0.1500	0.0225	0.1500		3.1358	2.4897	2.1767
	0.200	0.2000	0.0400	0.2000		3.1251	2.4701	2.1623
	0.250	0.2500	0.0625	0.2500		3.1013	2.4531	2.1467
4	0.100	0.1000	0.0100	0.0010		3.0702	2.5111	2.2522
		0.1000	0.0100	0.1000				
	0.150	0.1500	0.0225	0.0034		3.0231	2.5141	2.2231
		0.1500	0.0225	0.1500				
	0.200	0.2000	0.0400	0.0080		3.0089	2.4913	2.2178
		0.2000	0.0400	0.2000				
	0.250	0.2500	0.0625	0.0156		2.9853	2.4863	2.2056
		0.2500	0.0625	0.2500				
3	-0.100	-0.1000	0.0100	-0.1000		2.4866	2.2601	2.0423
	-0.150	-0.1500	0.0225	-0.1500		2.4229	2.2208	2.0078
	-0.200	-0.2000	0.0400	-0.2000		2.3307	2.1498	1.9807
	-0.250	-0.2500	0.0625	-0.2500		2.2456	2.0898	1.9506
4	-0.100	-0.1000	0.0100	-0.0010		2.6863	2.4456	2.1656
		-0.1000	0.0100	-0.1000				
	-0.150	-0.1500	0.0225	-0.0034		2.5974	2.4112	2.1567
		-0.1500	0.0225	-0.1500				
	-0.200	-0.2000	0.0400	-0.0080		2.5741	2.4009	2.1491
		-0.2000	0.0400	-0.2000				
	-0.250	-0.2500	0.0625	-0.0156		2.5569	2.3721	2.1299
		-0.2500	0.0625	-0.2500				

ρ = 0.100, 0.125, 0.200 and 0.250 respectively. The method was terminated whenever the actual probability was found to be very close to the nominal probability. The convergence of the trial and error method was quick. Only three trials provided the reported percentile points. Note that in computation of the percentile points, it is necessary to compute several infinite series (cf. equation (4.6)). These infinite series converged at different rate. For example, the convergence of the single infinite series was achieved by considering the first 32 terms. Similarly, convergence achieved for the double, triple and quadra infinite series by using r_1 = 21, r_2 = 25; r_1 = 17, r_2 = 18, r_3 =20; and r_1 = 15, r_2 = 17, r_3 = 19,r_4 = 21 respectively.

4.4 Third case : Heteroscedastic but unequally (positively or negatively) correlated normal variables

4.4.1 Application to Antedependence Models

Antedependence models, defined as a more general family of autoregressive models, are often used in socio-economic studies. For a set of n variables in a given order, sth order antedependence is said to hold (cf. Gabriel (1962)) if each variable given at least s preceeding variables in the order, is independent of the remaining variables. Note that due to the finite order antedependence, stationarity restrictions for the first and second order moments are not at all required. This is quite contrary to the standard autoregressive models in time series analysis where the restrictions on variances and correlations are necessary. Thus, in repeated measurement experiments, inferences on the parameters of such antedependence models can be made, even if nonstationarity occurs. To illustrate the situation, Albert (1992) considered medical research data where a growth variable (such as height or weight) is measured on each individual repeatedly over time. For such data, the model often becomes nonstationary as the variance and correlation parameters appearing in such a model can be seen to depend on time.

Let $\mathbf{Y} = (Y_1, \ldots, Y_i, \ldots Y_n)' \sim N_n(0, \Sigma_{(s)})$ be a vector of n-dimentional repeated observations on a single experimental unit through n time periods, where $\Sigma_{(s)}$ indicates the presence of sth ($s > 0$) order antedependence. The s^{th} order ante-dependence model is defined (cf. Albert (1992)) as

$$\begin{aligned} Y_1 &= \delta_1 \eta_1 \\ Y_i &= \sum_{l=1}^{s_i} a_{i,i-l} Y_{i-l} + \delta_i \eta_i \quad , \quad i = 2, \ldots, n \end{aligned} \tag{4.9}$$

where $s_i = min(s, i-1)$, $a_{i,i-l}$, $i = 2, \ldots, n$, $l = 1, \ldots, s_i$, $i > j$, are antedependent parameters, δ_i, $i = 1, 2, \ldots, n$ are n-scale parameters, and the errors η_i are independent and normally distributed with zero mean and unit variance. The above model (4.9) may be expressed as

$$Y = A\eta$$

where

$$A = \Gamma D^*$$

$$\text{with} \quad \Gamma^{-1} \begin{cases} = 1 & l = 1 \\ = -a_{i,i-l} & 1 \neq l \neq s_i \\ = 0 & \text{otherwise} \end{cases}$$

$$\text{and} \qquad D^* \quad = \quad diag(\delta_1, \ldots, \delta_n)$$

It then follows that

$$Disp(Y) \quad = \quad \Sigma_{(s)} = AA' = \Gamma D^* D^{*'} \Gamma' \tag{1.10}$$

To understand the nonstationarity in this an edependence process, let us first consider the special case $s = 1$. The variance-covariance matrix of Y in this case reduces to

$$\Sigma_{(1)} = \begin{pmatrix} \sigma_{11(1)} & \sigma_{12(1)} & \cdots & \sigma_{1n(1)} \\ \sigma_{21(1)} & \sigma_{22(1)} & \cdots & \sigma_{2n(1)} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1(1)} & \sigma_{n2(1)} & \cdots & \sigma_{nn(1)} \end{pmatrix}$$

where

$$\begin{aligned} \sigma_{ii(1)} &= a^2_{i,i-1}\sigma_{(i-1)(i-1)(1)} + \delta_i^2 \quad , \; i = 1, 2, \ldots, n \;\; \text{with} \;\; a_{1,0} = 0 \\ \sigma_{ij(1)} &= a_{j,j-i}\sigma_{i(j-i)(1)} \quad , \; i < j \end{aligned}$$

Notice that all the variance and covariances are functions of different scale and antedependent parameters, those may vary with regard to the change in time. Thus, this antedependece model does not require any variance or covariance stationarity.

Similar situations as for $s = 1$, will arise for other $s > 1$ cases too. For example, for $s = 2$, one may write the variance-covariance matrix of Y, $\Sigma_{(s)} = \{\sigma_{ij(s)}\}$, in terms of the antedependent parameters showing the non-stationarity of the components of Y.

For the sake of simplicity, we, however, present this for the case with $n = 3$. From Equations (4.9), we see that

$$\begin{aligned} Y_1 &= \delta_1\eta_1 \\ Y_2 &= a_{2,1}Y_1 + \delta_2\eta_2 \\ Y_3 &= a_{3,2}Y_2 + a_{3,1}Y_1 + \delta_1\eta_1 \end{aligned}$$

Then it is readily seen that

$$\sigma_{11(2)} = \text{Var}(Y_1) = \text{Var}(\delta_1\eta_1) = \delta_1^2$$

Similarly, one obtains

$$\sigma_{12(2)} = E(Y_1Y_2) = E(Y_1(a_{2,1}Y_1 + \delta_2\eta_2)) = a_{2,1}E(Y_1^2) = a_{2,1}\delta_1^2,$$

$$\sigma_{13(2)} = E(Y_1Y_3) = E(Y_1(a_{3,2}Y_2 + a_{3,1}Y_1 + \delta_3\eta_3)) = a_{3,2}\sigma_{12(2)} + a_{3,1}\sigma_{11(2)} = a_{3,2}a_{2,1}\delta_1^2 + a_{3,1}\delta_1^2,$$

$$\sigma_{22(2)} = \sigma_{22(2)} = E(Y_2^2) = E((a_{2,1}Y_1 + \delta_2\eta_2)^2) = a_{2,1}^2E(Y_1^2) + \delta_2^2E(\eta_2^2) = a_{2,1}^2\delta_1^2 + \delta_2^2,$$

and

$$\sigma_{23(2)} = E(Y_2Y_3) = a_{3,2}\sigma_{22(2)} + a_{3,1}\sigma_{12(2)} = a_{3,2}(a_{2,1}^2\delta_1^2 + \delta_2^2) + a_{3,1}a_{2,1}\delta_1^2$$

As in the case for $s = 1$, the variances and covariances for the second ($s = 2$) order antedependence model also depend on different scale and antedependent parameters, showing the nonstationarity among the components of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$. In the similar fashion, we may show the nonstationarity for the case with $n > 3$ and appropriate s.

In a repeated measurements experiment, there are many situations where antedependence models are used. For example, we consider the calf data analyzed by Kenward (1987). In this problem, the main object is to compare two or more methods for controling the intestinal parasites in cattle. During the grazing season, from spring to autumn, cattle can ingest roundworm larvae, which have developed from eggs previously deposited on the pasture in the faeces of infected cattle. Once infected an animal is deprived of nutrients and its resistance to other disease is lowered, which in turn can greatly affect its growth. In order to control the disease, an infected calf was assigned to a particular treatment. For monitoring the effects of a treatment for the disease, the response of interest, weight, is recorded for an infected calf at n time points and it is examined whether the maximum of these weights ($y_{(n)}$) are less than a standard weight (h) of an uninfected calf of same age (at the initial level of the experiments). That is, we require to compute the probability $\Pr(Y_{(n)} \leq h)$, for known h, which indicates the failure of the treatment. Alternatively, one may find the probability $\Pr(Y_{(1)} \geq h)$ to see whether the treatment is working effectively. Here the observations $y_1, y_2, \ldots, y_n$ will most likely be a realization of the sample $Y = (Y_1, Y_2, \ldots, Y_n)$ that follow the antedependence (nonstationary) model given in (4.9), as the weights are likely to vary with repeated time (equally or unequally spaced).

Under the assumption that time points are far apart from each other such that the correlations are small, we may directly use the probability density function developed in (3.27) to compute the distribution function of $Y_{(n)}$, that is, $\Pr(Y_{(n)} \leq h)$ just by using σ_i^2 for $\sigma_{ii(s)}$ and ρ_{ij} for $\sigma_{ij(s)}/\{\sigma_{ii(s)}\sigma_{jj(s)}\}^{1/2}$, $i, j = 1, 2, \ldots, n$. The distribution function of $Y_{(n)}$, in such cases, reduces to

$$\begin{aligned} G_n^{**}(h, \rho_{ij}, \sigma_i^2) \;\simeq\; & U_n^*\Phi(h) + D_n^* \sum_{i<j}^{n} \Lambda_{ij(kl)}^{1100}(h) - Q_n^* \sum_{i=1}^{n} \Lambda_{i(ikl)}^{2000}(h) \\ & + S_n^* \sum_{i<j}^{n} \Lambda_{ij(kl)}^{2200}(h) + T_n^* \sum_{i\neq j\neq k}^{n} \Lambda_{ijk(l)}^{2110}(h) \\ & + M_n^* \sum_{i\neq j\neq k\neq l}^{n} \Lambda_{ijkl}^{1111}(h) \qquad (4.11) \end{aligned}$$

where the coefficients $U_n^*, D_n^*, Q_n^*, S_n^*, T_n^*$ and M_n^* are defined as in theorem 3.1, and $\Phi(h)$

and λ's are defined as in equations (4.3) and (4.6) respectively.

4.4.1.1 Percentile Points of $Y_{(n)}$ for antedependence model

4.4.1.1.1 Computation of a correlation structure

Before computing the percentile points of the maxima, that we first compute the correlation coefficients among the repeated observations those are generated following an antedependence model of order $s = 1$ and 2 having the variance-covariance matrix discussed in the previous section. In this numerical computation, we consider $n = 3$ and 4, and scale parameters of the antedependence model $\delta_i = 1$ for all i. We also consider the values of the antedependent parameters $a_{i,i-l}$ as $a_{i,i-l} \leq 0.250$ for $i = 2, \ldots, n, l = 1, \ldots, s_i, i > l$ so that all possible values of ρ_{ij} are small, namely, $\rho_{ij} \leq 0.250$ for $i \neq j$. More specifically, the values of ρ's and variances for selected values of a's are shown in Table 4.4. In the next section, we compute the percentile points of the maxima for this antedependence correlations set-up.

4.4.1.1.2. Computation of 5% Critical Values of $Y_{(n)}$: An Application of Small Correlations Approach

We now compute the 95% percentile points of $Y_{(n)}$ for $n = 3$ and 4 for the antedependent correlation structures with heteroscedastic variances reported in Table 4.4. This computation is done by using the distribution function $G_n^{**}(h, \rho_{ij}, \sigma_i^2)$ given in equation (4.10). In the manner similar to the homoscedastic normal variable cases, the trial and error method was used to compute the percentile points of the maxima. This trial and error method requires the initial values of h, a critical value of Y_n. In selecting such values for the cases where $\rho_{ij} \leq 0.1$, for example, we have chosen h from Gupta (1973) for the case with $\rho = 0.1$. The percentile points are shown in Table 4.5.

Note that the h values computed for $\rho_{ij} \leq \rho_0$ generally different from h values obtained

Table 4.4: The variances and correlations pattern for antedependence models for selected values of α and n with $\delta_i = 1$ $(i = 1, \ldots, n)$ and $s = 1$ and 2

n	s	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{3,2}$	σ_1^2	σ_2^2	σ_3^2	ρ_{12}	ρ_{13}	ρ_{23}
		$\alpha_{4,2}$	$\alpha_{4,3}$		σ_4^2			ρ_{14}	ρ_{24}	ρ_{34}
3	1	0.0679		0.1000	1.0	1.0046	1.0100	0.0677	0.0068	0.0997
		0.1500		0.1267	1.0	1.0225	1.0164	0.1483	0.0189	0.1271
		0.1765		0.2000	1.0	1.0311	1.0412	0.1738	0.0346	0.19901
		0.2500		0.2301	1.0	1.0625	1.0563	0.2425	0.0559	0.2308
3	2	0.1000	0.0675	0.0812	1.0	1.0100	1.0123	0.0995	0.0752	0.0878
		0.1278	0.0978	0.1312	1.0	1.0163	1.0303	0.1268	0.1129	0.1425
		0.2000	0.1111	0.1624	1.0	1.0400	1.0469	0.1961	0.1403	0.1832
		0.2134	0.1111	0.2308	1.0	1.0455	1.0789	0.2087	0.1543	0.2195
4	1	0.0875		0.0657	1.0	1.0077	1.0044	0.0872	0.0057	0.0658
			0.1000		1.0104			0.0006	0.0066	0.0997
		0.1175		0.1347	1.0	1.0138	1.0181	0.1167	0.0157	0.1343
			0.1500		1.0229			0.0023	0.0201	0.1497
		0.2000		0.1765	1.0	1.0400	1.0320	0.1961	0.0347	0.1771
			0.1596		1.0260			0.0056	0.0284	0.1601
		0.2156		0.2512	1.0	1.0465	1.0660	0.2108	0.0525	0.2488
			0.1961		1.0409			0.0104	0.0494	0.1984
4	2	0.1000	0.0671	0.0789	1.0	1.0100	1.0118	0.0995	0.0745	0.0855
		0.0834	0.0992		1.0184			0.0156	0.0915	0.1059
		0.1134	0.0856	0.1371	1.0	1.0129	1.0290	0.1127	0.0997	0.1455
		0.0978	0.1292		1.0306			0.0238	0.1157	0.1432
		0.1198	0.0966	0.1371	1.0	1.0224	1.0325	0.1481	0.1154	0.1505
		0.1145	0.1792		1.0529			0.0372	0.1395	0.1944
		0.1568	0.1073	0.1831	1.0	1.0246	1.0520	0.1549	0.1326	0.1969
		0.1145	0.2292		1.0791			0.0473	0.1561	0.2482

Table 4.5: The SCA based 95% percentile points of the maxima for antedependence data with order $s = 1, 2$ and selected ρ_{ij}, σ_i^2 and n

n	s	σ_1^2	σ_2^2	σ_3^2	σ_4^2	ρ_{12} ρ_{14}	ρ_{13} ρ_{24}	ρ_{23} ρ_{34}	h
3	1	1	1.0046	1.0100		0.0677	0.0068	0.0997	2.1598
		1	1.0225	1.0164		0.1483	0.0189	0.1271	2.1423
		1	1.0312	1.0112		0.1738	0.0346	0.1990	2.1357
		1	1.0625	1.0562		0.2425	0.0559	0.2308	2.1283
3	2	1	1.0100	1.0123		0.0995	0.0752	0.0878	2.1512
		1	1.0163	1.0303		0.1268	0.1129	0.1125	2.1398
		1	1.0400	1.0469		0.1961	0.1403	0.1931	2.1257
		1	1.0455	1.0789		0.2087	0.1543	0.2495	2.1167
4	1	1	1.0077	1.0044	1.0104	0.0872	0.0057	0.0658	2.2409
						0.0006	0.0066	0.0997	
		1	1.0138	1.0184	1.0229	0.1167	0.0157	0.1343	2.2345
						0.0023	0.0201	0.1497	
		1	1.0400	1.0320	1.0260	0.1961	0.0317	0.1771	2.2256
						0.0056	0.0284	0.1601	
		1	1.0465	1.0660	1.0409	0.2108	0.0525	0.2489	2.2043
						0.0101	0.0491	0.1984	
4	2	1	1.0100	1.0118	1.0184	0.0995	0.0745	0.0854	2.2335
						0.0156	0.0915	0.1059	
		1	1.0129	1.0290	1.0306	0.1126	0.0997	0.1455	2.2226
						0.0238	0.1157	0.1432	
		1	1.0224	1.0325	1.0529	0.1481	0.1153	0.1505	2.2177
						0.0371	0.1395	0.1944	
		1	1.0246	1.0520	1.0794	0.1549	0.1326	0.1969	2.2112
						0.0473	0.1561	0.2482	

in Gupta (1973) for $\rho = \rho_0$. More specifically, our calculations show that for $n = 3$, the h's for the cases with unequal correlations are greater than the h's for the cases with equal correlatons. On the other hand, for $n = 4$, some of the h values for the cases with unequal correlations are greater than the h values for the cases with equal correlations and some of them are less than the h's for the cases with equal correlations. In the next section, we examine the performance of the proposed approximation (small correlations approach) by conducting a small simulation study.

4.4.1.2 Verification of Critical Values : A Simulation Study

To examine the accuracy of the critical values shown in Table 4.5, in this subsection, we conduct a small simulation study. To do this, we generated 5000 clusters of sizes $n = 3, 4$ from normal distributions with zero mean and the variance-covariance matrix of the antedependence model of order $s = 1, 2$ (discussed in section 4.3.1). Note that in generating the clusters, we have considered the same mean and the same variance-covariance matrix that were used to compute the percentile points h given in Table 4.5 for different cases. We now postulate as the null hypothesis (H_0) that the sample does not contain any extreme observation, where any observations greater than or equal to h value (taken from Table 4.5) is considered as an extreme observation. Under a given set-up that is for a selected n and a set of ρ's for these n repeated observations, we first observe whether the maximum of y_{li} is greater than or equal to the percentile points h where y_{li} is the ith ($i = 1, \ldots, n$) observation in the lth ($l = 1, 2, \ldots, 5000$) cluster. For a given set-up, we then compute the proportion of the simulation runs which satisfy $max_{1 \le i \le n}(y_{li}) \ge h$, where $max_{1 \le i \le n}(y_{li})$ is the maxima of the n observation in the lth cluster. These proportions, commonly called as the size of the test for testing the H_0, are reported in column four of Table 4.6 for nominal $\alpha = 0.05$. Here, nominal $\alpha = 0.05$

means that h was chosen from Table 4.5 such that $Pr\{Y_{(n)} \leq h\} = 0.95$. Note that in general, the actual size of the test appears to be close to 5% in almost all selected cases.

We also verify the performance of our small correlation approach in computing the critical values, as was done in the previous simulation study, under the alternative hypothesis H_a : there is one extreme observation in the cluster. For the purpose, we first generate 5000 clusters of sizes n from normal distributions with same mean (zero) and the same variance-covariance matrix of the antedependence model of order $s = 1, 2$ as under the case of H_0. We then add $\theta = 4$ and 5 with a pre-selected ith ($i = 1, \ldots, n$) observation in the l cluster. More specifically, we generate the new i'th ($i' = 1, \ldots, n$) observation for the lth cluster such that $y^*_{li'} = y_{li} + \theta$ for $i' = i$, and $y^*_{li'} = y_{li'}$ for $i' \neq i$. When $\theta = 0$, the ith observation is not an extreme observation. Now, similar to the case for the size of the test, we compute the proportion of simulation runs which satisfies $max_{1 \leq i \leq n}\{y_{li}\} \geq h$, where $max_{1 \leq i \leq n}\{y_{li}\}$ is the maxima of the n observation in the lth cluster. These proportions under the different situations with $\theta = 4$ and 5 are shown, respectively, in columns five and six of Table 4.6, which, infact, indicate the powers of the test for testing H_0, that is, there is no extreme observation in the sample. Note that the computed powers for both cases with $\theta = 4$ and 5 appear to quite high.

4.4.2 Computation of Critical Values for Heteroscedastic Case: Small Correlations Approach Versus Bound Approximation

In the linear regression analysis, the maximum studentized residual test statistic and the maximum normed residual test statistic are widely used for the detection of a sin-

Table 4.6: Sizes and powers of the test at 5% level of significance for selected h and n for testing H_0 that there is no extreme observation in the sample, based on 5000 simulations

n	s	h	sizes	powers for $\theta = 5$	powers for $\theta = 4$
3	1	2.1598	0.0466	0.9966	0.9664
		2.1423	0.0488	0.9970	0.9670
		2.1357	0.0504	0.9963	0.9668
		2.1283	0.0530	0.9970	0.9672
3	2	2.1512	0.0482	0.9980	0.9690
		2.1398	0.0500	0.9980	0.9698
		2.1257	0.0506	0.9982	0.9706
		2.1167	0.0510	0.9982	0.9712
4	1	2.2409	0.0536	0.9956	0.9668
		2.2345	0.0546	0.9954	0.9664
		2.2246	0.0564	0.9956	0.9668
		2.2043	0.0608	0.9958	0.9680
4	2	2.2335	0.0512	0.9954	0.9662
		2.2226	0.0532	0.9954	0.9666
		2.2177	0.0559	0.9954	0.9666
		2.2112	0.0580	0.9954	0.9672

gle outlier or influential observation. For these two statistics, the exact critical values are not easy to compute. Ellenberg (1973, 1976) has approximated the critical values of the maximum studentized residual test only by using the first Bonferroni bounds. Similar to the studentized residual test statistic, Stefansky (1971, 1972) has developed the bounds for the percentage points for the maximum normed residual test statistic under the assumption that errors in the linear regression model are homoscedastic. It is known that for this homoscedastic case, these bounds approximation work well. There is, however, no adequate discussion in the literature about the performance of bound approximation for the heteroscedastic case. Furthermore, the accurate computations for bound approximation require the computation of the joint probability which may not be easy to compute, see for example Chu and Sutradhar (1995).

In this section, we compare the performance of bound approximation with our small correlations approach, in computing the percentile points or p-values for the maxima. We do this for general unequal correlations cases with both equal and unequal variances. For this purpose, we confined our discussion to the nonregression situation, for simplicity, and we first compute the h values such that $Pr\{Y_{(n)} \geq h\} = \alpha$ by using our small correlations approach, as in the previous sections. These h values are then used to examine the performance of the Bonferroni bounds approximations. The upper and lower bounds, as functions of h values, are defined as follows :

$$LB(h, \sigma_i^2) = \sum_{i=1}^{n} Pr(Y_i \geq h) - \sum_{i<j}^{n} Pr(Y_i \geq h, Y_j \geq h) \qquad (4.12)$$

and

$$UB(h, \sigma_i^2) = \sum_{i=1}^{n} Pr(Y_i \geq h) \qquad (4.13)$$

where $E(Y_i^2) = \sigma_i^2$, and Y_i and Y_j are correlated such that $E(Y_i Y_j) = \rho_{ij}$ with $E(Y_i) = 0$.

Now by using Z_i for $\frac{Y_i}{\sigma_i}$, we obtain the lower bound as

$$LB(h,\sigma_i^2) = \sum_{i=1}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}) - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j})$$

$$= \sum_{i=1}^{n}[1 - F(\frac{h}{\sigma_i})] - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j}) \quad (4.14)$$

$$(4.15)$$

and the upper bound is given by

$$UB(h,\sigma_i^2) = \sum_{i=1}^{n}[1 - F(\frac{h}{\sigma_i})] \quad (4.16)$$

with $$Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j}) = \int_{\frac{h}{\sigma_j}}^{\infty}\int_{\frac{h}{\sigma_i}}^{\infty} f(z_i, z_j; \rho_{ij}) dz_i dz_j \quad (4.17)$$

which is cumbersome to compute directly. Abrahamson (1965) seems to mention about the solution of this integration for selected values of $\frac{h}{\sigma_i}$, $\frac{h}{\sigma_j}$ and ρ_{ij} but it's solution is not available from his paper.

Similar to the case when $E(Y_i^2) = \sigma_i^2$ for all i, we also obtain the lower and upper bounds, for the cases with $E(Y_i^2) = \sigma^2$ $(i = 1, 2, \ldots, n)$ as

$$LB(h,\sigma^2) = n[1 - F(\frac{h}{\sigma})] - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma}, Z_j \geq \frac{h}{\sigma})$$

$$= n[1 - F(\frac{h}{\sigma})] - \sum_{i<j}^{n} \Phi_2(\frac{h}{\tau}, \frac{h}{\sigma}, \rho_{ij}) \quad (4.18)$$

and $$UB(h,\sigma^2) = n[1 - F(\frac{h}{\sigma})] \quad (4.19)$$

For $\Phi_2(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij})$, we obtain an expression provided by Greig (1967) as

$$\Phi_2(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij}) = [1 - (1 - \rho_{ij})^{\frac{1}{2}}]\Phi_1(\frac{h}{\sigma}) + (1 - \rho_{ij})^{\frac{1}{2}}]\Phi_1^2(\frac{h}{\sigma})$$

with $$\Phi_1(\frac{h}{\sigma}) = [1 - F(\frac{h}{\sigma})]$$

Next, to compute the p-values of percentile points h such that $Pr\{Y_{(n)} \geq h\} = 0.05$ for general unequal correlations with equal and unequal variance cases,

Table 4.7: The SCA based critical values h for selected p-values and the corresponding Bonferroni upper and lower bounds for selected ρ_{ij} and n for homoscedastic normal variable case

n	σ^2	ρ_{12}	ρ_{13}	ρ_{23}	h	p-value	LB(h)	UB(h)
3	1.10	0.1000	0.0100	0.1000	2.2667	0.050291	0.055770	0.059006
		0.1500	0.0225	0.1500	2.2438	0.049813	0.057373	0.062053
		0.2000	0.0400	0.2000	2.1989	0.049611	0.061697	0.068413
		0.2500	0.0625	0.2500	2.1532	0.048262	0.066191	0.075141
	2.15	0.1000	0.0100	0.1000	4.4297	0.050230	0.055812	0.059051
		0.1500	0.0225	0.1500	4.3918	0.048976	0.056985	0.061624
		0.2000	0.0400	0.2000	4.3115	0.048059	0.060795	0.067389
		0.2500	0.0625	0.2500	4.2101	0.048583	0.066080	0.075312
	3.20	0.1000	0.0100	0.1000	6.6001	0.050666	0.055519	0.058736
		0.1500	0.0225	0.1500	6.5012	0.049813	0.058190	0.063287
		0.2000	0.0400	0.2000	6.3912	0.049611	0.061948	0.068697
		0.2500	0.0625	0.2500	6.2591	0.048906	0.066415	0.075703
	4.25	0.1000	0.0100	0.1000	8.7789	0.050065	0.055112	0.058296
		0.1500	0.0225	0.1500	8.5913	0.050759	0.059898	0.064845
		0.2000	0.0400	0.2000	8.4478	0.050188	0.063327	0.070264
		0.2500	0.0625	0.2500	8.2796	0.050286	0.067605	0.077097

Table 4.8: The SCA based critical values h for selected p-values and the corresponding Bonferroni upper bounds and p-values for selected ρ_{ij} and n for heteroscedastic case

n	σ_1^2	σ_2^2	σ_3^2	ρ_{12}	ρ_{13}	ρ_{23}	h	p-value	UB(h)
2	1.00	3.75		0.100			6.3004	0.0192	0.0465
				0.150			6.4231	0.0494	0.0434
				0.200			6.5989	0.0498	0.0392
				0.250			6.8901	0.0491	0.0331
	4.25	5.75		0.100			10.1389	0.0498	0.0475
				0.150			10.2409	0.0506	0.0454
				0.200			10.4915	0.0505	0.0408
				0.250			10.8109	0.0507	0.0355
	2.00	12.00		0.100			20.2169	0.0491	0.0460
				0.150			20.4231	0.0503	0.0444
				0.200			20.9649	0.0506	0.0403
				0.250			21.7832	0.0506	0.0347
3	1.00	1.26	2.50	0.068	0.007	0.100	3.9368	0.0509	0.0585
				0.118	0.019	0.127	3.5423	0.0490	0.0808
				0.174	0.035	0.199	3.3287	0.0511	0.0958
				0.243	0.056	0.231	2.9427	0.0499	0.1305
	1.10	2.15	3.20	0.100	0.010	0.100	5.2734	0.0482	0.0568
				0.150	0.023	0.150	4.9719	0.0505	0.0705
				0.200	0.040	0.200	4.6598	0.0495	0.0787
				0.250	0.063	0.250	4.4409	0.0496	0.1021

we used the distribution functions $G_n^{**}(h, \rho_{ij}, \sigma^2)$ and $G_n^{**}(h, \rho_{ij}, \sigma_{ij}^2)$ given in equations (4.8) and (4.9) respectively. For the same percentile points h, we also compute the upper and lower bounds by using the Bonferroni bounds approximations in equations (4.18) - (4.19) for homoscedastic normal variable cases. In the heteroscedastic normal variable cases, the computation of the lower bound is slightly complicated. Consequently, we have used the upper bound given by (4.16) to compare the bound approximation with our SCA based results. It is interesting to note that our numerical computations show that in some situations the upper bounds are seen to be lower than 0.05, indicating that the lower bounds calculations are not necessary in such cases. In other situations, however, the lower bound calculations would have been much better representative bounds than the upper bounds, but they were not calculated because of the technical difficulty as mentioned above. The results are reported, respectively, as in Table 4.7 and Table 4.8 for equal and unequal variance cases.

Note that in this homoscedastic and heteroscedastic normal variable cases, the bounds for different percentile points h based on the Bonferroni approach are deviated from nominal probability at 0.05. Furthermore, it was found that this deviation increases as variances and correlations increase in general. But the corresponding p value for the same percentile point based on our small correlations approach were found to be very close to the nominal probability 0.05.

Chapter 5

Summary and Some Topics for Further Research

5.1 Summary

Correlated data arise in many applications in statistics. In this thesis, we have discussed order statistics inferences for normal random variables with a general correlation structure, where correlations can be unequal or equal, positive or negative. More specifically, we have provided the distribution of the rth order statistic for any $r = 1, \ldots, n$, n being the sample size or number of correlated variables. Special attention was given to the derivations for the distributions of maxima and minima. For all of these derivations, we have adopted a small correlations based Taylor's series approach, that is, our results are valid for general correlation structures but the absolute magnitude of correlations should be small (not exceeding 0.25). This small correlations approach will have applications to many areas, mainly in clustered data analysis. This is because, as argured in the thesis, in familial clustered data, the cluster sizes are generally small and the correlations between the observations are generally small too.

For positive equi-correlated cases, our results have been compared with the existing results due to Gupta (1973), and others. It was found that the proposed approach works quite well where correlations are small, as expected. For unequal positive and negative correlated cases, the proposed small correlations approach also works quite well, which has been verified by a limited simulation study. For these unequal correlations cases, we have discussed two special situations. In the first situation, homoscedastic normal variables with unequal correlations have been considered and the percentile points of the maxima were derived for a well known AR(1) correlation process. In the second situation, we have considered the heteroscedastic random variables with unequal correlations and similar results were derived for antedependence (nonstationary) process.

Note that to compute the percentile points of the maxima or minima in these cases, one may also use the well-known Bonferroni bounds approximation. We have compared our results with these approximations and found that the Bonferroni bounds approximation does not work well for the heteroscedastic cases, whereas our small correlations approach works well for both homoscedastic and heteroscedastic cases.

5.2 Topics for Further Research

We remark that in the present thesis, we have developed a small correlations approach to find the percentile points of the distribution of a single order statistic, which has applications to certain clustered familial data. In practice, however, there are other situations, where the correlations among the variables may be high (cf. Kenward (1987)). To develop methodologies to handle high correlations appear to be extremely difficult, and further investigations are needed.

Second, in this thesis, it was assumed that the scale parameters (correlations as well as variances) are known. For the situations where these parameters are unknown, one needs to obtain consistant estimates of these parameters and carefully study the effects of estimation on the required distributions.

Furthermore, the present methodology may be extended to the analyses of the linear models with several covariates, but this is beyond the scope of the present study.

Chapter 6

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