FIXED POINT THEOREMS IN UNIFORM SPACES

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by

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ABSTRACT

A mapping F of a metric space X into itself is said to satisfy a Lipschitz condition with Lipschitz constant K if $d(F(x), F(y)) \leq K\, d(x, y)$, $(x, y \in X)$. If this condition is satisfied with a Lipschitz constant K such that $0 \leq K < 1$ then F is called a contraction mapping. If we let $K = 1$ the mapping is called non-expansive, and if $K = 1$ and we have a strict inequality it is called contractive.

In this thesis we give a survey of the various definitions offered for non-expansive, contractive and contraction mappings in uniform spaces. In particular we study the following definition of a $\mathcal{U}$-contractive mapping given by Casesnoves) [3]. DEFINITION: If $(E, \mathcal{U})$ is a complete uniform space and F a map of E into itself such that $g = (F, F)$ is the extension of F to the product space $E \times E$, then F is said to be $\mathcal{U}$-contractive, provided the following conditions are satisfied.

(a) $\forall\ V \in \mathcal{U}$, $g(V) \subset V$

(b) $\forall\ V, \forall\ W \in \mathcal{U}$, $k \in N$, $\forall\ p > 0$, $\forall\ n \geq k$
$g^n(V) 0 g^{n+1}(V)\ 0 \ldots 0\ g^{n+p}(V) \subseteq W.$

We consider also sequences of contraction mappings in metric and uniform spaces. In metric spaces we prove a theorem for a sequence of contraction mapping of a complete ϵ - chainable metric space. In uniform spaces we prove the following theorem and then show how it may be used to prove other results for sequences of mappings in uniform spaces.

<u>THEOREM</u>: Let $(E, \mathcal{U})$ be a complete uniform space and F_k a $\mathcal{U}$-contractive mapping from E into itself, with fixed points U_k $(k = 1, 2, \dots)$. Suppose $\lim_{k\to\infty} F_k(x) = F(x)$ for every $x \in E$, where F is a $\mathcal{U}$-contractive mapping from E into itself. Then $\lim_{k\to\infty} U_k = U$, where U is a fixed point of F.

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CHAPTER I

Introduction

In the latter part of the nineteenth century a lot of work was being done on distance-related concepts in a number of specific "spaces" which were not "spaces" in the usual sense, for example; "spaces" in which the typical "point" might be a curve or a function.

In 1906, Maurice Fréchet suggested in his doctoral thesis that this work might be done more economically by considering a single, abstract, but restricted concept of "distance" defined for pairs of equally abstract points and developing its properties once and for all. He suggested and explored several alternate ways of doing this, but his best proposal was that of a metric space. In such a space, distance-related concepts such as continuity and convergence could be defined and interpreted in a natural way.

The process of generalization did not terminate with metric spaces, for within a short time, men such as Riesz, Hausdorff, and Fréchet himself observed that the notion of distance could be replaced with the notion of "neighbourhood" and that continuity, perhaps the principal metric space property, could be considered equally well in terms of "neighbourhoods". Hausdorff, in 1914, laid down certain conditions that these neighbourhoods must satisfy and called the resulting space a topological space.

The greatest advantage enjoyed by a topological space over a metric space is the fact that a topological space does not depend on the system of real numbers, or for that matter, on any other more specialized mathematical system. However, in moving from metric spaces to topological spaces some important concepts were lost, for example; uniform continuity, uniform boundedness and Cauchy nets. Efforts were made to develop theories in which these and similar ideas could be worked out without suffering from the limitations of metric spaces.

One idea was to introduce a generalized metric space, that is: $d : X \times X \to [0, \infty]$ where X is a non-empty set, satisfying the usual axioms for a metric space, i.e. $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$. [18].

Another idea was that of proximity spaces, (for a discussion of the theory of proximity spaces see Thron [29]) but the most useful of all was the concept introduced by A. Weil [30], in 1937, of a uniform space.

See Kelley [14] for the following definitions and terminology.

A uniform space is defined as follows: A uniformity for a set X is a non-viod family $\mathcal{U}$ of subsets of $X \times X$ such that:

(a) each member of $\mathcal{U}$ contains Δ ; (where $\Delta = [(x, x) : x \in X]$ called the diagonal);

(b) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) If $U \in \mathcal{U}$, then $V \circ V \subset U$ for some V in $\mathcal{U}$;

(d) If U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

(e) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

If $\mathcal{U}$ satisfies the condition

$$\bigcap [U : U \in \mathcal{U}] = \Delta$$

then $\mathcal{U}$ is called an Hausdorff (or separated) uniformity. The elements of a uniformity are sometimes called entourages, and the pair $(X, \mathcal{U})$ is called a <u>uniform space</u>.

A subfamily β of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ iff each member of $\mathcal{U}$ contains a member of β .

Now given a metric space (X, d) one can define a uniformity for X by letting $V_\varepsilon = \{(x, y) : d(x, y) < \epsilon\}$. It can also be shown that every uniform space is a topological space. Uniform spaces therefore will lie somewhere between metric and topological spaces.

Uniform continuity and Cauchy nets can be defined on a uniform space in the following way (Kelley [14]).

If f is a function on a uniform space $(X, \mathcal{U})$ with values in a uniform space $(Y, \mathcal{V})$, then f is a uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ iff for each V in $\mathcal{V}$ the set $\{(x, y) : (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$.

A net $\{S_n, n \in D\}$ in the uniform space $(X, \mathcal{U})$ is a Cauchy net iff for each member U of $\mathcal{U}$ there is N in D such that $\{S_m, S_n\} \in U$ whenever both m and n follow N in the ordering of D.

One of the best known theorems in connection with the mappings of a metric space X into itself is the Banach contraction principle stated below.

<u>THEOREM 1</u> : If f maps the complete metric space X into itself and if

there exists λ such that $0 \leq \lambda < 1$ and

(1) $d(f(x), f(y)) < \lambda[d(x, y)]$ for all x and y in X,

then there exists a unique point x in X such that $f(x) = x$.

A mapping satisfying (1) is called a contraction and λ is called the contraction constant for f with respect to d.

Because of the simplicity and usefulness (see for example, Kolmogorov and Fomin [16]) of Theorem 1, various generalizations and localizations of it are given which in one way or other relax the restrictions on (1) (Edelstein [11], Rakotch [23] and Naimpally [21]). Several authors have also examined non-expansive mappings which satisfy

(2) $d(f(x), f(y)) \leq d(x, y)$.

Some of these are listed in the bibliography (Edelstein [9], Edrei [12]).

Another approach has been to consider mappings which satisfy

(3) $d(f(x), f(y)) \geq d(x, y)$

called expansive mappings (Edelstein [9]).

The previous condition insures the existence of fixed or periodic points in certain cases, but most results obtained from (3) are of quite different nature. For example, Brown and Comfort [2] proved the following.

THEOREM 2 : Suppose X is compact and metric and suppose f is one-to-one mapping satisfying (3), then f is, in fact, an isometry on X. That is $d(f(x), f(y)) = d(x, y)$ for all x and y in X.

The concept of contractions has also been made meaningful in spaces more general than metric (Davis [6] and Diaz and Margolis [7]). However, as contraction and non-expansive mappings are uniformly continuous it is not possible to define them in ordinary topological spaces, but they can be defined in uniform spaces.

Rhodes [24], in 1955, gave first definitions in uniform spaces, when he defined non-expansive and expansive mappings, as an aid for the generalization of isometries to uniform spaces. He considered the fact that if we let X be a metric space with metric d and let

$$U_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}$$

then two points x, y of X near of order U_ε imply $d(x, y) \leq \varepsilon$, therefore the images $f(x)$, $f(y)$ of the points x, y under a non-expansive map f are also near of order U_ε. Rhodes then defined non-expansive mappings for uniform spaces in the following way.

(4) A transformation f of a uniform space E with basis of vicinities β into itself is said to be non-expansive if, for every pair of points x, y of E and every vicinity V of β, $(x, y) \in V$ implies $(f(x), f(y)) \in V$, i.e. $(f, f)V \subset V$.

Obviously a non-expansive map is uniformly continuous.

Expansive mappings were defined in the following way.

(5) A transformation f of a uniform space E with basis of vicinities β into itself is an expansive if, for every pair of images $f(x)$, $f(y)$ of points x, y of E and every vicinity V of β, $(f(x), f(y)) \in V$ implies $(x, y) \in V$.

In 1959, Brown and Comfort [2] used (4) and (5) to prove some further results concerning isometries in uniform spaces. For example, they were able to prove a generalization of Theorem (2).

The first fixed point theorem in uniform spaces, using a modified version of (4), was given, in 1963, by Kammerer and Kasriel [17]. Other results using the definitions of Kammerer and Kasriel were given by Edelstein [10] and Naimpally [20]. Knill [15] using a different definition for the non-expansive map in uniform spaces, gave some further results.

In 1965, Casesnoves [3] added an extra condition to (4) and gave a definition for contraction mappings in uniform spaces. His exact definition is stated below.

If $(E, \mathcal{U})$ is a uniform space and f a map of E into itself, such that $g = (f, f)$ is the extension of f to the product space $E \times E$. Then f is said to be "$\mathcal{U}$-contractive" provided the following conditions are satisfied

(a) $\forall V \in \mathcal{U}$, $g(V) \subset V$

(6) (b) $\forall V$, $\forall W \in \mathcal{U}$, $\exists k \in N$, $\forall p > 0$, $\forall n \geq k$,

$$g^n(V) 0\, g^{n+1}(V) 0 \ldots 0\, g^{n+p}(V) \subseteq W.$$

By $g^n(V)$ we mean the n-fold iterations of V by g. Using this definition Casesnoves was able to prove the Banach Contraction Principle in uniform spaces.

R-Salinos [25] and most recently Chandler [4] have given different definitions of contraction mappings in uniform spaces and proved fixed point theorems.

In Chapter II, of this dissertation, we give a survey of the fixed point theorems proven for non-expansive and contraction mappings in uniform spaces; and using definition (6) we prove some further results.

In Chapter III we consider sequences of contraction mappings, again we give a survey of what has already been done in uniform spaces, and using the definition of Casesnoves give some further results.

References throughout this dissertation are given by a number in brackets indicating a particular article or book in question. Definitions and terminology, unless otherwise indicated, are taken from Kelley [14]. A complete list of references arranged in alphabetical order is given at the end of this dissertation.

CHAPTER II

NON-EXPANSIVE AND CONTRACTION MAPPINGS OF A UNIFORM SPACE

In this chapter, given a map F of a uniform space E into itself, we shall be interested in seeking the conditions on F and E sufficient to insure the existence and uniqueness of a fixed point of F in E .

We begin by stating several definitions which were first given by Brown and Comfort [2] ; Kammerer and Kasriel [17] .

Let $(E, \mathcal{U})$ be a uniform space and let β be a basis for the uniformity.

2.1 DEFINITION: β is said to be open if each of its elements are open in $E \times E$.

2.2 DEFINITION: β is said to be ample if; whenever $(x, y) \in U \in \beta$; there is a $W \in \beta$ for which $(x, y) \in W \subset \overline{W} \subset U$.

2.3 DEFINITION: Let $U \in \beta$. Then a U-chain is any finite set of points x_o, $x_1, \ldots, x_n$ in E such that $(x_{i-1}, x_i) \in U$, $i = 1, 2, \ldots, n$. We shall say in such a case that x_o and x_n are joined by a U-chain. The uniform space $(E, \mathcal{U})$ is said to be U-chainable if for each pair (x, y) of its points there exists a U-chain joining x and y.

The above definition in uniform space is a generalization of the ε-chain concept for metric spaces. A formal definition of an ε-chain in metric spaces will be given in Chapter III.

As a means of comparison we shall list as 2.4 an abreviated form of definition 1.5 and then as 2.5 state the definition of a β-contractive map as introduced by Kammerer and Kasriel.

2.4 DEFINITION: Let $(E, \mathcal{U})$ be a uniform space and β be a basis for $\mathcal{U}$. A function $F : E \to E$ is said to be a contraction, if $(F(x), F(y)) \subset U$ whenever $(x, y) \in U \in \beta$.

2.5 DEFINITION: Let $(E, \mathcal{U})$ be a uniform space and β be a basis for $\mathcal{U}$. A function $F : E \to E$ is said to be β-contractive, provided that for each $U \in \beta$ and $(x, y) \in U$ $(x \neq y)$ there exists a $W \in \beta$ such that $(F(x), F(y)) \in W \subset U$ and $(x, y) \notin W$.

Definition 2.5 is a more restrictive definition than 2.4 and can be thought of as a generalization of the metric $d(F(x), F(y)) < d(x, y)$ called a contractive mapping (see Edelstein [8]) where as 2.4 is a generalization of $d(F(x), F(y)) \leq d(x, y)$ the non-expansive mapping definition.

Kammerer and Kasriel, using Definition 2.5, were able to prove the following theorem which is a generalization of a theorem given by Edelstein [8] for metric spaces.

2.6 THEOREM: Let $(E, \mathcal{U})$ be a Hausdorff uniform space and let β be an open ample basis for the uniformity of E. If $F : E \to E$ is β-contractive and is such that the image of E under some iterate of F is compact, then

(a) The set of periodic points in X is a nonempty finite set $A = \{x_0, x_1, \ldots, x_n\}$ so that for some positive integer p, $F^p(x_i) = x_i$ for each x_i in A. Furthermore, for each x in E, there exists an

$x_i \in A$ such that $\lim_{n\to\infty} F^{pn}(x) = x_i$.

(b) Suppose E is U-chainable, $U \in \beta$. Then A reduces to a single point. [Hence F has a unique fixed point x_0, and for each $x \in X$, $\lim_{n\to\infty} F^n(x) = x_0$.]

The fact that in the above theorem, if A consists of a single point then F has a unique fixed point depends on a lemma, which for completeness shall be included here together with its proof.

2.7 LEMMA: If F is a continuous mapping of a set into itself and if, for some positive integer k , F^k has a unique fixed point, then F has a unique fixed point.

PROOF: Denote by Z the unique fixed point of F^k. Since

$$F(Z) = F(F^k(Z)) = F^{k+1}(Z) = F^k(F(Z))$$

it follows that $F(Z)$ is a fixed point of F^k and so $F(Z) = Z$ since Z is unique. Thus F possesses a unique fixed point, and a fixed point of F is necessarily a fixed point of F^k and so is unique.

The next results, using Definitions 2.4 and 2.5 , were given by Edelstein [10]. In this paper Edelstein generalizes some fixed point theorems, which he had proven in metric spaces, to uniform spaces. In so doing he gives a generalization of Theorem 2.6.

2.8 DEFINITION: Let E^F denote the set of all points $x \in E$ with the property that x is a cluster point of $\{F^n(y)\}$ for some $y \in E$, where F is a mapping of the uniform space $(E, \mathcal{U})$ into itself.

2.9 DEFINITION: Let $(E, \mathcal{U})$ be a uniform space and β a base for $\mathcal{U}$.

We say that a mapping $F : E \to E$ is β-contractive at $x \in E$ if for each $U \in \beta$ and $(x, y) \in U$, $x \neq y$, a $W \in \beta$ exists so that $(Fx, Fy) \in W \subset U$ and $(x, y) \notin W$.

Clearly a mapping is β-contractive if, and only if, it is β-contractive for all $x \in E$.

2.10 THEOREM: Let $(E, \mathcal{U})$ be a uniform space and β an open ample base for $\mathcal{U}$. Let $F : E \to E$ be a contraction with respect to β. Suppose $x \in E^F$ and F is β-contractive at x. Then

(a) x is periodic under F, i.e. there exists a positive integer k such that $F^k(x) = x$;

(b) if $(x, y) \in U \in \beta$ and y is periodic under F then $y = x$;

(c) if $Z \in E$ and x is a cluster point of $\{F^n(Z)\}$ then $\{F^{nk}(Z)\}$, $n = 1, 2, \ldots$, converges and its limit is $F^{\ell}(x)$ for some $\ell = 0, 1, \ldots, k - 1$.

REMARK 1: If $(E, \mathcal{U})$ and β are as in theorem 2.10 and F is β-contractive then each $x \in E^F$ is periodic, i.e. each $x \in E^F$ is a fixed point of F.

2.11 THEOREM: Let F be a β-contractive mapping of a uniform space $(E, \mathcal{U})$ into itself with respect to an open ample base β. Then the set of all periodic points of F is closed. If E is compact then this is finite. Moreover for each $x \in X$ there is a periodic point Z and an integer k so that $\{F^{nk}(Z)\}$, $n = 1, 2, \ldots$, converges to Z.

PROOF: Note that the set of all periodic points of F is here precisely E^F. Let x be an accumulation point of this set. It suffices to show that x is a cluster point of $\{F^n(x)\}$. Let, then, $U \in \beta$ be arbitrary and n a positive integer. Let $y \in W[x] \cap E^F$ where $W \in \beta$, $W \subset V \cap V^{-1}$ and $V \circ V \subset U$. Then $(x, y) \in W$ and, since $F^k(y) = y$ for some k, we have

$$(F^{nk}(x), F^{nk}(y)) = (F^{nk}(x), y) \in W$$

thus $(F^{nk}(x), x) \in W \circ W^{-1} \subset U$, i.e. $F^{nk}(x) \in U[x]$. Therefore x is a cluster point of $\{F^n(x)\}$ as asserted.

If, now, E is compact then E^F is compact too. (Since F is continuous.) The family $\{U[x] / x \in E^F\}$ is an open cover. This cover contains a finite subcover and by part (b) of Theorem 2.10 each element of this subcover contains one point of E^F only. Thus E^F is finite.

Since E is compact the final part of this theorem follows immediately from part (c) of Theorem 2.10.

REMARK 2: Theorem 2.11 is essentially the same as part (a) of Theorem 2.6. The only difference being, in Theorem 2.6, the author requires that the uniform space be Hausdorff and some iterate of E under F be compact. In Theorem 2.11 the uniform space E is assumed to be compact. In both cases they are using the fact that the continuous image of a compact space is compact to prove that the set of periodic points under F is compact and hence finite. The different assumptions arrive from the different techniques used to prove this fact.

2.12 THEOREM: Let $(E, \mathcal{U})$ be a U-chainable uniform space for some $U \in \beta$ where β is an open ample base for $\mathcal{U}$. Suppose F is a contraction mapping of E into itself with respect to β which is β-contractive at $x \in E^F$.

Suppose, further, that $\{F^n(y)\}$, $n = 1, 2, \ldots$, has a cluster point whenever $y \in U[x]$. Then $F(x) = x$ and x is unique with this property.

PROOF: It suffices to show that the set of all periodic points reduces to a singleton. Suppose y, $y \neq x$, is periodic and let n be the smallest integer with the property that a U-chain exists of the form

$$x = x_o, x_1, \ldots, x_n = y .$$

Let $F^k(x) = x$, $F^\ell(y) = y$. Since F is a contraction it follows that $\{F^{mk\ell}(x_i)\}$, $i = 0, 1, \ldots, n$, is a U-chain for all $m = 0, 1, \ldots$. The fact that β is ample clearly implies that all cluster points of $\{F^{mk\ell}(x\ )\}$ belong to U. It then follows from part (b) of Theorem 2.10 that the only cluster point of this sequence is x. From part (c) of the same theorem it then follows that $x = \lim_{m\to\infty} F^{mk\ell}(x_i)$. This in turn can be seen to imply that for suitable $m;(x, F^{mk\ell}(x_2)) \in U$. For such m a U-chain exists in which n can be replaced by $n' \leq n - 1$ since $x = x_o$, $F^{mk\ell}(x_2)$, $F^{mk\ell}(x_3)$, $\ldots$, $x_n = y$ is such a chain. This contradicts the definition of n, thus proving that x is a unique periodic - hence a fixed (Lemma 2.7) - point under F.

REMARK 3. If we allow E to be a countably compact uniform space with an ample base β for the uniformity $\mathcal{U}$, E to be U-chainable, and F to be β-contractive, then all assumptions of the above theorem are satisfied and the conclusion holds. Because a compact space is also countably compact, then part (b) of Theorem 2.6 satisfies the conditions of Theorem 2.12 and is therefore a weaker result.

The final work, using Definition 2.5, which we shall include here was given by Naimpally [20]. In this paper Naimpally proves Theorem 2.6, in the light of the generalized metric, for the special case having a base β as defined below.

2.13 DEFINITION: Let (X, d) be a generalized metric space. For any real $\varepsilon > 0$ let $U_\varepsilon = \{(x, y) \in X \times X / d(x, y) < \varepsilon\}$. Then the set $\beta_1 = \{U_\varepsilon / \varepsilon > 0\}$ is a base for a uniformity $\mathcal{U}_1$ of X under d.

2.14 LEMMA: The base β_1 is ample and the uniform space $(X, \mathcal{U}_1)$ is Hausdorff.

PROOF: If $(x, y) \in U_\varepsilon \in \beta_1$, i.e. $d(x, y) < \varepsilon$ for some $\varepsilon > 0$, let $d(x, y) = \lambda$ and consider $W = \{(x, y) \in X \times X / d(x, y) < \frac{\varepsilon + \lambda}{2}\}$. Then $\overline{W} = \{(x, y) \in X \times X / d(x, y) \leq \frac{\varepsilon + \lambda}{2}\}$ and

$$(x, y) \in W \subset \overline{W} \subset U_\varepsilon .$$

Thus β_1 is a ample base for $\mathcal{U}_1$.

Now the $\bigcap\{U_\varepsilon / \varepsilon > 0\} = \bigcap\{(x, y) \in X \times X / d(x, y) < \varepsilon\} = \{(x, y) \in X \times X / d(x, y) = 0\} = \Delta$.

Thus $(X, \mathcal{U}_1)$ is Hausdorff.

For the following X will denote a Hausdorff uniform space with a base β_1 for its uniformity induced by a generalized metric d and the terms Cauchy sequence, complete etc., will have the usual meanings with reference to d.

2.15 DEFINITION: A function $F : X \to X$ is said to be ε-contractive iff $d(Fx, Fy) \leq d(x, y)$ for all $x, y \in X$ such that $0 < d(x, y) < \varepsilon$.

2.16 LEMMA: If X is compact and F is an ε-contractive self mapping of X then F has a periodic point.

2.17 DEFINITION: Let β be a base for the uniformity $\mathcal{U}$ of X and $U \in \beta$. A function $F : X \to X$ is said to be U-contractive iff for each $(x, y) \in X$ $(x \neq y)$, $(x, y) \in U$ implies there exists a $W \in \beta$ such that $(Fx, Fy) \in W \subset U$ and $(x, y) \notin W$.

If F is U-contractive for all $U \in \beta$ then it is β-contractive.

REMARK 4: If we let $\beta_1 = \beta$ then this definition is equivalent to Definitions 2.14 of an ε-contractive mapping for a suitable $\varepsilon \geq 0$, $\varepsilon \in R$. We note also the difference between Definitions 2.9 and 2.14, In 2.9 Edelstein allows an $x \in X$ to be fixed and F to be β-contractive for all $(x, y) \in V \in \beta$ where $x \neq y$. In 2.14 F is allowed to be β-contractive for a given $U \in \beta$.

2.18 THEOREM: Let X be a Hausdorff uniform space with a basis β_1 obtained from a generalized metric d. For some $U \in \beta$ if $F : X \to X$ is U-contractive and $F^k[X]$ is compact then F has a periodic point. Moreover the set of all periodic points of F equals $A = \bigcap_{n=1}^{\infty} F^n[X]$ which is finite.

PROOF: U is obtained from some ε belonging to R with $\varepsilon > 0$ and so F is ε-contractive. The restriction of F to $F^k[X]$ is an ε-contractive self-mapping and so by Lemma 2.15 F has a periodic point in $F^k[X]$ and hence in X.

The final part of the theorem follows from the fact $A = \bigcap_{n=1}^{\infty} F^n[X]$ is compact, $F[A] = A$ and for any two distinct points $(p, q) \in A$, $d(p, q) \geq \varepsilon$.

2.19 <u>THEOREM</u>: If in Theorem 2.18 X is U-chainable then F has a unique fixed point.

<u>PROOF</u>: Since $A \subset X^F$, A itself is ε-chainable for some ε belonging to R, $\varepsilon > 0$ which corresponds to U. But for $p, q \in A$, $p \neq q$ $d(p, q) \geq \varepsilon$ and so A must contain only one point which is the unique fixed point of F.

Because Theorem 2.6 was the first fixed point theorem proven for uniform spaces, the papers which followed were, as we have seen, dealing with one generalization or another of it. The first to diverge from it and prove an entirely different fixed point theorem was due to Knill [15]. This paper also gave a new definition of the non-expansive map for uniform spaces as stated below.

2.20 <u>DEFINITION</u>: Let $(E, \mathcal{U})$ be a uniform space. A function F of E into itself is non-expansive if for any entourage U of $\mathcal{U}$ there is a closed entourage V such that

$$F\,V \subset \mathrm{Int}(V) \subset U.$$

Here $\mathrm{Int}(V)$ = interior of V and $F\,V = \{(Fx, Fy)/(x, y) \in V\}$.

The function F is a uniform non-expansive if for any entourage $U \in \mathcal{U}$ there are entourage V and W such that

$$F\,V \circ W \subset V \subset U.$$

Knill makes uses of this definition to give a contraction principle for uniform spaces. He then shows how the contraction principle may be used to give a simple proof, in uniform spaces, of two theorems from metric spaces due to Edelstein [8] and Rakotch [23]. A complete proof of Knill's contraction principle will be presented here.

2.21 LEMMA: Let $(E, \mathcal{U})$ be a uniform space. A function F of E into itself is a uniform non-expansive map iff for every entourage U of $\mathcal{U}$ there are symmetric entourages V and W such that

$$W \circ (F(W \circ V \circ W)) \circ W \subset V \subset U.$$

2.22 COROLLARY: Any uniform non-expansive map of a uniform space $(E, \mathcal{U})$ is a non-expansive map.

2.23 DEFINITION: A uniform space $(E, \mathcal{U})$ is well-chained if for every pair x, y of points of E and any entourage U of $\mathcal{U}$ there is a positive integer n such that $(x, y) \in U^n (U^n = U \cdot U \circ U \ldots)$.

2.24 THEOREM: (Uniform contraction principle). A uniform contraction F of a sequentially complete well-chained uniform space $(E, \mathcal{U})$ leaves exactly one point of E fixed.

PROOF: As in the metric case it is sufficient to show that for any point x of E, the sequence $(F^n(x), n = 1, 2, \ldots)$ is a Cauchy sequence. The limit point x_o of this sequence is then the fixed point of F. Let x be any point of E. Before showing that $(F^n(x), n = 1, 2, \ldots)$ is a Cauchy sequence we need to show that for any $U \in \mathcal{U}$ there is a positive integer N such that:

$$(F^n(x) , F^{n+1}(x)) \in U \quad \text{for all } n \geq N. \tag{1}$$

Now from Lemma 2.21 it follows that there are symmetric entourages V and W such that

$$F (V \circ W) \subset V \subset U \quad \text{and} \quad F W \subset W^n$$

We now show that for all $n \geq 1$,

$$F^n(V \circ W^n) \subset V \tag{2}$$

For $n = 1$ it is true by our choice of V and W. Suppose (2) is true for some $n \geq 1$. Then because F is continuous we have

$$F^{n+1}(V \circ W^{n+1}) \subset F^n_\circ(F(V \circ W) \circ F(W^n)) \subset F^n(V \circ W^n) \subset V.$$

(Since $F(V \circ W) \subset V \circ W$ and $FW^n \subset W^n$.)

Thus by induction on n, (2) is true for positive integer n.

To prove (1) observe that since E is well chained, there is a positive integer N such that $(x, F(x)) \in W^n$. Thus for $n \geq N$,

$$(F^n(x) , F^{n+1}(x)) \in F^n(V \circ W^N) \subset F^n(V \circ W^n) \subset V \subset U,$$

which proves (1).

Now apply (1) to W to choose an integer $N' \geq 0$ such that for $n \geq N'$, $(F^n(x) , F^{n+1}(x)) \in W$. We claim that for any pair of integers $m, n > N'$

$$(F^m(x) , F^n(x)) \in V \tag{3}$$

i.e. $(F^n(x) , n = 1, 2, \ldots)$ is a Cauchy sequence, since $V \subset U$ and U was an arbitrary element of $\mathcal{U}$. Since V is symmetric it suffices to prove (3) for $n = m + k$ where $k \geq 0$. For $k = 0$ it is true since V contains the diagonal (Δ) of E. Suppose (3) is true for all m and for some $n = m + k$ and $k \geq 0$; then $(F^{m-1}(x), F^m(x)) \in W$ by assumption and $(F^m(x), F^{m+k}(x)) \subset V$ by the induction hypothesis. Hence $(F^{m-1}(x), F^{m+k}(x)) \in V \circ W$ for all $m > N'$, and so $(F^m(x), F^{m+k+1}(x)) \in F(V \circ W) \subset V$ for all $m > N'$. Thus (3) is true by induction on $k = n - m$ for all $m > N'$ and all $n \geq m$, which was what we wanted to show.

Knill also proved the following lemma.

2.25 LEMMA: Let $(E, \mathcal{U})$ be a uniform space such that the topology induced on E by $\mathcal{U}$ is compact. Then a self map of E is non-expansive iff it is uniform non-expansive.

The following corollary of Theorem 2.24 follows from the above lemma and the fact that for a compact space E, E is well-chained iff E is connected.

2.26 COROLLARY: If F is a non-expansive map of a compact connected space E then F leaves exactly one point of E fixed.

The following theorems have been proven in metric spaces.

2.27 THEOREM: (Edelstein): Let (X, d) be a compact connected metric space and let a be a positive real number. If F is a function of X into itself such that $d(F(x), F(y)) < d(x, y)$ for all x, y in X such that $d(x, y) < a$, then F has a unique fixed point in X.

2.28 THEOREM: (Rakotch): Let (X, d) be a complete ε-chainable metric space, let a be a positive real number and let ρ be a monotone decreasing function of the interval $(0, a]$ into the interval $[0, 1)$. Suppose F is a function of X into itself such that whenever x, y are in X and $0 < d(x, y) \leq a$, then $d(F(x), F(y)) < \rho(d(x, y)) \cdot d(x, y)$. Then F has a unique fixed point in X.

REMARK 5. Suppose (X, d) is a metric space and F is a self map of X such that $d(F(x), F(y)) < d(x, y)$ whenever $d(x, y) < a$ for some fixed a. Now consider the uniform space $(X, \mathcal{U}d)$ where Ud is the uniformity for X generated by d in the usual way. Then if $Ud \in \mathcal{U}d$ i.e. $U_d = \{(x, y)/d(x, y) < \varepsilon, \ 0 < \varepsilon \leq a\}$ and we let $V_d = \{(x, y)/d(x, y) < \varepsilon/2\}$ then $V_d \subset U_d$ and we have for all $(x, y) \in V_d$

$$d(F(x), F(y)) < d(x, y).$$

Then $F(Vd) \subset \mathrm{Int}(Vd) \subset Ud$ and F is therefore a non-expansive map on $(X, \mathcal{U}d)$. Thus the Corollary 2.26 extends Edelstein's theorem to arbitrary compact connected spaces.

To see that Knill's uniform contraction principle includes Rakotch's theorem we need the following.

2.29 DEFINITION: A uniform non-expansion of a metric space (X, d) is a self map F of X such that for some number $a > 0$ and every $e \in (0, a]$ there is a real number $r = r(e)$ less than 1 such that if x, y are points of X and $d(x, y) < e$, then $d(F(x), F(y)) < r \cdot e$.

2.30 PROPOSITION: If F is a uniform non-expansive map of a metric space (X, d) then F is a uniform non-expansive map of the uniform space $(X, \mathcal{U}d)$ where $\mathcal{U}d$ is the uniform structure on X induced by d.

2.31 COROLLARY: Rakotch's theorem.

PROOF: We need only to show that for (X, d) and F as in the statement of Theorem 2.28, F is a uniform non-expansion of the metric space (X, d). Let ρ be a monotone decreasing function of the interval $(0, a]$ into $[0, 1)$ such that if x, y are points of X and $0 < d(x, y) \leq a$, then $d(F(x), F(y)) < \rho(d(x, y) \cdot d(x, y))$. Suppose $e \in (0, a]$. Then let $r = \max(\rho(e/2), 1/2)$. If $d(x, y) < e$, then either $d(x, y) = 0$ in which case $d(F(x), F(y)) = 0 \leq r \cdot e$, or $0 < d(x, y) \leq e/2$ in which case $d(F(x), F(y)) < \varepsilon/2 \leq r \cdot e$, or finally $\varepsilon/2 < d(x, y) < e$, in which case

$$d(F(x), F(y)) < \rho(d(x, y)) \cdot d(x, y) \leq \rho(\varepsilon/2) \cdot e \leq r \cdot e.$$

Thus in all cases $d(F(x), F(y)) \leq r \cdot e$ if $0 \leq d(x, y) < e$.

REMARK 6: It is often more convenient, as shown above, to prove a theorem for metric space by working with an equivalent uniform space, having the uniformity induced by the metric d. Another example of this technique is given at the end of this chapter, when we introduce a theorem by Chandler [4].

We note also that Davis [6] has proven a fixed point theorem for a well-chained topological space, however this theorem is straightforwardly included in Theorem 2.24.

We move now to contraction mappings and recall the definition of a $\mathcal{U}$-contractive mapping as given by Casesnoves.

2.32 DEFINITION: If $(E, \mathcal{U})$ is a uniform space and F a map of E into itself, such that $g = (F, F)$ is the extension of F to the product space $E \times E$, then F is said to be $\mathcal{U}$-contractive, provided the following conditions are satisfied.

(a) $\forall V \in \mathcal{U}$, $g(V) \subset V$

(b) $\forall V$, $\forall W \in \mathcal{U}$, $\exists k \in N, \ni \forall p > 0$, $\forall n \geq k$,
$g^n(V) 0\, g^{n+1}(V)\, 0 \ldots 0\, g^{n+p}(V) \subseteq W$.

This definition differs in two ways from the foregoing definitions of non-expansive mappings in uniform spaces.

(1) Here the mapping is defined on the elements of the uniformity directly, rather than on its base. There is no discrepancy here as a base defines the uniformity completely (see Kelley [14] (p 176-177)), it is sufficient to work with its base.

(2) We have in this definition the extra condition (b) which is designed to replace in uniform space, the Lipschitz constant for contraction

mappings ... -ic spaces (see Definition 1.1).

2.33 PROPOSITION: Let $(E, \mathcal{U})$ be a uniform space, and $F: E \to E$ be $\mathcal{U}$-contractive. Then F is uniformly continuous.

PROOF: Since F is $\mathcal{U}$-contractive $g(V) \subset V$, $\forall\ V \in \mathcal{U}$. Now for any V in $\mathcal{U}$ there exists an entourage U of $\mathcal{U}$ such that $g(U) \subset V$, namely V itself.

Thus F is uniformly continuous.

2.34 THEOREM: (contraction principle). Let $(E, \mathcal{U})$ be a complete uniform space, $F : E \to E$ be $\mathcal{U}$-contractive then F has a unique fixed point.

PROOF: As in Theorem 2.24 we start by proving that the sequence of iterates $\{F^n(x),\ n = 0, 1, 2, \ldots\}$ is a Cauchy sequence.

Let $(x, F(x)) \in V$. Then $g(x, F(x)) = (F(x), F^2(x)) \subset g(V)$ by the very definition of F. Continuing we have;

$$(F^n(x)\ ,\ F^{n+1}(x)) \subset g^n(V).$$

Now if $m > n$ then we have $(F^n(x), F^{n+1}(x)) \subset g^n(V)$, $(F^{n+1}(x), F^{n+2}(x)) \subset g^{n+1}(V), \ldots, (F^{m-1}(x), F^m(x)) \subset g^{m-1}(V)$.

Thus $(F^n(x), F^{n+1}(x)) \circ (F^{n+1}(x), F^{n+2}(x)) \circ \ldots \circ (F^{m-1}(x), F^m(x)) \subset g^n(V) \circ g^{n+1}(V) \circ \ldots \circ g^{m-1}(V)$, i.e.

$$(F^n(x), F^m(x)) \subset g^n(V) \circ g^{n+1}(V) \circ \ldots \circ g^{m-1}(V).$$

By (b) of Definition 2.32, given any $W \in \mathcal{U}$ and choosing $n \geq k$, $m > n$ then for m and n satisfying these conditions we have

$$(F^n(x), F^m(x)) \subset W.$$

Thus $(F^n(x)$, $n = 0, 1, 2, \ldots)$ is a Cauchy sequence.

Now as E is complete, each Cauchy sequence has a limit, if this limit is Z then

$$Z = \lim_{n\to\infty} F^n(x) = F \lim_{n\to\infty} [F^{n-1}(x)] = F(Z)$$

holds because F is continuous. Therefore Z is a fixed point of F.

Next it is required to show that the fixed point is unique. To do this we show that the Cauchy sequences obtained by beginning with any two arbitrary points x, y of E are equivalent.

Let $(x, y) \in V$ then $(F(x), F(y)) \in g(V) \subset V$. It follows that $(F(x), F(y)) \in g(V)$ and $(F^2(x), F^2(y)) \in g(V)$ and therefore $(x, y), (F(x), F(y)), (F^2(x), F^2(y)) \in g(V)$ (Since $\Delta \in V$ and $g(V)$ also $(x, y) \in V$, $(y, y) \in g(V)$ we have $(x, y) \in g(V)$, similarly for the others). Continuing this procedure with n, m such that $m > n$, we have $(F^n(x), F^n(y)), (F^{n+1}(x), F^{n+1}(y)) \in g^n(V)$ and $(F^{m-1}(x), F^{m-1}(y)),(F^m(x), F^m(y)) \in g^{n-1}(V)$ thus

$$(F^n(x), F^n(y)), (F^m(x), F^m(y)) \in g^n(V)\ 0 \ldots 0\ g^{m-1}(V)$$

which proves that the sequence $\{F^n(x), F^n(y)\}$, $n = 0, 1, \ldots$ is a Cauchy sequence, and therefore the two Cauchy sequences are equivalent.

It follows that we cannot have more than one solution of equation (1), for if Z and y are both fixed points of F then $\{F^n(Z), F^n(y)\}$ cannot by Cauchy, contrary to above. Thus F has a unique fixed point.

<u>REMARK</u> 7: We note that even though Theorems 2.24 and 2.34 are similar in both statement and proof, Theorem 2.34 appears to be the better result. In the case of Theorem 2.24, Banach's contraction principle is, strictly

speaking, not generalized since the space must be well-chained.

2.35 <u>COROLLARY</u>: If $(E, \mathcal{U})$ is a uniform space such that some iterate F^k of F is $\mathcal{U}$-contractive then F has a unique fixed point.

<u>PROOF</u>: Since F^k is $\mathcal{U}$-contractive it has a unique fixed point x_o, say. Thus

$$x_o = \lim_{n\to\infty} F^{nk}(x_o) = F\,[\lim_{n\to\infty} F^{nk-1}(x_o)] = F(x_o) .$$

Thus F has a fixed point and uniqueness follows from Lemma 2.7.

Using Theorem 2.35, we prove the following two theorems in uniform spaces. The theorems were originally given in metric spaces by Chu and Diaz [5]. We would like to give these theorems in uniform space.

2.36 <u>THEOREM</u>: Consider $F : E \to E$ and if

(a) $(E, \mathcal{U})$ a complete uniform space and
$(E, \mathcal{V})$ a uniform space;

(b) The uniformity U is smaller than the uniformity V i.e. each entourage of U also belongs to V ;

(c) F is continuous on $(E, \mathcal{U})$ and $\mathcal{V}$-contractive on $(E, \mathcal{V})$.

Then F has a unique fixed point in E.

<u>PROOF</u>: Consider x_o , $F(x_o)$, $F^2(x_o)$, ... the iterations of F for some $x_o \in E$. Now since F is $\mathcal{V}$-contractive in $(E, \mathcal{V})$, by the same method as used in the proof of Theorem 2.34, it follows that the sequence is Cauchy in $(E, \mathcal{V})$.

Now because each entourage U of $\mathcal{U}$ also belongs to $\mathcal{V}$, $\{F^n(x_o), n = 1, 2, \ldots\}$ is a Cauchy sequence in $(E, \mathcal{U})$. But $(E, \mathcal{U})$ is complete thus $\{F^n(x_o), n = 1, 2, \ldots\}$ converges to some point Z in E and again by the same method as used in Theorem 2.34 we can prove that Z is unique.

Since F is continuous on $(E, \mathcal{U})$ we have

$$Z = \lim_{n\to\infty} F^{n+1}(x_o) = F \lim_{n\to\infty} F^{n}(x_o) = F(Z) \quad , \quad \text{i.e.}$$

Z is a unique fixed point of F.

2.37 <u>THEOREM</u>: Let T be a mapping from the complete uniform space $(E, \mathcal{U})$ into itself and let K be another map also defined on E into itself, such that K possesses a right inverse K^{-1}. Then T has a unique fixed point if $K^{-1}TK$ is $\mathcal{U}$-contractive.

<u>PROOF</u>: Since $K^{-1}TK$ is a $\mathcal{U}$-contractive it has a unique fixed point, say x_o. Thus $K[K^{-1}TK(x_o)] = K(x_o)$, $\therefore TK(x_o) = K(x_o)$. $K(x_o)$ is therefore a fixed point for T.

To prove uniqueness we assume that $K(x_o)$ and $K(x_1)$ are two fixed points for T. i.e.

$T(K(x_o)) = K(x_o)$ and $T(K(x_1)) = K(x_1)$, therefore

$$K^{-1}TK(x_o) = K^{-1}K(x_o) = x_o \quad \text{and}$$

$$K^{-1}TK(x_1) = K^{-1}K(x_1) = x_1$$

contradicting the fact that $K^{-1}TK$ has a unique fixed point.

2.38 <u>COROLLARY</u>: Let T and K be mapping from a complete uniform space $(E, \mathcal{U})$ into itself. If K possesses a right inverse and T is $\mathcal{U}$-contractive, then $K^{-1}TK$ has a unique fixed point.

<u>PROOF</u>: For some $x_o \in E$ we have

$$K^{-1}TK(x_o) = K^{-1}T[K x_o] = K^{-1}K x_o$$

$$= x_o$$

Thus $K^{-1}TK$ has a fixed point x_o.

To prove uniqueness we assume that $K^{-1}TK$ has two fixed points, say x_o and x_1, then $K^{-1}TK(x_o) = x_o$, thus $T(K(x_o)) = K(x_o)$ and $K^{-1}TK(x_1) = x_1$, thus $T(K(x_1)) = K(x_1)$,
But this is a contradiction since T has a unique fixed point.

R - Salinas [25] has defined a contraction mapping for uniform spaces and offered,without proof,a contraction principle somewhat similar to that of Casesnoves. He then uses this theorem to prove several short theorems involving fixed points, which, in one way or another, expands the stated theorem. His proofs, however, are purely topological and do not depend upon the unproved theorem. We offer here two of the more important theorems and prove one of them.

The theorem which R - Salinas has stated, without proof, will now be given below, and the definition of his contraction mapping shall be included in the statement of the theorem. We note that both the statement of the theorem and the definition of the mapping are much more involved than that of Casesnoves.

2.39 <u>THEOREM</u>: Let $(E, \mathcal{U})$ be a complete uniform space and $\{U_n, n = 1, 2, \ldots\}$ a sequence of elements of $\mathcal{U}$ such that for each $U \in \mathcal{U}$ one can find an $n_o \geq 0$ so that

$$U_n^{\,p} = U_{n+p} \circ U_{n+p-1} \circ \cdots \circ U_{n+1} \circ U_n \subset U \qquad (1)$$

for all $n \geq n_o$ and $p \geq 0$ and

$$\overline{\bigcup_{n \geq 0} U_o^{\,n}} \subset U^* \text{ , where } U^* \in \mathcal{U}. \qquad (2)$$

Let T be a continuous map of $U^*[x_o]$ into E with the property that $(x, y) \in U_n$ and $\{x, y\} \subset U^*[x_o]$ imply $(T(x), T(y)) \in U_{n+1}$ for each $n \geq 0$.

Then, if $x \in U_o[x_o]$ and $(x, T(x)) \in U_1$ the sequence $\{T^n(x), n = 1, 2, \ldots\}$ converges to a point x^*, which is a fixed point of T, and therefore the existence of one such point x assures the existence of at least one fixed point x^* of T. Furthermore, if x_1 and x_2 are two fixed points of T, belonging to $U^*[x_o]$ and such that $(x_1, x_2) \in U_o$ it follows that $x_1 = x_2$.

As in the case of Theorem 2.34 the following is an obvious corollary.

2.40 COROLLARY: With the same conditions as Theorem 2.39, if from $(x, y) \in U^*$ and $\{x, y\} \subset U^*[x_o]$ it follows that $(T^n(x), T^n(y)) \in U_o$ for some $n \geq 0$ dependent on (x, y), then there exists at most, one fixed point of T belonging to $U^*[x_o]$.

REMARK 8: If $U^*[x_o]$ is closed and T is a continuous map of $U^*[x_o]$ into itself, such that $TU^*[x_o] \subset U^*[x_o]$ one can suppress the condition (2) in Theorem 2.39 as well as in the above corollary.

NOTATION: In what follows:

1. E will be a compact Hausdorff space;

2. T will be a continuous map of E into itself;

3. $S_a x$ will be the collection of accumulation (or limit) points of $\{T^n(x), n = 1, 2, \ldots\}$ for $x \in E$. It is obvious that x is a fixed point of T if and only if $S_a x = \{x\}$.

2.41 THEOREM: There exists at least one closed subset $E_m \neq \phi$ of E, invarient w.r.t. T (i.e. $TE_m = E_m$), which contains no other closed subset $F \neq \phi$ such that $TF \subset F$.

If E_m is a set with these properties it can only happen that:

(1) E_m is finite and consists of n_o points.

Then if $x \in E_m$ it follows that $T^{n_o}x = x$ and

$$E_m = \{T^n(x) \ / \ 0 \leq n < n_o\} \quad \text{i.e. } S_a x = E_m.$$

or (2) E_m is infinite. Then, for each $x \in E_m$, $T^n(x) \neq x$ for all $n = 1, 2, \ldots$ and $S_a x = x_m$ consequently E_m is a perfect set.

PROOF: Keeping in mind that each decreasing chain of closed subsets (non-empty) of E, which are transformed into themselves by T, have an intersection with these properties, it follows from Zorn's lemma that there exists a closed subset $E_m \neq 0$ with $TE_m \subset E_m$ which contains no other closed subset $F \neq 0$ with $TF \subset F$.

The other assertions of the theorem are easily proved; for example, to prove that $S_a x = E_m$ for each $x \in E_m$, it suffices for $S_a x$ to be a closed subset (non-empty) of E_m, for $x \in E_m$ with $TS_a x \subset S_a x$ it follows that $S_a x = E_m$. In particular, then $TE_m = TS_a x = S_a x = E_m$.

2.42 THEOREM: Let $B = \{B_n, n = 0, 1, \ldots\}$ be a decreasing sequence of neighborhoods of the diagonal Δ of E^2 with

$$\bigcap_o^\infty \overline{B}_n = \Delta$$

and $TB_n[x] \subset B_{n+1}[x]$ for each $x \in E$ and $n \geq 0$. Then

$$E_o = \{S_a x \ / \ x \in E\}$$

is a finite set such that each component (see Kelley [14] p. 54-55) of E

contains at most one point of E_o.

2.43 COROLLARY: If in addition to the conditions satisfying Theorem 2.42 E is connected, the sequence $\{T^n(x), n = 1, 2, \ldots\}$ converges to a point x^* independent of $x \in E$ and thus x^* is a unique fixed point of T.

PROOF: Trivial, since if E is connected, then it is its only component, $S_a x$ therefore, contains only one point.

Finally we give another definition for the contraction map in uniform spaces and use it to prove the necessary contraction principle, both of which are due to Chandler [4] . In this case the uniform space is restricted to the class having a countable symmetric base. The theorem is, however, important; because a uniform space as described above, is equivalent to a metric space and thus a result proven here can be immediately transferred to metric spaces.

2.43 DEFINITION: Let $(E, \mathcal{U})$ be a uniform space. A mapping $F : E \to E$ is $\mathcal{U}$-contracting provided there is a collection of symmetric sets $\{V_n\}, n \in Z$, cofinal in $\mathcal{U}$(with respect to the ordering $U_1 \geq U_2$ if and only if $U_1 \subseteq U_2$) which satisfy

(1) $V_i \subseteq V_j$ if $i \subseteq j$, $\bigcap_{n \in Z} V_n = \Delta$, $\bigcup_{n \in Z} V_n = E \times E$,

(2) for each $n = 1, 2, \ldots$ there is an integer $p(n) > 0$ such that $\{p(n) / n \in Z\}$ is bounded and $V_{n-p(n)} \circ V_{n-p(n)} \subseteq V_n$,

(3) if $(x, y) \in V_n$ then $(F(x), F(y)) \in V_{n-1}$.

2.44 LEMMA: If $F : E \to E$ is $\mathcal{U}$-contracting then F has at most one fixed point.

2.45 LEMMA: If $F : E \to E$ is $\mathcal{U}$-contracting then so is any iterates, F^p, of F and also F is uniformly continuous.

2.46 THEOREM: Let $F : E \to E$ be $\mathcal{U}$-contracting where $(E, \mathcal{U})$ is a complete uniform space. Then there is exactly one $x_o \in E$ for which $F(x_o) = x_o$.

PROOF: Let $P = \max \{p(n) / n \in Z\}$ and let x be an arbitrary point of E. Let g denote the pth iterate of F. Rename, if necessary, the V_n so that $(x, g(x)) \in V_o$. Then

$$(g(x), g^2(x)) \in V_{-p} \quad , (g^2(x), g^3(x)) \in V_{-2p} \quad , \cdots ,$$
$$(g^n(x), g^{n+1}(x)) \in V_{-np} \quad , \ldots, (g^{n+q}(x), g^{n+q+1}(x)) \in V_{-(n+q)p}.$$

Thus $(g^n(x), g^{n+q+1}(x)) \in V_{-np} \circ V_{-(n+1)p} \circ \ldots \circ V_{-(n+q+1)p} \circ V_{-(n+q)p}$.

Now $V_{-(n+q)p} \subsetneq V_{-(n+q-1)p}$ (since $V_{-(n+q)p} \subset V_{-(n+q)p+1}$ $\subset \ldots \subset V_{-(n+q)p+p} = V_{-(n+q-1)p})$, so that
$V_{-(n+q-1)p} \circ V_{-(n+q)p} \subsetneq V_{-(n+q-1)p} \circ V_{-(n+q-1)p} \subsetneq V_{-(n+q-2)p}$.

Consequently, we see that

$V_{-np} \circ V_{-(n+1)p} \circ \ldots V_{-(n+q-1)p} \circ V_{-(n+q)p} \subsetneq V_{-np} \circ V_{-np} \subsetneq V_{-(n-1)p}$.

For each $U \in \mathcal{U}$ there is an N such that if $(n - 1)p > N$ then $V-(n-1)p \subsetneq U$ since $\{V_n, n \in Z\}$ is cofinal in $\mathcal{U}$.

Thus, if $n > N/p+1$ and $q \geq 0$, we have $(g^n(x), g^{n+q+1}(x)) \in V_{-(n-1)p} \subsetneq U$. Therefore $\{g^n(x), n = 1, 2, \ldots\}$ is a Cauchy sequence in $(E, \mathcal{U})$.

Let $x_o = \lim g^n(x)$. Since g is uniformly continuous we have $g(x_o) = g(\lim g^n(x)) = \lim g^{n+1}(x) = x_o$ and so x_o is a fixed point of g. However,

$$g(F(x_o)) = F(g(x_o)) = F(x_o).$$

Thus $F(x_o)$ is also a fixed point of g. We conclude that $F(x_o) = x_o$.

2.47 <u>COROLLARY</u>: (Banach.) If $F : E \to E$ where E is a complete metric space (metric d) and $d(F(x), F(y)) \leq \alpha d(x, y)$ for some $\alpha \in [0, 1)$ and all $x, y \in E$, then F has a unique fixed point.

<u>PROOF</u>: If $\alpha = 0$ then F is a constant mapping and so has a unique fixed point. If $\alpha \neq 0$ then in $E \times E$ we define
$V_n = \{(x, y) / d(x, y) < \alpha^{-n}\}$, $n = 0, 1, \ldots$.
Then $\{V_n\}$, $n \in Z$ shows that F is $\mathcal{U}$-contracting.

We show, finally, how one can use Theorem 2.46 to give a simple proof for a theorem in metric spaces by using the equivalent uniform space.

2.49 <u>THEOREM</u>: (Edelstein). If $F : X \to X$ is a (ε, α) - uniformly locally contractive $(d(F(x), F(y)) \leq \alpha d(x, y)$ when $d(x, y) < \varepsilon$, $\alpha \in [0, 1)$, and $\varepsilon > o)$ where (X, d) is a complete metric space and if for each $(x, y) \in X \times X$ there is an integer $n > 0$ such that $d(F^n(x), F^n(y)) < \varepsilon$, then F has a unique fixed point.

<u>PROOF</u>: Define

$$V_{-n} = \{(x, y) / d(x, y) < \alpha^n \varepsilon\}, \quad n = 0, 1, 2, \ldots$$

and

$$V_n = \{(x, y) / (F^n(x), F^n(y)) \in V_o\}, \quad n = 1, 2, \ldots$$

(If $\alpha = 0$ define

$$V_o = \{(x, y) / d(x, y) < \varepsilon\}$$

and

$$V_{-n} = \{(x, y) / d(x, y) < \varepsilon 2^{-n}\}, (n = 1, 2, \ldots).$$

Then $\{V_n\}_{n \in Z}$ shows that F is $\mathcal{U}$-contracting.

CHAPTER III

Sequences of Contraction Mappings

We recall the Banach contraction principle states that a contraction mapping from a complete metric space to itself leaves exactly one point fixed. One may ask : If X is a complete metric space, does the convergence of a sequence of contraction mappings to a contraction mapping T imply the convergence of the sequence of their fixed points to the fixed point of T? The first solution to this problem was offered to Bonsall [1] in the following way.

3.1 <u>THEOREM</u>: Let X be a complete metric space. Let T and T_n be contraction mappings of X into itself for $n = 1, 2, \ldots$, with the same Lipschitz constant $K < 1$, and with fixed points u, u_n $(n = 1, 2, \ldots)$ respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$. Then $\lim_{n\to\infty} u_n = u$.

In this chapter we consider some extensions of Theorem 3.1 and also state and prove a theorem for a sequence of contraction mappings for a complete ϵ-chainable metric space. In the later part of the chapter we use the definition of a $\mathcal{U}$-contractive mapping, given by Casesnoves [3], to prove a theorem in uniform spaces similar to Theorem 3.1.

In the statement of Theorem 3.1 it is assumed that T is a contraction mapping. We now show that this condition is superfluous as it can be proven from the remaining statement of the theorem.

3.2 <u>LEMMA</u> : Let X be a complete metric space and let $T_n (n = 1, 2, \ldots)$ be contraction mappings of X into itself with the same Lipschitz constant

$K < 1$. Suppose $\lim\limits_{n\to\infty} T_n x = Tx$ for each $x \in X$, where T is a mapping from X into itself. Then T is a contraction mapping.

<u>PROOF</u> : Since $K < 1$, is the same Lipschitz constant for all n,

$$|T(x) - T(y)| = \lim_{n\to\infty} |T_n(x) - T_n(y)| \leq K|x - y|.$$

Thus T is a contraction mapping with contraction constant K, and as such has a unique fixed point.

We now state Theorem 3.1 in the modified form and give for it a proof due to Singh [28] which is simpler than that given by Bonsall [1].

3.3 <u>THEOREM</u> : Let X be a complete metric space and let $\{T_n, n = 1, 2, \ldots\}$ be a sequence of contraction mappings with the same Lipschitz constant $K < 1$, and with fixed points $u_n (n = 1, 2, \ldots)$. Suppose that $\lim\limits_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point u and $\lim\limits_{n\to\infty} u_n = u$.

<u>PROOF</u> : From Lemma 3.2 it follows that T has a unique fixed point u.

Since the sequence of contraction mappings converges to T, therefore for a given $\epsilon > 0$, there exists an N such that $n \geq N$ implies

$$d(T_n(u) , T(u)) \leq (1 - K)\epsilon ,$$

where K is the contraction constant. Now for $n \geq N$,

$$\begin{aligned} d(u, u_n) &= d(T(U) , T_n(U_n)) \\ &\leq d(T(u) , T_n(u)) + d(T_n(u) , T_n(u_n)) \\ &\leq (1 - K)\epsilon + K\, d(u, u_n) \end{aligned}$$

Thus $(1 - K)d(u, u_n) \leq (1 - K)\epsilon$. Now since $0 < K < 1$, we have

$$d(u, u_n) < \epsilon, \quad n \geq N$$

and so $\lim_{n\to\infty} u_n = u$.

The above theorem emits a useful corollary.

3.4 <u>COROLLARY</u> : Let X be a complete metric space and let T_n $(n = 1, 2, \ldots)$ be contraction mappings of X into itself with Lipschitz constants $K_n (n = 1, 2, \ldots)$ such that $K_{n+1} \leq K_n < 1$ for each n, and with fixed points U_n $(n = 1, 2, \ldots)$. Suppose that $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point U and $\lim_{n\to\infty} U_n = U$.

<u>PROOF</u> : Now $\lim_{n\to\infty} |T_n(x) - T_n(y)| \leq \lim_{n\to\infty} K_n |x - y|$ and since $K_{n+1} \leq K_n$ for all n, it follows that $\lim_{n\to\infty} K_n < 1$. Hence

$$T(x) = \lim_{n\to\infty} T_n(x)$$

is a contraction mapping. Moreover K_1 will be a Lipschitz constant for all T_n $(n = 1, 2, \ldots)$. Thus the proof follows by replacing K by K_1 in the above theorem.

3.5 <u>EXAMPLE</u> : Consider $T_n : [0, 2] \to [0, 2]$ such that $T_n(x) = 1 + \frac{x}{n+1}$. $(n = 1, 2, \ldots)$.

Now $\lim_{n\to\infty} T_n(x) = 1 = T(x)$ for every $x \in [0, 2]$. The Lipschitz constants are $K_n = \frac{1}{n+1}$ $(n = 1, 2, \ldots)$. Thus $K_1 = 1/2$ will serve the purpose for all mappings to be contraction. The fixed points are

$$U_n = \frac{n+1}{n}; \ (n = 1, 2, \ldots),$$

Now $\lim_{n\to\infty} U_n = 1$, and 1 is the unique fixed point for T.

If in the above corollary the Lipschitz constants are such that $K_{n+1} \geq K_n$ $(n = 1, 2, \ldots)$ the result is, in gnereal, false.

3.6 EXAMPLE: Consider $T_n : E' \to E'$ such that $T_n x = p + \frac{nx}{n+1}$ $(n = 1, 2, \ldots)$, $p > 0$. (Where $E' = (-\infty, +\infty)$).

The Lipschitz constants are $K_n = \frac{n}{n+1}$; $(n = 1, 2, \ldots)$.

The fixed points are $U_n = (n+1)P$; $(n = 1, 2, \ldots)$.

Now $\lim\limits_{n\to\infty} T_n(x) = p + x = T(x)$ for every $x \in E'$.

That is under the mapping T every point of E' has been translated by a distance P, thus T does not have a fixed point. Moreover,

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} (n+1)p = \infty \quad \text{and} \quad \infty \notin E'.$$

Also

$$\lim_{n\to\infty} |T_n(x) - T_n(y)| = |p + x - p - y| = 1\,|x - y|, \quad (x, y \in E').$$

Thus, T is not a contraction mapping.

Singh [26] has proven a corollary similar to Corollary 3.4 by replacing the condition $K_n \leq K_{n+1}$ by $K_n \to K < 1$ for all $n = 1, 2, \ldots$.

3.7 DEFINITION: Let (X, d) be a metric space and $\epsilon > 0$. A finite sequence $x_o, x_1, \ldots, x_n$ of points of X is called an ε-chain joining x_o and x_n if

$$d(x_{i-1}, x_i) < \epsilon, \quad (i = 1, 2, \ldots, n).$$

The metric space is said to be ε-chainable if, for each pair (x, y)

of its points, there exists an ε-chain joining x and y.

Edelstein [8] proved the following theorem.

3.8 <u>THEOREM</u>: Let T be a mapping of a complete ε-chainable metric space (X, d) into itself, and suppose that there is a real number K with $0 \leq K < 1$ such that

$$d(x, y) < \epsilon \Rightarrow d(T(x), T(y)) \leq Kd(x, y).$$

Then T has a unique fixed point U in X, and $U = \lim_{n\to\infty} T^n x_o$ where x_o is an arbitrary element of X.

In the above theorem Edelstein has taken an ε-chainable metric space and has considered contraction mapping. In [27] we proved a theorem by considering a sequence of such mappings.

3.9 <u>THEOREM:</u> Let (X, d) be a complete ε-chainable metric space, and let T_n $(n = 1, 2, \ldots)$ be mappings of X into itself, and suppose that there is a real number K with $0 \leq K < 1$ such that

$$d(x, y) < \epsilon \Rightarrow d(T_n(x), T_n(y)) \leq Kd(x, y) \text{ for all } n.$$

If $U_n (n = 1, 2, \ldots)$ are the fixed points for T_n and $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$, then T has a unique fixed point U and $\lim_{n\to\infty} U_n = U$.

<u>PROOF:</u> (X, d) being ε-chainable we define for $x, y \in X$,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{p} d(x_{i-1}, x_i)$$

where the infimum is taken over all ε-chains $x_o, x_1, \ldots, x_p$ joining $x_o = x$ and $x_p = y$. Then d_ε is a distance function on X satisfying

(i) $d(x, y) \leq d_\varepsilon(x, y)$

(ii) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$.

From (ii) it follows that a sequence $\{x_n\}$, $x_n \in X$ is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect to d. Hence, (X, d) being complete, (X, d_ε) is also a complete metric space. Moreover $T_n(n = 1, 2, \ldots)$ are contraction mappings with respect to d_ε. Given $x, y \in X$, and any ε-chain $x_o, x_1, \ldots x_p$ with $x_o = x, x_p = y$, we have $d(x_{i-1}, x_i) < \varepsilon (i = 1, 2, \ldots p)$, so that $d(T_n(x_{i-1}), T_n(x_i)) \leq K_d(x_{i-1}, x_i) < \varepsilon$ $(i = 1, 2, \ldots, p)$. Hence $T_n(x_o), \ldots, T_n(x_p)$ is an ε-chain joining $T_n(x)$ and $T_n(y)$ and

$$d_\varepsilon(T_n(x), T_n(y)) \leq \sum_{i=1}^{p} d(T_n(x_{i-1}), T_n(x_i))$$

$$\leq K \sum_{i=1}^{p} d(x_{i-1}, x_i)$$

$x_o, x_1, \ldots, x_p$ being an arbitrary ε-chain, we have

$$d_\varepsilon(T_n(x) , T_n(y)) \leq Kd_\varepsilon(x, y).$$

Now since $T_n(n = 1, 2, \ldots)$ are contraction mappings with respect to d_ε and (X, d_ε) is a complete metric space, then $T(x) = \lim_{n\to\infty} T_n(x)$ is a contraction mapping with respect to d_ε. Moreover T has a unique fixed point U and $\lim_{n\to\infty} U_n = U$ by Theorem 3.3.

This unique fixed point is given by

$$\lim_{m\to\infty} d_\varepsilon(T^m(x_o), U) = 0 \quad \text{for } x_o \in X \text{ arbitrary.}$$

But (i) at the beginning of the proof implies

$$\lim_{m\to\infty} d(T^m(x_o) , U) = 0.$$

Nadler [19] gave a generalization of Theorem 3.3 in the following way.

3.10 <u>THEOREM</u>: Let (X, d) be a locally compact metric space, let $T_n : X \to X$ be a contraction with fixed point a_n for each $n = 1, 2, \ldots$, and let $T_o : X \to X$ be a contraction with fixed point a_o. If the sequence $\{T_n, n = 1, \ldots\}$ converges pointwise to T_o, then the sequence $\{a_n, n = 1, 2, \ldots\}$ converges to a_o.

Further results for sequences of contractive mappings and fixed points have been given by Fraser and Nadler [13].

Finally, using Definition 2.32 of a $\mathcal{U}$-contractive mapping we give a generalization of Theorem 3.1 in uniform spaces.

3.11 <u>THEOREM</u>: Let $(E, \mathcal{U})$ be a complete uniform space and F_k a $\mathcal{U}$-contractive mapping from E into itself, with fixed points U_k $(k = 1, 2, \ldots)$. Suppose $\lim\limits_{k\to\infty} F_k(x) = F(x)$ for every $x \in E$, where F is a $\mathcal{U}$-contractive map from E into itself. Then $\lim\limits_{k\to\infty} U_k = U$ where U is the unique fixed point of F.

<u>PROOF</u>: Consider
$$\lim_{n\to\infty}\left[\lim_{k\to\infty} F^n{}_k(x)\right] = \lim_{n\to\infty}\left[F^n(x)\right] \tag{1}$$

Now because F is a $\mathcal{U}$-contractive it follows that $F^n(x)$ is a Cauchy sequence and as such converges; thus by Theorem 2.34 we have,

$$\lim_{n\to\infty}\left[F^n(x)\right] = U$$

Now $$\lim_{n\to\infty}\left[\lim_{k\to\infty} F^n{}_k(x)\right]$$

$$= \lim_{k\to\infty} \left[U_k \right] , \qquad (2)$$

Since $F_k^{\,n}(x)$, $n = 1, 2, \ldots$ is $\mathcal{U}$-contractive with fixed points for each k and thus $\lim_{n\to\infty} F_k^n(x) = U_k$.

Now combining (1) and (2) we have

$$\lim_{k\to\infty} U_k = U.$$

Using Theorem 3.11 we offer also the following theorem.

3.12 THEOREM: Consider $T_n(k = 1, 2, \ldots)$ and T mappings from E into itself and if

(a) $(E, \mathcal{U})$ a complete uniform space and

$(E, \mathcal{V})$ a uniform space;

(b) the uniformity U is smaller than the uniformity V, i.e. each entourage of U also belongs to V ;

(c) T_k and T are continuous on $(E, \mathcal{U})$ and $\mathcal{V}$-contractive on $(E, \mathcal{V})$.

Then if $(U_k, k = 1, 2, \ldots)$ and U are the fixed points of $T_k(k = 1,2,\ldots)$ and T respectively and if $\lim_{k\to\infty} T_k(x) = T(x)$ for every $x \in E$, we have,

$$\lim_{k\to\infty} U_k = U.$$

PROOF: It follows from Theorem 2.36, the sequence of iterates $T_k^{\,n}$ $(n = 1, 2, \ldots)$ of $T_k(k = 1, 2, \ldots)$ and T^n of T are Cauchy in $(E, \mathcal{U})$ and as such converge to the unique fixed points of T_k and T respectively

i.e.

$$\lim_{n\to\infty} T^n{}_k(x) = U_k \quad (k = 1, 2, \ldots) \quad \text{and}$$

$$\lim_{n\to\infty} T^n(x) = U$$

where $x \in E$. Thus from Theorem 3.11 it follows that

$$\lim_{k\to\infty} U_k = U.$$

3.13 REMARK: In the proof of Theorem 3.11 the contraction mapping principle of Casesnoves was used only to show that the sequence of iterates F^n_k $(n = 1, 2, \ldots)$ of $F_k (k = 1, 2, \ldots)$ and F^n of F are Cauchy and as such converge. As the convergence of the Cauchy sequence is also used by Knill and Chandler, it is possible to give a generalization of their contraction principle similar to Theorem 3.11. In the case of Chandler the theorem is as follows.

3.14 THEOREM: Let $(E, \mathcal{U})$ be a complete uniform space and F_k a $\mathcal{U}$-contracting mapping from E into itself, with fixed points $U_k (k = 1, 2, \ldots)$. Suppose $\lim_{k\to\infty} F_k(x) = F(x)$ for every $x \in E$, where F is a $\mathcal{U}$-contracting mapping from E into itself. Then $\lim_{k\to\infty} U_k = U$.

The proof of the above theorem follows the same pattern as that of Theorem 3.11. We note that the difference between Theorem 3.11 and 3.14 is, by virtue of the definition of a $\mathcal{U}$-contracting mapping, the uniform space in Theorem 3.14 has a countable symmetric base.

We conclude by showing that Theorem 3.1 follows as an easy Corollary of Theorem 3.14.

3.15 COROLLARY: Theorem 3.1.

PROOF: If $K = 0$ then F_k, $(k = 1, 2, \ldots)$ and F are constant mapping and thus since $\lim_{k\to\infty} F_k(x) = F(x)$ for every $x \in X$ it follows that $\lim_{k\to\infty} U_k = U$.

If $K \neq 0$ then in $E \times E$ define

$V_n = \{(x, y)/d(x, y) < \alpha^{-n}\}$, $n \in Z$. Then $\{V_n\}$ $n \in Z$ showes that

$F_k (k = 1, 2, \ldots)$ and F are $\mathcal{U}$-contracting and hence by Theorem 3.14,

$\lim_{k\to\infty} U_k = U$.

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