A STUDY OF THE EXPLANATION USED BY ELEMENTARY SCHOOL STUDENTS IN THE INTERPRETATION OF THE FRACTION CONCEPT, AND ADDITION AND SUBTRACTION OF FRACTION

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by

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ABSTRACT

The purpose of this study was to analyze the explanations given by grade six students in their interpretations of the fraction concept and the addition and subtraction of fractions.

A series of 52 questions dealing with the fraction concept and the addition and subtraction of fractions were developed, tested, and analyzed. Twenty students were randomly selected and were interviewed and audiotaped. The questions included problems at the physical, pictoral and symbolic levels. Questions were presented using both part of a set interpretation of a fraction and part of a whole interpretation of a fraction. The questions used in the study were of two different types. Type 1 questions were of the form where the number in the set was the same as the denominator. Type 2 questions were of the form where the number in the set was a multiple of the denominator.

An analysis was carried out to determine the interpretations of the fraction concept and the addition and subtraction of fractions. Where incorrect interpretations were found, they were analyzed to hypothesize their nature and cause.

There were very few incorrect interpretations of the fraction concept and the addition and subtraction of fractions when the students were given Type 1 questions. The only consistent incorrect interpretation was that three students explained the fraction in terms of a relationship between the parts of a set rather than as a relationship between the parts and the total number in the set.

There were many incorrect interpretations of the fraction concept and the addition and subtraction of fractions when the students were given the Type 2 questions. The most commonly used incorrect interpretation was an attempt by the students to reduce the Type 2 questions to Type 1 questions. The students explained their procedures as if the questions were Type 1.

There were very few computational errors made by the students and therefore it was pointless to determine if the errors were systematic.

The incorrect interpretations were not found to be systematic. Students gave consistent incorrect interpretations to particular questions but they were not consistent in using the same incorrect interpretations for all the questions.

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CHAPTER I

INTRODUCTION

Background

Educators and researchers such as Streefland (1978) and West (1978) have agreed that fractions can be one of the difficult topics in the elementary mathematics curriculum. For example, Streefland noted:

> Names like one-third, one-fourth, and so on are not self-evident to children. The meaning third, fourth, . . . refers to the ordinal aspect of natural numbers which might outvote the tender concept of fraction in the beginning when it is still developing mentally. (p. 53)

Streefland indicated the necessity of emphasizing and understanding the relationship between equivalent parts, quantities that result from the subdivision of an arbitrary unit, and the fractional names used to describe these parts. This understanding develops over time and involves many aspects of the fraction concept.

West (1978) suggested that the road through mathematics is beset with difficulties where the learner stumbles along the way to the attainment of mathematical skills. One of these difficulties deals with the fraction concept. He stated:

> Another problem for many children is finding a fractional part of a set, even though the same children may find that identifying a fractional part of a region is easy. Many texts try to ease the way by giving illustrations of sets arranged in rectangular arrays. Unfortunately the real world is not so accommodating. (p. 20)

Piaget (1953) suggested that fractions are introduced into the curriculum at a time when the child is at the concrete operations stage. These operations require the use of mental logic and the student's logical thinking is based on actions he performs. When a student makes a mistake the error he has made should be looked at in terms of his own logical thought. This is particularly evident in the reasoning the student uses as he attempts to comprehend the concept of equivalent fractions. Given two equivalent fractions such as 1/5 and 2/10, the student knows that the numerical values of two and ten are greater than those of one and five, and consequently believes 2/10 to be greater than 1/5. Using his own logical reasoning the student is correct due to his past experiences with whole numbers.

Retkin (1978) suggested that the most confusing part of comprehending a fraction is that when we say third we write the numeral three, for fourth we write four, and so on. For children, 1/3 is just two numbers, one number above the line and the other number below the line. It is no wonder children get the symbol confused. Retkin suggested this would rarely happen if the written word were used in the denominator of the fraction and if figures or shapes were provided with the names of the fractions written as suggested (e.g., using a square with four parts write the fraction with the numeral one in the numerator and the word fourth in the denominator). This strategy for introducing the fraction should continue to be used until children are very familiar with the fraction concept.

Ellerbruch and Payne (1978) suggested that many interpretations are needed before a student can completely understand the fraction concept. These interpretations include fractions as measures of regions and segments, fractions as parts of sets, and fractions as equivalent classes of ordered pairs of natural numbers. From their research and classroom experiences, Ellerbruch and Payne suggested that the introduction of the fraction be based on a measurement model because it seemed most natural and useful for children. The use of such a model would require the picture or illustration to be processed by the student and then related to the fraction.

The fraction concept can be challenging and difficult for students to comprehend. The reasons for these difficulties need to be investigated. It was proposed that the answer to this question could be found through an analysis of the explanations given by students as they worked with fractions.

Purpose of the Study

The major purpose of the study was to analyze the explanations given by students in the interpretation of the fraction concept and the addition and subtraction of fractions. The study also involved an examination of the computational errors that were found in the addition and subtraction of fractions. Where incorrect interpretations were found to exist it was proposed to determine if any systematic patterns existed in the responses of those students who gave incorrect interpretations and made computational errors. Specifically, the study investigated the following questions.

1) What interpretations of the fraction concept and the addition and subtraction of fractions are made by students?

2) What computational errors are made by students in the addition and subtraction of fractions?

3) Are these incorrect interpretations and computational errors systematic?

Significance of the Study

In the National Assessment of Education Progress (1978) the importance of the fraction concept within the structure of mathematics and its application to everyday life are stressed. The specific results of the study indicated that the major problem in adding fractions arose when students were asked to add fractions with unlike denominators. The authors of the study suggested this could be avoided by focusing more on the total sequencing of ideas rather than on each individual algorithm. They summarized their findings by stating:

> Competency with fractions is important. A quality instructional program should strive to ensure that every student has frequent opportunities to become competent. Emphasis should be placed on sound development of fractions with all algorithms connecting the main idea of the initial development. (p. 42)

Joy (1976) supported the inclusion of fractions in the curriculum and suggested that one of the main points of concern was the methods used to teach fractions. Joy stated the need to introduce the fraction concept through methods that relate the fraction to the everyday world of the child. The use of such a model would become a way for removing one of the stumbling blocks that students encounter when working with fractions.

Cable (1976) suggested two reasons for teaching fractions. These reasons included the need of fractions for comparison purposes and educators have a committment to teach fractions because of their place in many areas of work. Dana (1978) stated that the most important reason for retaining and using the common fraction was that the common fraction describes situations in the child's world.

It was not the intent of this study to provide a case for the inclusion of fractions in the curriculum. However, an analysis of the literature indicated that fractions do play a significant role in mathematics and hence an investigation of the fraction concept and related operations is an important area for research.

Scope and Limitations

The limitations of this study arose from three main areas. Firstly, the nature and size of the sample which limited the generalizability of the results. The sample involved only students at the grade six level attending two rural Newfoundland schools. Results might have been different if the study were carried out with students living in urban communities, other rural communities, or from different age and grade levels. The sample size ($n = 20$) was kept small due to the method used to collect the data. Each student was individually audio-taped and interviewed when answering the questions.

Secondly, the method for collecting the data must be considered. This individualized method of interviewing and audiotaping was new for these students. By being exposed to a novel situation students might have responded differently than they normally would have in a regular classroom situation.

Thirdly, the presence of the investigator must be considered. The students had no previous contact with the investigator and the interviews were held with only the investigator and student present. The students might have responded differently if the interviews were carried out in a regular classroom situation and in the presence of an investigator with whom the students had prior contact.

Definition of Terms

For purposes of this study the following term is defined:

Fraction: a numeral of the form a/b where $a/b > 0$, $b > a$, $a \neq 0$, and $b \neq 0$. The following fractions were used in the study: thirds, fourths, fifths, sixths, sevenths, eighths, ninths, tenths, and twelfths.

CHAPTER II

REVIEW OF THE LITERATURE

In recent years the topic of student thought processes involved in the interpretation of concepts has received the attention of many researchers, especially those involved in the field of mathematics education. Even though the topic is not new, in recent years it has become the focus of much attention as educators and researchers have directed their attention to the identification of the thought processes rather than to the actual solution of a problem. Once the thought processes have been identified it becomes a basis for remediation.

Radatz (1979) stated that many mistakes made in mathematics are not the result of carelessness but rather a misunderstanding or misinterpretation of the various subskills that are prerequisite to performing computations and solving problems. It was proposed that an understanding of students' incorrect interpretations was best accomplished through an analysis of their explanations. These explanations were arrived at by having students use a "think aloud technique". In this technique the students verbalized the procedure used to obtain the solution. These statements were audiotaped and used at a later date for analysis.

The review of the literature will be presented in two parts. In the first part research on the "think aloud technique" and its relationship to understanding incorrect interpretations is presented. In the second part research on the fraction concept and addition and subtraction of fractions is presented.

Research on "Thinking Aloud"

While finding errors is a necessary part of the diagnostic procedure, what is more relevant is the diagnosis of what caused the incorrect interpretation to occur. Researchers have differed in regard to the actual format to use to conduct a diagnosis. Researchers such as Cox (1975) and Englehardt (1977) suggested that initially the algorithm be broken down into levels of computational skills, administer the test, and apply criteria for the types of errors. Following analysis and tentative diagnosis, the student is then asked to explain the procedure used to find the solution.

The example provided by Lankford (1974) illustrated the "think aloud technique" that was proposed for this study. Rather than break the algorithm down into various levels, administer the test, and use criteria for errors, Lankford held interviews at various times throughout the year. He found that oral interviews were an effective way to examine the thought processes which students used in performing mathematical exercises. Once the interview was completed and an analysis carried out, Lankford hypothesized the thought processes used by the students. From this he decided whether or not the strategy used by the student required changing or reinforcing.

Walton (1975) conducted an exploratory study to determine whether or not there are patterns in the thinking processes used by average or below average ability mathematics students. A tape recording was made of each student "thinking aloud" as he solved the problem with 50% of the students being asked guiding questions. Dalton concluded that patterns do exist in the thinking processes of students but that the effect of prompting students through the use of guiding questions was inconclusive.

Kantowski (1974) conducted an exploratory study of processes involved in solving mathematical problems under heuristic instruction. At the end of the study a posttest was administered in which Kantowski obtained verbal protocols of solutions of problems by having students "think aloud" as they solved problems. The protocols were recorded on cassette tapes. She found that in a high percentage of the students who had been provided with a goal-oriented heuristic, that the use of the "think aloud technique" indicated regular patterns of analysis and synthesis in the solutions given by the students. However, there were periods of silence in which the students did not verbalize his thoughts. Kantowski (1975) and Gagné and Smith (1962) suggest that this is a serious limitation since the "think aloud technique" does not provide information about all the processes used by the students.

An analysis of student explanations through the "think aloud technique" can provide valuable information to educators as a way of remediating the problems that students encounter. A review of the literature indicated that an analysis of students' incorrect interpretations can be accomplished through a study of the explanations given by students. It was proposed that this approach be utilized for this study.

By integrating knowledge of the curriculum with individual differences in children, this approach of analyzing incorrect interpretations in relation to student explanations should give some indication of how a concept might be made easier for students to comprehend. The next part of the review focuses on the research dealing with the fraction. Firstly, the research dealing with the fraction and its operations will be presented. Secondly, the research dealing with the teaching of fractions and its operations will be presented.

Research on Fractions

Campbell (1974) and Wagner (1975) studied the development of the fraction concept in young children. The data from their studies not only supported the gradual development of the fraction concept but also indicated that if a correct sequence is used it is possible to introduce the fraction concept earlier. The results of the studies indicated that children in grades kindergarten to two were able to determine the number of fractional parts in a whole and that understanding occurred in the order of one-half, one-third, and then one-fourth.

Campbell (1974) and Carpenter, Coburn, Reys and Wilson (1978) stressed the need for the continued use of manipulatives, especially when the fraction concept is being introduced. The results from Campbell's work with primary grade students implied that it was less than half as effective to use pictures rather than use physical objects. Carpenter et al. (1976) suggested that the time we choose to use manipulatives is crucial. The earlier fractions are introduced then the use of manipulatives must be increased.

Carpenter et al. (1976) and Peck, Jencks and Chatterly (1980) presented views on why 12 to 14-year-olds were still unable to compute with fractions. They suggested that students were not viewing fractions as representing quantities but as numbers to be combined. In many instances students merely performed a mechanical process with little understanding of when and why it needed to be applied. They also found that students were unable to determine the reasonableness of their answers. The authors attributed this to the fault of the models employed to teach fractions to the students. There was no relationship between the models and the real world. This was especially true for the models used to teach addition and subtraction of fractions.

Knifong and Holtan (1976) analyzed the written work of sixth graders to determine the typical errors made by students to provide teachers with a factual basis for planning instruction. Following analysis they were successful in categorizing the computational errors made by students in working with fractions. They found the most common errors were of the type 3/4 + 1/2 computed to be 4/6; 2/3 x 150 calculated as 150 ÷ 3 = 50; and inverted responses where a solution 4/5 was written as 5/4.

Hiebert and Tonnessen (1978) investigated a strategy to introduce the fraction concept earlier than is presently done. In their study of students in the age range of five to eight years, they found that children were capable of learning the fraction concept earlier and they were very specific in how this earlier learning of fractions could be accomplished. The data collected revealed that students could perform better if set/subset tasks were used to help develop the fraction concept rather than using length or area tasks for this purpose.

Owens (1977) presented an alternative view which conflicted with the findings of the research revealed by Hiebert and Tonnessen. Owens concluded that neither age nor grade had any relationship to either the learning of area measurement or of the fraction concept. He stated that children in grades three and four were equally capable of using the area model to learn the fraction concept. It is worthy to note that Hiebert and Tonnessen's study dealt with students in grades kindergarten to two while Owen's study was conducted using grade three and four students. It may be that once students are past the first three years of school then area tasks can be used when working with the fraction concept. Owens concluded that the area concept does not have an effect on fraction learning which utilizes an area model as a basis for instruction.

Brown (1979) and Tucker (1978) suggested that the use of everyday activities and materials with which the students are familiar should be the basis on which to develop the fundamental concepts of a fraction, namely visual recognition of parts and wholes. These materials and experiences can be utilized to bridge the gap between experience, vocabulary, and verbalization. Brown and Tucker found this approach contributed to the development of the fraction concept while revealing the real-world application of the exercises. Brown summarized his approach as "a good mind stretcher for the students' conceptualization of fractions" (p. 8).

Galloway (1975) studied the use of manipulatives and focused on the use of the number line to develop the fraction concept and the difficulties primary grade students encountered with the fraction. She indicated that even though all students were able to use concrete objects to identify fractions the lower ability primary grade student had much difficulty in naming and writing numerals for fractional parts. She found that the use of the ruler and number line activities proved very difficult even for 9 and 10-year-olds. She suggested that the use of the number line activities for developing the fraction concept be postponed until the student has a good understanding of the fraction concept.

Joy (1976) and Stenger (1971) studied different approaches for teaching elementary school children how to add and subtract fractions. The results of both studies indicated that the least common denominator approach proved very difficult for students to use but when an equivalent fraction approach was used students did better on immediate and delayed posttests. They implied that it may not be the concept of addition and subtraction of fractions that is totally responsible for students' difficulties but rather the approach or strategy used to teach addition and subtraction of fractions. In Stenger's study the results from the immediate posttest showed that low IQ students who used the equivalent fraction approach scored higher than the high IQ students who used the least common denominator approach.

Summary

The research studies reviewed contain some major implications for this study. Firstly, that the use of the "think aloud technique" is very effective in diagnosing student problems. Secondly, fractions are important and the meaning of a fraction is a key component in the development of fractions and associated operations. Thirdly, even though there are many problems with operations on fractions the interpretation of a fraction plays a major role. Fourthly, physical interpretations of fractions seem to be significant in the teaching of fractions and important in terms of interpretation of fractions.

CHAPTER III

DESIGN OF THE STUDY

In this chapter a description of the design used to carry out the study is discussed. This information is presented as follows: population and sample, design of the instruments, procedure, analysis of data, and finally the pilot study.

Population and Sample

The sample for this study consisted of 20 grade six students from two rural elementary schools. Both schools were within a 40 kilometre radius of a medium sized urban centre of approximately 100 000 people. These schools were comparable in facilities and programs, with the average class size being approximately 25 students. Subjects for the study were randomly selected from class lists provided by the classroom teachers. All the students were unfamiliar to the investigator and had no previous contact with him.

This particular sample of students was selected for the following reasons: 1) it was easy to contact the schools since the investigator taught in a third elementary school in the same area; 2) both schools were feeder schools for the same high school and it was reasonable to assume the programs were compatible; and 3) the principals and classroom teachers had offered their assistance and co-operation.

The Instrument

The instrument consisted of a series of 52 questions designed to test students' comprehension of the fraction concept and addition and subtraction of fractions. The exercises were based on those used in the First Mathematics Assessment conducted by the National Assessment of Educational Progress and those included in the textbook, Investigating School Mathematics, Level 6 (1974).

The questions involved two interpretations of a fraction with some questions utilizing the part of a whole model while the remaining questions utilized the part of a set model. Bruner's levels of concept development were also considered in the structuring of the questions. The method of presentation included the use of physical, pictorial, and symbolic representations of fractions. In the physical form students were provided with blocks which they could manipulate to help arrive at the solution. In the pictorial form students were provided with pictures which they were to color to illustrate their understanding of fractions. In the symbolic form students were required to give the answers without the help of manipulatives or visual aids. This resulted in the framework presented in Figure 1.

	Concept	Addition	Subtraction
Physical	$A1_s/A1_w$	$B1_s/B1_w$	$C1_s/C1_w$
Pictorial	$A2_s/A2_w$	$B2_s/B2_w$	$C2_s/C2_w$
Symbolic	A_3	B_3	C_3

S = part of a set
W = part of a whole

FIGURE 1. Two interpretations of fractions.

For example, the code $A1_w$ refers to questions that relate to the fraction concept and are presented physically as part of a whole. The code $C2_s$ refers to questions that relate to the subtraction of fractions and are presented pictorially as part of a set.

The questions used in this study were of two different types. Type 1 questions were of the form where the number of blocks corresponded to the denominator of the fraction. For example, the student was given a set of six blocks and asked to find 1/6 of the set. Type 2 questions were of the form where the number of blocks was a multiple of the denominator in the fraction. For example, the student was given a set of 10 blocks and asked to find

2/5 of the set. Figure 2 indicates the number of Type 1 and Type 2 questions for each of the problem types used in the study.

Problem Types	Type 1 Questions	Type 2 Questions
$A1_s$	4	1
$A1_w$	4	1
$A2_s$	4	1
$A2_w$	4	1
$B1_s$	1	1
$B1_w$	1	1
$B2_s$	1	1
$B2_w$	1	1
$C1_s$	2	2
$C1_w$	2	2
$C2_s$	2	2
$C2_w$	2	2

FIGURE 2. Number of Type 1 and Type 2 questions for each problem type used in the study.

All fractions were restricted to a value between zero and one. It was felt that the use of mixed or improper fractions would inhibit the interpretation of the explanations given by the students.

Sample questions used in the study are found in Figure 3 while a complete list of the questions can be found in Appendix A.

$A1_w$	What fraction of the square is green?
$B2_s$	In this picture there is a set of 4 circles. If you add 1/4 of the circles to 2/4 of the circles, what fraction of the set of circles do you have?
$C1_w$	This circle has 10 parts. If 2/5 of the parts of the circle is subtracted from 4/5 of the parts of the circle, what fraction is left?

FIGURE 3. Sample questions used in main study.

<u>Procedure</u>

Each student was taken to a room where only the student and the investigator were present. Upon entering the room the purpose of the session and the presence of the tape recorder were explained to the student. If the student had any questions regarding the interview or the tape recorder every attempt was made to answer the question and make the student feel at ease. When these formalities were completed the actual interview began.

The questions were presented in the following format. Questions dealing with the fraction concept were presented first, followed by questions dealing with the addition of fractions, and finally questions dealing with the subtraction of fractions. Each question was read for the student and also written on a file card for the student to read. After allowing sufficient time and if the student indicated difficulty in obtaining the answer, one standard follow-up question was given, namely "How many parts are in the set or whole?" If the student was still unable to respond, the next question was begun. For all questions the student was required to give the solution and explain the procedure used to arrive at the solution.

Analysis

Initially, the responses to all 52 questions were analyzed individually for each student. The answer to each question was determined as being correct or incorrect and then followed by an analysis of the explanation to determine if the student provided a correct interpretation. If the interpretation was incorrect a note was made regarding the explanation the student used to interpret the fraction or operation.

The three questions stated in Chapter I were analyzed as follows:

Question 1) What interpretations of the fraction concept and the addition and subtraction of fractions are made by students?

The analysis for the question was divided into three sections to include the interpretation of the 1) fraction concept, 2) addition of fractions, and 3) subtraction of fractions. For each of these sections the interpretation given by each student was recorded. If the interpretation was incorrect, the explanation given by the student was analyzed to hypothesize the nature and underlying cause of the incorrect interpretation and then categorized. Since the literature contained very limited information on the nature of the misconceptions no a priori categorization was formulated.

The following criteria was used to determine when a student's response indicated an incorrect interpretation. The student had to give the same incorrect interpretation in at least 50% of the items appropriate to a given situation. For example, using Figure 2, there are four Type 1 questions in the problem type $A1_s$. If a student gave the same incorrect response in at least two of these items then the student's explanation was considered to be an incorrect interpretation. Where there was only one item for each problem type (e.g. $B2_s$) the student's explanation was considered to be an incorrect interpretation if the student responded incorrectly.

Question 2) What computational errors are made by students in the addition and subtraction of fractions?

The analysis for the question was divided into two sections to include errors in the 1) addition and 2) subtraction of fractions. The answers given by the student were determined as correct or incorrect. For the answers that were incorrect an analysis was carried out so that the types of errors could be categorized.

Question 3) Are the incorrect interpretations and computational errors systematic?

Again the analysis was divided into three sections to include 1) fraction concept, 2) addition of fractions, and 3) subtraction of fractions. Once the incorrect interpretations and errors were found, the responses were analyzed further to examine the explanations that led to the incorrect interpretations and errors. Since this was an exploratory question and no a priori criteria for systematic interpretations were established, it was proposed to examine the data for possible patterns of incorrect interpretations and errors.

Pilot Study

The pilot study was carried out several months prior to the actual study using seven grade six students. It was conducted in the same area as the main study but in a different school. The purposes of the study were 1) to determine if the instrument was a suitable device to help gather the necessary information, and 2) to familiarize the investigator with the technique of interviewing. The questions and procedure used in the pilot were very similar to those used in the main study.

The students' responses were analyzed to determine if any changes were necessary before the main study was undertaken. Appropriate changes were made and this led to the following modifications. The order of several questions was changed because it was felt that the way in which the questions were sequenced might have effected the students' responses. Some questions were reworded because they provided too much information at the expense of having students do less thinking on their own. Several questions were deleted because of their similarity to other questions. Other questions were added in order to gather additional information that was felt necessary. For example, initially all subtraction items were worded in the following way: This is a set of 10 rectangles. How much more is 7/10 of the set of rectangles than 3/10 of the set of rectangles? To be certain that a student's incorrect response was not due to problem solving, additional items were included and worded in the following way: This is a set of 10 squares. If 1/5 of the squares is subtracted from 3/5 of the squares, what fraction is left? Finally, a specific method for asking the questions was required for the main study. This was the standard method that was outlined in the procedure. In this way the questioning was consistent for each student interview.

As a final check on the questions and the interviewing technique the modified procedures were tested. The results of this final piloting were satisfactory and the main study was then undertaken.

CHAPTER IV

ANALYSIS OF RESULTS

In this chapter the data collected from the study are presented. The data are based on the responses of 20 students to the 52 questions used in the study. The data are presented in three different sections corresponding to the three research questions. Each section is subdivided into three parts corresponding to the subcomponents relevant to the fraction concept, the addition of fractions, and the subtraction of fractions.

Research Question 1: What interpretations of the fraction concept and the addition and subtraction of fractions are made by students?

Fraction Concept

Part of a set--physical presentation. An example of a Type 1 question was where the student was given a set of four squares, one of the squares colored green, and asked to state what part of the set was colored green. There were 13 students who were able to state the correct solution and use the squares to give a correct explanation of their procedure. The most common incorrect interpretation was to compare the three white blocks to the green block. A complete list of the incorrect interpretations is given in Table 1.

TABLE 1

Students' incorrect interpretations of the fraction concept when fractions are presented physically as part of a set using Type 1 questions

Number of Students	Explanations
3	Students compared the 1 green square to the other 3 squares and responded that 1/3 was the solution.
1	Student placed the green square over the other 3 squares to illustrate the answer.
1	Student used the 1 green square to represent the numerator. To get the denominator he subtracted the green square from the total number of squares.
2	Students stated that they did not understand the question and therefore did not give an answer.

An example of the Type 2 question was where the student was given a set of 6 blocks and asked to find 2/3 of the set. There were two students who were able to state the correct solution and use the blocks to give a correct explanation of their procedure. The most common incorrect interpretation was to use the denominator to determine group size and subtract one from the group so that only 2 blocks remained. This group of two blocks was called 2/3. A complete list of the incorrect interpretations is given in Table 2.

TABLE 2

Students' incorrect interpretations of the fraction concept when fractions are presented physically as part of a set using Type 2 questions

Number of students	Explanations
4	Students made a group of 3, removed 1, and suggested the remaining 2 was 2/3.
3	Students made a group of 2 and suggested this was 2/3.
3	Students made 2 groups of 3, selected 2 parts from 1 group, and suggested this was 2/3.
3	Students made 2 groups of 3 and said this was 2/3.
1	Student selected 2 blocks and replied 2/3 is half the set.
1	Student placed 2 blocks over 3 blocks to illustrate 2/3.
1	Student made 2 groups of 3 and selected 2 parts from each group to get 2/3.
1	Student made 3 groups of 2, selected 1 group, and called it 2/3.
1	Student stated he did not understand the question and therefore could not provide any explanation.

Part of a whole--physical presentation. An example of a Type 1 question was where the student was given a rectangle with eight parts and asked to find 3/8 of the rectangle. There were 17 students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. The most common incorrect interpretation was to compare three parts of the rectangle to the remaining five parts of the rectangle. A complete list of the incorrect interpretations is given in Table 3.

TABLE 3

Students' incorrect interpretations of the fraction concept when fractions are presented physically as part of a whole using Type 1 questions

Number of Students	Explanations
2	Students compared 3 parts of the rectangle to the remaining 5 parts of the rectangle.
1	Student placed 3 parts of the rectangle over the remaining 5 parts of the rectangle to illustrate the fraction.

An example of the Type 2 question was where the student was given a rectangle with 10 parts and asked to find 2/5 of the rectangle. There were two students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. The most common incorrect interpretation was to use the denominator to determine group size and use the numerator to determine what part of only one group to select. A complete list of the incorrect interpretations is given in Table 4.

TABLE 4

Students' incorrect interpretations of the fraction concept when fractions are presented physically as part of a whole using Type 2 questions

Number of Students	Explanations
6	Students made 2 groups of 5 and selected 2 parts from 1 group to get 2/5.
4	Students made 2 groups of 5 and called this 2/5.
2	Students placed 2 parts of the rectangle over 5 parts of the rectangle to get 2/5.
2	Students made 2 groups of 5 and selected 2 parts from each group.
1	Student selected 5 parts of the rectangle and called this 2/5.
1	Student selected 2 parts of the rectangle, then selected 5 other parts of the rectangle, and called this 2/5.
1	Student made 2 groups of 5, selected 1 group, and called it 2/5.
1	Student stated he did not understand the question and therefore did not give an answer.

<u>Part of a set--pictorial presentation</u>. An example of a Type 1 question was where the student was given a picture of a set of six circles and asked to color 2/6 of the set. There were 18 students who were able to state the correct solution and use the circles to give a correct explanation of their procedure. The only incorrect interpretation is given in Table 5.

TABLE 5

Student's incorrect interpretation of the fraction concept when fractions are presented pictorially as part of a set using Type 1 questions

Number of Students	Explanation
2	Students compared 2 circles to the remaining 4 circles to get the answer.

An example of the Type 2 question was where the student was given a picture of a set of eight circles and asked to color 1/4 of the set. There were four students who were able to state the correct solution and use the circles to give a correct explanation of their procedure. The most common incorrect interpretation was to color the number of parts in the set suggested by the number in the denominator. A complete list of the incorrect interpretations is given in Table 6.

TABLE 6

Students' incorrect interpretations of the fraction concept when fractions are presented pictorially as part of a set using Type 2 questions

Number of Students	Explanations
6	Students colored 4 out of 8 circles to get 1/4.
5	Students made 2 sets of 4 and colored 1 part of 1 set of circles.
2	Students colored 1 circle in the set and called it 1/4.
1	Student colored 1 circle and compared it to 4 other circles.
2	Students made 2 sets of 4 and colored 1 set to get 1/4.

<u>Part of a whole--pictorial presentation</u>. An example of a Type 1 question was where the student was given a picture of a rectangle with five parts and asked to color 1/5 of the rectangle. There were 17 students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. The only incorrect interpretation is given in Table 7.

TABLE 7

Students' incorrect interpretation of the fraction concept when fractions are presented pictorially as part of a whole using Type 1 questions

Number of Students	Explanation
3	Students compared 1 part of the rectangle with the other 4 parts of the rectangle to get the answer.

An example of the Type 2 question was where the student was given a picture of a square with 12 parts and asked to color 1/4 of the square. There were four students who were able to state the correct solution and use the parts of the square to give a correct explanation of their procedure. The most common incorrect interpretation was to color the number of parts in the square suggested by the number in the denominator. A complete list of the incorrect interpretations is given in Table 8.

TABLE 8

Students' incorrect interpretations of the fraction concept when fractions are presented pictorially as part of a whole using Type 2 questions

Number of Students	Explanations
6	Students colored 4 parts of the square and said this was 1/4.
2	Students made 3 groups of 4 and colored 1 part of 1 group to get 1/4.
2	Students colored 1 part of the square and said this was 1/4.
2	Students made 3 groups of 4, colored 1 group, and said this was 1/4.
1	Student colored 1 part of the square and compared it to 4 other parts of the square.
1	Student made a group of 4, colored 1 part of the group and called it 1/4.
1	Student colored 11 parts and suggested the other uncolored part was 1/11.
1	Student made 3 groups of 4 and colored 1 part of each group. Even though the response was correct, this student was unable to explain a correct understanding of the fraction concept.

Symbolic presentation. An example of these questions was where the student was asked to state the meaning of the fraction 1/5 or to write the fraction one-third. There were 17 students who were able to give a correct explanation for the meaning of the fraction and write the fraction stated by the investigator. However, 11 of these students gave a very limited explanation of the meaning of the fraction. The explanations were limited to removing the parts of a set (e.g., the fraction 1/5 meant removing 1 part from a set of 5) or coloring the parts of a set (e.g., the fraction 1/5 meant having a set of 5 objects and coloring 1 object in the set.). These students were not able to generalize beyond this narrow concept of the fraction. The three incorrect interpretations are given in Table 9.

TABLE 9

Students' incorrect interpretations of the fraction concept when fractions are presented symbolically

Number of Students	Explanations
1	Student stated that a fraction meant the total number indicated by the sum of the numerator and the denominator.
1	Student stated that a fraction meant subtracting the top line from the bottom line and drawing a line in between them.
1	Student did not provide any explanation of the meaning of a fraction.

Addition of Fractions

Part of a set--physical presentation. An example of a Type 1 question was where the student was given a set of six blocks and asked to compute 3/6 + 1/6. There were 12 students who were able to state the correct solution and use the blocks to give a correct explanation of their procedure. The most common incorrect interpretation was to add the fractions mentally and not be able to use the blocks to explain a correct procedure for finding the solution. A complete list of the incorrect interpretations is given in Table 10.

TABLE 10

Students' incorrect interpretations of addition of fractions when fractions are presented physically as part of a set using Type 1 questions

Number of Students	Explanations
3	Students figured out the answer mentally, then selected 4 blocks but could not explain it.
1	Student selected 3 blocks, then re-selected 1 out of these 3 blocks, added and got 4/6.
1	Student added the denominators and could not use the blocks.
1	Student selected 3 blocks, then selected 2 more blocks, added and got 5/6.

(cont'd.)

Number of Students	Explanations
1	Student suggested that there should be 9 blocks so she could place 3 blocks over 6 blocks to illustrate the answer.
1	Student added together every number on the fild card.

An example of the Type 2 question was where the student was given a set of 10 blocks and asked to compute 1/5 + 2/5. There was one student who was able to state the correct solution and use the blocks to give a correct explanation of his procedure. Two common incorrect interpretations occurred. The student would either ignore the denominator and build sets to correspond to the two numerators or use the denominators to determine group size and the numerators to determine what part of each group to select. A complete list of the incorrect interpretations is given in Table 11.

TABLE 11

Students' incorrect interpretations of addition of fractions when fractions are presented physically as part of a set using Type 2 questions

Number of Students	Explanations
4	Students selected 1 block, then selected 2 blocks, added and got 3/5.
4	Students made 2 groups of 5, then selected 1 from 1 group and 2 from the other group, added and got 3/5.
1	Student made 2 groups of 5, selected 2 parts from each group, then selected 1 additional part, added and got 3/5.
1	Student placed 1 block over 4 blocks to get 1/5, and placed 2 blocks over 3 blocks to get 2/5.
1	Student could not use the blocks and found the incorrect answer mentally by first adding numerators and then adding denominators.
1	Student placed 2 blocks over 5 blocks and could not proceed any further.
1	Student made 2 groups of 5 and suggested there should be another similar group to get 3/5.
1	Student made 3 uneven groups, removed 2 groups and called the remaining group 3/5.
1	Student selected 3 blocks and could not proceed any further.
1	Student selected 5 blocks, subtracted 2 and said this is 2/5. Then he added 1 block from the other group of 5 to get 3/5.

(cont'd.)

Number of Students	Explanations
1	Student made 2 groups of 5, took 3 from 1 group and suggested this was 3/5.
2	Students stated that they did not understand the question and therefore did not give an answer.

<u>Part of a whole--physical presentation</u>. An example of a Type 1 question was where the student was given a rectangle with 9 parts and asked to compute 4/9 + 1/9. There were 15 students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. The most common incorrect interpretation was to select four parts of the rectangle, then reselect one of these parts, and add. A complete list of the incorrect interpretations is given in Table 12.

TABLE 12

Students' incorrect interpretations of addition of fractions when fractions are presented physically as part of a whole using Type 1 questions

Number of Students	Explanations
3	Students selected 4 parts of the rectangle, then selected 1 of the parts already chosen, added and got 5/9.
1	Student added every number in the question to get 23 as the answer.
1	Student placed 1 part over 8 parts to get 1/9 and then placed 4 parts over 5 parts to get 4/9.

An example of the Type 2 question was where the student was given a square with eight parts and asked to compute 2/4 + 1/4. There was one student who was able to state the correct solution and use the parts of the square to give a correct explanation of his procedure. The most common incorrect interpretation was to ignore the denominator, build sets to correspond to the two numerators, and add. A complete list of the incorrect interpretations is given in Table 13.

TABLE 13

Students' incorrect interpretations of addition of fractions when fractions are presented physically as part of a whole using Type 2 questions

Number of Students	Explanations
9	Students selected 1 part, then selected 2 parts, added and got 3/4.
3	Students made 2 groups of 4 but suggested there should be another similar group to get 3/4.
1	Student made a group of 3 and a group of 4, added and got 3/4.
1	Student made 2 groups of 4, selected 3 parts from 1 group to get 3/4.
1	Student placed 2 parts over 2 parts and called it 2/4. Then he placed 1 part over 3 parts to get 1/4, added and got 3/4.
1	Student made 2 groups of 4 and selected 2 from each group. He then selected 1 more from each group to get a total of 6.
1	Student made 4 groups of 2, selected 2 groups and then selected 1 part from another group, added and got 3/4.
1	Student selected 4 parts of the square and could not proceed any further.
1	Student stated that he did not understand the question and therefore did not give an answer.

<u>Part of a set--pictorial presentation</u>. An example of a Type 1 question was where the student was given a picture of a set of four circles and asked to compute 2/4 + 1/4. There were 18 students who were able to state the correct solution and use the parts of the set to give a correct explanation of their procedure. The two incorrect interpretations are given in Table 14.

TABLE 14

Students' incorrect interpretations of addition of fractions when fractions are presented pictorially as part of a set using Type 1 questions

Number of Students	Explanations
1	Student added the denominators to find the solution and was not able to color the circles or explain a procedure for finding the solution.
1	Student colored 2 circles, then colored 1 of these 2 circles, added, and got 3/4.

An example of the Type 2 question was where the student was given a picture of a set of 10 squares and asked to compute 2/5 + 1/5. There were two students who were able to state the correct solution and use the parts of the set to give a correct explanation of their procedure.

Two common incorrect interpretations occurred. The students would either color the number of squares indicated by the sum of the two numerators or use the denominators to determine group size and color the number of squares suggested by the two numerators. A complete list of the incorrect interpretations is given in Table 15.

TABLE 15

Students' incorrect interpretations of addition of fractions when fractions are presented pictorially as part of a set using Type 2 questions

Number of Students	Explanations
8	Students colored 1 square, then colored 2 more squares, added and got 3/5.
4	Students made 2 groups of 5, colored 2 squares in one group, colored 1 square in the other group, added and got 3/5.
1	Student made 2 groups of 5, colored 1 square in each group, then colored 2 more squares in each group, added and got 3/10.
1	Student made a group of 5, called this group 1/5, then added it to the other group of 5 to get 2/5.
1	Student made 2 groups of 5, colored 2 squares in each group, colored 1 more square in each group, added and got 6/10.
1	Student made 2 groups of 5, colored 1 square and then 2 more squares in the same group, added and got 3/5.

(cont'd.)

Number of Students	Explanations
1	Student made 2 groups of 5 but could not give any further explanation.
1	Student colored 3 squares, and then added the denominators.

Part of a whole--pictorial presentation. An example of a Type 1 question was where the student was given a picture of a square with six parts and asked to compute 3/6 + 1/6. There were 19 students who were able to state the correct solution and use the parts of the square to give a correct explanation of their procedure. The only incorrect interpretation is given in Table 16.

TABLE 16

Students' incorrect interpretation of addition of fractions when fractions are presented pictorially as part of a whole using Type 1 questions

Number of Students	Explanation
1	Student colored 3 parts of the square, added 1/6 + 3/6, and got 4/12 as the answer.

An example of the Type 2 question was where the student was given a picture of a rectangle with 10 parts and asked to compute 2/5 + 1/5. There were two students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. Two common incorrect interpretations occurred. The students would either color the number of parts of the rectangle suggested by the sum of the two numerators or use the denominator to determine group size and color the number of parts suggested by the two numerators. A complete list of the incorrect interpretations is given in Table 17.

TABLE 17

Students' incorrect interpretations of addition of fractions when fractions are presented pictorially as part of a whole using Type 2 questions

Number of Students	Explanations
10	Students colored 2 parts of the rectangle, then colored 1 other part of the rectangle, added and got 3/5.
3	Students made 2 groups of 5, colored 1 part of the first group, then colored 2 parts of the second group, added and got 3/5.
1	Student made 2 groups of 5, colored 1 part of the first group, then colored 2 parts of the second group, added and got 3/10.

(cont'd.)

Number of Students	Explanations
1	Student made 2 groups of 5, colored 1 part in each group, then colored 2 other parts in each group, added and got 3/5.
1	Student made 2 groups of 5, colored 1 part and then colored 2 parts in the same group, added and got 3/5.
1	Student colored 3 parts of the rectangle and then added the denominators.
1	Student stated he did not understand the question and therefore could not provide any explanation.

Symbolic presentation. An example of these questions was where the student was given the fractions 4/7 and 2/7 and asked to add the fractions and explain a procedure for addition of fractions. There were 18 students who were able to state the correct solution but only four of these students were able to explain correctly their procedure for finding the solution. A complete list of the incorrect interpretations is given in Table 18.

TABLE 18

Students' incorrect interpretations of addition of fractions when fractions are presented symbolically

Number of students	Explanations
14	Students merely used a rule with little or no understanding of why the numerators were added and the denominators were not added. The reasons given by the students for not adding denominators included 1) because the denominators are the same; and 2) because the denominator is greater than the sum of the two numerators.
2	Students added the denominators when finding the sum of the two fractions.

Subtraction of Fractions

Part of a set--physical presentation. An example of a Type 1 question was where the student was given a set of 10 rectangles and asked to compute 7/10 - 3/10. There were 10 students who were able to state the correct solution and use the rectangles to give a correct explanation of their procedure. The most common incorrect interpretation was to select seven rectangles, then select the remaining three rectangles, and subtract. A complete list of the incorrect interpretations is given in Table 19.

TABLE 19

Students' incorrect interpretations of subtraction of fractions when fractions are presented physically as part of a set using Type 1 questions

Number of Students	Explanations
6	Students selected 7 parts, then selected the remaining 3 parts, subtracted, and got 3/10.
1	Student made 5 groups of 2, removed 3 groups and suggested the remaining 2 groups was the answer.
1	Student removed 7 parts and suggested that the remaining 3 parts was the answer.
1	Student found the answer mentally, then used it to make a group of 4, then made a group of 3 and suggested that 4/10 was the answer.
1	Student stated that he did not understand the question and therefore did not give an answer.

An example of the Type 2 question was where the student was given a set of 10 blocks and asked to compute 3/5 - 1/5. There were two students who were able to state the correct solution and use the blocks to give a correct explanation of their procedure. The most common incorrect interpretation was to ignore the denominators and subtract the two numerators. A complete list of the incorrect interpretations is given in Table 20.

TABLE 20

Students' incorrect interpretations of subtraction of fractions when fractions are presented physically as part of a set using Type 2 questions

Number of Students	Explanations
11	Students selected 3 squares, then selected 1 other square, subtracted and got 2/5.
1	Student selected 3 squares, selected 1 of these 3 squares, subtracted and got 2/3.
1	Student could not use the squares, mentally subtracted the numerators, and then subtracted the denominators.
1	Student made 2 groups of 5, selected 1 group, and called it 1/5. The student then selected 3 squares from the other group, subtracted 1, and got 2/5 as the answer.
1	Student made 2 groups of 5, selected 3 squares from 1 group, then selected the remaining 2 squares in that group, subtracted these and got 2/5.
1	Student made 2 groups of 5, selected 3 squares from each group, subtracted 1 from each to give a total of 4 or 4/10.
1	Student made 2 groups of 5, selected 3 squares from 1 group, subtracted 1 square to get 2 squares left. These 2 squares were then added to 2 other squares from the second group to get 4/5 as the answer.
1	Student stated that he did not understand the question and therefore did not give an answer.

<u>Part of a whole--physical presentation</u>. An example of a Type 1 question was where the student was given a square with 12 parts and asked to compute 9/12 - 5/12. There were 14 students who were able to state the correct solution and use the parts of the square to give a correct explanation of their procedure. A complete list of the incorrect interpretations is given in Table 21.

TABLE 21

Students' incorrect interpretations of subtraction of fractions when fractions are presented physically as part of a whole using Type 1 questions

Number of Students	Explanations
1	Student removed 10 parts, said there are 2 parts left and the answer is 2/12.
1	Student found the answer mentally but was not able to use the parts of the square to find the answer.
1	Student selected 9 parts of the square, removed 5 parts and had 4 left. This 4 was placed with the remaining 3 parts and got 7/12 for an answer.
1	Student selected 9 parts of the square but could not proceed any further.
2	Students stated that they did not understand the question and therefore did not give any answer.

An example of the Type 2 question was where the student was given a circle with 10 parts and asked to compute 4/5 – 2/5. There were two students who were able to state the correct solution and use the parts of the circle to give a correct explanation of their procedure. The most common incorrect interpretation was to ignore the denominators and subtract the numerators. A complete list of the incorrect interpretations is given in Table 22.

TABLE 22

Students' incorrect interpretations of subtraction of fractions when fractions are presented physically as part of a whole using type 2 questions

Number of Students	Explanations
6	Students made a group of 4, removed 2 from the group, and got 2/5.
1	Student made a group of 4 and a group of 2, subtracted and got 2/5.
4	Students made 2 groups of 5, selected 4 parts from 1 group and 2 parts from the other group, subtracted these parts and got 2/5.
1	Student added 2 parts and 4 parts to get 6. He stated there is 4 parts left so the answer is 4/10.
1	Student made 2 groups of 5, and stated this was 2/5.

(cont'd.)

Number of Students	Explanations
2	Students made 2 groups of 5, selected 4 parts from one group, subtracted 2 parts and got 2/5.
1	Student made 2 groups of 5, removed 1 part from each group and stated there were 4 parts left in each group. He then subtracted 2 from each group of 4 to have a total of 4 parts remaining which he called 4/10.
2	Students stated that they did not understand the question and therefore did not give an answer.

<u>Part of a set--pictorial presentation</u>. An example of a Type 1 question was where the student was given a picture of a set of six circles and asked to compute 5/6 − 1/6. There were nine students who were able to state the correct solution and use the parts of the set to give a correct explanation of their procedure. The most common incorrect interpretation was to color five circles, then color the other circle, and subtract. A complete list of the incorrect interpretations is given in Table 23.

TABLE 23

Students' incorrect interpretations of subtraction of fractions when fractions are presented pictorially as part of a set using Type 1 questions

Number of Students	Explanations
1	Student colored 5 circles, then colored the other circle, subtracted and got 3/6.
1	Student colored 5 circles and could not proceed any further.
1	Student colored 5 circles, said there is 1 circle left and this 1 circle means 1/6.
1	Student arrived at the answer mentally and could not use the circles.
3	Students colored 5 circles, then colored the other circle, subtracted and got 4/6.
2	Students suggested there was not enough circles to find the answer but then proceeded to find the answer mentally.
2	Students stated that they did not understand the question and therefore did not give an answer.

An example of the Type 2 question was where the student was given a picture of a set of eight triangles and asked to compute 3/4 - 1/4. There were two students who were able to state the correct solution and use the parts of the set to give a correct explanation of their procedure. The most common incorrect interpretation was to ignore the denominators, color the number of triangles suggested by the two numerators, and subtract. A complete list of the incorrect interpretations is given in Table 24.

TABLE 24

Students' incorrect interpretations of subtraction of fractions when fractions are presented pictorially as part of a set using Type 2 questions

Number of Students	Explanations
5	Students colored 3 triangles, then colored 1 more triangle, subtracted the triangles and got 2/4.
4	Students colored 3 triangles, removed 1 triangle and got 2/4.
2	Students made 2 groups of 4, colored 3 triangles in one group, colored 1 triangle in the other group, subtracted and got 2/4.
1	Student made 2 groups of 4 and called these groups 2/4.
1	Student colored 4 triangles, then colored 1 more triangle, subtracted the triangles and got 3/4.
1	Student made 2 groups of 4 and colored 1 triangle in each group. The student suggested there were 3 left in each group to give an answer of 6.
1	Student made 2 groups of 4, colored 1 in one group and colored 3 in the other group to give a total of 4. The student subtracted this 4 from the total set of 8 triangles to get an answer of 4.

(cont'd.)

Number of Students	Explanations
1	Student made 2 groups of 4, colored 3 triangles in one group, subtracted 1 of these triangles and got 2/4.
1	Student made 2 groups of 4, colored 3 triangles in one group, removed the other triangle in the same group and got 2/4.
1	Student colored any 2 triangles, then colored 4 other triangles, subtracted and got 2/4.

<u>Part of a whole--pictorial presentation</u>. An example of a Type 1 question was where the student was given a picture of a square with seven parts and asked to compute 5/7 - 2/7. There were 13 students who were able to state the correct solution and use the parts of the square to give a correct explanation of their procedure. The most common incorrect interpretation was to color five parts of the square and then subtract from the five parts the remaining two parts of the square. A complete list of the incorrect interpretations is given in Table 25.

TABLE 25

Students' incorrect interpretations of subtraction of fractions when fractions are presented pictorially as part of a whole using Type 1 questions

Number of Students	Explanations
4	Students colored 5 parts, then colored the other 2 parts, subtracted and got 3/7.
1	Student colored 5 parts and could not proceed any further.
1	Student colored 5 parts and suggested the remaining 2 parts was the answer.
1	Student stated that he did not understand the question and therefore did not give an answer.

An example of the Type 2 question was where the student was given a picture of a rectangle with 12 parts and asked to compute 5/6 - 1/6. There were two students who were able to state the correct solution and use the parts of the rectangle to give a correct explanation of their procedure. Two common incorrect interpretations occurred. The students would either color the number of parts of the rectangle suggested by the two numerators and subtract or color the number of parts of the rectangle suggested by the larger numerator and subtract the number of parts suggested by the smaller numerator. A complete list of the incorrect interpretations is given in Table 26.

TABLE 26

Students' incorrect interpretations of subtraction of fractions when fractions are presented pictorially as part of a whole using Type 2 questions

Number of Students	Explanations
5	Students colored 5 parts of the rectangle, then colored 1 other part of the rectangle, subtracted and got 4/6.
5	Students colored 5 parts of the rectangle, subtracted 1 from the 5 parts and got 4/6.
1	Student made 2 groups of 6 and could not proceed any further.
2	Students made 2 groups of 6, colored 5 parts in one group, colored 1 part in the other group, subtracted the parts and got 4/6.
1	Student made 2 groups of 6, removed 1 part from each group to have 5/6 left in each group. The student subtracted 1 more part from each group to get 4/6 left which made a total of 8.
1	Student found the answer mentally and colored 4 parts of the rectangle.
1	Student made a group of 6, colored 1 part and suggested that there was 5/6 left.
1	Student made 2 groups of 6, colored 5 parts in one group, removed 1 of these parts to get 4/6.
1	Student stated that he did not understand the question and therefore did not give an answer.

Symbolic presentation. An example of these questions was where the student was given the fractions 5/9 and 1/9 and asked to subtract the fractions and explain a procedure for subtraction of fractions. There were 19 students who were able to state the correct solution but only five of these students were able to explain correctly their procedure for finding the solution. A complete list of the incorrect interpretations is given in Table 27.

TABLE 27

Students' incorrect interpretations of subtraction of fractions when fractions are presented symbolically

Number of Students	Explanations
14	Students used a rule to find the correct answer. When asked why they did not subtract the denominators three different reasons were given. They included: 1) if you do, you will get zero; 2) the difference between the numerators is less that the denominator; and 3) because the denominators are the same.
1	Student subtracted the denominators when finding the answer.

Research Question 2: What computational errors do students make in the addition and subtraction of fractions?

The majority of students committed very few computational errors in the addition and subtraction of fractions regardless if the questions were Type 1 or Type 2. The most consistent computational error was either adding or subtracting denominators. There were two students who added the denominators and one of these two students who also subtracted the denominators. Figure 4 indicates the number of questions correctly computed for each item type used in the study.

Even though the data in Figure 4 indicated that students could correctly add and subtract fractions, the majority of students were not able to correctly model and interpret the fractions physically, pictorially, and symbolically. For most item types students were not able to use the materials to give a correct explanation of their procedure.

	Item Types	Number of Students	Number of Questions	Total Number of Questions	Total Number of Questions Correctly Computed
ADDITION					
Type 1	Physical	20	2	40	37
Type 1	Pictorial	20	2	40	38
Type 2	Physical	20	2	40	37
Type 2	Pictorial	20	2	40	33
	Symbolic	20	2	40	36
SUBTRACTION					
Type 1	Physical	20	4	80	68
Type 1	Pictorial	20	4	80	72
Type 2	Physical	20	4	80	69
Type 2	Pictorial	20	4	80	73
	Symbolic	20	2	40	38

FIGURE 4. Number of questions computed correctly for each item type used in the study.

Research Question 3: Are the incorrect interpretations and computational errors systematic?

A study of the data outlined in Figure 4 indicated that there were very few computational efrors. Therefore, this research question will focus only on the incorrect interpretations. For purposes of this study the incorrect interpretations are presented in two parts. Part one deals with the incorrect interpretations that were given for Type 1 questions. Part two deals with the incorrect interpretations that were given for Type 2 questions.

A study of the results indicated that students gave very few incorrect interpretations of the fraction concept and addition and subtraction of fractions when Type 1 questions were given. However, there were three students who gave consistent incorrect interpretations for the Type 1 questions that dealt with the fraction concept. For example, when given a square with one of its four parts colored green, these three students stated that 1/3 of the square was green. When these students were given the addition and subtraction Type 1 questions, their incorrect interpretations were not consistent. They gave different incorrect interpretations which varied for the addition and subtraction questions.

When these three students were presented with the four Type 2 questions that dealt with the fraction concept, their incorrect interpretations were consistent within each of the questions but not consistent across the questions. The students were consistent in giving different incorrect interpretations of the fraction concept and their incorrect interpretations were not consistent with those they had given for the Type 1 questions.

There were many incorrect interpretations given by the students for all the Type 2 questions. Although there were many incorrect interpretations, there was little evidence of a systematic application of an incorrect interpretation. However, a study of the students' explanations indicated the use of two common incorrect interpretations that were given consistently by students to explain their procedures. The most common incorrect interpretation was to reduce the Type 2 questions to Type 1 questions. Consequently, students explained their procedures in relation to the Type 1 questions.

There were other students who were consistent in using the total number of parts in the set to explain their procedures. For example, given a set of 10 blocks and asked to compute 2/5 + 1/5, these students were consistent in making 2 groups of 5. However, they were not consistent in explaining their procedures for using the 2 groups of 5. Some of these students selected 2 parts from one group and one part from the other group. Some students selected 2 parts and 1 part from the same group. Other students selected 2 parts and 1 part from each of the groups. Once again the students demonstrated that they were consistent in giving different incorrect interpretations.

In the next chapter the reasons students were consistent in giving these incorrect interpretations are discussed.

CHAPTER V

SUMMARY, DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

In this chapter a summary of the study, a discussion of the results, the conclusions drawn from the study, and the recommendations for future research are presented.

Summary

The purpose of the study was to determine the explanations used by grade six students in the interpretation of the fraction concept, and the addition and subtraction of fractions. To examine these thought processes three research questions were asked: What interpretations of the fraction concept and the addition and subtraction of fractions are made by students? What computational errors are made by students in the addition and subtraction of fractions? Are the incorrect interpretations and computational errors systematic? To investigate these questions a series of 52 items on the fraction concept, and the addition and subtraction of fractions was developed, tested and analyzed.

The study was conducted using 20 grade six students from two rural elementary schools in Newfoundland. The students were randomly selected and were individually interviewed and audiotaped. Students were presented with 52 questions in three different formats to include physical, pictorial, and symbolic presentations of fractions. Some of the fractions were presented using part of a set interpretation of a fraction while other questions were presented using part of a whole interpretation of a fraction. The questions used in the study were of two different types. Type 1 questions were of the form where the number in the set was the same as the number in the denominator. Type 2 questions were of the form where the number in the set was a multiple of the denominator.

Initially, the responses to all 52 questions were analyzed for each individual student to determine if the student's interpretation was correct or incorrect. If the interpretation was incorrect a note was made regarding the process the student used to interpret the fraction or operation.

The analysis of question one was divided into three sections to include the interpretation of the fraction concept, the addition of fractions, and the subtraction of fractions. For each of these sections the interpretation given by the student was recorded. If the interpretation was incorrect, the explanation used by the student was analyzed to hypothesize the nature and cause of the incorrect interpretation and categorized. The responses relating to research question two were then analyzed so that the types of errors in the addition and subtraction of fractions could be categorized.

A study of research question three was carried out based on the responses to the first two research questions. Once the systematic incorrect interpretations and errors were found, the responses were further analyzed to determine the explanations that led to the incorrect interpretations and errors. Where faulty explanations were found to exist it was proposed to determine if any systematic patterns existed in the responses of those students who gave incorrect interpretations and computational errors.

Discussion

A study of the results indicated that there were differences in the explanations used by students in the interpretation of the fraction concept and the addition and subtraction of fractions. These differences varied depending on whether the questions given were Type 1 or Type 2. Consequently, the discussion is presented in two sections to allow for this difference.

Students had little difficulty in interpreting the meaning of a fraction when Type 1 questions were given. When the questions were presented physically and pictorially the students were able to state the correct solution and use the materials to give a correct explanation of their procedures. Furthermore, the students had little difficulty in explaining their procedures for the addition and subtraction Type I questions. There were several students who found the answer mentally and then used it to explain a procedure for computing the fractions. This suggested that the students did have the skills to compute fractions but had little understanding of what they were doing. These students were unable to use the materials to find the solution. This was possibly due to these students' limited experiences with such materials.

When the addition and subtraction questions were presented symbolically the students had little difficulty in computing the fractions. In most instances students found the answer mentally without using pencil and paper. This indicated that students had had prior experiences with fractions presented in this way. However, a study of the results indicated that students were using a rule, remembered it well, and were unable to explain why fractions were computed in this way. When questioned about the meaning of the fraction concept (e.g., 2/7) most students gave a very limited explanation. They related 2/7 to parts of a pie or some geometric shape and were unable to extend the meaning to various other places in life.

The students did not give consistent incorrect interpretations of the meaning of the fraction when Type 2 questions were given. The most common explanation was to reduce the Type 2 questions to Type 1 questions. The students paid no attention to the fact that the number in the set was a multiple of the denominator. Rather, they used part of the set to explain their procedures as if the questions were Type 1. These students continued to use this procedure to explain the addition and subtraction Type 2 questions.

A possible explanation for this occurrence is that the students were unable to rename a whole set into a group of equivalent parts or they did not sense any kind of a relationship between a whole set and the same set divided into equivalent parts. The students did not perceive the whole set as keeping its quantity of parts. They perceived a renamed fraction as being a smaller group to work with. Rather than express their answer as part of a whole set, the students concentrated only on part of the set and explained their answer in terms of that part.

There were other students who did not attempt to reduce the Type 2 questions to Type 1 questions and were consistent in explaining that the fractions must be expressed in relation to the whole set. However, they were not consistent in explaining their procedures for using all the parts of the set. For example, given a set of 10 blocks and asked to compute 2/5 + 1/5, these students consistently made 2 groups of 5. The students were not consistent in their procedures after the groups were selected. Some of these students selected 2 parts from one group and 1 part from the other group. Some students selected 2 parts and 1 part from the same group. Other students selected 2 parts from both groups and 1 part from both groups.

A study of the results showed little evidence of a systematic application of an incorrect interpretation. However, it did show that there were students who were consistent in their incorrect interpretation of particular questions and that these students were consistent in making different interpretations of other questions. As previously discussed, the reason for this was possibly due to the fact that the students attempted to reduce the Type 2 questions to Type 1 questions.

The results of the study also showed that for both Type 1 and Type 2 questions that only those students who were able to explain the physical and pictorial exercises were then able to give a correct answer and explanation of the symbolic exercises. There were no students who were able to explain the symbolic exercises without being able to explain the physical and pictorial exercises. This indicated that understanding of physical and/or pictorial presentations of fractions is prerequisite to correct interpretation of symbolic presentations of fractions.

Conclusions

The conclusions in this study are limited to grade six level students. It may be possible to generalize to other grade levels and to other mathematical concepts but due to the nature of the study the investigator is cautious about doing so.

In summary, the analysis led to the following conclusions:

1. Students were able to interpret the meaning of a fraction when Type 1 questions were given. The students also had little difficulty in explaining their procedures for addition and subtraction of fractions if Type 1 questions were given.

2. Most students (18 out of 20) showed very little understanding of what they were doing when they explained their procedures for the Type 2 questions that related to the fraction concept and the addition and subtraction of fractions. In most cases they tried to reduce the Type 2 questions to Type 1 questions.

3. Students had much difficulty in using the physical and pictorial materials to explain their procedures for finding the solution. Materials were provided but the students were unable to use them to give a correct explanation. However, this was true only for the Type 2 questions and not the Type 1 questions.

4. The students demonstrated that they were able to compute correctly but they were not able to explain their procedures.

5. There was little evidence of a systematic application of an incorrect interpretation. There were students who gave consistent incorrect interpretations for particular questions used in the study but they did not consistently use this incorrect interpretation for all the questions. Rather, the students were consistent in giving different incorrect interpretations of other questions.

6. The pictorial activities proved easier than the physical for the students to explain. Students correctly explained 77% of the pictorial activities while they correctly explained 65% of the physical activities. This was possibly a reflection of the type of approach (i.e., illustrations on the blackboard) that the students had more experience with.

7. The results showed that only those students who were able to explain correctly the physical and pictorial exercises were then able to give a correct explanation of the symbolic exercises. There were no students who were able to explain correctly the symbolic exercises without first being able to explain correctly the physical and pictorial exercises. This indicated that understanding of physical and/or pictorial presentations of fractions for students in this study was prerequisite to correct understanding of symbolic presentations of fractions.

8. The use of the "think aloud technique" appeared to be an effective strategy to determine how students interpret, add, and subtract fractions. The use of the technique can be used to help gather valuable information which can not be so readily obtained from pencil and paper tests.

Recommendations

As a result of this study, the following recommendations are made for further research and applications for teachers:

1. A similar study should be conducted with a larger sample and at a period of time closer to when fractions are taught in school. This study was conducted in the fall of the year and the subjects had been introduced to fractions about five months earlier. A study should be carried out at the end of the school year in which fractions have been introduced. In this way it would be possible to control the effect of factors such as summer vacation and lapse of time upon students' performance.

2. Similar studies should be conducted with fraction concepts (i.e., equivalent fractions or multiplication of fractions) to determine if these findings are generalizable to these concepts.

3. The emphasis given to symbolic activities when working with fractions needs to be re-examined. A study of the results indicated that students need ample opportunities to work with physical and pictorial presentations of fractions before being introduced to symbolic presentations of fractions. If symbolic activities are overstressed at the expense of not using sufficient physical and pictorial activities, then students will be performing a skill with little understanding of what they are doing.

4. Educators and teachers should not assume that because students are able to give the correct solution to fractional exercises that the students have an understanding of the fraction concept and its computations. If teachers wish to find out what students know about fractions, it is not sufficient to use only a written test. Valuable and helpful information can be gained from situations where students are given the opportunities to explain their thoughts, ideas, and procedures for finding solutions. This can help reveal not only what students know about fractions but also what they do not know about fractions.

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APPENDIX A

QUESTIONS USED IN THE STUDY

$A1_s$ 1. What fraction of the whole set of blocks is green?

2. Show me 1/3 of the set of blocks.

3. What fraction of the whole set of blocks is blue?

4. Show me 4/9 of the set of blocks.

5. The set has 6 blocks. Show me 2/3 of the set.

$A1_w$ 1. What fraction of the square is green?

2. Shoe me 1/5 of the rectangle.

3. What fraction of the square is blue?

4. Show me 3/8 of the rectangle.

5. The rectangle has 10 parts. Show me 2/5 of the rectangle.

$A2_s$ 1. In this picture what fraction of the set of figures are triangles?

2. Show me 1/4 of this set.

3. In this picture what fractions of the set of figures are circles?

4. Show me 2/6 of this set of circles.

5. This set has 8 circles. Show me 1/4 of the set.

$A2_w$ 1. In this picture what fraction of the circle is colored blue?

2. Show me 1/5 of this rectangle.

3. In this picture what fraction of the circle is colored red?

4. Show me 3/10 of this square.

5. The square has 12 parts. Show me 1/4 of this square.

A3

1. What does the fraction 1/5 mean?

2. Write the fraction one-third.

3. What does the fraction 2/7 mean?

4. Write the fraction five-eighths.

$B1_s$

1. Here is a set of blocks. If you add 3/6 of the blocks to 1/6 of the blocks, what fraction of the set of blocks do you have?

2. Here is a set of 10 blocks. If you add 2/5 of the blocks to 1/5 of the blocks, what fraction of the set of blocks do you have?

$B1_w$

1. Here is a rectangle with 9 parts. If you add 4/9 of the rectangle to 1/9 of the rectangle, what fraction of the rectangle do you have?

2. Here is a square with 8 parts. If you add 2/4 of the square to 1/4 of the square, what fraction of the square do you have?

$B2_s$

1. In this picture there is a set of 4 circles. If you add 1/4 of the circles to 2/4 of the circles, what fraction of the set of circles do you have?

2. In this square there is a set of 10 squares. If you add 1/5 of the squares to 2/5 of the squares, what fraction of the set of squares do you have?

$B2_w$

1. Here is a picture of a square with 6 parts. If you add 1/6 of the parts to 3/6 of the parts, what fraction of the square do you have?

2. Here is a picture of a rectangle with 10 parts. If you add 2/5 of the rectangle to 1/5 of the rectangle, what fraction of the rectangle do you have?

B3

1. Add: 1/4 + 2/4

2. Add: 4/7 + 2/7

$C1_s$

1. This is a set of 8 circles. If 3/8 of the circles is subtracted from 5/8 of the circles, what fraction is left?

2. This is a set of 10 squares. If 1/5 of the squares is subtracted from 3/5 of the squares, what fraction is left?

3. This is a set of 10 rectangles. How much more is 7/10 of the set of rectangles than 3/10 of the set of rectangles?

4. This is a set of 12 squares. How much more is 3/4 of the set of squares than 1/4 of the set of squares?

$C1_w$

1. Here is a rectangle with 6 parts. If 2/6 of the parts of the rectangle is subtracted from 5/6 of the parts of the rectangle, what fraction is left?

2. This circle has 10 parts. If 2/5 of the parts of the circle is subtracted from 4/5 of the parts of the circle, what fraction is left?

3. Here is a square with 12 parts. How much more is 9/12 of the square than 5/12 of the square.

4. Here is a rectangle with 10 parts. How much more is 4/5 of the rectangle than 2/5 of the rectangle?

$C2_s$

1. This is a picture of a set of 5 squares. If 2/5 of the set of squares is subtracted from 4/5 of the set of squares, what fraction is left?

2. This is a picture of a set of 8 triangles. If 1/4 of the set of triangles is subtracted from 3/4 of the set of triangles, what fraction is left?

3. This is a picture of a set of 6 circles. How much more is 5/6 of the set of circles than 1/6 of the set of circles?

4. This is a picture of a set of 10 squares. How much more is 4/5 of the set of squares than 2/5 of the set of squares.

$C2_w$

1. This is a picture of a circle with 6 parts. If 1/6 of the parts of the circle is subtracted from 5/6 of the parts of the circle, what fraction is left?

2. This is a picture of a rectangle with 8 parts. If 1/4 of the parts of the rectangle is subtracted from 3/4 of the parts of the rectangle, what fraction is left?

3. This is a picture of a square with 7 parts. How much more is 5/7 of the parts of the square than 2/7 of the parts of the square?

4. This is a picture of a rectangle with 12 parts. How much more is 5/6 of the parts of the rectangle than 1/6 of the parts of the rectangle?

C3

1. Subtract: 5/9 - 1/9

2. Subtract: 7/8 - 5/8