AN INVESTIGATION INTO VAN HIELE LEVELS AND PROOF-WRITING OF LEVEL I GEOMETRY STUDENTS IN NEWFOUNDLAND

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AN INVESTIGATION INTO THE VAN HIELE LEVELS AND PROOF-WRITING ACHIEVEMENT OF LEVEL I GEOMETRY STUDENTS IN NEWFOUNDLAND

by

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A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Master of Education

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Memorial University of Newfoundland
August 1983

Abstract

The main purpose of this study was to determine the van Hiele levels and proof writing achievement of grade 10 students in Newfoundland. Another aspect of the study was to determine the relationship between students' van Hiele levels and their proof-writing achievement. Other concerns of this study included trying to determine any sex-related differences between van Hiele levels and proof-writing achievement.

The sample consisted of 201 students on the Avalon Peninsula in the province of Newfoundland. Each student was administered two tests. One test attempted to determine the students' van Hiele level; the other test was given to determine the students' proof-writing achievement. The tests were administered over a two-week period in latter part of May, 1983.

The average van Hiele level was found to be 2.29 and 1.45 using the 3 of 5 and 4 of 5 criteria respectively for the classical van Hiele theory. The average van Hiele levels for the modified van Hiele theory were found to be 2.22 and 1.45 using the 3 of 5 and 4 of 5 criteria respectively.

A significant relationship was found between the students' assigned van Hiele levels and their proof-writing achievement. No sex-related differences were found between students' van Hiele levels and their proof-writing achievement.

Acknowledgements

The writer wishes to express his appreciation to all those who provided assistance in the preparation and conduction of this study.

Special appreciation is extended to Dr. Frank Riggs, my advisor, for his constant interest, cooperation, and assistance.

Sincere gratitude is also extended to Dr. A. Brace and Dr. D. Drost for their willingness to serve on my examining committee.

Appreciation is expressed to the many principals and teachers for their cooperation during the administering of the tests for the study.

A special word of thanks goes to the writer's wife, Joy, and his daughter, Melissa, for their support and encouragement which made it possible to devote the time necessary to complete the study.

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CHAPTER I

THE PROBLEM

Introduction and Statement of the Problem

Deductive proofs are, at present, formally introduced in Newfoundland schools in the grade nine 'advanced' mathematics program which uses the text Geomtery (Moise & Downs, 1982). Most high school students do not study proofs in geometry until doing 'academic' mathematics at Level I (grade 10) in the text Math Is Geometry - Grade 10 (Ebos, F.; Tuck, B.; Hatcher, G.; Drost, D., 1981). Senk (1982) stated "An understanding of the concept of proof and the facility to write proofs are fundamental to success in the study of higher mathematics" (p. 1). She also said that it is believed writing geometry proofs is an area in which students experience little success. Hoffer (1981) and Freudenthal (1973) indicated that high school geometry includes much more than proof and that too much emphasis is being placed on doing formal proofs. This study concerns the question of student readiness to reason deductively and to write geometric proofs and it will also explore the question as to whether or not formal proofs are introduced too early for students to understand them.

The van Hiele theory, which was developed by two Dutch mathematicians in the late 1950's, has been used to try to explain why students have difficulty with high school geometry, particularly proof. The van Hiele theory contains three main components; (1) five sequential thought levels; (ii) properties of these levels; (iii) phases of learning. Usiskin (1982) stated:

> The van Hiele theory has been applied to explain why many students have difficulty with higher order cognitive processes, particularly proof, required for success in high school geometry. It has been theorized that students who have trouble are being taught at a higher van Hiele level than they are at or ready for. The theory also offers a remedy: go through the sequence of levels in a specific way. (p. 1)

In this study the van Hiele levels and the proof-writing achievement of Level I (grade 10) students in Newfoundland are investigated. (Level I students referred to in this study are equivalent to grade 10 students. This is due to the renaming of grades 10, 11 and 12 to Levels I, II, and III in the revised high school system in Newfoundland. To avoid any confusion with the van Hiele levels, herein Level I students will be referred to as grade 10 students.)

Purpose of Study

This study was carried out to determine the van Hiele level of grade 10 geometry students and to investigate its relationship to the proof-writing achievement of these students. In order to do this, the following questions were considered:

1) What are the van Hiele levels of geometry students at the end of grade 10?

2) What are the proof-writing achievements of students at the end of grade 10?
3) What is the relationship between the van Hiele level of thinking and proof-writing achievement?
4) Are there sex-related differences in van Hiele levels or in proof-writing achievement?

Major Hypotheses

1) There is no significant relationship between the students van Hiele levels and their proof-writing achievement.
2) There is no significant difference between the van Hiele levels of male and female grade 10 students.
3) There is no significant difference between proof-writing achievement of male and female grade 10 students.
4) The relationship between the males' van Hiele levels and proof-writing achievement is not significantly different from the relationship between the females' van Hiele levels and proof-writing achievement.

Significance of Study

The van Hieles hypothesized that students studying geometry have to pass through a series of five levels of geometric thought (Freudenthal, 1973). They also believed that the students must pass through these levels consecutively and that an organized sequence of five phases of learning enables the student to move from one level to the next (Hoffer, 1982). They also claimed that a student at a particular level cannot understand what is being discussed at any level above this level in the sequence (Senk, 1982).

The van Hiele levels of geometric thought have been used to explain why many students have difficulty with geometry, particularly with geometric proof (Wirzup, 1976; Hoffer, 1982; Usiskin, 1982). As Hoffer stated:

> One major purpose of the levels is to recognize obstacles that are presented to the students. If a problem that requires vocabulary, concepts, or thinking at level n confronts students, who are at level n-1, the students are unable to make progress on the problem with expected consequences such as frustration, anxiety, even anger. (p. 2)

Many current geometry courses are thought by some to have been taught at a higher van Hiele level than most students have attained (Wirzup, 1976, Hoffer, 1982). Usiskin (1982) stated that the van Hiele theory also offers a remedy: "go through the sequence of levels in a specific way" (p. 1).

This study attempted to gather initial base data on the van Hiele levels and proof-writing achievement from a sample of high school geometry students in Newfoundland and attempted to indicate if students are able to write proofs with understanding.

Definitions

The first six definitions concerning the van Hiele levels are taken from Usiskin (1982).

1) Weighted Sum: for the van Hiele tests, a student is assigned a weighted sum score in the following manner (see Table III, Chapter 3).

1 point for meeting criterion on items 1-6	(Level 1)
2 points for meeting criterion on items 6-10	(Level 2)
4 points for meeting criterion on items 11-15	(Level 3)
8 points for meeting criterion on items 16-20	(Level 4)
16 points for meeting criterion on items 21-25	(Level 5)

2) Classical van Hiele level (i.e., the level if the entire theory is considered):

Level	corresponds to	Weighted Sum
0		0
1		1
2		3
3		7
4		15
5		31

3) Modified van Hiele level (the level if level 5 is excluded from consideration):

Level	Weighted Sum
0	0 or 16
1	1 or 17
2	3 or 19
3	7 or 23
4	15 or 31

4) No-fit - means that a student has satisfied the indicated criterion at some level n but not at all levels below n. For the van Hiele theory fitting is a verification of a student going through the levels in order. This idea satisfies Property 1 of the theory, that the student at

level n satisfy the criterion not only at that level but also at all preceding levels. (see Table IV, Chapter 3).

5) Criterion - There are two criteria regarding the assignment of a student to a van Hiele level; 3 of 5 and 4 of 5. Both of these are investigated in this study. The 4 of 5 criterion will be called the stricter criterion. The 3 of 5 criterion means the students got 3 correct of 5 multiple choice items on that particular level. The stricter criterion requires the student to get 4 correct of each group of five items corresponding to each level.

6) 0 to 4 marking scale for proof tests (Malone, J., Douglas, G., Kissan, B., and Mortlock, R., 1980). Senk (1982) used a scale based on criteria developed by Malone et al. (1980). (see grading procedure, Appendix C) These criteria are:

0 - noncommencement - no work or only meaningless work was done

1 - approach - some meaningful work was done, but an early impasse was reached

2 - substance - sufficient detail indicated that the students proceeded toward a rational solution, but major errors invalidate the proof

3 - results - minor errors in an otherwise valid proof

4 - completion - a valid proof with no minor errors.

7) Student Category - Students involved in this study were placed into five different categories. These categories are used because students doing advanced mathematics in grade 10 most likely did advanced mathematics in grade 9, thus having two years of writing geometric proofs. Students in grade 10 mathematics who have done the academic program in grade 9 are doing geometric proofs for the first time. These five student categories are:

Category 1: Advanced mathematics grade 9 to advanced mathematics grade 10

Category 2: Advanced mathematics grade 9 to academic mathematics grade 10

Category 3: Academic mathematics grade 9 to advanced mathematics grade 10

Category 4: Academic mathematics grade 9 to academic mathematics grade 10

Category 5: Basic mathematics grade 9 to academic mathematics grade 10.

Scope and Limitations

This study is essentially a descriptive study in that no control group was used. Two tests were given to each student in the sample (see Appendices A and B). One of the tests was used to determine the van Hiele level of the student; the other was used to determine the student's proof-writing achievement. The tests used are those from the CDASSG project (Cognitive Development and Achievement in Secondary School Geometry; Usiskin, 1982).

In order to get an accurate picture of the student's van Hiele level and achievement in writing proofs, the experimenter administered both tests to students involved. This ensured that each student was given the same amount of time for each test and questions from students were answered with consistency. Due to this fact, the sample was limited to school districts in Newfoundland east of Clarenville. Cost and time were the other contributing factors to the choice of sample. This limited generalizations to this area and not to the whole province.

During the 1982-83 school year in Newfoundland, schools were closed for 15 days due to breakdown in conciliation talks with the government. This corresponds to 15 hours of lost mathematics instruction time. The testing was started nine days after the students returned to classes. Some students attitudes may have been nonfavorable to the testing due to the fact that it had no promotional value and most had to prepare for school finals.

Description of the Study

The main purpose of the study was to determine the relationship between the student's van Hiele level and the student's proof-writing achievement. Sex-related questions on capabilities of writing proofs and assigned van Hiele levels were also investigated.

Two hundred and one grade 10 students, both advanced and academic, took part in this study. Each student was administered two tests. The van Hiele test determined the student's van Hiele level; the other test determined the student's proof-writing achievement. All tests were corrected by the researcher after the correcting procedure for each proof was verified by another experienced high school mathematics teacher and a university professor. The data from these two instruments were used to answer the questions and test the hypotheses previously stated in this chapter.

Outline of the Report

A review of selected relevant literature is presented in Chapter II. A description of the design of the study, the instruments, and the methods used to analyse data are included in Chapter III. The results of data analysis is contained in Chapter IV. Chapter V includes a summary of the study, discusses the results and contains some implications for further research.

CHAPTER II

REVIEW OF RELATED LITERATURE

Introduction

The van Hiele theory as it is known today was developed by two high school teachers in the Netherlands in separate doctoral dissertations at the University of Utrecht in 1957. These two teachers were Dina van Hiele-Geldof and her husband Pierre Marie van Hiele. Dina van Hiele-Geldof died shortly after the completion of her dissertation; Pierre van Hiele has been the person who has explained their theory in Europe and North America.

The van Hieles were influenced by Piaget in their formulation of thought levels in geometry. They noticed that many problems presented to children are often above the child's level of thinking. The vocabulary and properties presented in the problem or needed in answering the problem are at a certain level of thought while the child's geometric thought is below this level (Hoffer, 1982).

The van Hiele Theory

The van Hiele theory deals with levels of thought development in geometry and phases of learning that need to take place in order to move from one level to the next. The theory attempts to explain why students have so much difficulty with geometry.

1. Thought Levels

Five (5) sequential levels of mental development were identified by the van Hieles. Wirzup (1976) gave an extensive overview of these levels using descriptions of the Russian research, arguments presented by Freudenthal (1973), and major points as stated by Pierre van Hiele in 1959.

Hoffer, who has done major research with the van Hiele theory, visited with van Hiele in the Netherlands and wrote about these levels in 1981. Hoffer offers a summarized description of each level. The names of each level are also according to Hoffer (1981, pp. 13-14).

Level 1 - Recognition

The student learns some vocabulary and recognizes a shape as a whole. For example, at this level a student will recognize a picture of a rectangle but likely will not be aware of many properties of rectangles.

Level 2 - Analysis

The student analyses properties of figures. At this level a student may realize that the opposite sides and possibly even the diagonals of a rectangle are congruent but will not notice how rectangles relate to squares or right triangles.

Level 3 - Ordering

The student logically orders figures and understands interrelationships between figures and importance of accurate definitions. At this level a student will

> understand why every square is a rectangle but may not be able to explain, for example, why the diagonals of a rectangle are congruent.
>
> Level 4 - Deduction
>
> The student understands the significance of deduction and the role of postulates, theorems and proof. At this level a student will be able to use the SAS postulate to prove statements about rectangles but not understand why it is necessary to postulate the SAS condition and how the SAS postulate connects the distance and angle measures.
>
> Level 5 - Rigor
>
> The student understands the importance of precision in dealing with foundations and interrelationships between structures. This most advanced level is rarely reached by high school students. At this level a student understands, for example, how the parallel postulate (Euclidean) relates to the existence of rectangles and that in non-Euclidean geometry, angles do not exist.

With regard to the last level, Usiskin (1982) concluded from his study; "In the form given by the van Hieles, level 5 either does not exist or is not testable. All other levels are testable" (p. 79).

2. Properties of Levels

In 1958-59 the van Hieles identified properties of the levels to which Usiskin (1982) assigned various names. Wirzup (1976) also discussed these properties which may contribute to a better understanding of the thought levels. The following description is from Usiskin (1982, p. 5).

Property 1 - Fixed Sequence

A student cannot be at a van Hiele level n without having gone through level n-1.

Property 2 - Adjacency

At each level of thought what was intrinsic in the preceding level becomes extrinsic in the current level.

Property 3 - Distinction

Each level has its own linguistic symbols and its own network of relationships connecting these symbols.

Property 4 - Separation

Two persons who reason at different levels cannot understand each other.

Property 5 - Attainment

The learning process leading to complete understanding at the next higher level has five phases, approximately but not strictly sequential, entitled: inquiry, directed orientation, explanation, free orientation and integration.

(Note: This property of attainment is also known as the phases of learning.)

3. Phases of Learning

Usiskin (1982) considered these specific and detailed explanations called 'phases of learning' as a fifth property of the levels which he called 'attainment'. The van Hieles maintained that in order to attain the next level the student must go through these phases of learning. This very significant aspect of the theory is described in detail by Hoffer (1982). He compares these five phases of learning to Polya's principle of consecutive phases consisting of exploration, formalization and assimilation. He also compares these phases to the Dienes' learning cycle. The following is Hoffer's description of each phase (1982, pp. 5-6).

> Phase 1 - Inquiry
>
> The teacher engaged the students in (two way!) conversations about the objects of the study to be pursued. The teacher learns how the students interpret the words and gives the students some understanding of what topic is to be studied. Questions are raised and observations made that use the vocabulary and objects of the topic and set the stage for further study.
>
> Phase 2 - Directed Orientation
>
> The teacher carefully sequences activities for student exploration by which students begin to realize what direction the study is taking, and they become familiar with the characteristic structures. Many of the activities in this phase are one-step tasks which elicit specific responses.
>
> Phase 3 - Expliciting
>
> The students with minimal prompting by the teacher and building on previous experiences refine their use of the vocabulary and express their opinions about the inherent structures of the study. During this phase, the students begin to form the system of relations of the study.
>
> Note: This phase has been incorrectly translated as Explanation by other writers. It is essential here that students make the observations explicitly rather than receive lectures (explanations) from the teacher.

Phase 4 - Free Orientation

The students now encounter multi-step tasks or tasks that can be completed in different ways. They gain experience in finding their own way or resolving the tasks. By orienting themselves many of the relations between the objects of the study become explicit to the students.

Phase 5 - Integration

The students now review the methods at their disposal and form an overview. The objects and relations are unified and internalized into a new domain of thought. The teacher aids this process by providing global surveys of what the students already know, being careful not to present new or discordant ideas,

"At the close of the fifth phase the new level of thought is attained" (Hoffer, 1982, p. 5). Hoffer want on to say the van Hiele theory could be applied to other topics other than geometry. He gave examples of five levels using logic, geometric transformations and real numbers. Usiskin (1982) stated that wide applicability is an appealing characteristic of this theory.

Brief Historical Perspective

The Russians investigated this theory between 1960 and 1964. The mathematics educators at the Soviet Academy of Pedagogical Sciences researched and experimented with the van Hiele theory and confirmed its validity. It is the van Hiele theory that has formed the basis for designing the new Soviet geometry curriculum (Wirzup, 1976).

This theory of levels of geometric thought went unnoticed in Western Europe and the United States until the early seventies. Freudenthal's major work, Mathematics as an Educational Task (1973), brought the theory to the attention of Western Europe. His work together with the work of the Soviets brought the theory to the attention of Wirzup who was responsible for first presenting the idea in the United Stated in 1974. In his paper Wirzup (1976) described the breakthroughs in Russia in the teaching of geometry and the van Hiele theory.

In 1978, Coxford analyzed Wirzup's article in detail and concluded that at least three types of studies are needed so that we can better understand the cognitive structure of children;

> i) carefully documented longitudinal case studies of individual children
>
> ii) substantial data gathered by age sampling to compare cognitive structure and developmental stages
>
> iii) analysis of the effects of the type and amount of instruction taking place.
>
> (pp. 329-330)

Presently, there are three major studies in the United States that deal with the van Hiele theory. The three needs as stated by Coxford are partially fulfilled by these three studies.

1) The Oregon Project: Assessing Children's Development in Geometry (Burger, 1982).

> The purpose of the study was to investigate the extent to which the van Hiele levels serve as a model to assess student understanding of geometry. (Hoffer, 1982)

This study was sponsored by National Science Foundation (NSF), headed by Burger, and lasted from September 1979 through February 1982. The study involved clinical interviews using various tasks and scripts with over 70 students in grades 1 to 12 in three states. From these interviews a list was made of reasoning phenomena that could be interpreted as possible indicators of a particular van Hiele level. This project found that the same script could be used with children of all age levels, thus developing a prototype instrument to use for longitudinal case studies. Hoffer (1982) stated that this project relates to Coxford's call for longitudinal case studies.

2) The Brooklyn Project: Geometric Thinking Among Adolescents in Inner City Schools. (Geddes, 1981).

> The purpose of the study was to determine whether the van Hiele model describes how students learn geometry and how the model can be interpreted in the context of American curriculum and environment -- in particular, in the context of minority students (grades 6 and 9) and teachers of grades 6 and 9 in a large urban area (Brooklyn, New York). (Hoffer, 1982, p. 16)

This study was also sponsored by the NSF and lasted from November 1979 through January 1982 and was directed by Dorothy Geddes. The study involved the development and implementation of four instructional modules based on the van Hiele levels and phases. To do this, several textbooks were evaluated for content and activities in relation to van Hiele model. This project satisfies Coxford's third need of analyzing the effects of the type and amount of instruction taking place in the context of formal schooling.

3) <u>The Chicago Project: Cognitive Development and Achievement in Secondary School Geometry.</u> (CDASSG, Usiskin, 1982).

> The purpose of the study was to determine the effects of the student's stage of cognitive development and performance on a test of mathematics prerequisites have on student achievement in standard geometry concepts and proof. (Hoffer, 1982, p. 17)

This project was sponsored by the National Institute of Education (NIE) for the duration of July 1979 through June 1982. This three-year project was the most comprehensive of the three and was headed by Usiskin. This project entailed administering four different tests (1. Entering Geometry Student Test; 2. Van Hiele Geometry Test; 3. Proof Test; 4. Geometry Achievement Test) to approximately 2900 high school geometry students in six different states. The first three tests were constructed by the staff of this project. (It should be noted that the van Hiele Geometry test and the proof test of the CDASSG project are the ones that were administered in the current study.) The Chicago project gathered data on students' thought levels and performance in geometry, thus meeting Coxford's second suggestion for gathering substantial data by age sampling to compare cognitive structures and developmental stages (Hoffer, 1982).

Some conclusions of these and other studies are examined next when discussing implications of the van Hiele theory.

Implications of the van Hiele Theory

The major implication of this theory is that it allows us to investigate problem areas such as writing geometry proofs. The theory has been applied to explain why students have difficulty with these higher order processes. As Wirzup (1976) states:

> The use of these levels permits us to isolate (and study) the essential aspects of the development of geometric thought from the large complex of interrelated factors characterizing the development of thinking in general. (p. 79)

Hoffer (1982) in his article "Geometry Is More Than Proof" (1981) mentioned some points regarding students' trouble with proof. "Some students say that they 'got by' geometry by memorizing proofs" (p. 13). This quote implies that proof is being taught as an algorithm leading to only short-term memory and little understanding. Freudenthal supported this idea;

> As long as the child is not able to reflect on its own activity, the higher level remains inaccessible. The higher level operation can then, of course, be taught as algorithm though with little lasting consequence. This has been proved by the failure of teaching fractions. (1973, p. 130)

Thus, the theory has major implications for achievement in writing proofs. Writing proofs may be taught as an algorithm so the student may not really understand it, yet on a short-term basis can memorize enough to get through a high school geometry course. This problem of students memorizing proofs or learning to write 'almost correct' proofs by imitating their teacher is supported by many other mathematical educators. Kline (1973) makes this very point when he stated:

> The concept of proof is fundamental in mathematics and so in geometry the students have the opportunity to learn one of the great features of the subject. But since the final deductive proof of a theorem is usually the end result of a lot of guessing and experimenting and often depends on an ingenious scheme which permits proving the theorem in proper logical sequence, the proof is not necessarily a natural one, that is, one which would suggest itself readily to the adolescent mind. Moreover, the deductive argument gives no insight into the difficulties that were overcome in the original creation of the proof. Hence, the student cannot see the rationale and he does the same thing in geometry that he does in algebra. He memorizes the proof. (p. 7)

Rosskopf and Exner (1957) suggested, "The writers are convinced that average mathematics student never really learns what a proof is, but rather learns how to write correct proofs through imitation of his instructor and his textbooks and by adjusting his efforts to their authority" (p. 274).

Hoffer (1981), as does Wirzup (1976), claimed that the present experiences in elementary and junior high school mathematics are insufficient to enable the student to write proofs. Hoffer (1981) stated we are devoting too much time to formal proofs and that this takes away from developing other geometric skills such as visual, verbal, drawing, logical, and applied skills. These statements support the idea that formal proofs are started too early for the students' mental development; they are just not ready! Hoffer (1981) and Wirzup (1976) stated that the high school geometry course is taught at a higher van Hiele level than most students have attained and the student at the lower level cannot understand what is being discussed at a higher level. The majority of students having difficulty at high school geometry are probably at the first van Hiele level while the course being taught is at the fourth level (Wirzup, 1976; Hoffer, 1981). Wirzup (1976) stated:

> The majority of our high school students are at the first level of development in geometry, while the course they take demands the fourth level of thought. It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continually later on. (p. 96)

Teachers who teach at a higher level than the students have attained and are not aware of the lower levels at which the students are operating will experience little success. Senk (1983) stated, "teachers who reason at the higher levels, if they are not sensitive to the lower levels of thinking, will not understand their students" (p. 36). Usiskin (1982) calls this the Property of Separation, in the van Hiele theory; "Two persons who reason at different levels cannot understand each other" (p. 5). Senk (1983) stated "according to the van Hieles, no matter how precise the teacher's explanations may be, these students will not comprehend them, until the students have first moved through levels 2 and 3" (p. 36).

Usiskin (1982) also supported these claims with conclusions dealing with the van Hiele level and proof-writing achievement:

> In geometry classes that have studied proof, the van Hiele levels of most students toward the end of the school year are too low to afford a high likelihood of success in geometry proof. (p. 83)

This conclusion implies that some students who have studied proof for a year do not understand proofs since their van Hiele level is either 1 or 2. They are being taught above the level they have attained. Usiskin (1982) went on to conclude:

> In geometry classes that study proof, the fall van Hiele levels of over half the students are too low to afford even a 2-in 5 chance of success at proof. (p. 84)
>
> Taken together, [both previous conclusions], support the claims of Izaak Wirzup and Alan Hoffer that many if not most students in the United States enter geometry at van Hiele levels that are too low to insure success and that the geometry course, as presently taught, does not improve their understanding (as measured by van Hiele levels) enough to get that success. (p. 84)

Usiskin also claims that students can be misplaced into a proof-oriented geometry course when they only have a 50 percent chance of being successful in that course.

> Using van Hiele levels as the criterion, almost half of geometry students are placed in a course in which their chances of being successful at proof are only 50-50. (p. 85)

This statement implies offering a non-proof alternative course to high school geometry students in order that misplacements may be minimized.

Sharon Senk, who was also a staff member of the CSASSG project, made various conclusions (using the same data as the project) concerning proof-writing achievement.

1. About 70% of the students can do simple proofs requiring only one deduction beyond those made from the given.

2. Achievement is considerably lower on proofs requiring auxiliary lines or longer chains of reasoning.

3. After a full year of a geometry course with proof, only about half the students can do more than simple proofs.

4. Writing proofs is not an "all" or "nothing" task. Among the half of the population that can do more than simple proofs, there is a wide range of proof-writing achievement.

5. Both van Hiele level and achievement on standard content correlate highly and significantly with proof-writing achievement

(Senk, 1982, pp. 7-8)

Senk's third conclusion is based upon the fact that on each form of the proof test there was one very simple proof (Usiskin, 1982). The fourth conclusion that proof is not an "all" or "nothing" task contradicts a belief held by some geometry teachers that students either 'get' proofs or they 'don't' (Senk, 1982). According to Senk, the van Hieles' hypotheses that only at level 3 do students begin to understand deductive proof, and that only students who have reached levels 4 or 5 are able to write their own proofs, are generally supported by the data and conclusions (1982, p. 11). Also, Wirzup's (1976) assertion that most students enter the high school geometry course at either of the first two van Hiele levels is supported by this data.

Roberta Dees, (1982), another staff member of the CDASSG project, analysed sex differences in geometry achievement. The main conclusion was that the ability to learn geometry, from facts through proof, is equal between the sexes. This is contrary to popular belief that boys are superior to girls in mathematical problem solving ability.

These are a few of the major implications and conclusions concerning the van Hiele theory and proof-writing achievement of high school geometry students.

Summary

The van Hiele theory, containing 5 geometric thought levels, properties and phases of learning was developed in 1957 and first brought to the attention of North Americans by Wirzup in 1974. Since then, three major projects on developing and verifying this theory have taken place. Some of the major implications of the theory deal with students' abilities to write geometric proofs. Various mathematics educators believe proofs are memorized by the student rather than learned with understanding. Also, writing proofs may be taught as an algorithm which again leads to little understanding. The van Hiele theory has been used to try and explain the difficulty students encounter with writing geometric proofs. The explanation the theory offers is that students need to be at a certain level of geometric thought

(Level 4) in order to understand geometric proofs and most students are at a lower level. The theory contains certain phases of learning that can enable a student to go from one level to the next.

CHAPTER III

DESIGN OF THE STUDY

Sample

The population included all academic and advanced mathematics students in grade 10 in all school districts in Newfoundland, east of Clarenville. Ten schools were chosen randomly from the districts and one class of grade 10 mathematics students was chosen randomly from each school. Seven of the ten schools chosen agreed to participate in the study. The sample consisted of 201 students, 79 girls and 122 boys in five different categories. A breakdown of the sample with respect to sex and category is given in Table I.

Category 1 represented those students who have done advanced mathematics in grades 9 and 10. Students in category 2 did advanced mathematics in grade 9 but did the academic mathematics program in grade 10. Category 3 students did the academic mathematics in 9 and switched to the advanced program in grade 10. Category 4 represented the largest majority of students. These students have done the academic mathematics program in both 9 and 10. Category 5 was added for those students who chose to do the academic program in 10 after having completed the basic mathematics in grade 9. Totals in Table I show that most students (92%) belong either to category 1 or 4 so only these categories will be considered in the analyses.

Table I

Sex and Category of Sample

	Male	Female	Totals	
Category 1	53	13	66	(32.8%)
Category 2	5	1	6	(3.0%)
Category 3	5	0	5	(2.5%)
Category 4	54	65	119	(59.2%)
Category 5	5	0	5	(2.5%)
Totals	122 (60.7%)	79 (39.3%)	201	

n = 201

Each student was administered two tests, the van Hiele level test and the proof test. The order of administering the tests were alternated each time to avoid any bias in the testing.

Instrumentation

The two tests administered in this study were from the CDASSG project and were used with permission (Usiskin, 1982). The van Hiele level geometry test (Appendix A) contains 25 sequential multiple choice items and was constructed by the staff of the project. The first five questions deal with level one, the second five items with level two and so on. The other test that was used in this study is a proof test (Appendix B) that was designed by Sharon Senk under the auspices of the CDASSG project (1979). This test consisted of six items; two short answer questions and four full proofs. The first item concerned filling in four blank spaces in a geometric proof. In the second item, the student had to draw the figure, state what is given and what is to be proved from a given statement; the student did not have to prove this statement. Items three to six were four full geometric proofs. The proof tests included items covering congruent triangles, parallel lines, quadrilaterals and similarity which the staff of the project determined were the most widely-used topics in most geometry texts.

There are three similar forms of the proof test that were used in the CDASSG project. The item analysis of the three proof tests (Table II) showed that some of the items were more difficult than others. The results in this item analysis were used to choose the six items which make up the proof test for this study. If the items were close in terms of the percentage of students having them correct, then one item was chosen randomly for use in the present study. For example, item 2 on each form of the proof exam had relatively close percentages of students scoring 3 or 4. On form 2, 78% scored 3 or 4 and on form 3, 66% scored either 3 or 4. Therefore item 2 of the test used in this study was chosen at random from these three items.

Table II

Proof Tests - Item Analysis

(Usiskin, 1982)

		Percentage Scoring						
Form	Item	0	1	2	3	4	mean	s.d.
1	1	1	8	9	23	35	2.36	1.61
	2	8	8	24	50	10	2.48	1.04
	3	17	8	4	9	63	2.95	1.57
	4	34	15	19	5	26	1.76	1.60
	5	23	20	20	18	19	1.90	1.43
	6	47	41	7	1	5	0.77	0.99
2	1	1	14	15	19	52	3.07	1.14
	2	8	5	10	31	47	3.03	1.21
	3	19	22	5	5	67	2.98	1.61
	4	20	20	12	9	38	2.25	1.60
	5	40	11	7	31	11	1.62	1.52
	6	57	18	7	13	13	0.99	1.42
3	1	11	22	17	24	26	2.32	1.36
	2	10	7	17	20	46	2.86	1.33
	3	31	8	15	10	37	2.14	1.69
	4	40	5	4	4	47	2.14	1.89
	5	37	19	12	8	24	1.63	1.60
	6	34	25	6	12	24	1.66	1.60

* Items chosen for the proof test of this study.

Also, items that involved the idea of similarity were eliminated due to the fact that this concept, in terms of doing proofs, is not covered in the grade 10 academic geometry text.

There was extensive piloting of both tests by the CDASSG project to determine their validity and reliability. The following Kuder-Richardson reliability coefficients were found for both the fall and spring van Hiele tests (considered as five 5-item tests); Level 1 - (.74) and (.79); Level 2 - (.82) and (.88); Level 3 - (.88) and (.88); Level 4 - (.43) and (.69); Level 5 - (.38) and (.65). Usiskin stated that "the low reliabilities at levels 4 and 5 may be a byproduct of the lack of specification of the van Hiele theory at these levels" (1982, p. 29). The reliability coefficients for the proof test ranged from .79 to .88 (Senk, 1982). It was also found that 35 minutes was enough time to complete each test. This alloted time of 35 minutes was also found to be appropriate for grade 10 Newfoundland mathematics students.

Grading Procedure

The investigator in the present study evaluated all tests that were administered. For the 25 multiple choice van Hiele level test, a student was assigned a weighted sum score in the following manner (see Definition #1, Chapter I):

1 point for meeting criterion on items 1-5 (Level 1)

2 points for meeting criterion on items 6-10 (Level 2)

4 points for meeting criterion on items 11-15 (Level 3)

8 points for meeting criterion on items 16-20 (Level 4)

16 points for meeting criterion on items 21-25 (Level 5)

The criterion used was either 3 out of 5 correct or 4 out of 5 correct.

The proof tests were graded using a scale of 0 to 4 designed by Malone et al. (1980). Senk adapted this scale and set specific criteria for each proof item (see Grading Procedure, Appendix C). Senk's (1982) criteria for items 1, 2, and 5 were used unmodified. The investigator using the same scale, designed specific criteria for items 2, 4 and 6. The grading criteria were verified by a university professor and an experienced high school mathematics teacher. Ten tests from the pilot were chosen at random and corrected independently by the writer and these two aforementioned mathematics educators. There was little disagreement and never more than one point. When disagreement occurred, the item was graded again, and an agreement was then reached on the students' score. At the end of the grading of the proof tests, 25 tests were chosen at random and scores were then verified. Scores on these 25 tests were checked again and found to be consistent.

Pilot

Both of these tests were piloted in order to determine if language level, content, time and choice of proof items were appropriate for grade 10 geometry students in Newfoundland. The pilot consisted of two classes; one advanced and one academic. There were 51 students, 29 females and 22 males. The time of 35 minutes for each test was found to be appropriate for grade 10 Newfoundland students. As a result of the pilot it was determined that Item 6 on the proof test was too difficult for the majority of grade 10 mathematics students. The mean on Item 6 was 1.16 on a grading scale of 0 to 4. This item was replaced by another proof (Item 6, Appendix B) which was thought to be more appropriate for the majority of grade 10 students in Newfoundland.

Statement of Analyses Used

This study was concerned with four questions and four hypotheses relating to the van Hiele levels and proof-writing achievement of grade 10 students in Newfoundland. These questions and hypotheses, along with the corresponding statistical analysis, are given below.

Question 1

What are the van Hiele levels of geometry students at the end of grade 10?

This question was answered by administering the van Hiele level geometry test (Appendix A) to 201 grade 10 students on the Avalon Peninsula. This test contained 25 multiple choice items; the first 5 items correspond to van Hiele level 1, items 6-10 correspond to van Hiele level 2 and so on. Both 3 of 5 criterion and 4 of 5 criterion were used in the analyses. Each student was assigned a weighted sum and then assigned to the corresponding van Hiele level. A breakdown of each weighted sum with the corresponding assigned van Hiele levels is given in Table III. Both the classical van Hiele theory and the modified van Hiele theory (level 5 omitted) are investigated in the analyses.

This first question was answered in four main parts: percentage of those who 'fit' the van Hiele theory; the distribution of all students into the van Hiele theory; the overall mean van Hiele level; and the mean van Hiele levels of those students who studied academic mathematics in both grades 9 and 10 (category 4) and those who studied advanced mathematics in both 9 and 10 (category 1)

Question 2

What are the proof writing capabilities of students at the end of grade 10?

This question was answered by administering a proof test (Appendix B) to all students involved in this study.

Table III

Breakdown of Weighted Sums with Corresponding Assigned van Hiele Levels

Weighted Sum	Level 1	Level 2	Level 3	Level 4	Level 5	van Hiele levels classical	van Hiele levels modified
0	-	-	-	-	-	0	0
1	1	-	-	-	-	1	1
2	0	2	-	-	-	No-fit	No-fit
3	1	2	-	-	-	2	2
4	-	-	4	-	-	No-fit	No-fit
5	1	-	4	-	-	"	"
6	-	2	4	-	-	"	"
7	1	2	4	-	-	3	3
8	-	-	-	8	-	No-fit	No-fit
9	1	-	-	8	-	"	"
10	-	2	-	8	-	"	"
11	1	2	-	8	-	"	"
12	-	-	4	8	-	"	"
13	1	-	4	8	-	"	"
14	-	2	4	8	-	"	"
15	1	2	4	8	-	4	4
16	-	-	-	-	16	No-fit	0
17	1	-	-	-	16	"	1
18	-	2	-	-	16	"	No-fit
19	1	2	-	-	16	"	2
20	-	-	4	-	16	"	No-fit
21	1	-	4	-	16	"	"
22	-	2	4	-	16	"	"
23	1	2	4	-	16	"	3
24	-	-	-	8	16	"	No-fit
25	1	-	-	8	16	"	"
26	-	2	-	8	16	"	"
27	1	2	-	8	16	"	"
28	-	-	4	8	16	"	"
29	1	-	4	8	16	"	"
30	-	2	4	8	16	"	"
31	1	2	4	8	16	5	4

All tests were graded on a scale of 0 to 4 (Malone, et al., 1980). All exams were evaluated by the investigator after setting down grading criteria for each item on the proof test (See Grading Procedure, Appendix C).

In answering this question an analysis for each item was used which showed the number and percentages of student scores and the mean and standard deviation for each item. The means and standard deviation of scores on the proof test were also calculated for students' categories 1 and 4.

Question 3

What is the relationship between the van Hiele level of thinking and proof-writing achievement?

This question was answered by determining the Pearson correlation coefficients between each item on the proof test and average on the proof test, with the assigned van Hiele levels and weighted sum for both criteria. The Pearson correlations were also determined between the average score on the proof test and the van Hiele levels for categories 1 and 4.

Question 4

Are there sex-related differences in van Hiele levels or in proof-writing achievement?

This question was answered by determining the following; the mean van Hiele levels and standard deviations for both the female and male students; the average score and standard deviations on the proof test for both female and male students; and the Pearson correlations between individual average scores on the proof test and the van Hiele levels for both sexes using both criterion. This was done for all students together and separately for students in categories 1 and 4.

Hypothesis 1

There is no significant relationship between the students' van Hiele levels and their proof writing achievement.

For this hypothesis all Pearson correlations that were calculated were tested with a Z-test using Fisher's Z-transformation of the correlation at the level of significance of .001.

Hypothesis 2

There is no significant difference between the van Hiele levels of male and female grade 10 students.

This hypothesis was tested using a t-test to see if the average van Hiele level for males was significantly different from the average van Hiele level for males. The mean van Hiele levels for males and females in categories 1 and 4 were also tested. The level of significance was chosen to be .01.

Hypothesis 3

There is no significant difference between the proof-writing achievement of male and female grade 10 students.

A t-test was also used to test the significance of this difference between the average score on the proof test for both males and females. The proof test averages for males and females were also tested for significance in categories 1 and 4. Again, a level of significance of .01 was used.

Hypothesis 4

The relationship between the males' van Hiele levels and proof-writing achievement is not significantly different from the relationship between the females' van Hiele levels and proof-writing achievement.

A Z-test using Fisher's Z-transformation was used to test this hypothesis. The level of significance used was .01. All Pearson correlations that were calculated in answering Question 4 were tested.

Summary

In this chapter the experimental design of the study has been presented. Statement of types of data analyses were given to indicate where the data to be used in Chapter IV was obtained. Only descriptive statistics were used to answer the four questions. The data collected is presented in the next chapter and discussed in Chapter V.

CHAPTER IV

THE RESULTS OF THE STUDY

The main purpose of the study was to investigate the relationship between the van Hiele levels and proof-writing achievement of grade ten students in Newfoundland. In this chapter, the results of the analysis of the data relating to the four main questions and hypotheses are presented. The discussion of results and implications are given in Chapter V.

Question 1

What are the van Hiele levels of geometry students at the end of grade 10?

The van Hiele levels are numbered 1 to 5 in this paper (Level 1 being the first level, Level 2 being the second level and so on). The level 0 used in this report refers to those students who are not operating at the first level or basic level. A no-fit refers to a student who cannot be assigned a level.

In Table IV the percentages of students who 'fit' the van Hiele theory are given. Using the 3 of 5 criterion 67% of students were classifiable into a classical van Hiele level and 83% into a modified van Hiele level. Using the 4 of 5 criterion (the stricter criterion) 90% of students were classifiable into a classical van Hiele level and 93% into a modified van Hiele level. Using the modified van Hiele theory 16% more students were classified than the classical theory using the 3 of 5 criterion and 3% more using the stricter criterion. This is partly due to the fact that when level 5 is included (classical theory), there are only six weighted sums that can be used to assign a student to a level. This idea is stated in Property One of the van Hiele theory called Fixed Sequence which states "A student cannot

Table IV

Number and Percentage of Students
Who 'Fit' the van Hiele Theory

	N	%
3 of 5 criterion		
Classical	135	67.2
Modified	167	83.1
4 of 5 criterion		
Classical	182	90.5
Modified	187	93.0

be at van Hiele level n without having gone through level n-1" (Usiskin, 1982, p. 5). These six weighted sums and their assigned levels are:

Weighted Sum	Assigned Classical Level
0	0
1 = 1	1
3 = 1 + 2	2
7 = 1 + 2 + 4	3
15 = 1 + 2 + 4 + 8	4
31 = 1 + 2 + 4 + 8 + 16	5

So according to the property of Fixed Sequence, a student has to meet the criterion on a certain level and all preceding levels in order to be assigned that level.

If we delete level 5 (modified theory), which is worth 16 points, then the number of weighted sums that can assign a van Hiele level are doubled:

Weighted Sum	Assigned Modified Level
0 or 16	0
1 or 17	1
3 or 19	2
7 or 23	3
15 or 31	4

A breakdown of each weighted sum with the corresponding assigned van Hiele levels is given in Table III in Chapter III.

In Table V and Table VI the distribution of all students into the van Hiele levels using each criterion is given. There are distinct changes in the number of students at levels 0 and 1 between the 3 of 5 criterion and

Table V

Number and Percentages of Students
at Each van Hiele Level Using
the 3 of 5 Criterion

	Classical		Modified	
	N	%	N	%
Level				
0	4	2.0	5	2.5
1	31	15.4	38	18.9
2	48	23.9	58	28.9
3	33	16.4	47	23.4
4	12	6.0	19	9.5
5	7	3.5	-	-
No-fit	66	32.8	34	16.9
Total	201	100.0	201	100.0

Table VI

Number and Percentages of Students at Each van Hiele Level Using the 4 of 5 Criterion

	Classical		Modified	
	N	%	N	%
Level				
0	31	15.4	31	15.4
1	77	38.3	79	39.3
2	45	22.4	46	22.9
3	21	10.4	23	11.4
4	7	3.5	8	4.0
5	1	0.5	-	-
No-fit	19	9.5	14	7.0
Total	201	100.0	201	100.0

the stricter criterion. The number of students in Level 0 increases from 4 students to 31 students using the classical theory; an increase of 13% of those classifiable. The increase is similar with the modified theory; from 5 students to 31 students which corresponds to an increase of 13% of those classifiable. The number of students in Level 1 increases from 31 to 77, an increase of 23% of those classifiable in the classical theory. The increase in the modified theory is from 38 to 79 corresponding to a 20% increase of those classifiable. The number of students in Level 2, 3, and 4 decreases slightly going from the 3 of 5 criterion to the stricter criterion.

The fact that more people "fit" with the modified theory than the classical means that many students reached the criterion on Level 5 items but not on one or more of the lower level items. This raises many questions about the validity of Level 5 items and the existence of a levels that are beyond the scope of this investigation.

The mean van Hiele levels are reported in Table VII. There is little difference between the means found using the classical and modified theories. Using the 3 of 5 criterion, the average van Hiele level of those classifiable is 2.29 (Level 2) for the classical theory and 2.22 (Level 2) for the modified theory. The mean van Hiele level using the stricter criterion drops slightly to 1.45 for both the classical and modified theories.

Table VII

Means and Standard Deviations of
Assigned van Hiele Levels

3 of 5 criterion	n	$\bar{x}$	s.d.	t-value	2-tail prob.
Classical	135	2.29	1.16	2.71	.0001
Modified	167	2.22	1.06		
4 of 5 criterion					
Classical	182	1.45	1.06	1.00	.0001
Modified	167	1.45	1.04		

It is important to realize in this study there were two main categories of students. Category 1 represents those who have studied advanced mathematics in grade 9 and in grade 10, thus having done two full years of geometric proofs. Category 4 represents students who have studied the academic mathematics program in both grades 9 and 10. It was expected the average van Hiele level to be higher for the advanced students than those doing the academic program and this was the case in this study, but only by a little less than one level. The number of students, the average van Hiele level and standard deviation for each category is given in Table VIII. Since category 1 (Advanced mathematics grade 9 to advanced mathematics grade 10) and category 4 (academic mathematics grade 9 to academic mathematics grade 10) contain the majority of mathematics students it was of

Table VIII

Means and Standard Deviations of the van Hiele Levels in Categories 1 and 4 with results of t-test

	Classical					Modified				
	n	$\bar{x}$	s.d.	t-value	2-tail prob.	n	$\bar{x}$	s.d.	t-value	2-tail prob.
<u>3 of 5 criterion</u>										
Category 1	41	2.85	1.25	3.55	.0001	56	2.66	0.94	3.72	.0001
Category 4	82	2.10	1.04			98	2.05	1.00		
<u>4 of 5 criterion</u>										
Category 1	55	1.98	1.01	4.44	.0001	59	1.97	1.00	4.44	.0001
Category 4	111	1.23	1.03			111	1.24	1.01		

interest to note the difference between their average in the van Hiele levels. Using the 3 of 5 criterion, the mean van Hiele levels for students in category 1 were 2.85 (classical) and 2.66 (modified). The mean levels in category 4 were 2.10 (classical) and 2.05 (modified). Using the stricter criterion, the mean van Hiele levels in category 1 were 1.98 (classical) and 1.97 (modified) and in category 4 were 1.23 (classical) and 1.24 (modified). The difference between the mean van Hiele levels for students in categories 1 and 4 was found to be significant at the .001 level for both criteria.

Question 2

What are the proof-writing capabilities of students at the end of grade 10?

An item analysis of the proof test and the number and percentages of students scores on each item is contained in Table IX. The overall mean on the proof test, using the scale of 0 to 4, was 2.23 with a standard deviation of 1.12. The mean score of each item is reported in the item analyses. Item 5 had the lowest mean of 1.38 and Item 4 had the highest mean of 2.91. It was also interesting to note the mean on the proof test for each category (Table X). Category 1 students who have had two years of writing geometric proofs should do better than those in category 4 who have had only one year of writing proofs. The mean on the proof test for

Table IX

Proof Test - Item Analysis

	Score											
	0		1		2		3		4		$\bar{x}$	s.d.
Item	N	%	N	%	N	%	N	%	N	%		
1	20	10.0	48	23.9	41	20.4	36	17.9	56	27.9	2.30	1.36
2	46	22.9	20	10.0	54	26.9	33	16.4	48	23.9	2.09	1.46
3	20	10.0	28	13.9	31	15.4	17	8.5	105	52.2	2.79	1.45
4	25	12.4	25	12.4	13	6.5	19	9.5	119	59.2	2.91	1.51
5	78	38.8	51	25.4	23	11.4	15	7.5	34	16.9	1.38	1.48
6	61	30.3	24	11.9	29	14.4	45	22.4	42	20.9	1.92	1.55

Table X

Number, Means and Standard Deviation of Scores on Proof Test for Each Category with results of t-test

	n	$\bar{x}$	s.d.	t-value	2-tail prob.
Category					
1	66	2.56	1.00	2.78	.006
4	119	2.08	1.16		

category 1 was 2.56 only slightly better than the mean on the proof test for category 4 students which was 2.08. However, the mean proof scores for students in categories 1 and 4 using a t-test were found to be significantly different at the .01 level of significance.

Question 3

What is the relationship between the van Hiele level of thinking and proof-writing achievement?

The correlations between the proof scores for each item and weighted sums and van Hiele levels, using both the 3 of 5 and 4 of 5 criteria are reported in Table XI. The average score on the proof test correlated best overall with the levels ranging from .43 to .53 using 3 of 5 criterion.

Table XI

Pearson Correlation Coefficients Between the Weighted Sums, Classical and Modified Levels with Each Item and Average Score on the Proof Test*

	P1	P2	P3	P4	P5	P6	PAVG
<u>3 of 5 criterion</u>							
weighted sum	.37	.34	.34	.25	.30	.39	.43
classical levels	.43	.37	.39	.32	.41	.52	.53
modified levels	.38	.37	.35	.31	.39	.48	.49
<u>4 of 5 criterion</u>							
weighted sum	.37	.34	.31	.28	.39	.43	.46
classical levels	.38	.33	.37	.27	.43	.45	.48
modified levels	.39	.31	.38	.27	.44	.45	.49

* All correlation coefficients significant at the .001 level.

The average score on the proof test also correlated higher than the individual items with the van Hiele levels using the stricter criterion. The range is from .40 to .49. The classical theory generally yielded higher correlations as did the 3 of 5 criterion. The fourth item on the proof test correlated the lowest using both criterion, ranging from .25 to .32 for the 3 of 5 criterion and from .27 to .28 for the stricter criterion.

The correlations between the van Hiele levels and average score on the proof test for students in categories 1 and 4 are given in Table XII. The correlation coefficients for category 1 were found to be slightly higher than the correlations for category 4 in all cases except for the modified levels using both criterion.

In all cases the correlation coefficient were statistically significant at the .001 level. Hypothesis 1 was then tested to see if any relationship existed between the students' van Hiele levels and their proof-writing achievement.

Hypothesis 1

There is no significant relationship between the students' van Hiele levels and their proof-writing achievement.

All Pearson correlation coefficients calculated were found to be significant at the .001 level except for those calculated between the average score on the proof test for

Table XII

Pearson Correlations Between the Weighted Sum, Classical and Modified Levels with the Average Score on the Proof Test for Categories 1 and 4 *

	Average on Proof Test	
	Cat 1	Cat 4
3 of 5 criterion		
weighted sum	.48	.41
classical level	.58	.49
modified level	.46	.46
4 of 5 criterion		
weighted sum	.48	.42
classical level	.48	.45
modified level	.44	.46

* All correlation coefficients significant at the .001 level.

females in category 1 and the van Hiele levels of both classical and modified levels. This was due to the fact of the small number of 13 female students in category 1. These correlation coefficients were significant at the .01 level and are found in the second column in Table XIII.

This null hypothesis was rejected, thus there exists some relationship between the students' van Hiele levels and their proof-writing achievement. A student with higher van Hiele levels tends to score higher on the proof-test.

Question 4

Are there sex-related differences in van Hiele levels or in proof-writing achievement?

The mean van Hiele level for the males was slightly higher than the mean females' van Hiele level in all cases. The mean levels for the females, using the 3 of 5 criterion, found in Tables XIV and XV were 2.19 (classical) and 2.05 (modified). The corresponding mean van Hiele level for the males was found to be 2.35 (classical) and 2.33 (modified). Using the stricter criterion, the average van Hiele level for the females was 1.26 (classical) and 1.25 (modified), for the males the average level was 1.57 (classical) and 1.59 (modified). Note that the average van Hiele levels differed only slightly using the classical and modified theories. Breaking down category 1 students into males and females, it was found that the males' average van Hiele level is again slightly higher.

Table XIII

The Pearson Correlations Between the Average Score on the Proof Test for Males and Females and the Classical and Modified van Hiele Levels*

	Average on Proof Test					
	Females			Males		
	Overall	Cat. 1	Cat. 4	Overall	Cat. 1	Cat. 4
3 of 5 criterion						
weighted sum	.36	.49	.35	.48	.46	.46
classical levels	.45	.58	.44	.59	.57	.59
modified levels	.42	.60	.42	.55	.41	.58
4 of 5 criterion						
weighted sum	.43	.54	.42	.50	.47	.45
classical levels	.51	.51	.52	.49	.47	.42
modified levels	.52	.51	.52	.49	.42	.44

* All correlation coefficients were significant at the .001 level except for the females in category 1 (second column) which were significant at the .01 level.

Table XIV

Breakdown of Mean van Hiele levels for both Males and Females
with results of t-test using the 3 of 5 criterion

		Classical					Modified				
		n	$\bar{x}$	s.d.	t-value	2-tail prob.	n	$\bar{x}$	s.d.	t-value	2-tail prob.
Overall	Female	52	2.19	1.55	0.77	0.445	64	2.05	1.06	1.75	0.87
	Male	82	2.35	1.16			103	2.33	0.98		
Category 1	Female	12	2.67	1.16	0.61	0.547	12	2.58	1.00	0.32	0.762
	Male	29	2.93	1.31			43	2.68	0.93		
Category 4	Female	39	2.08	1.13	0.17	0.865	51	1.94	1.04	1.14	0.257
	Male	43	2.12	0.96			47	2.17	0.94		

Table XV

Breakdown of Mean van Hiele levels for both Males and Females with results of t-test using the 4 of 5 criterion

		Classical					Modified				
		n	$\bar{x}$	s.d.	t-value	2-tail prob.	n	$\bar{x}$	s.d.	t-value	2-tail prob.
Overall	Female	73	1.26	1.11	1.94	0.058	73	1.25	1.06	2.20	0.029
	Male	109	1.57	1.01			114	1.59	1.01		
Category 1	Female	12	1.83	1.03	0.57	0.569	12	1.83	1.03	0.51	0.610
	Male	43	2.02	1.01			47	2.00	1.00		
Category 4	Female	60	1.15	1.10	0.94	.351	60	1.13	1.0[illegible]	1.22	0.222
	Male	51	1.33	0.93			52	1.37	0.95		

The overall average score on the proof test for both females and males, found in Table XVI, using all categories together, are relatively the same. When the categories were broken down the situation changed. In category 1 the male average was 2.62 which was higher than the female average of 2.28. In category 4, it reversed, the female average of 2.26 was higher than the male average of 1.86.

The correlations between the average on the proof test and the classical and modified levels are given in Table XIII. Overall the correlation was higher with the males using the 3 of 5 criterion but with the stricter criterion male and female correlations were relatively the same. The range was from .36 to .52 for females and .45 to .59 for the males. The highest correlations for the females were with the classical (.51) and modified (.52) levels using the stricter criterion. The highest correlation for the males was with the classical level (.59) using the 3 of 5 criterion. In category 1, the females' mean proof score correlated higher with their van Hiele levels than did the males. This was due to the fact there were only 13 female students in category 1. The highest correlation for the males in category 1 was again with the classical level (.57) using 3 of 5 criterion. In category 4, the males mean score on the proof test correlated higher using the 3 of 5 criterion but the females mean score correlated higher than

Table XVI

Mean Scores on the Proof Test Between Males and Females
Overall and in Categories 1 and 4
with results of t-test

Variable	Group	N	Mean	Std. Dev.	t-value	2-tail prob.
Overall	Female	[illegible]9	2.27	1.12	0.43	.666
	Male	122	2.20	1.12		
Category 1	Female	13	2.28	0.99	-1.09	.279
	Male	53	2.62	1.01		
Category 4	Female	65	2.26	1.16	1.85	.067
	Male	54	1.87	1.14		

the males using the 4 of 5 criterion. The highest correlations for the males in category 4 were with the classical (.59) and modified (.58) levels using the 3 of 5 criterion. The highest correlations for the females were with the classical (.52) and modified (.52) levels using the stricter criterion.

The following three hypotheses were checked for any significant sex-related differences between the van Hiele levels and proof writing achievement of grade 10 students.

Hypothesis 2

There is no significant difference between the van Hiele levels of male and female grade 10 students.

The results of the t-test used to test this hypothesis are reported in Tables XIV and XV. The average van Hiele levels were consistently higher for males than females in both the classical and modified theories using both criteria. These differences in mean van Hiele levels between males and females were found not to be significantly different. Also, when the male and female students were separated into categories 1 and 4, the t-values calculated were not significant at the 0.05 level. Tables XIV and XV give a breakdown of the results of the t-test in categories 1 and 4 respectively. The t-values for category 1 students ranged from 0.61 to 0.51 and for category 4 the t-values ranged from 1.22 to 0.17. None of the t-values were significant at the 0.05 level thus no adequate grounds exist for rejecting this null hypothesis. Therefore, no significant difference exists between the males' van Hiele levels and the females' van Hiele levels.

Hypothesis 3

There is no significant difference between proof-writing achievement of male and female grade 10 students.

The results of the t-test used to test the above hypothesis are found in Table XVI. The t-values calculated for all students and for students in categories 1 and 4 were not significant at the chosen .05 level of significance. Thus [illegible] grounds exist for rejecting this null hypothesis. Therefore, no significant difference exists between the proof-writing achievement of male and female grade 10 students.

Hypothesis 4

The relationship between the males' van Hiele levels and proof-writing achievement is not significantly different from the relationship between the females' van Hiele levels and proof-writing achievement.

All relevant pairs of Pearson correlation coefficients between the average score on a proof test for male and females and the van Hiele levels found in Table XIII were tested for significance. A Z-test using Fisher's Z-transformation for testing the significance between correlations coefficients was used to test this hypothesis. The Z-test values for the overall correlations ranged from -1.09 to .388. For category 1 the range of Z-test values was .041 to .912 and for category 4, the values ranged from -1.03 to .83. In order to be significant at the .01 level of significance a Z-test value of ±2.57 was required. Since all Z-test values calculated were less than the required level, no grounds exist for rejecting this null hypothesis. Thus, the relationship between the males' van Hiele levels and proof-writing achievement is not significantly different from the relationship between the females' van Hiele levels and proof-writing achievement.

Summary

The data collected in the study relative to the four questions and hypotheses stated in Chapter I have been presented in this chapter. The discussion of the results and implications are given in the following chapter.

CHAPTER V

CONCLUSIONS AND IMPLICATIONS

In this study, it was attempted to determine the van Hiele levels and proof-writing achievement of a sample of grade 10 students in Newfoundland. It was also tried to determine if a relationship exists between a student's van Hiele level and his/her proof-writing achievement. Also investigated were the percentage of students who 'fit' the van Hiele theory. Both the classical and modified (Level 5 omitted) theories were considered. There were two main categories of students involved in this study; Category 1 students who have had two years of doing formal geometric proofs and category 4 students who have had only one year of writing proofs. Most analyses used were done with all students together and separately with students in categories 1 and 4.

The van Hiele theory includes five sequential levels of geometric reasoning as well as five properties of these levels. It also offers a series of five phases of learning that can enable a student to go from one level to the next. The main claim of the van Hiele theory for this study is that students need to be at level 4 to be able to write proofs with understanding and most grade 10 students are only at level 1 or 2.

The Study

The population under investigation in this study was all advanced and academic mathematics students in all school districts east of Clarenville for the school year 1982-1983. A sample was chosen randomly from this population and contained 201 students. Students were stratified into male and female students and again further stratified into categories 1 and 4.

The questions and hypotheses put forth in this study were tested using a van Hiele level test and a proof-writing achievement test adopted from the CDASSG project. These instruments were found to be both reliable and valid for this study. A pilot study of 51 students was conducted before the start of the main research. Both tests were administered and corrected by the investigator of this study. Grading criteria were set before evaluation of the students' tests took place and randomly checked. Students were assigned a van Hiele level according to a weighted sum. The average on the proof scores was then correlated with the students' van Hiele level. The results of the questions and hypotheses stated in Chapter I with results given in Chapter IV are discussed in the following section of this chapter.

Discussion of the Results

The results of this study are discussed in this chapter and include comparisons to the results of the CDASSG project as stated by Usiskin (1982) and Senk (1983).

Number of Students Who 'Fit' the Theory

Usiskin (1982) found that 68% of students were classifiable into a classical van Hiele level using the 3 of 5 criterion. In this study relatively the same percentage of 67% was found even though the numbers of students involved in each study were drastically different. Usiskin's study contained as many as 2057 students, this study had a sample size of 201, yet the results in most cases were the same. Usiskin (1982) found 79% to fit the classical van Hiele theory on the stricter criterion. This study found 90% to fit the classical van Hiele theory using the 4 of 5 criterion. For the modified theory, Usiskin found 87% to fit using the 3 of 5 criterion and 86% using the stricter criterion. This study found similar results of 83% for the 3 of 5 criterion and 93% for 4 of 5 criterion. Indeed, it appears that the modified theory gives a better fit than the classical theory. That is, the deletion of level 5 does enable more students to be classified.

It is discouraging to note that when using the 3 of 5 criterion only 6% (classical) and 9.5% (modified) were actually at the deduction level even after at least one full year of doing geometric proofs. These percentages dropped more when the 4 of 5 criterion was used; 3.5% (classical) and 4.0 (modified) were at the deduction level at the end of grade 10. These results bring up the question of instruction time. The van Hiele theory is based upon instruction time as Usiskin reported in his comparison with Dina van Hiele's results.

> Dina van Hiele (1957) reports having been able to lead students from level 1 to 3 in 70 lessons, 20 lessons to go from Level 1 to Level 2 and 50 more lessons to go from Level 2 to Level 3. (1982, p. 39)

So it seems the amount of instruction time before grade 10 in geometry is a very important factor. This also brings out the idea of a developmental approach to geometry from grades K-12 based upon the van Hiele levels. This has major implications for the changing of mathematics curriculum in all grade levels.

The Mean van Hiele Level

The mean van Hiele level for this study was determined to be 2.29 and 2.22 for the classical and modified levels respectively using the 3 of 5 criterion. Usiskin found the average van Hiele level for the students in the spring to be 2.73 for the classical and 2.55 for the modified with the 3 of 5 criterion. The average van Hiele levels for this study using the 4 of 5 criterion were 1.45 for both the classical and modified theory. Usiskin's average van Hiele levels were 1.77 and 1.79 for the classical and modified levels respectively. The mean van Hiele levels are consistently higher in Usiskin's study than in this present study. There are some important factors to be considered with this comparison. In the United States, secondary school geometry is studied in a single year, normally in the tenth grade (Usiskin, 1982). In Newfoundland geometry and algebra are both taught in grade 10 mathematics, so instruction time of geometry differs greatly. Age is another important factor; the United States students have the option of doing this geometry course in any year of secondary school. In Usiskin's study 47% of the students were at the age of 15 which is the average age of grade 10 students in Newfoundland. Thirty-four percent of the students involved in CDASSG project were 16 years of age and over (Usiskin, 1982, p. 95).

The mean van Hiele level for grade 10 students in Newfoundland were therefore consistently lower than those found in the CDASSG project. These low van Hiele levels indicate that the majority of grade 10 Newfoundland students are not ready for a study of deductive reasoning.

In this study the average van Hiele level for the advanced mathematics students (category 1) was only slightly better than that for the academic mathematics students (category 4). The difference of average levels in each category was approximately 0.7 a level. This was possible due to the fact that one of the seven schools involved did not offer the advanced mathematics programs. This made it necessary for advanced students in this school to do the academic program. Another reason for the small difference in average van Hiele levels between category 1 and 4 was possibly due to the smaller sample size of 66 in category 1.

It was also found in this study that no significant difference existed between the van Hiele levels of male and female students in either category.

Wirsup's Claim

Wirzup's claim (1976) stated that "The majority of our high school students are at the first level of development in geometry, while the course they take demands the fourth level of thought." (p. 96). The data found in this study does support the first part of this claim. The latter part of this claim was not investigated in this study. The average van Hiele levels ranged from 1.45 to 2.29 depending upon the chosen criterion.

Proof-Writing Achievement

The average score on the proof test for this study was determined to be 2.23 and the results from the CDASSG project were similar with a proof average on the same six proof items of 2.34. It was anticipated that the category 1 students would do much better than category 4 students since these students have been writing proofs for 2 years while the students in category 1 have only 1 year of writing geometric proofs. The means for category 1 and category 4 are 2.56 and 2.08 respectively using a grading scale of 0 to 4. Again, this may be due to the smaller numbers in category 1 or the fact that one school did not offer the advanced mathematics program which would likely inflate the mean. There was no significant difference found between the male and female students' scores on the proof test in both categories which is contrary to some studies that have shown males to be superior to girls in mathematical problem solving ability.

The Relationship Between van Hiele Levels and Proof-Writing Achievement

All correlations that were calculated, except for females in category 1 as explained in Chapter IV, were significant at the .001 level. Usiskin's correlations were similar, ranging from .57 to .62 between the proof totals and the spring van Hiele levels. Hence, the higher the van Hiele level of the student, the greater his or her chance of being able to write geometric proofs. If a student has a low van Hiele level, success at writing proofs seems highly unlikely. Correlations between the van Hiele level and proof-writing achievement were only slightly higher for those students in category 1 over those in category 4. Again, there were no significant differences between the males' van Hiele level and proof-writing relationship and the relationship of the females' van Hiele level and proof-writing achievement. Usiskin found the same conclusions concerning equality among the sexes except for the spring van Hiele levels which were consistently different favoring the male students.

Summary of the Discussion of Results

The following is a summary of the results as discussed in the previous section:

1) There is a large majority of students who are classifiable into either of the classical and modified theories with more students being classifiable into the modified theory.

2) The average van Hiele levels for the classical van Hiele theory were 2.29 and 1.45 using the 3 of 5 and 4 of 5 criterions respectively.

3) The average van Hiele levels for the modified theory were 2.22 and 1.45 using the 3 of 5 and 4 of 5 criterions respectively.

4) There exists a significant relationship between the students' assigned van Hiele level and their proof-writing achievement.

5) There is no significant difference between the van Hiele levels of male and female grade 10 students.

6) There is no significant difference between proof-writing achievement of male and female grade 10 students.

7) The relationship between the males' van Hiele levels and proof-writing achievement is not significantly different from the relationship between the females' van Hiele levels and proof-writing achievement.

Implications

The discussion of the results has suggested that the majority of grade 10 students are not at the appropriate van Hiele level to ensure a likelihood of success at writing proofs. Hoffer (1981), as does Wirzup (1976), claimed that the present experiences in elementary and junior high school mathematics are insufficient to enable most students to write geometric proofs later in high school. This study partly supports this idea of systematic geometry instruction before high school. Geometry should be possibly taught using a developmental approach if we desire students who have more geometry knowledge and are highly successful at writing proofs. The van Hiele theory offers a remedy. Level 1 geometric concepts could be introduced at the primary level and the concepts of levels 2 and 3 could be developed through elementary and junior high. This would most likely enable students to enter high school at level 3 so they may have a high chance of success at writing proofs with understanding. If this developmental, systematic direction does not take place, then a non-proof alternative geometry course could be offered. Presently, the majority of students in grade 10 geometry are not highly successful at writing proofs.

The results of this study also have implications for the teacher who could try to determine the levels at which students are operating, making it possible to match content and teaching methods. The teaching methods must be adjusted to meet the level at which the student is operating, to enable the student to learn with understanding and not by memorizing.

These implications and opinions based on the results of this study are only a few of many that should be investigated. One of the major implications for further research deals with the development, implementation and testing of a Grades K - 12 geometry course based on the van Hiele levels. Some other questions that need to be answered by further research are:

1) Would a prerequisite course in logic lead to an improvement of proof-writing achievement?

2) Should proofs be included in the high school geometry course and should an alternative non-proof geometry course be offered for those who have little success in writing proofs?

3) Would a developmental geometry course from K - 12 based on the van Hiele levels produce a majority of students who could write geometric proofs with understanding?

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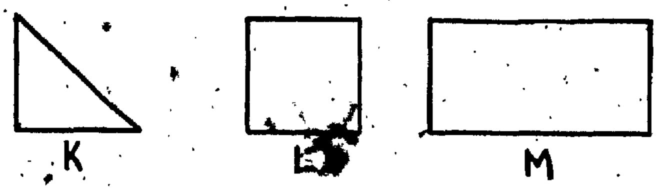
APPENDIX A

VAN HIELE GEOMETRY TEST

(CDASSG PROJECT, Usiskin, 1982)

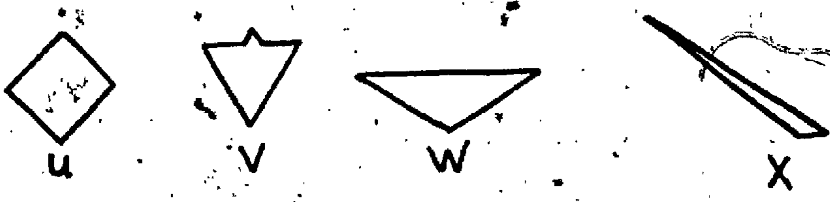
VAN HIELE GEOMETRY TEST

1. Which of these are squares?



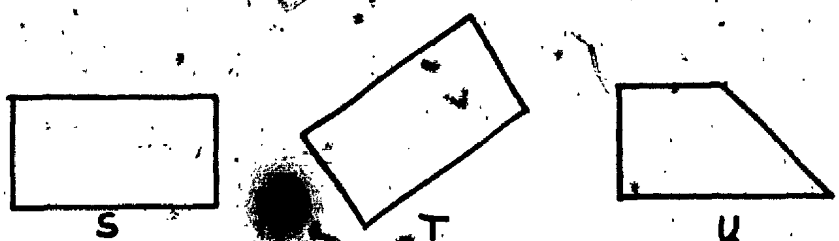
(A) K only
(B) L only
(C) M only
(D) L and M only
(E) All are squares.

2. Which of these are triangles?



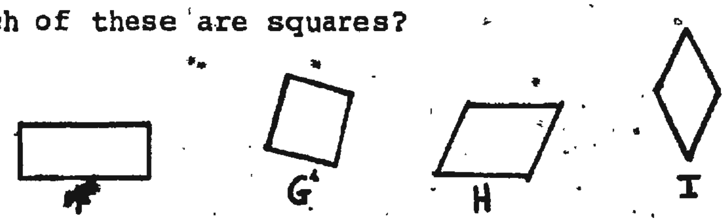
(A) None of these are triangles.
(B) V only
(C) W only
(D) W and X only
(E) V and W only

3. Which of these are rectangles?



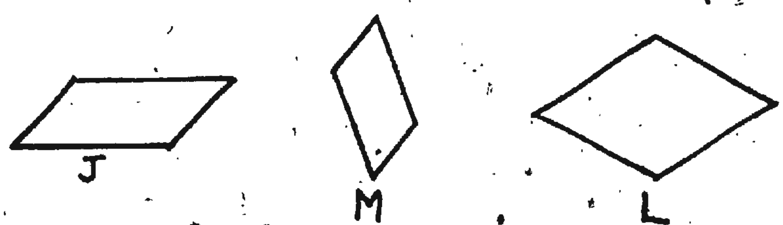
(A) S only
(B) T only
(C) S and T only
(D) S and U only
(E) All are rectangles.

4. Which of these are squares?



(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

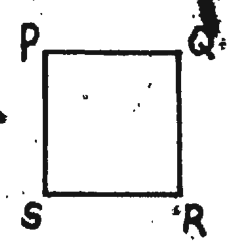
5. Which of these are parallelograms?



(A) J only
(B) L only
(C) J and M only
(D) None of these are parallelograms.
(E) All are parallelograms.

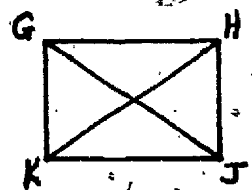
6. PQRS is a square.

Which relationship is true in all squares?



(A) $\overline{PR}$ and $\overline{RS}$ have the same length.
(B) $\overline{QS}$ and $\overline{PR}$ are perpendicular.
(C) $\overline{PS}$ and $\overline{QR}$ are perpendicular.
(D) $\overline{PS}$ and $\overline{QS}$ have the same length.
(E) Angle Q is larger than angle R.

7. In a rectangle GHJK, $\overline{GJ}$ and $\overline{HK}$ are the diagonals.

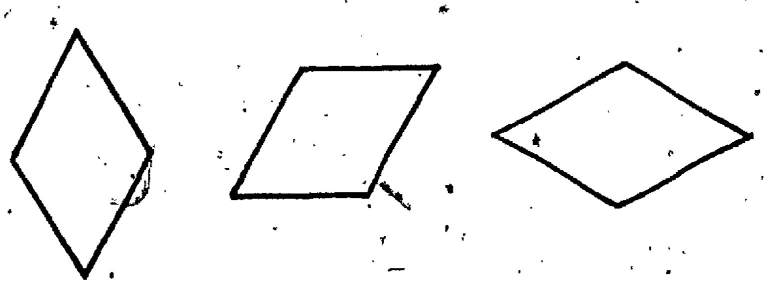


Which of (A) - (D) is not true in every rectangle?

(A) There are four right angles.
(B) There are four sides.
(C) The diagonals have the same length.
(D) The opposite sides have the same length.
(E) All of (A) - (D) are true in every rectangle.

8. A rhombus is a 4-sided figure with all sides of the same length.

Here are three examples.



Which of (A) - (D) is not true in every rhombus?

(A) The two diagonals have the same length.
(B) Each diagonal bisects two angles of the rhombus.
(C) The two diagonals are perpendicular.
(D) The opposite angles have the same measure.
(E) All of (A) - (D) are true in every rhombus.

9. An isosceles triangle is a triangle with two sides of equal length. Here are three examples.

Which of (A) - (D) is true in every isosceles triangle?

(A) The three sides must have the same length.

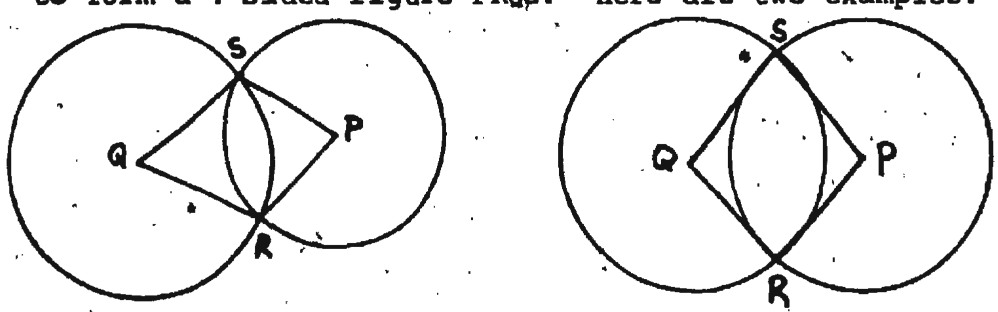
(B) One side must have twice the length of another side.

(C) There must be at least two angles with the same measure.

(D) The three angles must have the same measure.

(E) None of (A) - (D) is true in every isosceles triangle.

10. Two circles with centers P and Q intersect at R and S to form a 4-sided figure PRQS. Here are two examples.



Which of (A) - (D) is not always true?

(A) PRQS will have two pairs of sides of equal length.

(B) PRQS will have at least two angles of equal measure.

(C) The lines $\overline{PQ}$ and $\overline{RS}$ will be perpendicular.

(D) Angles P and Q will have the same measure.

(E) All of (A) - (D) are true.

11. Here are two statements.

Statement 1: Figure F is a rectangle.

Statement 2: Figure F is a triangle.

Which is correct?

(A) If 1 is true, then 2 is true.
(B) If 1 is false, then 2 is true.
(C) 1 and 2 cannot both be true.
(D) 1 and 2 cannot both be false.
(E) None of (A) - (D) is correct.

12. Here are two statements.

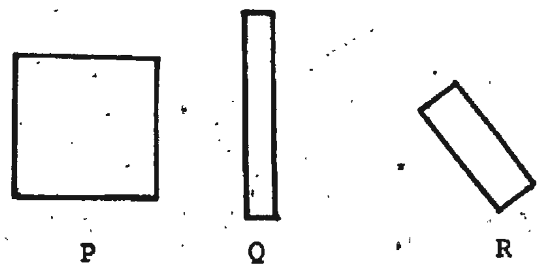
Statement S: ABC has three sides of the same length.

Statement T: In ABC, B and C have the same measure.

Which is correct?

(A) Statements S and T cannot both be true.
(B) If S is true, then T is true.
(C) If T is true, then S is true.
(D) If S is false, then T is false.
(E) None of (A) - (D) is correct.

13. Which of these can be called rectangles?



(A) All can.
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only

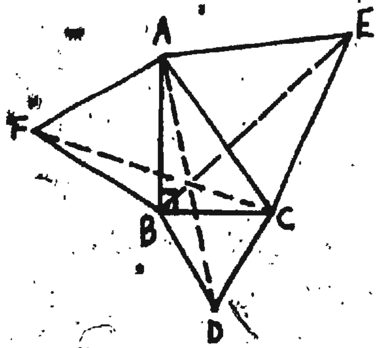
14. Which is true?

(A) All properties of rectangles are properties of all squares.
(B) All properties of squares are properties of all rectangles.
(C) All properties of rectangles are properties of all parallelograms.
(D) All properties of squares are properties of all parallelograms.
(E) None of (A) - (D) is true.

15. What do all rectangles have that some parallelograms do not have?

(A) opposite sides equal
(B) diagonals equal
(C) opposite sides parallel
(D) opposite angles equal
(E) none of (A) - (D)

16. Here is a right triangle ABC. Equilateral triangles ACE, ABF, and BCD have been constructed on the sides of ABC.



From this information, one can prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common. What would this proof tell you?

(A) Only in this triangle drawn can we be sure that $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common.

(B) In some but not all right triangles, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common.

(C) In any right triangle, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common.

(D) In any triangle, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common.

(E) In any equilateral triangle, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common.

17. Here are three properties of a figure.

Property D: It has diagonals of equal length.

Property S: It is a square.

Property R: It is a rectangle.

Which is true?

(A) D implies S which implies R.

(B) D implies R which implies S.

(C) S implies R which implies D.

(D) R implies D which implies S.

(E) R implies S which implies D.

18. Here are two statements.

    I. If a figure is a rectangle, then its diagonals bisect each other.

    II. If the diagonals of a figure bisect each other, the figure is a rectangle.

Which is correct?

(A) To prove I is true, it is enough to prove that II is true.

(B) To prove II is true, it is enough to prove that I is true.

(C) To prove II is true, it is enough to find one rectangle whose diagonals bisect each other.

(D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.

(E) None of (A) - (D) is correct.

19. In geometry:

(A) Every term can be defined and every true statement can be proved true.

(B) Every term can be defined but it is necessary to assume that certain statements are true.

(C) Some terms must be left undefined but every true statement can be proved true.

(D) Some terms must be left undefined and it is necessary to have some statements which are assumed true.

(E) None of (A) - (D) is correct.

20. Examine these three sentences.

(1) Two lines perpendicular to the same line are parallel.

(2) A line that is perpendicular to one of two parallel lines is perpendicular to the other.

(3) If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines m and p are perpendicular and lines n and p are perpendicular. Which of the above sentences could be the reason that line m is parallel to line n?

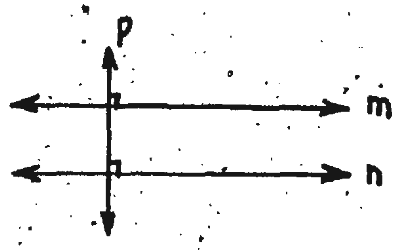
(A) (1) only

(B) (2) only

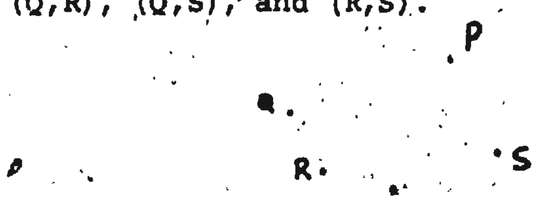
(C) (3) only

(D) Either (1) or (2)

(E) Either (2) or (3)



21. In F-geometry, one that is different from the one you are used to, there are exactly four points and six lines. Every line contains exactly two points. If the points are P, Q, R and S, the lines are (P,Q), (P,R), (P,S), (Q,R), (Q,S), and (R,S).



Here are how the words "intersect" and "parallel" are used in F-geometry. The lines (P,Q) and (P,R) intersect at P because (P,Q) and (P,R) have P in common.

The lines (P,Q and (R,S) are parallel because they have no points in common.

From this information, which is correct?

(A) (P,R) and (Q,S) intersect.

(B) (P,R) and (Q,S) are parallel.

(C) (Q,R) and (R,S) are parallel.

(D) (P,S) and (Q,R) intersect.

(E) None of (A) - (D) is correct.

22. To trisect an angle means to divide it into three parts of equal measure. In 1847, P.L. Wantzel proved that, in general, it is impossible to trisect angles using only a compass and an unmarked ruler. From his proof, what can you conclude?

(A) In general, it is impossible to bisect angles using only a compass and an unmarked ruler.

(B) In general, it is impossible to trisect angles using only a compass and a marked ruler.

(C) In general, it is impossible to trisect angles using any drawing instruments.

(D) It is still possible that in the future someone may find a general way to trisect angles using only a compass and an unmarked ruler.

(E) No one will ever be able to find a general method for trisecting angles using only a compass and an unmarked ruler.

23. There is a geometry invented by a mathematician J in which the following is true:

The sum of the measures of the angles of a triangle is less than 180°.

Which is correct?

(A) J made a mistake in measuring the angles of the triangle.

(B) J made a mistake in logical reasoning.

(C) J has a wrong idea of what is meant by "true".

(D) J started with different assumptions than those in the usual geometry.

(E) None of (A) - (D) is correct.

24. Two geometry books define the word rectangle in different ways. Which is true?

(A) One of the books has an error.

(B) One of the definitions is wrong. There cannot be two different definitions for rectangle.

(C) The rectangles in one of the books must have different properties from those in the other book.

(D) The rectangles in one of the books must have the same properties as those in the other book.

(E) The properties of rectangles in the two books might be different.

25. Suppose you have proved statements I and II.

I. If p, then q.
II. If s, then not q.

Which statement follows from statements I and II?

(A) If p, then s.

(B) If not p, then not q.

(C) If p or q, then s.

(D) If s, then not p.

(E) If not s, then p.

VAN HIELE GEOMETRY TEST

STUDENT ANSWER SHEET

NAME: ____
last first middle

BIRTHDATE: ______________________ SEX: Female ___
d m y
Male ___

SCHOOL: ____

MATHEMATICS CATEGORY (tick only one)

1. ☐ Advanced Mathematics grade 9 to Advanced Mathematics 1201
2. ☐ Advanced Mathematics grade 9 to Academic Mathematics 1203
3. ☐ Academic Mathematics grade 9 to Advanced Mathematics 1201
4. ☐ Academic Mathematics grade 9 to Academic Mathematics 1203
5. ☐ Basic Mathematics grade 9 to Academic Mathematics 1203

CIRCLE THE CORRECT ANSWER

1.	A	B	C	D	E	14.	A	B	C	D	E
2.	A	B	C	D	E	15.	A	B	C	D	E
3.	A	B	C	D	E	16.	A	B	C	D	E
4.	A	B	C	D	E	17.	A	B	C	D	E
5.	A	B	C	D	E	18.	A	B	C	D	E
6.	A	B	C	D	E	19.	A	B	C	D	E
7.	A	B	C	D	E	20.	A	B	C	D	E
8.	A	B	C	D	E	21.	A	B	C	D	E
9.	A	B	C	D	E	22.	A	B	C	D	E
10.	A	B	C	D	E	23.	A	B	C	D	E
11.	A	B	C	D	E	24.	A	B	C	D	E
12.	A	B	C	D	E	25.	A	B	C	D	E
13.	A	B	C	D	E						

USE OTHER SIDE FOR ANY ROUGH WORK

APPENDIX B

PROOF TEST

(CDASSG PROJECT, Usiskin, 1982)

CDASSG GEOMETRY TEST

Name ______________________ School ______________
Last First

Your Birthdate ____ ____ ____ Today's date __________
Mon. Day Year

DIRECTIONS:

1. You will have 35 minutes to complete this test. Take your time but do not spend too much time on any one question.

2. All answers should be written on these pages. If you need more space, use the other side of one of the pages.

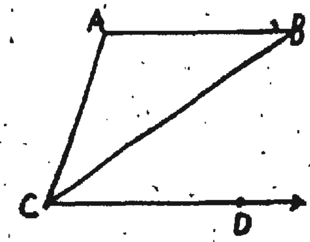
3. Work on a question even if you cannot answer it completely, because partial credit will be given.

4. You may use abbreviations for names of theorems.

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

CDASSG GEOMETRY TEST

1. Write statements and reasons to complete this proof.



GIVEN: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$AB = AC$

PROVE: $\overrightarrow{CB}$ bisects $\angle ACD$

Statements	Reasons
1. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.	Given.
2. $\angle B \cong \angle BCD$.	______________
3. $AB = AC$	Given.
4. ______________	Base angles of an isosceles triangle are congruent (equal in measure).
5. ______________	Transitive property or substitution
6. ______________	Definition of angle bisector

2. Statement: If a line passes through the midpoints of two sides of a triangle, it is parallel to the third side of that triangle.

Suppose you wished to prove the above statement. In the space provided:

1. Draw and label a figure.
2. Write, in terms of your figure, what is given and what is to be proved.

FIGURE:

GIVEN:

TO PROVE:

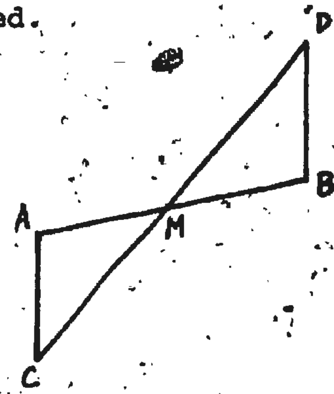
DO NOT PROVE THE STATEMENT. GO ON TO THE NEXT PAGE.

3. Write this proof in the space provided.

GIVEN: M is the midpoint of $\overline{AB}$.

M is the midpoint of $\overline{CD}$.

PROVE: $\triangle ACM \cong \triangle BDM$



GO ON TO THE NEXT PAGE.

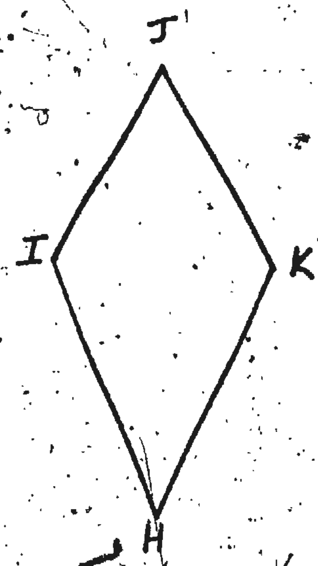
4. Write this proof in the space provided.

GIVEN: Quadrilateral HIJK

HI = HK

IJ = JK

PROVE: $\angle I \cong \angle K$

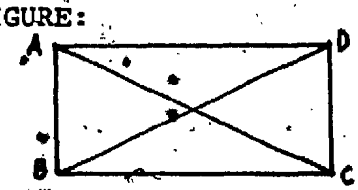


GO ON TO THE NEXT PAGE.

5. Here is the theorem you have had. Complete its proof in the space provided.

Theorem: The diagonals of a rectangle are congruent.

FIGURE:



GIVEN: ABCD is a rectangle.

PROOF:

TO PROVE: $\overline{AC} \cong \overline{BD}$

GO ON TO THE NEXT PAGE.

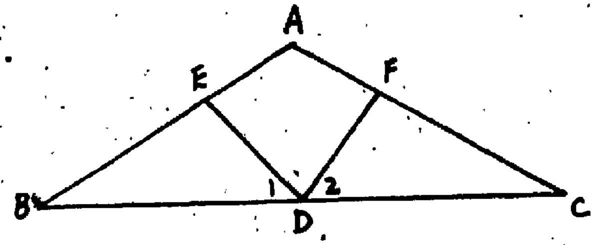
6. Write this proof in the space provided.

GIVEN: D is the midpoint of $\overline{BC}$.

$\angle 1 \cong \angle 2$

$\overline{DE} \cong \overline{DF}$

PROVE: $\triangle$ ABC is isosceles.



THIS IS THE LAST PAGE. IF YOU HAVE TIME, YOU MAY GO BACK TO PREVIOUS PAGES.

APPENDIX C

GRADING POLICY AND GRADING CRITERIA FOR EACH PROOF ITEM

(Adopted in part from Senk, 1983)

CDASSG Proof Tests--Grading Policy

The following is the grading policy for the proof items as stated by Senk and Usiskin (1981):

Each item on the tests is graded on a scale from 0 to 4.

Item 1: The grade is the number of blanks correctly completed.

Item 2: The grade is given as follows: 2 points for given; 1 point for "to prove"; 1 point for figure.

Items 3-6: The following broad criteria (from John A. Malone et al., "Measuring Problem Solving Ability," in Problem Solving in School Mathematics, edited by Stephen Krulik and Robert Reys, 1980 Yearbook of the National Council of Teachers and Mathematics, Reston, VA: NCTM, 1980) are being applied.

Score	Solution Stage
0	Noncommencement The student is unable to begin the problem or hands in work that is meaningless.
1	Approach The student approaches the problem with meaningful work, indicating some understanding of the problem, but an early impasse is reached.
2	Substance Sufficient detail demonstrates that the student has proceeded toward a rational solution, but major errors or misinterpretations obstruct the correct solution process.
3	Result The problem is very nearly solved; minor errors produce an invalid final solution.
4	Completion An appropriate method is applied to yield a valid solution.

CDASSG PROOF TESTS

Grading Conventions

The following grading conventions are adopted from Senk (1983):

ALL ITEMS

1. There is no penalty for
   a. spelling errors.
   b. art, poetry or other comments.
   c. misusing or ignoring the distinctions between
      (1) congruence and equality
      (2) an angle and its measure.
      (3) a theorem and a postulate.
      (4) notation for segment, line, ray or length of a segment.

2. There will be penalties for
   a. errors in notation, such as
      (1) naming an angle by its vertex, when three letters are necessary.
      (2) writing vertices in non-corresponding order in statements about similarity or congruence.
      (3) failing to write whether three letters indicate an angle or a triangle, e.g., ABC ≅ XYZ. DO NOT PENALIZE student if she/he forgets symbol only once in a statement, e.g., △ABC ≅ XYZ.
   b. errors in vocabulary or names of therems, such as
      (1) calling an angle pair supplementary instead of complementary.
      (2) giving as a reason "// lines → ≅ alt. int. ∠'s" instead of "≅ alt. int. ∠'s → // lines."
   c. arithmetic or algebraic errors.

3. A student will not be penalized more than once per item for a given error.

ITEM 2

1. If the figure is topologically correct, give 1 point. Errors in labels will be caught when grading GIVEN and TO PROVE.

2. Give 1 of 2 possible points for the GIVEN if student has gotten part, but not all, of the hypothesis correct.

Grading Conventions (Cont'd)

ITEMS 3 to 6

1. A student's score will not be influenced by
   a. marks made on the diagram.
   b. whether or not the GIVEN was restated.

2. For geometry proofs, errors in logical reasoning are major errors; errors in notation, vocabulary, names of reasons, arithmetic or algebra are minor errors.

3. Specific examples describing each score are:

   0 - student writes nothing, writes only the given, or writes invalid or useless deductions.

   1 - student writes at least 1 useful valid deduction with reason

   2 - student either

   a. deduces approximately half the steps necessary for a valid proof, and then either stops or writes irrelevant or incorrect information thereafter,

   or b. writes a complete proof in which the last half is correct, but in which an early step contains an error in logical reasoning.

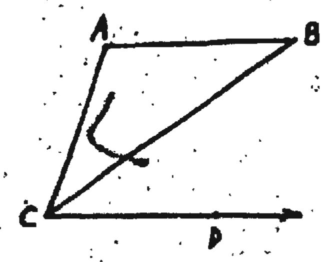
   3 - student writes a proof in which all steps follow logically from the previous ones, but in which there are minor errors.

   4 - student writes a valid proof with at most one error in notation. More than one error in notation, or even one error in the name of a figure or reason will lower score to a 3.

4. Readers will have to be sensitive to curtailment, i.e., skipping steps will not automatically result in a lower score.

Criteria for Grading Proofs

Item 1. Write statements and reasons to complete this proof.



GIVEN: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$AB = AC$

PROVE: $\overleftrightarrow{CB}$ bisects $\angle ACD$.

Statements	Reasons
1. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.	Given.
2. $\angle B \cong \angle BCD$.	______________
3. $AB = AC$	Given.
4. ______________	Base angles of an isosceles triangle are congruent (equal in measure).
5. ______________	Transitive property or substitution.
6. ______________	Definition of angle bisector.

Grading Criteria

The grade is the number of blanks correctly completed.

Criteria for Grading Proofs

Item 2.

> Statement: If a line passes through the midpoints of two sides of a triangle, it is parallel to the third side of that triangle.

Suppose you wished to prove the above statement. In the space provided:

1. Draw and label a figure.
2. Write, in terms of your figure, what is given and what is to be proved.

FIGURE:

GIVEN:

TO PROVE:

DO NOT PROVE THE STATEMENT. GO ON TO THE NEXT PAGE.

Grading Criteria

a) FIGURE (1 point): If figure is topologically correct, give 1 point; figure must also be labeled.

b) TO PROVE (1 point): If stated correctly, give 1 point.

c) GIVEN (2 points):

(i) If stated as AD = DB and AE = EC rather than D is the midpoint of AB and E is the midpoint of AC, no points awarded. (This was considered an interpretation of the given.)

(ii) If stated as D and E are midpoints (not mentioning the lines), give 1 point.

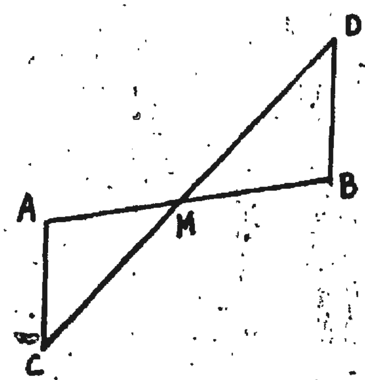
Criteria for Grading Proofs

Item 3. Write this proof in the space provided.

GIVEN: M is the midpoint of $\overline{AB}$.

M is the midpoint of $\overline{CD}$.

PROVE: $\triangle ACM \cong \triangle BDM$



Grading Criteria (adopted from Senk, 1983)

Item 1-3

Response		Score
a) Either AMC = BMD or AM = BM, & CM = DM	with correct reason	a) 1
b) Both AMC = BMD and AM = BM, & CM = DM	with correct reasons	b) 2
c) Either a valid proof with minor errors such as letters of the vertical angles permuted or more than one irrelevant, but correct, statement; or AM = BM, CM = DM and $\triangle ACM \cong \triangle BDM$ by SAS, without mentioning the vertical angles.		c) 3

Criteria for Grading Proofs

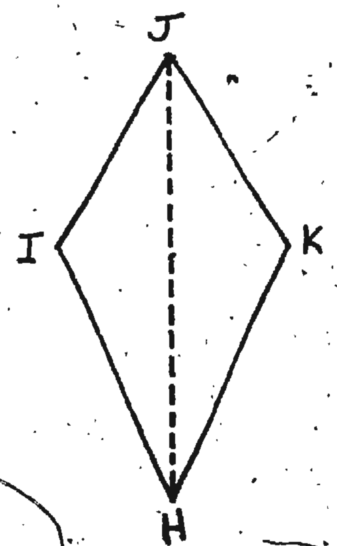
Item 4. Write this proof in the space provided.

GIVEN: Quadrilateral HIJK

HI = HK

IJ = JK

PROVE: ∠I ≅ ∠K



Grading Criteria

1. Congruent Triangles Strategy

Response		Score
a)	Using line $\overline{JH}$ (not necessary to draw it) or Using line $\overline{JH}$ with $\overline{JH} \cong \overline{JH}$ and correct reason	a) 1
b)	Using line $\overline{JH}$ and $\Delta IJH = \Delta KJH$ and correct reason	b) 2
c)	Using line $\overline{JH}$ with either (i) $\Delta IJH \cong \Delta KJH$, ∠I ≅ ∠K with correct reasons (ii) $\overline{JH} \cong \overline{JH}$, ∠I ≅ ∠K with correct reasons	c) 3

Criteria for Grading Proofs

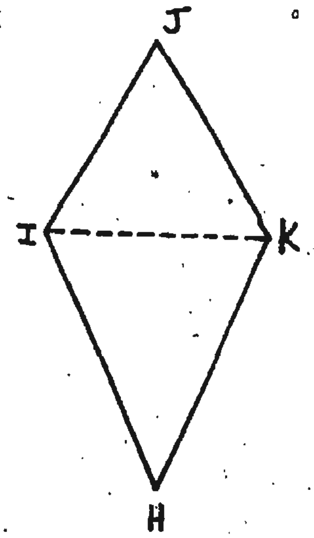
Item 4. Write this proof in the space provided.

GIVEN: Quadrilateral HIJK

HI = HK

IJ = JK

PROVE: ∠I ≅ ∠K



2. Angle Addition Strategy

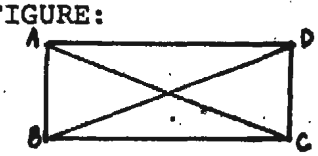
Response	Score
a) Using line $\overline{IK}$ (not necessary to draw it) or Using line $\overline{IK}$ and stating △IJK and △IHK are isosceles with correct reasons.	a) 1
b) Using line $\overline{IK}$ and stating base angles are congruent with correct reasons. i.e. ∠JIK = ∠JKI ∠HIK = ∠HKI	b) 2
c) Using line $\overline{IK}$ with: ∠JIK = ∠JKI ∠HIK = ∠HKI ∠JIK + ∠HIK = ∠JKI + ∠HKI } with correct reasons	c) 3
d) Using line $\overline{IK}$ with: JIK = ∠JKI ∠HKI = ∠HKI ∠I = ∠K } with correct reasons	d) 4

Criteria for Grading Proofs

Item 5. Here is a theorem you have had. Complete its proof in the space provided.

Theorem: The diagonals of a rectangle are congruent.

FIGURE:



GIVEN: ABCD is a rectangle.

PROOF:

TO PROVE: $\overline{AC} \cong \overline{BD}$

Grading Criteria (adopted from Senk, 1983)

Response	Score
a) one correct deduction about sides or right angles with reason, or two correct deductions with "almost correct" reasons	a) 1
b) any one of (1) - (3) below	b) 2
(1) two correct deductions, one about ≅ sides, the other about consecutive right angles, with reasons	
(2) 2 non-overlapping right Δ's ≅ by SAS, followed by ≅ diagonals by CPCTC	
(3) 2 overlapping right Δ's ≅ by faulty reasoning followed by ≅ diagonals by CPCTC	
c) overlapping Δ's ≅ via correct reasoning, with notational error(s)	c) 3

Criteria for Grading Proofs

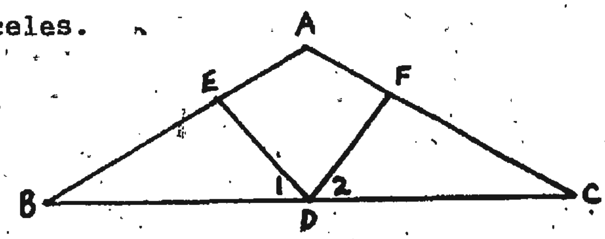
Item 6. Write this proof in the space provided.

GIVEN: D is the midpoint of $\overline{BC}$.

$\angle 1 \cong \angle 2$

$\overline{DE} \cong \overline{DF}$

PROVE: ΔABC is isosceles.



Grading Criteria

Response				Score
a)	$\overline{BD} = \overline{CD}$ with correct reason			a) 1
b)	(i) ΔEBD $\cong$ ΔFCD with correct reasons			b) 2
or	(ii)	$\overline{BD} \cong \overline{CD}$ $\angle B \cong \angle C$ ΔABC is isosceles	with only one incorrect reason	
c)	(i)	$\overline{BD} = \overline{CD}$ $\angle B \cong \angle C$ ΔABC is isosceles	with correct reasons	c) 3
or	(ii)	ΔEBD $\cong$ ΔFCD $\angle B \cong \angle C$ ΔABC is isosceles	with correct reasons	

APPENDIX D

BEHAVIORS AT EACH VAN HIELE LEVEL

(CDASSG PROJECT, Usiskin, 1982)

BEHAVIORS AT EACH VAN HIELE LEVEL

(Usiskin, 1982, pp. 9-12)

Level 1 (their base level, level 0)

(P.M., 1958-59)

1. "Figures are judged according to their appearance."

2. "A child recognizes a rectangle by its form, shape.

3. . . . and the rectangle seems different to him from a square."

4. "When one has shown to a child of six, a six year old child, what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is able to produce those figures without error on a geoboard of Gattegno, even in difficult situations."

5. "A child does not recognize a parallelogram in a rhombus."

6. "The rhombus is not a parallelogram. The rhombus appears ... as something quite different."

(P.M., 1968)

7. "When one says what one calls a quadrilateral whose four sides are equal a rhombus this statement will not be enough to convince the beginning student (from which I deduce that this is his level 0) that the parallelograms which he calls squares are part of the set of rhombuses."

(P.M., 1979)

8. (on a question involving recognition of a tilted square as a square) "basic level, because you can see it."

Level 2 (their first level)

(P.M., 1957)

1. "He is able to associate the name 'isosceles triangle' with a specific triangle, knowing that two of its sides are equal, and draw the subsequent conclusion that the two corresponding angles are equal."

Level 2 (cont.)

(Dina, 1957; P.M. and Dina, 1958)

2. ". . . a pupil who knows the properties of the rhombus and can name them, will also have a basic understanding of the isosceles triangle = semirhombus."

3. "The figures are the supports (lit. 'supports' in French) of their properties."

4. "That a figure is a rectangle signifies that it has four right angles, it is a rectangle, even if the figure is not traced very carefully."

5. "The figures are identified by their properties. (E.g.) If one is told that the figure traced on the blackboard possesses four right angles, it is a rectangle, even if the figure is not traced very carefully."

6. "The properties are not yet organized in such a way that a square is identified as being a rectangle."

(P.M., 1959)

7. "The child learns to see the rhombus as an equilateral quadrangle with identical opposed angles and inter-perpendicular diagonals that bisect both each other and the angles."

8. (a middleground between this and the next level) "Once the child gets to the stage where it knows the rhombus and recognizes the isoceles triangle for a semi-rhombus, it will also be able to determine offhand a certain number of properties of the equilateral triangle".

9. "Once it has been decided that a structure is an 'isosceles triangle' the child will also know that a certain number of governing properties must be present, without having to memorize them in this special case."

(P.M., 1976)

10. "The inverse of a function still belongs to the first thought level."

11. "Resemblance, rules of probability, powers, equations, functions, revelations, sets-with these you can go from zero to the first thought level."

Level 3 (their second level)

(Dina, 1957)

1. "Pupils . . . can understand what is meant by 'proof' in geometry. They have arrived at the second level of thinking."

(P.M., 1957)

2. "He can manipulate the interrelatedness of the characteristics of geometric patterns."

3. "e.g., if on the strength of general congruence theorems, he is able to deduce the equality of angles or linear segments of specific figures."

(P.M., 1958-59)

4. "The properties are ordered (lt. 'ordonnent'). They are deduced from each other: one property precedes or follows another property."

5. "The intrinsic significance of deduction is not understood by the student."

6. "The square is recognized as being a rectangle because at this level definitions of figures come into play."

(P.M., 1959)

7. "The child . . . (will) recognize the rhombus by means of certain of its properties, . . . because, e.g., it is a quadrangle whose diagonals bisect each other perpendicularly."

8. "It (the child) is not capable of studying geometry in the strictest sense of the word."

9. "The child knows how to reason in accordance with a deductive logical system . . . this is not however, identical with reasoning on the strength of formal logic."

Level 3 (cont.)

(P.M., 1976)

10. "The question about whether the inverse of a function is a function belongs to the second thought level."

11. "The understanding of implication, equivalence, negation of an implication belongs to the second thought level."

(P.M., 1978)

12. "They are able to understand more advanced thought structure, such as: 'the parallelism of the lines implies (according to their signal character) the presence of a saw, and therefore (according to their symbolic character) equality of the alternate interior angles'."

13. "I (the student) can learn a definition by heart. No level. I can understand that definitions may be necessary: second level."

14. ". . . . you know what is meant by it (the use of 'some' and 'all') second level."

Level 4 (their third level)

(P.M., 1957)

1. "He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations. For example: if he can distinguish between a proposition and its reverse" (sic. meaning our converse)

(Dina, 1957)

2. "We can start studying a deductive system of propositions, i.e., the way in which the interdependency of relations is effected. Definitions and propositions now come within the pupils' intellectual horizon."

3. "Parallelism of the lines implies equality of the corresponding angles and vice versa."

Level 4 (cont.)

(P.M. and Dina, 1958)

4. "The pupil will be able, e.g., to distinguish between a proposition and its converse."

5. "It (is) . . . possible to develop an axiomatic system of geometry".

(P.M., 1958-59)

6. "The mind is occupied with the significance of deduction, of the converse of a theorem, of an axiom, of the conditions necessary and sufficient."

(P.M., 1968)

7. ". . . one child tell him (the student) that in a proof, it is really a question of knowing whether these theses are true or not, or rather of the relationship between the truth of these theses and of some other. Without their understanding such relationships we cannot explain to the student that one has to have recourse to axioms." (I induced the level from the first part of this statement; he never identified the level.)

Level 5 (their fourth level)

(Dina, 1957)

1. "A comparative study of the various deductive systems within the field of geometrical relations is . . . . reserved for those, who have reached the fourth level . . . ".

(P.M. and Dina, 1958)

2. "Finally at the fourth level (hardly attainable in secondary teaching) logical thinking itself can become a subject matter."

3. "The axiomatics themselves belong to the fourth level."

Level 5 (cont.)

(P.M., 1958-59)

4. "One doesn't ask such questions as: what are points, lines, surfaced, etc.? . . . Figures are defined only by symbols connected by relationships. To find the specific meaning of the symbols, one must turn to lower levels where the specific meaning of these symbols can be seen."