A STUDY OF THE EFFECTIVENESS OF A TEACHER DEVELOPED UNIT ON PROOF IN SECONDARY SCHOOL GEOMETRY

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ABSTRACT

The purpose of this study was to design, implement and evaluate a unit on proof in geometry, incorporating a review of fundamental concepts, an organizational scheme called a "Dictionary" and some problem-solving strategies that could be applied to proof. Answers were sought to the following questions: Does the use of the instructional unit result in significantly different achievement on an immediate and delayed posttest of concepts and skills? And, is the effect on achievement different for repeaters than for non-repeaters?

This study was conducted using 54 Grade 10 matriculation geometry students from an all-male regional high school in urban Newfoundland. These students were assigned to one of two treatment conditions. The experimental group was taught a unit on proof using the instructional unit developed for this study, whereas the control group followed the program outlined in the Grade 10 geometry text used in Newfoundland when this study was carried out. Classes were held once a day for a total of 40 class sessions.

To determine the students' achievement, two tests were administered. The first, the immediate posttest, was given at the end of eight weeks instruction and a 2-period review session. The second, the delayed posttest was given two months later.

The data were collected and analyzed using a two-factor analysis of variance procedure. The major findings of the study were: (1) There were no significant differences in achievement on the immediate and delayed posttests between treatment groups; (2) non-repeaters scored

significantly higher than repeaters on the immediate posttest but not on the delayed posttest; and (3) there was no significant interaction between treatment received and the grade status of students.

On the basis of these findings, it was concluded that the experimental treatment was no more effective than the textbook approach in promoting higher achievement in geometry at the Grade 10 level.

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This thesis is dedicated to my wife, Marie and son, Steven, whose days and nights were so lonely and who on so many occasions understood.

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CHAPTER I

STATEMENT OF THE PROBLEM

> Secondary school geometry is frequently described as "the most disliked subject" and, as often, as "the most poorly taught subject" in the high school mathematics curriculum. Whether these designations are true or false, it is certainly true that the problem of instruction in geometry is multi-faceted. The problem includes questions on the nature of geometry, the way students learn it, and the optimum teaching approaches. (Farrell, 1971)

Usiskin (1980) reported that the single reason most commonly given for teaching geometry is that geometry provides an example of a mathematical system. Historically, it was the first and has been the most influential mathematical system. In earlier times the goal of geometry courses was to train the mind, to develop critical thinking, and to teach the student to reason. Because studies from the 1930's to the 1950's showed that the unenriched standard course did not help critical thinking or reasoning ability, the rhetoric had to be changed. Now, Usiskin contends, geometry is the place where the student supposedly learns how mathematics is developed. It is the place where the student is asked to do what mathematicians presumably do, that is, prove theorems (p. 413).

Williams (1974) maintained that to fully understand the nature of a mathametical system, students must be exposed to the concept of mathematical proof. He commented: ". . . students can only fully appreciate what a mathematical system is if they understand what it means to prove something in mathematics" (p. 5).

> Ask students what they like most and what they dislike most about their geometry course. . . . To what they dislike, there is one strong answer: proof. For most teachers, proof is the most important concept—the raison d'être—of the geometry course. . . . Thus, the most hated and feared idea is the most important. (Usiskin, 1980)

Bowling (1977) commented:

> For a variety of reasons, a person's first contact with proof is usually frustrating. Often it is not clear why proofs are necessary in the first place, and one of life's little mysteries remain, 'How did the teacher know to write this step after that one?' Unfortunately, students frequently adopt the rule, 'Ask what proofs we're responsible for, memorize the steps, and hope for partial credit'. (p. 506)

More unfortunately, Bowling contended, this algorithm is usually quite sufficient.

Background

Wooldridge (1978) reported difficulties with the high school geometry program in Newfoundland. He wrote: ". . . not all is well in the geometry program in Newfoundland. Some of the problems are related to the curriculum while others have pedagogical bases" (p. 10). Some of the difficulty being experienced in the geometry program is due to the fact that the geometry content is often skipped in the elementary grades.

Wooldridge (1978) commented:

> . . . the practice of omitting geometry topics in the elementary and junior high grades deprives students of proper development prior to the formal geometry course, and thus contributes to poor performance in the high school geometry course. (p. 10)

McGrath (1979) reported difficulties with geometry throughout the sixth, seventh and eighth grades. In a survey of elementary mathematics teachers, most teachers indicated that they did not have the time to cover the geometry because of other topics that had to be covered. They further indicated that the amount of material in the geometry section was so overwhelming that they were not sure what to cover and, as a result, ended up doing very little or none at all. McGrath concluded that "while the approach of selecting appropriate material is not necessarily bad . . . , it may result in gaps in the child's experiences that can lead to more serious problems in later grades" (p. 70).

Roberts (1980) reported that for the school year 1978-79, elementary school teachers in the Province of Newfoundland spent less than one-half the recommended time on geometry. Roberts concluded:

> It is evident that the required time is not being spent on geometry in our elementary schools, and as a result, the children will continue to experience difficulties with geometry at the secondary level. (p. 82)

A model for students' development in learning geometry has been described by the Van Hieles. Mathematics educators, methodologists, and psychologists at the Soviet Academy of Pedagogical Sciences, between 1960 and 1964 have verified the validity of the van Hieles' assertions and principles (Wirszup, 1976). According to the van Hiele model, the learner, assisted by appropriate instructional experiences, passes through levels of thinking beginning with recognition of shapes as a whole, progressing to discovery of properties of figures and informal reasoning about these figures, and culminating in a rigorous study of axiomatic geometry (Geddes, Fuys, Lovell & Fischler, 1981). Progression from level-to-level can only be accomplished through careful instruction and levels cannot be skipped (Burger, 1980).

The majority of our high school students, according to Wirszup (1976), are at the first level of development in geometry, while the course they take demands the fourth level of thought. He commented: "It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continuously later on" (p. 96).

Piaget (1964) maintained that a child can receive valuable information via language or via education directed by an adult only if he is in a state where he can understand this information.

> That is, to receive the information, he must have a structure which enables him to assimilate this information. This is why you cannot teach higher mathematics to a five year old. He does not yet have structures which enable him to understand. (p. 180)

Piaget suggests that children pass through various stages of logical operations (Lester, 1975). Studies by Lowell (1961), Case and Collinson (1962), Randall (1967), and others indicate that many adolescents are somewhere in a transitional stage between the concrete and formal operational stages. Lawson and Renner (1975) have documented that secondary school students who are still reasoning at the concrete level are able to learn very little, if any, of what is taught in an abstract verbal way.

In discussing adolescents' transition to a new stage, Ausubel (1964) commented:

> It is quite possible, . . . , that prior intuitive understanding of certain concepts during childhood could facilitate their learning and stabilize their retention when they are taught at a more formal, abstract level during adolescence. (p. 261)

Niedermeyer (1968), in discussing Gagné's experiments on the hierarchial nature of learning commented:

> What Gagné is saying . . . is that some things must be learned before others - the concepts that comprise a principle must be acquired before the learner can show his understanding of the principles, etc. (p. 313)

Kane (1975) has suggested that in order to engage in proof-making, a student must have at least an informal knowledge about logic. As a very minimum, Kane says that students must have experiences with the logical connectives, truth values of statements and the laws of logic. Wooldridge (1978) maintained that ". . . without some instruction in logic and the nature of proof, there is little wonder that students are unable to prove theorems" (p. 11).

Traditionally, the two-column proof in statement-reason format has been the form of proof to which the student is first introduced. Hallerburg (1971) saw this form as probably more of a hindrance than a help to the average or below average student. He pointed out two areas of difficulty. First, in about half of the steps, the step is a particular application of what Hallerburg called the basic H.I.C.--the statement of the step is the Conclusion, the reason for the step is the Implication, and the Hypothesis probably appears as the statement in the preceding step, or, if a compound hypothesis, it may be found scattered in various of the preceding steps. This reverse order is confusing and the student often feels that he should first of all come up with a statement and then he should find a reason to justify it. Second, a more basic criticism of the two-column proof is in the manner in which a single reason may intermingle "the form logic and content of the proof." When a single reason in a text's two-column proof states, "by steps 5 and 8 above, the definition of isosceles triangles and substitution," the student may very well wonder what he will have to include in his reasons when he is developing his own proof of a new theorem (p. 205).

The undue emphasis placed on the content aspect of geometry by the present high school matriculation geometry text is seen as a further problem. Wooldridge (1978) commented:

> There is a great deal of discussion as to whether there is enough emphasis on the proof aspect in the matriculation course. Certainly, the textbook does not do an adequate job there. (p. 10)

The difficulties suggested previously are that too many students in the matriculation course have not attained many of the fundamental concepts of geometry and they experience great difficulty with proof.

Statement of the Problem

It has been argued above that many secondary school geometry students have not attained many of the fundamental concepts of geometry and are experiencing great difficulty with proof. The purpose of this study was to design and test a unit on proof in geometry incorporating

(1) a review of fundamental concepts;

(2) an organizational scheme, using some basic principles of logic;

(3) some problem-solving strategies that can be applied to proof;

(4) examples and exercises based on the above; and

(5) a test measuring skills and concepts taught during the designated instruction period.

A second purpose was to compare the effects of the instructional unit on the achievement of students who were repeating their geometry course with those students who were attempting the course for the first time.

Specifically, answers were sought to the following questions:

(1) Does the use of the instructional unit have any effect on achievement in geometry?

(2) Does the use of the instructional unit have any effect on retention?

(3) Is the effect on achievement different for repeaters than for non-repeaters?

Rationale

Every effort to construct curricula in mathematics demands decisions about structuring the content and designing and ordering instructional tasks. Heimer (1969) commented that there are few precise, coherent, and empirically testable sets of hypotheses for identifying and guiding those instructional decisions. There are no well-formulated theories of instruction. In short, there are no adequate teaching algorithms (p. 495).

Cooney and Henderson (1972) declared that one of the concerns of teachers is helping students organize the concepts, facts, and principles that they learn. A question of interest to mathematics educators in particular, as seen by Cooney and Henderson, is how to find methods of instruction which prove effective in helping students structure their knowledge. Cooney and Henderson maintained that information concerning the most effective means of organizing knowledge was not available.

There is substantial evidence to support the general theory of the hierarchical structure of knowledge. Gagné (1968) demonstrated that new skills and knowledge emerge from lower-order knowledge and that there is a significant amount of positive transfer from each successive subordinate level to the next higher level in a hierarchical ordering of such levels. Gagné (1963) commented: ". . . the most important difference among learners in their ability to perform a final task resides in their possession of this subordinate knowledge " (p. 624).

According to Gagné (1963) learning of any particular capability requires the retention of other particular items of subordinate knowledge. The learner acquires a new item of knowledge largely because he is able to integrate previously acquired principles into new principles, and he cannot do this unless he really knows these previously learned principles.

Proof making, according to Kane (1975), may be viewed as a complex terminal behavior. We should set out to teach it by systematically teaching the prerequisite behaviors one by one and by teaching how these behaviors may be sequenced (p. 90).

What are these prerequisite behaviors? Greeno (1978) reported that detailed analyses of problem solving in geometry have indicated that three kinds of knowledge are involved: (a) knowledge for pattern recognition, that is being able to identify the similarities and differences between geometric objects, such as points, line segments, angles, and so forth, (b) knowledge of the propositions used to make inferences to prove theorems, for example, "corresponding angles are congruent," or "the sum of the angles of a triangle is 180 degrees"; and (c) strategic knowledge involved in using this kind of information in order to set goals, form plans and, in general, to organize activity on the problems. In the context of classroom instruction, in teaching, and in textbooks, the first two types of knowledge, according to Glaser (1979), are explicitly taught but the third is not. Carpenter, Coburn, Reyes and Wilson (1975) reported that successful performance in applying geometric relationships depends, to a large extent, on a sound initial development of the basic concepts and terminology in the early elementary grades (p. 449). As Holloway (1960) contended:

> . . . any system of mathematical information imparted to children without their having had adequate experience of the right kind actually impoverishes their development, though it may at first give it a spurious maturity. (p. 6)

As cited earlier, there is evidence to show that the basic concepts and terminology used in geometry are not being developed in the elementary grades (Wooldridge, 1978; McGrath, 1978; Roberts, 1970). In light of this situation, a review of basic definitions, theorems and concepts of elementary geometry was viewed as the first prerequisite to successful proof-making.

Williams (1974) proposed the possibility that the deductive nature of mathematical proof can only be understood by students who have been exposed to the basic ideas of mathematical logic. He suggested that students need to be exposed to a number of basic principles including the notion of a syllogism, the transitive law of implication, the law of the excluded middle, the idea that an implication and its contraposition are logically equivalent, and the difference between an implication and its converse or inverse (p. 11).

Kane (1975) has suggested that in order to engage in proof-making, a student must have at least an informal knowledge about logic. However, as Kane contended, it may develop that to engage in proof-making demands only the most informal knowledge about logic (p. 91).

Williams (1974) suggested that the basic principles of logical reasoning be made explicit to students in the context of the mathematics they learn, to aid students in understanding what proof and deductive inference in mathematics really mean (p. 11). Van Engen (1970) contended that to understand proof as exhibited in secondary mathematics, it is essential to (1) have a good understanding of what an <u>if-then</u> statement means, and (2) recognize the role of selected inference patterns.

Allendoerfer (1957) presented a procedure used to establish most theorems in mathematics. In carrying out a deductive process two steps labeled "substitution" and "detachment" are used. The Rule of Substitution states that at any point we may substitute one proposition for another while the Law of Detachment states the following: If we have established that (1) the implication $p \rightarrow q$ is true, and (2) that the proposition p is true, then we may conclude that the proposition q is true. The Law of the Syllogism and the Rule of Detachment are frequently combined into one step in our reasoning. For example: If we can establish that (1) $p \rightarrow q$ is true, and (2) $q \rightarrow r$ is true, we can conclude that $p \rightarrow r$ is true. This process can be carried out in a repetitive fashion and gives us results such as: If we can establish that (1) $p \rightarrow q$ is true, (2) $q \rightarrow r$ is true, (3) $r \rightarrow s$ is true, and (4) $s \rightarrow t$ is true, we can conclude that $p \rightarrow t$ is true. Finally, in a case like this, we can conclude that t is true if we know that p is true.

In the current study, a number of "sequences", using the above procedure, were developed as an instructional strategy in presenting this second prerequitie to successful proof-making, an informal knowledge of logic.

Lesh (1977) maintained that to develop effective instructional materials, it is important to present ideas in a form that will be most understandable to children. Van Engen (1953) saw understanding as an organizational process. "Understanding is more nearly a process of integrating concepts--placing them in a certain sequence according to a set of criteria" (p. 76). These concepts, according to Reed (1946), if logically learned are learned more quickly and are remembered longer than are concepts illogically learned. Commenting on this fact Stroud (1946) said:

> Material high in associative value is for that reason comparatively easy to learn and for the same reason easily recalled, relearned or recognized afterward. Logical material, material capable of meaningful organization or reduction to some kind of system, comes within the operations of transfer of training, operations that facilitate recall as well as learning. (p. 538)

Accepting van Engen's thesis on the formation of concepts, a "dictionary" incorporating definitions, theorems and logical inference schemes was developed as part of this study to be used as an instructional strategy in presenting this third prerequisite to successful proof-making, an organizational process.

In order for concepts and other information that require understanding to be included in one's conceptual knowledge, some sort of heuristic problem-solving capabilities must exist. Simon (1975) provided insight about the relation of heuristics and conceptual knowledge. "Understanding requires, in addition, the acquisition of a host of heuristic problem-solving capabilities, some of which are peculiar to the given subject, but others of which have a wider range of application" (p. 14).

Scandura (1971) maintained that in order to prove most theorems, indeed to successfully engage in complex deductive reasoning of any sort, a student must know more than just rules of inference, or even a large number of relatively complex logical procedures. The student must also have higher order rules available by which he can combine known inference rules and other logical procedures into new forms, "that is, so that he can create" (p. 194).

According to Scandura, one type of higher order rule, that is frequently used in constructing proofs is closely associated with the heuristic: "Work backward from the conclusion" (p. 193). In this case the learner attempts to derive a procedure for generating the conclusion from the premises, that is, to construct a proof he first selects an inference rule which yields the conclusion. He then tries to derive a logical procedure, using this or other higher order rules, which yields the input of the first rule selected. Presumably, the subject continues in this way until he either succeeds or the whole approach breaks down (p. 194). Support for this idea is also given by Henderson and Pingry (1953). They commented:

> The student should learn that when he faces a situation for which he has no immediate solution he can profitably direct his thinking by starting with the "to prove" or "conclusion" and, saying to himself, "If I show this I will first have to prove this. This in turn requires that I know . . . ," until the given data and conclusion are linked logically. (p. 239)

Duncker (1945) introduced the concept of a "search model" that is useful in understanding the psychological process of solving a problem. The search model evolves as the individual clarifies the problem. It bridges the gap between what is given and what is required, and serves for a period of time to direct or channel the individual's deliberation. His region of search consists of the mathematical concepts and generalizations he has learned. Part of the teacher's work according to Henderson and Pingry (1953), consists in helping students conceptualize functional search models as they clarify problems (p. 240). This strategic problem-solving knowledge was seen as a fourth prerequisite to successful proof-making. Working backwards was used as the main strategy, in this study, for proofs using the "dictionary" as a search model.

When one considers a final performance to be learned, we find, according to Gagné (1968) that it can be analyzed into a number of subordinate topics which must first be mastered before the final task can be attained. In proof-making these prerequisite skills are: (1) knowledge of the basic definitions, theorems and concepts of elementary geometry; (2) an informal knowledge of logic; (3) an organizational process for this knowledge; and (4) some problem-solving strategies that can be employed.

Definition of Terms

For the purpose of this study, the following definitions are stipulated:

1. Dictionary:

The list of definitions, axioms, postulates and theorems that were discussed in the instructional unit. The material in the dictionary was organized under the following headings: (1) Definitions; (2) Theorems; (3) Sequences A--information one can conclude from a figure without a "Given"; and (4) Sequences B--conclusions that can be drawn from a "Given." Specific examples are presented in Chapter III of this report.

2. Grade Status:

The term grade status is used to refer to students as being repeaters or non-repeaters.

3. Repeater:

Any student who was repeating the present Grade 10 geometry course because of failure in that course in the previous year.

4. Non-repeaters:

Any student who was not repeating the present Grade 10 geometry course.

Delimitations

The following delimitations were placed on this study:

1. Only one school comprising all male students was used in this study, thereby limiting its generalizability.
2. The results of this study are delimited in their generalizability to the area under investigation--geometry, and further limited to the grade level of the students involved in the study--Grade 10.

Organization of the Report

The background and statement of the problem, the rationale for the study, the definition of terms used, and a brief description of the delimitations of the study have been presented in this chapter.

A review of the literature follows in Chapter II with an emphasis on theories of learning of the van Hieles, Piaget, and Gagné. Research findings on each of these theories are presented.

In Chapter III the design of the study, including a brief description of the sample and population, the procedures used in the investigation, a description of the instructional unit, the statement of hypotheses, limitations imposed by the design, the statistical tests used in the analysis and significance levels accepted are all outlined. Also included in this chapter is a description of a secondary study conducted along with the main investigation.

Results of the study are presented in Chapter IV and a discussion of these results is presented in Chapter V. The study is summarized and conclusions drawn from the study are presented. Recommendations evolving from the study conclude the report.

CHAPTER II

REVIEW OF RELATED LITERATURE

Introduction

In this chapter, the learning theories of the van Hieles', Piaget, and Gagné, and the related research are reviewed. At the end of the chapter the three theories are discussed together, as they relate to the learning of elementary geometry.

The van Hiele Model of Intellectual Development

Of particular significance to this study is a theory of learning which has formed the basis for designing the new Soviet curriculum and methods of teaching geometry in the U.S.S.R. (Wirszup, 1976). This model was developed by two Dutch mathematics educators, Pierre van Hiele and his wife, Dina van Hiele-Geldof.

According to the van Hieles, learning is a discontinuous process. Pierre van Hiele (1958) found:

> . . . jumps in the learning curve that reveal the presence of levels. The learning process has stopped. Later on it will start itself once again. In the meantime, . . . the teacher does not succeed in further explanation of the subject. He, . . . seem(s) to speak a language which cannot be understood by the pupils who have not yet reached the new level. They might accept the explanation of the teacher, but the subject taught will not sink into their minds. The pupil himself feels helpless; perhaps he can imitate certain actions, but he has no view of his own activity until he has reached the new level. (p. 75)

In the van Hiele model, the learner, assisted by appropriate instructional experiences, passes through five levels.

The learner cannot achieve one level of thinking without having passed through the previous levels.

Level 0. Recognition. This initial level is characterized by the perception of geometric figures in their totality as entities. Figures are judged according to their appearance. Students do not see the parts of the figure, nor do they perceive the relationships among components of the figures or among the figures themselves. They cannot even compare figures with common properties with one another. The children who reason at this level distinguish figures by their shape as a whole.

Level 1. Analysis. The student who has reached this level begins to discern the components of the figures; he also establishes relationships among these components and relationships between individual figures. At this level, he is therefore able to make an analysis of the figures perceived. This takes place in the process and with the help of observations, measurements, drawings, and model-making. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. At this stage the figures act as the bearers of their properties, and the student recognizes the figures by their properties. However, these properties are still not connected with one another. For example, the pupil notices that in both the rectangle and the parallelogram the opposite sides are equal to one another, but he does not yet conclude that a rectangle is a parallelogram (Wirszup, 1976).

Level 2. Ordering. Students who have reached this level of geometric development establish relations among the properties of a figure and among the figures themselves. At this level there occurs a logical ordering of the properties of a figure and of classes of figures. The student is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level the student still does not grasp the meaning of deduction as a whole. Although, a student at level 2 can follow a deductive argument, can supply parts of the argument, and recognizes the role of "proof," he still does not grasp its meaning in an axiomatic sense. For example, he does not see the need for definitions and basic assumptions, and he cannot yet establish interrelationships between networks of theorems (Geddes et al., 1981).

Level 3. Deduction. At the fourth level, the students grasp the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the students' understanding of the role and the essence of axioms, definitions, and theorems; of the logical structure of a proof, and of the analysis of the logical relationships between concepts and statements.

Level 4. Rigor. At this level one attains an abstraction from the concrete nature of objects and from the concrete meaning of the relations connecting these objects. A person at this level develops a theory without making any concrete interpretation. Here, geometry acquires a general character and broader applications. For example, several objects, phenomena or conditions serve as "points" and any set of "points" serves as a "figure" and so on (Wirszup, p. 79). This most advanced level, according to Hoffer (1981), is rarely reached by high school students.

The van Hieles made certain observations about the general nature of these levels of thinking and their relationship to teaching. First, at each level there appears in an extrinsic manner what was intrinsic in the preceding level. At level 0 the figures were determined by their properties, but someone who is thinking at this level is not conscious of these properties. Second, each level has its own languages, its own set of symbols and its own network of relations uniting these symbols. A relation which is correct "at one level" can reveal itself to be incorrect at another (Geddes et al., 1981). Third, two people who reason at different levels cannot understand each other. Neither can follow the thought processes of the other. As Geddes et al. (1981) commented: "Many failures in teaching geometry result from a language barrier--the teacher using the language of a higher level than is understood by the student" (p. 5). Fourth, the maturation process which leads to a higher level unfolds in a characteristic way; one can distinguish several phases. It is then possible and desirable for the teacher to encourage and hasten it (Wirszup, 1976).

The phases which lead from any level to the next higher level of thought are, according to van Hiele (1958), as follows:

(1) Information

(2) Directed Orientation

(3) Explanation

(4) Free Orientation

and (5) Integration

Further details of these phases are given in Wirszup (1976).

As a result of this fifth phase, the new level of thought is reached. The student arranges a network of relations which connect with the totality of the domain explored. This new domain of thought, which has acquired its own intuition, has been substituted for the earlier domain of thought which possessed an entirely different intuition (Wirszup, 1976).

Van Hiele (1981) contended that the introduction of proof in geometry is not possible before students have reached the second level of development and to reach that level, at least three years of study in geometry are required. Van Hiele sees structure as being the essence of proof. If a student has forgotten the proof he can discover it again by reading the structure and by generalization. Most proofs, according to van Hiele (1981), are not given in this way. He commented:

> Usually the structure from which the proof can be read is omitted and therefore problem solving has become an art in itself. We have not learned to find the structure from which the result can be read but we have learned to perform a series of tricks to get the demanded result. (p. 5)

Van Hiele maintained that only after having found the structure does a student have the feeling that he has really understood.

A person who has to make a decision when working with quite new material does not come to a conclusion by reasoning but rather by reading it from a structure (van Hiele, 1981, p. 2).

As Wirszup (1976) commented:

> One could say that the basis of human knowledge consists in this: Man appears in a position to uncover a structure in all material, no matter how disordered it is, and this structure is perceived in the same way by many people - as a result of the conversation on this subject in which they can engage. (p. 83)

The majority of our high school students, according to Wirszup, are at the first level of development in geometry, while van Hiele

(1981) maintained that the introduction of proof in geometry is not possible before the students have reached the second level of development. However, progression from level to level can be accomplished through careful instruction, and is not directly related to age.

At present, research on the van Hiele model is limited. Wirszup (1976) reported that psychologists at the Soviet Academy of Pedagogical Sciences have verified the validity of van Hiele's assertions and principles, while other authors (Hoffer, 1981; van Dormolen, 1977; Berger, 1980; and Geddes et al., 1981) have discussed the van Hiele model. Geddes et al. (1981) are presently involved in a two-year investigation of "Geometric Thinking Among Adolescents in Inner City Schools." This study will focus on an analysis of current U.S.A. geometry curriculum materials according to the van Hiele levels and the development and validation of instructional modes based on the van Hiele model. At present, only preliminary results have been reported.

Burger (1981) investigated whether the van Hiele levels can serve as a model of student development in geometry. The following preliminary findings were reported: (1) Some students in grades 1-8 conceive of geometric shapes primarily in terms of visual cues; (2) some students use a form of "quasi-analytical" reasoning when considering geometrical shapes; (3) some students form an abstract conception of a shape as an example of a reference set of properties. Consequently, they reason that every shape with the properties in the set is of the given type; (4) some students, particularly those studying secondary geometry, are able to form correct deductive arguments, but without understanding the different roles of postulates and theorems in geometry; and (5) few, if any students seem to understand the idea of a mathematical system. However, given the small sample of 50 students, these conclusions, suggested Burger, must only be considered as indications of potentially discernible differences in the types of reasoning used by school children in the context of geometry.

In conclusion, it is quite possible because of the activities used in the present study, that repeaters who are at a low van Hiele level, may have a better opportunity to progress through these levels.

Piaget's Learning Theory of Intellectual Development

Educators and mathematicians agree that the study of proof should enter the school curriculum at the earliest possible time consistent with children's intellectual development. In Piagetian theory there are four phases of growth which characterize intellectual development: the sensori-motor stage, the preoperational stage, the stage of concrete operations, and the stage of formal operations. Studies by Lovell (1961), Case and Collinson (1962), Randall (1967) and others indicate that many adolescents are in a transitional state between the concrete and the formal operational stage. This review will be limited to a discussion of these two stages; the concrete and the formal operational.

In this context an operation, according to Piaget (1958), is a reversible internalizable action which is bound up with others in an integrated structure. It is a means for mentally transforming data about the real world so that they may be used in the solution of problems. An operation may be carried out externally, in the manipulation of categories (concrete) or propositions (formal). Concrete operations are mental actions which organize observed or experienced reality and do not necessarily involve the actual manipulation of tangible objects.

As the child grows and his ability for logical thinking develops, he uses concrete operations with increasing facility and on more complex problems. However, when he is confronted with a problem in which he must isolate one variable and hold one or more other variables constant, or in which he must think of all possible combinations and systematically exclude some of them, his thought system is inadequate; he cannot solve the problem until he is able to reason with propositions and hypotheses. The ability to consider the possible as well as the given and to use combinatorial analysis in solving problems distinguishes formal from concrete operational thought (Howe, 1974).

Farrell (1969) listed four characteristics of formal operations. First, at the formal level, the possible is considered as including reality as a subset with the result that hypotheses may proceed from non-observed and non-experienced phenomena. This characteristic of the formal state, the ability to imagine the possible as containing the real, frees the adolescent from the restrictions of his senses and sets the scene for hypothetico-deductive thinking. The formal operational child has the capacity to use formal operations but is not compelled to do so. He may revert to any of the earlier modes of thinking as they now issue from the transformed cognitive structures, for earlier stages are not eradicated but integrated into later stages.

The first characteristic of formal operations, the changed relation of the real to the possible, is dependent for existence upon the presence of the second characteristic, the potential for combinational analysis. In the concrete stage, the child faced with a multiple variable situation usually is limited to trying one to many correspondences or to testing unsystematically other possible correspondences, but the adolescent able to employ combinational analysis can consider all possible combinations of variables in a systematic manner. This ability is a necessary condition for generating all possibilities and so determines the shift in the orientation toward the real and the possible.

The third characteristic is the hypothetico-deductive property. The adolescent's reasoning is less "This is true, therefore. . ." and more "If this were true then. . . ." This kind of reasoning is essential if the possible is to include the real in the set of hypotheses. It also follows hand-in-hand with the ability to systematically check all possible combinations.

Finally, formal operations are characterized by propositional thinking. The elements manipulated by the adolescent are propositions, statements containing raw data, but not the data itself. In other words, the older child may utilize concrete operations of the earlier stage by organizing reality into classes, ordering them and so on, but then he proceeds to form propositions using these results and to operate on the proposition via conjunction, disjunction, implication, negation and equivalence. This type of thinking is what Piaget calls second-degree thinking, operations which result in statements about statements. For example, the recognition of the equality of two ratios constitutes the elaboration of a second-order relation. An analogy of the form "3 is to 12 as 5 is to 20" involves a certain relationship between the first two terms, a certain relationship between the third and fourth terms, and the establishment of an identity relationship between these two relationships (Lovell, 1971).

The four characteristics of formal operations outline the manner in which the adolescent thinks. Presented with a new situation, he begins by classifying and ordering the concrete elements of the situation. The results of these concrete operations are divested by their intimate ties with reality and become simply propositions which the adolescent may combine in various ways. Using combinational analysis, the student regards the totality of combinations as hypotheses which need to be verified and rejected or accepted. However, these powerful tools of formal operations are not always employed by the adolescent. In the case of completely new material the process of transforming reality into concrete relations and those into propositions must occur before propositions are available for formal thought; and even in situations which do not require this initial step, a particular adolescent may not utilize formal thinking (Farrell, 1969).

"Proof making," according to Kane (1975), refers to the activities undertaken to create a sequence of statements, each one following inescapably from the preceding ones, with the final statement asserting that which was to be proved. The terms deductive thinking or necessary inference signify the proof-making task. This type of hypothetico-deductive thinking is characteristic of the formal operational stage.

Research Related to Piaget's Theory

In any problem-solving activity, especially those activities identified with proof, logical reasoning plays an important role. However, there is conflicting evidence concerning the development of logical reasoning abilities. According to Piaget (1958) children of ages 11 to 13 years are able to handle certain formal operations, for example, implication and exclusion, successfully, but they are not able to set up an exhaustive method of proof. This ability to deal with premises that require hypothetico-deductive reasoning is not present until the child is approximately 14 to 15 years of age (Lester, 1975). However, Suppes (1966) and Hill (1961) supported Hazlitt's (1930) claim that there is no relationship between logical reasoning and age beyond that imposed by the lack of experience. Their studies suggest that children in the age range 6 through 8 years are able to recognize valid conclusions derived from hypothetical premises.

Piaget recognized that the realization of the possibilities at a given stage "can be accelerated or retarded as a function of cultural and educational conditions" (Inhelder & Piaget, 1958, p. 337). However, Piaget cautioned that the individual must be in a state of readiness to assimilate these contributions, and this readiness is a function of maturation of individual cerebral mechanisms (p. 338).

Several investigators have undertaken programs aimed at improving logical thinking through instruction. Fishbein, Pampu, and Minzat (1970) gave direct instruction to 60 subjects at ages 10, 12 and 14 on a formal operational task of combinational analysis and reported that the instruction was effective in improving the students' ability to do the tasks. Tomlinson-Keasey (1972) tested sixth grade girls, university women students, and middle-aged women on three tasks, followed this with training, posttesting immediately after training, and again after an interval of one week. She found that the training produced increases in conceptual level on the three tasks but did not produce transfer to other tasks (Howe, 1974).

There have been a few attempts to teach logic *per se* (Donaldson, 1963; Suppes & Binford, 1965; McSloon, 1969; Miller, 1969). For the most part these studies reported that children have learned principles of logic. But, according to Gregory and Osborne (1975), those studies that reported comparative data for students not receiving formal instruction in logic, reported equivalent growth between pretest and posttest administration.

Ennis and Paulus (1965) conducted a study dealing with "deductive logic in adolescence." The forms of reasoning used included affirmation of the antecedent (modus ponens), inversion, conversion, contraposition, and transitivity. Of these, the two principles expressing the basic fallacies (conversion and inversion) were the most difficult at ages 10 to 12, but there was a marked improvement over the period studied, ages 10 to 18. Among the validity principles, contraposition and transitivity were of medium difficulty at ages 10 to 12. There was little improvement over the years relative to contraposition but considerable improvement, without training, relative to transitivity (Carroll, 1975). Ennis and Paulus suggested that logic might be learned by the students who did not receive formal instruction because of other school influences, which would include teacher effectiveness in terms of content presentation and interaction with students (Gregory & Osborne, 1975).

Methods of teaching employing formal and concrete operations have been investigated by Sheehan (1970). Sheehan classified a sample of 60 students as concrete or formal operational, and assigned them at random to a concrete or formal-operational training group. The outcome was that formal operational students showed greater gains, regardless of mode of instruction, that achievement was more durable for formal-operational students and that concrete level instruction was more effective for all students (Howe, 1974).

The content or subject matter of the task has increasingly come to be recognized as a factor in the ability to solve problems. Lunzer (1973) has suggested, and Stone and Ausubel (1969) have produced evidence, that the structure of a problem may not be the determining factor in whether it will be solved. Piaget (as cited in Howe, 1974) suggested the possibility that in life situations an adult might use formal operations in the area of his work and not in other areas. As Howe commented:

> The majority of high school students are probably not able to use formal operations except, perhaps, in a small number of situations. It is certainly a mistake to assume that even upper-level secondary students, except those who are very able, have access to formal operations for the solution of most problems. (p. 10)

In summary, Piaget found that a child passes through four distinct stages of mental growth. The order in which a child progresses through these stages is fixed, but his rate of progress is not fixed. The transition from one stage to the next can be hastened by enriched experiences and good teaching (Adler, 1966).

An examination of research involving the logical reasoning abilities of young children revealed that these abilities may be far superior to their ability to put an argument in written form (Lester, 1975). This evidence, according to Lester, suggests that the ability to create the essence of mathematical proofs may be superior to their ability to write proofs.

Vast differences exist between adolescents in intellectual achievement and ability (Howe, 1974). Instruction cannot possibly bring about learning unless it takes these differences into account. According to Howe, a popular notion is that we have to wait until an adolescent becomes "formal operational" and then begin certain kinds of instruction. On the contrary, she contends, we should be devising instructional methods and promoting attitudes that will help students move forward from wherever they are (p. 14).

Gagné's Learning Theory

Gagné (1968) postulated that the design of an instructional situation is basically a matter of designing a sequence of topics, and that acquisition of new knowledge depends upon the recall of old knowledge. Presented in this section of the review is a discussion of Gagné's Learning Hierarchy theory.

Gagné (1977) described a model of human learning based upon the notion of cumulative learning. It is a model which proposes that new learning depends primarily upon the combining of previously acquired and recalled learned entities, as well as upon their potentialities for transfer of learning.

Acquisition of new knowledge, according to Gagné (1968), depends upon the recall of old knowledge. The learning of any particular capability requires the retention of other particular items of subordinate knowledge--not just any knowledge or knowledge in some general sense. The learner acquires a new item of knowledge largely because he is able to integrate previously acquired principles into new principles, and he cannot do this unless he knows these previously learned principles. The design of an instructional situation is basically a matter of designing a sequence of topics (p. 164).

Gagné (1977) suggested that any human learning task may be analyzed into a set of component tasks which are distinct from each other in terms of the experimental operations needed to produce them. Thus, the presence or absence of these task components effects positive transfer to the final performance. Following this reasoning, according to Heimer (1969), it is necessary to identify the component tasks of a final performance, ensure achievement of each of the tasks, and arrange the total learning situation in a sequence which ensures optimal mediational effects from one component to another (p. 496).

Research Related to Theory

There is substantial evidence to support the general theory of the hierarchical structure of knowledge. Several authors have demonstrated that new skills and knowledge emerge from lower order knowledge and that there is a significant amount of positive transfer from each successive subordinate level to the next higher level in a hierarchical ordering of such levels. Gagné, Mayor, Gertens, and Paradise (1962) reported a study in which the hypothesis that a final behavior of adding integers depends upon the attainment of a hierarchy of subordinate behaviors was tested. The results of this study showed that with few exceptions (3%), learners who were able to learn the capabilities higher in the hierarchy also knew how to do the tasks reflected by the simpler rules lower in the hierarchy.

Wiegand (as cited in Gagné, 1968) conducted a study of subordinate skills in a science problem. A learning hierarchy was constructed indicating hypothesized prerequisite capabilities for this task. The experiment confirmed the hypothesis that learning of initially missing subordinate skills produced marked positive transfer in the learning of a complex problem-solving skill. Other studies (Gagné, 1962; Gagné & Paradise, 1961) support the conclusion that the attainment of any behavior in a learning hierarchy depends upon the achievement of the relevant supporting behaviors.

Merrill (1965) reported a study that seemingly offers contrary evidence. He tested the hypothesis that learning and retention of a hierarchical task are facilitated by mastering each successive component of the hierarchy before continuing in the instructional program. Merrill ensured mastery by channeling a student who erred on any particular component into a two-stage correction review procedure. The results of his study indicated that it is not necessary to master one level before proceeding to the next. However, Briggs (1968) suggested that Merrill's task analysis might be faulty.

Gagné (1963) stated that the design of an instructional situation is basically a matter of designing a sequence of topics. There is evidence to suggest that optimal learning sequences exist. Brown (1970) indicated that subjects using materials sequenced according to learning hierarchies performed better than subjects using materials whose sequence was scrambled, relative to time to complete the instructional program, to errors made on the program, and to performance on a criterion test of complex problem-solving skills. Brown concluded that when a sequence involves tasks that are complex, ordering of problem-solving behaviors is an important factor in learning even for bright and relatively mature learners. In summarizing research on sequencing mathematical tasks, Miller (1969) concluded that mastery of individual subtasks in a hierarchy can be achieved in several ways, including learning from randomly ordered sequences, but that logical sequencing appeared best in terms of overall efficiency and effectiveness.

An analysis of the literature on learning hierarchies and their role in the development of presentation sequences makes the following conclusions seem tenable. There are no well-defined algorithms for producing learning hierarchies; the connection between the (logical) structure of the knowledge and the associated learning hierarchy has not yet been adequately explored; and the role of learning hierarchies in the development of a presentation sequence is unclear (Heimer, 1969, p. 499).

Kane (1975) stated that proof-making may be viewed as a complex terminal behavior. Associated with it is a set of prerequisite or subsidiary behaviors, and we should set out to teach it by systematically teaching the prerequisite behaviors one by one and by teaching how these behaviors may be combined one with another (p. 90). This procedure was adopted in the development of the instructional unit tested in the present study.

Summary of Literature Review

The research reviewed in this chapter focused on theories of learning as postulated by the van Hieles, Piaget, and Gagné. Bruner (1964) stated that any theory of teaching must be concerned with how best to learn what one wishes to teach, rather than with a descriptive analysis of teaching. Adler (1966) supported this position in claiming that teaching, if it is to be effective, must be based on an adequate theory of learning. However, Gage (1964) claimed that teaching embraces far too many kinds of behaviors and activities to be a proper subject of a single theory. Furthermore, commented Aichele and Reys (1974), the selection of a model should be a function of the instructional objectives and classroom situations with which one is dealing.

The research reviewed in this chapter contains several common themes, the first of these being learner readiness. Van Hiele maintained that the introduction of proof in geometry is not possible before students have reached the second level of development and to reach that level at least three years of study in geometry are required. As stated earlier, hypothetico-deductive thinking is characteristic of the formal operational stage. According to Piaget children of ages 11 to 13 years are able to handle certain formal operations, for example, implication and exclusion, successfully but they are not able to set up an exhaustive method of proof until approximately 14 to 15 years of age. However, Suppes and Hill claimed that there is no relationship between logical reasoning and age beyond that imposed by lack of experience. As Gagné maintained, the most important difference among learners in their ability to perform a task resides in their possession or lack of possession of subordinate knowledge.

A second theme is the concept of "levels" or "stages" of development. Both Piaget and the van Hieles postulated that students pass through certain "stages" or "levels" of intellectual development. Without claiming that these two concepts are synonymous, both Piaget and van Hiele agree that the order of these "stages" and "levels" are fixed, but that rate of progress is not. The transition from one "stage" to another is dependent on maturation while the transition from one "level" to another can be hastened or retarded as a function of instruction.

Flavell (1963) summarized Piaget's views on the implication of the stages of development for educating children:

> In trying to teach a child some general principle or rule, one should so far as it is feasible parallel the development process of internalization of actions, that is the child should first meet with the principle in the most concrete and action-oriented content possible.

However, as Adler (1968) pointed out, Piaget's use of the word "concrete" should not be confused with the uses of the word in everyday speech. For Piaget, the concrete operations of a person are mental operations with propositions about some real system of objects and relations that the person perceives. What is concrete or not concrete in this sense is relative to the person's past experience and his mental maturity.

The van Hieles' research indicates that for students to function adequately at one of the advanced levels, they must have mastered large chunks of the prior levels. This development which leads to a higher geometric level proceeds under the influence of learning and therefore depends on the content and methods of instruction.

According to Gagné, when we consider a final performance to be learned, we find that it can be analyzed into a number of subordinate subjects which must first be mastered before the final task can be attained. The basic principles of instructional design in Gagné's theory consists of first, identifying the component tasks of a final performance; then ensuring that each of these component tasks is fully achieved; and finally, arranging the total learning situation in a sequence which will ensure optimal mediational effects from one component to another.

As cited earlier, the purpose of this study was to design and test a unit on proof in geometry. An analysis of the literature has shown that there are no well-formulated theories of instruction. However, this study is an attempt to integrate many of the ideas of the van Hieles', Piaget and Gagné.

CHAPTER III

PROCEDURE AND STATISTICAL DESIGN

The design of this study and the procedures employed in conducting the research are presented and elaborated upon in this chapter. More specifically, the following topics are discussed: (1) population and sample; (2) experimental design; (3) procedure; (4) the instructional unit; (5) instruments; (6) limitations; (7) hypotheses; and (8) statistical tests and significance levels.

Population and Sample

The population for this study consisted of all students enrolled in the Grade 10 matriculation geometry program, at an all-male high school in the St. John's urban area, who were not currently taking Chemistry or Physics. There were a total of 91 students in four classes. These students were assigned to classes by the principal. Students had not been assigned to classes in any particular manner. A few students had been assigned to particular classes in an attempt to avoid discipline problems. One of these classes consisted of only 11 students and was therefore excluded from the present study. Of the remaining three classes one was randomly selected to pilot the instruments used in this study, and the remaining two classes were used as the sample.

This particular school was chosen because the investigator was a member of the teaching staff, and the subjects were therefore readily available. Tenth grade classes were chosen because an integral part of the tenth-grade geometry course is formal proof.

Experimental Design

The design for this study was a 2 x 2 factorial design using the independent variables treatment and grade status. This design is summarized in Figure 1.

Grade Status / Treatment	Repeater	Non-repeater
Dictionary	Dependent Variable	
Control	Achievement	

Figure 1. Factorial design employed.

The design was a modification of Design 6 as described by Campbell and Stanley (1963). Design 6 is a Posttest-Only Control Group Design which controls for threats to internal validity such as history, maturation, testing, instrumentation, regression, selection, mortality, and the interaction effects of these factors.

Procedure

The two classes that comprised the sample for this study were randomly assigned to one of two treatment groups. Both groups were taught a unit on proof in geometry, specifically proofs involving congruent triangles. The experimental group was taught this material using an instructional unit developed by the researcher. The instructional unit consisted of the compiling of a "dictionary" of basic facts and principles necessary for proving triangles congruent and the use of an analysis-synthesis strategy of proof. A discussion of the instructional unit is presented later in this chapter.

The control group was taught the same material on proofs involving congruent triangles following the program outlined in the textbook, Modern Basic Geometry (Jurgensen, Maier & Donnelly, 1973). The control group did not use the dictionary or write their proofs in the analysis-synthesis format.

Class sessions consisted of one 40-minute class each day over a period of approximately eight weeks. All possible attempts were made to keep the instruction time and duration approximately equal for both groups. However, it is noted that since this unit was taught as part of the regular school curriculum and in a school situation it was not entirely possible to have a completely controlled situation.

After completion of the unit on proof, both groups were given a review session, which lasted for two class sessions. Three days after the review session all subjects were given a posttest on the concepts and skills taught in the unit. The posttest is included in Appendix A. These tests were scored by the researcher.

To further test the effects of both treatments on achievement over time, a delayed posttest was given approximately two months after termination of instruction. This instrument is included in Appendix B. Between the two tests the teachers continued with their regular instruction of other topics to be covered in the Grade 10 geometry curriculum.

Two points need discussion at this time. Firstly, the delayed posttest was not as comprehensive as the immediate posttest. The retention test was part of a more comprehensive term exam. Therefore, in light of this constraint, only a limited attempt can be made to compare the two. Secondly, the regular instruction of other topics between tests reinforced the material taught in the unit on congruent triangles, as proving triangles congruent was an integral part of this material.

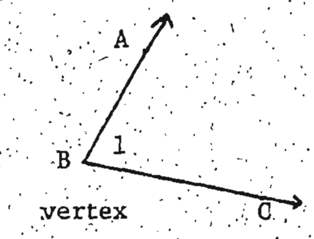
All results of both immediate and delayed posttests were tabulated and analyzed using the appropriate statistical tests to be discussed later in this chapter.

The Instructional Unit

The instructional unit for this study was a unit on using congruent triangles to prove geometric deductions. The unit was designed to incorporate (1) a review of fundamental concepts; (2) an organizational scheme, using some basic principles of logic; (3) some problem-solving strategies that could be applied to proof; (4) examples and exercises based on the above; and (5) a test measuring skills and concepts taught. The principal part of the unit was the compiling of a "dictionary" of basic facts and principles necessary for proving triangles congruent. The dictionary was used to organize information in a logical way, under the following headings: (1) Definitions; (2) Theorems; (3) Sequences A—information one can conclude from a figure without a "Given"; and (4) Sequences B—conclusions that can be drawn from a "Given." A discussion of the development of the dictionary follows, with examples under each heading.

(1) Definitions: Presented in this section was a review of definitions necessary for proving triangles congruent. If a student at a later time, could not remember the meaning of a particular definition, he could refer to his dictionary to refresh his memory. For example:

<u>Definition:</u> An angle is the union of two rays with a common endpoint.



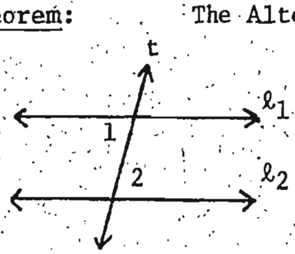
can be written as: $\angle B$

$\angle ABC$ or $\angle CBA$

or $\angle 1$

(2) Theorems: The purpose of this section was to list, in an if . . . then . . . format, theorems that are frequently used in the Grade 10 geometry program. For example:

<u>Theorem:</u> The Alternate Interior Angle Theorem.

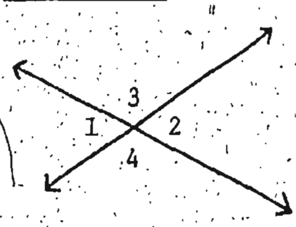


(a) If $\ell_1 \parallel \ell_2$ then $\angle 1 \cong \angle 2$

(b) If $\angle 1 \cong \angle 2$ then $\ell_1 \parallel \ell_2$

(3) Sequences A: The majority of proofs in the Grade 10 matriculation program are direct proofs employing the use of diagrams. The purpose of this section was to list conclusions that can be drawn if given a figure. For example:

<u>Sequences A:</u>

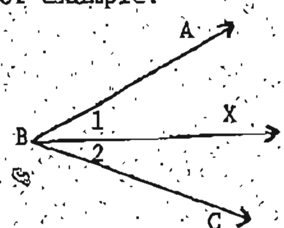


	Statement	Reason
(a)	$\angle 1 \cong \angle 2$	Vertical $\angle s$
(b)	$\angle 3 \cong \angle 4$	Vertical $\angle s$

It should be noted that this example is not meant to be exhaustive as one can also conclude information about supplementary angles from the above figure.

(4) Sequences B: Robinson (1963) commented that the aim of a proof in mathematics is to demonstrate that some given statement holds in the axiomatic system under consideration. The strategy of the direct proof is to show that the given statement is deducible from the axioms. The direct proof, according to Robinson, proceeds from the axioms by repeated application of the Law of Detachment. However, the two-column form of proof does not make this very clear. As cited earlier, Hallerburg (1971) supported the above conclusion.

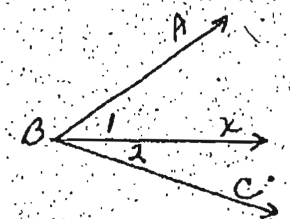
The purpose of this section of the dictionary was to list conclusions arrived at by using the laws of logic, arising from a premise presented to students as information "Given" about a certain figure. For example:



Given: $\overrightarrow{BX}$ bisects $\angle ABC$

1. If a ray bisects an angle then it forms two congruent angles; (this is the definition of an angle bisector)	1. If P, then Q
2. $\overrightarrow{BX}$ bisects $\angle ABC$	2. P is true
3. Therefore, $\angle 1 \cong \angle 2$	3. Therefore, Q is true

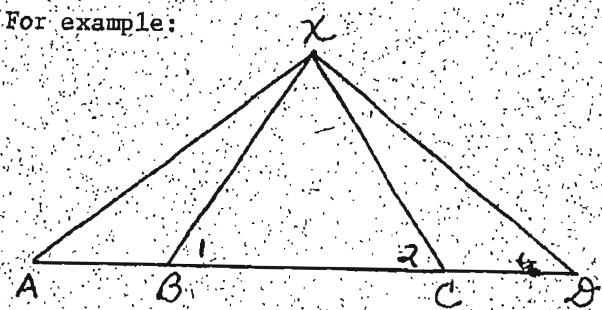
The syllogism presented on the right above, which illustrates the Law of Detachment was used to develop each of the sequences in this section of the dictionary. However, this logic terminology was not presented to the students. Instead, the argument was written in a form to parallel a two-column proof. The above syllogism was presented in the following form:



	Statement	Reason
1.	$\overrightarrow{BX}$ bisects $\angle ABC$	1. Given
2.	$\angle 1 \cong \angle 2$	2. def. $\angle$ bisector

Given: $\overrightarrow{BX}$ bisects $\angle ABC$.

Incorporated into this instructional unit was a strategy of proof that could be used by the students. This strategy was "work backward from the conclusion." For example:



Given: $\overline{AX} \cong \overline{DX}$

$\overline{AB} \cong \overline{CD}$

Prove: $\angle 1 \cong \angle 2$

Analysis: I can prove	If I can prove
1. $\angle 1 \cong \angle 2$	$\overline{BX} \cong \overline{CX}$
2. $\overline{BX} \cong \overline{CX}$	$\triangle XAB \cong \triangle XDC$

Plan of Attack: Show $\triangle XAB \cong \triangle XDC$

It should be noted that in the synthesis of the proof for this particular exercise the dictionary could be used to draw a conclusion from $\overline{AX} \cong \overline{DX}$.

Proof:

	Statement	Reason
1.	$\overline{AX} \cong \overline{DX}$	1. Given
2.	$\angle A \cong \angle D$	2. Isos. $\triangle$ th.
3.	$\overline{AB} \cong \overline{CD}$	3. Given
4.	$\triangle XAB \cong \triangle XDC$	4. S.A.S. and by 1, 2 and 3
5.	$\overline{BX} \cong \overline{CX}$	5. C.P.C.T.C. and by 4
6.	$\angle 1 \cong \angle 2$	6. Isos. $\triangle$ th.

The procedure outlined above was used as a strategy in solving many of the proof exercises in this unit. The analysis, however, was not required as part of the solution.

In summary, the unit was designed as an organizational scheme to be used by the students in studying proof in geometry. Students who used this instructional unit comprised the group called "Dictionary" and those in the control group, "Non-Dictionary."

Instruments

Two tests were designed by a panel of high school geometry teachers to test the stated behavioral objectives of the instructional unit. The objectives are listed in Appendix C. To establish the validity of the tests, a specialist in the field of mathematics education was consulted.

The stability method (Ahman & Glock, 1967, p. 315) was used to test the reliability of the immediate posttest. According to the stability method, a test is administered to a group of students once; then after a certain time interval it is administered a second time to the same group of students. A coefficient of reliability is computed from the two sets of test scores. This correlation coefficient to test the reliability of the test was found by computing a Pearson product moment correlation coefficient. The test was given to a sample of 26 students enrolled in the matriculation program but not a part of this study. The correlation coefficient computed for this test was 0.90.

The split-half method was used to test the reliability of the delayed posttest. Two subtests were found and scores obtained on the two halves were correlated. The half-test reliability coefficient for the delayed posttest was 0.74. Using the Spearman-Brown formula the corrected reliability coefficient was 0.85. The tests can be found in Appendices A and B.

Limitations

In Chapter I it was noted that the study was delimited to Grade 10 level male students and to the subject of geometry. In addition, there were other factors which potentially limit the generalizability of the results.

The results of this study were limited by the degree of teacher effect present. Two different teachers were used in this study. Researchers such as Palardy (1969) and Brophy and Good (1970) have reported that teacher variables do affect student achievement.

The results may also be limited by such variables as general ability, school achievement, and the time of day of instruction. Because intact classes were used in this study, the researcher did not have control over these variables. However, the students that comprised the sample were enrolled in the same program, studied the same courses, and were assigned to classes by the principal at the beginning of the school year.

Hypotheses

The following null hypotheses were tested in this study:

1. There is no significant difference in achievement in geometry between students receiving regular classroom instruction and those using the instructional unit (Dictionary) on an immediate posttest of mathematical skills and concepts.

2. There is no significant difference in achievement between repeaters and non-repeaters on an immediate posttest of mathematical skills and concepts.

3. There is no significant interaction between the grade status of students and the treatment received on an immediate test of mathematical skills and concepts.

4. There is no significant difference in achievement in geometry between students receiving regular classroom instruction and those using the instructional unit (Dictionary) on a delayed posttest of mathematical skills and concepts.

5. There is no significant difference in achievement between repeaters and non-repeaters on a delayed posttest of mathematical skills and concepts.

6. There is no significant interaction between the grade status of students and the treatment received on a delayed posttest of mathematical skills and concepts.

Statistical Test and Significance Levels

Results of the immediate and delayed posttest were subjected to a two-way analysis of variance. The computer program in Statistical Package For the Social Sciences (1970), titled ANOVA, was used for the analysis. All stated hypotheses were tested at the 0.05 significance level.

In Chapter IV of this report a complete analysis of the results using the statistical tests and significance levels as outlined in this chapter is reported. These results are discussed in Chapter V.

CHAPTER IV

ANALYSIS OF RESULTS

In this chapter the data are examined in terms of the stated hypotheses. The chapter is divided into three sections: (1) immediate posttest results; (2) delayed posttest results; and (3) secondary study.

Immediate Posttest Results

In this section the data from the immediate posttest are presented. The following three hypotheses were tested using a two-way analysis of variance procedure.

Hypothesis 1: There is no significant difference in achievement in geometry between students receiving regular classroom instruction and those using the instructional unit (Dictionary) on an immediate posttest of mathematical skills and concepts.

Hypothesis 2: There is no significant difference in achievement between repeaters and non-repeaters on an immediate posttest of mathematical skills and concepts.

Hypothesis 3: There is no significant interaction between the grade status of students and the experimental treatment received on an immediate test of mathematical skills and concepts.

For Hypothesis 1, an F-ratio of 1.88 was obtained which was not significant at the 0.05 level and therefore the hypothesis was not rejected. It was concluded then that there was no significant difference in achievement between the two treatment groups. The analysis of variance results are reported in Table 1.

TABLE 1

ANALYSIS OF VARIANCE OF IMMEDIATE POSTTEST RESULTS

Source	SS	DF	MS	F	Significance Level
Treatment	115.50	1	115.50	1.88	0.18
Grade Status	514.08	1	514.08	8.39	0.01
Treatment by Grade Status	2.87	1	2.87	0.05	0.83
Error	3,063.04	50	61.26		
Total	3,713.70	53	70.07		

Students in the Dictionary group obtained a mean score of 35.33, whereas students in the Non-Dictionary group obtained a mean score of 32.17. Although the mean score was higher for the Dictionary group, this difference was not significant. A detailed breakdown of mean scores on the immediate posttest are presented in Table 2.

For Hypothesis 2, as shown in Table 1, an F-ratio of 8.39 was obtained which was significant at the 0.01 level of significance and therefore the hypothesis was rejected. It was concluded then that there was a significant difference in achievement between repeaters and non-repeaters. The non-repeaters scored significantly higher on the immediate posttest. Non-repeaters obtained a mean score of 36.05 whereas repeaters obtained a mean score of 29.29.

TABLE 2

MEAN ACHIEVEMENT SCORES ON THE IMMEDIATE POSTTEST BY TREATMENT GROUP AND GRADE STATUS

		Treatment		
		Non-Dictionary	Dictionary	Total
Grade Status	Non-repeater	34.56* N=16	37.19 N=21	36.05 N=37
	Repeater	27.38 N=8	31.00 N=9	29.29 N=17
	Total	32.17 N=24	35.33 N=30	33.93 N=54

*Maximum = 56.00

For Hypothesis 3, an F-ratio of 0.05 was obtained which was not significant at the 0.05 level and therefore the hypothesis was not rejected. It was concluded then that there was no significant interaction between the grade status of students and the treatment received. A graphic representation of this interaction is presented in Figure 2.

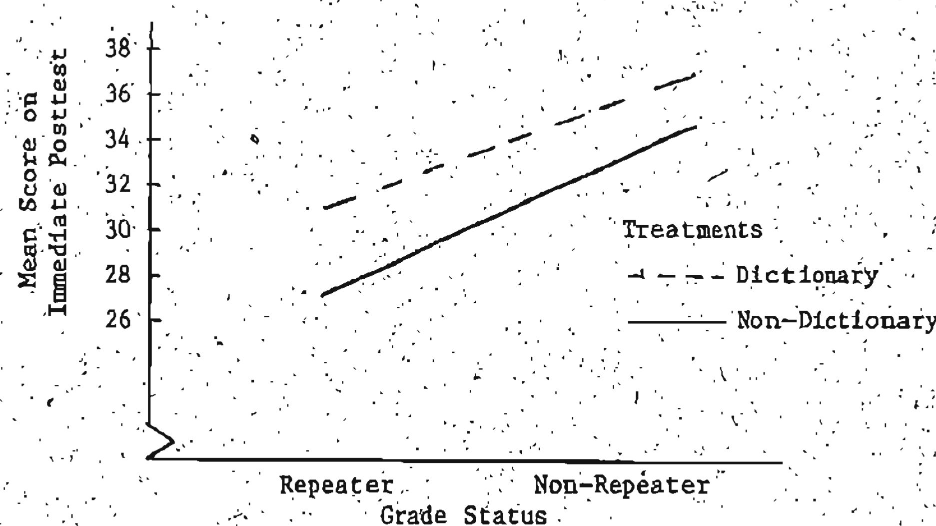


Figure 2. "Treatment by Grade Status" Interaction on the immediate posttest.

Delayed Posttest Results

In this section the data from the delayed posttest are presented. The following three hypotheses were tested using a two-way analysis of variance procedure.

Hypothesis 4: There is no significant difference in achievement in geometry between students receiving regular classroom instruction and those using the instructional unit (Dictionary) on a delayed posttest of mathematical skills and concepts.

Hypothesis 5: There is no significant difference in achievement between repeaters and non-repeaters on a delayed posttest of mathematical skills and concepts.

Hypothesis 6: There is no significant interaction between the grade status of students and the experimental treatment received on a delayed posttest of mathematical skills and concepts.

The analysis of variance results are reported in Table 3.

TABLE 3

ANALYSIS OF VARIANCE OF DELAYED POSTTEST RESULTS

Source	SS	DF	MS	F	Significance Level
Treatment	7.72	1	7.72	0.78	0.38
Grade Status	31.91	1	31.91	3.24	0.08
Treatment by Grade Status	8.09	1	8.09	0.82	0.37
Error	493.03	50	9.86		
Total	541.92	53	10.22		

For Hypothesis 4, an F-ratio of 0.78 was obtained which was not significant at the 0.05 level and therefore the null hypothesis was not rejected. It was concluded then that there was no significant difference in achievement between the two treatment groups.

Students in the Dictionary group obtained a mean score of 13.40, whereas students in the Non-Dictionary group obtained a mean score of 12.58. Although the mean score was higher for the Dictionary group, this difference was not significant. A detailed breakdown of mean achievement scores on the delayed posttest are presented in Table 4.

TABLE 4

MEAN ACHIEVEMENT SCORES ON THE DELAYED POSTTEST BY TREATMENT GROUP AND GRADE STATUS

		Treatment		
		Non-Dictionary	Dictionary	Total
Grade Status	Non-repeater	13.44* N=16	13.67 N=21	13.57 N=37
	Repeater	10.88 N=8	12.78 N=9	11.88 N=17
	Total	12.58 N=24	13.40 N=30	13.04 N=54

*Maximum = 18.

For Hypothesis 5, and F-ratio of 3.24 was obtained which was not significant at the 0.05 level and therefore the hypothesis was not rejected. It was concluded then that there was no significant difference in achievement between repeaters and non-repeaters. Non-repeaters obtained a mean score of 13.57, whereas repeaters obtained a mean score of 11.88. Although the mean scores were higher for non-repeaters, this difference was not significant.

For Hypothesis 6, and F-ratio of 0.82 was obtained which was not significant at the 0.05 level and therefore the hypothesis was not rejected. It was conoluded then that there was no significant interaction between the grade status of students and the experimental treatment received. A graphic representation of this interaction is presented in Figure 4.

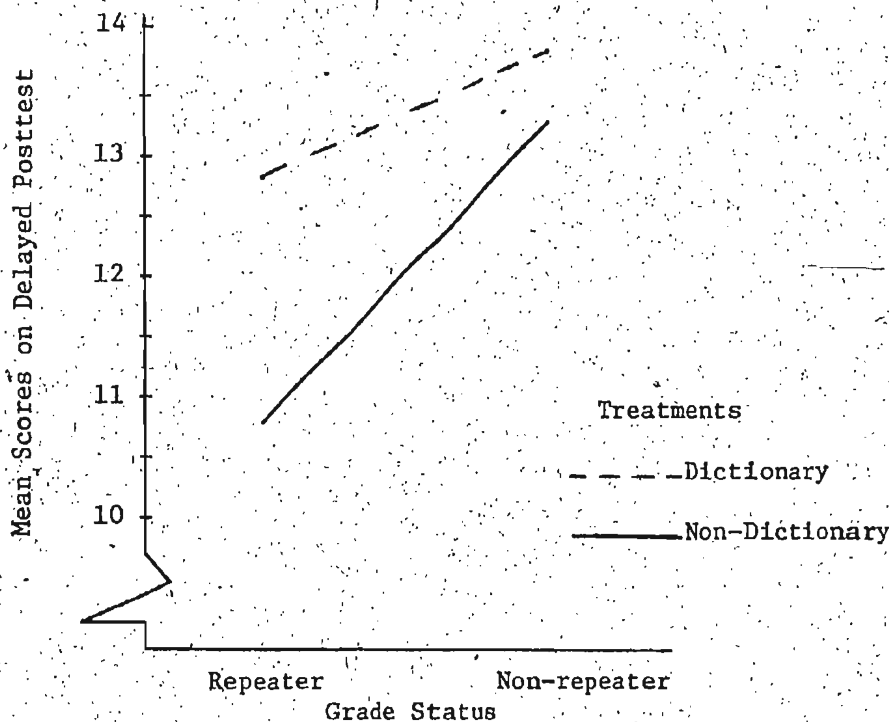


Figure 3. "Treatment by Grade Status" Interaction on the delayed posttest.

Secondary Study

Along with the main study, a secondary study was conducted with a third group, comprising only repeaters, studying geometry in the school. Williams (1974), Kane (1975), Wooldridge (1978) and others have suggested that in order to engage in successful proof-making students need to be exposed to the basic ideas of mathematical logic including the notion of a syllogism, the transitive law of implication and the law of the excluded middle.

In the main study the experimental treatment referred to as "Dictionary" incorporated each of the principles of mathematical logic previously mentioned. However, these principles were not made explicit to the students during instruction. A second experimental unit was developed to include each aspect of the experimental treatment "Dictionary" and to include instruction in the notion of a syllogism, the transitive law of implication, and the law of the excluded middle. This treatment is referred to as Dictionary and Logic for the statistical analysis discussed later in this section. A description of this unit is included in Appendix D.

The performance of the subjects in this secondary study was compared with that of the repeaters in the Dictionary and Non-Dictionary groups from the main study. All of these repeaters had been pretested prior to instruction, and posttested immediately after instruction of the respective units.

The following hypothesis was tested. There is no significant difference in achievement between groups receiving different experimental treatments. To test this hypothesis an analysis of covariance procedure was used with the pretest as a covariate.

Three important limitations of this secondary study warrant discussion. First, the results may be limited because of small group instruction. The "Dictionary and Logic" group was a small group of nine students, whereas the "Dictionary" and "Non-Dictionary" groups were part of larger classes. Second, the results may be limited because of teacher effectiveness. Two of the three groups, only, were taught by the same teacher. Third, the generalizability of the results of this secondary study are limited because of the small number of students in each experimental group.

In Table 5, the results of the analysis of covariance performed on the data in this secondary study are summarized. An F-ratio of 5.96 was obtained which was significant at the 0.01 level of significance and therefore the hypothesis was rejected. It was concluded then that there was a significant difference in achievement between treatment groups.

TABLE 5

ANALYSIS OF COVARIANCE OF ACHIEVEMENT ON THE POSTTEST BY TREATMENT WITH ACHIEVEMENT ON THE PRETEST AS A COVARIATE

Sources of Variation	SS	DF	MS	F	Significance Level
Covariate Ach. PRE	12.76	1	12.76	0.37	0.55
Main Effects					
Treatment	415.67	2	207.84	5.96	< .01
Error	767.57	22	34.89		
Total	1,196.00	25	47.84		

To determine where the significant difference between the treatments lay, a Scheffé test was carried out. Due to the fact that the Scheffé procedure is more rigorous than multiple t-tests, it was decided to employ a less rigorous significance level (.10) as suggested by Scheffé (1959). The analysis of data, summarized in Table 5, showed that the effect of the covariate was not significant.

Therefore, the actual group means were used in the calculation of the Scheffé test.

The results of this analysis indicated that students in the Dictionary and Logic group achieved significantly higher results than students in the Non-Dictionary group ($p < .10$). However, achievement differences between students in the Non-Dictionary and Dictionary groups, and those in the Dictionary and Dictionary and Logic groups were not significantly different.

A detailed summary of mean scores on the pretest and posttest by treatment group is presented in Table 6. Table 6 shows an increase in mean scores from pretest to posttest for all three treatment groups, with the Dictionary and Logic group showing the largest increase of 20.55 followed by the Dictionary group with 11.67 and the Non-Dictionary group with 2.13.

TABLE 6

MEAN ACHIEVEMENT SCORES ON THE PRETEST AND POSTTEST BY TREATMENT GROUPS OF SECONDARY STUDY

	Treatment			
	Non Dictionary	Dictionary	Dictionary and Logic	Total
Pretest	25.25 N=8	19.33 N=9	13.67 N=9	19.19 N=26
Posttest	27.38 N=8	31.00 N=9	34.22 N=9	31.00 N=26
Mean Gain	2.13	11.67	20.55	

In Chapter V, a summary of the main study is given. The results are discussed in light of the major questions stated earlier in this report. Conclusions are drawn and some recommendations for further research are presented.

CHAPTER V

SUMMARY, DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

In this chapter, a summary of the main study is given. The results are discussed in light of the major questions stated earlier in this report. Conclusions are drawn and some recommendations for further research are presented.

Summary of the Study

The purpose of this study was to design, implement, and evaluate a unit on proof in geometry, incorporating a review of fundamental concepts, an organizational scheme called a "Dictionary" and some problem-solving strategies that could be applied to proof. Answers were sought to the following questions:

(1) Does the use of the instructional unit (Dictionary) have any effect on achievement in geometry?

(2) Does the use of the instructional unit have any effect on retention?

(3) Is the effect on achievement different for repeaters than for non-repeaters?

This study was conducted using 54 Grade 10 matriculation geometry students from an all-male regional high school in urban Newfoundland. These students were assigned to one of two treatment groups and all students were taught a unit on proof in geometry. The experimental group (Dictionary) was taught the material using an instructional unit developed for this study. The control group

(Non-Dictionary) was taught this material following the program outlined in the Grade 10 geometry text used in Newfoundland when the study was carried out.

Two tests were administered to determine student achievement. The first, the immediate posttest, was given at the end of eight weeks' instruction and a review session. The second, the delayed posttest was given two months later to measure retention of the material covered during instruction. Both of these tests were constructed by a panel of high school geometry teachers and were designed to test whether the behavioral objectives of the unit had been achieved. The data were collected and analyzed using a two-factor analysis of variance procedure.

The major findings of the study were:

(1) There were no significant differences in achievement on the immediate and delayed posttests between treatment groups;

(2) Non-repeaters scored significantly higher than repeaters on the immediate posttest but not on the delayed posttest; and

(3) There was no significant interaction between the variables grade status and treatment.

<u>Discussion of Results</u>

The analysis of data indicated that there were no significant differences in achievement between treatment groups on either the immediate or delayed posttest. One possible explanation for these results is the degree of teacher effect present. Teacher variables, as researchers such as Palardy (1969) and Brophy and Good (1970) agree, do affect student achievement. Because two teachers were used in this study, there was no guarantee that the treatments were entirely different.

Kane (1975) stated that proof-making may be viewed as a complex terminal behavior. Associated with it is a set of prerequisite or subsidiary behaviors, and we should set out to teach it by systematically teaching the prerequisite behaviors one by one and by teaching how these behaviors may be combined one with another (p. 90). This procedure was adopted in the development of the instructional unit tested in the present study. In proof-making the prerequisite skills were seen as (1) knowledge of the basic definitions, theorems and concepts of elementary geometry; (2) an informal knowledge of logic; (3) an organizational process of categorizing this knowledge and some problem-solving strategies that can be employed. Miller (1969), in summarizing research on sequencing mathematical tasks, concluded that mastery of subtasks in a hierarchy can be achieved in several ways. It is possible that subjects in both the experimental and control group mastered each of the above mentioned prerequisite skills.

These findings are consistent with those of Gregory and Osborne (1975) and Ennis and Paulus (1965). Gregory and Osborne (1975), in reviewing studies reporting comparative data for students receiving and not receiving formal instruction in logic, reported equivalent growth between pretest and posttest administrations for both groups.

Ennis and Paulus (1965), on finding non-significant differences between students receiving or not receiving instruction in logic, suggested that logic might be learned by the students who did not receive formal instruction because of other school influences which would include teacher effectiveness in terms of content presentation and interaction with students. It may be the case that in this study students in both treatment groups mastered each of the above-mentioned prerequisite skills.

The analysis further indicated that mean scores for students in the Dictionary group tended to be higher than those in the control group. These results suggest that the instructional unit may be used as an alternate approach to teaching this unit on proof.

The analysis of data indicated that non-repeaters scored significantly higher than repeaters on the immediate but not on the delayed posttest. This result was not surprising. Kolesnik (1970) reported that previous academic achievement measures are strong predictors of subsequent academic performance. More specifically, Somerton (1976) found previous mathematics achievement the best predictor of subsequent performance in mathematics. Given these relationships and the fact that the repeaters used in this study had shown low mathematics achievement in the past, one would expect non-repeaters to obtain significantly higher results.

The difference in test results may be due to the nature of the tests. The immediate posttest evaluated all of the material covered during instruction. However, the delayed posttest was part of a more comprehensive term examination. It should be noted that although not significant, $p < .10$ for the delayed posttest.

Along with this study a secondary study was conducted using 26 Grade 10 matriculation geometry students who were repeating this course. The purpose of the secondary study was to incorporate into the original unit Dictionary, instruction in some of the basic concepts of logic and to evaluate the effects on achievement.

Students were assigned to one of three treatment groups. Students in the Non-Dictionary group received instruction following the program outlined in the Grade 10 geometry text at the time of this study. Students in the Dictionary group were taught the material using an instructional unit developed for this study. Students in the Dictionary and Logic group received instruction in logic and were taught the material using the instructional unit mentioned above.

Both a pretest and posttest were administered to determine student achievement. The data were analyzed using an analysis of covariance procedure.

The analysis indicated that there were significant differences in achievement between treatment groups on the posttest. It was found that there was a significant difference between the Dictionary and Logic group and the Non-Dictionary group. The results of this secondary study were encouraging in light of the results of the main study that the use of the Dictionary did not result in significant differences in achievement. However, when instruction in logic accompanied the Dictionary there were significant differences in achievement. It should be noted, however, that because of the sample size, further research is needed to substantiate these results.

The results of the analysis further indicated increases from pretest to posttest for all three treatment groups, with the Dictionary and Logic group showing greater increases than the Dictionary and Non-Dictionary groups. These results suggest that use of the Dictionary accompanied by instruction in logic may be a remedial approach to teaching proof to students that have demonstrated poor mathematical achievement in the past.

Conclusions

In summary, the data analysis for this and the accompanying secondary study led to the following conclusions:

(1) There were no significant differences in achievement in geometry between students receiving regular classroom instruction and those using the instructional unit Dictionary on both the immediate and delayed posttests. Thus the experimental treatment Dictionary was no more effective than the textbook approach in promoting higher achievement in geometry at the Grade 10 level.

(2) There were significant differences in achievement between non-repeaters and repeaters with the former scoring higher on both the immediate and delayed posttests.

(3) When the Dictionary was supplemented with instruction in logic, significant differences in achievement resulted with repeaters. However, because of the small sample size, this result is inconclusive.

Recommendations

As a result of this study, the following recommendations are made for further research:

(1) A similar study should be conducted with a larger sample. All three treatments should be used, especially to ascertain the effects of instruction in logic plus the Dictionary on the achievement of non-repeaters. The sample should also include females.

(2) A similar study should be conducted using one teacher for all groups thereby eliminating teacher variables.

(3) To examine the hierarchial nature of the sequences used in the study, a similar study should be conducted, employing some testing of subordinate skills developed during the unit.

(4) To determine if the findings of this study are generalizable to other topics, either in geometry or algebra which include proof, the Dictionary, accompanied by instruction in logic, should be expanded and similar studies should be conducted with these topics.

(5) The findings of the secondary study indicate that the incorporation of instruction in logic into the existing curriculum could potentially relieve some of the difficulties with proof experienced by students. Teachers who presently are not giving instruction in logic should re-examine this policy. Furthermore, teachers should not ignore the importance of review. The findings of this study indicate that an extensive review of prerequisite knowledge for studying proof in geometry leads to increased achievement. Teachers should not assume that students possess these prerequisite skills as evidence to the contrary exists.

BIBLIOGRAPHY

Adler, A. Mental growth and the art of teaching. The Arithmetic Teacher, 1966, 13, 576-584.

______. Mathematics and mental growth. New York: The John Day Company, 1968.

Allendoerfer, C.B. Deductive methods in mathematics. In Allendoerfer et al. (Eds.), Insights Into Modern Mathematics Twenty-Third Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1957, 65-99.

Ausubel, D.P. The transition from concrete to abstract cognitive functioning: theoretical issues and implications for education. Journal of Research in Science Teaching, 1964, 2, 261-266.

Beilin, H. The training and acquisition of logical operations. In M.R. Rosskopf, L.P.Steffe, and S. Taback (Eds.), Piagetian Cognitive-Development Research and Mathematical Education. Washington, D.C.: The National Council of Teachers of Mathematics, 1971.

Betz, W. The transfer of training with particular reference to geometry. In W.D. Rieve (Ed.), The Teaching of Geometry, Fifth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1930, 149-198.

Bloom, B.S. Learning for mastery. U.C.L.A. Evaluation Comment, 1968, 1, 49-71.

Bowling, M.J. Word chains: a simulation of proof. The Mathematics Teacher, 1977, 70, 506-508.

Briggs, L.J. Sequencing of instruction in relation to hierarchies of competence. Pittsburg: American Institutes for Research, 1968.

Brophy, J.E., & Good, T.L. Teacher's communication of differential expectations for children's classroom performance: some behavioral data. Journal of Educational Psychology, 1970, 61, 365-374.

Brown, J.L. Effects of logical and scrambled sequences in mathematical materials on learning with programmed instruction materials. Journal of Educational Psychology, 1970, 61, 41-45.

Brumfield, C. Conventional approaches using synthetic euclidean geometry. In K.B. Henderson (Ed.), Geometry in the Mathematics Curriculum. Thirty-Sixth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1973, 95-115.

Bruner, J.S. Some theories on instruction illustrated with reference to mathematics. In Theories of Learning and Instruction, 63rd Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press, 1964.

Burger, W.F. Developing an informal geometry program for middle school. A paper presented at the 1980 Annual Meeting of the N.C.T.M., Seattle, Washington.

_______. Thought level in geometry: Interim Report of the Study Assessing Children's Development In Geometry. Prepared for NSF-SEDR Project Directors' Meeting, Washington, D.C., 1981.

Callahan, L.G., & Robinson, M.L. Task-Analysis procedures in mathematics instruction of achievers and underachievers. School Science and Mathematics, 1973, 73, 578-584.

Campbell, D.T., & Stanley, J.C. Experimental and quasi-experimental design for research. Chicago: Rand McNally and Co., 1963.

Carpenter, T.P., Coburn, T.G., Reys, R.E., & Wilson, J.W. Results and implications of the NAEP mathematics assessment: elementary school. Arithmetic Teacher, 1975, 22 (6), 438-450.

Carroll, C.A. Low achievers' understanding of logical inference forms. In M.F. Rosskopf (Ed.), Children's Mathematical Concepts: Six Piagetian Studies in Mathematics Education. New York: Teachers College Press, 1975, 173-208.

Case, D., & Collinson, J.M. The development of formal thinking in verbal comprehension. Research Journal of Education and Psychology, 1962, 32, 103-111.

Cooney, T.J., & Henderson, K.B. Ways mathematics teachers help students organize knowledge. Journal for Research in Mathematics Education, 1972, 2 (1), 21-31.

Dean, J.E., & Moore, G.E. Functional Mathematics-Intermediate Four. Toronto: John C. Winston Co. Ltd., 1958.

Donaldson, M. A study of children's thinking. London: Tavistock Publications, 1963.

Duncker, Karl. On problem-solving. Psychological Monographs, 1945, 58, 1-111.

Dunne, J.A. Discovery, creativity, and school mathematics: a review of research. Educational Review, 1976, 28, 102-117.

Ennis, R.H. Children's ability to handle Piaget's propositional logic: a conceptual critique. Review of Educational Research, 1975, 45 (1), 1-41.

_______, & Paulus, D.H. Critical thinking readiness in grades 1-12; phase 1: Deductive logic in adolescence. Ithaca: Cornell University, 1965.

Evans, K.M. Attitudes and Interests in Education. London: Routledge and Kegan Paul, 1965.

Farrell, M.A. Geometry - a psycho-logical learning strategy. School Science and Mathematics, 1971, 71, 139-142.

_______. The formal stage: a review of research. Journal of Research and Development in Education, 1969, 3, 111-118.

Fawcett, H.P. The Nature of Proof. Thirteenth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1938, 1-15.

Fishbein, E., Pampu, I., & Minzat, I. Effects of age and instruction on combinatory ability in children. British Journal of Educational Psychology, 1970, 40, 261-270.

Flavell, J.H. The developmental psychology of Jean Piaget. Princeton, N.J.: D. van Nostrand, 1963.

Gage, N.L. Theories of teaching. In Theories of Learning and Instruction, 63rd Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press, 1964.

Gagné, R.M. Learning and proficiency in mathematics. The Mathematics Teacher, 1963, 56, 620-626.

_______. The acquisition of knowledge. Psychological Review, 1962, 69, 355-365.

_______. Learning hierarchies. Educational Psychologist, 1968, 6 (1), 3-6.

_______. The Conditions of Learning. New York: Holt, Rinehart and Winston, 1965.

_______. The Conditions of Learning (2nd ed.). New York: Holt, Rinehart and Winston, Inc., 1970.

_______, Mayor, J.R., Garstens, H.L., & Paradise, N.E. Factors in acquiring knowledge of a mathematical task. Psychological Monographs, 1962, 76 (526).

_______, & Paradise, N.E. Abilities and learning sets in knowledge acquisition. Psychological Monographs, 1961, 75 (518).

_______. The Conditions of Learning (3rd ed.). New York: Holt, Rinehart and Winston, Inc., 1977.

Geddes, D., Fuys, D., Lovett, C.J., & Fischter, R. An Investigation of the Van Hiele Model of Thinking in Geometry Applied to Inner City Adolescents. A paper presented at the 1981 Annual Meeting of the N.C.T.M., St. Louis, Missouri, 1981.

Glaser, R. Trends and research questions in psychological research on learning and schooling. Educational Researcher, 1979, 8, 6-14.

Greeno, J.G. Understanding and procedural knowledge in mathematics instruction. Educational Psychologist, 1978, 12, 262-283.

Gregory, J.W., & Osborne, A.R. Logical reasoning ability and teacher verbal behavior within the mathematics classroom. Journal for Research in Mathematics Education, 1975, 6, 27-36.

Hallerburg, A.E. A forum of proof. The Mathematics Teacher, 1971, 64 (3), 203-214.

Hazlitt, V. Children's thinking. British Journal of Psychology, 1930, 20, 354-361, as cited in Lester, F.K., Developmental aspects of children's ability to understand mathematical proof. Journal for Research in Mathematics Education, 1975, 6 (1), 15-25.

Heimer, R.T. Conditions of learning in mathematics: sequence theory development. Review of Educational Research, 1969, 39 (4), 493-508.

Henderson, K.B., & Pingry, R.E. Problem-solving in mathematics. In H.F. Fehr (Ed.), The Learning of Mathematics. Its Theory and Practice. Twenty-First Yearbook of The National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1953.

Hill, S.A. A study of logical abilities of children. Unpublished doctoral dissertation as cited in Lester, 1975.

______. A study of logical abilities of children. Unpublished doctoral dissertation, Stanford University, 1961.

Hoffer, A. Geometry is more than proof. The Mathematics Teacher, 1981, 74, 11-18.

Holloway, G.E.T. An introduction to the child's conception of geometry. London: Routledge and Kegan Paul, 1960.

Howe, A. Formal operational thought and the high school science curriculum. Paper presented at the annual meeting of the National Association for Research in Science Teaching. Chicago, April, 1974.

Kane, R.B. On the proof-making task. The Mathematics Teacher, 1975, 68 (2), 89-94.

Jurgensen, R.C., Maier, J.E., & Donnelly, A.J. Modern basic geometry. Ontario: Thomas Nelson & Sons (Canada) Limited, 1973.

Kolesnik, W.B. Educational Psychology. New York: McGraw Hill, Inc., 1970.

Lawson, A.E. The development and validation of a classroom test of formal reasoning. Journal of Research in Science Teaching, 1978, 15, 11-24.

Lesh, R. Recent cooperative research concerning the acquisition of spatial and geometric ideas. Journal for Research in Mathematics Education, 1977, 7, 317-320.

Lester, F.K. Developmental aspects of children's ability to understand mathematical proof. Journal for Research in Mathematics Education, 1975, 6 (1), 15-25.

Lovell, K. A follow-up study of Inhelder and Piaget's the growth of logical thinking. British Journal of Psychology, 1961, 57, 143-153.

Lunzer, E. Some points of Piagetian theory in the light of experimental criticism. Journal of Child Psychology and Psychiatry, 1960, 1, 191-202.

MacLean, W.B., Mumford, D.L., Bock, R.W., Hazell, D.N., & Kaye, G.A. Secondary School Mathematics - Grade 10. Copp, Clarke, 1958.

McAloon, S.M. An exploratory study on teaching units in logic to grades three and six. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, Michigan, 1969.

McGrath, J. A study of some of the most difficult topics identified in teaching mathematics in grades four to eight in three elementary schools. Unpublished manuscript, Memorial University of Newfoundland, 1979.

Merrill, M.D. Correction and review on successive parts in learning a hierarchical task. Journal of Educational Psychology, 1965, 56, 225-234.

Miller, H.R. Sequencing and prior information in linear programmed instruction. AV Communication Review, 1969, 17, 63-76.

Niedermeyer, F.C. The relevance of frame sequence in programmed instruction: an adddition to the dialogue. AV Communication Review, 1968, 16, 301-317.

Polardy, J. What teachers believe - what children achieve. Elementary School Journal, 1969, 69, 370-374.

Phillips, E.R., & Kane, R.B. Validating learning hierarchies for sequencing mathematical tasks in elementary school mathematics. Journal for Research in Mathematics Education, 1973, 3 (5), 141-151.

Piaget, J. Development and learning. Journal of Research in Science Teaching, 1964, 2, 176-186.

______, & Inhelder, B. The growth of logical thinking from childhood to adolescence. New York: Basic Books, 1958.

Randall, D.L. Examination of the hereditary assumptions underlying Piaget's theory of the development of intellectual structure. Unpublished doctoral dissertation, University of Colorado, 1967.

Reed, H.B. Factors influencing the learning and retention of concepts. Journal of Experimental Psychology, 1946, 36, 71-87, 166-179, 252-261.

Reeve, W.D. The teaching of geometry. In W.D. Reeve (Ed.), The Teaching of Geometry, Fifth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1930, 1-29.

Renner, J.W., & Lawson, A.E. Intellectual development in preservice elementary school teaching: an evaluation. Journal of College Science Teaching, 1975, 5 (2), 157-164.

Roberge, J.J. Recent research on the development of children's comprehension of deductive reasoning schemes. School Science and Mathematics, 1972, 72, 197-200.

Roberts, B. A study of the time spent on teaching geometry. Unpublished manuscript, Memorial University of Newfoundland, 1980.

Robinson, E. Strategies of proof. The Mathematics Teacher, 1963, 56, 637-645.

Robinson, G.E. The role of elementary school mathematics. The Arithmetic Teacher, 1966, 13 (1), 3-10.

Robitaille, D. Why are we teaching high school geometry? Saskatchewan Mathematics Teachers Society News/Journal, 1973, 11 (4), 22-26.

Scandura, J.M. A theory of mathematical knowledge: can rules account for creative behavior? Journal for Research in Mathematics Education, 1971, 2, 183-196.

Scheffé, H. A method for judging all contrasts in the analysis of variance. Biometrika, 1953, 40, 87-104.

______. The analysis of variance. New York: John Wiley and Sons, 1959.

Shaw, M.C., & McCaren, J.T. The onset of academic underachievement in brighter children. Journal of Educational Psychology, 1960, 51, 103-108.

Sheehan, D.J. The effectiveness of concrete and formal instructional procedures with concrete and formal-operational students. Unpublished doctoral dissertation, State University at Albany, 1970.

Simon, H.A. Learning with understanding. Mathematics Education Report. ERIC Information Analysis Center for Science, Mathematics and Environmental Education, The Ohio State University, June, 1975.

Smith, D.E., & Reeve, W.D. The teaching of junior high school mathematics. New York: Ginn & Co., 1926.

Somerton, E.J. The predictive ability of certain criteria of success in grade ten honours mathematics. Unpublished manuscript, Memorial University of Newfoundland, 1977.

Stone, M.A., & Ausubel, D. The intersituational generality of formal thought. Journal of Genetic Psychology, 1969, 115, 169-180.

Stroud, J.B. Psychology in education. New York: Longmans, Green and Co., 1946.

Suppes, P. Mathematical concept formation in children. American Psychologist, 1966, 21, 139-150.

_______, & Binford, F. Experimental teaching of mathematical logic in the elementary school. Arithmetic Teacher, 1965, 12 (3), 187-195.

Tomlinson-Keasey, C. Formal operations in females from 11 to 54 years of age. Developmental Psychology, 1972, 6 (2), 364.

Usiskin, Z. What should not be in the algebra and geometry curricula of average college-bound students? The Mathematics Teacher, 1980, 73, 413-424.

van Engen, H. The formation of concepts. In H.F. Fehr (Ed.), The Learning of Mathematics: Its Theory and Practice. Twenty-first Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1953, 69-98.

_______. Strategies of proof in secondary mathematics. The Mathematics Teacher, 1970, 63 (12), 637-645.

van Dormolen, J. Learning to understand what giving a proof really means. Educational Studies in Mathematics, 1977, 27-34.

van Hiele, P.M., & van Hiele-Geldof, D. A method of initiation into geometry at secondary schools. In H. Freudenthal (Ed.), Report on Methods of Initiation into Geometry. J.B. Walters, Groningen, 1958.

van Hiele, P.M. Levels of thinking, how to meet them, how to avoid them. Paper presented at the Research Pre-session sponsored by the Special Interest Group for Research in Mathematics Education of AERA and the Research Advisory Committee of NCTM.

Williams, E.R. Logic, proof, and the role of rigor in school mathematics. Teaching Mathematics, 1974, 2 (2), 5; 11.

Wirszup, A. Breakthrough in the psychology of learning and teaching geometry. In J.L. Martin (Ed.), Space and Geometry: Papers from a Research Workshop. Columbus: ERIC Center for Science, Mathematics and Environmental Education, 1976.

Wooldridge, G.K. Geometry: some areas of concern. Teaching Mathematics, 1978, 5 (2), 10-11.

APPENDIX A

IMMEDIATE POSTTEST

Directions: Place the answers to each question on the ruled paper provided.

1. In $\triangle XYZ$, $m\angle X = 40$, and $m\angle Y = 100$. What is the $m\angle Z$? 1. ____________

2. $\triangle ABC$ is an equilateral triangle. What is $m\angle B$? 2. ____________

3. The complement of $\angle LKM$ is $\angle MKN$. If $m\angle LKM = 60$, what is $m\angle MKN$? 3. ____________

4. If $\angle R \cong \angle S$ and $\angle S \cong \angle T$, what property tells you that $\angle R \cong \angle T$? 4. ____________

5. The four postulates that can be used for proving two triangles congruent are SSS, SAS, ASA and __?__. 5. ____________

6. If $\overrightarrow{QR}$ bisects $\angle SQT$ and $M\angle RQT = 25^{\circ}\ 17'$, what is $m\angle SQT$? 6. ____________

7. $\triangle ABC$ contains two sides that are congruent. What is the name given to triangles like $\triangle ABC$? 7. ____________

8. On $\overline{PQ}$, point M lies between P and Q. Therefore, PM + MQ = __?__. 8. ____________

9. In plane, P, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ and $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$. Therefore $\overleftrightarrow{AB}$ __?__ $\overleftrightarrow{EF}$. 9. ____________

10. $\angle B$ and $\angle M$ are vertical to each other. $m\angle B = 3x$ and $m\angle M = x + 40$. Therefore, x = __?__. 10. ____________

11. The measure of $\angle Y$ is 20. $\angle X$ is a supplement of $\angle Y$. What is $m\angle X$? 11. ____________

12. What is the sum of the measures of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ if ABCD is a quadrilateral? 12. ____________

13. Point B lines in plane P outside $\overleftrightarrow{AC}$. How many lines can be drawn through B and parallel to $\overleftrightarrow{AC}$? 13. ____________

14. In any right triangle the side opposite the right angle is called the __?__. 14. ____________

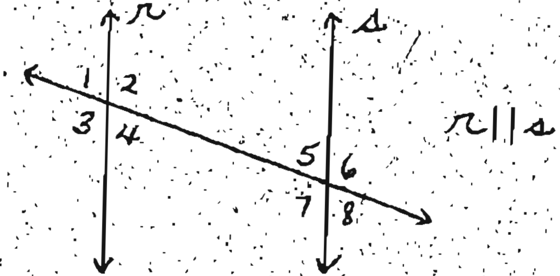
15. (a) Examine the figure and tell why:

(i) $\angle 5 \cong \angle 8$

(ii) $\angle 1 \cong \angle 8$

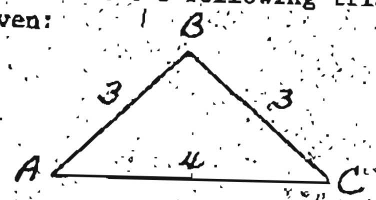
(iii) $\angle 4 \cong \angle 5$

(iv) $\angle 3 \cong \angle 7$



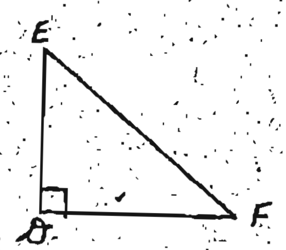
(b) Give a name to each of the following triangles according to the information given:

Example:



ABC is an isosceles triangle.

(i)



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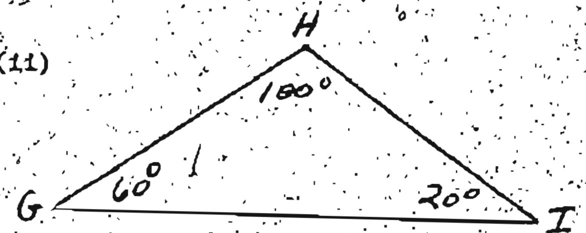

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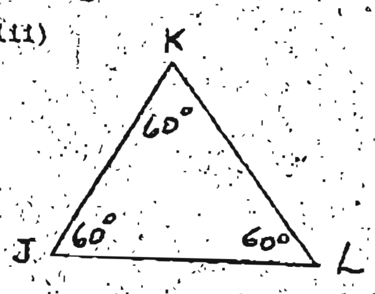

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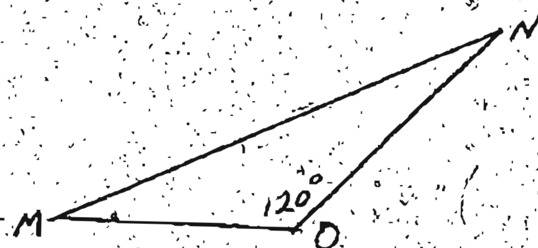


(c) What postulate is illustrated by each equation?

(i) $ab = ba$

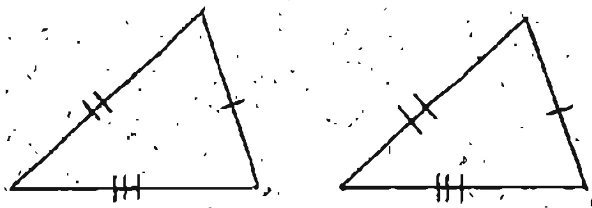
(ii) $a(b + C) = ab + ac$

(iii) If $a = b$ and $b = c$, then $a = c$

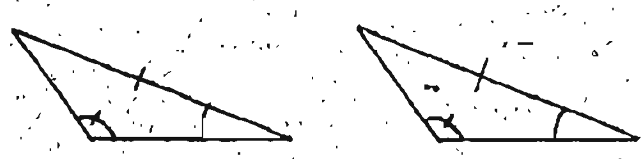
(iv) If $a = b$, then $a - c = b - c$

16. Examine each pair of triangles that follow and tell which postulate of theorem that you would use to prove them congruent, according to the information given. For right triangles, use only right triangle postulates or theorems.

(i)



(ii)



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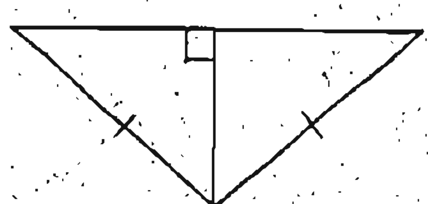
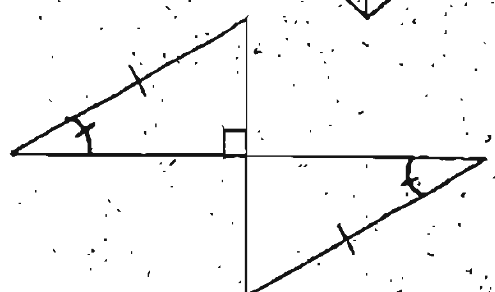
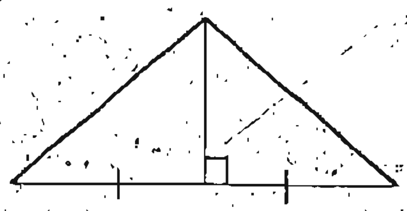


(iv)

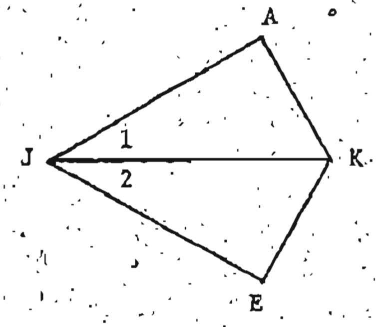


(v)

(vi)



17. Complete the following proof by providing reasons for the given statements:



Given: $\overline{JA} \perp \overline{KA}$

$\overline{JE} \perp \overline{KE}$

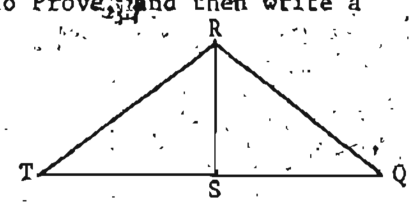
$\angle 1 \cong \angle 2$

To Prove: $\triangle JAK \cong \triangle JEK$

Proof:

Statements	Reasons
1. $\overline{JK} \cong \overline{JK}$	1.
2. $\angle 1 \cong \angle 2$	2.
3. $\overline{JA} \perp \overline{KA}$; $\overline{JE} \perp \overline{KE}$	3.
4. $\angle A$ is a rt. $\angle$; $\angle E$ is a rt. $\angle$	4.
5. $\triangle JAK$ and $\triangle JEK$ are rt. $\triangle$s.	5.
6. $\triangle JAK \cong \triangle JEK$	6.

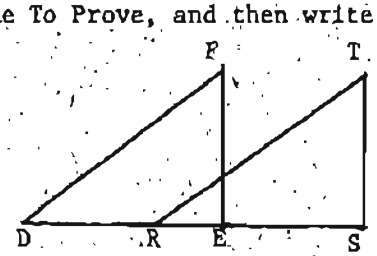
18. (a) Copy the Figure, the Given, and the To Prove, and then write a proof in two-column form.



Given: $\overline{RS} \perp \overline{TQ}$; $\overrightarrow{RS}$ bisects $\angle TRQ$
RT = 40; RQ = 40

To Prove: $\triangle RST \cong \triangle RSQ$

(b) Copy the Figure, the Given, and the To Prove, and then write a proof in two-column form.

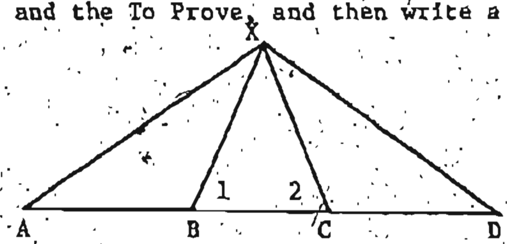


Given: $\overline{DF} \cong \overline{RT}$; $\overline{DE} \cong \overline{RS}$

$\overline{FE} \perp \overline{DS}$; $\overline{TS} \perp \overline{DS}$

To Prove: $\triangle DEF \cong \triangle RST$

(c) Copy the Figure, the Given, and the To Prove, and then write a proof in two-column form:



Given: $\overline{AX} \cong \overline{DX}$; $\overline{AB} \cong \overline{DC}$

To Prove: $\angle 1 \cong \angle 2$

APPENDIX B

DELAYED POSTTEST

Directions: Place the answers to each question on the ruled paper provided.

1. If $\triangle XYZ \cong \triangle PQR$, then $\overline{RP} \cong$ ___?___.

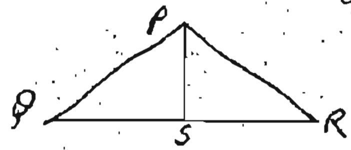
2. The measure of each angle of an equilateral triangle is ___?___.

3. In $\triangle ABC$, if $\angle B \cong \angle C$, then $\overline{AC} \cong$ ___?___.

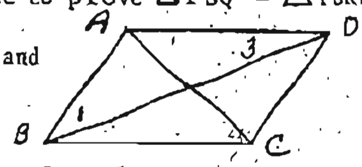
4. In an isosceles triangle, the measure of one of the base angles is 46. What is the measure of the vertex angle?

5. Given $\overline{PS} \perp \overline{QR}$

   $\overline{PQ} \cong \overline{PR}$



   Which postulate or theorem would you use to prove $\triangle PSQ \cong \triangle PSR$?



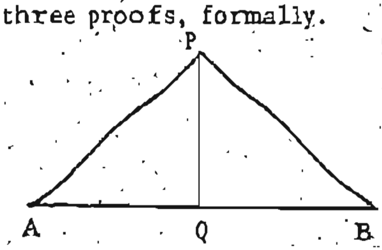
6. In the figure, if $\overline{AB} \parallel \overline{CD}$, $m\angle 1 = 58$ and $m\angle CDA = 86$, then $m\angle 3 =$ ___?___.

7. Do any two of the following three proofs, formally.

   (a) Given: $\overline{PQ} \perp \overline{AB}$

   $\overline{AQ} \cong \overline{BQ}$

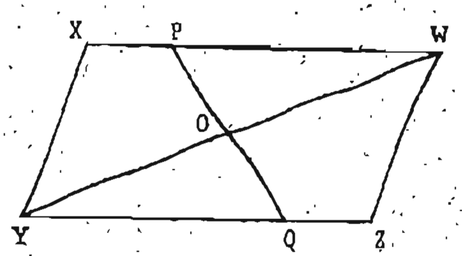
   Prove: $\overline{PA} \cong \overline{PB}$



   (b) Given: $\overline{XW} \parallel \overline{YX}$

   $\overline{OY} \cong \overline{OW}$

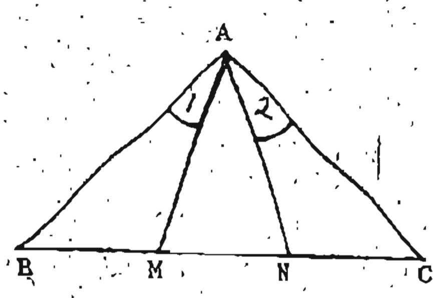
   Prove: $\triangle POW \cong \triangle QOY$



   (c) Given: $\overline{AB} \cong \overline{AC}$

   $\angle 1 \cong \angle 2$

   Prove: $\overline{AM} \cong \overline{AN}$



APPENDIX C

INSTRUCTIONAL OBJECTIVES

1. Given the statement $\triangle ABC \cong \triangle ETK$, the student will be able to write three statements about congruent sides and three statements about congruent angles of these triangles.
2. The student will be able to distinguish between the statement $\triangle ABC \cong \triangle ETK$ and the statement $\triangle ABC \cong \triangle EKT$.
3. The student will be able to state and apply the S.S.S., the S.A.S., the A.S.A., the A.A.S., and the H.L.R. Postulates.
4. The student will be able to state and apply the L.L.R., the H.A.R. and the L.A.R. theorems.
5. The student will be able to state four ways to prove triangles congruent.
6. The student will be able to state and identify four ways to prove right triangles congruent.
7. Given a diagram and sufficient information, the student will be able to apply an appropriate method to prove two triangles congruent.
8. Given the statements for a particular proof, the student will be able to supply the reasons.
9. The student will be able to select two triangles to prove congruent as a means of proving that two segments or two angles are congruent.
10. The student will be able to prove that two segments or two angles are congruent, using the method outlined in objective number 9.
11. The student will be able to state and apply the isosceles triangle theorem.
12. The student will be able to identify each of the definitions, and theorems used in the instructional unit.

APPENDIX D

The instructional unit called "Dictionary + Logic" was designed to give explicit instruction in some of the basic principles of logic. These principles were demonstrated under the following headings: (1) the Law of the Syllogism; (2) the If-Then Relationship; and (3) Deductive Proofs involving Applications of the Law of Detachment.

The purpose of this first section on logic was to introduce students to the idea of a syllogism and certain valid and invalid inference schemes as illustrated by the following exercise:

Exercise: Consider the following syllogism:

Many policemen are Irish.
Michael O'Shea is Irish.
Michael O'Shea is a policeman.

Identify the major premise, the minor premise, and the conclusion, and decide if the conclusion is valid.

The purpose of the second section, the if-then relationship, was to introduce students to writing syllogisms in the if-then form. For example, the syllogism, All dogs are vertebrates: Fido is a dog: Fido is a vertibrate, could be written: if all dogs are vertebrate and Fido is a dog, then Fido is a vertebrate.

The third section introduced the student to deductive reasoning and proofs applying the Law of Detachment. For example: If we are given the following statements:

1. If it is raining, then it is cold.

2. It is raining.

then the inescapable consequent of these two statements is the conclusion, It is cold. In the above argument we say we have proved or deduced the statement: It is cold. That is, we have proved that the derived sentence follows logically from the given sentences.

In symbols the argument may be represented as follows:

Premises: 1. $P \rightarrow Q$

2. P

Logical Consequent: Q (Law of Detachment)

An example of a proof applying the Law of Detachment follows:

Given: 1. $P \rightarrow Q$

2. P

3. $Q \rightarrow R$

Prove R

Proof:

Statements	Reasons
1. $P \rightarrow Q$	1. Given
2. P	2. Given
3. Q	3. Law of Detachment and by 1 and 2
4. $Q \rightarrow R$	4. Given
5. R	5. Law of Detachment and by 3 and 4

These proofs were then related to the "Dictionary" used in the main part of this study. The difference between the Dictionary group and the Dictionary + Logic group was that the latter were explicitly shown how each sequence in the "Dictionary" was developed.

The materials used in teaching the basic principles were adapted from Secondary School Mathematics - Grade Ten (MacLean, W.B., Mumford, D.L., Bock, R.W., Hazell, D.N., and Kaye, G.A., 1964) and Functional Mathematics - Intermediate Four (Dean, J.E., and Moore, G.E., 1958).

CHAPTER I

SYMBOLS AND ABBREVIATIONS USED IN GEOMETRY

SYMBOLS

$\angle$, $\angle s$	angle, angles
$\odot$, $\odot s$	circle, circles
$\equiv$	congruent, is congruent to
$=$	equals, is equal to
$\neq$	(is) not equal to
$\parallel\mkern-3mu\mid$	(is) similar to
$\therefore$	therefore
$\because$	since, because
$\parallel$	(is) parallel to
$\perp$	(is) perpendicular (to)
$\triangle$, $\triangle s$	triangle, triangles
$\parallel$gm.	parallelogram
$^\circ$, $'$, $''$	degrees, minutes, seconds
$>$	is greater than
$<$	is less than
$\geq$	is equal to or greater than
$\leq$	is less than or equal to
AB^2	the square drawn on the line AB
α, β	alpha, beta — Greek letters used for the measure of angles.
γ, δ	gamma, delta
θ, ϕ	theta, phi

ABBREVIATIONS

adj.	adjacent
alt.	alternate
Ax.	axiom
comp.	complementary
Const.	construction
Cor.	corollary
corr.	corresponding
Def.	definition
Ex.	exercise
ext.	exterior
Fig.	figure
Hyp.	hypothesis
Iden.	identity
int.	interior
isos.	isosceles
opp.	opposite
Post.	postulate
Prop.	proposition
quad.	quadrilateral
rect.	rectangle
req.	required
sq.	square
st.	straight
subst.	substitution
supp.	supplementary
trans.	tranversal
trap.	trapezium, trapezoid
vert.	vertical (ly)

s.a.s. = s.a.s. If two triangles have two sides and the contained angle of one triangle respectively equal to two sides and the contained angle of the other triangle, the triangles are congruent.

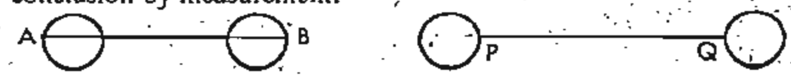
a.s.a. = a.s.a. If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, the triangles are congruent.

s.s.s. = s.s.s. If three sides of one triangle are respectively equal to the three sides of another triangle, the triangles are congruent.

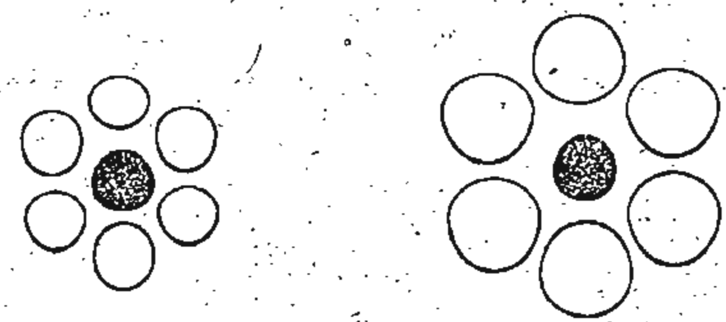
3 BASIC ASSUMPTIONS REASONING AND PROOF

39 YOU CAN'T BELIEVE YOUR OWN EYES!

Many students when first introduced to deductive geometry fail to see the need for *proof*. They are prepared to accept the equality of the base angles of an isosceles triangle because, on examination, the angles *look* equal. Yet it is well-known that two witnesses of an automobile accident, both equally honest and sincere, may give widely different versions of what they saw. It would seem that the statement, "I saw it with my own eyes," is open to question if one attempts to use it as a foundation for proof. This is true in geometry as in other phases of life. Consider the following diagrams. Use your own eyes to arrive at a conclusion and then test this conclusion by measurement.



Which is longer, line segment AB or line segment PQ?



Which of the grey circles is the larger?

Examine the diagrams on the following page.

(i) In the given figure taken as a whole, which is greater, the vertical dimension or the horizontal dimension?

(ii) Which line segment appears to be longer, MN or HK? Which actually is?

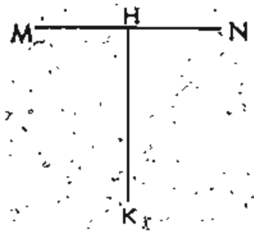
(iii) Are PQ and AB segments of straight lines or of curves?

(iv) How does the length of AK compare with that of KB?

(v) Rank the line segments in grey, x, y, z, in order of length.

(vi) Which of the lines below the grey rectangle is a continuation of PQ?

(i)



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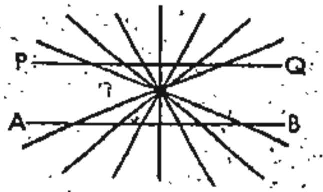

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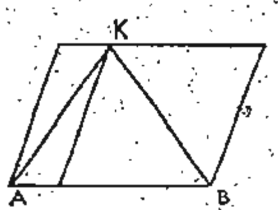

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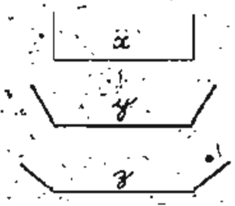

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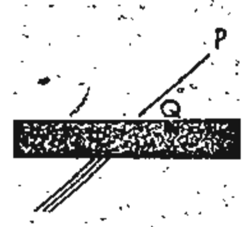


(vi)

The illustrations given above are all well-known optical illusions. Doubtless other examples will occur to you, such as the effect obtained when a pencil standing in a glass half full of water is viewed from the side. These illusions have practical applications. In architecture, stone columns are made slightly convex so that they will *appear* straight. Without analysing the reason, most tall girls instinctively avoid dresses which feature vertical stripes in the material. For the same reason a short chubby person will avoid materials with a prominent horizontal stripe. Of course we know that a dress cannot actually make a tall person taller, or a stout person fatter, but our eyes deceive us. Therefore, in geometry, it is not sufficient to say that two angles are equal, or two lines are equal, because they *look* equal. *We must have proof.*

In the two preceding chapters we have emphasized the fact that conclusions arrived at from inductive reasoning are only *probable* conclusions or *possible* conclusions and have not been proven. There are two reasons for this: (i) the conclusions depend upon measurement and observation, neither of which can be absolutely accurate, and (ii) it is

impossible to examine inductively *all possible cases* before arriving at a conclusion. As soon as a single exception is found to a conclusion arrived at by inductive reasoning, that conclusion must be false.

An interesting example of the hidden dangers in inductive reasoning occurs in the study of perfect numbers, a subject which interested the Greeks even before the time of Euclid. A perfect number is a whole number which is equal to the sum of all its possible integral factors, including 1, but not including the number itself. Thus,

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

Therefore 6, 28 and 496 are perfect numbers. About four hundred years after Euclid, Nicomachus of Gerasa noticed that there was one perfect number with one digit, one with two digits and one with three digits. He proceeded to find a fourth perfect number, 8128, which has four digits. He therefore concluded that there should be one perfect number with five digits, one with six digits and so on. This seems reasonable, but the generalization is *not* true, for the next perfect number after 8128 is 33555336.

40 INDUCTION AND DEDUCTION

The word *induction* comes from the Latin *in, duco,* (I lead into), so induction is the method of reasoning in which we consider a number of cases and arrive at (or "lead into") a general rule. Thus, if each member of the class measures each of the interior angles of a triangle and finds that their sum is approximately 180°, we might reason inductively that the sum of the angles of a triangle is 180°. In Euclidean geometry this is correct, but since no measurement is really exact, we cannot say that we have *proven* the statement.

In *deduction* (Latin, lead away from), we reason from the general statement to a particular case. Thus, if we have established (proven) that the sum of the interior angles of a triangle is always 180°, and we meet a new triangle, we may say, quoting our general statement as authority, that the interior angles of this triangle will also have a sum of 180°. Thus the system of deductive geometry consists of establishing a number of general statements (propositions) for which we must develop proofs without the use of measurement, and then applying these principles to prove relations in particular problems.

As we pointed out in the section on why we study geometry, it is in the thinking which leads to these proofs, and in the analysis of the problems presented, that the great value of deductive geometry lies. The basic thought-processes of geometry can be applied in other fields. They should enable the student to be critical of the assumptions made in arguments advanced either by himself or by others in the fields of education, politics, social welfare, advertising, international relations and so on. When the announcer in dulcet tones affirms that "More people clean their teeth with Kleendent toothpaste than any other kind," the student trained in analytical thinking will ask many questions. What proof have we that the statement of the announcer is true? If it is true, why *do* more people use Kleendent? Is it better or is it cheaper, or is it better advertised? Because a large number of people do a certain thing, does that prove it is the best thing to do? Does the value of brushing one's teeth lie in the brushing or in the particular dentifrice used?

The subject matter of geometry provides us with good material for developing a thinking process, because the elements of geometry (such as lines, points, areas and so on) are not calculated to arouse emotions and prejudices which might interfere with our reasoning.

41 ELEMENTARY LOGIC—THE SYLLOGISM

Logic is an analysis of the deductive thought process. It presupposes that we can all recognize valid reasoning when it is reduced to its essentials. The skeleton of deductive reasoning is the *syllogism*, a form of argument which consists of a *major premise*, a *minor premise* and a *conclusion*. The following is an example.

Major premise: All dogs are vertebrates.

Minor premise: Fido is a dog.

Conclusion: Therefore, Fido is a vertebrate.

It should be pointed out that logic is more concerned with the *validity* of the argument than with the *truth* of the conclusion. There is a difference between being valid and being true. If you are shopping at the grocteria and the cashier happens to push the wrong key on the cash register, she may ask you to pay more than the true value of your purchases as indicated by the prices on the articles. When you object, we might imagine the following conversation.

Cashier (to adding machine): You gave me the wrong total.

Adding Machine (to cashier): I gave you a *valid* total for the amounts you gave me. You gave me *wrong* amounts.

Logic is concerned with the fact that from the given amounts a *valid* total was obtained, even if it was not the *true* amount. In the exercise which follows, we are concerned only with arriving at a conclusion which is valid on the basis of the given assumptions, and not with the truth or falsity of the assumptions.

Exercise 17

Each of the following questions consists of three statements. Irrespective of the truth of the two initial statements, decide in each question whether or not the third statement is a *valid* conclusion from the other two.

1. All birds have wings.
All flies have wings.
All birds are flies.

3. All fish need water.
I need water.
I am a fish.

5. Each spider has eight legs.
Each team of horses has eight legs.
Each team of horses is a spider.

7. All shelters in which people live are dwellings.
The shelter in which I live is a tent.
A tent is a dwelling.

9. Mary is a girl.
Mary has short hair.
All girls have short hair.

11. All four-sided figures are quadrilaterals.
A rhombus is a four-sided figure.
A rhombus is a quadrilateral.

2. All fish are vertebrates.
A salmon is a fish.
A salmon is a vertebrate.

4. The only animals that need water are fish.
I am an animal that needs water.
I am a fish.

6. All figures with only three sides are triangles.
This figure is a figure with only three sides.
This figure is a triangle.

8. All ducks can swim.
All fish can swim.
Fish are ducks.

10. Students eat sunflower seeds.
Birds eat sunflower seeds.
Birds are students.

12. All fish live in the water.
A whale lives in the water.
A whale is a fish.

13. The only animals living in water are fish.
A whale is an animal living in water.
A whale is a fish.

14. All school rooms have windows.
This room has windows.
This room is a school room.

15. All Canadians between six and sixteen years of age must attend school.
John attends school.
John must be between six and sixteen years of age.

16. All Canadians between six and sixteen years of age must attend school.
Roy is a Canadian who is ten years old.
Roy must attend school.

17. A square is a figure with four sides.
A trapezium is a figure with four sides.
A trapezium is a square.

18. All people who live in Vancouver live in British Columbia.
Mr. Jones lives in British Columbia.
Mr. Jones lives in Vancouver.

19. Any triangle with all sides different is a scalene triangle.
This triangle has all sides different.
This triangle is a scalene triangle.

20. All people who live in Montreal live in Quebec.
M. Letendre lives in Montreal.
M. Letendre lives in Quebec.

42 CONDITIONS FOR A VALID CONCLUSION

In Exercise 17 each question consisted of three statements. Each statement was a complete sentence, with a *subject* and a *predicate*. Read the statements again, picking out the subject and the predicate in each statement.

Some of the conclusions in Exercise 17 were valid and some invalid. If you will examine the questions in which the conclusions were valid, you will find that they have the following in common.

1. The major premise makes a general statement about a subject.
2. The subject of the minor premise is a *part* of the subject of the major premise.
3. The conclusion involves the *subject* of the minor premise and the *predicate* of the major premise.

In brief, what is true for the whole is true for a part of it; but what is true of part of a subject is true only of that part of it.

Exercise 18

A

1. Consider the following syllogisms.

(a) Many policemen are Irish.
Michael O'Shea is Irish.
Michael O'Shea is a policeman.

(b) All policemen are Irish.
Michael O'Shea is Irish.
Michael O'Shea is a policeman.

(c) All policemen are Irish.
Michael O'Shea is a policeman.
Michael O'Shea is Irish.

(i) In which of the above is the major premise *not* a general statement. Is the conclusion valid?

(ii) In two of the above, the subject of the minor premise is *not* a part of the subject of the major premise. Which two? Are the conclusions valid?

(iii) Which of the above is the only valid syllogism?

B

Refer to the answers you gave to the questions in Exercise 17.

2. List the numbers of the syllogisms in which the subject of the minor premise is a part of the subject of the major premise.

Compare this list with the numbers of the syllogisms which you considered valid.

3. List the numbers of the syllogisms which contain *two* general statements, rather than one general and one particular.

Compare this list with your list of valid syllogisms.

4. In question 4, Exercise 17, while the reasoning is *valid* the conclusion is not *true*. Why is it not true?

Find a second syllogism in Exercise 17 in which you believe the conclusion is valid but *not* true.

5. In question 15, could John be twenty years old? Which test of a valid syllogism does this one violate?

6. The syllogisms in questions 18 and 20 appear to be much alike. What is the difference between them which makes one conclusion valid and the other invalid?

43 SUMMARY—VALIDITY AND TRUTH OF CONCLUSIONS

1. The truth of a conclusion depends on:

 (i) the truth of the premises.
 (ii) the validity of the logic.

2. The logic will *not* be valid

 (i) if the major premise is not a *general* statement, that is, if there are any exceptions to it.
 (ii) if the minor premise repeats the predicate instead of the subject of the major premise.
 (iii) if the minor premise fails to be a part of the *subject* of the major premise.
 (iv) if the conclusion does not contain the *subject* of the *minor* premise and the *predicate* of the *major* premise.
 (v) if the syllogism contains two major premises.
 (vi) if the syllogism contains two minor premises.

Exercise 19

State whether or not the following syllogisms are valid. If the conclusion is not valid, tell why it is not.

1. All quadrilaterals have four and only four sides.
 A trapezium has exactly four sides.
 A trapezium is a quadrilateral.
2. An isosceles triangle has two equal sides.
 An equilateral triangle has two equal sides.
 An equilateral triangle is isosceles.
3. A pentagon has five, and only five sides.
 A hexagon has five sides.
 A hexagon is a pentagon.
4. A parallelogram has both pairs of opposite sides parallel.
 A square has both pairs of opposite sides parallel.
 A square is a parallelogram.
5. Right angles, and only right angles, have their arms perpendicular.
 Angle A is a right angle.
 The arms of angle A are perpendicular.
6. Angles which are right angles are equal.
 $\angle x = \angle y$
 $\angle x$ and $\angle y$ are right angles.

7. Angles which are straight angles are equal.
 $\angle x$ and $\angle z$ are straight angles.
 $\angle x$ and $\angle z$ are equal.
8. Any figure with equal sides and equal angles is a regular polygon.
 A square has equal sides and equal angles.
 A square is a regular polygon.
9. Any figure with equal sides and equal angles is a regular polygon.
 A rhombus has equal sides.
 A rhombus is a regular polygon.
10. The sum of two supplementary angles is 180°.
 $\angle A$ and $\angle B$ are supplementary.
 The sum of $\angle A$ and $\angle B$ is 180°.

44 THE "IF—THEN" RELATIONSHIP

In everyday speech the syllogism frequently takes the form of a sentence containing an "if" clause, followed by a second clause beginning with "then." Thus, the syllogism *All dogs are vertebrates: Fido is a dog: Fido is a vertebrate*, could be written: *If all dogs are vertebrates and Fido is a dog, then Fido is a vertebrate.* Notice that the order of major premise, minor premise and conclusion, is maintained. The same sentence could be written: *if Fido is a dog, and all dogs are vertebrates, then Fido is a vertebrate.* This form is considered dangerous because the minor premise has been put first.

Most propositions in geometry are stated in the if-then form. For example, *if two straight lines intersect,* (*then*) *the vertically-opposite angles are equal.* The *if-clause* contains what is given and the *then-clause* states what is to be proven. The if-clause is called the *hypothesis* and the then-clause is the *conclusion.*

Example 1: Rewrite the following statement in the standard syllogistic form, then as an if-then relationship: *This is a good space suit, so it must be air-tight.*

(i) Syllogistic form: All good space suits must be air-tight.
 This is a good space suit.
 This space suit is air-tight.

(ii) *If-then* form: If this is a good space suit, then it must be air-tight.

Given (hypothesis): this is a good space suit.
Conclusion: then it must be air-tight.

Example 2: Rewrite the following syllogism in the *if-then* form and state its hypothesis and conclusion:

All right angles are equal.
$\angle A$ and $\angle B$ are right angles.
$\angle A = \angle B$.

(i) *If* $\angle A$ and $\angle B$ are right angles (and all right angles are equal) *then* $\angle A = \angle B$.

Given (hypothesis): $\angle A$ and $\angle B$ are right angles.
Conclusion: $\angle A = \angle B$.

Example 3: Write the following sentence in the *if-then* form: *A student who lives in British Columbia lives in Canada.*
If a student lives in British Columbia, *then* he lives in Canada.

Exercise 20

B

Rewrite each of the following sentences in the if-then form. State the hypothesis and the conclusion.

1. Any animal with four legs is a quadruped.
2. Two lines which are parallel to a third line are parallel to each other.
3. Any solid which is lighter than water will float in water.
4. The diagonals of a rectangle are equal.
5. An equilateral triangle is also equiangular.
6. Any animal which breathes with gills gets its oxygen from water.
7. A triangle is a figure bounded by three straight lines.
8. Complements of the same angle are equal.
9. ~~In an isosceles triangle,~~ the base angles are equal.

D

45 INITIAL ASSUMPTIONS

In a sequence of reasoning we must begin with certain initial assumptions. Having accepted these assumptions, we must be prepared to accept the conclusions which follow from them. For many years it was assumed that the best fisherman in a given village was the one who caught the most fish. This may have been true in pioneer Canada, but it was an assumption which had disastrous effects on the game-fish population of our lakes. Today the best fisherman is a true conservationist who limits both the

number and size of his catch. In other words, if the initial assumptions are changed, the conclusions must also change.

In geometry we argue that because certain statements are true, certain valid and true conclusions must follow. It is apparent that in such a chain of reasoning, the very first statements cannot be proved. We must therefore make certain initial assumptions. In geometry, these assumptions are of four kinds:

1. Certain terms and ideas which are accepted without definition (point, line, etc.).
2. Defined terms.
3. Axioms—accepted without proof.
4. Geometric postulates—assumptions made as the basis for geometric reasoning.

Once we have decided upon a set of initial assumptions we must be careful to see that we correctly apply the laws of logic to these assumptions. When we are reasoning about things with which we are familiar, such as circles, it is very easy to jump to conclusions which are based on experience rather than on a chain of reasoning. Similarly, when we are asked to think about things which are *not* familiar, we are apt to give stock answers in terms of things with which we are familiar.

Some of the ideas above will be clarified in the two sections which follow.

46 HAVE SPACE SUIT—WILL TRAVEL

The next frontier to be conquered by man is that of space. The establishment of a space station will be followed by trips to the moon and perhaps to other planets. A recent news item tells of a company which is already selling deeds to land on Mars. What assumptions are being made by this company? What assumptions are being made by those purchasing these deeds?

Most of the distances, speeds and times connected with space travel which we now accept are the result of the deductive process. The distance to the sun, or to the moon, has never been directly measured. For that matter, neither has the circumference of the earth. Yet as early as 240 B.C. Eratosthenes, the librarian at the University of Alexandria, with the aid of geometry, calculated the circumference of the earth as 25,000 miles. One of the assumptions made by Eratosthenes was that the sun is so far away that its rays may be considered parallel for any two points reasonably close together on the earth's surface. He also assumed that the earth was sufficiently large that its curvature over a distance of approximately

500 miles could be neglected. On the basis of these assumptions and some deductions concerning triangles and circles which the Greeks had proved, he made his amazingly accurate calculation.

It is this kind of deductive reasoning which has "measured" for us all the astronomical distances we now take for granted. It is this kind of reasoning which suggests some of the new assumptions which must be the basis of life on a space station. To insure that the space station remains permanently circling the earth and does not fall and disintegrate, its centrifugal force must equal the gravitational pull of the earth. This means that the space station, and all its passengers, will in effect be weightless.

Let us test your deductive thinking on the basis of *one* new postulate. Suppose you are the first high school student to visit a space station. You are now in the kitchen-cafeteria of the station. It is much like a modern lunch counter, but one thing is different. THERE IS NO GRAVITY.

The purpose of this section is not to train a new generation of interplanetary travellers, but to convince you of three things:

(1) How easy it is to substitute the familiarity of experience for logical reasoning.

(2) The importance of deductive reasoning in scientific and mathematical fields.

(3) The importance of basic assumptions.

Can you adjust your deductive thinking on the basis of the one new postulate "There is no gravity on a space station"? Try the following exercise.

Exercise 21

On the basis of the postulate *There is no gravity on a space station*, state which of the following conclusions are valid and which are invalid, if the safety and comfort of the passengers is considered.

1. All furniture must be fastened to the floor.
2. The passengers must be strapped to their seats.
3. The cook must never lift the lid from a kettle of boiling soup.
4. All diners must remember to place their knives and forks back on their plates before the waitress removes the dishes.
5. No drinking from cups is possible.
6. All dishes containing food must have lids that fasten down.
7. You must not spit on the floor.

8. The guests must be strapped into their beds at night.
9. Cups must be heated before hot coffee is poured into them.
10. While on the space station, passengers should be supplied with lead insoles for their shoes.
11. All liquids must be kept in closed containers.
12. The officer in charge will take care that all passengers do not stand on the same side of the space station at the same time.

In trying to decide which of the above conclusions follow from the assumption of weightlessness, you undoubtedly found your sense of reasoning competing with other things which you believed to be true. Most of the students who try to answer these questions feel that lead insoles would undoubtedly help "anchor" the passengers. But if the lead is weightless, what is the only possible conclusion?

If now you leave the space station and continue to the moon, can you profit by the deductions made on the station, or does an entirely new set of conditions (major premises) prevail? In the exercise which follows you are asked to supply the conclusion in a number of if-then relationships.

Exercise 22

1. If rocket missiles are streamlined only because of air resistance, and there is no air between the space station and the moon, then . . . .
2. If there is no oxygen on the moon, and humans need oxygen, any human who goes to the moon must then . . . .
3. If winds are horizontal movements of air, and there is no air on the moon, then . . . .
4. If our atmosphere keeps us from getting unbearably hot in the daytime, and there is no atmosphere on the moon, then . . . .
5. If daylight is caused by the scattering of light from dust particles in the air, and there is no air on the moon, then . . . .
6. If weather changes are caused by unequal heating of the atmosphere, and there is no atmosphere on the moon, then . . . .
7. If a boy's weight of 150 lb. on earth is due to the pull of the earth's gravity on that mass, and if the gravity on the moon is one-sixth the gravitational force on earth, then . . . .
8. If our atmosphere protects us from the damage caused by cosmic rays, and there is no atmosphere on the moon, then . . . .
9. If the tractors planned for exploring the moon use water-cooled engines, and there is no water on the moon, then . . . .

CHAPTER II

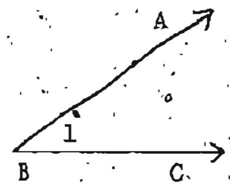
INITIAL ASSUMPTIONS

This chapter is a review of concepts necessary for proving $\triangle$'s $\cong$.

The object is to have the student understand and learn these. (No attempt at formal defs. ths. etc.)

Definitions

An angle is the union of two rays with a common endpoint, called the vertex.

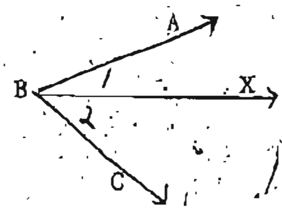


$\angle B$

$\angle ABC$ or $\angle CBA$

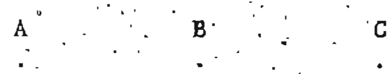
$\angle 1$

Adjacent angles are angles that have a common vertex and a common side but no interior points in common.



$\angle 1$ and $\angle 2$ are adjacent angles.

Betweeness of points



B lies between A and C if (i) A, B and C are distinct points on a line and (ii) AB + BC = AC.

Types of Angles

One Angle

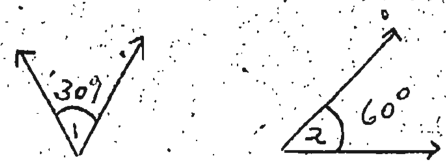
(i) If $m\angle 1$ is less than 90^o, $\angle 1$ is an acute angle.

(ii) If $m\angle 1 = 90^0$, $\angle 1$ is a right angle.

(iii) If $m\angle 1$ is greater than 90^0, then $\angle 1$ is an obtuse angle.

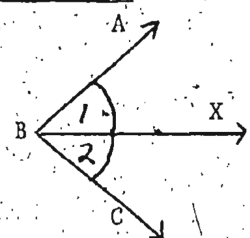
Two Angles

(i) If $m\angle 1 + m\angle 2 = 90^0$, $\angle 1$ and $\angle 2$ are complementary. We can also say that $\angle 1$ is a complement of $\angle 2$ or $\angle 2$ is a complement of $\angle 1$.

Example: 

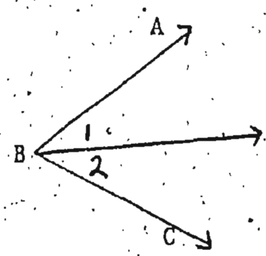
(ii) If $m\angle 1 + m\angle 2 = 180^\circ$, $\angle 1$ and $\angle 2$ are supplementary.

Angle Bisector

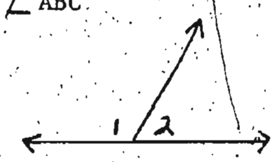


If $\angle 1 \cong \angle 2$ then $\overrightarrow{BX}$ bisects $\angle ABC$

The Angle Addition Postulate

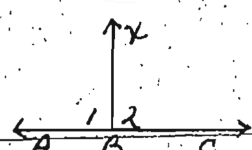


$m\angle 1 + m\angle 2 = m\angle ABC$



$\angle 1 + \angle 2$ are supplementary - Exterior sides form opposite ray.

Perpendicular Lines



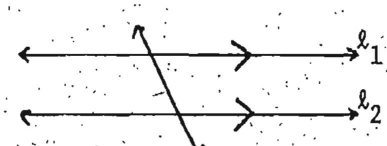
If $\angle 1 \cong \angle 2$ then $\overrightarrow{BX} \perp \overleftrightarrow{AC}$

Theorem: If $\overrightarrow{BX} \perp \overleftrightarrow{AC}$ then $\angle 1$ and $\angle 2$ are right $\angle$ s.

If $\angle 1$ and $\angle 2$ are right $\angle$ s then $\overrightarrow{BX} \perp \overleftrightarrow{AC}$.

Parallel Lines

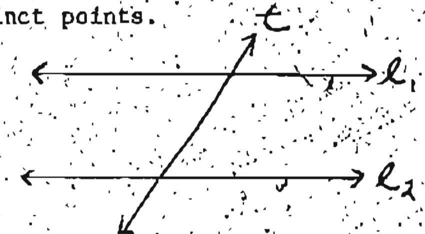
Two lines in the same plane that have no points in common are called parallel lines.



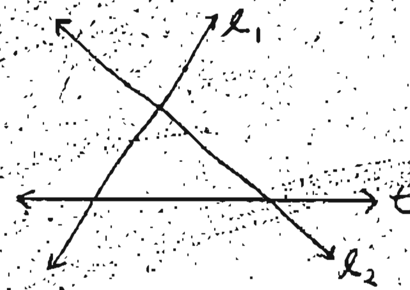
$\ell_1 \parallel \ell_2$

a) Transversal

A transversal is a line intersecting two other lines in two distinct points.

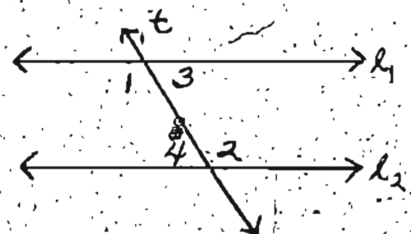


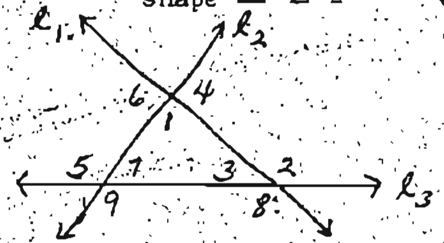
t is a transversal.



b) Alternate Interior Angles

shape





$\angle 1$ and $\angle 2$

$\angle 3$ and $\angle 4$

$\angle 1$ and $\angle 2$)
) ℓ_1 is a transversal
$\angle 3$ and $\angle 4$)

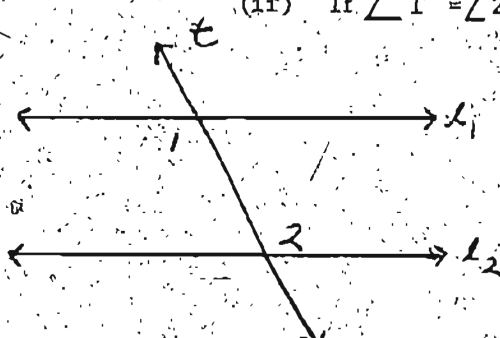
$\angle 1$ and $\angle 5$)
) ℓ_2 is a transversal
$\angle 6$ and $\angle 7$)

$\angle 7$ and $\angle 8$)
) ℓ_3 is a transversal
$\angle 3$ and $\angle 9$)

c) The Alternate Interior Angle Theorem (Alt. Int. $\angle$ th)

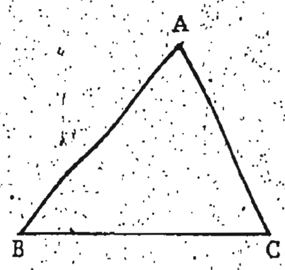
(i) If $\ell_1 \| \ell_2$ then $\angle 1 \cong \angle 2$

(ii) If $\angle 1 \cong \angle 2$ then $\ell_1 \| \ell_2$



Triangles

A triangle is the union of three segments determined by 3 non-collinear points.



$\triangle$ ABC
$\triangle$ CAB
$\triangle$ BAC
etc.

Classify by sides

3 sides $\cong$ $\rightarrow$ equilateral
2 sides $\cong$ $\rightarrow$ isosceles
no sides $\cong$ $\rightarrow$ scalene

Classify by angles

All angles less than 90^{o} $\rightarrow$ acute angled
One angle $= 90^{o}$ $\rightarrow$ right angled
One angle greater than 90^{o} $\rightarrow$ obtuse angled

CHAPTER III

INTRODUCTION TO DEDUCTIVE REASONING

Sentences, Sentential Connectives

In the study of inductive reasoning, we learned how to make a probable inference or conjecture from the examination of particular cases. We also learned that inductive reasoning does not provide proof that a conjecture is true. Mathematical proof demands a type of reasoning known as deductive reasoning or logical deduction. To begin with, we must become acquainted with the language of logic.

In logic, each English sentence has a form which is given an identifying name. For example, the sentence:

It is raining.

is a basic type of sentence. In logic such a sentence is called a simple sentence or an atomic sentence.

Definition: A sentence, in logic, is a statement which is either true or false, but not both.

If we combine two simple sentences by means of a connecting word, the resulting sentence is no longer a simple sentence. It is called a compound sentence or a molecular sentence. For example, the sentence:

It is raining and it is cold.

is a compound sentence. The word <u>and</u> combines the two simple sentences:

1. It is raining.
2. It is cold.

It should be noted that the connecting word <u>and</u> is not part of either simple sentence. It merely connects the two simple sentences to produce a compound sentence. Thus, the word <u>and</u> in logic is called a sentential connective.

The basic sentential connectives are <u>and</u>, <u>or</u>, <u>if</u> . . . <u>then</u> . . . , <u>not</u>. The use of each of these connectives together with the simple sentences:

1. It is raining.

2. It is cold.

to produce compound sentences is illustrated in the following:

(i) Connective: <u>and</u>

It is raining <u>and</u> it is cold.

This sentence is called the conjunction of the two simple sentences.

(ii) Connective: <u>or</u>

It is raining <u>or</u> it is cold.

This sentence is called the disjunction of the two simple sentences.

(iii) Connective: <u>if</u> . . . <u>then</u> . . .

<u>If</u> it is raining, <u>then</u> it is cold.

This sentence is called a conditional sentence or an implication. It should be noted that the connectives: <u>and</u>, <u>or</u>, <u>if</u> . . . <u>then</u> . . . , control or modify two sentences.

(iv) Connective: <u>not</u>

It is <u>not</u> raining.

This sentence is called the negation of the sentence, "It is raining." It should be noted that the connective <u>not</u> controls or modifies only one sentence.

The basic forms of compound sentences are:

(i) ( ) and ( ).

(ii) ( ) or ( ).

(iii) If ( ) then ( ).

(iv) not ( ).

where the parenthesis may be filled by simple or compound sentences.

In logic, capital letters, such as P, Q, R, S, T, A, B, are used to represent sentences. Thus, if P represents the sentence, It is raining, and Q represents the sentence, It is cold, the compound sentence,

(It is raining) and (it is cold).

may be represented symbolically as:

(P) and (Q)

or simply, P and Q.

In general, if A and B represent any two sentences, the basic compound sentences may be represented symbolically as:

(i) A and B.

(ii) A or B.

(iii) If A then B; (or) A implies B; (or) $A \rightarrow B$.

(iv) Not A; (or) $\sim A$.

Exercise

(A)

Classify each of the following sentences as simple or compound:

1. Today is Monday.
2. Mathematics is the Queen of the Sciences.
3. Camping is fun.
4. Governments fold at the polls.
5. Time does not stand still.
6. John enjoys music, and he has a large record collection.
7. $4x = 3y$.
8. The talk on how to make an interesting speech was uninteresting.
9. The better team does not always win.
10. $x \doteq y$ and $y \neq z$.

11. If $x - y = 0$, then $x = y$.

12. If two lines are parallel, then they do not have a common point.

13. 2 is rational and $\sqrt{2}$ is irrational.

14. He is intelligent.

15. Scientists are not eccentric.

16. Nero fiddled.

17. I drove to the cottage and hitch-hiked back.

(B)

In each of the following sentences make a list of connectives, if there are any:

18. Today is Monday, and tomorrow is Tuesday.

19. $2x + 3 = 9$ or $5x + 7 = 22$.

20. If $a, b \in N$, then $\frac{a}{b} \in N$.

21. $\{0\}$ is not a null set.

Deductive Reasoning

The process of making necessary conclusions from accepted statements by applying accepted rules of logical inference is called deductive reasoning.

If we are given the following statements:

1. If it is raining, then it is cold.
2. It is raining.

then the inescapable consequent of these two statements is the conclusion:

It is cold.

The two given statements are the premises of the argument, and the conclusion is the logical consequent of the premises.

The rule of logical inference which permits us to make this conclusion from these statements is called the Law of Detachment. Thus, by this law, if we are given any implication and precisely the "if clause" of this implication, we can detach the "then clause" and state it as a logical consequent.

In the above example, we say we have proved or deduced the statement: It is cold. That is, we have proved that the derived sentence follows logically from the given sentences.

The complete argument may be arranged as follows:

Premises: 1. If it is raining, then it is cold.

2. It is raining.

Logical consequent: It is cold. (Law of Detachment)

In symbols the argument may be represented as follows:

Premises: 1. $P \to Q$

2. P

Logical consequent: Q (Law of Detachment)

It should be noted that the Law of Detachment ensures that if P and $P \to Q$ are true (assumed or previously proved statements), then Q is a true statement.

The implication in the above is sometimes referred to as the major premise of the argument; the second statement is referred to as the minor premise; together the two premises form the hypothesis of the argument.

Example: State the logical consequent, if there is one, for each of the following sets of premises:

(i) 1. If logic is easy, then he will master it.

2. Logic is easy.

(ii) 1. If logic is easy, then he will master it.

2. He will master it.

(iii) 1. If $x - 3 = 0$ or $x + 2 = 0$, then $x = 3$ or $x = -2$.

2. $x - 3 = 0$ or $x + 2 = 0$.

(iv) 1. If $x = -2$, then $x^2 = 4$.

2. $x^2 = 4$.

Solution: (i) Logical consequent: He will master it. (Law of Detachment)

(ii) No logical consequent, since we are not given precisely the "if clause" of the implication.

(iii) Logical consequent: $x = 3$ or $x = -2$ (Law of Detachment)

(iv) No logical consequent, since we are not given precisely the "if clause" of the implication.

Deductive Proofs Involving Several Steps

Most deductive arguments consist of more than one application of the Law of Detachment. This is illustrated in the following example:

Premises: 1. If it is snowing, then it is cold.

2. It is snowing.

3. If it is cold, then I will stay at home.

Conclusion: I will stay at home.

This conclusion may be shown to be a logical consequent of the premises by applying the Law of Detachment twice. Thus, from statements 1 and 2 we obtain the conclusion, It is cold, by a first application of the law. Then by combining this conclusion with statement 3, the Law of Detachment brings us to the logical consequent, I will stay at home.

A formal deductive proof involving more than one application of the Law of Detachment follows:

Hypothesis: 1. $P \to Q$

2. P

3. $Q \to R$

Conclusion: R

Proof:

Statements	Authorities
1. $P \rightarrow Q$	1. Hypothesis
2. P	2. Hypothesis
3. Q	3. Law of Detachment 1, 2
4. $Q \rightarrow R$	4. Hypothesis
5. R	5. Law of Detachment 3, 4

A formal deductive proof is a succession of statements which lead to the desired conclusion. Each statement is either a premise or a statement derived directly by a rule of inference. The proof is arranged in two columns: statements in the first column and authorities for these statements in the second column. The process of building a formal deductive proof is called synthesis.

Example 1. Complete the following formal deductive proof; compare your proof with the complete proof on page

Hypothesis: $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$, A

Conclusion: D

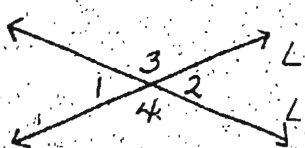
Proof:

Statements	Authorities
1. $A \rightarrow B$	1.
2. A	2.
3. B	3.
4. $B \rightarrow C$	4.
5. C	5.
6. $C \rightarrow D$	6.
7. D	7.

CHAPTER III

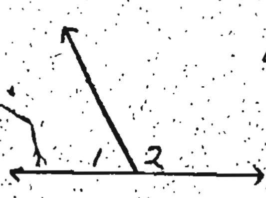
SEQUENCES

Sequence: A. Information you can conclude from a figure without being given anything else.

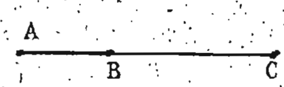
1. 

$\angle 1 \cong \angle 2$ Vertical $\angle$ s

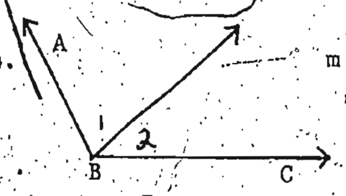
$\angle 3 \cong \angle 4$ Vertical $\angle$ s

2. 

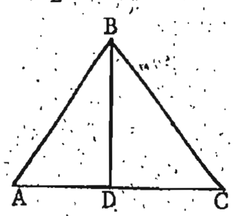
$\angle$ 1 and $\angle$ 2 are supp. Ext. sides form opp. rays

3. 

AB + BC = AC Def. betweeness

4. 

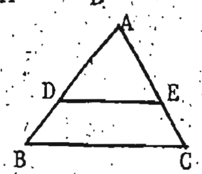
$m\angle 1 + m\angle 2 = m\angle ABC$ A.A.P.

5. 

$\overline{BD} \cong \overline{BD}$

or Reflexive

BD = BD

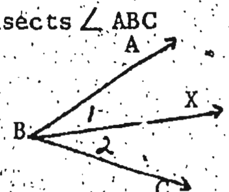
6. 

$\angle A \cong \angle A$ Reflexive

B. What conclusions can be drawn from a given

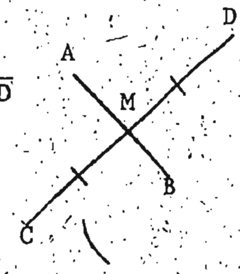
1. Given

BX bisects $\angle$ ABC



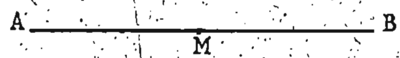
Statement	Reason
(i) BX bisects $\angle$ ABC	Given
(ii $\angle 1 \cong \angle 2$	Def. $\angle$ bisector

2. Given: $\overline{AB}$ bisects $\overline{CD}$



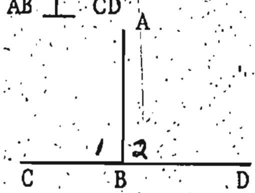
Statement	Reason
(i) $\overline{AB}$ bisects $\overline{CD}$	(i) Given
(ii) $\overline{CM} \cong \overline{MD}$	(ii) Def. seq. bisector

3. M midpt. of $\overline{AB}$



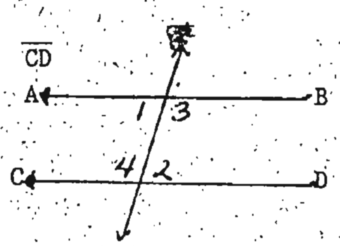
Statement	Reason
(i) M midpt. $\overline{AB}$	(i) Given
(ii) $\overline{AM} \cong \overline{MB}$	(ii) Def. midpt.

4. $\overline{AB} \perp \overline{CD}$



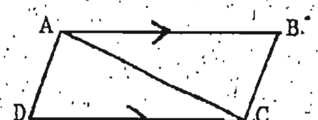
Statement	Reason
(i) $\overline{AB} \perp \overline{CD}$	(i) Given
(ii) $\angle 1$ and $\angle 2$ are rt. s	(ii) lines from rt. s
(iii) $\angle 1 \cong \angle 2$	(iii) All rt. $\angle$s are $\cong$

5a. $\overline{AB} \parallel \overline{CD}$



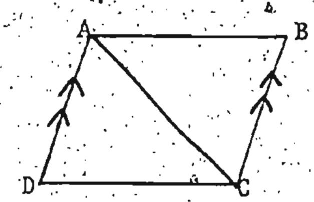
Statement	Reason
(i) $\overline{AB} \parallel \overline{CD}$	(i) Given
(ii) $\angle 1 = \angle 2$ or	(ii) Alt. Int. $\angle$ th
(ii) $\angle 3 = \angle 4$	(ii) Alt. Int. $\angle$ th

5b. $\overline{AB} \parallel \overline{CD}$



Statement	Reason
(i) $\overline{AB} \parallel \overline{CD}$	(i) Given
(ii) $\angle BAC = \angle DCA$	(ii) Alt. Int. $\angle$ th

5c. $\overline{AD} \parallel \overline{BC}$



Statement	Reason
(i) $\overline{AD} \parallel \overline{BC}$	(i) Given
(ii) $\angle DAC = \angle ACB$	(ii) Alt. Int. $\angle$ th

CHAPTER IV

CONGRUENT TRIANGLES

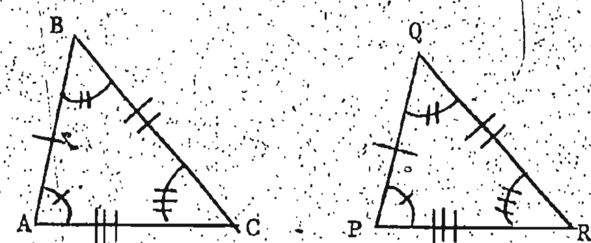
Def. Two triangles, $\triangle ABC$ and $\triangle PQR$ are congruent if and only if there exists a one-to-one correspondance between their vertices $ABC \longleftrightarrow PQR$ such that corresponding angles are congruent

$\angle A \cong \angle P$
$\angle B \cong \angle Q$
$\angle C \cong \angle R$

and corresponding sides are congruent.

We write $\triangle ABC \cong \triangle PQR$.

$\overline{AB} \cong \overline{PQ}$
$\overline{BC} \cong \overline{DR}$
$\overline{AC} \cong \overline{PR}$



So if $\triangle ABC \cong \triangle PQR$

then $ABC \longleftrightarrow PQR$

$\angle A \cong \angle P \quad \overline{AB} \cong \overline{PQ}$

$\angle B \cong \angle Q \quad \overline{BC} \cong \overline{QR}$

$\angle C \cong \angle R \quad \overline{AC} \cong \overline{PR}$

Exercises: List the 7 bits of information, we know if Given:

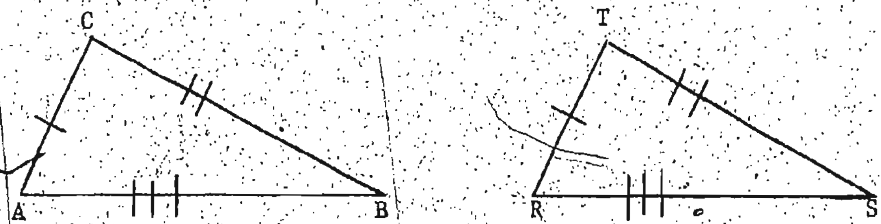
1) $\triangle ABC \cong \triangle DEF$

2) $\triangle PQR \cong \triangle XYZ$

Proving Triangles Congruent:

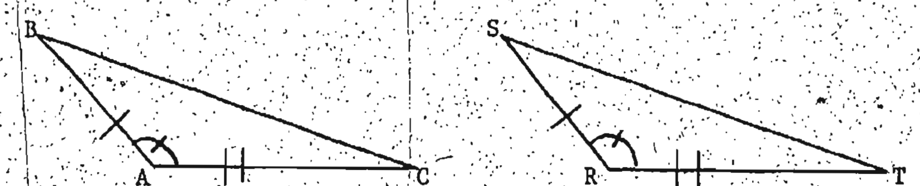
Methods:

1. The S.S.S. Postulate



IF $\overline{AB} \cong \overline{RS}$)
)
$\overline{BC} \cong \overline{ST}$) then $\triangle ABC \cong \triangle RST$ (S.S.S.)
)
$\overline{AC} \cong \overline{RT}$)

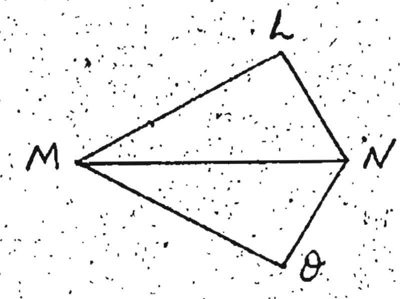
2. The S.A.S. Postulate



If $\overline{AB} \cong \overline{RS}$)
)
$\angle A \cong \angle R$) then $\triangle ABC \cong \triangle RST$ (S.A.S.)
)
$\overline{AC} \cong \overline{RT}$)

Using S.S.S. & S.A.S.

1.

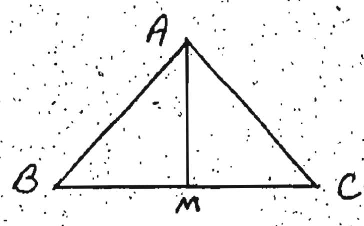


Given: $\overline{LM} \cong \overline{MO}$

$\overline{LN} \cong \overline{ON}$

Prove: $\triangle LMN \cong \triangle OMN$

2.

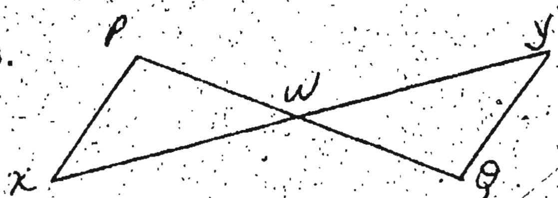


Given: $\overline{AB} \cong \overline{AC}$

M midpt. of $\overline{BC}$

Prove: $\triangle ABM \cong \triangle ACM$

3.

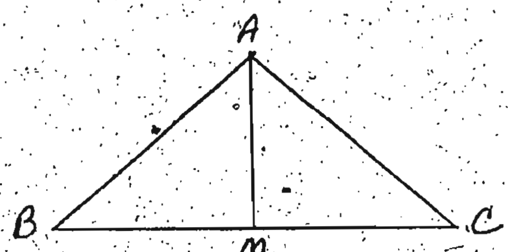


Given: $\overline{PQ}$ bisects $\overline{XY}$

$\overline{XY}$ bisects $\overline{PQ}$

Prove: $\triangle PXW \cong \triangle QYW$

4.

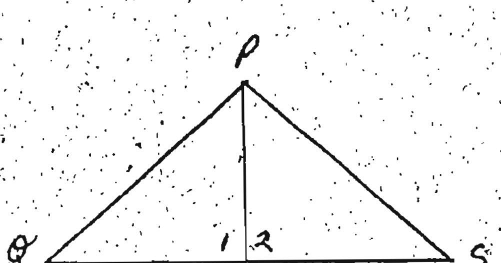


Given: $\overline{BM} \cong \overline{MC}$

$\overline{AM} \perp \overline{BC}$

Prove: $\triangle ABM \cong \triangle ACM$

5.



Given: R midpt. of $\overline{QS}$

$\angle 1 \cong \angle 2$

Prove: $\triangle PQR \cong \triangle PSR$

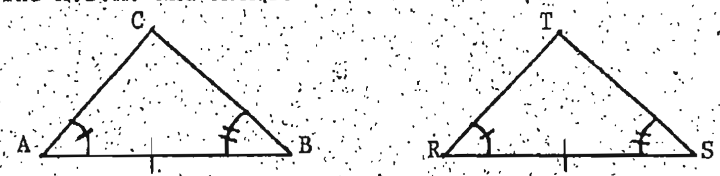
6.

Given: $\overline{DC} \parallel \overline{AB}$

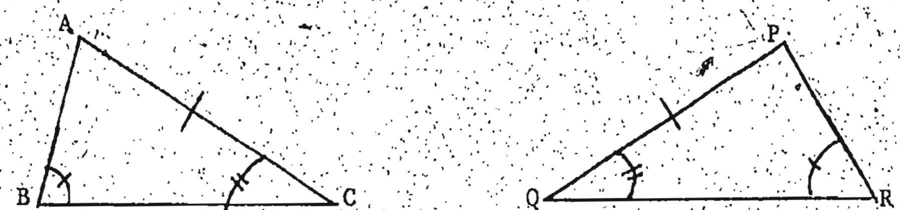
$\overline{DC} \cong \overline{AB}$

Prove: $\triangle ABC \cong \triangle CDA$

The A.S.A. and A.A.S. Postulate



If $\angle A \cong \angle R$)
$\overline{AB} \cong \overline{RS}$) then ABC = RST (A.S.A.)
$\angle B = \angle S$)

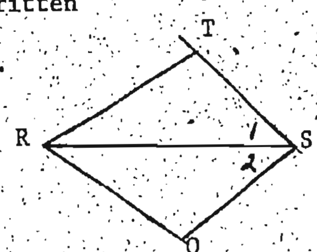


If $\angle B \cong \angle Q$)
$\angle C \cong \angle R$) then $\triangle ABC \cong \triangle PRQ$ (A.A.S.)
$\overline{AC} \cong \overline{PQ}$)

Exercises p. 155. #1-16 Oral.

Written

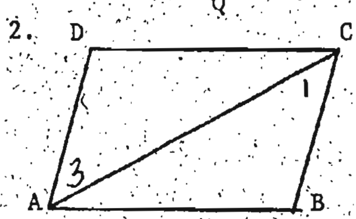
1.



Given: $\overrightarrow{SR}$ bisects $\angle TSQ$
$\angle T \cong \angle Q$

Prove: $\triangle RTS \cong \triangle RQS$

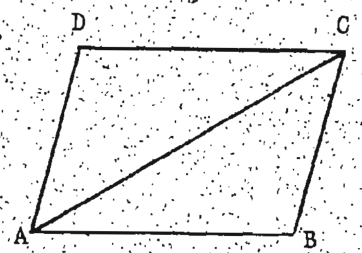
2.



Given: $\angle B \cong \angle D$
$\angle 1 \cong \angle 3$

Prove: $\triangle ABC = \triangle CDA$

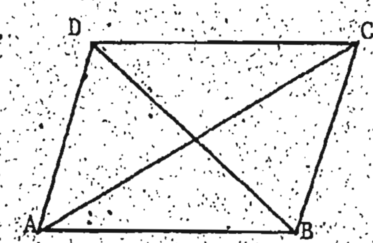
3.



Given: $\overline{DC} \parallel \overline{AB}$; $\overline{AD} \parallel \overline{BC}$

Prove: $\triangle ABC \cong \triangle CDA$

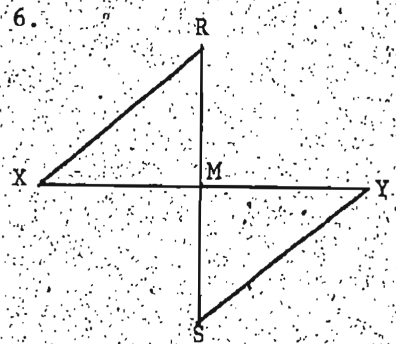
4.



Given: $\overline{DC} \parallel \overline{AB}$; $\overline{AB} \cong \overline{CD}$

Prove: $\triangle ABX \cong \triangle CDX$

5 & 6.



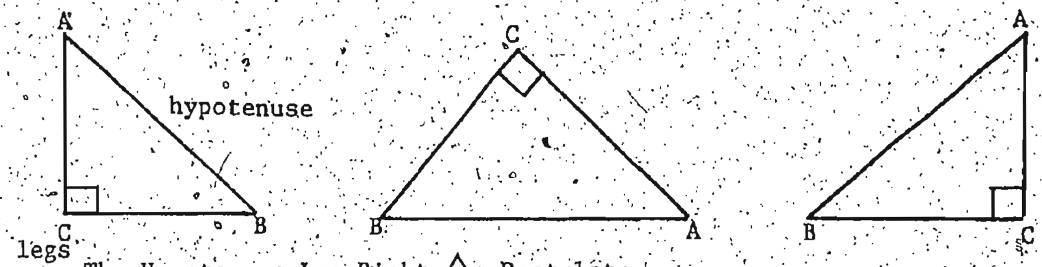
Given: $\overline{XR} \parallel \overline{SY}$

$\overline{XR} \cong \overline{SY}$

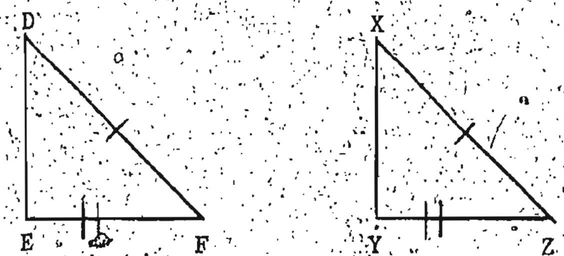
Prove: $\triangle XRM \cong \triangle YSM$

Use two methods to prove this.

The H.L.R. and L.L.R. Postulates



The Hypotenuse-Leg Right △ Postulate



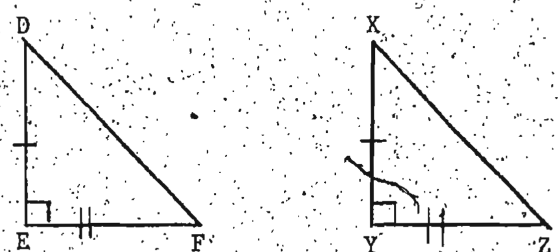
If (i) △DEF & △XYZ are rt. △'s

(ii) $\overline{DF} \cong \overline{XZ}$ (H)

(iii) $\overline{EF} \cong \overline{YZ}$ (L)

then △DEF ≅ △XYZ H.L.R.

The Leg-Leg Right Postulate



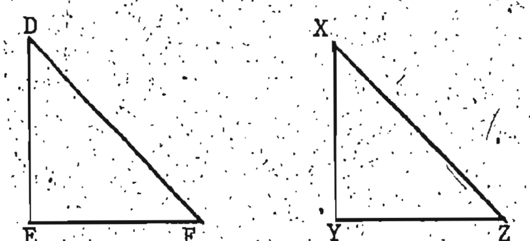
If (i) DEF & XYZ are rt. △ 's

(ii) $\overline{DE} \cong \overline{XY}$ (L)

(iii) $\overline{EF} \cong \overline{YZ}$ (L)

then △DEF ≅ △XYZ L.L.R.

Sequence for right triangles



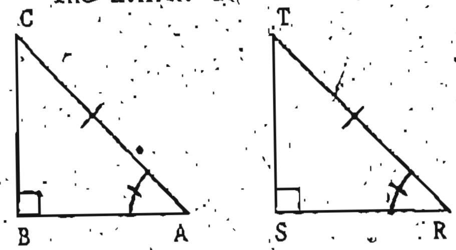
1. $\overline{DE} \perp \overline{EF}$; $\overline{XY} \perp \overline{YZ}$	1. Given
2. E & Y are rt. ∠'s.	2. ⊥ lines form rt. ∠
3. △DEF & △XYZ are rt. △'s.	3. def. rt. △'s.

Given: $\overline{DE} \perp \overline{EF}$; $\overline{XY} \perp \overline{YZ}$

Exercise Text p. 148, #1-8 Oral

p. 150, #7-36.

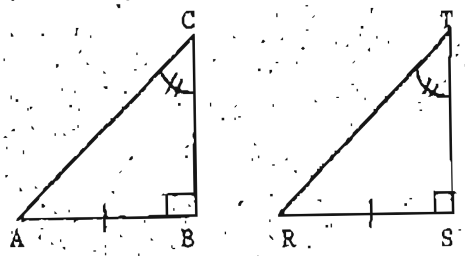
The H.A.R. and L.A.R. Theorem



If (i) $\triangle$ ABC & $\triangle$ RST are rt. $\triangle$'s
(ii) $\angle$ A $\cong$ $\angle$ R
(iii) $\overline{AC} \cong \overline{RT}$

then $\triangle$ ABC $\cong$ $\triangle$ RST

H.A.R.



If (i) $\triangle$ ABC & $\triangle$ RST are rt. $\triangle$'s
(ii) $\angle$ C $\cong$ $\angle$ T
(iii) $\overline{AB} \cong \overline{RS}$

then $\triangle$ ABC $\cong$ $\triangle$ RST

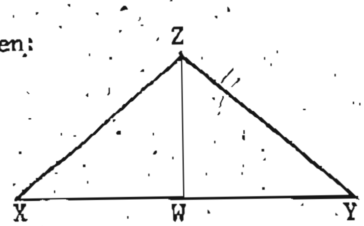
L.A.R.

Exercises p.159. #1-8 Oral.

Use only H.A.R. & L.A.R. in the following exercises.

Written:

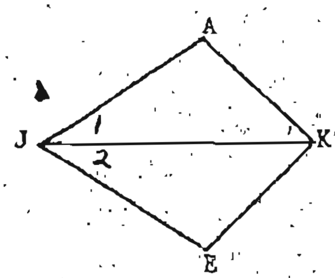
1.



Given: $\overline{ZW} \perp \overline{XY}$; $\angle X \cong \angle Y$

Prove: $\triangle$ XWZ $\cong$ $\triangle$ YWZ

2.

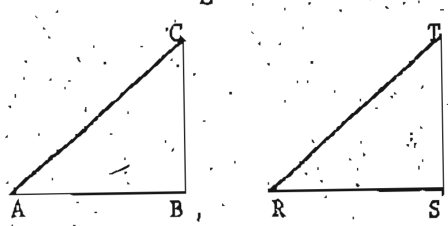


Given: $\overline{JA} \perp \overline{AK}$; $\overline{JE} \perp \overline{KE}$

$\angle 1 \cong \angle 2$

Prove: $\triangle$ JAK $\cong$ $\triangle$ JEK

3.



Given: $\overline{AB} \perp \overline{BC}$; $\overline{RS} \perp \overline{ST}$

$\overline{AC} \cong \overline{RT}$; $\angle A \cong \angle R$

Prove: $\triangle$ ABC $\cong$ $\triangle$ RST

4.

4. Given: $\overline{RT} \perp \overline{TQ}$; $\overline{QS} \perp \overline{RS}$

$\overline{RN} \cong \overline{QN}$

Prove: $\triangle RNT \cong \triangle QNS$

5. Given: $\overline{RT} \perp \overline{TQ}$; $\overline{QS} \perp \overline{RS}$

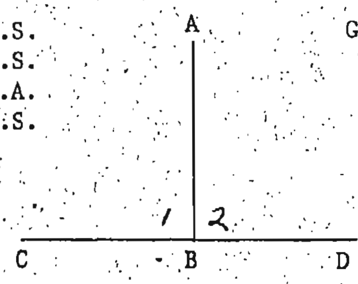
$\overline{SN} \cong \overline{NT}$

Prove. $\triangle RNT = \triangle QNS$

Summary: Proving Triangles Congruent

A. 4 Methods

S.S.S.
S.A.S.
A.S.A.
A.A.S.

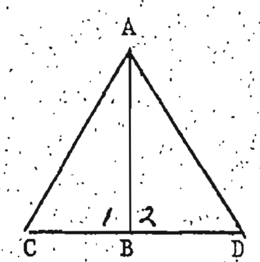


Given: $\overline{AB} \perp \overline{CD}$

1. $\overline{AB} \perp \overline{CD}$	1. Given
2. $\angle 1$ & $\angle 2$ are rt. $\angle$'s.	2. $\perp$ lines form rt. $\angle$'s.
3. $\angle 1 \cong \angle 2$	3. All rt. $\angle$'s are $\cong$

B. 4 Methods

H.L.R.
L.L.R.
H.A.R.
L.A.R.



Given: $\overline{AB} \perp \overline{CD}$

1. $\overline{AB} \perp \overline{CD}$	1. Given
2. $\angle 1$ + $\angle 2$ are rt. $\angle$'s.	2. $\perp$ lines form rt. $\angle$'s.
3. $\triangle ABC$ & $\triangle ABD$ are rt. $\triangle$'s.	3. def. rt. $\triangle$.

Note: 1) When using Methods B our perpendicular lines sequence changes, i.e., we must say right triangles.

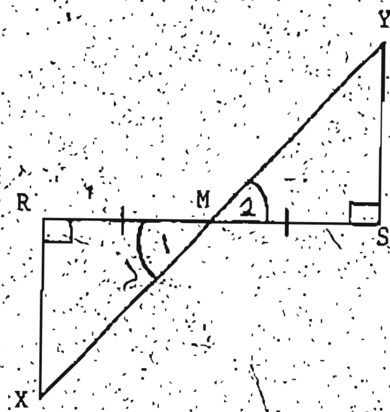
2) If you are asked to prove that two triangles are congruent, you can use any of the 8 methods.

Selecting a Method

Given: $\overline{XR} \perp \overline{RS}$;

$\overline{YS} \perp \overline{RS}$

Prove: $\triangle XRM \cong \triangle YSM$

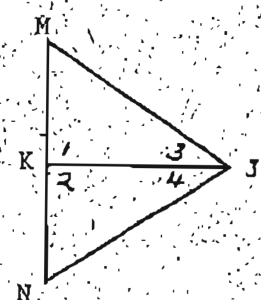


Two possibles: (i) A.S.A.

(ii) L.A.R.

Exercise: 1) Prove by (i) & (ii)

2)



(Rewrite Prove: $\triangle JKM$

$\cong \triangle JKN$

a) Given: $\overline{JK} \perp \overline{NM}$

$\angle M \cong \angle N$

Prove: $\overline{MK} \cong \overline{NK}$

b) Given: $\overline{JK} \perp \overline{MN}$

$\angle 3 \cong \angle 4$

Prove: $\overline{NJ} \cong \overline{MJ}$

End of Inst. Unit.

Proving Corresponding Parts Congruent

One way to prove that two segments or two angles are $\cong$ is:

1. Identify 2 $\triangle$'s in which the sides or $\angle$'s are corr. parts.

2. Prove the two triangles are $\cong$.

3. State the two parts are $\cong$, using, as the reason,

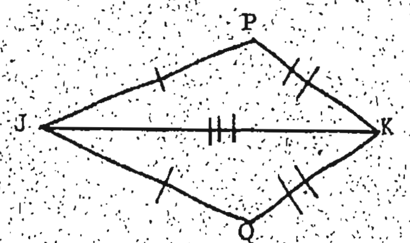
"Corr. parts of $\cong$ $\triangle$'s are $\cong$" C.P.C.T.C.

Example:

Given: $\overline{JP} \cong \overline{JQ}$

$\overline{PK} \cong \overline{QK}$

Prove: $\angle P \cong \angle Q$



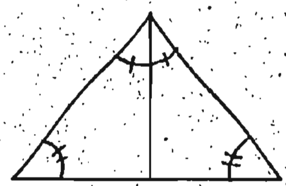
Plan of Attack: Prove: $\triangle PJK \cong \triangle QJK$ (S.S.S.)

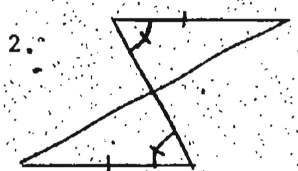
Proof:	1. $\overline{JP} \cong \overline{JQ}$	1. Given
	2. $\overline{PK} \cong \overline{QK}$	2. Given
	3. $\overline{JK} \cong \overline{JK}$	3. Reflexive
	4. $\triangle PJK \cong \triangle QJK$	4. S.S.S. and by 1, 2 & 3
	5. $\angle P \cong \angle Q$	5. By 4 and C.P.C.T.C.

Exercises p. 178, #21-28.

Quiz:

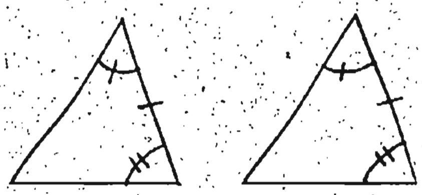
State whether you would use the S.S.S., S.A.S., A.S.A., or A.A.S. Postulate to prove each pair of triangles congruent.

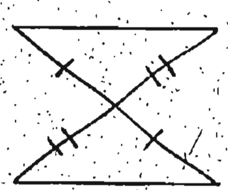
1.


2.


1. ______________________

2. ______________________

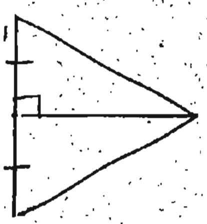
3.


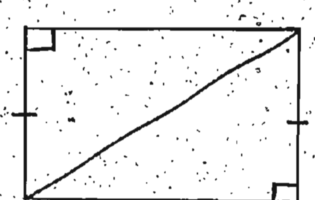
4.


3. ______________________

4. ______________________

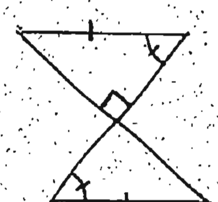
State whether you would use the L.L.R., H.L.R., L.A.R., or H.A.R., theorem to prove each pair of triangles congruent.

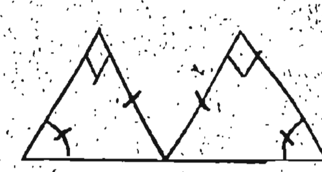
5.


6.


5. ______________________

6. ______________________

7.


8.


7. ______________________

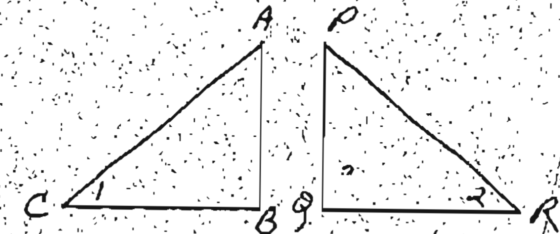
8. ______________________

NAME: ______________________________

Given: $\overline{AB} \perp \overline{BC}$; $\overline{PQ} \perp \overline{QR}$

$\overline{AC} \cong \overline{PR}$; $\overline{BC} \cong \overline{QR}$

Prove: $\angle 1 \cong \angle 2$



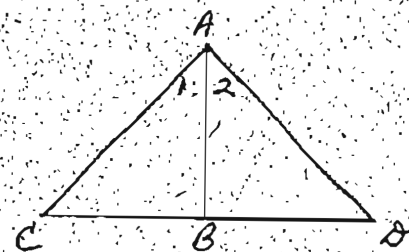
Proof:

Statements	Reasons

Given: $\overline{AB} \perp \overline{CD}$

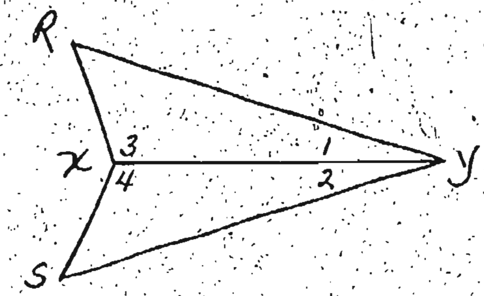
$\angle C \cong \angle D$

Prove: $\angle 1 \cong \angle 2$



Proof:

Statements	Reasons

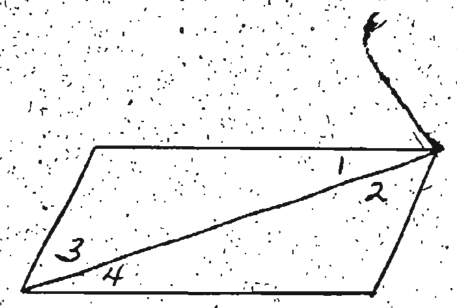


Given: $\overrightarrow{YX}$ bisects $\angle RYS$

$\overline{RY} \cong \overline{SY}$

Prove: $\triangle RXY \cong \triangle SXY$

Proof: Statements	Reasons



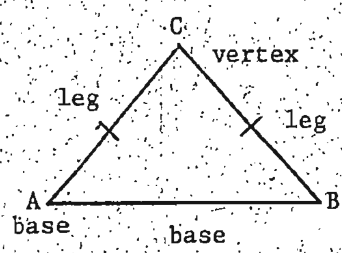
Given: $\overline{DC} \parallel \overline{AB}$; $\overline{AD} \parallel \overline{BC}$

Prove: $\overline{AB} \cong \overline{CD}$

Proof: Statements	Reasons

CHAPTER V

Isosceles Triangles



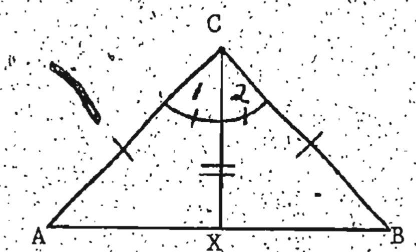
Def: Any triangle with two sides congruent is called isosceles.

If $\overline{AC} \cong \overline{BC}$ then $\triangle ABC$ is isos.

Theorem:

Given: $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$

Prove: $\angle A \cong \angle B$



Const. Draw the bisector of $\angle C$. Let X be the pt. where it intersects $\overline{AB}$.

Proof:	1. $\overrightarrow{CX}$ bisects $\angle ACB$	1. Given
	2. $\angle 1 \cong \angle 2$	2. def. $\angle$ bisector
	3. $\overline{AC} \cong \overline{BC}$	3. Given
	4. $\overline{CX} \cong \overline{CX}$	4. Reflexive
	5. $\triangle ACX \cong \triangle BCX$	5. S.A.S.
	6. $\angle A \cong \angle B$	6. C.P.C.T.C.

If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

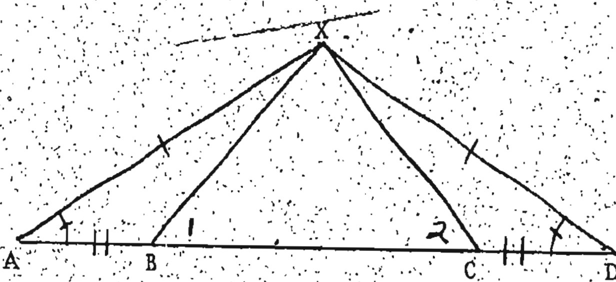
Exercises p. 181. #1-8 Oral

p. 183. #21-26.

Given: $\overline{AX} \cong \overline{DX}$

$\overline{AB} \cong \overline{CD}$

Prove: $\angle 1 \cong \angle 2$



Plan of Attack: Show $\triangle XAB \cong \triangle XDC$.

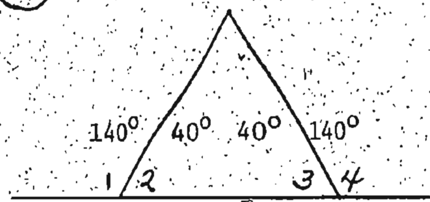
Proof:

1. $\overline{AX} \cong \overline{DX}$	1. Given
2. $\angle A \cong \angle D$	2. Isos. $\triangle$ th.
3. $\overline{AB} \cong \overline{CD}$	3. Given
4. $\triangle XAB \cong \triangle XDC$	4. S.A.S.
5. $\overline{BX} \cong \overline{CX}$	5. C.P.C.T.C.
6. $\angle 1 \cong \angle 2$	6. Isos. $\triangle$ th.

Analysis:

I can prove	If I can prove
1. $\angle 1 \cong \angle 2$	1. $\overline{BX} \cong \overline{XC}$
2. $\overline{BX} \cong \overline{XC}$	2. $\triangle XAB = \triangle XDC$

Analysis leads to a plan of attack.



$\angle 2 \cong \angle 3$

$\angle 1$ is a supp. of $\angle 2$

$\angle 4$ is a supp. of $\angle 3$

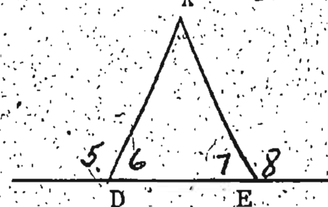
$\angle 1 \cong \angle 4$ supp. of $\cong$'s are $\cong$.

Sequence:

1. $\angle 2 \cong \angle 3$ — 1.
2. $\angle 1$ & $\angle 2$ are supp. — 2. Exterior sides form opp. rays
   $\angle 3$ & $\angle 4$ are supp.
3. $\angle 1 \cong \angle 4$ — 3. Supp. of $\cong$'s are $\cong$.

Exercise:

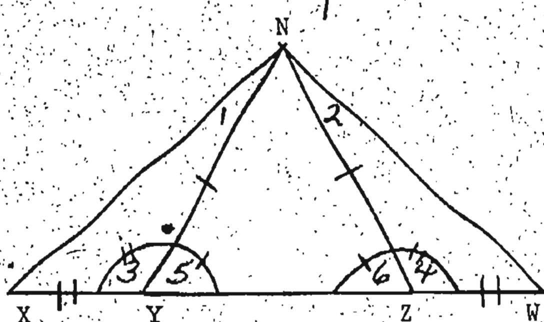
A.



Given: $\overline{DX} \cong \overline{EX}$

Prove: $\angle 5 \cong \angle 8$

B. p. 185, #41 & 42



Given: $\overline{NY} \cong \overline{NZ}$

$\overline{XY} \cong \overline{ZW}$

Prove: $\angle 1 \cong \angle 2$

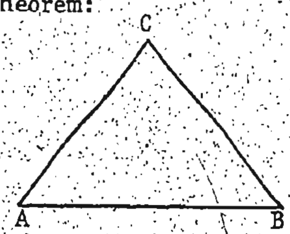
Analysis: $\angle 1 \cong \angle 2$ if $\triangle NXZ \cong \triangle NWZ$

Plan of Attack: Prove $\triangle NXY = \triangle NWZ$

Proof:

1. $\overline{NY} \cong \overline{NZ}$	1. Given
2. $\angle 5 \cong \angle 6$	2. Isos. $\triangle$ th.
3. $\angle 3$ & $\angle 5$ are supp. $\angle 4$ & $\angle 6$ are supp.	3. Exterior sides form opp. rays
4. $\angle 3 = \angle 4$	4. Supp. of $\cong$ $\angle$'s are $\cong$
5. $\overline{XY} \cong \overline{ZW}$	5. Given
6. $\triangle NXY \cong \triangle NWZ$	6. S.A.S. & by 1, 4 & 5
7. $\angle 1 \cong \angle 2$	7. C.P.C.T.C. & by 6.

Theorem:



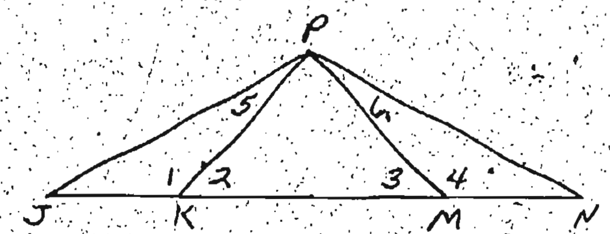
(i) If $\overline{AC} \cong \overline{BC}$ then $\angle A \cong \angle B$

(ii) If $\angle A \cong \angle B$ then $\overline{AC} \cong \overline{BC}$

<u>The Isosceles Triangle Theorem</u> (<u>Isos. $\triangle$ th</u>)

Stencil Exercises #1-6.

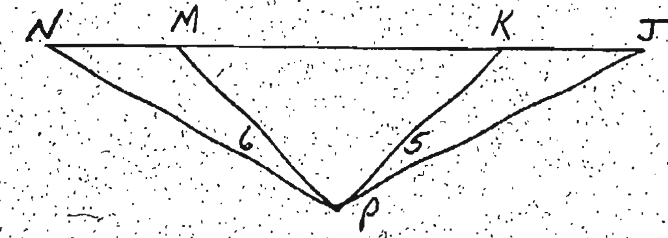
Grade 10 Geometry Isosceles △



Given: $\overline{JK} \cong \overline{MN}$

$\overline{PJ} \cong \overline{PN}$

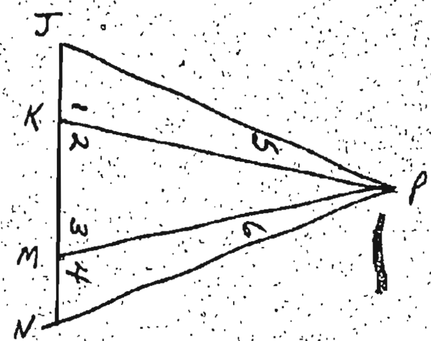
Prove: 5 = 6



Given: $\overline{JK} \cong \overline{NM}$

$\angle J \cong \angle N$

Prove: $\angle 5 \cong \angle 6$

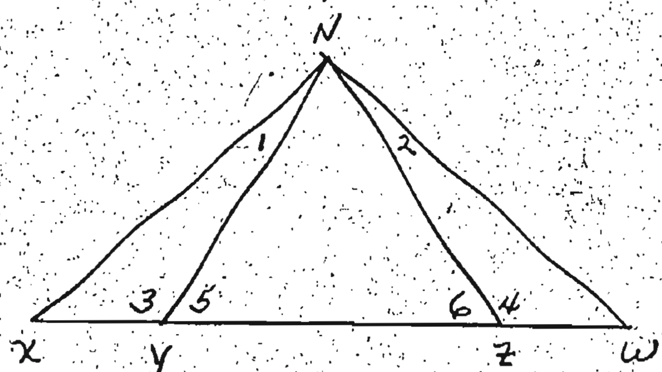


Given: $\angle 2 \cong \angle 3$

$\angle 5 \cong \angle 6$

$\overline{PJ} \cong \overline{PN}$

Prove: $\overline{JK} \cong \overline{MN}$



Given: $\angle 5 \cong \angle 6$

$\overline{XY} \cong \overline{ZW}$

Prove: $\overline{NX} \cong \overline{NW}$

Given: $\overline{NY} \cong \overline{NZ}$

$\angle 1 \cong \angle 2$

Prove: △ NXW is isos.

Given: $\angle 3 \cong \angle 4$

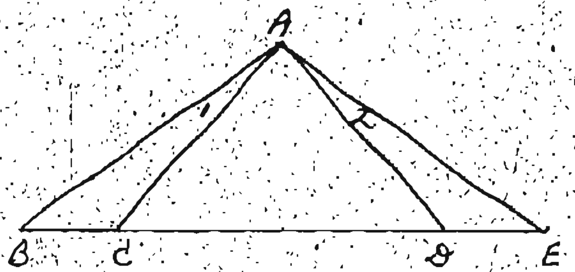
$\overline{XY} \cong \overline{ZW}$

Prove: △ NXW is isos.

(use this diagram for 4-6)

Geometry Grade 10 Matric Quiz #2 - Isosceles Triangles

1.



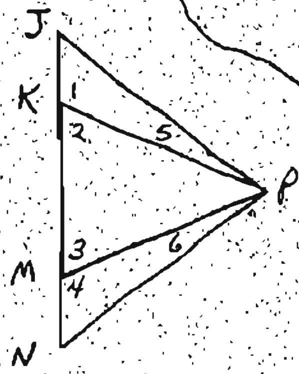
Given: $\overline{AB} \cong \overline{AE}$

$\overline{BQ} \cong$ [redacted]

Prove: $\angle 1 \cong \angle 2$

Proof: Statements	Reasons

2.



Given: $\angle 2 \cong \angle 3$

$\angle 5 \cong \angle 6$

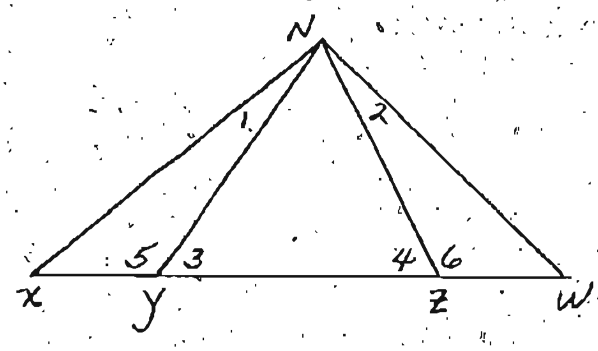
Prove: $\overline{JK} \cong \overline{MN}$

Proof: Statements	Reasons

3. Given: $\overline{NY} \cong \overline{NZ}$

$\angle X \cong \angle W$

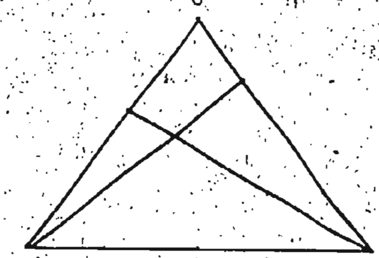
Prove: $\angle 1 \cong \angle 2$



Proof: Statements	Reasons

SUMMARY

Now the direct proof proceeds from the axioms by repeated application of the valid pattern of reasoning. This is what we mean when we say that the strategy is to show that a statement is deducible from the axioms. However, the classical two-column form for a proof does not make this very clear. So let us recast a simple direct proof into a form which shows the use of Modus ponens:



Given: $\triangle ABC$ with $\overline{AB} \cong \overline{CB}$

$\overline{CX} \perp \overline{AB}$ at X; $\overline{AY} \perp \overline{CB}$ at Y

Prove: $\overline{BX} \cong \overline{BY}$

1. If the sides of an angle are perpendicular to each other, then the angle is a right angle. — If P, then Q

   $\overline{CX} \perp \overline{AB}$ at X;
   $\overline{AY} \perp \overline{CB}$ at Y — P is true

   $\angle$ BXC and BYA are right angles — $\therefore$ Q is true

2. If one of the angles of a triangle is a right angle, then the triangle is a right triangle. — If P, then Q

   In $\triangle$BXC, $\angle$BXC is a right angle, and in $\triangle$BYA, $\angle$ BYA is a right angle — P is true

   $\triangle$ BYA and $\triangle$BXC are right triangles — $\therefore$ Q is true

3. If two right triangles have h.a. = h.a., then those two triangles are congruent. — If P, then Q

In right $\triangle$ BXC and BYA, it is given that $\overline{AB} \cong \overline{CB}$ and $m(\angle B) = m(\angle B)$

P is true

$\triangle BXC \cong \triangle BYA$

$\therefore$ Q is true

4. If two triangles are congruent, then all the corresponding parts have the same measure. — If P, then Q

$\triangle BXC \cong \triangle BYA$

P is true

$\overline{BX} \cong \overline{BY}$

$\therefore$ Q is true

The above form not only verifies that our reasoning was valid, but also shows that the deduction proceeded from the axioms. For implication statements 1, 2, and 4 are definitions of the system; statement 3 is a previously proved theorem. Thus definitions 1, 2 and 4, together with the axioms previously used to prove 3 are the subset of axioms of the system from which we have exhibited a deduction of our "theorem." This is the strategy of the direct proof.