ON THE ORDER STATISTICS FROM CORRELATED NORMAL DISTRIBUTION

KRISHNA KANTA SAHA
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By

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Abstract

The inferences for the order statistics for normal random variables with a general correlation structure, where correlations can be unequal or equal, positive or negative are discussed in this thesis. Specifically, based on a small correlations approach, we, first, develop the joint density function of the order statistics under the general correlation set-up. We, then, provide an approximation for the distribution of a single order statistic under the same correlation set-up. Special attention is given to the derivations for the distributions of the maxima and minima. The computational aspects of the distribution of the maxima, for example, are discussed in details for the homoscedastic equi-correlation, homoscedastic unequal correlations, and heteroscedastic unequal correlations cases. The applications of the proposed small correlations approach to compute the percentile points of the maxima are shown for the homoscedastic correlated normal variables following a stationary auto-regressive process of order one, and for the heteroscedastic correlated normal variables following a nonstationary antedependence model. Furthermore, the small correlations approach for the maxima is compared with the Bonferroni bounds approximation for unequally homoscedastic and heteroscedastic correlated normal variables.
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Chapter 1

Introduction

1.1 Motivation of the problem

Over the last few decades, order statistics inferences for the independent normal variables have been widely discussed in the literature. For example, we refer to David (1981), Galambos (1987), Balakrishman & Cohen (1991), Reiss (1989), Gumbel (1958), Barnett & Lewis (1978) for such inferences. In practice, order statistics problems arise in many practical situations, for example, in the context of flood control in a given place. In a flood control problem, let $Y_1, Y_2, \ldots, Y_j, \ldots, Y_n$ be the yearly water levels of a river for $n$ years, where $Y_j$ denotes the water level of the river for the $j$th year. For this kind of yearly data, it is reasonable to assume that they are mutually independent. In order to take any remedial measure in preventing the future flood spread, it is important to study the pattern of the maximum water level of the river. On the other hand, if we want to use the river as a source of electrical energy in order to maintain a minimum level, then our
interest would be to study the pattern of minimum water level of the river. In notations, the former problem requires to find $Pr(Y_{(n)} \leq a)$, and the latter problem requires to find $Pr(Y_{(1)} \geq b)$, where $Y_{(n)} = \max(Y_j, 1 \leq j \leq n)$, $Y_{(1)} = \min(Y_j, 1 \leq j \leq n)$ and $a$ & $b$ are the specified water levels.

But, in practice, there are also many situations where the observations may be correlated but still follow the Gaussian distribution. For example, consider the widespread production of rice in Hungary (say), for which a minimum amount of rainfall per week is needed. Similarly, consider the production of potatoes in West Africa (say) where the production may be affected if the daily high temperatures exceed a maximum amount of daily temperature. Here, let $Y_j, j = 1, 2, \ldots, n$ be the amount of rainfall in Hungary or the maximum temperature in West Africa during the $j$th, $j = 1, 2, \ldots, n$ week or day. It then follows that in the rice production problem one would study the pattern of $Y_{(1)} = \min(Y_j), 1 \leq j \leq n$ and in the potato production problem one would be interested to study the pattern of $Y_{(n)} = \max(Y_j), 1 \leq j \leq n$. Note, however, that in both of these problems, as the observations are collected successively over time, week or day, it is reasonable to assume that the original observations $Y_1, Y_2, \ldots, Y_n$ are correlated random variables. Consequently, these problems reduce to the order statistics problems where observations are correlated.

Furthermore, under the cases when variables are correlated, most of the order statistics problems discussed so far in the literature belong to equi-correlated case. For such analyses, we refer to Gupta, Pillai & Steck (1961), Gupta, Nagel & Panchapakesan (1973), Owen & Steck (1962), Hoffman & Saw (1975), Rawlings (1976), among others. Gupta, Pillai and Steck (1964) considered the distributions of a linear function and ratio of two linear functions of order statistics from an equally correlated set of normal random
variables. Later on, Gupta, Nagel & Panchapakesan (1973) have studied the distribution theory of the maxima which arises in the context of ranking and multiple comparison problems. More specifically, these authors have discussed some general distribution theory for certain order statistics from correlated normal random variables with a special correlation structure $\rho_{ij} = \rho \geq 0$, where $\rho_{ij}$ is the correlation coefficient between $Y_i$ and $Y_j$, for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.

The order statistics inferences for the equi-correlated normal random variables, was also studied by Owen and Steck (1962). In their study, they have shown how the marginal moments and product moments of the order statistics may be obtained from the corresponding moments and product moments for the independence case, $\rho = 0$. Rawlings (1976) studied the distribution of the maxima for such equi-correlated random variables. More specifically, Rawlings extended the distribution of the maxima of one group of $k$-dimensional equi-correlated variables studied by Gupta (1973) to the case of $m$ independent groups of $k$-dimensional equicorrelated random variables.

Unlike the above authors, Hoffman and Saw (1975) attempted to include the negative equicorrelated case in finding the distribution function of the maxima that requires certain integrations in the complex domain, which may not be easy in general.

There has also been a few studies on the order statistics inferences, where correlations may be unequal. But, they were done for very special correlations structure. For example, by expressing a multivariate probability integral as a power series of the univariate probability integral, Greig (1967) has provided an approximation to the distribution of extreme values in correlated normal population, for a very special situation when the correlation matrix has dominant elements adjacent to the leading diagonal with $\rho_{i+1, i} \neq 0$ but $\rho_{ij} = 0$ for all $i \neq j$ & $j \neq i + 1$. 
With regard to the detection of outliers in a simple regression set up, Ellenberg (1973, 1976) has used Bonferroni inequalities approach to compute the limits for the probability of maxima of standardized least squares residuals. Here in this problem, the residuals have unequal correlations based on the structure of the design matrix involved in the linear model.

Observe that all the studies mentioned above deal with either equicorrelated or special types of unequal correlated structures. But, as in reality, the normal variables can be unequally positively or negatively correlated, in this thesis, we deal with such general correlation structures and study the order statistics inferences for such cases. The specific plan of the thesis is as follows:

1.2 Objective of the Thesis

1. In chapter two, we provide detailed background of order statistics problems for correlated normal data.

2. Chapter 3 concerns the inferences for the order statistics obtained from unequal positively or negatively correlated normal variables. In Section 3.1, based on a small correlations approach, we develop the joint density function for the order statistics for correlated normal variables with general correlation structure. An approximate marginal distribution of a single order statistic is simplified in Section 3.2. In Section 3.3, we provide the distributions of maxima and minima as two special cases.

3. The computational aspects for the percentile points of the maxima for normal
variables with general correlation structures are given, in chapter 1, for three special situations. First, in Section 1.2, we discuss the computation of the percentile points of the maxima for homoscedastic equi-correlated (positive or negative) normal variables. As our results are obtained based on small correlations approach, we compare them with the existing results due to Gupta (1973) for positive equi-correlated cases. Our results appear to agree quite well with those in Gupta (1973) for small correlations. In Section 1.3, the percentile points of the maxima are computed for the homoscedastic but unequally correlated (positive or negative) normal variables case. This we have done in the context of auto-regressive process of order one, where variances (of the variables) are equal and correlations are unequal following a decaying pattern for increasing lags. Next, in Section 1.4, we discuss the computation of the percentile points of the maxima for the heteroscedastic but unequally positively or negatively correlated normal variables case. Unlike the last case, we have done this in the context of antedependence (nonstationary) models. To assess the adequacy of our small correlations approach for the maxima in this case, a limited simulation study is also carried out.

Furthermore, we compare our approach with the well-known Bonferroni bounds approximation for both homoscedastic and heteroscedastic cases.

1. Chapter 5 contains the summary of the present work and provides some suggested topics for further research.
Chapter 2

Background of Order Statistics

Problems For Correlated Normal Data

In order to make inferences for order statistics from an equally correlated set of normal random variables, Owen and Steck (1962) showed that the moments and product moments of the order statistics for normal variables for any $\rho$, $\rho$ being the equi correlation coefficient between any two variables, can be obtained from the corresponding moments and product moments of the order statistics for independent ($\rho = 0$) normal variables. Suppose that $X_1, X_2, \ldots, X_n$ are independently and normally distributed random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$, for all $i = 1, 2, \ldots, n$. Also suppose that $X_0$ is another standardized normal variable but with $E(X_0X_i) = 0$, for $\rho > 0$ and $E(X_0X_i) = -(-\rho)^{\frac{1}{2}}/(1 - \rho)^{\frac{1}{2}}$, for $\rho < 0$. Let $Y_1, \ldots, Y_i, \ldots, Y_n$ be the $n$ correlated
random variables such that \( E(Y_i) = 0, \ E(Y_i^2) = 1 \) and \( E(Y_iY_j) = \rho \), for all \( i \neq j \). Now to obtain the marginal as well as product moments of these correlated random variables \( Y_i \ (i = 1, 2, \ldots, n) \). Owen & Steck expressed \( Y_i \ (i = 1, 2, \ldots, n) \) as a function of the standard normal random variables \( X_0, X_1, \ldots, X_n \), given by

\[
Y_{(i)} = \rho^{\frac{1}{2}} X_0 + (1 - \rho)^{\frac{1}{2}} X_{(i)}
\]  \hspace{1cm} (2.1)

where \( X_{(i)} \) and \( Y_{(i)} \) be, respectively, the \( i \)th order statistic of the sample \((X_1, \ldots, X_n)\) and \((Y_1, \ldots, Y_n)\). Based on this transformation, they obtained the characteristic function of \( Y_i \) as well as the joint characteristic function of \( Y_i \& Y_j, i \neq j \), which were then exploited to compute

\[
E(Y_{(i)}) = (1 - \rho)^{\frac{1}{2}} E(X_{(i)}).
\]
\[
E(Y_{(i)}^2) = \rho + (1 - \rho)E(X_{(i)}^2).
\]
\[
E(Y_{(i)}^3) = 3\rho(1 - \rho)^{\frac{1}{2}} E(X_{(i)}) + (1 - \rho)^{\frac{3}{2}} E(X_{(i)}^3).
\]
\[
E(Y_{(i)}^4) = 3\rho^2 + 6\rho(1 - \rho)E(X_{(i)}^2) + (1 - \rho)^2 E(X_{(i)}^4).
\]

and

\[
E(Y_{(i)} Y_{(j)}) = \rho + (1 - \rho)E(X_{(i)} X_{(j)}).
\]

These authors provided the means, standard deviations, and the third and the fourth central moments of \( Y_{(i)} \), in tabular form, for selected values of \( n \) and \( \rho \).

Using the above transformation (2.1), Gupta, Pillai & Steck (1964) have derived the distributions of the linear function \( Z = \sum_{i=1}^{n} a_i Y_{(i)} \). More specifically, they obtained the distribution function of \( Z \) in terms of the distribution function of \( \sum_{i=1}^{n} a_i X_{(i)} \) given by

\[
P_r\{Z \leq z\} = P_r\{\sum_{i=1}^{n} a_i X_{(i)} \leq -\left[p/(1 - \rho)\right]^{\frac{1}{2}}(\sum_{i=1}^{n} a_i)X_0 + z/(1 - \rho)^{\frac{1}{2}}\}.
\]

(2.2)

for \( \rho \geq 0 \), and
\[
Pr\{Z \leq z\} = Pr\left\{ \sum_{i=1}^{n} a_i n_i \leq -[\rho/(1-\rho)]^{1/2} \left( \sum_{i=1}^{n} a_i \right) X_0 + z/(1-\rho)^{1/2} \right\}, \quad (2.3)
\]
for \( \rho < 0 \).

There are, however, some practical situations where correlations may be different. For example, in familial analysis, it may be necessary to study the order pattern among \( n \) family members, where the variable under consideration for these family members may be unequally correlated. Let \( \rho_{ij} \), for all \( i \neq j \) be the correlation coefficient between \( ith \) & \( jth \) members of the family. For such a general situation with correlation structure \( E(Y_iY_j) = \rho_{ij} \ (i \neq j) \), Gupta et al. (1961) also obtained the distribution function of the range \( W = \max Y_i - \min Y_i \) of correlated normal random variables for \( n = 3, 4 \) based on the \( V \)-function described by Nicholson (1943). For the trivariate case, the distribution function for \( W \) was given as

\[
Pr(W \leq w) = -2\left[ V\left( \frac{w}{a_{12}}, \frac{w\theta_{12}}{a_{12}\sqrt{(1 - \theta_{12}^2)}} \right) + V\left( \frac{w}{a_{13}}, \frac{w\theta_{13}}{a_{13}\sqrt{(1 - \theta_{13}^2)}} \right) + V\left( \frac{w}{a_{23}}, \frac{w\theta_{23}}{a_{23}\sqrt{(1 - \theta_{23}^2)}} \right) \right], \quad (2.1)
\]

where \( V(l, m) \) is the \( V \)-function described by Nicholson (1943), and

\[
a_{ij} = \sqrt{2}(1 - \rho_{ij})^{1/2},
\]

\[
\theta_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23},
\]

\[
\theta_{13} = -(1 + \rho_{23} - \rho_{12} - \rho_{23})/a_{12}a_{13},
\]

\[
\theta_{23} = -(1 + \rho_{12} - \rho_{13} - \rho_{23})/a_{13}a_{23},
\]

and \( \rho_{ij} \) are the correlation coefficients.
Later on, Gupta, Nagel & Panachakesan (1973) presented the cumulative distribution function of maxima, but for equal correlations $\rho \geq 0$. Based on the transformation $Y_{(i)} = \rho^{\frac{1}{2}} X_0 + (1 - \rho)^{\frac{1}{2}} X_{(i)}$ as discussed above, they derived the distribution function of $Y_{(n)} = \max_x(Y_i, 1 \leq i \leq n)$ as

$$F_n(h; \rho) \equiv \Pr \{ Y_{(n)} \leq h \} = \int_{-\infty}^{h} \Phi^{n-1}((y\rho^{\frac{1}{2}} + H)/(1 - \rho)^{\frac{1}{2}}) \phi(y)dy \quad (2.5)$$

where $\Phi(y)$ and $\phi(y)$ are, respectively, the cumulative distribution function and the density function of a standardized normal random variable $Y$. Further, they provided the percentage points of $Y_{(n)}$, namely, the values of $H$ satisfying $F_n(H; \rho) = 1 - \alpha$ for selected values of $\alpha$ and $\rho$ in the form of tables. Note, however, that this approach does not permit one to compute the distribution function of $Y_{(n)}$ for $\rho < 0$ as well as for unequal $\rho$'s.

In a $n$ variate equi-correlated situation, Rawlings (1976) considered the probability integral for each of the $s \leq n$ variates to have magnitude less than $h$ and the remaining $n - s$ variates to have their magnitudes more than $h$. Following Gupta (1963) and Curnow & Dunnett (1962), Rawlings (1976) computed the probability that $Y_1$ to $Y_s$ fall below $h$ and $Y_{s+1}$ to $Y_n$ fall above $h$, given by

$$L_n(h; s, n - s, \rho) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{h} \cdots \int_{-\infty}^{h} \phi_n(y_1, \ldots, y_n; \rho) dy_1 \cdots dy_s \right] dy_{s+1} \cdots dy_n$$

$$= \int_{-\infty}^{\infty} \phi(x)[\Phi(w)]^{s}[1 - \Phi(w)]^{n-s}dx \quad (2.6)$$

where

$$\Phi(w) = \int_{-\infty}^{w} \phi(v)dv,$$

with

$$w = \frac{h + \rho^{\frac{1}{2}} x}{(1 - \rho)^{\frac{1}{2}}}.$$
Suppose there are \( m \) independent groups or clusters with \( n \) equicorrelated normal variables in each. For example, in a familial data, there may be \( m \) independent full-sib families each with \( n \) sibs having equal correlation \( \rho \). In such situations, Rawlings (1976) has then used the above probability \( L_n(\frac{n}{n - s}, \rho) \) to compute the probability density function for the \((n_0 - i)th\) order statistic, \( Y_{(n_0 - i,n,m,p)} \), given by

\[
 f(Y_{(n_0 - i,n,m,p)}) = \sum_{j=1}^{2^i} N(p, u_m, \{u_j\}) P_{(g, u_m, \{u_j\})} \\
\]

where \( n_0 = nm \), and

\[
 N(p, u_m, \{u_j\}) = \frac{\prod_{j=1}^{p} \binom{u}{u_j}}{B(p + 1, m - p) B(u_m + 1, n - u_m)} \\
 P_{(g, u_m, \{u_j\})} = \left[ L_n(g; n, 0, \rho) \right]^{n - p - 1} \left[ \prod_{j=1}^{p} L_n(g; n - u_j, u_j, \rho) \right] \\
\times \left[ L_{n-1}(w; u_m - 1, u_m, \rho^*) \right]
\]

with \( B(\cdot) \) as the usual beta function.

\[
w = g \left( \frac{1 - \rho}{1 + \rho} \right)^{\frac{1}{2}},
\]

and \( \rho^* = \rho / (1 + \rho) \)

\( u_j \) be the number of variates in subset \( j \) which are greater than \( g \) and \( p \) is the number of subsets having at least one variable greater than \( g \). In his paper, he tabulated the expectations of order statistics in correlated samples of \( m \) independent sets of \( k \) equicorrelated multinormal variates for selected \( i + 1, n, m \) and \( \rho \).
For handling negative equicorrelations, Hoffman and Saw (1975) provided a computationally feasible method for finding the cumulative distribution of the \( r \)th ranked of a set of equicorrelated normal variables. More specifically, they computed \( P(Y_{i(r)} \leq h) \), \( r = 1, 2, \ldots, n \) based on Tchebychev-Hermite and Legendre polynomial. The probability was given by

\[
P_{r, h}(h, \rho) = P(Y_{i(r)} \leq h) = \sum_{k=0}^{m} d_k(r, u) J_k(h, \rho)
\]

where

\[
d_k(r, u) = \frac{n! (2k + 1)!}{(k!)^2 (n + k + 1)!} \phi_k(r, u)
\]

and \( J_k(h, \rho) = \int_{-\infty}^{\infty} L_k(2P(\alpha + \theta u) - 1) z(u) du \)

with \( \alpha = h / (1 - \rho)^{1/2} \), \( \theta = \rho^{1/2} \), \( z(u) \) being the standard normal ordinate, \( \phi_k(r, u) \) being the \( k \)th order Tchebychev-Hermite polynomial on \( r \in \{0, 1, \ldots, n\} \) scaled, \( L_k(r) \) being the \( k \)th order Legendre polynomial in \( r \), and \( P(\alpha + \theta u) \) being a suitable complex function where \( \theta \) is complex. Note that the probability obtained as above by Hoffman & Saw (1975) is also valid for equal positive correlations. For \( 0 \leq k \leq 10 \) and \( r = m = 10 \), they tabulated the values of \( d_k(r, u) \) and \( J_k(h, \rho) \) for selected \( h \) and \( \rho \). Using these values, they obtained \( P_{10,10}(0, -1/9) = 0.000000; P_{10,10}(0, 1/8) = 0.008939; P_{10,10}(2, 1/8) = 0.809761; P_{10,10}(1, 1/2) = 0.160560; P_{10,10}(2, 1/2) = 0.866909 \).

For a very special case of general (equal or unequal) correlation structures, Greig (1967) developed an approximate formula for the moments of the smallest values in a correlated normal random variables with \( E(Y_i; Y_{i+1}) = \rho_{i,i+1} \neq 0 \), and \( E(Y_i; Y_j) = 0 \), for all
\( i \neq j \& j \neq i + 1 \). Since the application of the normal multivariate integral to calculate the exact moments of the minima is cumbersome, Greig provided first an approximate expression for multivariate probability integral \( \Phi_n \) in terms of \( \Phi_1 \) as

\[
\Phi_n(y_1, \ldots, y_n; \rho_{ij}) = \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \phi_n(u_1, \ldots, u_n; \rho_{ij}) \, du_1 \cdots du_n
\]

\[
\simeq \Phi_1(y) \prod_{i=1}^{n-1} \left[ (1 - (1 - \rho_{i,i+1})^{\frac{1}{2}} (1 - \Phi_1(y)) \right] \tag{2.9}
\]

where \( \phi_n(u_1, \ldots, u_n; \rho_{ij}) \) is the multivariate normal density, and

\[
\Phi_1(y) = \int_{y}^{\infty} \phi_1(u) \, du \tag{2.10}
\]

with \( \phi_1(u) \) as the density of a standard normal random variable. Utilizing this expression, the author, then, obtained an approximate result for the moments for the minima as

\[
\mu_{ns} \simeq \prod_{i=1}^{n-1} \left[ 1 - (1 - \rho_{i,i+1})^{\frac{1}{2}} (1 - \mu_{s}') \right] \tag{2.11}
\]

where \( \mu_{ns} \) is the \( s \)th moment of the smallest and \( \mu_{s}' = n \int_{-\infty}^{\infty} y^s \phi_1(y) \{ \Phi_1(y) \}' \, dy \). The above approach taken by Greig (1967) does not appear to be realistic. This is because, in reality, the other off-diagonal elements may not be negligible, although they were neglected in this approach.

Note that order statistics inference is also essential in the linear regression analysis, mainly, for the detection of outliers or influential observations. For example, consider a simple linear regression model

\[
Y = X\beta + \epsilon \tag{2.12}
\]

where \( Y = (Y_1, Y_2, \ldots, Y_n) \) is a \( n \times 1 \) response variable, \( X \) is a known design matrix of order \( n \times k \), \( \beta \) is a \( k \times 1 \) vector of unknown parameters and \( \epsilon \) is a \( n \times 1 \) error variable.
distributed as $\epsilon \sim N(0, \sigma^2 I_n)$. $I_n$ being the $n \times n$ identity matrix. To test for the presence of a single outlier in a linear regression model, the maximum studentized residual test statistic defined as $R_n = \max | \epsilon_i / s_i |$ is widely used. Here, $\epsilon_i = y_i - X_i^T \hat{\beta}$, $X_i^T$ is the $i$th ($i = 1, 2, \ldots, n$) row of the design matrix $X$, $\hat{\beta} = (X^T X)^{-1} X^T y$ is the least square estimate of $\beta$, and $s_i^2$ is the $i$th diagonal element of $V(\epsilon) = (I_n - V)\hat{\sigma}^2$, the estimated variance-covariance matrix of the residuals, with $V = X(X^T X)^{-1} X^T$ and $\hat{\sigma}^2 = \epsilon^T \epsilon / (n - k) = y^T (I_n - V) y / (n - k)$, where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$ is the $n \times 1$ residual vector. This $R_n$ statistic is recommended for use in situations where the variances of the individual residuals are expected to vary a great deal among themselves. There is another statistic, namely, the maximum normed residual test statistic defined as $R_n^* = \max | \epsilon_i / \hat{\sigma} |$ which is also frequently used but for the situations when all residuals have a common variance. Note, however, that the exact critical values for these two statistics are not available.

Ellenberg (1973, 1976) has used the first Bonferroni bounds in approximating the critical values of the maximum studentized residual test statistic $R_n = \max | \xi_i |$, where $\xi_i = \epsilon_i / s_i$ ($i = 1, 2, \ldots, n$) is the standardized least squares residuals. These Bonferroni bounds may be simplified as

$$nP_r(\xi_i > C'_a) = \sum_{i \leq j} P_r(\xi_i > C'_a \mid \xi_j > C'_a) \leq P_r(\max \xi_i > C'_a) \leq n P_r(\xi_i > C'_a)$$

(2.13)

with $C'_a$ as any critical constant. For the maximum normed residual test statistic $R_n^*$, Stefansky (1971, 1972) developed the bounds for the percentage points and it was shown how successive improvements could be made to the initial upper and lower bounds. It was also shown that in many situations the first upper (or lower) bound is either equal to or extremely close to the exact percentage point, the first upper bound for the 100(1 - $\alpha$)
percentage point of $R^*_n$ is being given by $[(n - k)F^*/(n - k - 1 + F^*)]^{1/2}$, where $F$ is the $100(1 - \alpha/n)$ percentage point of the $F^*$ distribution with 1 and $n - k - 1$ degrees of freedom.

In small sample cases, these $R_n$ and $R^*_n$ tests are not equivalent. It has been shown in Sutradhar (1996) [see also Sutradhar & Chu (1995)] through a simulation experiment that the maximum normed residual test is more powerful than the maximum studentized residual test, irrespective of the situations whether outlier arises due to slippage of the mean or inflation of the variance of the data. Consequently, between the two tests, it was recommended in Sutradhar (1996) to use the maximum normed residual test statistic for detecting a single mean-shifted or variance-inflated outlier in the linear models with fixed designs. Note, however, that as discussed in Sutradhar and Chu (1995), the first upper (or lower) bound for the critical value of $R^*_n$ found in Stefansky (1971, 1972) may be quite liberal in approximating the exact critical value of this statistic itself, especially when the residuals are heteroscedastic for certain choices of the design matrix.

It then follows from the above findings that in certain situations when variances of the random variables under consideration are unequal and when one is interested to find the p-value of a test statistic similar to $R^*_n$, the application of the first Bonferroni bounds may not be a good approximation to the exact p-value. Consequently, it seems quite appropriate to seek for alternative ways to calculate the critical value for the test statistics such as maxima, minima or general order statistics for normal correlated variables with unequal (or equal) variances. Motivated by this, we, in the present thesis, generalize the distributions of order statistics for the equi-correlated (as well as certain special case of unequally correlated) normal variables [cf. Gupta, Pillai & Steck (1964), Greig (1967), Gupta, Nagel & Panchapakesan (1973), Rawlings (1976), among others].
to the case where heteroscedastic normal variables have positive or negative unequal correlations. We adopt a small correlations approach to achieve this goal.
Chapter 3

Order Statistics From Unequal Correlated Normal Variables: A Small Correlation Approach

3.1 Joint Density Function

Let $Y_1, Y_2, \ldots, Y_n$ be normally distributed with mean zero and variance covariance matrix $\Sigma$, where $\Sigma = D^2 R D^2$
where \( \rho_{ij} \) in the \( R \) matrix is the correlation coefficients between \( Y_i \) & \( Y_j \), and \( \sigma_i^2 \) in the diagonal matrix \( D \) is the variance of \( Y_i \). Now we assume that \( \rho_{ij} \)'s are small in magnitude. This assumption about the small correlations is reasonable for many practical situations, for example, in clustered regression problems, where the within cluster correlations are usually small. In such cases (for example in familial data), the sample size \( n \) is usually small too.

Let \( Y(1), Y(2), \ldots, Y(n) \) be the order statistics of the random sample \( Y_1, Y_2, \ldots, Y_n \). The primary goal of this chapter is to derive the general marginal distribution of the \( r \)th order statistic, \( Y(r) \) \((r = 1, 2, \ldots, n)\). In order to do this, we require the joint probability density function of the order statistics \( Y(1), Y(2), \ldots, Y(n) \) which we will derive directly from the joint p.d.f. of the original variables \( Y_1, Y_2, \ldots, Y_n \) under the assumption that \( \rho_{ij} \)'s are small for all \( i \neq j \). For the derivation, we, first, approximate the joint probability density function (p.d.f.) of the original random variables \( Y_1, Y_2, \ldots, Y_n \)

\[
\begin{align*}
\frac{1}{\left[ 2\pi \right]^{n/2} |\Sigma|^{1/2}} e^{-1/2 y^\top \Sigma^{-1} y} = f(y_1, y_2, \ldots, y_n; \Sigma)
\end{align*}
\]

where \( Y = (y_1, y_2, \ldots, y_n)' \) is a \( n \times 1 \) random vector, by its Taylor's series expansion about \( \rho_{ij} = 0 \). More specifically, for small \( \rho_{ij} \)'s, the Taylor's series expansion of the joint p.d.f. of the original variables, \( f(y_1, y_2, \ldots, y_n; \Sigma) \), when evaluated at \( \rho_{ij} = 0 \), is given

\[
\begin{align*}
\begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \ldots & \rho_{1n} \\
\rho_{21} & 1 & \rho_{23} & \ldots & \rho_{2n} \\
\rho_{31} & \rho_{32} & 1 & \ldots & \rho_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \rho_{n3} & \ldots & 1
\end{pmatrix}
\end{align*}
\]
by

\[
\begin{align*}
\tilde{f}(y_1, y_2, \ldots, y_n; \Sigma) & \simeq \left[ U_n + \sum_{i<j}^n V_{ij} y_i y_j - \frac{1}{2} \sum_{i=1}^n V_i y_i^2 + \frac{1}{2} \sum_{i<j}^n W_{ij} y_i^2 y_j^2 \\
& + \sum_{i \neq j \neq k}^n W_{ijk} y_i y_j y_k + \sum_{i \neq j \neq k \neq l}^n W_{ijkl} y_i y_j y_k y_l \right] \\
& \times \tilde{f}(y_1, y_2, \ldots, y_n; D)
\end{align*}
\]

where

\[
\begin{align*}
f(y_1, y_2, \ldots, y_n; D) &= \frac{1}{(2\pi)^{n/2} |D|^{1/2}} \left( \frac{1}{2} \right)^{n/2} \\
U_n &= 1 + \frac{1}{2} \sum_{i<j}^n \rho_{ij}^2 \\
V_{ij} &= \frac{1}{\sigma_i \sigma_j} \left( \rho_{ij} - \sum_{k \neq i, j}^n \rho_{ik} \rho_{jk} \right) \\
V_i &= \frac{1}{\sigma_i^2} \sum_{i \neq j}^n \rho_{ij}^2 \\
W_{ij} &= \frac{\rho_{ij}^2}{\sigma_i^2 \sigma_j^2} \\
W_{ijk} &= \frac{\rho_{ij} \rho_{ik} \rho_{jk}}{\sigma_i \sigma_j \sigma_k} \\
ad W_{ijkl} &= \frac{\rho_{ijkl} + \rho_{ikl} \rho_{jk} + \rho_{ik} \rho_{jl}}{\sigma_i \sigma_j \sigma_k \sigma_l}
\end{align*}
\]

with \( D = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \).

Let \( Y(1) \leq \ldots \leq Y(n) \) be the correlated order statistics of \( Y(1) \ldots Y(n) \). Then, given the realizations of the order statistics to be \( y(1) \leq \ldots \leq y(n) \), the original variables \( Y_i \) \((i = 1, 2, \ldots, n)\) are constrained to take on the values \( y(n) \) which yields the same expression for the similar terms in equation (3.2) for all \( n! \) permutations \((i_1, i_2, \ldots, i_n)\) of \((1, 2, \ldots, n)\). Consequently, we may obtain the joint probability density function \( g^*(y(1), y(2), \ldots, y(n); \Sigma) \) of \( Y(1), Y(2), \ldots, Y(n) \) as given in Theorem 3.1.
THEOREM 3.1. Let the approximate joint density function \( f(y_1, y_2, \ldots, y_n) \) of \( Y_1, Y_2, \ldots, Y_n \) be given by (3.2). Then the joint p.d.f. \( g^*(y_{(1)}, y_{(2)}, \ldots, y_{(n)}) \) of \( Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)} \) is given by

\[
g^*(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; \Sigma) \approx \left[ U_n^* + D_n^* \sum_{i<j} y_{(i)} y_{(j)} - Q_n^* \sum_{i=1}^n y_{(i)}^2 - S_n^* \sum_{i<j} y_{(i)} y_{(j)}^2 \right. \\
\left. + T_n^* \sum_{i \neq j \neq k} y_{(i)}^2 y_{(j)} y_{(k)} + M_n^* \sum_{i \neq j \neq k \neq l} y_{(i)} y_{(j)} y_{(k)} y_{(l)} \right]
\times f(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; D) \tag{3.3}
\]

where

\[
U_n^* = n! U_n \\
D_n^* = 2(n-2)! \sum_{i<j} V_{ij} \\
Q_n^* = \frac{1}{2} (n-1)! \sum_{i=1}^n V_i \\
S_n^* = (n-2)! \sum_{i<j} W_{ij} \\
T_n^* = 2(n-3)! \sum_{i \neq j \neq k} W_{ijk} \\
M_n^* = 4!(n-1)! \sum_{i \neq j \neq k \neq l} W_{ijkl}
\]

and \( U_n, V_{ij}, V_i, W_{ij}, W_{ijk} \) and \( W_{ijkl} \) being given by (3.2).

Derivation of Theorem 3.1 : For simplicity, we start with \( n = 2 \). It follows by (3.2) that for \( n = 2 \), the joint p.d.f. for the original variables \( Y_1, Y_2 \) is given by

\[
f(y_1, y_2; \Sigma) \approx \left[ U_2 + V_{12} y_1 y_2 + \frac{1}{2} W_{12} y_1^2 y_2^2 - \frac{1}{2} V_1 y_1^2 - \frac{1}{2} V_2 y_2^2 \right]
\times f(y_1, y_2; D) \tag{3.4}
\]
where

\[ f(y_1, y_2; D) = \frac{1}{(2\pi)^{\frac{1}{2}} |D|} \left( y_1^2 + \frac{1}{2} \right)^{-\frac{1}{2}} \left( y_2^2 + \frac{1}{2} \right)^{-\frac{1}{2}} \]

\[ U_2 = 1 + \frac{\rho_{12}^2}{\sigma_1 \sigma_2} \]

\[ V_{12} = \frac{\rho_{12}}{\sigma_1 \sigma_2} \]

\[ V_1 = \frac{\rho_{12}}{\sigma_1} \]

\[ V_2 = \frac{\rho_{12}}{\sigma_2} \]

and \[ W_{12} = \frac{\rho_{12}^2}{\sigma_1^2 \sigma_2^2} \]

Now, consider the set \( A \) which is the union of the two mutually disjoint sets \( A_1 \) \( \{(y_1, y_2); y_1 < y_2\} \) and \( A_2 = \{(y_1, y_2); y_2 < y_1\} \). There are two of these sets because we can arrange \( y_1, y_2 \) in precisely \( 2! = 2 \) ways. Let \( B \) be the set of the order statistics defined as \( B = \{(y_{(1)}, y_{(2)}); y_{(1)} < y_{(2)}\} \). It follows that there exists one-to-one transformations that map each of \( A_1, A_2 \) onto the set \( B \). Inversely, in set \( B \), \( y_{(1)} = y_1, y_{(2)} = y_2 \) for the points in \( A_1 \), and \( y_{(1)} = y_2, y_{(2)} = y_1 \) for the points in \( A_2 \). The absolute value of the jacobian of the undertaking transformation for each set of \( A \) is 1. Thus the joint p.d.f. of order statistics \( Y_{(1)}, Y_{(2)} \) may be written as

\[
\begin{align*}
g_2^2 \left( y_{(1)}, y_{(2)}; \Sigma \right) & \simeq \left[ U_2 + V_{12} y_{(1)} y_{(2)} + \frac{1}{2} V_{12}^2 y_{(1)}^2 y_{(2)}^2 + \frac{1}{2} V_1 y_{(1)}^2 y_{(1)}^2 - \frac{1}{2} V_2 y_{(2)}^2 \right] \times f \left( y_{(1)}, y_{(2)}; D \right) + \left[ U_2 + V_{12} y_{(1)} y_{(2)} + \frac{1}{2} V_{12}^2 y_{(1)}^2 y_{(2)}^2 \right. \\
& \quad - \frac{1}{2} V_1 y_{(1)}^2 - \frac{1}{2} V_2 y_{(1)}^2] f \left( y_{(2)}, y_{(1)}; D \right) \\
& = \left[ 2! U_2 + 2(2 - 1)! V_{12} y_{(1)} y_{(2)} - \frac{(2 - 1)!}{2} \left( \sum_{i=2}^{2} V_i \right) \sum_{i=2}^{2} y_{(i)} \right. \\
& \quad + (2 - 2)! W_{12} y_{(1)}^2 y_{(2)}^2] f \left( y_{(1)}, y_{(2)}; D \right) 
\end{align*}
\]
\[ U_2^* = 2!U_2 \]
\[ D_2^* = 2(2 - 2)!W_12 = 2(2 - 2)! \sum_{i<j} V_{ij} \]
\[ Q_2^* = \frac{(2 - 1)!}{2} \sum_{i=1}^{2} V_i. \]

and
\[ S_2^* = (2 - 2)!W_{12} = (2 - 2)! \sum_{i<j} W_{ij} \]

Similarly, for \( n = 3 \), the joint p.d.f. of \( Y_1, Y_2, Y_3 \) obtained from equation (3.2) is given by
\[
f(y_1, y_2, y_3; \Sigma) \approx \left[ U_3 + \sum_{i<j}^3 V_{ij}y_iy_j - \frac{1}{2} \sum_{i=1}^{3} V_iy_i^2 + \frac{1}{2} \sum_{i<j} W_{ij}y_i^2y_j^2 \right. \\
\left. + \sum_{i\neq j\neq k} W_{ijk}y_i^2y_jy_k \right] f(y_1, y_2, y_3; D) \tag{3.6} \]

where
\[
f(y_1, y_2, y_3; D) = \frac{1}{(2\pi)^{3/2} |D|^{1/2}} e^{-\frac{1}{2}Y^TD^{-1}Y} \]
\[ U_3 = 1 + \frac{1}{2} \sum_{i<j} \rho_{ij}^2 \]
\[ V_{ij} = \frac{1}{\sigma_i\sigma_j} \left( \rho_{ij} - \sum_{k\neq i\neq j}^3 \rho_{ik}\rho_{jk} \right) \]
\[ V_i = \frac{1}{\sigma_i^2} \left( \sum_{j\neq i}^3 \rho_{ij}^2 \right) \]
In this case, the realizations \( y_1, y_2, y_3 \) of the original random variables are rearranged in ascending order of magnitude in \( 3! = 6 \) ways. Now, let \( E' \) and \( F' \) denote two sets, one for the original variables and one for order statistics respectively, where the set \( E' \) is the union of the six mutually disjoint sets as \( E_1 = \{(y_1, y_2, y_3): y_1 < y_2 < y_3\}, E_2 = \{(y_1, y_2, y_3): y_2 < y_1 < y_3\}, E_3 = \{(y_1, y_2, y_3): y_1 < y_3 < y_2\}, E_4 = \{(y_1, y_2, y_3): y_3 < y_1 < y_2\}, E_5 = \{(y_1, y_2, y_3): y_2 < y_3 < y_1\}, \) and \( E_6 = \{(y_1, y_2, y_3): y_3 < y_2 < y_1\}, \) and \( F' = \{(y_1, y_2, y_3): y_1 < y_2 < y_3\} \). Similar to the case \( n = 2 \), we can make an one-to-one transformation from each disjoint set of \( E' \) onto the set \( F' \) and \( \mid J \mid = 1 \) for each set of \( E' \). For simplicity, we, first, compute the ordered function for each term in equation (3.6) considering the above transformations for each disjoint set of \( E' \). So, for \( V_3 f(y_1, y_2, y_3; D) \) in equation (3.6), we obtain the ordered function as

\[
m(y_1, y_2, y_3; D) = V_3[f(y_1, y_2, y_3; D) + f(y_2, y_1, y_3; D)]
+ f(y_1, y_2, y_3; D) + f(y_2, y_3, y_1; D)
+ f(y_3, y_1, y_2; D) + f(y_3, y_2, y_1; D)]
= 3! V_3 [f(y_1, y_2, y_3; D)]
\] (3.7)

The ordered functions for the first term of \( \sum_{i<j}^3 V_{ij} y_i y_j f(y_1, y_2, y_3; D) = V_{12} y_1 y_2 + V_{13} y_1 y_3 + V_{23} y_2 y_3 f(y_1, y_2, y_3; D) \) in (3.6) is given by

\[
h_1(y_1, y_2, y_3; D) = V_{12}[y_1 y_2 f(y_1, y_2, y_3; D) + y_2 y_1 f(y_2, y_1, y_3; D)]
+ y_1 y_3 f(y_1, y_3, y_2; D) + y_3 y_1 f(y_3, y_1, y_2; D)
+ y_3 y_2 f(y_3, y_2, y_1; D) + y_2 y_3 f(y_2, y_3, y_1; D)]
\]
\[
= 2V_{12} \sum_{i<j}^3 y_{(i)}y_{(j)}f(y_{(1)}y_{(2)}y_{(3)}; D)
\]

Similarly, we may derive the second and third terms as follows

\[
h_2(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2V_{13} \sum_{i<j}^3 y_{(i)}y_{(j)}f(y_{(1)}y_{(2)}y_{(3)}; D),
\]

and

\[
h_3(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2V_{23} \sum_{i<j}^3 y_{(i)}y_{(j)}f(y_{(1)}y_{(2)}y_{(3)}; D)
\]

Combining the last three expressions, we obtain the ordered function of \(\sum_{i<j}^3 V_{ij}y_{ij}f(y_{1}, y_{2}, y_{3}; D)\) as

\[
h(y_{(1)}, y_{(2)}, y_{(3)}; D) = h_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + h_2(y_{(1)}, y_{(2)}, y_{(3)}; D)
\]

\[
+ h_3(y_{(1)}, y_{(2)}, y_{(3)}; D)
\]

\[
= 2 \left\{ \sum_{i<j}^3 V_{ij} \right\} \sum_{i<j}^3 y_{(i)}y_{(j)}f(y_{(1)}y_{(2)}y_{(3)}; D)
\]  (3.8)

Similarly, we obtain the ordered function for each term of \(\sum_{i<j}^3 W_{ij}y_{i}^2y_{j}^2f(y_{1}, y_{2}, y_{3}; D)\) as \([W_{12}y_{1}^2y_{2}^2 + W_{13}y_{1}^2y_{3}^2 + W_{23}y_{2}^2y_{3}^2]f(y_{1}, y_{2}, y_{3}; D)\) in (3.6) as

\[
i_1(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{12} \sum_{i<j}^3 y_{(i)}^2y_{(j)}^2f(y_{(1)}y_{(2)}y_{(3)}; D)
\]

\[
i_2(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{13} \sum_{i<j}^3 y_{(i)}^2y_{(j)}^2f(y_{(1)}y_{(2)}y_{(3)}; D)
\]

and

\[
i_3(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{23} \sum_{i<j}^3 y_{(i)}^2y_{(j)}^2f(y_{(1)}y_{(2)}y_{(3)}; D)
\]

respectively, yielding the ordered function of \(\sum_{i<j}^3 W_{ij}y_{i}^2y_{j}^2f(y_{1}, y_{2}, y_{3}; D)\) as

\[
i(y_{(1)}, y_{(2)}, y_{(3)}; D) = i_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + i_2(y_{(1)}, y_{(2)}, y_{(3)}; D)
\]

\[
+ i_3(y_{(1)}, y_{(2)}, y_{(3)}; D)
\]

\[
= 2 \left\{ \sum_{i<j}^3 W_{ij} \right\} \sum_{i<j}^3 y_{(i)}^2y_{(j)}^2f(y_{(1)}y_{(2)}y_{(3)}; D)
\]  (3.9)
In the manner similar to the derivation of (3.8) & (3.9), the ordered functions for each term of \( \sum_{i<j} V_i y_i^2 f(y_1, y_2, y_3; D) = [V_1 y_1^2 + V_2 y_2^2 + V_3 y_3^2] f(y_1, y_2, y_3; D) \) in (3.6) are obtained, respectively, as

\[
\begin{align*}
  j_1(y_{(1)}, y_{(2)}, y_{(3)}; D) &= V_1 y_{(1)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) + y_{(2)}^2 f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\
  &+ y_{(3)}^2 f(y_{(3)}, y_{(1)}, y_{(2)}; D) + y_{(2)}^2 f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\
  &= 2V_1 \sum_{i<j} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  j_2(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_2 \sum_{i<j} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  j_3(y_{(1)}, y_{(2)}, y_{(3)}; D) &= 2V_3 \sum_{i<j} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D),
\end{align*}
\]

which are exploited to compute the combined ordered function of \( \sum_{i<j} V_i y_i^2 f(y_1, y_2, y_3; D) \) as

\[
\begin{align*}
  j(y_{(1)}, y_{(2)}, y_{(3)}; D) &= j_1(y_{(1)}, y_{(2)}, y_{(3)}; D) + j_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &+ j_3(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &= 2 \left[ \sum_{i=1}^{3} V_i \right] \sum_{i=1}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &= 2 \left[ \sum_{i=1}^{3} V_i \right] \sum_{i=1}^{3} y_{(i)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \quad (3.10)
\end{align*}
\]

In the similar way, we also obtain the ordered function for each term of \( \sum_{i<j} W_{ijk} y_i^2 y_j y_k f(y_1, y_2, y_3; D) = [W_{112} y_1^2 y_2 y_3 + W_{122} y_1 y_2^2 y_3 + W_{123} y_1 y_2 y_3^2] f(y_1, y_2, y_3; D) \) in (3.6), respectively, as

\[
\begin{align*}
  k_1(y_{(1)}, y_{(2)}, y_{(3)}; D) &= W_{112} y_{(1)}^2 y_{(2)} y_{(3)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &+ W_{122} y_{(1)} y_{(2)}^2 y_{(3)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &+ W_{123} y_{(1)} y_{(2)} y_{(3)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &= W_{112} \sum_{i<i<j} y_{(i)}^2 y_{(j)} y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &+ W_{122} \sum_{i<i<j} y_{(i)} y_{(j)}^2 y_{(k)} f(y_{(1)}, y_{(2)}, y_{(3)}; D) \\
  &+ W_{123} \sum_{i<i<j} y_{(i)} y_{(j)} y_{(k)}^2 f(y_{(1)}, y_{(2)}, y_{(3)}; D) \quad (3.11)
\end{align*}
\]
\[ +y_{(2)}^2y_{(1)}y_{(3)}f(y_{(2)}, y_{(1)}, y_{(3)}; D) \\
+ y_{(1)}^2y_{(3)}y_{(2)}f(y_{(1)}, y_{(3)}, y_{(2)}; D) \\
+ y_{(2)}y_{(3)}y_{(1)}f(y_{(2)}, y_{(3)}, y_{(1)}; D) \\
+ y_{(2)}^2y_{(3)}y_{(2)}f(y_{(2)}, y_{(3)}, y_{(2)}; D) \\
+ y_{(3)}^2y_{(1)}y_{(2)}f(y_{(3)}, y_{(1)}, y_{(2)}; D) \]

\[ = 2W_{1123} \sum_{i<j}^{3} y_{(i)}^2y_{(j)}y_{(k)}f(y_{(1)}, y_{(2)}, y_{(3)}; D), \]

\[ k_2(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{1223} \sum_{i<j}^{3} y_{(i)}^2y_{(j)}y_{(k)}f(y_{(1)}, y_{(2)}, y_{(3)}; D). \]

and \[ k_3(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2W_{1233} \sum_{i<j}^{3} y_{(i)}^2y_{(j)}y_{(k)}f(y_{(1)}, y_{(2)}, y_{(3)}; D). \]

By using the above expressions, we obtain the ordered function of \( \sum_{i<j}^{3} W_{i,j,k}y_{i}^2y_{j}y_{k} \)

\[ \times f(y_{(1)}, y_{(2)}, y_{(3)}; D) \]

\[ k(y_{(1)}, y_{(2)}, y_{(3)}; D) = k_1(y_{(1)}, k_{(2)}, y_{(3)}; D) + k_2(y_{(1)}, y_{(2)}, y_{(3)}; D) \]

\[ + k_3(y_{(1)}, y_{(2)}, y_{(3)}; D) \]

\[ = 2 \left\{ \sum_{i<j}^{3} W_{i,j,k} \right\} \sum_{i<j}^{3} y_{(i)}^2y_{(j)}y_{(k)}f(y_{(1)}, y_{(2)}, y_{(3)}; D) \quad (3.11) \]

Next, combining the results of equations (3.7)-(3.11) for the five terms in equation (3.6), we obtain the joint p.d.f. of \( Y_{(1)}, Y_{(2)}, Y_{(3)} \) as

\[ g_{\Sigma}(y_{(1)}, y_{(2)}, y_{(3)}; \Sigma) \simeq [6U_3 + 2 \left\{ \sum_{i<j}^{3} V_{ij} \right\} \sum_{i<j}^{3} y_{(i)}y_{(j)} - \frac{1}{2} \left\{ \sum_{i=1}^{3} V_i \right\} \sum_{i=1}^{3} y_{(i)}^2 \]

\[ + \frac{1}{2} \left\{ \sum_{i<j}^{3} W_{i,j,k} \right\} \sum_{i<j}^{3} y_{(i)}^2y_{(j)}y_{(k)}^2 + 2 \left\{ \sum_{i\neq j \neq k}^{3} W_{i,j,k} \right\} \]

\[ \times \sum_{i\neq j \neq k}^{3} y_{(i)}^2y_{(j)}y_{(k)}f(y_{(1)}, y_{(2)}, y_{(3)}; D) \]

\[ = [U_3 + D_3 \sum_{i<j}^{3} y_{(i)}y_{(j)} - Q_3 \sum_{i=1}^{3} y_{(i)}^2 + S_3 \sum_{i<j}^{3} y_{(i)}^2y_{(j)}^2] \]
\[ + T_3^* \sum_{i \neq j \neq k} y_{(i)} y_{(j)} y_{(k)} f(y_{(i)}, y_{(j)}, y_{(k)}; D) \]  

(3.12)

where

\[ \begin{align*}
U_3^* &= 3U_3 \\
P_3^* &= 2 \sum_{i < j}^3 V_{ij} = 2(3 - 2)! \sum_{i < j} V_{ij} \\
Q_3^* &= \frac{1}{2} 2 \sum_{i = 1}^3 V_i = \frac{1}{2} (3 - 1)! \sum_{i = 1}^3 V_i \\
S_3^* &= \frac{1}{2} 2 \sum_{i < j}^3 W_{ij} = (3 - 2)! \sum_{i < j} W_{ij},
\end{align*} \]

and \[ T_3^* = 2 \sum_{i \neq j \neq k}^3 W_{i,j,k} = 2(3 - 3)! \sum_{i \neq j \neq k} W_{i,j,k} \]

As done in the last case, we now start with \( n = 4 \). For \( n = 4 \), the joint p.d.f. of the original variables \( Y_1, Y_2, Y_3, Y_4 \) by (3.2) is

\[ f(y_1, y_2, y_3, y_4; \Sigma) \simeq \left[ U_4 + \sum_{i < j}^4 V_{ij} y_i y_j - \frac{1}{2} \sum_{i = 1}^4 V_i y_i^2 + \frac{1}{2} \sum_{i < j}^4 W_{ij} y_i y_j^2 \right. \]

\[ \left. + \sum_{i \neq j \neq k}^4 W_{i,j,k} y_i y_j y_k + W_{1234} y_1 y_2 y_3 y_4 \right] \]

\[ \times f(y_1, y_2, y_3, y_4; D) \]  

(3.13)

where

\[ f(y_1, y_2, y_3, y_4; D) = \frac{1}{(2\pi)^{2|D|} |D|^{1/2}} e^{-\frac{1}{2} y' D^{-1} y} \]

\[ \begin{align*}
U_4 &= 1 + \frac{1}{2} \sum_{i < j}^4 \rho_{ij} \\
V_{ij} &= \frac{1}{\sigma_i \sigma_j} \left( \rho_{ij} - \sum_{k \neq i \neq j}^4 \rho_{ik} \rho_{jk} \right) \\
V_i &= \frac{1}{\sigma_i^2} \left( \sum_{j \neq i}^4 \rho_{ij}^2 \right)
\end{align*} \]
\[
W_{ij} = \frac{\rho_{ij}}{\sigma_i \sigma_j} \\
W_{ijk} = \frac{\rho_{ijk}}{\sigma_i \sigma_j \sigma_k}
\]

and \[W_{1234}^* = \frac{\rho_{12} \rho_{23} + \rho_{13} \rho_{24} + \rho_{14} \rho_{23}}{\sigma_1 \sigma_2 \sigma_3 \sigma_4}\]

Now, similar to the cases for \(n = 2,3\), we consider \(Y_1, Y_2, Y_3, Y_4\) and \(Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)}\) as the sets of original and ordered variables respectively. Here the set of original variables is the union of the 24 mutually disjoint sets because of \(4! = 24\) permutations of \((1, 2, 3, 4)\). By making one-to-one transformation from each disjoint set of original variables to the set of order statistics, we can obtain the ordered function for each term of (3.13) in the way similar to that for the case \(n = 3\). After a straightforward algebra, the ordered functions for \(U_4 f(y_1, y_2, y_3, y_4; D), \sum_{i < j} V_{ij} y_i y_j f(y_1, y_2, y_3, y_4; D), \sum_{i = 1}^4 V_i y_i^2 f(y_1, y_2, y_3, y_4; D), \sum_{i < j} W_{ij} y_i y_j^2 f(y_1, y_2, y_3, y_4; D)\) and \(\sum_{i \neq j \neq k} W_{ijk} y_i y_j y_k f(y_1, y_2, y_3, y_4; D)\) are, respectively, simplified as

\[
m^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4! U_4 f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \\
h^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4 \left\{ \sum_{i < j} V_{ij} \right\} \left\{ \sum_{i < j} y_i y_j \right\} \\
\quad \times f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \\
j^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 6 \left\{ \sum_{i = 1}^4 V_i \right\} \left\{ \sum_{i = 1}^4 y_i^2 \right\} f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \\
i^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 4 \left\{ \sum_{i < j} W_{ij} \right\} \left\{ \sum_{i < j} y_i y_j^2 \right\} \\
\quad \times f(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) \\
\text{and } k^*(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D) = 2 \left\{ \sum_{i \neq j \neq k} W_{ijk} \right\} \left\{ \sum_{i \neq j \neq k} y_i y_j y_k \right\} \\
\quad \times f(y_{(1)}, y_{(2)}, y_{(3)} y_{(4)}; D)
\]

For the last term \(W_{1234}^* y_1 y_2 y_3 y_4 f(y_1, y_2, y_3, y_4; D)\) in equation (3.13), we get the same
product function of the set of order statistics \( Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)} \) for each disjoint set of original variables \( Y_1, Y_2, Y_3, Y_4 \) which are exploited to compute the ordered function of this last term as

\[
\nu^*(y_{(1)}, y_{(2)}, y_{(3)}; D) = 2! y_{(1)} y_{(2)} y_{(3)} y_{(4)} f(y_{(1)}, y_{(2)}, y_{(3)}; D)
\]  

(3.19)

Based on the equations (3.14)-(3.19) for the six terms in equation (3.13), we then obtain the joint p.d.f. of \( Y_{(1)}, Y_{(2)}, Y_{(3)}, Y_{(4)} \) given by

\[
g_4^*\left(y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D\right) = \frac{1}{2} \left( \sum_{i<j} V_{ij} \right) \sum_{i<j} y_{(i)} y_{(j)} - \frac{1}{2} \sum_{i=1}^4 v_i \sum_{i=1}^4 y_{(i)}^2 \\
+ \frac{1}{2} \left( \sum_{i<j} W_{ij} \right) \sum_{i<j} y_{(i)}^2 y_{(j)}^2 + 2 \left( \sum_{i, j, k \neq l} W_{ijkl} \right) \\
\times \sum_{i \neq j \neq k} y_{(i)}^2 y_{(j)} y_{(k)} + 21 W_{1234} \left( y_{(1)} y_{(2)} y_{(3)} y_{(4)} ; D \right) \\
\times f\left( y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}; D \right)
\]

(3.20)

\[
\begin{align*}
U_4^* &= 21 U_4 = 4! U_4 \\
D_4^* &= 4 \sum_{i<j} V_{ij} = 2(4 - 2)! \sum_{i<j} V_{ij} \\
Q_4^* &= \frac{1}{2} 6 \sum_{i=1}^4 V_i = \frac{1}{2} (4 - 1)! \sum_{i=1}^4 V_i \\
S_4^* &= \frac{1}{2} 4 \sum_{i<j} W_{ij} = (4 - 2)! \sum_{i<j} W_{ij}
\end{align*}
\]
Finally, following the patterns for the joint probability density functions in the equations (3.5), (3.12) & (3.20) for \( n = 2, 3, 4 \) respectively, one may easily obtain, in general, the joint p.d.f. of \( Y_1, Y_2, \ldots, Y_n \) as given in Theorem 3.1.

3.2 Approximation to the Distribution of a Single Order Statistic

We now turn to the distribution theory of a single order statistic under the assumption that \( p_{ij} \)'s are small. An approximation to the distribution of \( Y_{(r)} \) \((1 \leq r \leq n)\), the \( r \)th order statistic, is provided in Theorem 3.2.

**THEOREM 3.2.** Let \( Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)} \) be the order statistics with joint p.d.f. as given in Theorem 3.1. Then the marginal density function of the \( r \)th order statistic, \( Y_{(r)} \) is given by

\[
y_{(r)}(y_{(r)}) \approx U_{n}^{*}(y_{(r)}) + D_{n}^{*} \sum_{i < j}^{n} \lambda_{ij}^{(1)}(y_{(r)}) - Q_{n}^{*} \sum_{i = 1}^{n} \lambda_{i}^{(2)}(y_{(r)}) + S_{n}^{*} \sum_{i < j}^{n} \lambda_{ij}^{(2)}(y_{(r)})
\]

\[
+ T_{n}^{*} \sum_{i \neq j \neq k}^{n} \lambda_{ijk}^{(3)}(y_{(r)}) + M_{n}^{*} \sum_{i \neq j \neq k \neq l}^{n} \lambda_{ijkl}^{(4)}(y_{(r)}), \quad -\infty \leq y_{(r)} \leq \infty
\]  

(3.21)

where \( U_{n}^{*}, D_{n}^{*}, Q_{n}^{*}, S_{n}^{*}, T_{n}^{*} \), \& \( M_{n}^{*} \) are defined as in Theorem 3.1. Further in (3.21),

\[
\phi(y_{(r)}) = \frac{1}{(r - 1)!(n - r)!} [F(y_{(r)})]^{r-1} [1 - F(y_{(r)})]^{n-r} f(y_{(r)})
\]

with \( F(y_{(r)}) = \int_{-\infty}^{y_{(r)}} f(x)dx \), \( f(x) \) being the p.d.f. of normal variable.
and for example, for $i < r \& j, k > r$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = f(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \cdot t_i, t_j = 1, 2
$$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = f(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \cdot t_i, t_j, t_k = 1, 2
$$

and for $i < r, j = r \& k > r$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = y_{(r)}^{t_i} f(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \cdot t_i, t_j = 1, 2
$$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = y_{(r)}^{t_i} f(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) \cdot t_i, t_j, t_k = 1, 2
$$

with

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = \int_{a=1}^{t_i} \prod_{a=1}^{t_i} y_{[a]}^{t_i} f(y_{[1,r-1]}) dy_{[1,r-1]}, \ t_i = 1, 2 \text{ for } a = i, j \tag{3.22}$$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = 0 \text{ for } a \neq i, j
$$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = \int_{a=n(-1)r+1}^{t_i} \prod_{a=n(-1)r+1}^{t_i} y_{[a]}^{t_i} f(y_{[r+1,a]}) dy_{[r+1,a]}, \ t_i = 1, 2 \text{ for } a = i, j, k \tag{3.23}
$$

$$
\lambda_{ij(kl)}^{t_i t_j t_k}(y(r)) = 0 \text{ for } a \neq i, j, k
$$

where $f^{2,r}$ and $f_{r,n-1}$ represent the multiple integrals $f_{x_1}^{y_{r-1}} \ldots f_{x_n}^{y_{r-1}}$ and $f_{y_{r+1}}^{y_{n+1}} \ldots f_{y_{n-1}}^{y_{n-1}}$ respectively and

$$
f(y_{[1,r-1]}) \equiv f(y_{(1)}) f(y_{(2)}) \ldots f(y_{(r-1)})
$$

$$
f(y_{[r+1,a]}) \equiv f(y_{(a)}) f(y_{(a-1)}) \ldots f(y_{(r+1)})
$$

$$
dy_{[1,r-1]} \equiv dy_{(1)} dy_{(2)} \ldots dy_{(r-1)}
$$

$$
dy_{[r+1,a]} \equiv dy_{(a)} dy_{(a-1)} \ldots dy_{(r+1)}
$$

**Derivation of Theorem 3.2**: We start with the approximate joint probability density function of all $n$ order statistics in (3.3) and then integrate out the variables
$Y_{(1)}, \ldots, Y_{(r-1)}, Y_{(r+1)}, \ldots, Y_{(n)}$ in order to derive the marginal density function of the $r$th order statistic, $Y_{(r)}$ ($1 \leq r \leq n$) as

$$g_r^*(y_{(r)}) = \int_{r,n-1}^{2,r} g^*(y_{(1)}, \ldots, y_{(n)}; \Sigma_i y_{[r+1,n]} y_{[1,r-1]}$$

$$= \int_{r,n-1}^{2,r} \left[ U_n^* + D_n^* \sum_{i \leq j} y_{(i)} y_{(j)} - Q_n^* \sum_{i=1}^{n} y_{(i)}^2 + Q_n^* \sum_{i \leq j} y_{(i)}^2 \right]$$

$$+ T_n^* \sum_{i \neq j \neq k} y_{(i)}^2 y_{(j)} y_{(k)} + M_n^* \sum_{i \neq j \neq k \neq l} y_{(i)} y_{(j)} y_{(k)} y_{(l)}$$

$$\times f(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; D)dy_{[r+1,n]}dy_{[1,r-1]}$$

(3.24)

Now, by computing the integration for each term in equation (3.24), we obtain the probability density function as in the theorem. The steps for the integrations are given below. For the first term, we simplify the integral as

$$\int_{r,n-1}^{2,r} f(y_{(1)}, \ldots, y_{(n)}; D)dy_{[r+1,n]}dy_{[1,r-1]}$$

$$= \left\{ \int_{r,n-1}^{2,r} f(y_{[r+1,n]}dy_{[1,r-1]} \right\} \left\{ \int_{r,n-1}^{2,r} f(y_{[r+1,n]}dy_{[r+1,n]} \right\} f(y_{(r)})$$

$$= \frac{1}{(r-1)!} [F'(y_{(r)})]^{r-1} \frac{1}{(n-r)!} [1 - F'(y_{(r)})]^{n-r} f(y_{(r)})$$

(3.25)

which is $\varphi(y_{(r)})$, as defined in (3.21).

Next, for $i < r$ and $j > r$, the integral in the second term may be expressed as

$$\int_{r,n-1}^{2,r} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D)dy_{[r+1,n]}dy_{[1,r-1]}$$

$$= f(y_{(r)}) \left\{ \int_{r,n-1}^{2,r} y_{(i)} f(y_{[r+1,n]}dy_{[1,r-1]} \right\} \left\{ \int_{r,n-1}^{2,r} f(y_{[r+1,n]}dy_{[r+1,n]} \right\}$$

(3.26)

Now, by using $t_i = 1$, $t_j = 1$ and $t_a = 0$ for $a \neq i, j$, the integral in (3.26) may further be expressed as

$$\int_{r,n-1}^{2,r} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D)dy_{[r+1,n]}dy_{[1,r-1]}$$
\[
= \int^{2r} \prod_{n=1}^{\min(b,r-1)} y_{(n)}^{s} f(y_{[r-1]}^{s}) dy_{[r-1]}
\times \left\{ \int_{r,n-1} y^{s+1}_{(n)} f(y_{[r+1,n]}^{s}) dy_{[r+1,n]} \right\}
= \lambda^{U(k_1)}_{U(k_2)}(y_{(r)}) \lambda^{U(k_1)}_{U(k_2)}(y_{(r)})
\]

which is \( \lambda^{U(k_1)}_{U(k_2)}(y_{(r)}) \).

Similarly, for \( i, j < r \), we obtain

\[
\int^{2r} \int_{r,n-1} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D) dy_{[r+1,n]} dy_{[1,r-1]} = \lambda^{U(k_1)}_{U(k_2)}(y_{(r)}) \lambda^{U(k_1)}_{U(k_2)}(y_{(r)})
\]

and for \( i < r \) and \( j = r \), we obtain

\[
\int^{2r} \int_{r,n-1} y_{(i)} y_{(j)} f(y_{(1)}, \ldots, y_{(n)}; D) dy_{[r+1,n]} dy_{[1,r-1]} = y_{(r)} f(y_{(r)}) \lambda^{U(k_1)}_{U(k_2)}(y_{(r)}) \lambda^{U(k_1)}_{U(k_2)}(y_{(r)})
\]

In the manner similar to the computations of the above integral, we may obtain the expressions for the remaining integrals in (3.24).

### 3.3 Special Cases: Distributions of Maxima and Minima

The maximum and the minimum order statistics in samples of size \( n \) are of special interest in numerous practical applications. An approximation to the distributions of the
maximum and minimum order statistics follows from equation (3.21). The probability density functions of the maxima and minima are, respectively, given by:

\[
g_n^*(y_{(n)}) = F_n^* \phi(y_{(n)}) + D_n^* \sum_{i<j}^n \lambda_{i,j,k,l}^{1100}(y_{(n)}) - Q_n^* \sum_{i=1}^n \lambda_{i,j,k,l}^{2200}(y_{(n)}) + S_n^* \sum_{i<j}^n \lambda_{i,j,k,l}^{2200}(y_{(n)})
\]

and

\[
g_l^*(y_{(1)}) = F_l^* \phi(y_{(1)}) + D_l^* \sum_{i<j}^n \lambda_{i,j,k,l}^{1100}(y_{(1)}) - Q_l^* \sum_{i=1}^n \lambda_{i,j,k,l}^{2200}(y_{(1)}) + S_l^* \sum_{i<j}^n \lambda_{i,j,k,l}^{2200}(y_{(1)})
\]

In equation (3.27),

\[
f(y_{(n)}) = \frac{1}{(n-1)!}[F(y_{(n)})]^{n-1} f(y_{(n)}).
\]

and for example, for \(i, j, k < n\),

\[
\lambda_{i,j,k,l}^{1100}(y_{(n)}) = f(y_{(n)}) \lambda_{i,j,k,l}^{1100}(y_{(n)}), \quad l_i, l_j = 1.2
\]

\[
\lambda_{i,j,k,l}^{1110}(y_{(n)}) = f(y_{(n)}) \lambda_{i,j,k,l}^{1110}(y_{(n)}), \quad l_i, l_j, l_k = 1.2
\]

and for \(i, k < n \quad \& \quad j = n\),

\[
\lambda_{i,j,k,l}^{1010}(y_{(n)}) = g_{(e)}^{(i)} f(y_{(n)}) \lambda_{i,j,k,l}^{1010}(y_{(n)}), \quad l_i, l_j = 1.2
\]

\[
\lambda_{i,j,k,l}^{1001}(y_{(n)}) = g_{(e)}^{(i)} f(y_{(n)}) \lambda_{i,j,k,l}^{1001}(y_{(n)}), \quad l_i, l_j, l_k = 1.2
\]

with

\[
\lambda_{i,j,k,l}^{1001}(y_{(n)}) = \int_{y_{(1)}^{[n-1]}}^{y_{(n)}^{[n-1]}} \prod_{a=1}^{n-1} y_{(a)}^{a} f(y_{[1,n-1]}) dy_{[1,n-1]} - 1.2 \text{ for } a = i, j \quad \& \quad l_{a} = 0 \text{ for } a \neq i, j
\]
for example.

Similarly, in equation (3.28), we have

$$
\alpha(y_{(1)}) = \frac{1}{(n-1)!} \left[ 1 - F(y_{(1)}) \right]^{n-1} f(y_{(1)}).
$$

and for example, for $i, j, k > 1$,

$$
\lambda_{ij(k)}^{(i_1j_1k_1)}(y_{(1)}) = f(y_{(1)}) \lambda_{ij(k)}^{(i_1j_1k_1)}(y_{(1)}), \quad l_i, l_j = 1, 2
$$

$$
\lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}) = f(y_{(1)}) \lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}), \quad l_i, l_j, l_k = 1, 2
$$

and for $j = 1$ and $i, k > 1$,

$$
\lambda_{ij(k)}^{(i_1j_1k_1)}(y_{(1)}) = g_y^{(i_1)} f(y_{(1)}) \lambda_{ij(k)}^{(i_1j_1k_1)}(y_{(1)}), \quad l_i, l_j = 1, 2
$$

$$
\lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}) = g_y^{(i_1)} f(y_{(1)}) \lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}), \quad l_i, l_j, l_k = 1, 2
$$

with

$$
\lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}) = \int_{y_{(1)} - 1}^{y_{(1)} - 2} \left[ \prod_{a=n}^{\lambda a} g_{(a)} y_{(a)} f(y_{[2,a]}) \right] dy_{[2,a]}, \quad l_a = 1, 2 \text{ for } a = i, j, k \quad (3.30)
$$

$$
\lambda_{ij(k)(l)}^{(i_1j_1k_1l_1)}(y_{(1)}) = 0 \text{ for } a \neq i, j, k
$$

for example.

Note that in the manner similar to that of (3.29) and (3.30), one may write the integrals for the remaining $\lambda$'s in (3.27) and (3.28). Further note that in (3.29) and (3.30), we just have expressed the specific $\lambda$'s in terms of integrals which we now simplify in the following section.
3.3.1 Computation of the Integral in (3.29) for general \( t_a \)

In practice, one requires the exact expressions for the above probability density functions of maxima and minima in equations (3.27) and (3.28). In order to do this, one needs to compute the integrations for \( \lambda_{ij(k)}^{t_j(t_k)(t_j(t_k))} (y_{(a)}) \) and \( \lambda_{ij(k)}^{t_j(t_k)(t_j(t_k))} (y_{(a)}) \) in equations (3.29) & (3.30) respectively for general \( t_a \). For the sake of simplicity, we show the integration technique below to obtain the result for \( \lambda_{ij(k)}^{t_j(t_k)(t_j(t_k))} (y_{(a)}) \) only for (3.29). More specifically, in this subsection, we compute the integral of the form

\[
I(y_{(a)}) = \int_{a}^{b} \prod_{n=1}^{m} y_{(a)}^n f(y_{[1,a-1]})dy_{[1,a-1]},
\]

for general \( t_a \).

3.3.1.1 Aids to Compute (3.31)

As shown in the following section, we need to evaluate two integrals

\[
\eta(x) = \int_{-\infty}^{x} u^t e^{-\frac{u^t}{2}} du \quad (3.32)
\]

and

\[
\xi(z) = \int_{-\infty}^{z} w^{t+d} e^{-\frac{w^t}{2}} dw \quad (3.33)
\]

for \( x, z \in \mathbb{R} \) to solve the integral in (3.31). In (3.32) and (3.33), \( t \) can take values 0 or 1 or 2, \( 0 < d < \infty \), and \( T \) is the sum of suitable number of \( t \). We, now, first solve these two integrals as in the following for positive \( x \) and \( z \).

It is well known that the incomplete gamma function \( I_\mu = \int_0^\mu s^{a-1} e^{-s} ds \) may be expressed in the form of a partial sum as

\[
I_\mu = \int_0^\mu s^{a-1} e^{-s} ds = \Gamma(a) \sum_{r=0}^{\infty} \frac{\mu^r e^{-\mu}}{r!}
\]

[cf. Gupta (1960, 1962) and Prescott (1974) ], which may be rewritten as

\[
I_\mu = \Gamma(a) \sum_{r=0}^{\infty} \frac{\mu^{r+a} e^{-\mu}}{(r+a)!}
\]
Now, \( x > 0 \), direct exploitation of this result yields

\[
\eta(x) = (-1)^l 2^{l+1} \frac{T^{l+1}}{2} \Gamma\left(\frac{l+1}{2}\right) + \sum_{r=0}^{\infty} \Gamma(\frac{r+1}{2}) \frac{\Gamma\left(\frac{r+1}{2}\right)}{2^{r+1} \sigma^{2r}(r + \frac{r+1}{2})!} x^{2r+1+1} \epsilon - \frac{\epsilon^2}{2r} \]

\[
= Q(t, \sigma^2) + \sum_{r=0}^{\infty} G(t, \sigma^2; r) x^{2r+1+1} \epsilon - \frac{\epsilon^2}{2r^2} \tag{3.31}
\]

where

\[
Q(t, \sigma^2) = (-1)^l 2^{l+1} \frac{T^{l+1}}{2} \Gamma\left(\frac{l+1}{2}\right)
\]

\[
G(t, \sigma^2; r) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2^{r+1} \sigma^{2r}(r + \frac{r+1}{2})!}
\]

and for \( z > 0 \),

\[
\xi(z) = (-1)^l 2^{l+1} \frac{T^{l+1}}{2} \Gamma\left(d + \frac{T+1}{2}\right) + \sum_{r=0}^{\infty} \frac{\Gamma\left(d + \frac{T+1}{2}\right)}{2^{r+1}(d + \frac{r+1}{2} + r)!} x^{2r+1+1} \epsilon - \frac{\epsilon^2}{2r^2} V^*
\]

\[
= Q(d, T, V^*) + \sum_{r=0}^{\infty} G(d, T, V^*; r) z^{2(r+1)+1} \epsilon - \frac{\epsilon^2}{2r^2} V^* \tag{3.35}
\]

where

\[
Q(d, T, V^*) = (-1)^l 2^{l+1} \frac{T^{l+1}}{2} \Gamma\left(d + \frac{T+1}{2}\right)
\]

\[
G(d, T, V^*; r) = \frac{\Gamma\left(d + \frac{T+1}{2}\right)}{2^{r+1}(d + \frac{r+1}{2} + r)!}
\]

Next, for \( x < 0 \) and \( z < 0 \), by similar operation, we have

\[
\eta(x) = Q(t, \sigma^2) + \sum_{r=0}^{\infty} G^*(t, \sigma^2; r) x^{2r+1+1} \epsilon - \frac{\epsilon^2}{2r^2} \tag{3.36}
\]

where

\[
G^*(t, \sigma^2; r) = (-1)^{l+1} G(t, \sigma^2; r)
\]

\[
\xi(z) = Q(d, T, V^*) + \sum_{r=0}^{\infty} G^*(d, T, V^*; r) z^{2(r+1)+1} \epsilon - \frac{\epsilon^2}{2r^2} V^* \tag{3.37}
\]
with

\[ G^*(d, T, V^*; r) = (-1)^{T+1} G(d, T, V^*; r) \]

The above integrations for \( \eta(z) \) and \( \xi(z) \) are done for general \( t, \sigma^2, d, T \) and \( V^* \). But, the integration in (3.31) requires the integration results for these functions for all possible \( t, \sigma^2, d, T \) and \( V^* \). To accommodate all these cases, we define

\[ Q_{i0} = Q(l_i, \sigma_i^2), \quad i = 1, 2, \ldots, n - 1 \]
\[ G_{i\sigma_k} = G(l_i, \sigma_i^2; r_k), \quad i, k = 1, 2, \ldots, n - 1 \]
\[ G_{i\sigma_k}^* = (-1)^{T+1} G_{i\sigma_k}, \quad i, k = 1, 2, \ldots, n - 1 \]

where

\[ \xi(l_i, \sigma_i^2) = (-1)^{l_i} 2^{l_i-1} \sigma^{l_i+1} \Gamma\left(\frac{l_i + 1}{2}\right) \]
\[ G(l_i, \sigma_i^2; r_k) = \Gamma\left(\frac{l_i + 1}{2}\right) \frac{1}{2^{l_i+1} \sigma^{2l_i} (r_k + \frac{l_i+1}{2})!} \]

as in (3.34).

Also define

\[ Q_{j, i} = Q(d_i, T; V^*_{ji}), \quad j = 1, 2, \ldots, n - 1, \quad i = 1, 2, \ldots, n - 2 \]
\[ G_{j, h, r_k} = G(d_i, T; V^*_{ji}; r_k), \quad j, h = 1, 2, \ldots, n - 1, \quad i = 1, 2, \ldots, n - 2 \]
\[ G_{j, h, r_k}^* = (-1)^{T+1} G_{j, h, r_k}, \quad j, h = 1, 2, \ldots, n - 1, \quad i = 1, 2, \ldots, n - 2 \]
\[ T_{ji} = \sum_{k=j-i}^{j-1} (l_k + 1) + l_j, \quad j = 1, 2, \ldots, n - 1, \quad i = 1, 2, \ldots, n - 2 \]
\[ V^*_{ji} = \sum_{k=j-i}^{j-1} \frac{1}{\sigma_k^2} + \frac{1}{\sigma_i^2}, \quad j = 1, 2, \ldots, n - 1, \quad i = 1, 2, \ldots, n - 2 \]

and

\[ C_n = \frac{1}{(2\pi)^{n/2} |D|^{1/2}} \]
where, as in (3.35),

\[ Q(d_i, T_{ji}, V_{ji}^*) = (-1)^{j_i} \frac{2^{\frac{d_i}{2}} \Gamma(\frac{d_i + \frac{l_i}{2} + 1}{2})}{V_{ji}^*} \]

\[ G(d_i, T_{ji}, V_{ji}^*; r_h) = \frac{V_{ji}^* \Gamma(d_i + \frac{l_i}{2} + 1)}{2\pi^4 (d_i + \frac{l_i}{2} + r_h)^{\frac{3}{2}}} \]

with

\[ d_i = \sum_{h=1}^{i} r_h, \quad i = 1, 2, \ldots, n - 2 \]

where \( r_h \) (\( h = 1, 2, \ldots, i \)) is the index value of \( i \) number of \( r_h \)'s used in the preceding consecutive \( G \) function in a particular product. For example, for \( G_{10r_1, Q_{2d_1}, G_{3w_2}, Q_{4d_4}} \), \( d_1 \) of \( Q_{2d_1} \) and \( Q_{4d_4} \) functions are \( r_1 \) and \( r_2 \) respectively, and for \( G_{10r_1, Q_{2d_1}, Q_{3d_1}, d_1 = r_1 + d_2} \) and \( d_2 = r_1 + r_2 \). Further, for positive \( Y_i \) (\( i = 1, \ldots, n \)) (implying \( x > 0 \) in (3.31), and \( z \cdot 0 \) in (3.35)) and for \( n_2 > n_1 \), let \( m H^{*{n_2}}(Q, G) \) denote a single combination of the product of \( n_1 \) 'G' functions and \( n_2 - n_1 \) 'Q' functions. As \( G \) and \( Q \) functions can be arranged in \( m_1 \), \( n_2 - q_{n_1, n_2} \) (say) possible ways to make such a product, for convenience of summation of all these product combinations, we label them as \( m H^{*(n_2)}_1(Q, G), \ldots, m H^{*(n_2)}_1(Q, G), \ldots, m H^{*(n_2)}_{q_{n_1, n_2}}(Q, G) \). For example, for \( n_1 = 3 \) and \( n_2 = 2 \), all possible combinations are

\[ m H^{*(2)}_1(Q, G) = Q_{10r_1} G_{2d_1} G_{3w_2} \]
\[ m H^{*(2)}_2(Q, G) = Q_{10r_1} Q_{2d_1} G_{3w_2} \]
\[ m H^{*(2)}_3(Q, G) = Q_{10r_1} G_{2d_1, r_2} G_{3w_2} \]

Note that without any lose of generality, one may label the second product \( G_{10r_1, Q_{2d_2}, G_{3w_1}} \) by \( m H^{*(2)}_1(Q, G) \) or \( m H^{*(2)}_3(Q, G) \).
In this case, \( d_1 = r_1, d_2 = r_1 + r_2 \) and in \( H_1^{d_2} (Q, G), G_{3d_4 r_2} = G(d_1, T_{31}, V_{31}; r_2) \), and for \( n_1 = 4 \) and \( n_2 = 2 \), all possible combinations are

\[
\begin{align*}
\text{4} H_1^{d_2} (Q, G) &= Q_{00} Q_{20} G_{3d_4 r_2} G_{4d_4 r_2} \\
\text{4} H_2^{d_2} (Q, G) &= Q_{00} Q_{20} r_1 Q_{3d_4 r_2} G_{4d_4 r_2} \\
\text{4} H_3^{d_2} (Q, G) &= Q_{10} Q_{2d_4} G_{3d_4 r_2} Q_{4d_4} \\
\text{4} H_4^{d_2} (Q, G) &= Q_{00} Q_{2d_4 r_1} Q_{3d_4 r_2} Q_{4d_4} \\
\text{4} H_5^{d_2} (Q, G) &= Q_{10} Q_{2d_4 r_1} Q_{3d_4 r_2} Q_{4d_4} \\
\end{align*}
\]

Here \( d_1 \) may be \( r_1 \) or \( r_2, d_2 = r_1 + r_2 \) and in \( H_1^{d_4} (Q, G), G_{4d_4 r_2} = G(d_1, T_{41}, V_{41}; r_2) \).

Note that any \( G_{jd_4 r_2} \) or \( Q_{jd_4} \) function will appear in the product combination only if it is preceded by a \( G \) function. More specifically, in any product combination, \( G_{jd_4 r_2} \) function will be preceded by \( (j - 1) \) \( 'Q' \) or \( 'G' \) functions and \( i \) number of \( G \) functions.

Similarly, in any product combination, \( Q_{jd_4} \) function will also be preceded by \( (j - 1) \) \( 'Q' \) or \( 'G' \) functions and \( i \) number of \( G \) functions. Furthermore, note that, for the case when smaller order statistics take negative values, that is, \( Y_i < 0 \) (implying \( r < 0 \) in (3.36), and \( z < 0 \) in (3.37)), the \( G \) functions in each term will be replaced by corresponding \( G^* \) functions.

3.3.1.2 Expressions for (3.31) for \( n = 2, 3 \) and 4

In this subsection, we discuss the integration technique in details for special cases with \( n = 2, 3 \) and 1. By using the notations in (3.22), for \( n = 2 \), it follows from (3.31) that

\[
I(y_{(2)}) = \int_{-\infty}^{y_{(2)}} y_{(1)} f(y_{(1)}) dy_{(1)}
\]
\[
\begin{align*}
I(y_{(2)}) &= C_1 Q_{10} + \sum_{r_1=0}^{\infty} C_{10r_1} y_{(2)}^{2r_1+t_1+1} e^{-\frac{\phi_{r_1}^2}{2\sigma_1^2}} \\
&= \Delta_1^{(1)} + \sum_{r_1=0}^{\infty} \psi_1^{(1)} y_{(2)}^{2r_1+t_1+1} e^{-\frac{\phi_{r_1}^2}{2\sigma_1^2}} \tag{3.39}
\end{align*}
\]

where
\[
\Delta_1^{(1)} = C_1 Q_{10}
\]
and
\[
\psi_1^{(1)} = C_1 C_{10r_1}
\]
with
\[
Q_{10} = Q(I_1, \sigma_1^2)
\]
and
\[
C_{10r_1} = G(I_1, \sigma_1^2; r_1)
\]

Similarly, for \( n = 3 \), by using (3.22), the equation (3.31) yields
\[
I(y_{(3)}) = \int_{-\infty}^{y_{(2)}} \int_{-\infty}^{y_{(2)}} y_{(1)}^{t_1} y_{(2)}^{t_2} f(y_{(1)}) f(y_{(2)}) dy_{(1)} dy_{(2)}
\]
\[
= C_2 \int_{-\infty}^{y_{(2)}} \left\{ \int_{-\infty}^{y_{(1)}} y_{(1)}^{t_1} e^{-\frac{\phi_{r_1}^2}{2\sigma_1^2}} dy_{(1)} \right\} y_{(2)}^{t_2} e^{-\frac{\phi_{r_2}^2}{2\sigma_2^2}} dy_{(2)}
\]
where
\[
C_2 = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_1 \sigma_2}
\]
\[(1 + \gamma) \sum_{\frac{1-\epsilon}{1-\xi}} \frac{1}{1+\xi} = \sum_{\frac{1-\epsilon}{1-\xi}} \frac{1}{1+\xi} + 1 = \sum_{\frac{1-\epsilon}{1-\xi}} \frac{1}{1+\xi} = \sum_{\xi}
\]

Where

\[\left \{ \begin{array}{l}
(1-\epsilon) \sum_{\frac{1-\epsilon}{1-\xi}} \frac{(3)_{j}^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \end{array} \right \}
\]

\[\left \{ \begin{array}{l}
(1-\epsilon) \sum_{\frac{1-\epsilon}{1-\xi}} \frac{(3)_{j}^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \\
\sum_{\xi} \frac{\gamma_{x} - \gamma_{x}^{(1-\epsilon)}(\frac{(3)_{j})^{(i)}}{\xi} \Theta \sum_{\xi} + \end{array} \right \}
\]

Note that in \([1.3.4]\), the integrals are in the form \(\int_{(x)}^{(x)}\) for \(\epsilon = \eta\) with \(\eta = \mu\) in the case with \(\eta = \mu\) in the form \(\int_{(x)}^{(x)}\).

\[(\xi) \sum_{\frac{1-\epsilon}{1-\xi}} \frac{(3)_{j}^{(i)}}{\xi} \Theta \sum_{\xi} + \]

Now, by \([2.3.4](x)\), this integral reduces to
\[
\Delta^{(2)}_1 = C_2 Q_{10} Q_{20} = C_2 \prod_{i=1}^{2} Q_{i0}
\]
\[
\Theta^{(2)}_1 = C_2 Q_{10} G_{20 r_1} = C_2 \Delta^{(1)}_1 G_{20 r_1}
\]
\[
\Theta^{(2)}_2 = C_2 G_{10 r_1} Q_{20 r_1} = C_2 \Delta^{(1)}_2 G_{20 r_1}
\]
\[
\psi^{(2)}_1 = C_2 G_{10 r_1} G_{20 r_2} = C_2 \prod_{j=1}^{2} G_{j d_j - 1 r_j}, \quad d_0 = 0 \& d_1 = r_1
\]
with \[
Q_{20} = Q(t_2, \sigma^2)
\]
\[
Q_{2 d_1} = Q(d_1, T_{21}, V_{21}^2)
\]
and \[
G_{2 d_1 r_2} = G(d_1, T_{21}, V_{21}^2, r_2)
\]

Now, for the case \( n = 1 \), by (3.22), we write the integral from (3.31)

\[
I(y_{(4)}) = \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(4)}} y_{(1)} y_{(2)} y_{(3)} f(y_{(1)}) f(y_{(2)}) f(y_{(3)}) dy_{(1)} dy_{(2)} dy_{(3)}
\]
\[
= C_3 \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(4)}} y_{(1)} y_{(2)} y_{(3)} f(y_{(1)}) f(y_{(2)}) f(y_{(3)}) dy_{(1)} dy_{(2)} dy_{(3)}
\]
\[
= C_3 \int_{-\infty}^{y_{(4)}} \int_{-\infty}^{y_{(4)}} y_{(1)} y_{(2)} y_{(3)} f(y_{(1)}) f(y_{(2)}) f(y_{(3)}) dy_{(1)} dy_{(2)} dy_{(3)}
\]

where

\[
C_3 = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_1 \sigma_2 \sigma_3}
\]

Now, in the manner similar to that of \( n = 2 \) and 3 and by using the results of \( \eta(x) \) and \( \xi(z) \) in different steps, we obtain

\[
I(y_{(4)}) = C_3 Q_{10} Q_{20} Q_{30} + C_3 \sum_{r_1=0}^{\infty} \left\{ Q_{10} Q_{20} G_{30 r_1} y_{(4)}^{2 r_1 + 1} e^{-\frac{y_{(4)}^2}{2 \sigma_1^2}} + Q_{10} G_{20 r_1} Q_{3 d_1} \right\}
\]
\[
+ C_3 Q_{10} Q_{2 d_1} Q_{30} + C_3 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \left\{ C_{10 r_1} Q_{2 d_1} G_{30 r_2} y_{(4)}^{2 r_2 + r_1 + 1} e^{-\frac{y_{(4)}^2}{2 \sigma_1^2}} \right\}
\]
\[
+ C_3 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} \left\{ C_{10 r_1} C_{2 d_1 r_2} G_{3 d_2 r_3} y_{(4)}^{2(r_2 + r_3) + r_1 + 1} e^{-\frac{y_{(4)}^2}{2 \sigma_1^2}} \right\}
\]
\[
= C_3 \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} \left\{ C_{10 r_1} C_{2 d_1 r_2} G_{3 d_2 r_3} y_{(4)}^{2(r_2 + r_3) + r_1 + 1} e^{-\frac{y_{(4)}^2}{2 \sigma_1^2}} \right\}
\]
\[
\Delta_i^{(3)} + \sum_{r_1=0}^{\infty} \left\{ \frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sum_{k=3-1}^{3} \frac{1}{\sigma_k^2}} \right\} + \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} 3B^{(2)}(r_1, r_2) + \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} \psi_{1}^{(3)} y_{1}^{2(d_4+r_3)+T_{3}+1} e^{-\frac{y_{1}^{2}}{2\sigma_1^2}}
\]

where

\[
V_{31}^{*} = \frac{1}{\sigma_3^2} + \frac{1}{\sigma_2^2} = \frac{1}{\sigma_3^2} + \frac{1}{\sum_{k=3-1}^{3} \frac{1}{\sigma_k^2}}
\]

\[
V_{32}^{*} = \frac{1}{\sigma_3^2} + \frac{1}{\sigma_2^2} = \frac{1}{\sigma_3^2} + \frac{1}{\sum_{k=3-2}^{3-1} \frac{1}{\sigma_k^2}}
\]

\[
T_{31} = l_{3} + t_{2} + 1 = l_{3} + \sum_{k=3-1}^{3-1} (l_{k} + 1)
\]

\[
T_{32} = l_{3} + t_{2} + t_{1} + 1 = l_{3} + \sum_{k=3-2}^{3-1} (l_{k} + 1)
\]

\[
\Delta_i^{(3)} = C_{3}Q_{10}Q_{20}Q_{30} = C_{3} \prod_{i=1}^{3} Q_{3i}
\]

\[
\Theta_{1}^{(3)} = C_{3}Q_{10}Q_{20}G_{10}r_{1} = C_{3}Q_{10}G_{10}r_{1}
\]

\[
\Theta_{2}^{(3)} = C_{3}Q_{10}G_{10}r_{1}Q_{3i} = C_{3}Q_{10}G_{10}r_{1}Q_{3i}
\]

\[
\Theta_{2}^{(3)} = C_{2}Q_{10}r_{1}Q_{2d_{1}}Q_{30} = C_{3}Q_{10}r_{1}Q_{2d_{1}}Q_{30}
\]

\[
\psi_{1}^{(3)} = C_{3}Q_{10}r_{1}Q_{2d_{1}}r_{2}Q_{3d_{2}} = C_{3}Q_{10}r_{1}Q_{2d_{1}}Q_{3d_{2}}
\]

and

\[
3B^{(2)}(r_1, r_2) = C_{3} \sum_{i=1}^{q_{3}} 3H_{i}^{(2)}(Q, G)
\]

with

\[
q_{3} = C_{2} = 3
\]

and

\[
3H_{1}^{(2)}(Q, G) = 3H_{1}^{(2)}(Q, G) y_{1}^{2r_{2}+t_{3}+1} e^{-\frac{y_{1}^{2}}{2\sigma_1^2}}
\]

\[
3H_{2}^{(2)}(Q, G) = 3H_{1}^{(2)}(Q, G) y_{1}^{2(d_{4}+r_{2})+T_{3}+1} e^{-\frac{y_{1}^{2}}{2\sigma_1^2}}
\]

\[
3H_{3}^{(2)}(Q, G) = 3H_{1}^{(2)}(Q, G)
\]
where

\[ 3H^{\alpha \beta}(Q, i) = G_{10}r_1 Q_{2d_2} r_2, \]

\[ 3H^{\alpha \beta}(Q, i) = Q_{10} Q_{2d_2} r_2, \]

\[ 3H^{\alpha \beta}(Q, i) = G_{10} r_1 Q_{2d_2} r_2, \]

with \( Q_{30} = Q(t, \sigma \frac{a_2}{2}) \)

\[ Q_{3d_2} = Q(d_1, T_{31}, V^{*}_{11}) \]

\[ Q_{3d_2} = Q(d_2, T_{32}, V^{*}_{12}) \]

\[ G_{3d_1 r_1} = G(d_1, T_{31}, V^{*}_{11}; r_2) \]

and \( G_{3d_2 r_1} = G(d_2, T_{32}, V^{*}_{12}; r_3) \)

Here \( d_1 \) and \( d_2 \) will be \( r_1 \) and \( r_1 + r_2 \) respectively.

### 3.3.1.3 Expression for (3.31) for general \( n \)

It now follows from subsection 3.3.1.2 that for general \( n \), one obtains

\[
I(y_{(n)}) = \int_{y_{(n)}}^{2,n} \prod_{n=1}^{2,n} y_{(n)}^{\mu_n} f(y_{(n)}; y_{(n-1)}) dy_{(n)}
\]

\[
= C_{n-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y_1^{r_1} \cdots y_{n-1}^{r_{n-1}} e^{-\frac{\eta_{1}^{2}}{2}} \cdots e^{-\frac{\eta_{n-1}^{2}}{2}} dy_{(n-1)} \]

\[
= C_{n-1} \Delta_{(n-1)} + \sum_{r_1=0}^{\infty} \cdots \sum_{r_1=0}^{\infty} \left\{ \Theta_{1}^{(n-1)}(r_1) y_{(n)}^{r_1} e^{-\frac{\eta_{1}^{2}}{2}} \right\} + \sum_{l=2}^{\infty} \Theta_{l}^{(n-1)}(r_1)
\]

\[
+ \sum_{r_1=0}^{\infty} \cdots \sum_{r_2=0}^{\infty} \sum_{r_1+1}^{\infty} \cdots \sum_{r_m=0}^{\infty} n^{-1} B_{r_1, \ldots, r_m} \psi_{(n)}^{(n-1)}(r_1, \ldots, r_m) + \cdots
\]

\[
+ \sum_{r_1=0}^{\infty} \cdots \sum_{r_2=0}^{\infty} \sum_{r_1+2}^{\infty} \cdots \sum_{r_m=0}^{\infty} n^{-1} B_{r_1, \ldots, r_m} \psi_{(n-1)}^{(n-2)}(r_1, \ldots, r_m) + \cdots
\]

\[
\times y_{(n)}^{2(d_n+r_{n-1})+T_{(n-1)}+n-1} e^{-\frac{\eta_{1}^{2}}{2}} \psi_{(n-1)}^{(n-1)}(r_1, \ldots, r_m) \quad (3.15)
\]
where

\[ C'_{n-1} = \frac{1}{(2\pi)^{\frac{n-1}{2}} |D|^\frac{1}{2}} \]

\[ \Delta^{(n-1)}_i = C'_{n-1} \prod_{i=1}^{n-1} Q_{A_i} \]

\[ \psi^{(n-1)}_i(r_1, \ldots, r_{n-1}) = \sum_{j=1}^{n-1} C'_j \prod_{i=1}^{n-1} \Gamma_{D_i} \quad \text{with} \quad d_0 = 0 \]

\[ \Theta^{(n-1)}_i(r_1) = \sum_{j=1}^{n-1} \prod_{i=1}^{n-1} \Gamma_{D_i} \left[ Q_{(n-1)-i+2} \right]^{l_i} \left[ \Delta^{(n-1)-i+3} \right]^{l_i} \]

with \( l_{s_1} = 0 \) if \((n - 1) - l + 2 > n - 1\) and \( l_{s_1} = 1\), otherwise. Similarly \( l_{s_2} = 0 \) if \((n - 1) - l + 3 > n - 1\) and \( l_{s_2} = 1\), otherwise. Furthermore, in (3.15), for \( m = 2, 3, \ldots, n - 2\), we have

\[ n-1 B^{(m)}(r_1, \ldots, r_m) = C'_{n-1} \sum_{i=1}^{n-1} H_i^{(m)}(Q, G) \]

with

\[ q_{n-1,m} = n-1 C'_{m} \]

and

\[ n-1 H_i^{(m)}(Q, G) = n-1 H_i^{(m)}(Q, G) \left[ \frac{2^{d^*+r^*+T^*+V^*+1} - d_{G}}{2^{\frac{n+1}{2}}} \right] \]

where \( l_{s} = 1 \) if the product function \( n-1 H_i^{(m)} \) is ended by a \( G \) function (cf. section 3.3.1.1) and \( l_{s} = 0\), otherwise. For the case when \( l_G = 1, d^*, r^*, T^* \) and \( V^* \) in the square bracket \([\ ]\) are replaced by \( d, r, T \) and \( V \), respectively, where the latter functions are taken from the last \( G \) function in \( n-1 H_i^{(m)}(Q, G) \).

Note that one may exploit the integration result in (3.15) to obtain any \( \lambda \)'s necessary in (3.27) by putting appropriate values of \( t_{\alpha} \). For example, for \( \lambda^{I_{ij}}_{(kl)} \) \((y_{(n)})\) in (3.29), we require to put \( t_{\alpha} = 1.2 \) for \( a = i, j \) and \( t_{\alpha} = 0 \) for all \( a \neq i, j \) in equation (3.15).
Chapter 4

Computational Aspects and Applications

4.1 Computation of Percentile Points of $Y_{(n)}$

In this chapter, we provide several exact percentile points of the distribution of the maxima for the correlated normal variables under three different situations. In the first situation, we compute percentile points of the maxima for the equi-correlated (positive or negative) normal variables with zero mean and equal variance $\sigma^2$. Our results for positive correlations are verified with the results provided by Gupta (1973) among others. Also our results for negative correlation will supplement some of the results provided by Hoffman and Saw (1975). In the second situation, we provide the percentile points of the maxima for the homoscedastic but unequally (positively or negatively) correlated normal variables. These distributional results for the homoscedastic unequal correlated...
normal variables case may be applied to certain repeated measures data, for example, to any data sets, where the correlations of the data follow the correlation structure of a stationary Gaussian process, say, AR(1). In the last situation, we compute the percentile points of the maxima for the unequal positively or negatively correlated normal variables with heteroscedastic variances. In this chapter, we also compare the performance of the Bonferroni bounds approximations with our small correlations approach in computing the percentile points of the maxima for both homoscedastic and heteroscedastic cases.

4.2 First case: Homoscedastic equi-correlated (positive and negative) normal variables

Based on the transformation as in the equation (2.1), Gupta (1973) provided tables for the \((1 - \alpha)\) percentile points of \(Y_{(n)}\) for selected values of \(n\), \(\alpha\) and positive \(\rho\), where \(Y_{(n)}\) is the maxima of the \(n\) standardized normal random variables \(Y_1, Y_2, \ldots, Y_n\) having correlation matrix \(\rho J_n - \rho I_n\) with \(J_n\) as the \(n \times n\) unit matrix and \(I_n\) as the \(n \times n\) identity matrix. This type of transformations is not suitable to handle the negatively equi-correlated normal variables cases. Further more, this approach requires a difficult integration (cf. section 2.1) to compute the percentile points of the maxima. Instead of solving this integral, Gupta (1973) has used a numerical approximation to resolve this problem. To examine the performance of our approach, we exploit our method discussed in the last chapter and compare the percentile points of \(Y_{(n)}\) for \(\alpha = 0.05, 0.025 \& 0.01\) and \(\rho = 0.100, 0.125, 0.200 \& 0.250\) with those given in Gupta (1973).

For the equi-correlated normal variables \(Y_1, Y_2, \ldots, Y_n\) with \(E(Y_i) = 0\), \(E(Y_i^2) = \sigma^2\).
for all \( i = 1, 2, \ldots, n \) and \( \lambda(Y_i, Y_j) = \rho \), for all \( i \neq j \), we obtain the probability density function of the maxima by putting \( \rho_{ij} = \rho \) for all \( i \neq j \) and \( \sigma_i^2 = \sigma^2 \) for \( i = 1, 2, \ldots, n \), in (3.27). The density is given by

\[
g_n^*(y_n, \rho, \sigma^2) \approx U_n^* \phi(y_n, \sigma^2) + D_n^* \sum_{i<j}^{n} \lambda_{ij(\text{kl})}^i(y_n, \sigma^2) - Q_n^* \sum_{i=1}^{n} \lambda_{i(kl)}^i(y_n, \sigma^2) \\
+ S_n^* \sum_{i<j}^{n} \lambda_{ij(\text{kl})}^{ij(\text{kl})}(y_n, \sigma^2) + T_n^* \sum_{i \neq j \neq k}^{n} \lambda_{ij(kl)}(y_n, \sigma^2) \\
+ M_n^* \sum_{i \neq j \neq k \neq l}^{n} \lambda_{ijkl}^i(y_n, \sigma^2) 
\]

(1.1)

where

\[
U_n^* = n! \left\{ 1 + \frac{1}{2} n(n - 1) \rho^2 \right\} \\
D_n^* = n! \left\{ p - (n - 2) \rho \right\} \frac{1}{\sigma^2} \\
Q_n^* = n!(n - 1) \rho^2 \frac{1}{\sigma^2} \\
S_n^* = n! \rho^2 \frac{1}{2\sigma^4} \\
T_n^* = n! \rho^2 \frac{1}{\sigma^4} \\
M_n^* = 3n! \rho^2 \frac{1}{\sigma^4} \\

\phi(y_n, \sigma^2) = \phi(y_n) \bigg|_{\sigma_1^2 = \ldots = \sigma_n^2 = \sigma^2},
\]

and for example,

\[
\lambda_{ij(\text{kl})}^i(y_n, \sigma^2) = \lambda_{ij(\text{kl})}^i(y_n) \big|_{\sigma_1^2 = \ldots = \sigma_n^2 = \sigma^2}, \quad l_i, l_j = 1, 2
\]

with \( \phi(y_n) \) and \( \lambda_{ij(\text{kl})}^i(y_n) \)'s as in equation (3.27). Note that the coefficients \( U_n^*, D_n^*, Q_n^*, S_n^*, T_n^*, \) and \( M_n^* \) in (4.1) are the special cases of the coefficients defined in (3.27).
Next, to compute the percentile points of \( Y_{(n)} \), namely, the values of \( h \), we may easily compute the distribution function of maxima from (4.1) as given by

\[
G_{n}^\ast(h, \rho, \sigma^2) = \Pr \{ Y_{(n)} \leq h \}
\]

\[
= \int_{-\infty}^{h} g_n^\ast(y_{(n)}; \sigma^2, \rho)dy_{(n)}
\]

\[
\simeq I_n^\ast \Phi(h, \sigma^2) + D_n^\ast \sum_{i<j} A_{ij}^{1100}(h, \sigma^2) - Q_n^\ast \sum_{i=1}^{n} A_{ii}^{2000}(h, \sigma^2)
\]

\[+ S_n^\ast \sum_{i<j} A_{ij}^{2200}(h, \sigma^2) + T_n^\ast \sum_{i \not= j \not= k} A_{ikj}^{2110}(h, \sigma^2)
\]

\[+ M_n^\ast \sum_{i \not= j \not= k \not= l} A_{ijkl}^{1111}(h, \sigma^2)
\]

(4.2)

where

\[
\Phi(h, \sigma^2) = \int_{-\infty}^{h} \phi(y_{(n)}; \sigma^2)dy_{(n)}
\]

\[= \left[ \int_{-\infty}^{h} \phi(y_{(n)})dy_{(n)} \right]_{\sigma^2_1 = \ldots = \sigma^2_n = \sigma^2}
\]

\[= \Phi(h) |_{\sigma^2_1 = \ldots = \sigma^2_n = \sigma^2}
\]

with \( \Phi(h) = \int_{-\infty}^{h} \phi(y_{(n)})dy_{(n)} \)

\[= \frac{1}{n!} \left[ \ell'(h) \right]^n
\]

(4.3)

And for example,

\[
A_{i(jkl)}^{1100}(h, \sigma^2) = \int_{-\infty}^{h} \lambda_{i(jkl)}^{1100}(y_{(n)}; \sigma^2)dy_{(n)} \quad \text{if } i, j = 1, 2
\]

\[= \left[ \int_{-\infty}^{h} \lambda_{i(jkl)}^{1100}(y_{(n)})dy_{(n)} \right]_{\sigma^2_1 = \ldots = \sigma^2_n = \sigma^2}
\]

\[= \lambda_{i(jkl)}^{1100}(h) |_{\sigma^2_1 = \ldots = \sigma^2_n = \sigma^2}
\]

with \( \lambda_{i(jkl)}^{1100}(h) = \int_{-\infty}^{h} \lambda_{i(jkl)}^{1100}(y_{(n)})dy_{(n)} \) \quad \text{if } i, j = 1, 2

(4.4)

In equation (4.3), for \( i, j < n \) and \( t_a = 1.2 \) for \( a = i, j \) \& \( t_a = 0 \) for \( a \not= i, j \). \( \lambda_{i(jkl)}^{1100}(h) \)
is evaluated by using (3.27) & (3.29) given by

\[ L_{ij(k1)}(h) = \int_{-\infty}^{\infty} f(y(u)) Y_{ij(k1)}(y(u)) dy(u) \]

\[ = \int_{-\infty}^{\infty} y_i^n f(y(u)) \left\{ \prod_{a=1}^{n-1} \frac{y_a^{l_n} f(y_{[1,n-1]})}{dy(y_{[1,n-1]})} \right\} dy(u) \quad (1.5) \]

where \( l_n = 0 \)

Next, by using the results of \( \eta(x), \xi(z) \) and \( I(y(u)) \) in the equations (3.31), (3.35) and (3.43), respectively, we obtain

\[ L_{ij(k1)}(h) = \Delta \Delta^* \left( \sum_{r_1=0}^{\infty} \Theta_1^{(n)}(r_1) h^{2r_1 + n} e^{-\frac{r_1^2}{\sigma_n^2}} + \sum_{l=2}^{n} \Theta_l^{(n)} \right) \]

\[ + \sum_{r_1}^{\infty} \sum_{r_2}^{\infty} \sum_{r_{n-1}}^{\infty} B^{(2)}(r_1, r_2; h) + \ldots + \sum_{r_1}^{\infty} \ldots \sum_{r_{n-1}}^{\infty} B^{(n)}(r_1, \ldots, r_n; h) \]

\[ + \sum_{r_1}^{\infty} \ldots \sum_{r_{n-1}}^{\infty} \Psi^{(n)}(r_1, \ldots, r_{n-1}) \}

\[ \times \quad h^{2(d_n-1)+r_{n-1}+T_{n(n-1)+1}} e^{-\frac{h^2}{2 \Gamma_{n(n-1)}}} \quad (1.6) \]

where \( \Delta, \Theta^*, B^* \) and \( \Psi \) are defined as in equation (3.13) and \( t_n = 1, 2 \) for \( a = i, j, t_n = 0 \) for \( a \neq i, j \) and \( a = n \) in this case. Note that one may exploit the integration result in (4.1) to obtain any \( \Delta \)'s necessary in (1.2) by putting appropriate values of \( t_n \).

In order to examine the performance of the proposed procedure, we now compute the probability \( G_n^*(h, \rho, \sigma^2) \) given by (4.2) for selected values of \( \alpha = 0.010, 0.025 \) and 0.050, \( \rho = 0.100, 0.125, 0.200 \) and 0.250 as in Gupta (1973). Without loss of generality, we consider \( \sigma^2 = 1 \), and \( n = 2, 3 \& 4 \) and compute \( G_n^*(h, \rho, \sigma^2) \) by borrowing the values of \( h \) from Gupta (1973). These probabilities along with the cumulative probabilities computed by Gupta (1973) are shown in the Table 4.1. More specifically, for different
Table 4.1: The Actual Probabilities for the maxima of positive equi-correlated normal variables based on CTM and SCA for selected $\alpha$, $\rho$ and $n$ with $\sigma^2 = 1$, corresponding to the Nominal $100(1 - \alpha)$% probabilities with $\alpha = 0.01, 0.025$ and $0.05$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>n = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.010</td>
<td>0.025</td>
</tr>
<tr>
<td>0.100</td>
<td>$h$</td>
<td>2.5739</td>
</tr>
<tr>
<td></td>
<td>CTM</td>
<td>0.989799</td>
</tr>
<tr>
<td></td>
<td>SCA</td>
<td>0.989832</td>
</tr>
<tr>
<td>0.125</td>
<td>$h$</td>
<td>2.5736</td>
</tr>
<tr>
<td></td>
<td>CTM</td>
<td>0.989801</td>
</tr>
<tr>
<td></td>
<td>SCA</td>
<td>0.989812</td>
</tr>
<tr>
<td>0.200</td>
<td>$h$</td>
<td>2.5722</td>
</tr>
<tr>
<td></td>
<td>CTM</td>
<td>0.989802</td>
</tr>
<tr>
<td></td>
<td>SCA</td>
<td>0.989877</td>
</tr>
<tr>
<td>0.250</td>
<td>$h$</td>
<td>2.5709</td>
</tr>
<tr>
<td></td>
<td>CTM</td>
<td>0.989802</td>
</tr>
<tr>
<td></td>
<td>SCA</td>
<td>0.989906</td>
</tr>
<tr>
<td>$\rho \downarrow$</td>
<td>$n = 3$</td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>--------</td>
<td>-------</td>
</tr>
<tr>
<td>$\alpha = 0.010$</td>
<td>2.7105</td>
<td>2.3878</td>
</tr>
<tr>
<td>CTM</td>
<td>0.989801</td>
<td>0.974802</td>
</tr>
<tr>
<td>SCA</td>
<td>0.988399</td>
<td>0.974687</td>
</tr>
</tbody>
</table>

| $\alpha = 0.025$ | 2.7099 | 2.3829 | 2.1111 |
| CTM         | 0.989799 | 0.974805 | 0.949801 |
| SCA         | 0.988554 | 0.974806 | 0.949234 |

| $\alpha = 0.050$ | 2.7078 | 2.3829 | 2.1080 |
| CTM         | 0.989801 | 0.974807 | 0.949813 |
| SCA         | 0.991875 | 0.975631 | 0.952218 |

| $\alpha = 0.100$ | 2.7058 | 2.3795 | 2.1029 |
| CTM         | 0.989802 | 0.974807 | 0.949811 |
| SCA         | 0.991132 | 0.976198 | 0.951731 |

<table>
<thead>
<tr>
<th>$n=4$</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.010$</td>
<td>2.8041</td>
<td>2.1907</td>
<td>2.2276</td>
</tr>
<tr>
<td>CTM</td>
<td>0.989801</td>
<td>0.974804</td>
<td>0.949806</td>
</tr>
<tr>
<td>SCA</td>
<td>0.990062</td>
<td>0.975166</td>
<td>0.950091</td>
</tr>
</tbody>
</table>

| $\alpha = 0.125$ | 2.8034 | 2.1894 | 2.2255 |
| CTM         | 0.989799 | 0.974801 | 0.949801 |
| SCA         | 0.990824 | 0.975913 | 0.950789 |

| $\alpha = 0.200$ | 2.7078 | 2.3829 | 2.1080 |
| CTM         | 0.989802 | 0.974801 | 0.949814 |
| SCA         | 0.994899 | 0.975946 | 0.951433 |

| $\alpha = 0.250$ | 2.7083 | 2.4804 | 2.2116 |
| CTM         | 0.989802 | 0.974807 | 0.949801 |
| SCA         | 0.994904 | 0.975926 | 0.957025 |
h, Gupta’s and our results are shown under the heading C'TM (correlation transformation method) and SCA (small correlations approach) respectively.

It is clear from Table 4.1 that the SCA based actual probabilities are pretty close to the C'TM based actual probabilities of Gupta’s (1973), for the equi-correlated standard normal variables case for the equi-correlation coefficients $\rho \leq 0.250$. However, for $\rho > 0.250$, our SCA based results will not be the same as Gupta’s (1973) C'TM based results, which is expected. This is because, our approximation is developed based on small values of correlations. We, however, computed some of the probabilities for $\rho \geq 0.300$ and find that for $n = 3$ and $\rho = 0.300$, the SCA yields the actual probabilities 0.996865, 0.980392 and 0.961371 corresponding to 99%, 97.5% and 95% nominal probabilities respectively. Similarly, for $n = 4$, these actual probabilities were found to be 0.997539, 0.979913 and 0.961380 respectively, showing the departure from the nominal values.

For negative equi-correlations case, Gupta’s (1973) method given in (2.4) is not suitable to compute the percentile points of the maxima. For this case, however, Hoffman and Saw (1975) proposed an alternative method to compute the percentile points of the maxima but they did not provide detailed numerical results in the paper. While our main interest is to obtain the percentage points of the maxima for unequally (positively or negatively) correlated normal random variables, we still computed the percentile points of the maxima based on our method for several negative small equi-correlations $\rho = -0.100, -0.125, -0.200$ and -0.250. The results are shown in Table 4.2, which may be verified with the results of Hoffman and Saw (1975).

Note that the percentile points shown in table 4.2 are computed by considering the results for the positive equi-correlation provided by Gupta (1973) as the initial values. Moreover, it was found that the results in the above table do not remain the same
Table 4.2: The SCA based $100(1 - \alpha)\%$ percentile points of the maxima for negative equi-correlations and selected $\alpha$ and $n$ with $\sigma^2 = 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\rho$</th>
<th>$\alpha = 0.010$</th>
<th>$0.025$</th>
<th>$0.050$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.100</td>
<td>2.5123</td>
<td>2.2519</td>
<td>1.9878</td>
</tr>
<tr>
<td></td>
<td>-0.125</td>
<td>2.5067</td>
<td>2.2408</td>
<td>1.9707</td>
</tr>
<tr>
<td></td>
<td>-0.200</td>
<td>2.1956</td>
<td>2.2397</td>
<td>1.9667</td>
</tr>
<tr>
<td></td>
<td>-0.250</td>
<td>2.1811</td>
<td>2.2251</td>
<td>1.9513</td>
</tr>
<tr>
<td>3</td>
<td>-0.100</td>
<td>2.5405</td>
<td>2.3065</td>
<td>2.0907</td>
</tr>
<tr>
<td></td>
<td>-0.125</td>
<td>2.1799</td>
<td>2.2799</td>
<td>2.0509</td>
</tr>
<tr>
<td></td>
<td>-0.200</td>
<td>2.3778</td>
<td>2.1829</td>
<td>2.0218</td>
</tr>
<tr>
<td></td>
<td>-0.250</td>
<td>2.2858</td>
<td>2.1295</td>
<td>1.9837</td>
</tr>
<tr>
<td>4</td>
<td>-0.100</td>
<td>2.7279</td>
<td>2.4567</td>
<td>2.2156</td>
</tr>
<tr>
<td></td>
<td>-0.125</td>
<td>2.6801</td>
<td>2.3912</td>
<td>2.1967</td>
</tr>
<tr>
<td></td>
<td>-0.200</td>
<td>2.5178</td>
<td>2.3369</td>
<td>2.1613</td>
</tr>
<tr>
<td></td>
<td>-0.250</td>
<td>2.1765</td>
<td>2.2801</td>
<td>2.1409</td>
</tr>
</tbody>
</table>
4.3 Second Case: Homoscedastic but unequally (positively or negatively) correlated normal variables

There also exists a few methods to study certain specific inference problems for the order statistics of the unequally positively or negatively correlated normal variables. Based on the V-function described by Nicholson (1943), Gupta et al. (1964) have studied the distribution of the range, $Y_{(n)} - Y_{(1)}$, of unequally correlated normal variables for $n = 3$ and 4. For a very special case of unequal correlation structures such that $E(Y_i Y_j) = 0$, for all $i \neq j$ & $j \neq i + 1$ and non-zero $E(Y_i Y_{i+1}) = \rho_{i,i+1}$, Greig (1967) studied the distributional properties of the order statistics of the correlated normal random variables.

As discussed in chapter 3, we have studied the distributions of the order statistics of unequally (positively or negatively) correlated normal random variables. But this was done for small correlations. The small correlations among normal variables arise in many practical situations, in particular, in the context of cluster regression analysis. In cluster analysis, the cluster sizes are usually small. But for generality, we have, however, provided the theory in the last chapter for general $n$. For cluster regression analysis with small $\rho$'s among the observations within the cluster, we refer to Rao, Sutradhar and Yue (1993), and Wu, Holt and Holmes (1988), among others.
4.3.1 Application to AR(1) Models

We now consider a non-regression set-up where cluster observations may be generated by repetitions of measurements of single individual over a period of time. In such cases, it is likely that there will be a decay in the correlation with increasing time lags and the data may behave like an autoregressive normal process of order one given by

\[ y_t = \phi_1 y_{t-1} + \epsilon_t \]  

(1.7)

where \( \epsilon_t \) is independently and normally distributed random variables with mean zero and constant variance, \( \sigma^2 \), and \( \phi_1 \) is the parameter of the autoregressive process of order one. Here \( Y = [Y_1, \ldots, Y_t, \ldots, Y_n] \) is jointly normal with \( E(Y) = 0 \) and \( Disp(Y) = \sigma^2 \Upsilon \), for all \( i = 1, 2, \ldots, n \), \( \Upsilon \) being the \( n \times n \) non-singular matrix given by \( \Upsilon = (\phi_1^{t-t(t-1)} / (1 - \phi_1^2)) \), \( t, t' = 1, 2, \ldots, n \). Also we have the correlation matrix of \( Y \) as \( corr(Y) = \phi_1^{t-t(t-1)} \), \( t, t' = 1, 2, \ldots, n \).

Now, in order to compute the percentile points of \( Y_{(n)} \) for unequally correlated (positive or negative) normal variables with constant variances \( \sigma^2 \), we obtain the distribution function of the maxima from equation (3.27) by putting \( \sigma_i^2 = \sigma^2 \) in the manner similar to that of (4.2) and after some adjustments in the coefficients, we write the distribution function as

\[
P_r(Y_{(n)} \leq h) = \Phi_1^{**}(h, p_{ij}, \sigma^2) \\
\simeq U_n^* \Phi(h, \sigma^2) + U_n^* \sum_{i<j}^n \Lambda_{ij(kl)}^{1100}(h, \sigma^2) - Q_n^* \sum_{i=1}^n \Lambda_{i(kl)}^{2000}(h, \sigma^2) \\
+ S_n^* \sum_{i<j}^n \Lambda_{ij(kl)}^{2200}(h, \sigma^2) + T_n^* \sum_{i\neq j \neq k}^n \Lambda_{ijkl}^{1110}(h, \sigma^2) \\
+ M_n^* \sum_{i\neq j \neq k \neq l}^n \Lambda_{ijkl}^{1111}(h, \sigma^2) \]  

(4.8)
where

\[
I_n^* = n! \left\{ 1 + \frac{1}{2} \sum_{i<j} \rho_{ij}^2 \right\}
\]

\[
D_n^* = \frac{2(n-2)!}{\sigma^4} \sum_{i<j} \left\{ \rho_{ij} - \frac{1}{n \neq j \neq k} \rho_{ik} \rho_{jk} \right\}
\]

\[
Q_n^* = \frac{(n-1)!}{2\sigma^2} \sum_{i=1}^{n} \rho_{ij}^2
\]

\[
S_n^* = \frac{(n-2)!}{\sigma^4} \sum_{i<j} \rho_{ij}^2
\]

\[
T_n^* = \frac{2(n-3)!}{\sigma^4} \sum_{i<j} \rho_{ij} \rho_{ik}
\]

\[
M_n^* = \frac{4!(n-4)!}{\sigma^4} \sum_{i \neq j \neq k \neq l} \left( \rho_{ij} \rho_{kl} + \rho_{ik} \rho_{jl} + \rho_{il} \rho_{jk} \right).
\]

and \( \Phi(h, \sigma^2) \) and \( L(h, \sigma^2) \)'s are defined as in equations (4.3) and (4.1) respectively.

Next, we compute the 100(1 - \( \alpha \))\% percentile points of maxima for unequal positive or negative correlations for selected values of \( \alpha = 0.010, 0.025 \) and 0.050 based on \( G_{n}^*(h, \rho_{ij}, \sigma^2) \) given in (4.8). In our numerical computations, we actually considered a homoscedastic normal variable case with \( \text{var}(y_i) = 1 \) for all \( i \) but the correlation structures similar to AR(1) with \( \phi_1 = \pm 0.100, \pm 0.125, \pm 0.200 \) and \( \pm 0.250 \), and \( n = 3 \) and 4. For specific value of \( \phi_1 \), we may easily obtain the correlation coefficients \( \rho_{ij} = \phi_1^{i-j} \) between two repeated variables with lag \( |i-j| \) according to the correlation matrix of AR(1) model. For example, for \( \phi_1 = 0.250 \) and \( n = 4 \), the correlation coefficients are: \( \rho_{12} = 0.250, \rho_{13} = 0.0625, \rho_{14} = 0.0156; \rho_{23} = 0.250, \rho_{24} = 0.0625; \) and \( \rho_{34} = 0.250 \). The percentile points \( h \) for this situation along with the correlations are shown in Table 4.3. To compute the percentile points of the maxima for the selected values of \( \phi_1 = \pm 0.100, \pm 0.125, \pm 0.200 \) and \( \pm 0.250 \), the trial and error method was applied where the initial values of \( h \) were chosen from Gupta (1973) for the positive values of
Table 4.3: The SCA based 100(1 - α)% percentile points of the maxima for AR(1) data with unit variance and selected $\phi_1$, $\alpha$ and $n$, $\phi_1$ being the parameter of the AR(1) process.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi_1$</th>
<th>$\rho_{ij}$</th>
<th>$i &lt; j$</th>
<th>$i, j = 1, \ldots, n$</th>
<th>$\alpha$ = 0.010</th>
<th>0.025</th>
<th>0.050</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.100</td>
<td>0.1000</td>
<td>0.0100</td>
<td>0.1000</td>
<td>3.1151</td>
<td>2.5078</td>
<td>2.1858</td>
</tr>
<tr>
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\( p = 0.100, 0.125, 0.200 \) and 0.250 respectively. The method was terminated whenever the actual probability was found to be very close to the nominal probability. The convergence of the trial and error method was quick. Only three trials provided the reported percentile points. Note that in computation of the percentile points, it is necessary to compute several infinite series (cf. equation (4.6)). These infinite series converged at different rate. For example, the convergence of the single infinite series was achieved by considering the first 32 terms. Similarly, convergence achieved for the double, triple and quadra infinite series by using \( r_1 = 21, r_2 = 25; r_1 = 17, r_2 = 18, r_3 = 20; \) and \( r_1 = 15, r_2 = 17, r_3 = 19, r_4 = 21 \) respectively.

4.4 Third case: Heteroscedastic but unequally (positively or negatively) correlated normal variables

4.4.1 Application to Antedependence Models

Antedependence models, defined as a more general family of autoregressive models, are often used in socio-economic studies. For a set of \( n \) variables in a given order, \( s \)th order antedependence is said to hold (cf. Gabriel (1962)) if each variable given at least \( s \) preceding variables in the order, is independent of the remaining variables. Note that due to the finite order antedependence, stationarity restrictions for the first and second order moments are not at all required. This is quite contrary to the standard autoregressive models in time series analysis where the restrictions on variances and
correlations are necessary. Thus, in repeated measurement experiments, inferences on the parameters of such antedependence models can be made, even if nonstationarity occurs. To illustrate the situation, Albert (1992) considered medical research data where a growth variable (such as height or weight) is measured on each individual repeatedly over time. For such data, the model often becomes nonstationary as the variance and correlation parameters appearing in such a model can be seen to depend on time.

Let \( \mathbf{Y} = (Y_1, \ldots, Y_l, \ldots, Y_n)' \sim N_n(0, \Sigma(s)) \) be a vector of \( n \)-dimensional repeated observations on a single experimental unit through \( n \) time periods, where \( \Sigma(s) \) indicates the presence of \( s \)th \((s > 0)\) order antedependence. The \( s \)th order antedependence model is defined (cf. Albert (1992)) as

\[
\begin{align*}
Y_i &= \delta_i \eta_i \\
Y_i &= \sum_{l=1}^{s_i} a_{i,i-l} Y_{i-l} + \delta_i \eta_i \quad i = 2, \ldots, n 
\end{align*}
\]

(1.9)

where \( s_i = \text{min}(s, i-1) \), \( a_{i,i-l}, i = 2, \ldots, n, l = 1, \ldots, s_i, \ i > j \), are antedependent parameters, \( \delta_i, i = 1, 2, \ldots, n \) are \( n \)-scale parameters, and the errors \( \eta_i \) are independent and normally distributed with zero mean and unit variance. The above model (1.9) may be expressed as

\[
\mathbf{Y} = \mathbf{A} \eta
\]

where

\[
\mathbf{A} = \Gamma \mathbf{D}^* 
\]

with

\[
\Gamma^{-1} = \begin{cases} 1 & l = 1 \\ -a_{i,i-l} & l \neq l \neq s_i \\ 0 & \text{otherwise} \end{cases}
\]

\[
\mathbf{D}^* = \begin{cases} 0 & l = 1 \\ 1 & \text{otherwise} \end{cases}
\]
and \[ D^* = \text{diag}(\delta_1, \ldots, \delta_n) \]

It then follows that

\[
Di_N(y) = \Sigma_{(s)} = \Lambda \Lambda' = \Gamma D^* D^* \Gamma'
\]  \hspace{1cm} (1.10)

To understand the nonstationarity in this autocorrelation process, let us first consider the special case \( s = 1 \). The variance-covariance matrix of \( Y \) in this case reduces to

\[
\Sigma_{(1)} = \begin{pmatrix}
\sigma_{11(1)} & \sigma_{12(1)} & \cdots & \sigma_{1n(1)} \\
\sigma_{21(1)} & \sigma_{22(1)} & \cdots & \sigma_{2n(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n1(1)} & \sigma_{n2(1)} & \cdots & \sigma_{nn(1)}
\end{pmatrix}
\]

where

\[
\sigma_{ii(1)} = a_{i,i-1}^2 \sigma_{(i-1)(i-1)} + \delta_i^2, \quad i = 1, 2, \ldots, n \quad \text{with} \quad a_{1,0} = 0
\]

\[
\sigma_{ij(1)} = a_{i,j-i} \delta_{(j-i)}, \quad i < j
\]

Notice that all the variance and covariances are functions of different scale and antedependent parameters, those may vary with regard to the change in time. Thus, this antedependence model does not require any variance or covariance stationarity.

Similar situations as for \( s = 1 \), will arise for other \( s > 1 \) cases too. For example, for \( s = 2 \), one may write the variance-covariance matrix of \( Y \), \( \Sigma_{(s)} = \{ \sigma_{ij(s)} \} \), in terms of the antedependent parameters showing the non-stationarity of the components of \( Y \).
For the sake of simplicity, we, however, present this for the case with \( n = 3 \). From Equations (4.9), we see that

\[
\begin{align*}
Y_1 &= \delta_1 \eta_1 \\
Y_2 &= a_{2,1} Y_1 + \delta_2 \eta_2 \\
Y_3 &= a_{3,2} Y_2 + a_{3,1} Y_1 + \delta_1 \eta_1
\end{align*}
\]

Then it is readily seen that

\[\sigma_{11(2)} = \text{Var}(Y_1) = \text{Var}(\delta_1 \eta_1) = \delta_1^2\]

Similarly, one obtains

\[
\begin{align*}
\sigma_{12(2)} &= E(Y_1 Y_2) = E(Y_1(a_{2,1} Y_1 + \delta_2 \eta_2)) = a_{2,1} E(Y_1^2) + a_{2,1} \delta_1^2 \\
\sigma_{13(2)} &= E(Y_1 Y_3) = E(Y_1(a_{3,2} Y_2 + a_{3,1} Y_1 + \delta_1 \eta_3)) = a_{3,2} a_{2,1} \delta_1^2 + a_{3,1} \delta_1^2 \\
\sigma_{22(2)} &= \sigma_{22(2)} = E(Y_2^2) = E((a_{2,1} Y_1 + \delta_2 \eta_2)^2) = a_{2,1}^2 E(Y_1^2) + \delta_2^2 E(\eta_2^2) = a_{2,1}^2 \delta_1^2 + \delta_2^2
\end{align*}
\]

and

\[
\sigma_{23(2)} = E(Y_2 Y_3) = a_{3,2} \sigma_{22(2)} + a_{3,1} \sigma_{12(2)} = a_{3,2}(a_{2,1}^2 \delta_1^2 + \delta_2^2) + a_{3,1} a_{2,1} \delta_1^2
\]

As in the case for \( s = 1 \), the variances and covariances for the second \((s = 2)\) order antedependence model also depend on different scale and antedependent parameters, showing the nonstationarity among the components of \( \mathbf{Y} = (Y_1, Y_2, Y_3)' \). In the similar fashion, we may show the nonstationarity for the case with \( n > 3 \) and appropriate \( s \).

In a repeated measurements experiment, there are many situations where antedependence models are used. For example, we consider the calf data analyzed by Kenward (1987). In this problem, the main object is to compare two or more methods for controlling the intestinal parasites in cattle. During the grazing season, from spring to
autumn, cattle can ingest roundworm larvae, which have developed from eggs previously deposited on the pasture in the faeces of infected cattle. Once infected an animal is deprived of nutrients and its resistance to other disease is lowered, which in turn can greatly affect its growth. In order to control the disease, an infected calf was assigned to a particular treatment. For monitoring the effects of a treatment for the disease, the response of interest, weight, is recorded for an infected calf at $n$ time points and it is examined whether the maximum of these weights $(y_{(n)})$ are less than a standard weight $(h)$ of an uninfected calf of same age (at the initial level of the experiments). That is, we require to compute the probability $\Pr(Y_{(n)} \leq h)$, for known $h$, which indicates the failure of the treatment. Alternatively, one may find the probability $\Pr(Y_{(1)} \geq h)$ to see whether the treatment is working effectively. Here the observations $y_1, y_2, \ldots, y_n$ will most likely be a realization of the sample $Y = (Y_1, Y_2, \ldots, Y_n)$ that follow the antedependence (nonstationary) model given in (4.9), as the weights are likely to vary with repeated time (equally or unequally spaced).

Under the assumption that time points are far apart from each other such that the correlations are small, we may directly use the probability density function developed in (3.27) to compute the distribution function of $Y_{(n)}$, that is, $\Pr(Y_{(n)} \leq h)$ just by using $\sigma_r^2$ for $\sigma_{n(s)}$ and $p_{ij}$ for $\sigma_{i,j(s)}/\{\sigma_{i(j(s)}\sigma_{j(i(s)}}\}^{1/2}$, $i,j = 1,2,\ldots,n$. The distribution function of $Y_{(n)}$, in such cases, reduces to

$$
G_n^{*}(h, \rho_{ij}, \sigma_r^2) \approx L_n^{*}\Phi(h) + D_n^{*} \sum_{i \neq j} A_{ij(kh)}^{1100}(h) - Q_n^{*} \sum_{i=1}^{n} A_{i(kl)}^{2000}(h) \\
+ S_n^{*} \sum_{i \neq j} A_{ij(kl)}^{2200}(h) + T_n^{*} \sum_{i \neq j \neq k} A_{ij(kl)}^{2110}(h) \\
+ M_n^{*} \sum_{i \neq j \neq k \neq l} A_{ijkl}^{1111}(h)
$$

(4.11)

where the coefficients $L_n^{*}, D_n^{*}, Q_n^{*}, S_n^{*}, T_n^{*}$ and $M_n^{*}$ are defined as in theorem 3.1, and $\Phi(h)$
and $\Lambda$'s are defined as in equations (4.3) and (4.6) respectively.

### 4.4.1.1 Percentile Points of $Y_{(n)}$ for antedependence model

#### 4.4.1.1.1 Computation of a correlation structure

Before computing the percentile points of the maxima, that we first compute the correlation coefficients among the repeated observations those are generated following an antedependence model of order $s = 1$ and 2 having the variance-covariance matrix discussed in the previous section. In this numerical computation, we consider $n = 3$ and 4, and scale parameters of the antedependence model $\lambda_i = 1$ for all $i$. We also consider the values of the antedependent parameters $a_{i,i-l}$ as $a_{i,i-l} \leq 0.250$ for $i = 2, \ldots, n, l = 1, \ldots, s_i, i > l$ so that all possible values of $p_{i,j}$ are small, namely, $p_{i,j} \leq 0.250$ for $i \neq j$. More specifically, the values of $\rho$'s and variances for selected values of $a$'s are shown in Table 4.1. In the next section, we compute the percentile points of the maxima for this antedependence correlations set-up.

#### 4.4.1.1.2. Computation of 5\% Critical Values of $Y_{(n)}$: An Application of Small Correlations Approach

We now compute the 95\% percentile points of $Y_{(n)}$ for $n = 3$ and 4 for the antedependent correlation structures with heteroscedastic variances reported in Table 4.4. This computation is done by using the distribution function $G^* \hat{n}(h, \rho_{ij}, \sigma^2_i)$ given in equation (4.10). In the manner similar to the homoscedastic normal variable cases, the trial and error method was used to compute the percentile points of the maxima. This trial and error method requires the initial values of $h$, a critical value of $Y_{n}$. In selecting such values for the cases where $\rho_{ij} \leq 0.1$, for example, we have chosen $h$ from Gupta (1973) for the case with $\rho = 0.1$. The percentile points are shown in Table 4.5.

Note that the $h$ values computed for $\rho_{ij} \leq \rho_0$ generally different from $h$ values obtained
Table 4.1: The variances and correlations pattern for antedependence models for selected values of $\alpha$ and $n$ with $\delta_i = 1$ ($i = 1, \ldots, n$) and $s = 1$ and 2

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Table 4.5: The SCA based 95% percentile points of the maxima for antedependence data with order $s = 1, 2$ and selected $\rho_{ij}$, $\sigma_i^2$ and $h$

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in Gupta (1973) for \( \rho = p_1 \). More specifically, our calculations show that for \( n = 3 \), the \( h \)'s for the cases with unequal correlations are greater than the \( h \)'s for the cases with equal correlations. On the other hand, for \( n = 1 \), some of the \( h \) values for the cases with unequal correlations are greater than the \( h \) values for the cases with equal correlations and some of them are less than the \( h \)'s for the cases with equal correlations. In the next section, we examine the performance of the proposed approximation (small correlations approach) by conducting a small simulation study.

### 4.4.1.2 Verification of Critical Values: A Simulation Study

To examine the accuracy of the critical values shown in Table 4.5, in this subsection, we conduct a small simulation study. To do this, we generated 5000 clusters of sizes \( n = 3, 4 \) from normal distributions with zero mean and the variance-covariance matrix of the antedependence model of order \( s = 1, 2 \) (discussed in section 4.3.1). Note that in generating the clusters, we have considered the same mean and the same variance-covariance matrix that were used to compute the percentile points \( h \) given in Table 4.5 for different cases. We now postulate as the null hypothesis \((H_0)\) that the sample does not contain any extreme observation, where any observations greater than or equal to \( h \) value (taken from Table 4.5) is considered as an extreme observation. Under a given set-up that is for a selected \( n \) and a set of \( \rho \)'s for these \( n \) repeated observations, we first observe whether the maximum of \( y_{li} \) is greater than or equal to the percentile points \( h \) where \( y_{li} \) is the \( i \)th \((i = 1, \ldots, n)\) observation in the \( l \)th \((l = 1, 2, \ldots, 5000)\) cluster. For a given set-up, we then compute the proportion of the simulation runs which satisfy \( \max_{1 \leq i \leq n}(y_{li}) \geq h \), where \( \max_{1 \leq i \leq n}(y_{li}) \) is the maximum of the \( n \) observation in the \( l \)th cluster. These proportions, commonly called as the size of the test for testing the \( H_0 \), are reported in column four of Table 4.6 for nominal \( \alpha = 0.05 \). Here, nominal \( \alpha = 0.05 \).
means that $h$ was chosen from Table 4.5 such that $Pr\{Y(n) \leq h\} = 0.95$. Note that in general, the actual size of the test appears to be close to 5% in almost all selected cases.

We also verify the performance of our small correlation approach in computing the critical values, as was done in the previous simulation study, under the alternative hypothesis $H_a$: there is one extreme observation in the cluster. For the purpose, we first generate 5000 clusters of sizes $n$ from normal distributions with same mean (zero) and the same variance-covariance matrix of the antidependence model of order $s = 1, 2$ as under the case of $H_0$. We then add $\theta = 4$ and 5 with a pre-selected $i$th ($i = 1, \ldots, n$) observation in the $l$th cluster. More specifically, we generate the new $i$th ($i' = 1, \ldots, n$) observation for the $l$th cluster such that $y_{il'} = y_{ii} + \theta$ for $i' = i$, and $y_{il'} = y_{i'}$ for $i' \neq i$. When $\theta = 0$, the $i$th observation is not an extreme observation. Now, similar to the case for the size of the test, we compute the proportion of simulation runs which satisfies $\max_{1 \leq i \leq n}\{y_{il}\} \geq h$, where $\max_{1 \leq i \leq n}\{y_{il}\}$ is the maxima of the $n$ observation in the $l$th cluster. These proportions under the different situations with $\theta = 1$ and 5 are shown, respectively, in columns five and six of Table 4.6, which, in fact, indicate the powers of the test for testing $H_0$, that is, there is no extreme observation in the sample. Note that the computed powers for both cases with $\theta = 4$ and 5 appear to quite high.

4.4.2 Computation of Critical Values for Heteroscedastic Case:

Small Correlations Approach Versus Bound Approximation

In the linear regression analysis, the maximum studentized residual test statistic and the maximum normed residual test statistic are widely used for the detection of a sin-
Table 4.c: Sizes and powers of the test at 5% level of significance for selected \( h \) and \( n \) for testing \( H_0 \) that there is no extreme observation in the sample, based on 5000 simulations

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gle outlier or influential observation. For these two statistics, the exact critical values are not easy to compute. Ellenberg (1973, 1976) has approximated the critical values of the maximum studentized residual test only by using the first Bonferroni bounds. Similar to the studentized residual test statistic, Stefansky (1971, 1972) has developed the bounds for the percentage points for the maximum normed residual test statistic under the assumption that errors in the linear regression model are homoscedastic. It is known that for this homoscedastic case, these bounds approximation work well. There is, however, no adequate discussion in the literature about the performance of bound approximation for the heteroscedastic case. Furthermore, the accurate computations for bound approximation require the computation of the joint probability which may not be easy to compute, see for example Chu and Sutradhar (1995).

In this section, we compare the performance of bound approximation with our small correlations approach, in computing the percentile points or p-values for the maxima. We do this for general unequal correlations cases with both equal and unequal variances. For this purpose, we confined our discussion to the nonregression situation, for simplicity, and we first compute the \( h \) values such that \( Pr\{Y_{(n)} \geq h\} = \alpha \) by using our small correlations approach, as in the previous sections. These \( h \) values are then used to examine the performance of the Bonferroni bounds approximations. The upper and lower bounds, as functions of \( h \) values, are defined as follows:

\[
LB(h, \sigma_i^2) = \sum_{i=1}^{n} Pr(Y_i \geq h) - \sum_{i<j}^{n} Pr(Y_i \geq h, Y_j \geq h) \quad (4.12)
\]

and

\[
UB(h, \sigma_i^2) = \sum_{i=1}^{n} Pr(Y_i \geq h) \quad (4.13)
\]

where \( E(Y_i^2) = \sigma_i^2 \), and \( Y_i \) and \( Y_j \) are correlated such that \( E(Y_iY_j) = \rho_{ij} \) with \( E(Y_i) = 0 \).
Now by using $Z_i$ for $\frac{Y_i^2}{\sigma_i}$, we obtain the lower bound as

$$LB(h, \sigma_i^2) = \sum_{i=1}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}) - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j})$$

$$= \sum_{i=1}^{n}[1 - F\left(\frac{h}{\sigma_i}\right)] - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j})$$

(4.14)

(4.15)

and the upper bound is given by

$$UB(h, \sigma_i^2) = \sum_{i=1}^{n}[1 - F\left(\frac{h}{\sigma_i}\right)]$$

(4.16)

with

$$Pr(Z_i \geq \frac{h}{\sigma_i}, Z_j \geq \frac{h}{\sigma_j}) = \int_{\frac{h}{\sigma_i}}^{\infty} \int_{\frac{h}{\sigma_j}}^{\infty} f(z_i, z_j; \rho_{ij})dz_idz_j$$

(4.17)

which is cumbersome to compute directly. Abrahamson (1965) seems to mention about the solution of this integration for selected values of $\frac{h}{\sigma_i}$, $\frac{h}{\sigma_j}$ and $\rho_{ij}$ but it's solution is not available from his paper.

Similar to the case when $E(Y_i^2) = \sigma_i^2$ for all $i$, we also obtain the lower and upper bounds, for the cases with $E(Y_i^2) = \sigma^2$ ($i = 1, 2, \ldots, n$) as

$$LB(h, \sigma^2) = n[1 - F\left(\frac{h}{\sigma}\right)] - \sum_{i<j}^{n} Pr(Z_i \geq \frac{h}{\sigma}, Z_j \geq \frac{h}{\sigma})$$

$$= n[1 - F\left(\frac{h}{\sigma}\right)] - \sum_{i<j}^{n} \Phi_2\left(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij}\right)$$

(4.18)

and

$$UB(h, \sigma^2) = n[1 - F\left(\frac{h}{\sigma}\right)]$$

(4.19)

For $\Phi_2\left(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij}\right)$, we obtain an expression provided by Greig (1967) as

$$\Phi_2\left(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij}\right) = [1 - (1 - \rho_{ij})^{\frac{1}{2}}] \Phi_1\left(\frac{h}{\sigma}\right) + (1 - \rho_{ij})^{\frac{1}{2}}] \Phi_1^{*}\left(\frac{h}{\sigma}\right)$$

with

$$\Phi_1\left(\frac{h}{\sigma}\right) = [1 - F\left(\frac{h}{\sigma}\right)]$$

Next, to compute the p-values of percentile points $h$ such that $Pr\{Y_n \geq h\} = 0.05$ for general unequal correlations with equal and unequal variance cases,
Table 4.7: The SCA based critical values \( h \) for selected \( p \)-values and the corresponding Bonferroni upper and lower bounds for selected \( \rho_{ij} \) and \( n \) for homoscedastic normal variable case

<table>
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<th>( \sigma^2 )</th>
<th>( \rho_{12} )</th>
<th>( \rho_{13} )</th>
<th>( \rho_{23} )</th>
<th>( h )</th>
<th>( p )-value</th>
<th>LB(( h ))</th>
<th>UB(( h ))</th>
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Table 4.8: The SCA based critical values \( h \) for selected p-values and the corresponding Bonferroni upper bounds and p-values for selected \( \rho_{ij} \) and \( n \) for heteroscedastic case

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<th>( \sigma_3^2 )</th>
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<th>( \rho_{13} )</th>
<th>( \rho_{23} )</th>
<th>( h )</th>
<th>p-value</th>
<th>UB(h)</th>
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<td>0.250</td>
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<td>0.0196</td>
<td>0.1021</td>
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</table>
we used the distribution functions \( G_n(h, P_{ij}, \sigma^2) \) and \( G_n(h, P_{ij}, \sigma^{ij}_n) \) given in equations (4.8) and (4.9) respectively. For the same percentile points \( h \), we also compute the upper and lower bounds by using the Bonferroni bounds approximations in equations (4.18) - (4.19) for homoscedastic normal variable cases. In the heteroscedastic normal variable cases, the computation of the lower bound is slightly complicated. Consequently, we have used the upper bound given by (4.16) to compare the bound approximation with our SCA based results. It is interesting to note that our numerical computations show that in some situations the upper bounds are seen to be lower than 0.05, indicating that the lower bounds calculations are not necessary in such cases. In other situations, however, the lower bound calculations would have been much better representative bounds than the upper bounds, but they were not calculated because of the technical difficulty as mentioned above. The results are reported, respectively, as in Table 4.7 and Table 4.8 for equal and unequal variance cases.

Note that in this homoscedastic and heteroscedastic normal variable cases, the bounds for different percentile points \( h \) based on the Bonferonii approach are deviated from nominal probability at 0.05. Furthermore, it was found that this deviation increases as variances and correlations increase in general. But the corresponding \( p \) value for the same percentile point based on our small correlations approach were found to be very close to the nominal probability 0.05.
Chapter 5

Summary and Some Topics for Further Research

5.1 Summary

Correlated data arise in many applications in statistics. In this thesis, we have discussed order statistics inferences for normal random variables with a general correlation structure, where correlations can be unequal or equal, positive or negative. More specifically, we have provided the distribution of the $r$th order statistic for any $r = 1, \ldots, n$, $n$ being the sample size or number of correlated variables. Special attention was given to the derivations for the distributions of maxima and minima. For all of these derivations, we have adopted a small correlations based Taylor's series approach, that is, our results are valid for general correlation structures but the absolute magnitude of correlations should be small (not exceeding 0.25). This small correlations approach will have applications
to many areas, mainly in clustered data analysis. This is because, as argued in the thesis, in familial clustered data, the cluster sizes are generally small and the correlations between the observations are generally small too.

For positive equi-correlated cases, our results have been compared with the existing results due to Gupta (1973), and others. It was found that the proposed approach works quite well where correlations are small, as expected. For unequal positive and negative correlated cases, the proposed small correlations approach also works quite well, which has been verified by a limited simulation study. For these unequal correlations cases, we have discussed two special situations. In the first situation, homoscedastic normal variables with unequal correlations have been considered and the percentile points of the maxima were derived for a well known AR(1) correlation process. In the second situation, we have considered the heteroscedastic random variables with unequal correlations and similar results were derived for antedependence (nonstationary) process.

Note that to compute the percentile points of the maxima or minima in these cases, one may also use the well-known Bonferroni bounds approximation. We have compared our results with these approximations and found that the Bonferroni bounds approximation does not work well for the heteroscedastic cases, whereas our small correlations approach works well for both homoscedastic and heteroscedastic cases.

5.2 Topics for Further Research

We remark that in the present thesis, we have developed a small correlations approach to find the percentile points of the distribution of a single order statistic, which has applications to certain clustered familial data. In practice, however, there are other
situations, where the correlations among the variables may be high (cf. Kenward (1987)). To develop methodologies to handle high correlations appear to be extremely difficult, and further investigations are needed.

Second, in this thesis, it was assumed that the scale parameters (correlations as well as variances) are known. For the situations where these parameters are unknown, one needs to obtain consistent estimates of these parameters and carefully study the effects of estimation on the required distributions.

Furthermore, the present methodology may be extended to the analyses of the linear models with several covariates, but this is beyond the scope of the present study.
Chapter 6

References


