ERROR PATTERNS IN THE SIMPLIFICATION OF POLYNOMIAL EXPRESSIONS

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ERROR PATTERNS IN THE SIMPLIFICATION OF POLYNOMIAL EXPRESSIONS

by

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ABSTRACT

The purpose of this study was to investigate whether students made systematic and common errors when simplifying polynomials. In addition to this general question, the study investigated the relationship between the errors made in an algebraic context and a corresponding arithmetic context, whether errors were a function of grade, program, or sex, and whether differences existed between direct and indirect situations. Twenty-five students were randomly selected from eight groups representing a Grade (9 or 10), by Program (Matriculation or Honours), by Sex matrix, resulting in a total sample of 200 students in the analysis. Three tests, an algebra test, an arithmetic test, and a computation test were administered to intact classes within a 40-minute period. The 20-item computation test involved single operations with integers. The 32-item algebra and 20-item arithmetic tests included items involving exponential expressions, the distributive principle and grouping. These tests contained items requiring the same type of skills but the arithmetic test contained no variables.

The results indicated that 15 common, systematic errors were made in algebra. The common errors were found

in the categories of sign errors, wrong operation errors, distribution errors and exponent errors.

Most students who made common errors did so in one context only, either algebra or arithmetic, but not both. Most common errors arose in the direct mode, where only one step solutions were needed, rather than in the indirect mode, where a series of steps were necessary. The major difference found between grades was in the frequency of errors rather than the types of errors. The same was found when errors made by students in the matriculation programs were compared to those made by students in the honours programs. Only minor differences in performance were found between male and female students.

Implications for remediation, as well as for teaching in general, were discussed. Recommendations for further research in error analysis were also proposed.

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CHAPTER I

THE PROBLEM

Rationale for the Study

The study of polynomial expressions has long served an important role in the high school mathematics curriculum. Current algebra textbooks include many topics which deal with polynomials in one form or another. The program of studies adopted by all schools in the Province of Newfoundland suggests that a large percentage of the instructional time in algebra should relate to polynomial expressions. Within the study of algebra, familiarity with its symbolism is essential (National Advisory Committee on Mathematical Education, 1975, p. 138), and this symbolism is the essence of polynomials and their format. It is felt that the mathematical language itself should be well known in order to develop the mathematical concepts (Ailles, Norton & Steel, 1973, p. 2). Polynomials are the sentences and phrases of that mathematical language. Polynomial expressions form the "backbone" of algebra, and since algebra is accepted as a major part of the mathematics curriculum, it seems essential that students thoroughly understand polynomials and their characteristics. However, often the mathematics which children learn and retain differs greatly from the objectives and desires of mathematics educators (Carry, Lewis & Bernard, 1978, p. iii). Many students are unable to demonstrate the understanding of and familiarity with algebra for which teachers strive. Irregardless of the best efforts of students and teachers, children still make errors.

The analysis and documentation of errors in mathematics can be traced back to the early twenties when researchers, such as Myers (1924), observed that mistakes in computation were "persistent". More recently, researchers such as Budden (1972), Carry et al. (1978), Laursen (1978), Meyerson (1978), and Davis (1980) also indicated that errors occurred "consistently" in exercises involving polynomials. Roberts (1968), Engelhardt (1977), Carry et al. (1978), and Radatz (1979) were able to classify and categorize the "common" errors they found. Thus, there is evidence throughout the literature to suggest that errors made by students, whether they occur in algebra, arithmetic, or geometry, are both common and frequent.

Too often teachers underrate the important information inherent in students' mistakes and fail to realize the strategies used by students to obtain such solutions. Error analysis is a field of study which involves the investigation of the nature of errors and the processes behind them. During the late twenties, the study of "recurrent" errors was valued highly by investigators in the algebra field (Pease, 1929, p. 264). It was believed that success in teaching algebra depended primarily on the teacher's knowledge of typical difficulties which pupils faced while learning algebraic topics (Pease, 1929, p. 264). It has been shown by researchers who have delved into the field of error analysis that the study of errors does provide valuable insights pertaining to both the learning and teaching of mathematics. For instance, the careful examination of a child's errors reveals patterns which are quite logical to the child who developed them (Pincus, 1975, p. 184). Errors made by pupils are often systematic and are retained by children if remediation does not occur (Cox, 1975). Errors in algebra can sometimes be traced to difficulties in reading and fundamental arithmetic (Wattawa, 1927). Therefore, any knowledge of such systematic errors that can be obtained could prove to be valuable information for a teacher.

As indicated earlier, polynomials occupy a large part of high school algebra courses. Yet, few empirical studies were found which dealt with high school algebra, and even fewer still specifically examined errors in polynomial expressions. Therefore, an investigation into students' errors in the simplification of polynomial expressions was warranted.

In an attempt to provide an explanation of the errors found, any possible links between these algebraic errors and similar errors in arithmetic should be sought. In studies by Carry et al. (1978) and Wattawa (1927) the arithmetic-algebra connection surfaced. For instance, Wattawa found that children who had not developed certain fundamental arithmetic processes to a level of what she termed "automatic recall and application," demonstrated greater difficulty with beginning algebra courses. She believed that the link between arithmetic and algebra was so important that success in algebra depended on success in arithmetic. Yet, Carry et al. (1978) revealed that most of the college students who participated in their study did not view algebra as generalized arithmetic. Rather, algebra was recognized as a separate entity. Although the major emphasis of this study was on errors in the simplification of polynomials, corresponding arithmetic and computational items were included to permit an exploration into any links between algebraic and arithmetic processes.

Purpose of the Study

The main problem investigated concerned the types of common errors which grade nine and ten students commit when they deal with the addition, subtraction, and multiplication of monomials. These operations with monomial expressions arise in the introduction to algebraic expressions and serve as a basis for future algebraic topics such as the simplification of polynomials and rational expressions. In particular, the problem was to identify and classify any common systematic errors found in the given algebraic examples as well as to record the frequency with which these errors occurred. A secondary problem concerned the possible relationship between arithmetic errors and algebraic errors. This involved the need to investigate the existence of any common characteristics in the errors made in corresponding arithmetic and algebraic examples.

With respect to the problem, the following research questions were investigated.

1. Do students make systematic algebraic errors? What classifications appropriately describe these errors?

2. What common errors do grade nine and ten students commit when adding, subtracting, and multiplying monomials?

3. Do students who make systematic errors in algebra make the corresponding arithmetic errors and vice versa?

4. If a student makes a systematic direct error, does the student make the corresponding indirect error, and vice versa?

5. Do grade nine and ten students make the same errors or are there differences?

6. Within grades, are the errors made by students in the honours program different from, or

similar to, those made by students taking the matriculation mathematics program?

Scope and Limitations

The first limitation of this study arose from the sample selection. The sample was chosen from schools within a 20 km radius of a small, urban community of 100 000. It was drawn from students in intact classes in large high and junior high schools whose populations ranged from 300 to 1200 students. Since many schools in Newfoundland are smaller and often much farther from an urban center, this was seen as a limitation on the generalizability of the results.

A second limitation arose from the size and selection of the interview sample. This sample was relatively small because interviews were carried out on a one-to-one basis and involved audiotaping of each session. Since only 16 students were interviewed concerning the errors, the conclusions drawn may not be generalizable to errors made by other students. Also, the students were not randomly selected but were chosen based on the number of errors they made. Since this was an exploratory study and the interviews were to be used only to enhance the analysis, this bias was accepted as a limitation.

The method used to collect the data was seen as a third limitation. Students were required to complete three short tests within a prescribed time limit and this time limit may have affected performance. Students may have rushed through exercises and committed more errors than usual or students may have worked too slowly and not completed the exercises.

Definition of Terms

Certain terms or phrases used throughout the review of literature and the study are defined as follows:

Error:	Any incorrect procedure used to solve a problem.
Systematic Error:	An error which was made by a student on at least 50% of the occasions in which the student had an opportunity to make that error. Studies reviewed in Chapter I may use alternate criteria.
Common Error:	Any systematic error which was made by at least 10 children from the sample of 200.
Direct Error (Direct Mode):	Any error which was made in the first step of a solution.
Indirect Error (Indirect Mode):	Any error which was made in other than the first step of a solution.
Arithmetic Error:	An error which occurred when operating with numbers only. Errors with facts, operations, properties and computation were arithmetic.
Algebraic Error:	An error which occurred whenever variables were present. Errors with copying, properties, operations and solution procedures were algebraic.
Monomial:	An expression of the form ax^n where a was an integer and n was a whole number. For example, x, 3x, $-5x^3$, x^2 are monomials.

<u>Active Operation</u>:	The operation which was activated in order to simplify the expression. For example, both addition and multiplication are present in $2x + 3x$ but addition is the active operation, since to simplify it we add, $2x + 3x = (2 + 3)x = 5x$.
<u>Honours Program</u>:	A mathematics program designed for students with superior ability in mathematics. (Division of Instruction, Department of Education, Government of Newfoundland and Labrador, 1980-81).
<u>Matriculation Program</u>:	The core mathematics program designed for students with an average general ability in mathematics. (Division of Instruction, Department of Education, Government of Newfoundland and Labrador, 1980-81).
<u>Basic Program</u>:	A mathematics program designed for students with a low level of academic achievement in mathematics. (Division of Instruction, Department of Education, Government of Newfoundland and Labrador, 1980-81).

CHAPTER II

REVIEW OF RELATED LITERATURE

The purpose of this chapter is to review the literature concerned with error analysis. Research on errors in both algebra and arithmetic is included because the relationship between algebraic errors and arithmetic errors was investigated. The chapter is organized under four subheadings. First an overview of the error analysis research is given. Then, the literature pertaining to the rationale and methodology used for error analysis is reviewed. Next, the errors detected by experienced teachers are discussed. Finally, the error patterns reported in formal research findings are considered.

An Overview of Error Analysis Research

The earlier research studies in arithmetic, such as those done by Myers (1924), Brueckner and Elwell (1932), Brueckner (1935), and Grossnickle (1935, 1936) involved investigations of the "persistent" errors present in computation, and these authors simply listed the errors they found. Later, researchers such as Roberts (1968), Cox (1975b), and Engelhardt (1977) extended the earlier studies by classifying the errors into general error types or categories. The early researchers in algebra, such as Wattawa (1927) and Pease (1929), also reported "persistent" errors but their investigations covered a broad range of topics and they included a frequency count of the errors as part of their studies. More recent research by Davis and Cooney (1977), Sachar (1979), and Carry et al. (1980), included investigations of errors within the solutions to linear equations. They also examined adjunct topics and speculated as to the causes of the errors found.

Budden (1972), Laursen (1978), and Meyerson (1978), all experienced teachers, presented a variety of algebraic errors and commented upon their possible origins. Other researchers, including Davis, Jockusch, and McKnight (1978), Radatz (1979), and Carry et al. (1980) presented various models of the thinking process which were obtained through information processing methods and which often formed a basis for their studies.

In general, most of the literature involving algebra either dealt with errors present in polynomial exercises similar to those included in this study, or discussed errors which were relevant to the process of simplifying polynomials. For example, a "like term" error found by Davis and Cooney in the context of equation solving is relevant to the process of polynomial simplification. An example is provided in Figure 1.

$$8x + {}^{-}20 = 4$$
$$^{-}4 + 8x + {}^{-}20 = 4 - 4$$
$$4x = 20$$

FIGURE 1. An example of a "like term" error which arose in a child's solution to a linear equation (Davis & Cooney, 1977, p. 171).

Rationale and Methodology

In a survey of studies involving error analysis, Radatz (1980) indicated that since the 1970's interest and activity in this field of research had increased. Radatz (1979) reported that researchers were no longer limiting their error analyses to arithmetic computation. He claimed that interest in the diagnostic aspects of teaching, and criticisms of the traditional paradigms of empirical research have led to the acceptance and expansion of error analysis in mathematics education (pp. 163-164).

The need for alternative research paradigms in education was also supported by Davis et al. (1978) who pointed out that educational phenomena can never be understood in terms of numerical variables only. They suggested that since descriptive studies were well established in various other fields, then education ought not be an exception (pp. 11-12). In the field of error analysis the descriptive study paradigm has been employed by most researchers.

Further rationale for the study of error patterns can be drawn from the information inherent in children's mistakes. If the written work of a child is to provide useful information, it must be scored and <u>analysed</u> (Ashlock, 1972, p. 1). In fact, the careful examination of the kinds of errors children make reveals patterns which are quite logical to the child who made them (Pincus, 1975, p. 580). Researchers in this field have observed that the mistakes in a student's exercises often outline faulty procedures or strategies which the student has adopted to obtain the required answer. The character of a child's error contains as much information as the nature of a correct answer; both hold the keys to the child's thought processes. Therefore, error analysis is not only an alternative research paradigm, but is one which possesses rich potential in ascertaining the difficulties in learning mathematics.

Two particular methods have emerged within error analysis research. The most prominent technique involves a paper and pencil survey approach with the analysis performed after the fact. The second technique, which has gained more popularity in recent years, involves interviewing the student and observing the errors made. Rudnitsky, Breakeron, Jaworowski, and Puracchio (1980) reported that several researchers, including Erlwanger (1975), and Davis et al. (1978), used task-based interviews quite successfully to study the students' understanding of mathematics (p. 2). Often researchers, such as Lankford (1974) and Kent (1978a, 1978b), used interviews as a form of diagnosis. Lankford suggested a variety of procedures to follow for such interviews, including the suggestion that verbatim recordings be made, students be permitted to proceed their own way without corrections, erasures should not be allowed, and leading questions be avoided. Lankford also noted that a subject should never be hurried (pp. 31-32).

Carry et al. (1980) employed both paper and pencil tests and interviews. Children's comments were keyed to the written work by using video recordings during the sessions.

The advantage of the paper and pencil method lies in the time factor and the size of the sample which can be tested, while the interview method must be used on a one-to-one basis. The interview method, however, has more potential for determining causes of the learning difficulties while the paper and pencil method provides useful information for an initial assessment of areas of difficulties for large groups. If a combination of these methods is employed, the advantages of both techniques can be utilized.

Errors Detected by Experienced Teachers

Without undertaking any formal investigations, many teachers have reported errors which arise during classroom or homework activities. Some teachers examined very specific types of errors and speculated as to possible reasons for such mistakes. Laursen (1978), for instance, discussed errors which she believed originated when "students try to extend a shortcut method to other seemingly similar configurations" (p. 194). In particular, she reported errors made when children misapplied shortcuts for the crossmultiplication and cancellation of fractions as well as shortcuts for multiplying radicals. Some examples of such misapplications are presented in Figure 2.

Example 1. The rule for multiplying radicals,

$$\sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = ab,$$

is misapplied as follows:

$$\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2} = a + b$$

Example 2. The rule for crossmultiplication,

$$\frac{a}{b} = \frac{c}{d} \text{ where } ad = bc,$$

is misapplied as follows:

$$\frac{a}{b} + \frac{c}{d} = ad + bc.$$

FIGURE 2. Two examples of the "misapplication" errors, (Laursen, 1978, pp. 194-195).

Meyerson (1978) examined various solutions to quadratic equations in which the principle of zero products was misapplied. An example can be found in Figure 3.

$$
\begin{array}{ll}
x^2 - 5x + 6 = 6 & \\
(x - 2) \quad (x - 3) = 6 & \\
x - 2 = 6 & x - 3 = 6 \\
\quad x = 8 & \quad x = 9
\end{array}
$$

FIGURE 3. An example of the misapplication of the principle of zero products (Meyerson, 1980).

He reasoned that such errors occurred when specific mathematical procedures were learned without understanding of the origin or the application of the procedure (p. 49).

Budden (1972) reported errors made by his students in a boys' school. He classified the errors according to the faulty procedure he felt students used. Some of the types of errors that Budden discussed are in Figure 4.

1. Law of Universal Distributivity. A child distributes regardless of the operation or symbolism. For example:
   $$a\,(bc) = a \cdot b \cdot a \cdot c$$

2. Commutativity of Operations. A child assumes operations are cummutative. For example:
   $$(a + b)^2 = a^2 + b^2$$
   since the square of the sum equals the sum of the squares.

3. Confusion of Operations. A child fails to distinguish between operations. For example:
   $$(a \cdot b)^n = ab^n \text{ or } x^2 = 2x$$

4. Omission of Punctuation. A child omits or ignores parentheses going so far as to even introduce his/her own grouping schemes. For example:
   $$5 + 2(3 + 7) = 70$$

FIGURE 4. Error classifications suggested by Budden (1972).

These reports involved teachers and students at the secondary level and the subject in which the errors arose was algebra. Hence, these teachers have provided evidence that errors are made in the natural school environment, and these errors possess discernible common characteristics which permit classification.

Error Patterns Reported in Formal Research

Findings in formal research studies supported the error patterns found by experienced teachers. Descriptions and examples of the specific error types listed in the various studies are provided in Appendix A. This section's primary focus is on the conclusions and implications drawn from these studies.

The earliest research study reviewed was by Wattawa (1927). In this study, the oral and written class work and tests of a beginning class in algebra were examined for possible errors. Wattawa found that the most frequent errors were due either to a lack of a thorough knowledge of the fundamentals of arithmetic or to faulty reading. She explained that faulty reading, such as 'minus' read as 'plus' or 'z' read as 'y', led to incorrect copying and this, in turn, caused difficulty with written solutions. The relationship between arithmetic and algebraic errors was a major concern in her study, and Wattawa addressed it from several angles. She reported that the difficulty with subtraction found in arithmetic problems carried over to the work in algebra where the subtraction of polynomials was considered the most difficult operation. Wattawa further elaborated on this relationship by attempting to explain the link in terms of concentration levels. She stated that children, whose fundamentals of arithmetic were not automatic, had much more difficulty with algebra as they were unable to rely on 'reflex' for arithmetic aspects and concentrate solely on the algebra concepts. Students with insufficient knowledge of basic arithmetic still had to concentrate on the arithmetic involved and, therefore, could not concentrate on the algebra being developed. Of the 407 errors Wattawa recorded, 85.4% were errors in simple arithmetic, signs, copying, and reading. Other errors were due to the use of incorrect operations or the lack of comprehension.

While studying the relative difficulty of learning units found in the first year algebra text, Pease (1929) also classified errors in arithmetic and algebra. He did not investigate any direct link between the errors in these areas of mathematics but rather noted the frequency with which the errors arose. In particular, Pease reported that of the 43 000 errors found, 31% were functional, 22.9% were due to sign difficulty, 8.5% were exponent errors, and 8.2% were due to carelessness. In his study a "functional error" was defined as an error within the solution procedure as opposed to an incorrect calculation. Pease also distinguished between "literal numbers" and "numbers" and defined arithmetic errors as mistakes made with operations in the absence of "literal numbers". He implied that adding 2 and 3 was arithmetic while adding 2x and 3x was algebraic and the procedures to be followed were distinct.

Frequency of errors was also the focus of a study by Davis and Cooney (1977). The researchers concentrated on the errors made by regular and basic algebra students while they solved linear equations. In this investigation, more than one-half of the errors were attributed to miscalculations with the four basic operations or to incorrect applications of the rules for computing signed numbers. These researchers also discussed "process" (functional) errors. They found that the 'better' students in their sample committed more computational errors than process errors. Again in this study, the most common errors found in the algebraic topic chosen were attributable to difficulties with arithmetic.

The report by Davis et al. (1978) drew upon a variety of studies which were carried out to substantiate a theory of mathematical learning that the authors proposed. They, like Wattawa (1927), indicated that errors often resulted from misreading one's own notation due to the visual similarity of initial cues. These authors also pointed out that errors arose when components of procedures were so salient and automatic that they were virtually unknown and unrecognizable. In particular, Davis et al. referred to error types such as 'binary confusions'. This label was attached to all errors in which the general attributes of operations were adopted and the simpler operation was often used to replace the higher one. For example, if a child added when he was required to multiply, the error was classified as a "binary confusion".

Davis et al. also discussed two error types particularly relevant to the simplification of polynomials. First, they provided a lengthy explanation of a phenomena by which children did not distinguish between symbols and their meanings. This phenomena, together with any ambiguities in the language, often led to errors. For instance, "$\frac{3x}{x} = 2x$" was an example used to demonstrate the case where a child lacked sufficient knowledge of the symbolism and thus made a mistake. Second, errors were found in exercises where "like terms" were to be combined. In these examples students misinterpreted the necessary distinguishing characteristics and grouped terms with insufficient similarities. For example, "$2x$, $3x^2$ and $4x^3$" were combined as "like terms" since all of the expressions contained an "x". Davis et al. did not consider the arithmetic versus algebra issue. Instead they emphasized the misconceptions inherent in a lack of understanding of the language and symbols of mathematics as possible explanations of the algebraic errors found.

Difficulties inherent in the mathematical symbolism was also one aspect of the study by Sachar (1979). Sachar compared the errors generated on "literal equations" with those generated on equations with "numerical coefficients". The number of errors increased significantly when "literal" coefficients were involved. Sachar concluded that the complexity of the equation, which was indicated by the presence of literal coefficients, did change the frequency of the errors but it did not influence the type of errors made.

Carry et al. (1980) also investigated errors pertaining to equations. Two groups of college students, described as good and poor equation solvers, were involved and numerous categories of errors were proposed. These categories are included in Appendix A. The errors discovered in this study were "systematic" within a student's work and were "common" within the work of different students. From comments made by solvers, Carry et al. concluded that both the student's knowledge and execution of a procedure were faulty. For instance, several of the "operator errors" were described as examples in which incomplete knowledge or incorrect knowledge was overextended. That is, students "stretched" the pieces of knowledge they had in order to solve partially familiar situations. An example is provided in Figure 5.

The correct simplification

$$\frac{ax}{a} = x$$

is incorrectly extended to

$$\frac{a + x}{a} = x$$

FIGURE 5. Use of knowledge overextension.

It should be noted that this error category resembled the "misapplication" errors discussed by Laursen (1978).

Carry et al. also focused on the use of "generic operations" which were defined as operations based on general key notions. These authors claimed that if algebra was seen as an exercise in symbol manipulation, students often organized their knowledge of operators in a generic form. That is, students suppressed the restrictions on or the specifics of an operator and carried out general actions. For example, when addition and multiplication represented a generic combining operation, y + yz became 2yz, since the expression was read as "one y" and "one y" and "one z", that is, "two y's and one z" (pp. 52-53).

The bulk of "applicability errors" reported by Carry et al. involved the assignment of a false grouping to terms in an expression. It was indicated that the absence of parentheses was often overlooked and children imposed their own grouping order before simplifying. For example, given x + 2 (x + 1), students grouped x and 2 together and proceeded to multiply (x + 2) by (x + 1) · (p. 72).

Lewis (1980) used the results found by Carry et al. to discuss the knowledge required to solve equations in elementary algebra. He pointed out that in essence, a student who uses a generic operator simply drops some of the critical aspects of the operation required and works with a general notion of the required procedures. In this report, Lewis linked algebra and arithmetic together when he indicated that students often applied correct arithmetic procedures in similar algebraic examples and errors resulted. For instance, operations learned while doing fractions in arithmetic were applied to algebraic examples with fractions, resulting in an incorrect answer.

The idea of "generic operators" also surfaced when Kent (1978a) interviewed school children and adults attending remedial classes in the evening. For example, subtraction was described as a "decreasing" operation— an operation in which the solution is always smaller than the largest "subtrahend". Such generic operations lead to difficulties both in the execution and understanding of particular problems. For instance, when 15 - ⁻3 was assigned, students ignored the negative in front of the 3 and wrote 12 as the answer. They also had great difficulty in believing that 15 - ⁻3 was 18, since this answer was larger than 15 or 3, and this contradicted their generic operator.

In a second study by Kent (1978b), the main focus was on the students' misinterpretation of symbolism. Students often interpreted the symbol "xy" as a "number" whose "ones" digit was "y" and whose "tens" digit was "x". Few students in this study recognized "xy" as "x times y" where "x" and "y" represented different numbers. Such misconceptions led to many difficulties as students were unable to solve 3x + 2 = 14 since "thirty-blank" plus two could never equal 14.

As indicated earlier, since an arithmetic component was included in the present study, research involved with arithmetic topics was considered relevant. Thus far, most research reviewed involved studies in algebra at the high school level. The studies which follow were on arithmetic topics and the subjects were elementary school children, with the exception of those in Lankford (1972).

Lankford (1972) investigated errors which seventh graders made when they computed with whole numbers and fractions. Most of the errors students made with whole numbers were in subtraction and division, while the errors with fractions occurred in all operations. Few errors were due to poor recall of facts, and most of the errors found were process-oriented.

Roberts (1968) was one of the first researchers who classified the computational errors he found. The four categories he suggested, namely, "wrong operation", "computational error", "defective algorithm", and "random response" were later refined by Engelhardt (1977) who replicated Roberts' study with third and sixth graders. Both sets of categorizations are included in Appendix A. After subdividing the original four classes to eight categories, Engelhardt found that over 40% of the errors were made by the lowest quartile of students. He also indicated that students erred in the execution of the procedure rather than the recall of facts. He claimed that errors arose with the procedures because the procedures themselves were not meaningful to the students.

Many of the specific errors listed by Pincus (1975) could be classified under Engelhardt's broader categories. In the same study, however, Pincus revealed other types of errors which had been given little attention previously. He described errors which resulted from poor penmanship and alignment of numbers, as well as errors which arose from the failure to estimate or check answers.

Finally, a textbook for pre-service teacher education by Ashlock (1972) contained some relevant information concerning error analysis and remediation. Ashlock contended that erroneous procedures often produce correct answers which reinforce the child's actions and increases the difficulty of detecting the error pattern. He also indicated that the investigator's interpretation of an error strategy may differ from the strategy used by the child. This supports the need to interview the child who made the error in order to provide an accurate description of the child's thoughts. Ashlock pointed out that any error analysis which is to be useful has to be thorough and detailed. When discussing potential causes of students' errors, Ashlock claimed that often the instructional strategies used by a teacher lead a child to adopt erroneous strategies. One example demonstrated that confusion might arise if two algorithms were taught without changing the arrangements used. For instance, if a child was taught to add in columns and was then presented with the same example for multiplication, he might be inclined to multiply in columns. An example of such an error is given in Figure 6.

$$\begin{array}{r} 23 \\ +43 \\ \hline 66 \end{array} \qquad \begin{array}{r} 23 \\ \times 43 \\ \hline 89 \end{array}$$

FIGURE 6. Example of error in multiplication influenced by the arrangement of the items and previous knowledge.

Summary

The literature on error analysis provided evidence that students do make systematic errors and that many of the errors are common. The conclusions drawn from the research also provided support for the contention that common, systematic errors are classifiable. Furthermore, it was found that these classifications were verified in different studies.

This information had several implications for this study. The categories of errors found in the literature were related to those expected to occur in the simplification of polynomial expressions. Thus, these categories provided a working base for the development of a hypothetical set of error types used in the coding scheme for the present study. There was also support for the contention that a relationship existed between arithmetic and algebraic errors and this provided a foundation for the investigation of such a relationship between errors in the simplification of polynomials and corresponding errors in arithmetic.

CHAPTER III

METHODOLOGY

In this chapter, the methodology used in the investigation is described. Initially, the population and sample are defined, and then the pilot study is outlined. Following this the final instruments are described, and the procedures used in the main study are explained. Next, the coding scheme used to classify the errors is presented and the methods used to analyse the data are reported.

Population and Sample

The population consisted of students enrolled in grade nine and ten algebra classes. An initial sample of 19 intact classes was selected from six schools. Eight classes were grade nine matriculation, four were grade nine honours, four were grade ten matriculation and three were grade ten honours. A total of 573 students were tested and the average class size was approximately 30.

The schools' populations ranged from 300 to 1200 students and only two schools contained both grade levels. Three of the schools were junior high schools while one other included grades 10 and 11 only. All schools were within a 20 km radius of a small urban community of 100 000 people.

Two hundred of the 573 students were included in the sample for analysis. As shown in Figure 7, eight groups were formed based on the grade, sex, and the program of the students. Each group of 25 was randomly selected from the appropriate set of students in the original sample.

		PROGRAM			
		Matriculation		Honours	
		SEX			
		Male	Female	Male	Female
GRADE	9	25	25	25	25
	10	25	25	25	25

FIGURE 7. Distribution of students in the sample.

Pilot Study

The objectives of the pilot study were:

1. To ascertain the time limits necessary to allow students to complete the tests comfortably.

2. To check the difficulty of the complete tests as well as any particular items.

3. To observe any difficulties with the written and oral instructions.

4. To investigate whether systematic errors

did arise in order to determine the feasibility of the study.

Two intact mixed ability classes, one grade nine and one grade ten, were chosen from one school within the area designated for the main study. A total of 50 students were tested.

Using items similar to the exercises in Chapter 3 of Using Algebra (Travers, Dalton, Brunner & Taylor, 1979), four algebra tests were developed. Four arithmetic tests and one computation test containing items requiring the same type of skills as those on the algebra tests were also developed. Every student wrote the computation test but each algebra and arithmetic test was written by a quarter of each class. The time taken by a student to complete each test was recorded and any oral comments or reactions were noted by the investigator.

No student required more than 40 minutes to complete the three tests, thus the length of the instruments used in the main study was similar. The design of the computation and algebra tests posed no difficulties and the instructions were clear. However, the instructions used on the arithmetic tests were unclear, and students were unsure as to exactly what was expected.

Consequently, the arithmetic tests were refined and a sample of two matriculation classes was used to pilot the new versions. Based on these results, the appropriate instructions were chosen, and the final instruments were devised. Finally, since systematic errors were found among students' responses in the pilot study, the study was considered feasible.

Instruments

The final instruments consisted of an algebra, an arithmetic, and a computation test. The 20 computation items involved single operations with integers. To limit the length and complexity of this test, division was excluded. Also, due to the isomorphic relationship between addition and multiplication of wholes and addition and multiplication of positive integers, these operations were omitted as well. The 32 items on the algebra test involved single operations with exponential expressions, the distributive principle of multiplication over addition, and the grouping of like terms, with some particular items involving a combination of these procedures. The 20 arithmetic items were chosen to correspond to the algebraic items, resulting in similar skills being tested on the algebra and arithmetic tests. Large numbers were used on the arithmetic test to deter students from calculating to obtain a single numeral solution, and examples were included to alert students to the type of responses required. To avoid errors due to the order of operations, brackets were inserted in appropriate arithmetic items.

On both the algebra and arithmetic tests, space was provided for students to write several steps of a solution when desired. On all tests, there were two items for every skill in a given format, while other items required the same or a similar skill but in different formats. To illustrate this, examples from the algebra test are given in Figure 8, and copies of the instruments are included in Appendix B.

Example	Same skill and format	Same skill/ different format
$^{-}5p(2p-7)$	$^{-}7w(3w-6)$	$^{-}2w(3w+7) + {}^{-}3w(2-5w)$

FIGURE 8. Sample of items from the algebra test involving the same skill.

Procedure

The data were collected at the end of April since the teachers involved had indicated that all students would have completed the relevant material on polynomials at least one month earlier. All three tests were administered by the investigator or by an assistant who was thoroughly familiar with the procedures. Each test was assigned a maximum time limit to ensure that all students attempted all three tests within a 40 minute period. The algebra test, with a 20 minute limit, was assigned first followed by the arithmetic test with a 10 minute limit. The computation test was assigned in the last five minutes. Any students who did not require the maximum time to complete a test were permitted to write any subsequent tests without intermediate delays. If students finished all three tests before the 40 minute period had ended, they were permitted to check their work and to return to any omitted items.

All instructions pertaining to the content and solution methods were written on the tests. Technical instructions were given orally. Students were told to use pen or pencil and to write their solutions on the test papers. Erasures were not permitted and students were instructed to draw one line through the error before making any changes to the answers. Students' names were requested in order to match each of the three tests.

All tests were written during regularly scheduled mathematics periods. Prior to the testing the students were not aware of what content was to be tested, and no review related to the particular skills was carried out.

As a follow-up to the written tests, a select sample of students was interviewed on a one-to-one basis. Due to time constraints, tests were corrected but the errors had not been classified prior to the interviewing. The algebra tests written by students in the matriculation classes were sorted according to the number of errors.

Sixteen students from different schools, who had made the most errors in their group and who had, on initial inspection, made errors similar to other students, were interviewed individually in late May.

An audio tape was made of each interview session between the student and the investigator. In an interview, students were given blank test papers and were requested to complete particular items while repeating aloud the procedures used. If students did not verbalize their actions sufficiently, the interviewer asked questions to obtain explanations and clarifications. For example, if a student said "$10p^2 - 8$" as the description of the procedure, the investigator asked "How did you obtain the $10p^2$?" The session lasted 15-20 minutes on the average and was followed by an informal review of the errors for the students' benefit.

The comments made during these sessions served as one of the components utilized in the discussion and interpretation of the results obtained from the written data.

Coding Scheme

In order to investigate and classify error patterns, a coding scheme was developed. Using error types available in the literature as well as some errors found during the preliminary analysis of the pilot material, a list of general error categories was compiled. Based on the information gathered during a trial run of the analyses, refinements and modifications were made. Each general category was subdivided into specific error types. A description of the general categories is provided in this section, with detailed examples of specific error types contained in Table 18, Appendix C.

Ten general categories were used and a total of 111 specific error types were hypothesized. A brief description of each category follows.

1. Sign errors dealt with errors where students carried out the correct operations and procedures, and arrived at an answer correct in absolute value, but incorrect in sign.

2. Basic fact errors were errors in which the correct operation was followed but an addition, subtraction, or multiplication fact was recalled incorrectly.

3. Wrong operation errors included errors where the student performed a different operation from that required. For example, students who added when multiplication was the operation, committed wrong operation errors.

4. Distribution errors involved situations where a number was to be distributed and the student either failed to distribute or distributed incorrectly.

5. Grouping errors were errors in which students grouped terms and thereby changed the meaning of the expression.

6. The category labelled "incorrect operation symbols written" included situations where students wrote addition

symbols where multiplication symbols were required or vice versa. This category was limited to arithmetic errors since it was on the arithmetic test where students were instructed to indicate their procedures only and were asked not to compute a final answer.

7. The category "numerical bases multiplied" was for arithmetic test items only and it included errors where students multiplied the bases in an exponential expression.

8. Exponent errors encompassed all errors students made with exponents when they simplified exponential expressions.

9. Like term errors involved errors which students made when combining "unlike" terms as if they were "like" terms.

10. A miscellaneous category was included to provide codes for other errors which did not fall within the descriptions.

Individual coding sheets were designed by using these categories and the test items. An example of such a coding sheet and its use is contained in Appendix D.

Analysis of Data

For each of the eight cells in the design, a summary sheet was completed. On this sheet, records were kept of how many errors each student made in a particular error category. Further details of these summary sheets are included in Appendix E.

To facilitate between-group comparisons on common systematic errors, a final summary sheet was designed to record the number of students who made errors in each category and the frequency with which they made them. This summary sheet is available in Appendix F.

Before describing the analysis techniques used for each question, two key definitions are restated here. A "systematic error" was defined as an error which occurs on at least 50% of the occasions in which the student has the opportunity to make such an error. A "common error" was any systematic error which was made by at least 10 of the 200 students in the sample.

<u>Question 1</u>. Do students make systematic algebra errors? What classifications appropriately describe these errors?

An inspection of the final summary sheet was used to determine if any of the error types occurred in a systematic manner. The classifications of such error categories were noted.

<u>Question 2</u>. What common errors do grade nine and ten students commit when adding, subtracting, and multiplying polynomials?

By examining the final summary sheet, the total number of students who made each error systematically was determined. All error categories which fulfilled the criterion indicated earlier were noted as common errors.

Any group of errors which were conceptually related and satisfied the criterion as a set were considered for analysis as a set of errors.

> <u>Question 3</u>. Do students who make systematic errors in algebra, make the corresponding arithmetic errors and vice versa?

Each common systematic error in algebra was compared to its arithmetic counterpart to determine if the same students were making both errors.

> <u>Question 4</u>. If a student makes a systematic direct error, does the student make the corresponding indirect error, and vice versa?

The students who made direct and indirect errors within a category were compared to ascertain whether or not students made the error in both situations.

> <u>Question 5</u>. Do grade nine and ten students make the same errors or are there differences?

For each common error type, a comparison was made between grades and any errors which were grade specific were recorded.

> <u>Question 6</u>. Within grades, are the errors made by students in the honours program different from, or similar to, those made by students taking the matriculation mathematics program?

For each common error found, a comparison was made between students in different programs. Any errors which were specific to a particular group were reported.

CHAPTER IV

RESULTS

In order to present a complete report of the data, this chapter contains a variety of components. First, all necessary notation is explained. Next, the results are reported in the context of the general error categories. Tabulations of students who made specific error types within each general category are presented prior to the report on the errors in that category. Any hypothesized errors which did not occur systematically are noted. Then, an overall summary of the systematic algebraic errors is presented. Comparisons are made between groups based on grade, program, and sex. Any common errors in the category are reported in detail and comparisons are made between similar algebraic errors as well as any parallel arithmetic errors. Finally, a discussion of the results in terms of the research questions posed in Cahpter I is provided, with appropriate reference to the interview data.

Frequency counts were recorded for each hypothesized error type and the number of students who made each error type at particular frequencies was also tabulated. Any error type which occurred on at least 50% of the occasions in which the student had the opportunity to make the error was considered to be systematic. Any systematic error type which was made by a minimum of 10 students was called a common error. Tabulations were made according to grade, program, and sex to permit between group comparisons. Indirect and direct errors were coded separately in order to ascertain the situation in which the error occurred.

A total of 200 students, consisting of eight subgroups of 25, was included in the analysis. There were 10 general error categories, each of which was subdivided into specific error types. A total of 111 hypothesized error types was investigated. Each subcategory was assigned a three or four character descriptor. Each descriptor began with a letter which represented the test on which the error occurred, C for computation, A for arithmetic, or P for algebra (polynomials). A second letter was then used to indicate the general error category, as shown in Table 1.

TABLE 1

Letters Used to Represent General Categories

Category	Letter	Category	Letter
Sign Errors	S	Incorrect Symbolism	L
Basic Fact Error	F	Numerical Bases Multiplied	B
Wrong Operation	W	Exponent Errors	E
Distribution	D	Like Term Errors	T
Grouping	G	Miscellaneous	M

If an error type was related to an operation, a third letter was included to indicate whether it was addition (A), multiplication (M), or subtraction (S). The number which appeared at the end of each description indicated the position of the error type in the listing. To clarify these descriptions, two examples are provided in Figure 9.

EXAMPLE 1

AEM1

arithmetic exponent error multiplication first

i.e., AEM1 was the descriptor for the first exponent error in multiplication on arithmetic items.

EXAMPLE 2

PD3

algebra distribution error third

i.e., PD3 was the descriptor for the third distribution error on algebra items.

FIGURE 9. Examples of the descriptors used.

In this chapter, "arithmetic test" refers to that particular test used, while the word "arithmetic" refers to the arithmetic context as a whole, including items from both the arithmetic and computation tests.

Sign Errors

As shown in Table 2, 30 specific error types were hypothesized under the category of sign errors. Five of the algebraic errors, PSM5, PSA4, PSA6, PSS2, PSS3, and two

TABLE 2: (at back of this paper)

of the arithmetic errors, ASM1, ASM3, were not made systematically by any students. However, all of the 10 sign errors proposed for the computation items were made systematically by some students.

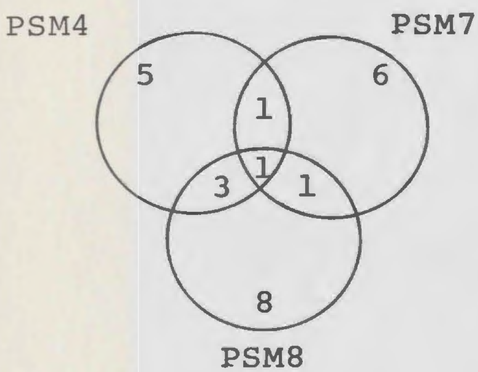
Systematic algebraic sign errors. Overall, 68 students made systematic sign errors on algebraic items. Forty-eight students made a single error type, while 14 students made two error types, three students made three error types, one student made four error types, another student made five error types, and one other student made seven error types. When between group comparisons were made, it was found that more students in grade nine made systematic, algebraic sign errors than students in grade ten. Fewer students in the honours program made such systemeric errors and no difference was found between the performance of males and females. These results are summarized in Table 3. A detailed description of each common algebraic sign error follows.

TABLE 3

Between Group Comparison (Algebra)--Sign Errors

	Grade		Program		Sex	
	9	10	M	H	M	F
Number of students who erred systematically	40	28	51	17	34	34

Common error types, PSM4 and PSM8, occurred within the general framework of problems involving the distributive principle for multiplication over subtraction. For example, in problems such as $^{-}2x\ (4x - 6)$ a student would write $^{-}12$ as the coefficient of the second term rather than 12. PSM7, in which the term being distributed was a negative integer rather than a monomial, while not common, was the only other systematic error of this general type. As indicated in the Venn diagram shown in Figure 10, 19 of the 25 students who made these errors made the error in only one of the three situations described.



ERROR	EXAMPLE
PSM4	$^{-}5p(2p - 7)$ where the coefficient of the second term was written as $^{-}35$.
PSM8	$^{-}3w(2 - 5w)$ where the coefficient of the second term was written as $^{-}15$.
PSM7	$^{-}3(7r - 2)$ where the second term was written as $^{-}6$.

FIGURE 10. Number of students making sign errors when distributing multiplication over subtraction.

These algebraic errors corresponded directly to ASM2 on the arithmetic test and indirectly to CSM1 on the computation test. When given problems such as $^{-}59(65 - 97)$ on the arithmetic test, 21 students wrote a negative second term, in this case $^{-}59 \cdot 97$. On the computation items, nine students said that problems like $^{-}4 \cdot {}^{-}21$ had a negative product, namely $^{-}84$. When ASM2 and CSM1 were considered together to represent this systematic, arithmetic sign error and PSM4, PSM7, and PSM8 were considered to represent the systematic, algebraic sign error, only nine students were found to have this sign error in both algebra and arithmetic. As shown in Figure 11, 16 students made this type of systematic error in arithmetic only and 16 others made it is algebra only.

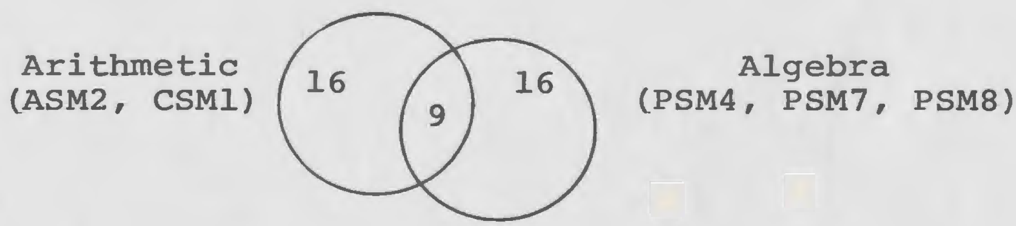
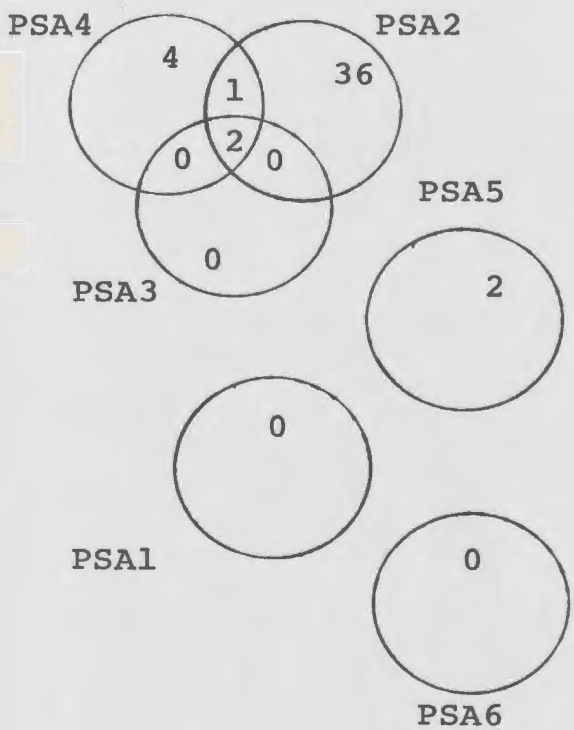


FIGURE 11. A comparison of algebraic and arithmetic sign errors involving the multiplication of two negative integers.

Common sign error, PSA2, involved the incorrect addition of coefficients in problems of the type $ax^n + bx^n$ where at least one of the coefficients was negative. For example, in problems such as $^{-}23x^2 + 12x^2$ students would write the correct magnitude of the coefficient but the incorrect sign, namely 11 in this case instead of $^{-}11$. Specifically, the error type PSA2 referred to a direct error of the form illustrated above, where the negative coefficient has a greater absolute value than the positive coefficient and the error was made in the first step of a solution. As shown in Figure 12, the other variants of coefficients did not result in many errors. Even the indirect error PSA5, which was identical to PSA2 but was made in a step other than the first step of the solution, was made by only two students and neither of them had made the direct error.



ERROR	EXAMPLE
PSA1	${}^-2x^2 + {}^-5x^2$ where the coefficient was written as 7.
PSA2	$5r + {}^-21r$ where the coefficient was written as 16r.
PSA3	${}^-6w^2 + 15w^2$ where the coefficient was written as $-9w^2$.
PSA4	Same as PSA1 but indirect.
PSA5	Same as PSA2 but indirect.
PSA6	Same as PSA3 but indirect.

FIGURE 12. Number of students who made systematic sign errors in addition.

The error category CSA2 on the computation test corresponded to the systematic, algebraic error PSA2. Given problems such as $27 + {}^-39$, students who wrote 12 for the answer were said to have made error CSA2. As indicated in the Venn diagram in Figure 13, only five students made this sign error systematically in both arithmetic and algebraic items.

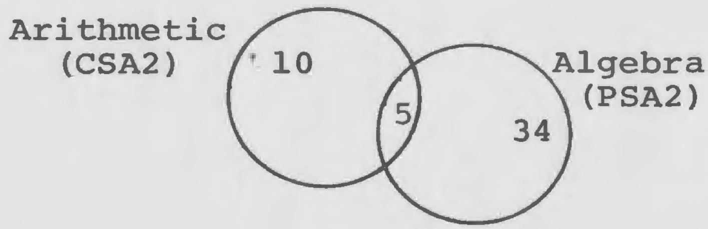
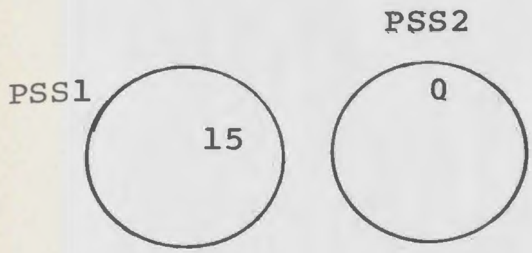


FIGURE 13. A comparison of the number of students who made systematic algebraic and arithmetic errors when adding constants of opposite signs.

Common error type, PSS1, involved the incorrect subtraction of coefficients in problems of the type $ax^n - bx^n$ where both coefficients are positive. For example, in problems such as $4p^2 - 6p^2$ students would write 2 instead of $^-2$ for the coefficient. Specifically, PSS1 referred to the error type where the subtrahend was larger than the minuend as illustrated in the above example. The other error type in this category, PSS2, did not appear systematically. As can be seen in Figure 14, any student who made a systematic sign error in subtraction did so only when the subtrahend was larger than the minuend.

The arithmetic error CSS4 corresponded directly to PSS1 and arose in problems such as 25-35 where students



ERROR	EXAMPLE
PSS1	$4p^2 - 6p^2$ where the coefficient of p^2 was written as 2.
PSS2	$8d^2 - 7d^2$ where the coefficient of d^2 was written as -1.

FIGURE 14. Number of students making systematic sign errors in subtraction.

wrote 10 for the answer. As shown in Figure 15, only four students made a systematic sign error in both arithmetic and algebraic items.

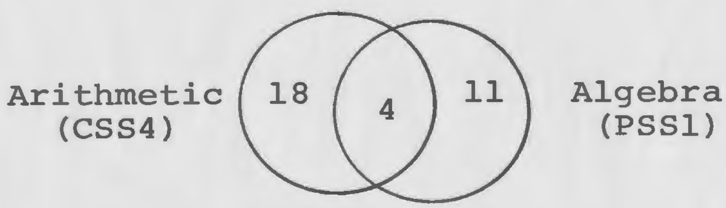


FIGURE 15. A comparison of the number of students who made systematic algebraic and arithmetic sign errors involving subtraction.

<u>Wrong Operation Errors</u>

As shown in Table 4, 24 specific error types occurring on the algebra and computation tests were hypothesized as wrong operation errors. Nine of the algebraic errors, PW1, PW2, PW5, PW8, PW9, PW10, PW11, PW12, and two of the computational errors, CW1, CW2, were not made systematically by any students.

TABLE 4: (at back of this paper)
(at back of this paper)

TABLE 4

Wrong Operation Errors

Grade and Sex	9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10H-F			TOTAL		
	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>
Frequency	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
Error Types																											
Wrong Operation*																											
CW1				3						1						1									5	0	0
CW2	4			2			2						3			3			1			3			18	0	0
CW3	7		2	4	3	1	3			4	1	2	1					1	1						20	4	6
CW4		5	4		3	3		1	3		3	6		5			3						2		0	22	16
CW5		5	5		5	5			1		2	2								1			1		0	14	13
CW6	2		1		1		1				1														3	2	1
PW1	1									1															2	0	0
PW2	2																								2	0	0
PW3								1																	0	1	0
PW4	1	1					1	1																	2	2	0
PW5																									0	0	0
PW6																	1								0	1	0
PW7		1									1														0	2	0
PW8							1																		1	0	0
PW9	1																								1	0	0
PW10	1						1			2						1									5	0	0
PW11																									0	0	0
PW12																									0	0	0
+PW13		3	1		2	1		1				1		1	1										0	7	4
+PW14		1	1		5				1					1			1								0	8	2
PW15	2			3			1			1															7	0	0
+PW16		10			8	1		2						5			5			7			3		0	40	1
+PW17		2	9		2	7			2		1	2		3	2		4	2		1	1			1	0	13	26
+PW18	1		3	2	2	3	1		1			3					1	1			1				4	3	12

+A common algebraic error.

No such errors were proposed for arithmetic since students were not permitted to calculate. Instead, the error category, "incorrect operation symbol written" was applied in the arithmetic test items. That is, a student would write down the incorrect symbol rather than carry out the wrong operation.

Systematic algebraic wrong operation errors. Overall, 70 students made wrong operation errors systematically on algebraic items. Of the 37 students who made multiple errors, 27 students made two error types, five others made three error types, and five more students made four error types. When between group comparisons were made, only a marginal difference could be found between the performance of males and females, with more males making errors. More grade nine students made systematic errors than grade ten students and fewer students in the honours program than in the matriculation program made systematic wrong operation errors in algebra. These comparisons are indicated in Table 5.

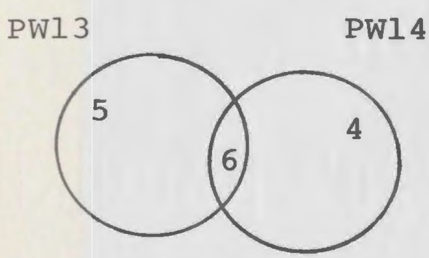
TABLE 5

Between Group Comparisons (Algebra)--Wrong Operation Errors

	Grade		Program		Sex	
	9	10	M	H	M	F
Number of students who erred systematically	47	23	45	25	38	32

A detailed description of each common algebraic wrong operation error follows.

Common wrong operation errors, PW13 and PW14, occurred in the general framework of problems where two binomials were incorrectly subtracted. For example, when students simplified problems such as (17x + 2) - (12x + 9), they added the coefficients of the like terms, in this case 29 would be the coefficient of the first term and 11 would be the second term. As indicated in Figure 16, six students made both errors systematically.



ERROR	EXAMPLE
PW13	$(17x + 2) - (12x + 9)$ was written as $29x - 11$.
PW14	$(8d^2 - 13) - (7d^2 - 4)$ was written as $15d^2 - 17$.

FIGURE 16. Number of students who made systematic, wrong operation errors with subtraction.

The arithmetic error, CW3, where students added when they were required to subtract, was the arithmetic error corresponding to the common algebraic errors mentioned above. Here, students would write 60 as the answer to problems like 25-35. However, as seen in Figure 17, no student made this type of error systematically on both arithmetic and algebraic items.

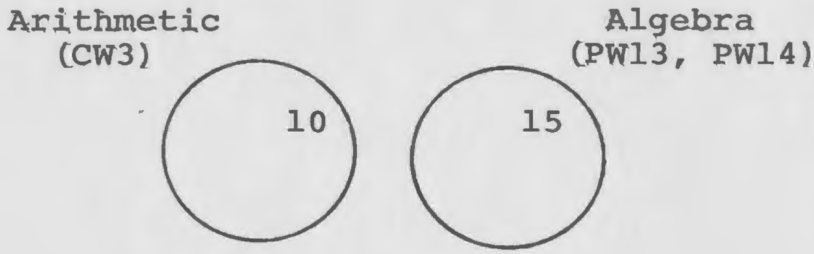
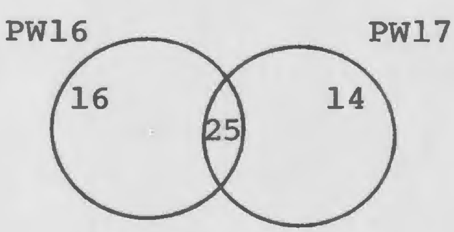


FIGURE 17. A comparison of the number of students who made wrong operation errors with subtraction on corresponding arithmetic and algebraic items.

Common wrong operation errors, PW16 and PW17, occurred in problems of the form $ax^n + bx^n$ where the two monomials were multiplied instead of added. For example, in problems such as $4x^2 + 7x^2$ students would simplify by writing $28x^4$. Specifically, error PW17 occurred in problems where $b = 1$ and $n = 1$. As indicated in the Venn diagram in Figure 18, 25 students made both error types systematically, but 30 other students made the systematic error in only one of the situations.



ERROR	EXAMPLE
PW16	$4x^2 + 7x^2$ was written as $28x^4$.
PW17	$13x + x$ was written as $13x^2$.

FIGURE 18. Number of students who systematically multiplied monomials when asked to add them.

These errors in algebra corresponded to the error CW2 in computation, where students given problems such as $18 + {}^{-}7$ wrote ${}^{-}126$ as the answer. As seen in Figure 19, no student made CW2 systematically.

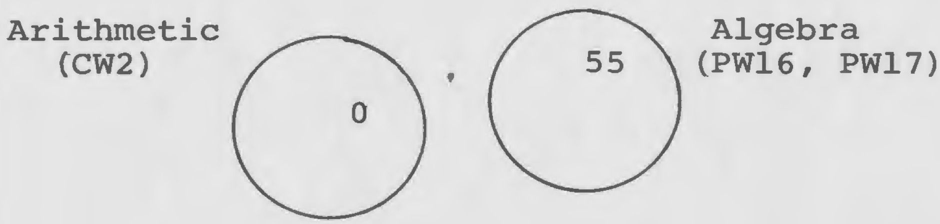


FIGURE 19. A comparison of the number of students who systematically multiplied instead of added on corresponding algebraic and arithmetic items.

The error, PW17, where "ax + x" was written as "ax^2", was directly related to the arithmetic error described as "$ab + b = ab^2$". The latter error was not hypothesized a priori and was inserted only after it occurred in several cases. In this arithmetic error, students would simplify problems such as $35 \cdot 789 + 789$ by writing $35 \cdot 789^2$. As shown in Figure 20, only 7 of the 59 students made these errors systematically in both arithmetic and algebraic items.

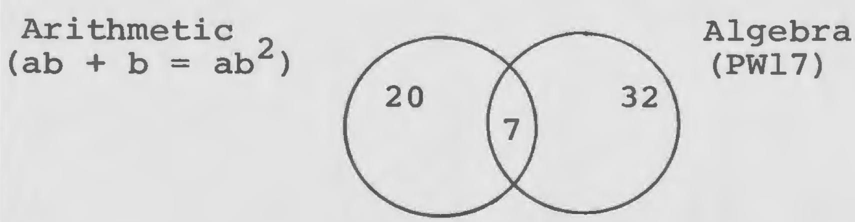


FIGURE 20. A comparison of the number of students who systematically made the errors "PW17" and "$ab + b = ab^2$".

Common wrong operation error, PW18, occurred in problems of the type $(ax^n \pm b) - (cx^n \pm d)$ where the binomials were multiplied instead of subtracted. For example, on problems such as $(7x + 2) - (12x + 9)$, 15 students multiplied the binomials and wrote variations of $84x^2 + 63x - 24x + 18$. No corresponding arithmetic or computation items were included in the tests and hence no comparisons could be made.

Distributive Errors

As shown in Table 6, 10 specific error types were hypothesized as distributive errors. Only one of the algebraic error types, PD5, was not made systematically by any students. All of the four distributive errors proposed for the arithmetic items were made systematically by some students. However, since problems involving a solution by the application of the distributive principle were not present in the computation test, no such errors were proposed for that test.

Systematic algebraic distributive errors. Overall, 35 students made systematic distributive errors on algebraic items. Thirteen of these students made systematic errors in two error types, and the remaining 22 students made systematic errors in only one error type. When between group comparisons were made, it was found that more grade nine students than grade ten students made distributive

TABLE 6: (at back of this paper)

TABLE 6

Distributive Errors

Grade and Sex	9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10H-F			TOTAL		
	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>
Frequency	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
Error Types																											
Distributive																											
AD1		4	3			2		1	2		1	1					1				3			4	0	7	15
AD3	1		1	3	1			1	1																4	2	2
AD4		4	2		2	2		1	1		1	1		1	9		2	6		3	1		1	3	0	15	25
AD5		1	2		2	2		2	1		2	2			1		5				1			2	0	12	11
PD1	3	2	2	2				1		2			2												9	3	2
PD2	2	1	4	2													1								4	2	4
+PD3		1	1		3	1		1			1	3		3	2		1	1							0	10	8
+PD4		1	1		1	4		1	2		2	1			1		1	1		1					0	7	10
PD5	3																								3	0	0
PD6		2	1																						0	2	1

+a common algebraic error

errors systematically and more students in matriculation than in the honours program made systematic distributive errors. A small difference was found between the performance of males and females, with more males making errors. The number of students in each group is presented in Table 7.

TABLE 7

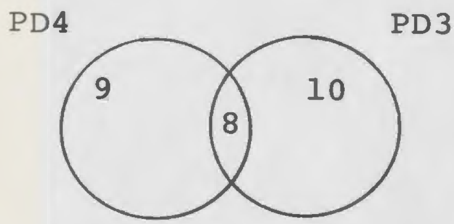
Between Group Comparisons (Algebra)--Distributive Errors

	Grade		Program		Sex	
	9	10	M	H	M	F
Number of students who erred systematically	25	10	25	10	20	15

A detailed description of each common algebraic distributive error follows.

Common distributive errors, PD3 and PD4, occurred within the general framework of problems involving the difference of two binomials where the distributive principle was applied incorrectly. For example, in problems such as $(4p^2 - 3) - (6p^2 - 5)$, some students would write $4p^2 - 3 - 6p^2 - 5$, resulting in an incorrect sign for the last term. Specifically, in both errors PD3 and PD4, the second binomial had an implied coefficient of one. As shown in

Figure 21, eight students made both errors systematically, while another 19 made this type of error systematically in only one of the two situations.



ERROR	EXAMPLE
PD3	$(8d^2 - 13) - (7d^2 - 4)$ was written as $8d^2 - 13 - 7d^2 - 4$
PD4	$(7x + 2) - (12x + 9)$ was written as $7x + 2 - 12 + 9$

FIGURE 21. Number of students who systematically failed to distribute correctly when subtracting binomials.

These algebraic error types corresponded to the distributive error, AD1, on the arithmetic items. For example, when given items such as 169 - (349 + 876), 22 students systematically wrote 169 - 349 + 876 as the response, thereby failing to distribute correctly. As shown in Figure 22, 25 students made this systematic distributive error only in algebra while 20 other students made it systematically in arithmetic only.

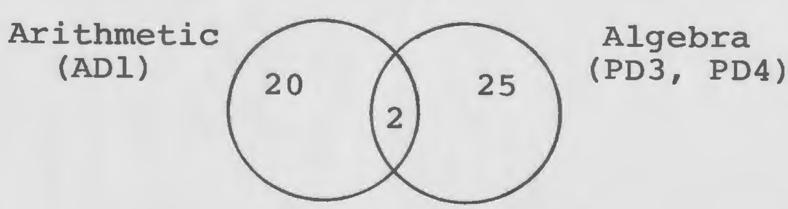
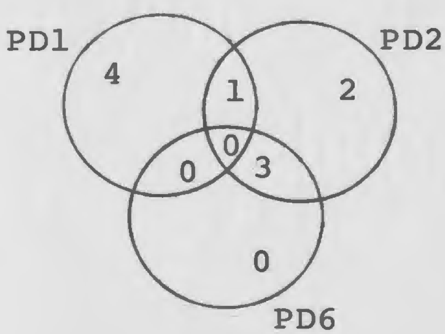


FIGURE 22. A comparison of the number of students who made systematic distributive errors when subtracting binomials on corresponding arithmetic and algebra items.

Error types, PD1, PD2, and PD6 constitute a conceptually related set of errors, and when considered together, can be considered as a common error. These error types were specific situations where only partial distribution was carried out. For example, when students were given $^-5p(2p - 7)$, they failed to multiply the second term by ^-5p and wrote -7 instead of $35p$. Specifically, PD1 and PD6 were error types in which the term being distributed was a negative integer instead of a monomial as in PD2. Other variants of this error were not hypothesized, and did not, in fact, occur. As shown in Figure 23, no student made all three error types.



ERROR	EXAMPLE
PD1	$^-6(13a + 8)$ was written as $^-78a + 8$
PD2	$^-7w(3w - 6)$ was written as $^-21w^2 - 6$
PD6	$^-8(7y + 9)$ was written as $7y + {}^-72$

FIGURE 23. Number of students who partially distributed, systematically.

These algebraic distributive errors corresponded to AD3 in the arithmetic test. Given problems such as $^-12(517 - 229)$, students would write 229 instead of $^-12 \cdot 229$ for the final term. As shown in Figure 24, 10 students made such algebraic errors systematically and four others made the arithmetic errors systematically but no student made errors in both contexts.

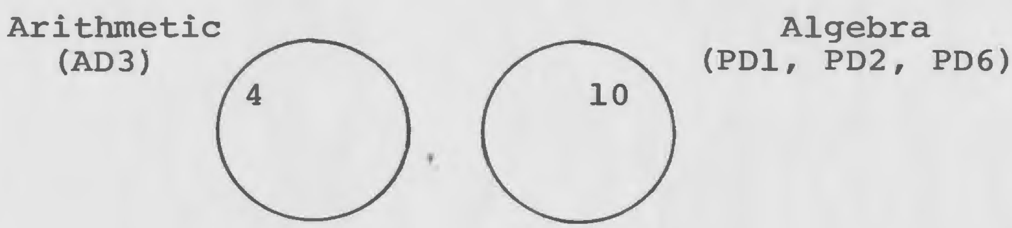


FIGURE 24. A comparison of the number of students who partially distributed a negative term systematically on corresponding arithmetic and algebraic items.

<u>Exponent Errors</u>

As shown in Table 8, 17 specific error types were hypothesized under the general category of exponent errors. Four of the algebraic errors, PEM5, PEM6, PEA4, PES1, were not made systematically by any students. However, all of the five exponent errors proposed for the arithmetic items were made systematically by some students. No exponent error types were appropriate for the computation test.

<u>Systematic algebraic exponent errors</u>. Overall, 37 students made systematic exponent errors in algebra. Three of these students made systematic errors in three error types, seven others made systematic errors in two error types, and the remaining 27 students made systematic errors in only one error type. When between group comparisons

TABLE 8: (at back of this paper)

TABLE 8

Exponent Errors

Grade and Sex	9M-M			9M-F			9H-M			9H-F			10M-M			10M-F			10H-M			10H-F			TOTAL		
	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>
Frequency	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
Error Types																											
Exponents																											
AEM1						2							1			1									2	0	2
AEM2						1																			0	0	1
AEM3		1																1		1			1		0	3	1
AEA1	2	1		1	1		1	1								1			1			1			7	3	0
AEA2		4	1		3	2					1									1			1		0	10	3
+PEM1			3					1	1					1						1	2		1	1	0	4	7
PEM2	2		2	1	2		1						1		1	1									6	2	3
PEM3	1	1					1				1					2			1	1					5	3	0
+PEM4	8	1	3	2	3	1	2		2	4		1	5		1	6		1	2			1			30	4	9
PEM5																									0	0	0
PEM6																									0	0	0
PEA1	4		2	2		2				1						2									9	0	4
PEA2	4	3	1	5	2		3	1					1	1		3	1		1			1			18	8	1
PEA3	3			1												2		1							6	0	1
PEA4																									0	0	0
PES1										1															1	0	0
PES2					1	1								1											0	2	1

+A common algebraic error

were made, it was found that more students in grade nine than students in grade ten made systematic exponent errors and fewer students in the honours program than those in the matriculation program made such systematic errors. A small difference was found between the performance of the males and females, with more males making errors. These comparisons are shown in Table 9.

TABLE 9

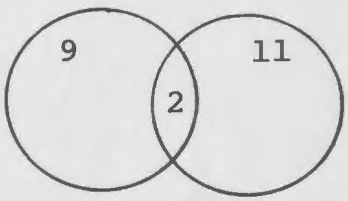
Between Group Comparisons (Algebra)--Exponent Errors

	Grade		Program		Sex	
Exponent Errors	9	10	M	H	M	F
Number of students who erred systematically	25	12	26	11	22	15

A detailed description of each common algebraic exponent error follows.

Common algebraic exponent errors, PEM1 and PEM4, involved the omission of an exponent in response to problems of the type $ax \cdot bx$. For example, given such problems as $8a \cdot 13a$, some students would write 104a and no explicit exponent was written. Specifically, PEM4 was the same type of error, but it occurred in the context of problems such as $^{-}5p(2p - 7)$ where students would write "p" rather than

"p^2" in the first term. As shown in Figure 25, 20 of the 22 students who made this error systematically did so in only one of the situations.



ERROR	EXAMPLE
PEM1	15b · 7b was written as 105b.
PEM4	$^{-}7w\ (3w - 6)$ where the first term was written as $^{-}21w$.

FIGURE 25. Number of students who made systematic exponent errors when multiplying expressions with implicit exponent of 1.

While no parallel arithmetic errors were hypothesized, these two algebraic exponent errors are similar to the arithmetic error, AEM1, where a number was multiplied by itself. Here, it was predicted that a student would write the number alone. For example, 231 · 231 would be written as 231. However, only two students made this error systematically, and as shown in Figure 26, neither of them made the algebraic errors.

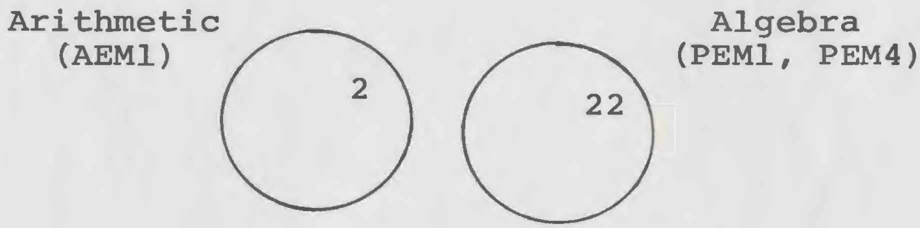
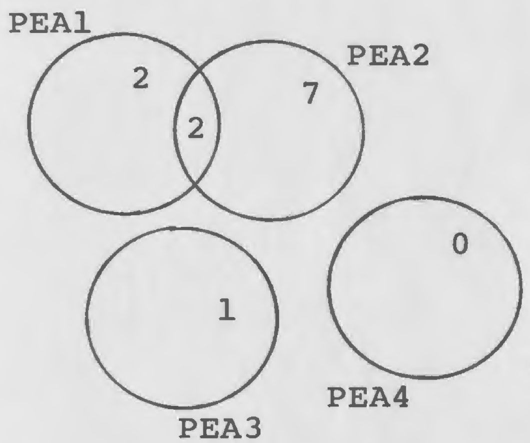


FIGURE 26. A comparison of the number of students who systematically omitted an explicit exponent when multiplying expressions with unwritten exponents of 1.

Error types, PEA1, PEA2, PEA3, and PEA4 constitute a conceptually related set of errors, and when considered together, they can be considered as a common error. These error types were specific situations where students added coefficients and exponents when given problems of the form $ax^n + bx^n$. In particular, PEA1 and PEA2 were the direct errors, and PEA3 and PEA4 were the indirect errors when $n = 1$ and $n = 2$, respectively. For example, given a problem such as $4x^2 + 7x^2$ students would write $11x^4$. As shown in Figure 27, 10 students made this type of error systematically in only one of the specific cases. Although the number of students making these errors systematically is small, the data indicated that students tended to make this systematic error in direct situations rather than indirect ones.



ERROR	EXAMPLE
PEA1	$2p + {}^{-}7p$ was written as ${}^{-}5p^2$.
PEA2	$4x^2 + 7x^2$ was written as $11x^4$.
PEA3	Same as PEA1 but indirect.
PEA4	Same as PEA2 but indirect.

FIGURE 27. Number of students who systematically added exponents when adding monomials.

The algebraic errors PEA1 and PEA2 corresponded to the arithmetic errors AEA1 and AEA2. These arithmetic exponent errors arose in problems such as ${}^{-}9 \cdot 18^2 + 17 \cdot 18^2$ where students would write $({}^{-}9 + 17)18^4$. As shown in Figure 28, however, only one student made such exponent errors systematically in both arithmetic and algebra, while 26 students made the systematic errors in only one of the contexts.

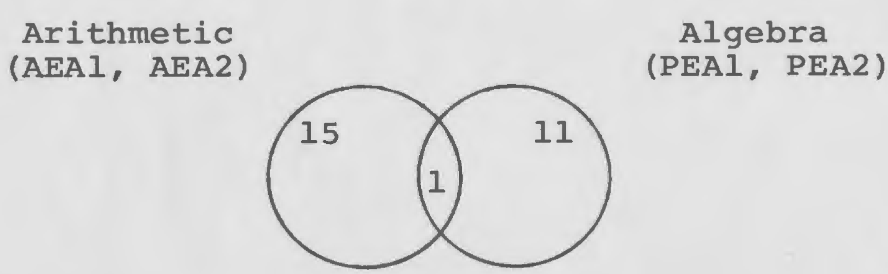


FIGURE 28. A comparison of the number of students who systematically added exponents when adding expressions in arithmetic and algebra.

Other Errors

The six other general categories were not reported in detail because none of the specific algebraic error types were made systematically by 10 or more students. The category "Basic Fact Error" did not occur systematically on any tests and neither did the category called "Incorrect Operation Symbol Written".

Ten students did make "Like Term Errors" systematically but no particular error type or set of error types was common. Five students made the like term error PT9, which involved addition of common terms without applying the necessary distributive principle. For example, in problems such as $5r + {}^{-}3\ (7r - 2)$, students would combine 5r and 7r, and $^{-}3$ and $^{-}2$ without distributing first. As shown in Figure 29, few students made the other variants in this category.

ERROR	EXAMPLE	NUMBER OF STUDENTS
PT1	$27b - 10 = 17b$	0
PT2	$15x^2 + 3 = 18x^2$	1*
PT3	same as PT1, but indirect	1
PT4	same as PT2, but indirect	1
PT5	$15x^2 + 3x = 18x^3$	2
PT6	same as PT5, but indirect	0
PT7	$15x^2 + 3x = 18x^2$	1*
PT8	same as PT7, but indirect	0
PT9	$5r + {}^{-}3(7r - 2) = 12r - 5$	5

*same student made both these errors systematically

FIGURE 29. Number of students who made systematic like term errors.

The only grouping error proposed for algebra, PG1, occurred in problems of the form $ax + {}^{-}b\ (cx - d)$ where the first two terms were incorrectly grouped together. Four students, systematically, wrote $(5r + {}^{-}3) \cdot (7r - 2)$ when given problems such as $5r + {}^{-}3\ (7r - 2)$. However, 12 students made the corresponding arithmetic error AG3, and only one of them made the error systematically in both arithmetic and algebra. As shown in Figure 30, 11 students wrote $(189 + {}^{-}21) \cdot (537 - 792)$ as a response to $189 + {}^{-}21\ (537 - 792)$ but did not group 10p and $^{-}3$ together in problems like $10p + {}^{-}3\ (6p + 8)$.

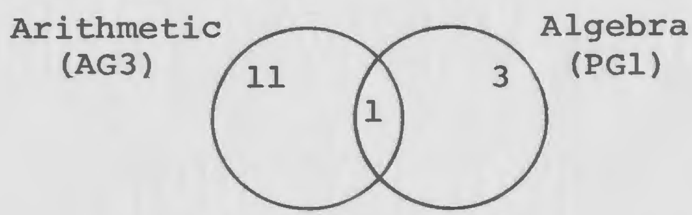


FIGURE 30. A comparison of the number of students who made the same systematic grouping error in algebra and arithmetic.

A commutativity error in subtraction, AM3, was prominent in arithmetic, yet no student made the corresponding algebraic error, PM3, systematically. In this error type students commuted terms in subtraction problems. For example, problems such as $(31 \cdot 340) + (71 \cdot 340) - 123$ were rewritten as $123 - (31 + 71)\ 340$, yet problems like $16d + 31d - 27$ were never written as $27 - 47d$. These errors were listed in the "miscellaneous" category.

The category of errors called "Numerical Bases Multiplied" was specifically related to arithmetic items and was made systematically by only two students.

All the common algebraic errors and all but two or three systematic errors found in the study had been hypothesized prior to the data analysis. Thus, although systematic errors did not occur in all hypothesized categories, the hypothesized error categories were considered to be appropriate descriptions of the systematic algebraic errors students made when simplifying polynomials.

An analysis of the data with respect to each research question is now considered. The data is presented for each question. Where appropriate, in the discussion which follows the questions, explanations of the results are suggested and the interview data is analyzed to provide support for these explanations or to suggest alternatives.

Research Questions

> Question 1. Do students make systematic algebraic errors? What classifications appropriately describe these errors?

To determine whether or not an error was systematic, a comparison was made between the number of times a student made the error and the number of times the student had an opportunity to make the error. As defined earlier, a systematic error was an error which occurred on at least

50% of the occasions in which the student had the opportunity to make that error. As shown in Table 19, in Appendix F, 77 of the 111 hypothesized errors were made systematically. Specifically, 42 of the 72 hypothesized algebraic errors were made in a systematic manner.

Overall, the classifications generated from the literature were appropriate descriptions of the error types found. No different error types occurred frequently enough to warrant alternative classifications, although some students did make unique errors. Thirteen specific algebraic error types were not present in any of the students' responses, and the remaining 17 algebraic errors occurred in an unsystematic fashion. Details are available in Appendix F.

Discussion. When analyzing and coding the data, systematic patterns were found in the students' responses. In many studies the criterion for a systematic error is one that occurs in more than 50% of the occasions on which it is possible rather than at least 50% of the occasions. Due to the breadth of this study and its exploratory nature, the "at least 50%" criterion was introduced to capture those errors which occurred in situations which arose when only two items were available. However, even when the more stringent criterion of more than 50% is applied, 79 of the 122 students who made systematic errors did so on more than 50% of the occasions. In particular, when the criterion of "more than 50%" was applied there were 30 algebraic errors which were made systematically. Thus, although in all the following discussions the word "systematic" applies to the "at least 50%" criterion, it appears that this less stringent criterion does not provide a serious limitation to the conclusions.

Because few discrepancies were found between the predicted errors and those which students made systematically, it seems reasonable to conclude that these hypothesized errors were adequate descriptions of the procedures students used to reach the "incorrect" response. The information gathered from the interviews generally supported this assumption.

For example, for the error PW14, it was predicted that when students simplified problems such as $(4p^2 - 3) - (6p^2 - 5)$, they would ignore the active operation of subtraction and proceed to add like terms, resulting in the answer $10p^2 - 8$. Students who were interviewed indicated that this was their procedure. These students explained their answer of "$10p^2 - 8$" with comments such as "add 4 and 6 to get 10, and 3 and 5 to get 8, because they're alike."

Although, as stated previously, the interview data generally supported the hypothesized descriptions, there were discrepancies. Two examples of these follow. For the error PSM4, it was predicted that students would multiply the negative terms ^-5p and $^-7$ in problems such as $^-5p\ (2p - 7)$ and would obtain a negative product, ^-35p, as the result.

Some students who were interviewed did write $^{-}35p$ but the procedure they used differed from the predicted one. These students did not change $2p - 7$ to $2p + {}^{-}7$ as expected, but instead they multiplied $^{-}5p$ by 7 to obtain $^{-}35p$. Thus, it seemed as if they ignored the definition of subtraction whereby $2p - 7$ would have been written as $2p + {}^{-}7$, and an error resulted.

Another discrepancy was also found between the predicted procedure for the error PW17 and the procedure students used during the interviews. It was predicted that students, given such problems as $13x + x$, would multiply the terms instead of adding them, and would write $13x^2$. However, in the interviews, no student who wrote $13x^2$ said they were multiplying. Some students did indicate that the "unwritten" coefficient of x caused problems and that "x^2" came from the fact that two x's were involved. This seemed to imply that students were obtaining $13x^2$ from $13x + x$ by a procedure where 13 + "an invisible value" was 13 and two x's means x^2.

It should be noted that only 16 students were interviewed and these students were requested to solve only particular items. Some students did not repeat the systematic errors they made during the written tests, therefore, their comments may not be indicative of the procedures used by all students in the sample or the population. Within these limitations, the information gathered from the interviews provides a basis for the conjecture of "possible" explanations of systematic errors.

> <u>Question 2</u>. What common errors do grade nine and ten students commit when adding, subtracting, and multiplying monomials?

At least 10 students out of the 200 tested had to commit a systematic error type before it was considered to be common. This same criterion was also used to determine if combinations of specific error types within a general category were common. As a result, 13 specific algebraic error types and two combinations of algebraic error types were considered to be common. A list of these common errors and their descriptions is contained in Table 10.

The common algebraic errors which were found belonged to four of the 10 general categories of errors, namely sign errors, wrong operation errors, distribution errors, and exponent errors. Although common error types occurred in these categories, only a few specific error types in each category were common. For example, only one of the six hypothesized algebraic sign errors in addition was made by at least 10 students.

<u>Discussion</u>. Common algebraic error types, PSM4 and PSM8, occurred in the general category of sign errors in multiplication. Specifically, the common error types were made when a negative monomial was to be distributed over a binomial involving subtraction. For example, given problems such as $^{-}7w\ (3w - 6)$, students would write -42 as

TABLE 10

Common Algebraic Errors

Error Type	Description of Error	Number of Students Grade 9	 Grade 10	 Total
PSM4	${}^-ax\ (bx - c)$ where ${}^-ax \cdot {}^-c = {}^-acx$	7	3	10
PSM8	${}^-ax\ (c - bx)$ where ${}^-ax \cdot {}^-bx = {}^-abx^2$	9	4	13
PSA2	$\left.\begin{array}{l} {}^-bx^n + ax^n \\ ax^n + {}^-bx^n \end{array}\right\} = (b - a)\ x^n \quad (b > a)$	23	16	39
PSS1	$ax^n - bx^n = (b - a)\ x^n \quad (b > a)$	10	5	15
PW13	$(ax^n + b) - (cx^n + d) = (a + c)\ x^n - (b + d)$	9	2	11
PW14	$(ax^n - b) - (cx^n - d) = (a + c)\ x^n - (b + d)$	9	1	10
PW16	$ax^n + bx^n = abx^{2n}$	29	12	41
PW17	$ax + x = ax^2$	31	8	39
PW18	$(ax^n \pm b) - (cx^n \pm d) = acx^{2n} \pm adx^n + bcx^{2n} \pm bd$	10	5	15
PD1 *	$a\ (bx \pm c) = abx \pm c$	4	1	5
PD2	$ax\ (bx \pm c) = abx^2 \pm c$	6	0	6
PD3	$(ax^n - b) - (cx^n - d) = ax^n - b - cx^n - d$	13	5	18
PD4	$(ax^n + b) - (cx^n + d) = ax^n + b - cx^n + d$	10	7	17
PEM1	$ax \cdot bx = abx$	5	6	11
PEM4	$ax\ (bx \pm c) = abx \pm acx$	11	2	13
PEA1 **	$ax + bx = (a + b)\ x^2$	4	0	4
PEA2	$ax^2 + bx^2 = (a + b)\ x^4$	7	2	9
PEA3	$ax + bx = (a + b)\ x^2$ (indirect)	0	1	1
PEA4	$ax^2 + bx^2 = (a + b)\ x^4$ (indirect)	0	0	0

*Both errors together constitute the partial distribution error type and it is this error type which is common.

**All four errors together denote the exponent error in addition, and it is this general error type which was common and no specific situation.

the coefficient of the second term. The other variations of errors in this category involved the multiplication of the negative and positive monomials, and none of these were common. It therefore seems as if some students had sign difficulties because the two negative terms were being multiplied.

This contention was only partially supported by the interview data. There were students in the interviews who used the definition of subtraction, in problems such as $^{-}7w\ (3w - 6)$, to change $3w - 6$ to $3w + {}^{-}6$ and who said that $^{-}7w$ times $^{-}6$ was $^{-}42w$. It appeared then that these students applied the incorrect rule that a negative integer times a negative integer is negative.

The procedures followed by other students, in the interviews, indicated an alternative rationale for making the same error. These students did not change $7w - 6$ to $7w + {}^{-}6$ but instead, multiplied $^{-}7w$ by 6. With this strategy students would get $^{-}42w$ as the second term, which would be correct for the particular product they calculated, but incorrect for the complete exercise. Thus, while these students obtained a final incorrect response, they did not appear to attend to the definition of subtraction.

Common algebraic error types, PSA2 and PSS1, occurred in the general category of sign errors in addition and subtraction. Specifically, the common sign error in addition was made when two monomials of opposite signs were being added, and the negative monomial possessed the largest absolute value. Similarly, the common sign error in subtraction was made when two positive monomials were subtracted, but the subtrahend was larger than the minuend. For example, given problems such as $^{-}23x^2 + 12x^2$ of $4p^2 - 6p^2$, students would write $11x^2$ and $2p^2$, respectively. These sign errors in addition and subtraction were considered together since they appeared to be conceptually related in that problems such as "$4p^2 - 6p^2$" require the "same" solution procedure as "$4p^2 + {}^{-}6p^2$". The other sign errors in subtraction and addition included different combinations of two signed monomials, but none of the errors made with these combinations were common. Initially it seemed as if the larger size of the negative monomial caused the difficulty and signs were overlooked in the response.

In the interviews, however, most students responded correctly. This indicated that the sign errors might have resulted because of the "testing milieu" rather than any particular incorrect strategy.

Common algebraic error types, PW13, PW14, PW16, PW17, PW18, belong to the general category of wrong operation errors. Specifically, wrong operation errors, PW13, PW14, were made when the like terms in two binomials which were to be subtracted were added instead. For example, in problems such as $(8d^2 - 13) - (7d^2 - 4)$ students would write $15d^2 - 17$. It was assumed that in such examples students did not attend to the active operation but rather used the presence of "like terms" to determine their operational procedures. As reported in Carry et al. (1980) and Kent (1978a), students seem to have developed a generic rule for "combining" like terms. This contention was supported by the interview data, where students who made such wrong operation errors described their actions as "adding like terms". Some students stated that all like terms are supposed to be added.

Errors, PW16, PW17, involved the multiplication of two monomials instead of adding them, as was required. For example, given problems such as $4x^2 + 7x^2$ or $13x + x$, students would write $28x^4$ or $13x^2$, respectively. When the particular items involved were examined, it was found that students made this error when the coefficients were relatively small and both were positive. Few students multiplied in the item $^-23x^2 + 12x^2$. As was indicated in Roberts (1968) and Engelhardt (1972), a possible explanation is that students appeared to be using the size and sign of the numbers to determine the operation required.

The information gathered during the interviews indicated that these two common error types were not as related as it was first thought. In the items of the type $4x^2 + 7x^2$, most students did not multiply the monomials as they had on the written test. One student who did multiply explained that she did not notice the addition sign and this would lend support to the contention that students may not be keying in on the active operation.

The categorization of wrong operation error for items like $13x + x$ was contradicted in the interviews. As indicated in the discussion of question 1, no student who was interviewed multiplied the terms $13x$ and x to get $13x^2$. Instead, students who made this error indicated that the "unwritten coefficient" in front of 'x' caused them difficulty. These students also explained that they wrote "x^2" simply because there were two "x's" present. That is, students in the interviews were adding when they obtained the answer $13x^2$. An alternative hypothesis to explain the procedure in these items seemed to be that the "unwritten" coefficient was considered to be "nothing" and 13 + "nothing" is 13, while "x^2" was used to denote the two x's in the sum.

Common wrong operation error, PW18, was made when two binomials were multiplied instead of subtracted. For example, for problems such as $(4p^2 - 3) - (6p^2 - 5)$, students would multiply $4p^2 - 3$ by $6p^2 - 5$ and obtain a variation of $24p^4 - 38p^2 + 15$ for their answers. A possible explanation here is that students were influenced more by the brackets and the presence of binomials than they were by the active operation of subtraction.

This explanation was supported by the interview data, since students who committed this error rationalized their procedure by the "fact" that "brackets mean you multiply."

Again, the number of students who used this rationale was limited, but no alternative was proposed since the other students were unsure as to why they multiplied.

Error types PD1, PD2 were conceptually related and when they were considered together, they constituted a common error. Two other common algebraic error types, PD3, PD4, were also present in the category of distributive errors. Specifically, the common distributive error types were made when students partially applied the distributive principle of multiplication over addition. For example, PD1, PD2 involved problems such as $^-7w\ (3w - 6)$ where students wrote $^-21w^2 - 42w$, while PD3, PD4 involved problems such as $(4p^2 - 3) - (6p^2 - 5)$ where students wrote $4p^2 - 3 - 6p^2 - 5$. Since few students made such partial distribution errors in both situations, it was assumed that students did not perceive the examples as the same and the procedures used to obtain a solution may not have included the distributive principle, per se. A possible explanation is that students, who could solve one set of examples but not the other, were applying general rules like "multiply everything in the brackets" or "remove the brackets", without any consideration for the principles involved.

The interview data neither supported nor contradicted this explanation. Students who solved the appropriate examples during the interview made different errors from those in question and thus were unable to provide any further information appropriate to these errors.

The final group of algebraic errors, PEM1, PEM4, PEA1, PEA2, PEA3, PEA4, occurred in the general category of exponent errors. Specifically, the common exponent errors, PEM1, PEM4, were made when both terms which were to be multiplied contained "unwritten" exponents. For example, given problems such as $8a \cdot 13a$ or $^-5p\ (2p - 7)$, students would write $104a$ and $^-10p + 35p$, respectively. The other variations of exponent errors in multiplication involved terms which contained at least one written exponent, for example, $p \cdot p^7$ or $11n^3 \cdot 2n^2$, and few common errors were made. Therefore, a possible explanation is that as long as one exponent was written, it served as a "cue" to initiate the proper algorithm. When neither exponent was explicit, students seemed not to use the appropriate rule.

Students in the interview sample failed to provide an "explicit" exponent in items which contained no "written" exponent. That is, students did not write an exponent in their responses. However, they did not explain their procedures for reaching such answers. Instead, all responses were direct without any intermediate comments. Therefore, little information as to why they "neglected" the exponent was available.

Exponent errors, PEA1, PEA2, PEA3, and PEA4, were conceptually related and when they were considered together they constituted a common error. Specifically, these errors involved the addition of monomials where the exponents and coefficients were added. For example, when given problems such as $4x^2 + 7x^2$ or $16d + 31d - 27$, students would write $11x^4$ or $47d^2 - 27$, respectively. A possible explanation is that students misapplied the rule used in multiplication of monomials, where it is correct to add exponents.

Some of the interview data seemed to support this explanation, while other comments provided an alternative rationale. Most students who were interviewed justified their procedure with the rule that "in addition, you add your exponents." One student went so far as to say, "you always add exponents". In both cases, it seemed that adding exponents was a rule students had adopted, and although no student made the comparison, it was plausible that the rule originated in multiplication.

However, other students who were interviewed indicated that the sign of the coefficients affected the procedure used with the exponents. For example, some students who added exponents in items such as $4x^2 + 7x^2$, where both coefficients were positive, would not add the exponents in items where one of the coefficients was negative, such as $-23x^2 + 12x^2$. These students indicated that the negative sign influenced them, and they "felt" like simply writing "x^2" as the variable in the answer. These students did not use the active operation of addition to activate their rule for exponents, but rather used the sign of the coefficient to determine whether or not the exponents should be combined.

> Question 3. Do students who make systematic errors in algebra make the corresponding arithmetic errors and vice versa?

The common algebraic errors, as well as some common arithmetic errors, were considered for this question. When students made common systematic errors in either algebra or arithmetic, they did not necessarily make the corresponding errors in the other context. The data concerning the number of students who made each common error type are summarized in Table 11. Details of the error types and in-depth comparisons were provided earlier in this chapter. A short description of each error type is also available in Appendix C.

As indicated in Table 11, the majority of students did not make corresponding errors on all tests. Some students made the error types on algebra only, while others made the same error types in arithmetic only.

The comparisons discussed in this question were made between the algebraic context and the arithmetic context. The "arithmetic context" included any applicable items on either the computation test or the arithmetic test, since both tests included numerical items only. The algebraic context involved the algebra test since it was the only test which included variables in its items. In

TABLE 11

Comparison of Arithmetic and Algebraic Errors

Error Types			Number of Students		
TESTS			CONTEXT		
Computation	Arithmetic	Algebra	Arithmetic* only	Algebra only	Both Arithmetic & Algebra
CSM1	ASM2	PSM4, PSM7, PSM8	16	16	9
CSA2	-	PSA2	10	34	5
CSA4	-	PSS1	18	11	4
CW3	-	PW13, PW14	10	15	0
CW2	-	PW16, PW17	0	55	0
-	$ab + b = ab^2$	PW17	20	32	7
-	AD1	PD3, PD4	20	25	2
-	AD1, AD4, AD5	PD3, PD4	67	17	10
-	AD3	PD1, PD2, PD6	4	10	0
-	AG3	PG1	11	3	1
-	AG4	PW17	34	31	8
-	AG2	PT1	13	0	0
	AEM1	PEM1, PEM4	2	22	0
	AEA1, AEA2	PEA1, PEA2 PEA3, PEA4	14	11	1

*Arithmetic refers to either the arithmetic test or the computation test.

the following discussion of this question "arithmetic test" refers to that specific test, while the word "arithmetic" alone refers to the arithmetic context as a whole, including items from both the arithmetic and computation tests.

Discussion. The results obtained in this study seem to support the conclusions of both Pease (1929) and Carry et al. (1980). Pease claimed that algebraic and arithmetic procedures were distinct, and the procedure used to add 2 + 3, say, was different from that used to add 2x + 3x, while Carry et al. (1980) indicated that students do not perceive algebra as generalized arithmetic.

Discussions with students supported this hypothesis, since students described arithmetic as the context in which you "compute" and algebra as the context in which you "simplify". Even teachers, who were shown the tests, indicated that students would more readily apply properties to algebraic items than to the items on the arithmetic test. In an attempt to determine the different perceptions students might have, each error type in Table 11 is discussed.

The first set of error types, CSM1, ASM2, PSM4, PSM7, PSM8, which were compared occurred in the general category of sign errors in multiplication. Specifically, these errors involved the multiplication of two negative terms. The following are examples of errors made by different students. On computation items such as $^{-}4 \cdot {}^{-}21$ some students would write $^{-}84$; on the arithmetic test items such as $^{-}12$ (517 - 229) some students would write $^{-}12 \cdot 517 - 12 \cdot 229$; and on the algebraic items such as $^{-}5p$ (2p - 7) some students would write $^{-}10p^2 - 35p$. As shown in Table 11, while some students made these errors in both contexts, there seemed to be no clear reason why other students would make these errors in only one context.

While the initial analysis suggested no clear explanations for the results, the interview data provided two possible explanations as to why students made the algebraic error only. One explanation was based on the fact that some students carried out a procedure in algebra which was unrelated to the arithmetic items. For example, some students incorrectly simplified $^{-}5p$ (2p - 7) because they calculated the second term, $^{-}35p$, by multiplying $^{-}5p$ by 7. This procedure would not be related to the one applied in the arithmetic context where items such as $^{-}4 \cdot {}^{-}21$ were given. Students knew the procedures for determining the sign of both products, but the terms chosen in the algebraic items led to errors in the exercise.

A second explanation was based on the methods used to simplify items on the arithmetic and algebra tests. In algebra, students simplified the given expressions by computing mentally first, before writing a final response. In the arithmetic test, students were not permitted to calculate and they copied the answer term by term. For example, given $^{-}5p$ (2p - 7) students would complete $^{-}5p$ times 2p and $^{-}5p$

times $^{-}7$ mentally, ending with $^{-}10p^2 - 35p$, while these same students, when given items such as $^{-}12\ (517 - 229)$, would write $^{-}12 \cdot 517 - {}^{-}12 \cdot 229$, term by term in order. In both circumstances, the "minus" sign indicated the operation, and a correct calculation of $^{-}5p \cdot {}^{-}7$ could still lead to an error. In the arithmetic, students may have written the correct answer only by chance, since few students wrote $^{-}12 \cdot 517 + 12 \cdot 229$, which would indicate more clearly that the "sign change" was recognized.

The interview data also provided a plausible rationale as to why some students would make the arithmetic error only. One student indicated that the example (a) on the arithmetic test was used as a model for the items involved. Using the example $158 - 7\ (651 + 318) = 158 - 7 \cdot 651 - 7 \cdot 318$, this student simplified $^{-}12\ (517 - 229)$ by writing $^{-}12 \cdot 517 - 12 \cdot 229$. Since no examples were given on the algebra test, it is possible that students would answer those items correctly, but when examples were present in arithmetic, these same students were influenced by them, and errors occurred from such misapplications.

The next two sets of error types, CSA2 and PSA2, CSA4 and PSS1, occurred under the general category of sign errors in addition and subtraction. These two sets are discussed together since they both involved two terms with opposite signs, where the negative quantity had the largest absolute value. The following are examples of errors made by different students. Given computation items such as $18 + {}^{-}39$ or $25 - 35$, some students would write 21 or 10, respectively, and for algebraic items such as ${}^{-}23x^2 + 12x^2$ or $4p^2 - 6p^2$, some students would write $11x^2$ or $2p^2$, respectively. To ascertain why some students might make these sign errors in algebra only, the particular items were compared. It was found that in all arithmetic items the negative term followed the positive one, while on the item which caused the most difficulty in algebra, the negative integer preceded the positive one. It appeared then that students may have been using a generic rule where the sign preceding the second term influenced the sign of the answer.

The interview data did not support this contention. Most students who made the algebraic error corrected themselves during the interview situation. Therefore, this suggested that students who made the algebra error only may have done so because of the "testing" situation.

A possible explanation as to why students made these sign errors in arithmetic only may have been because they used the sign of the first term to determine the sign of the answer. Again, the information gathered during the interviews did not support such a contention. Instead, students corrected their arithmetic sign errors, indicating that it may have been the "testing milieu" which led to the errors originally.

The set of errors, CW3, PW13, PW14, occurred in the general category of wrong operation errors in subtraction. Students who made these errors added instead of subtracted. In computation they added integers, while in algebra they added binomials. The following are examples of errors made by different students. Given computation items such as $^{-}40 - {}^{-}73$ or $25 - 35$, some students would write $^{-}113$ or 60, respectively, and given algebraic items such as $(17x \pm 2) - (12x \pm 9)$, some students would write $29x - 11$. A possible explanation why some students made such errors in algebra only, is that they applied a generic rule for "combining like terms" which would not be appropriate in the arithmetic context.

The interview data supported this explanation as students clearly indicated that they were adding like terms. Students explained that since the terms had variables, they were alike and therefore should be combined, and "combined" meant "added together". No student spoke of like terms in arithmetic. Instead, students seemed to use the signs of the numbers to determine the procedures.

A possible explanation why students made this wrong operation error in arithmetic only is because they applied a generic rule for "adding two negative integers". This was illustrated when most students made the computation error in items such as $^{-}40 - {}^{-}73$ rather than items such as $25 - 35$. Since the corresponding algebraic items did not explicitly contain these conditions, students were apt not to apply the same generic rule to the algebraic items.

This contention was supported by the interview data. Students who made the computational error rationalized their procedures with statements like, "when there are two negatives, you always add them." No such comments were made in algebra.

The set of error types, CW2, PW16, PW17, were also under the general category of wrong operation errors, but these error types involved the replacement of an addition operation with multiplication. Students made these wrong operation errors when they multiplied two terms instead of adding them. The following are examples of errors made by different students. On computation items such as $18 + {}^{-}7$ some students would write -126 and on algebraic items such as $4x^2 + 7x^2$ or $13x + x$, some students would write $28x^4$ or $13x^2$, respectively. When the items in both contexts were examined it was noted that all the computation items contained both a positive and a negative integer or two negative integers while all but one of the appropriate algebraic items involved two positive coefficients. A possible explanation why some students made the algebraic errors only, is because it was more "acceptable" to multiply positive values rather than negative ones.

This explanation was not directly supported by the interview data. Students who repeated this error during the interviews claimed they did not see the addition sign but they did not explain why they chose to multiply. Thus, there was no explanation as to why they made the algebraic error. However, the interviews concerning the items such as "$13x + x$", which were included in this error type, did provide an explanation as to why students would make an error here, without making the corresponding arithmetic error. These students did not multiply $13x$ by x to obtain $13x^2$ and therefore it would be consistent if they did not multiply when given items such as $18 + {}^{-}7$. This particular algebraic item is further discussed in the following set of errors.

The set of errors, PW17 and "$ab + b = ab^2$", were compared to each other because the items on which they were made were parallel. There was no general category as such in this case, but specifically, the errors involved the squaring of the "like" parts of the terms which were to be added. The following are examples of errors made by different students. In arithmetic items such as $35 \cdot 789 + 789$, some students would write $35 \cdot 789^2$ and, in the algebraic items such as $13x + x$, some students would write $13x^2$. Since these items are so similar, it was unclear as to any specific reason why students might make this error in one context and not the other. One possible explanation is that the major influence on this error was the different perceptions students have of algebra and arithmetic.

The interview data supported the contention that many students perceived these two items as unrelated. Some of the students were asked directly if they could see the relationship between $35 \cdot 789 + 789$ and $13x + x$, and all of them admitted that ordinarily they would not recognize any connection. To further interpret the processes involved, the comments of the students who were interviewed with respect to either item were examined.

It was observed that for some students, who made the error in algebra, but did not make it in arithmetic, the numerical characteristic of the latter items permitted them to check their work. One student, for example, responded originally with $35 \cdot 789^2$ but then realized that "squaring them means 'times'", so the student changed the answer to $35 \cdot 1578$, where the 789's were added. At this point, the student realized that this meant that 35 was being "times by both of the 789's" and "it wasn't suppose to." This student finally settled for $(35 \cdot 789) + 789$ for an answer. From such a session, it was apparent that, for at least some students, their knowledge of numbers permitted them to be critical of their responses, but their knowledge of algebra seldom permitted such critiques. This same student, for instance, wrote $13x^2$ immediately and saw no reason to change it.

The interview data also indicated that some students who erred on the arithmetic item only, did so because the instructions and item were unfamiliar. Many students who were interviewed indicated that this particular item was troublesome in arithmetic. They were all capable of calculating the item correctly, but when the instructions prevented this move, they were uncertain as to what could be done. Some students recognized that $35 \cdot 789 + 789$ was similar to the example $18 \cdot 120 + 33 \cdot 120$ but the absence of an explicit coefficient for the second number prevented them from using it as a model. No student described this item as $35 \cdot 789 + 1 \cdot 789$, but students who did the algebra correctly, often said "$13x + 1x$" before writing $14x$. Thus, for some students the unwritten "one" was quite acceptable in algebra but was not even considered in arithmetic.

The set of error types, AD1, PD3 and PD4, occurred under the general category of distributive errors. These errors occurred when a binomial was preceded by an unwritten "one" and the operation of subtraction. The following are examples of errors made by different students. Given items on the arithmetic test such as $169 - (349 + 876)$, some students would write $169 - 349 + 876$, and given algebraic items such as $(17x + 2) - (12x + 9)$, some students would write $5x + 11$ directly, or following $17x + 2 - 12x + 9$. As shown in Table 11, although some students made these errors in both contexts, there was no clear explanation as to why some students would make the algebraic error only.

While initial analysis provided no clear explanation, the interviews concerning the applicable items provided one possible reason. Students who were interviewed calculated directly in algebra and they omitted any possible intermediate steps. For example, when given (17x + 2) - (12x + 9), students would calculate "17x minus 12x" and "2 plus 9" and would write "5x + 11". Students did not indicate that they had removed the brakcets or distributed, and it seemed as if the "brackets" were "ignored" from the beginning. These same students wanted to calculate in the corresponding arithmetic items, too. Because the instructions did not permit any calculations, the students were unaware of what was expected. When it was suggested that they "remove" the brackets, most students did so correctly, implying that students perceived the role of brackets in algebra as different from their role in arithmetic.

A possible explanation as to why students made this distributive error in arithmetic only is because the instructions did not "seem" to apply in this situation. During the interviews, students indicated that these particular arithmetic items did not seem to "fit". Many students said that "if you can't calculate, there's nothing to do." When these students were requested to remove the brackets, some did it correctly as discussed earlier, but other students removed the brackets, literally. Students rewrote items such as 169 - (349 + 876) as 169 - 349 + 876 and commented that "it was the same as above; nothing has been changed." These students were unaware of the significance of brackets in the arithmetic context. Carry et al. (1980) indicated that parentheses were "abused" by students who often omitted or inserted them at random. This parentheses "problem" was highlighted further by a student who wrote (35) 789 + 789 to an earlier item. When asked what was meant by this, the student replied that both 789's were multiplied by 35. Other students wrote $19 + {}^{-}42\ (107)$ when they meant $(19 + {}^{-}42)\ 107$.

The set of error types, AD1, AD4, AD5, PD3, PD4, were an extension of those just discussed and also occurred under the general category of distributive errors. In this case, students were given items such as $169 - (349 + 876)$ and they would make one of three errors, namely, $169 - 349 + 876$ or $169 - 349 + 169 - 876$ or $169 \cdot 349 + 169 \cdot 876$. In the corresponding algebraic items, $(4p^2 - 3) - (6p^2 - 5)$, students would write ${}^{-}2p^2 - 8$. Since so many students made an arithmetic distributive error without making the corresponding algebraic one, it seemed as if the arithmetic item was perceived differently from the algebraic one. A possible explanation is that the distributive principle of multiplication over addition was misapplied only in the arithmetic items because of their structure. It would seem more plausible to perceive $169 - (349 + 876)$ as if it were $169\ (349 + 876)$ and write $169 \cdot 349 + 169 \cdot 876$ than it would be to write $(4p^2 - 3) \cdot 6p^2 - (4p^2 - 3) \cdot 5$ for $(4p^2 - 3) - (6p^2 - 5)$.

As it was previously indicated, students in the interviews did find the arithmetic items in this case to be confusing, and their reflex was to calculate. Students who made the latter arithmetic errors did not state directly that they were using the distributive principle. Thus, absolute support was not present. However, some students indicated that they used example (a), "$158 - 7\ (651 + 318) = 158 - 7 \cdot 651 - 7 \cdot 318$," which employed the distributive principle, in order to obtain the responses given. No student used such a model in algebra, but instead, "common" terms were combined "automatically".

The set of error types, AD3, PD1, PD2, PD6, also occurred under the general category of distributive errors. These error types were made when the distributive principle of multiplication over addition or subtraction was only partially completed. The following are examples of errors made by different students. Given items such as $^-12 \cdot (517 - 229)$, some students would write $^-12 \cdot 517 - 229$, while given items like $^-7w\ (3w - 6)$, other students would write $^-21w^2 - 6$. It was not clear at first why students would make this error in algebra only.

The interview data did not indicate why students made the algebraic distributive error and not the arithmetic error. No student explained the algebraic error and little discussion took place on it. Some students who did the arithmetic item correctly indicated that example (a), where "158 - 7 (651 + 318) = 158 - 7 · 651 - 7 · 318," did help them determine the procedure. No such example was available in algebra.

A possible explanation as to why students might make the arithmetic error only is because students would be more familiar with the multiplication of a binomial by a monomial than with the application of the distributive principle in arithmetic. During the interviews, most students corrected the 'partial distribution' error on the arithmetic items. Some students indicated that it was already simplified, and others indicated that if they could not calculate, it was unclear as to how it could be "simplified". When some students were given more direct instructions such as "rewrite" instead of "simplify", a correct response was made. Consequently, it seemed that the difficulty in arithmetic originated from the instructions. It appeared that "simplify" was an instruction much more appropriate for the algebraic circumstance than the arithmetic one.

The two error types, AG3, PG1, occurred under the general category of grouping errors. These particular errors were made when students imposed an incorrect grouping scheme which overruled the order of operations present. The following are examples of errors made by different students. When given items such as 189 + $^{-}21$ (537 - 792), some students would write (189 + $^-21$) (537 - 792) or 168 (537 - 792), and when given items such as 5r + $^-3$ (7r - 2), some students would write (5r + $^-3$) (7r - 2). A possible explanation is that students who erred only in algebra used the example (a) on the arithmetic test to help them reach a solution, since example (a), 158 - 7 (65 + 318) = 158 - 7 · 65 - 7 · 318 was similar to items like 189 + $^-21$ (537 - 792).

Some students in the interviews referred to the examples on the test in order to complete several particular problems on the arithmetic test. When students used the example they completed the distribution correctly but often made sign errors. Students who erred on 5r + $^-3$ (7r - 2) during the interviews indicated that they were uncertain as to what procedure to follow, so they often grouped 5r + $^-3$ and multiplied it by 7r - 2.

More students made this error in arithmetic than in algebra. A possible explanation is that the characteristics of the terms influenced the grouping procedure. It seemed that students would more likely group two numbers as in 189 + $^-21$ (537 - 792) than they would group unlike terms, as in 5r + $^-3$ (7r - 2). All students who were interviewed demonstrated clear recognition of "like" and "unlike" terms, and all of them were aware that you add "like" terms only. In the arithmetic items, students who wrote (189 + $^-21$) (537 - 792) indicated that ordinarily they would calculate and write 168 (537 - 792). Therefore, the parentheses were inserted only to satisfy the instructions. No student indicated he would add 5r and $^-3$ in 5r + $^-3$ (7r - 2) so there would be little reason to group these terms. The interview data indicated that the presence of "unlike" terms served as a deterrent to such an error and no such deterrent was present in the arithmetic items.

The two error types, AG4, PW17, occurred under two general categories. The arithmetic error occurred in the category of grouping errors and the algebraic error occurred in the category of wrong operations. When given items such as $35 \cdot 789 + 789$, students would write 35 (789 + 789), and given items such as 13x + x, students would write $13x^2$. This comparison was made to determine if students who made the arithmetic error were also saying 13 (x + x) where "x + x" was x^2. Since very few students made both these errors, it is possible that students who made the algebraic error only, did so for completely different reasons. That is, $13x^2$ was the response, but no incorrect grouping led to it.

As indicated earlier, the interviews supported this assumption, since most students wrote $13x^2$ for numerous reasons other than adding x and x separately. Some students indicated "x^2" was a way of saying "there's two x's," but at no time did they imply that the "x's" had been grouped.

It is possible that students would group only in arithmetic because in algebra 13x may not be interpreted as

13 times x but rather as a "term". Thus, $35 \cdot 789 + 789$ might be interpreted as 35 times 789 plus 789, "35 (789 + 789)", but the algebraic item would not be seen as 13 times x plus x, "13 (x + x)". No student who was interviewed read the item "13x + x" as "13 times x plus x". Instead, it was read as "13x plus x". The multiplication did not seem as obvious in algebra as it was in arithmetic, where the multiplication symbol was present. Also, it was noted earlier that students had difficulty with symbolism, especially the meaning of parentheses, and it is a possibility that students who wrote 35 (789 + 789) may not have meant what it says. This was not checked in the interviews, however.

The two error types, AG2, PT1, also occurred in two different categories. The arithmetic error type occurred in the general category of grouping errors, but the algebraic error type occurred in the general category of "like term errors". The grouping error involved the "over-distribution" of a common number, while the like term error involved the addition of "unlike" terms. The following are examples of errors made by different students. When given arithmetic items of the form $(31 \cdot 340) + (71 \cdot 340) - 123$, some students would write $(31 + 71 - 123) \cdot 340$; and when given algebraic items of the type $16d + 31d - 20$, some students would write 27d. It is possible that this was exclusively an arithmetic error because students were less apt to group "unlike terms" than they were to group numbers. It appears as if students' knowledge of algebra deters them from combining unlike terms, but their inexperience with "rewriting" arithmetic terms allows them to regroup since they do not realize they have made an error.

During the interviews, some students did comment that they did not know what to do with the "-123" in the item, "(31 · 340) + (71 · 340) - 123". Others commuted the expression in an attempt to make it match example (a), where "158 - 7 (651 + 318)" was rewritten. Such behavior suggested that students were unclear as to what to do with the arithmetic items, as was suspected. Such uncertainty was not visible in the algebra context. No student hesitated when asked to simplify "16d + 31d - 20" and most of them wrote "47d - 20" almost automatically.

The set of error types, AEM1, PEM1, PEM4, occurred under the general category of exponent errors in multiplication. These errors were made when two terms, whose exponents were unwritten "1's", were multiplied, and the product contained an unwritten "1" for its exponent. For example, when given 231 · 231, some students would write 231, and given 18x · 3x, some students would write 54x. A possible explanation as to why many students would make the algebra error only is because their knowledge of numbers would make the writing of 231 · 231 = 231 unacceptable while their knowledge of variables may not indicate any error in writing "x · x" as "x".

The interview data was partially supportive of this contention. Students who worked with this arithmetic item did receive automatic feedback as to its plausibility. For example, one student in the interview originally wrote "$2 \cdot 231$" but quickly changed it to "231^2" because the first value "would not be large enough". The students who were interviewed with respect to the algebraic items did not provide any further explanations. They often wrote "x" as an automatic response and one student who was questioned further wrote "x^2", saying "x times x is x^2 sure". This implied that the calculation was rule bound, since no explanation was based on the properties or meanings of the symbol "x".

The last set of errors, AEA1, AEA2, PEA1, PEA2, PEA3, PEA4, occurred under the general category of exponent errors in addition. These errors were made when the exponents of the terms being added were also added. The following are examples of errors made by different students. Given items such as $^-9 \cdot 18^2 + 17 \cdot 18^2$, some students would write $(^-9 + 17)\ 18^4$; while given items such as $4x^2 + 7x^2$, some students would write $11x^4$. As shown in Table 11, only one student made such an error in both contexts. However, it was unclear why students would make the algebraic error but not the arithmetic one as it seemed that the arithmetic items would be less familiar.

There was little interview data available on the arithmetic items, but one student who did these correctly in the interviews, modelled the given example accurately. This student exhibited doubt at first, then read the example (b) where $18 \cdot 120 + 33 \cdot 120 = (18 + 33) \cdot 120$, and proceeded to do this with all the applicable items, including those with the exponent equal to 2. It seems then that students could err in algebra, but with the aid of the examples complete the corresponding arithmetic items successfully.

A possible explanation as to why students would make the arithmetic error only is because the algebraic circumstances were more familiar and students would not transfer the algebraic procedures to the arithmetic context. There were few examples similar to the arithmetic items available in the current textbooks so the lack of familiarity was justified. The interview data provided no further information and it was unclear why students would still not apply the "algebraic" procedures in these circumstances.

Overall, though, the interviews did indicate that students did not recognize algebra as general arithmetic and procedures used in both contexts were believed to be distinct.

<u>Question 4</u>. If a student makes a systematic direct error, does the student make the corresponding indirect error, and vice versa?

A direct error occurred in the first step of a solution, while an indirect error occurred in a subsequent step. Indirect and direct errors were predicted in four of the general categories of errors. Students made systematic errors in the direct and indirect modes in only two of the possible four categories, namely, "sign errors" and "exponent errors". In both cases, the operation of addition was involved. Students did not make systematic indirect and direct errors, as predicted, in the "like term errors" and "wrong operation errors" categories.

As indicated in Table 12, most students made the direct errors and not the indirect ones, and no student made an error in both modes.

TABLE 12

Comparison of Students who made Direct and Indirect Errors

Direct Error	Number of Students	Indirect Error	Number of Students
PSA1	0	PSA4	5
PSA2	39	PSA5	2
PSA3	2	PSA6	0
PEA1	4	PEA3	1
PEA2	9	PEA4	0

Discussion. One general category in which students made systematic errors in the indirect and direct modes was sign errors in addition. The particular error types were made when students added signed coefficients and obtained the correct magnitude but the incorrect sign for the response. For example, a direct error was of the type where students wrote $11x^2$ for items such as $^-23x^2 + 12x^2$, and an indirect error was of the type where given items such as $5r + {}^-3\ (7r - 2)$, students would write $5r + {}^-21r + 6$, for the first step, but in the second step they would write $16r + 6$. A possible explanation is that students made the direct error and not the indirect one because "other" errors made in the "first" step of the indirect situation may have prevented the sign error from occurring in a subsequent step.

When the appropriate items were checked in the tests, it was found that this explanation was incorrect. Most students who made a systematic direct sign error in addition had the correct signs under similar circumstances in the indirect mode.

Another possible reason why the number of direct errors was greater than the number of indirect ones was simply because there were more items on the respective tests involving the direct mode. In the interviews, the students, who had made direct sign errors on the written test, responded correctly while redoing them. It appeared then that the testing "milieu" may have been responsible.

The second general category in which students made systematic errors in the direct and indirect modes was exponent errors in addition. These error types occurred when students added the exponents of the monomials which were to be combined. For example, a direct exponent error was made in such items as $4x^2 + 7x^2$ where students would write $11x^4$, and indirect exponent errors were made in items such as $^-6x\ (3x + 4) + {}^-3x\ (5x - 3)$ where students first wrote $^-18x^2 + {}^-24x + {}^-15x^2 + 9x$ and then wrote $^-33x^4 + {}^-15x^2$, or other variations. A possible explanation is that students would make only direct errors in this case because the adding of exponents when multiplying in the first step may serve as a deterrent to adding exponents again in the following step.

The interviews provided little information in this respect, since the direct and indirect situations were not openly discussed. However, when completing the item $^-5p\ (2p - 7)$ one student wrote $^-10p - 35p$ but would not add the two terms together as she did in questions like $16d + 31d - 20$, where the student wrote $47d - 20$. When inquiries were made, the student explained that these were two different situations. In the former, she had multiplied to obtain the terms so they remain separate, while in the latter, the terms were meant to be added. This implied then that procedures applied in original steps of a question are not necessarily acceptable in subsequent steps of a similar problem.

Question 5. Do grade nine and ten students make the same errors or are there differences?

When the common algebraic error types were examined for each grade, it was found that students in both grades made the same types of errors. The differences existed with regard to the frequency of the error in each grade, rather than the error itself.

As indicated in Table 13, most of the common algebraic errors were made by at least twice as many grade nine students as grade ten students. Only one error type was made by more grade ten students than grade nine students, and then only one extra student was involved.

Discussion. On a mathematics test of this type a narrow range of responses would be expected. As indicated earlier, only 15 out of the 111 predicted errors were common, and no error arose in just one grade. Furthermore, a decrease in the frequency of errors at the grade ten level might have been expected. A possible explanation is that with an "extra" year of experience with polynomials, grade ten students would be more familiar with polynomials and their properties.

It was noted that grade ten students employed factoring as part of their solution strategies and grade nine students did not. No common factoring errors were found, but some students did misapply this "factoring" strategy on such occasions as 16d + 31d - 27 where they would write (4d - 9) (4d + 3). A possible explanation is

TABLE 13

Common Algebraic Errors made by both Grade 9 and 10 Students

Error Type	Grade 9	Grade 10
PSM4	7	3
PSM8	9	4
PSA2	23	16
PSS1	10	5
PW13	9	2
PW14	9	1
PW16	29	12
PW17	31	8
PW18	10	5
PD1*	4	1
PD2	6	0
PD3	13	5
PD4	10	7
PEM1	5	6
PEM4	11	2
PEA1**	4	0
PEA2	7	2
PEA3	0	1
PEA4	0	0

*These two error types together constitute a "common" error type.
**These four error types together constitute a "common" error type.

that grade ten students were influenced by their experience with factoring which is often emphasized at this grade level. Again, there were no comments in the interviews which provided any further explanation of such procedures.

> Question 6. Within grades, are the errors made by students in the honours program different from, or similar to, those made by students taking the matriculation mathematics program?

When the common algebraic error types were examined for each program within each grade, it was found that students in both programs made similar errors. In grade nine, there was one case where no student in the honours program made the error (PSS1) but ten students in the matriculation program did so. In grade ten, there were also situations where errors were made by students in one program only, but each of these situations involved less than 6 out of 50 matriculation students.

The major difference existed with regard to the frequency of the errors, rather than the errors themselves. As indicated in Table 14, in both grades, more students in the matriculation program made the common algebraic errors than students in the honours program. One exception in grade nine occurred in the distributive error type, PD3, where $(4p^2 - 3) - (6p^2 - 5)$ was rewritten as $4p^2 - 3 - 6p^2 - 5$ by one more student in the honours program. There were two exceptions in grade ten, where more students in the honours

TABLE 14

Common Algebraic Errors which were made by Students in the Matriculation and Honours Programs at each Grade Level

Common Error Type	Grade Level 9 Program:*** M	H	10 M	H
PSM4	6	1	3	0
PSM8	7	2	3	1
PSM2	20	3	12	4
PSS1	10	0	4	1
PW13	7	2	2	0
PW14	7	2	1	0
PW16	19	10	2	10
PW17	20	11	5	3
PW18	8	2	4	1
PD1*	4	0	1	0
PD2	5	1	0	0
PD3	6	7	5	0
PD4	7	3	6	1
PEM1	3	2	1	5
PEM2	8	3	2	0
PEA1**	4	0	0	0
PEA2	6	1	2	0
PEA3	0	0	1	0
PEA4	0	0	0	0

*These two error types together constitute a common error type.

**These four error types together constitute a common error type.

***M = matriculation program; H = honours program.

program made the exponent error in multiplication (PEM1) and the wrong operation error (PW16). Ten students in the honours program, in grade ten, wrote $4x^2 + 7x^2$ as $28x^4$ while only two matriculation students did so, and five students in the honours program wrote $8a \cdot 13a$ as $104a$ while only one matriculation student did so.

Discussion. Since both mathematics programs include instruction on the simplification of polynomials, it was expected that students would make similar errors. Likewise, fewer students in the honours program were expected to make errors because these students are by definition "above average" mathematics students.

The fact that no student in the grade nine honours program made a sign error in subtraction when ten students in the grade nine matriculation program did, was not explained. No information from the programs or the interviews led to a clarification of this occurrence. As for the errors which seemed to be made exclusively by students in grade ten matriculation, the numbers were so low, namely 6 or less out of 50, it is suggested that they might have been random.

There was no clear reason available to explain why more students in the grade ten honours program would make the wrong operation error and exponent error as described earlier. No students from the honours program were interviewed, but their programs were examined. It was found that these topics are only covered briefly in a direct fashion, and emphasis is often placed on more "difficult" exercises. It is suggested that procedures used by the students in the more complex items may not be transferred to the "simpler" items as one might expect. There was no interview data to support this conjecture.

CHAPTER V

CONCLUSIONS, IMPLICATIONS, AND RECOMMENDATIONS

In this chapter, a brief overview of error analysis and a summary of the study are given. Then, the conclusions reached are summarized, and the implications for teaching suggested. Finally, some recommendations for further research are made.

Overview

Error analysis is considered to be a field in which the students' thought processes are determined through an analysis of the errors they commit. The premise of error analysis research has been that students' errors contain patterns which illustrate the incorrect strategies used to obtain the answer. Although research on errors made in algebra by high school students was limited, the research available supported the premise that "systematic" errors are made. Researchers were able to list or categorize the errors they found, as well as use the errors to describe the types of algebraic knowledge the students possessed. To describe the strategies used by their subjects, researchers such as Wattawa (1927), Davis et al. (1978), Carry et al. (1980), and Lewis (1980), used terms such as "operator gaps", "binary confusions", and "overextension of pieces of knowledge". Wattawa (1927) attributed the difficulty in algebra to the lack of fundamental arithmetic knowledge. Davis et al. (1978) proposed that students made errors if the procedures they used were so salient and automatic that they were applied without any conscious awareness by the student. Lewis (1980) indicated that learning procedures without meaning led to errors. Whatever the rationale, all researchers agreed that students' knowledge of algebra was incomplete.

The simplification of polynomials was chosen as the algebraic topic to investigate in this study since it serves as a basis for most other algebraic topics. Students in grade nine and ten mathematics programs were chosen as appropriate subjects. To control the instructional factor as much as possible, different classes were chosen within several different schools and the tests were administered directly by the investigator. Of the 573 students who were tested, 25 students were randomly selected from eight groups representing a Grade (9 or 10) by Program (Matriculation or Honours) by Sex (Male or Female) matrix, resulting in a total sample of 200 students in the analysis. Each student's errors were classified in terms of a proposed coding scheme, and any unique errors which were found infrequently were noted but were not formally classified.

As a follow-up to the written tests, 16 students were interviewed to aid in the interpretation of the results.

Conclusions

A systematic error was defined as an error which a student made on "at least 50%" of the occasions in which the student had the opportunity to make the error. The results indicated that grade nine and ten students do make systematic errors when simplifying polynomials. Since some of the skills tested were found in only two items, the criterion of "at least 50%" may at first glance be considered too low. However, when a more stringent criterion of "more than 50%" was applied, 79 of the 122 students who made systematic errors still qualified. In particular, when the "more than 50%" criterion was applied, there were 30 algebraic errors which were made systematically.

A common error was defined as a systematic error which was made by 10 or more students, irrespective of the grade or program. Fifteen common algebraic errors were present and three of these were made by about 20% of the sample. Thus, it was concluded that students in different grades, programs, and schools made both systematic and common errors.

No student made all the error types listed under any general category. For example, no individual student, who made an error, made all the "exponent errors"; rather each student made particular errors in exponent problems such as the errors made when multiplying exponential expressions which contained unwritten exponents. Similarly, no student made all the possible "wrong operation" errors, but many students "added binomials when they were required to subtract". The occurrence of such "specific" error types indicated the "incomplete" nature of the students' knowledge. For example, most students knew that expressions such as $11x^2 \cdot 2x^3$ were simplified by multiplying the coefficients and adding the exponents, yet many of these same students simplified $8x \cdot 13x$ by simply multiplying the coefficients, and writing $104x$. In this case, these students' knowledge of the multiplication of exponential expressions was "incomplete" since it did not include situations where the exponents were unwritten. "Incomplete" knowledge of this sort allowed many students to solve a major portion of exercises in each category correctly, while being insufficient for the correct completion of particular tasks. It was concluded that some students did not necessarily perceive the relationships which existed between "particular" tasks within a general category.

Furthermore, students did not necessarily perceive the relationships which existed between algebraic and corresponding arithmetic items. It seemed as if students perceived many of the procedures used with polynomials as distinct algebraic operations which were not applicable in arithmetic. As indicated in the previous chapter, most students in the interview sample saw no connection between items such as $13x + x$ and $35 \cdot 789 + 789$ and few students, who made errors, applied the same procedure in both. Because many students who performed correctly on algebraic tasks were unable to apply similar properties in arithmetic, it was concluded that students can carry out algebraic algorithms without knowing the properties which underlie the procedures. During the interviews, students used rules such as "adding exponents" and "combine like terms" but no student spoke of the "distributive principle" or the "commutative property". It appeared that not only were arithmetic and algebra considered as two distinct entities, but algebraic procedures employed in the simplification of polynomials were not overtly based on the "properties" involved.

For many students, algebra was an exercise in symbol manipulation and little concern was given to the meanings of those symbols. Students who were interviewed referred to "x^2" as "two x's", or "$13x + x$" was read "13x plus x" rather than "13 times x plus x". These students seemed to imply that they saw "letters" and were unaware of what those "letters" represented. Students failed to recognize the meaning of a "variable" and were willing to manipulate it in ways which were unacceptable for "numbers".

Differences also occurred in students' approaches to direct and indirect situations. As indicated in the results, more students made direct errors than indirect ones. Using the length of a solution as one criteria for difficulty, it would seem that the direct situations should have been easier. Thus, it was unexpected when students who made "direct" errors did not make the same error in the corresponding "indirect" situations. It seemed that when more than one step was involved in the solution, certain cues tended to deter students from errors which they had made when only one step was required. As Davies et al. (1978) proposed, procedures in the direct circumstances may have been so automatic that students would not doubt their behavior, but when the procedure was initiated in an indirect circumstance, other characteristics of the problem prevented students from reacting in the same way.

It was also possible that students did not perceive the direct and indirect situations as similar circumstances. For example, one student who was interviewed wrote $^{-}10p + 35p$ for $^{-}5p\ (2p - 7)$ but refused to combine the terms, as she had in the problem $16d + 31d - 27$ where she automatically wrote $47d - 27$. This student explained the two different approaches by saying that "because she had obtained $^{-}10p + 35p$ by multiplying $^{-}5p\ (2p - 7)$, the terms must remain separate" but "since there's no multiplication in $16d + 31d - 27$, all you can do is add".

It was found that fewer grade ten students than grade nine students as well as fewer students in the honours program than students in the matriculation program made systematic errors in algebra. However, only the frequency decreased, and the nature of the errors remained the same. It seemed that while an extra year of exposure to algebra for grade ten students reduced the number of systematic errors, it was not sufficient to eliminate them. Also, although students with above average ability in mathematics made fewer systematic errors than those with an average ability, errors were still made.

There was some general information obtained during the interviewing which was also noteworthy. For instance, some students showed confidence in their incorrect procedures, and often believed they were doing well. For example, one student who simplified $^{-}2w\ (3w + 7) + {}^{-}3w\ (2 - 5w)$ erroneously step by step, ended with the answer $8w - 7$ and the comment "I haven't been thinking this clear in a while". Other students who changed strategies for similar exercises often chose the more frequently utilized incorrect strategy over the correct one. For instance, a student who added exponents in all addition exercises, except one, chose to add the exponents there too when the discrepancy was pointed out. In most cases, it appeared that many of the students who were interviewed were unaware that errors had been committed.

Some students who were interviewed applied their knowledge from other areas of mathematics even when the necessary conditions were not met. For example, several students used the "transposition property" employed in equation solving to simplify situations such as $^{-}18x^2 + {}^{-}24x + {}^{-}15x^2 + 9x$", by writing "$^{-}18x^2 + 15x^2 + 24x + 9x$". Such misapplication of knowledge seemed to depend on the students' interpretation of the task at hand.

Many students rewrote $35 \cdot 789 + 789$ as $(35)\ 789 + 789$ which would be interpreted as meaning the same thing. However, when one student was asked what he meant by $(35)\ 789 + 789$, he replied "35 is multiplied by both 789's". This student wrote $(35)\ 789 + 789$, but he meant $35\ (789 + 789)$. Later, when asked to show that only one 789 was multiplied by 35, he wrote $35 \cdot 789^2$. This student's interpretation was clearly different from the standard interpretation attached to such symbols. Thus, it seemed that a student's perception of an initial exercise could be different from what was required, and errors could result.

Implications for Teaching

The results and conclusions in this study have several implications for teachers and authors of textbooks. The overall conclusion was that students, at these grade levels, do make systematic errors which contain "logical" patterns and teachers need to be aware of these errors, since they disclose important information concerning the failure strategies students have adopted. The analysis of errors in this study has implications for remediation procedures as well as for possible "preventive" techniques which might be useful.

It is important that teachers realize that although "common" errors were found, remediation might be more successful if the particular "incorrect" procedure that the student used is addressed. Even though ten students or more obtained the answer $^{-}10p^2 - 35p$ for $^{-}5p\ (2p - 7)$, for example, the procedures used by individuals were different. Consequently, individual remediation seems to be necessary.

Individual remediation becomes even more important when it is realized that the error types which occur are very "specific". For instance, to inform students who made the error, "$8a \cdot 13a = 104a$", that "exponents are supposed to be added when you multiply" would be of limited value, since most of these same students demonstrated in other questions that this rule is already known, but not applied here. Instead, remedial methods could be geared directly to the characteristics of the task which permit the error to occur. In this case, the unwritten exponent appeared to be the specific characteristic which should be attended to.

Besides using the errors found in this study for remediation purposes, it is important that teachers be aware of the possible difficulties and attempt to prevent their repetition in other students' work. In this study, most students were unaware of the link between arithmetic and algebra. Consequently, the procedures used in one context did not seem applicable in the other context. To help clarify the relationship between these two areas, the properties of numbers might be demonstrated in both. Where possible, attempts could be made to help students "understand" algebra in hopes of preventing students from seeing algebra as the mere manipulation of symbols. To increase the understanding of algebra, it may help if students are reminded that the "variable" represents a "numerical value" and it must always be treated as such. Examples and counterexamples might be used to demonstrate the necessary conditions needed for the application of algebraic procedures. This relationship between arithmetic and algebra might be further strengthened if students were encouraged to check their algebraic responses by using numerical replacements. "Checking" tended to be more "natural" in the arithmetic context, and as a consequence, students who wrote $35 \cdot 789^2$ for $35 \cdot 789 + 789$ were more suspicious than those who wrote $13x^2$ for $13x + x$.

Because many students failed to focus on the active operation to determine the procedure required, perhaps more emphasis could be placed on the "operator" during instruction. For example, when students add like terms, they might be encouraged to do so because the operation is addition and not because the like terms are present. In this respect, it may be helpful if the properties involved in carrying out the operation are emphasized. For instance, a student who simplifies $13x + x$ to $14x$ needs to realize that the distributive principle was used and it was not "magic".

Since students adopt their own interpretation of mathematical language, symbolism should not be taken for granted. It seems possible that if teachers become more aware that students often perceive the language differently, the explicit emphasis on symbols, per se, might help remediate errors.

Where possible, students could receive a balanced exposure to direct and indirect cases of particular skills. For example, overemphasizing the multiplication of monomials by binomials to the neglect of the product of monomials can lead to difficulties. In this study, students were able to simplify $^{-}2w\ (3w + 7) + {^{-}3w}\ (2 - 5w)$ in which they combined $^{-}6w^2 + 15w^2$ correctly, but when given items such as $^{-}6w^2 + 15w^2$ directly, students made errors. Therefore, it appeared that procedures used in the "more complex" task were not readily transferred to the "simpler" ones.

Errors in the distributive error category and the exponent error category were more prevalent in items which involved an unwritten coefficient or an unwritten exponent. For example, more students experienced difficulty with items such as $13x + x$ and $(17x + 2) - (12x + 9)$ than they did with items such as $2p + {}^{-}7p$ and ${}^{-}6x\ (3x + 4) + {}^{-}3x\ (5x - 3)$. This difficulty with the unwritten "one" seemed significant. Therefore, to improve students' ability to work with such expressions, teachers and authors of textbooks might consider writing $13x^1 + 1x^1$ instead of $13x + x$, at least until the student is definitely capable of functioning without it.

The general implication then is for teachers to be as attentive and empathetic to the students' perception of algebra as possible. The teacher should attempt to eliminate all possibilities for ambiguity and to emphasize the characteristics and restrictions of particular examples so that students are aware of the conditions necessary for the application of certain procedures.

Recommendations for Future Research

Due to the limitations of this study, and because there are some questions still left unanswered, there are several recommendations to be made for further research.

Because the literature found in error analysis at the high school level was limited, it is recommended that other studies of this nature be carried out at all grades from seven to twelve and with a variety of topics and courses. The aim of these studies should be to determine the difficulties students experience in high school mathematics.

Since the interviews used in this study, even though limited by the number and selection of subjects, were able to provide useful information concerning the students' thought processes, it is recommended that more in-depth interviewing of students at all grade levels be carried out and attention paid to the most common errors found.

It is recommended that other studies be carried out at these grade levels on this topic, but with refinements to the instruments so that additional items appropriate for a given error are included.

Further research concerning the indirect and direct modes of operating could also be valuable. In this respect, a greater balance should be maintained between the number of test items which involved these two modes.

Since many of the common errors found in this study concerned exercises involving either an unwritten coefficient or exponent, a more restricted study on this characteristic might prove informative.

While this study shed some light on the relationship between errors in arithmetic and algebra, many aspects still remain unexplained and are worthy of further research.

Research into the retention of systematic errors should be carried out, and if possible, the students tested here should be retested in future years to determine if their errors persist.

The role of instructions and examples should be investigated, since these factors seemed to influence students' behavior on the arithmetic instrument.

Overall, future research needs to be continued in the field of error analysis with subjects at the high school level. These students' perception and interpretation of mathematics ought to be determined, and teachers need to be aware of the difficulties and ambiguities that students experience.

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APPENDIX A

SUMMARY OF ERROR TYPES FOUND IN
REVIEW OF LITERATURE

Most researchers in error analysis provided some error classifications and often lists of specific error types. In this Appendix, the error types reported in the literature reviewed is presented in conjunction with a description and example. Wherever possible the examples and descriptions were taken directly from the researcher's report.

Those marked with an asterisk (*) indicate those errors considered applicable to the simplification of polynomials.

TABLE 15

Specific Error Types Reported in Available Studies

Researcher	Error Type	Description	Example
Wattawa (1927)	*Sign Errors	Incorrect sign distribution when parentheses are present	$6x - 4(x - 5) = 6x - 4x - 20$
	*Exponent Errors	Incorrect use of exponents	$t \cdot t = t$; $w^2 \cdot w^2 = w^2$
	Arithmetic Errors	Errors made when computing with numbers	$4^2 = 8$; $\frac{1}{4} \cdot \frac{1}{4} = 2\frac{1}{4}$ $(2x - 5)^2 = 4x - 10$
Pease (1929)	*Literal Number Errors	Ignoring literal number with no written number coefficient	$-a + -4a = -4a$
		Omitting the literal number from the sum	$-9a + 4a = -5$
		Misuse of $a - b$ as a monomial	$\begin{array}{r} (a - b) \\ \underline{-4(a - b)} \\ \underline{-3a - b} \end{array}$
	*Sign Errors	Like and unlike sign errors in subtraction and multiplication	$-3 \cdot -5 = -15$ $-3 \cdot 7 = 21$
	Transcribing Errors	Errors in copying, omission of terms or misarrangement of terms	
	*Errors with multiplication of monomials	Adds the numerical coefficients	$3x \cdot 2x = 5x^2$
		Multiplies exponents	$(2x^2)(3x^3) = 6x^6$
		Omits exponent in product	$(3x^2)(4x^3) = 12x$
		Ignores an unprinted exponent	$(2x)(3x^2) = 6x^2$
		Literal exponent applied to numerical coefficient	$(2a)(-4a^2) = 32a^3$
		Failure to combine literal numbers	$(2a)(-4ab) = -8aab$

(cont'd.)

Researcher	Error Type	Description	Example
Carry et al. (1980)	*Operator Errors	Errors which involved the deletion of elements from expressions. That is, operations like - and ÷ are identified as generic deletion operations	$\frac{a + x}{a} = x$
	*Recombination Errors	Errors arising from an interpretation of addition and multiplication as generic operations of combining	$y + yz = 2yz$, $x + x$, $x \cdot x$, $2x$ and x^2 all seen as combining two x's.
	Arithmetic Errors	Errors involving simple incorrect arithmetic on unsigned numbers	
	Operator Gaps	Solvers lacked certain operators and had trouble when such operators appeared	$p + prt$ was incomplete since subject did not recognize p as a factor
	*Applicability Errors	A correct operation was applied to an expression or equation that did not satisfy the conditions for application	$2 \cdot 3 + 6$ treated as if $2 \cdot (3 + 6)$
	*Execution Errors	Incomplete execution of a correct operation or the possibility of complete execution of an incorrect operation	$2(x + 1) = 2x + 1$

(cont'd.)

Researcher	Error Type	Description	Example
Roberts (1968)	*Wrong Operation	Pupil used an operation other than the one required	$8 \cdot 9 = 17$
	Obvious Computational Error	Pupil applied correct operation but recalled incorrect basic fact	$4 \times 9 = 32$
	Defective Algorithm	Correct operation used but errors, other than basic facts, made in carrying out the necessary steps	$8 - 13 = 5$
	Random Response	Response showed no discernible relationship to the problem	
Engelhardt (1977)	*Basic Fact Error	Errors in the recall of basic number facts	$5 \times 7 = 34$
	*Defective Algorithm	Pupil executed a systematic but erroneous procedure	$\begin{array}{r} 123 \\ \underline{\times 42} \\ \underline{186} \end{array}$
	Grouping Error	Errors due to lack of attention to the positional nature of the number system	$\begin{array}{r} 57 \\ \underline{93} \\ \underline{1410} \end{array}$
	*Inappropriate Inversion	Computation involved the reversal of some critical aspects of the solution procedure	$\begin{array}{r} 43 \\ \underline{-19} \\ \underline{36} \end{array}$
	Incorrect Operation	Pupil performed an operation other than the appropriate one	$6 \times 9 = 15$

(cont'd.)

Researcher	Error Type	Description	Example
Engelhardt (1977)	*Incomplete Algorithm	Pupil initiated an appropriate procedure but aborted it or left out critical steps	
	Identity Errors	Pupil showed confusion with operation of identities (0 and 1)	2 x 1 = 1
	Zero Errors	Pupil indicated difficulty with concept of zero	30 x 21 = 63
Pincus (1975)	Poor Understanding of the Meaning of Number and Place Value		
	Inadequate Mastery of Basic Facts		
	Poor Alignment of Digits in Columns		
	*Poor Penmanship		
	Failure to check Answer or Estimate		
	Disregard of Symbol		56 - 21 = 77

(cont'd.)

Researcher	Error Type	Description	Example
Radatz (1979)	Language Difficulties	Misunderstanding of semantics of mathematical texts led to errors	
	Difficulties in Obtaining Spatial Information	When performing a mathematical task, children were unable to obtain visual or spatial information	Boundaries in Venn Diagrams misinterpreted
	*Deficient Mastery of Prerequisite Skills, Facts and Concepts	Ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures for applying mathematical techniques and insufficient knowledge of necessary concepts and symbols	When asked to "double the smallest 3-digit number and add the largest 4-digit number", pupils wrote 111 + 111 = 222 222 + 9999 = 10221
	*Incorrect Associations or Rigidity of Thinking	Used cognitive operations when fundamental conditions of tasks had changed	9 x 60 = 560 5 x 13 = 63 6 x 60,00 = 36,000
	*Application of Irrelevant Rules or Strategies	Use of comparable rules or strategies from other areas	When asked to rotate a square, students folded them instead.

TABLE 16

Other Applicable Error Types Found in the Literature

Author	Error Type	Description	Example
Laursen (1978)	Extension of a shortcut procedure	Student applies a shortcut in situation where conditions are insufficient	$\sqrt{a^2+b^2} = \sqrt{a^2} + \sqrt{b^2} = a + b$
Budden (1972)	*Universal Distributivity of Operations	Distribution is carried out regardless of the operation or symbolism	$c\,(ab) = cacb$
	Commutativity of Operations	All operations assumed to be commutative	$(a + b)^2 = a^2 + b^2$
	Omission of Punctuation	Child omits or ignores parentheses or introduces own grouping schemes	$5 + 2\,(3 + 7) = 70$
Davis et al. (1980)	Misinterpretation of symbolism and distinguishing characteristics	Students do not distinguish symbols as perceived by mathematicians	$3x$, $2x^4$, $4x^5$ are like terms as they all have x and number in front or $\frac{3x + h}{h}$ same as $\frac{3xh}{h}$
Kent (1978a)	Misinterpretation of symbolism	Students interpret letters, numbers and operation signs as one group of symbols	$\frac{2xy^2 + 2y}{2} = xy^2 + 2y$

APPENDIX B

INSTRUMENTS

COMPUTATION TEST

SIMPLIFY:

1. $-13 \cdot 9 =$ ____________
2. $18 + {}^{-}7 =$ ____________
3. $13 - {}^{-}12 =$ ____________
4. $-40 - {}^{-}73 =$ ____________
5. $16 \cdot {}^{-}3 =$ ____________
6. $-27 - 32 =$ ____________
7. $-41 - {}^{-}21 =$ ____________
8. $27 + {}^{-}39 =$ ____________
9. $25 - 35 =$ ____________
10. $-18 + {}^{-}27 =$ ____________
11. $-4 \cdot {}^{-}21 =$ ____________
12. $-14 - 12 =$ ____________
13. $15 - {}^{-}18 =$ ____________
14. $16 - 33 =$ ____________
15. $-10 \cdot {}^{-}17 =$ ____________
16. $15 + {}^{-}41 =$ ____________
17. $-32 + {}^{-}8 =$ ____________
18. $-37 - {}^{-}52 =$ ____________
19. $56 + {}^{-}24 =$ ____________
20. $-19 - {}^{-}16 =$ ____________

ARITHMETIC TEST

SIMPLIFY, but do not make calculations to find a final answer. For example,

a) $158 - 7(651 + 318) = 158 - 7 \cdot 651 - 7 \cdot 318$

b) $18 \cdot 120 + 33 \cdot 120 = (18 + 33)\ 120$

1. $^{-}9 \cdot 18^2 + 17 \cdot 18^2$

2. $231 \cdot 231$

3. $19 \cdot 107 + {}^{-}42 \cdot 107$

4. $^{-}12\ (517 - 229)$

5. $(31 \cdot 340) + (71 \cdot 340) - 123$

6. $23 \cdot 666 + 51 \cdot 666$

7. $(58 \cdot 171) + (43 \cdot 171) - 516$

8. $13^6 \cdot 13^5$

9. $35 \cdot 789 + 789$

10. $392 \cdot 392$

(cont'd.)

11. $612 - (349 + 876)$

12. $^{-}59\,(65 - 97)$

13. $14 \cdot 376 + 376$

14. $189 + {}^{-}21\,(537 - 792)$

15. $156 \cdot 156$

16. $139 + {}^{-}5\,(487 - 632)$

17. $97 - (793 + 184)$

18. $107 \cdot 107^{3}$

19. $8^{4} \cdot 8^{7}$

20. $^{-}23 \cdot 57^{2} + 12 \cdot 57^{2}$

ALGEBRA TEST

SIMPLIFY:

1. $z^4 \cdot z^9$

2. $16d + 31d - 27$

3. $9m \cdot {}^{-}3$

4. $(4p^2 - 3) - (6p^2 - 5)$

5. ${}^{-}2w(3w + 7) + {}^{-}3w(2 - 5w)$

6. ${}^{-}5p\ (2p - 7)$

7. $4w \cdot 9w^5$

8. $8x \cdot {}^{-}7$

9. $11n^3 \cdot 2n^5$

10. ${}^{-}6\ (13a + 8)$

(cont'd.)

11. $^-6x(3x+4)+{}^-3x(5x-3)$

12. $5r + {}^-3\ (7r - 2)$

13. $(8d^2 - 13) - (7d^2 - 4)$

14. $10p + {}^-3\ (6p + 8)$

15. $8a \cdot 13a$

16. $n^3 \cdot n^5$

17. $2p + 5p^2 - 4 + 8p^2 + 5 + {}^-7p$

18. $^-8\ (7y + 9)$

19. $15b \cdot 7b$

20. $(17x + 2) - (12x + 9)$

21. $7w^4 \cdot 8w^7$

22. $13x + x$

(cont'd.)

23. $4x^2 + 7x^2$

24. $^-4x^2 + 7 + 2x + {}^-6 + 5x^2 - 3x$

25. $^-23x^2 + 12x^2$

26. $27b + 10b - 5$

27. $p \cdot p^7$

28. $^-7w\,(3w - 6)$

29. $3m \cdot 7m^3$

30. $17y + y$

31. $w \cdot w^5$

32. $(4w + 13) - (3w + 6)$

APPENDIX C

DESCRIPTION OF ERROR
CATEGORIES HYPOTHESIZED

Each error type was categorized using a three or four character code. Each code began with a letter which represented the test on which the error occurred, C for computation, A for arithmetic, or P for algebra (polynomials). The second character, also a letter, represented the category of error as shown in Table 17.

TABLE 17

List of Letters Used for Each Category

Category	Letter	Category	Letter
Sign Errors	S	Incorrect Symbolism Written	L
Basic Fact Errors	F	Numerical Bases Multiplied	B
Wrong Operation	W	Exponent Errors	E
Distribution Errors	D	Like Term Errors	T
Grouping Errors	G	Miscellaneous	M

If a third letter was present, it indicated the operation involved, and could be M for multiplication, S for subtraction, and A for addition. The final character was a digit which indicated the number of the error in a particular group represented by the previous characters. Two examples are provided in Figure 31.

Example 1:

AEM1

arithmetic exponent multiplication first

AEM1 means the first exponent error in multiplication on the arithmetic test.

Example 2:

PD3

algebra distribution third

PD3 means the third distribution error on the algebra test.

FIGURE 31. Two examples of the abbreviations used in the coding.

TABLE 18: (at back of this paper)

TABLE 18

DESCRIPTIONS AND EXAMPLES OF ERROR TYPES

Error Category	Code	Description	Example	Max.*
Sign Errors	CSM1	${}^{-}a \cdot {}^{-}b = {}^{-}(a \cdot b)$	${}^{-}5 \cdot {}^{-}6 = {}^{-}30$	2
	CSM2	${}^{-}a \cdot b = ab$ $a \cdot {}^{-}b = ab$	${}^{-}5 \cdot 6 = 30$ $5 \cdot {}^{-}6 = 30$	2
	ASM1	${}^{-}a\ (b - c) = ab + ac$	${}^{-}3\ (4 - 5) = 3 \cdot 4 + 3 \cdot 5$	4
	ASM2	${}^{-}a\ (b - c) = {}^{-}ab - ac$	${}^{-}3\ (4 - 5) = {}^{-}3 \cdot 4 - 3 \cdot 5$	4
	ASM3	${}^{-}a\ (b - c) = ab - ac$	${}^{-}3\ (4 - 5) = 3 \cdot 4 - 3 \cdot 5$	4
	PSM1	$ax \cdot {}^{-}b = abx$	$9m \cdot {}^{-}3 = 27m$	2
	PSM2	${}^{-}ax\ (bx \pm c)$ where ${}^{-}ax \cdot bx = abx^2$	${}^{-}5p\ (2p \pm 7) = 10p^2 \pm 35p$	4
	PSM3	${}^{-}ax\ (bx + c)$ where ${}^{-}ax \cdot c = acx$	${}^{-}5p\ (2p + 7) = {}^{-}10p^2 + 35p$	4
	PSM4	${}^{-}ax\ (bx - c)$ where ${}^{-}ax \cdot {}^{-}c = {}^{-}acx$	${}^{-}5p\ (2p - 7) = {}^{-}10p^2 - 35p$	4
	PSM5	${}^{-}a\ (bx \pm c)$ where ${}^{-}a \cdot bx = abx$	${}^{-}5\ (2p \pm 7) = 10p \pm 35$	4
	PSM6	${}^{-}a\ (bx + c)$ where ${}^{-}a \cdot c = ac$	${}^{-}5\ (2p + 7) = {}^{-}10p + 35$	3
	PSM7	${}^{-}a\ (bx - c)$ where ${}^{-}a \cdot {}^{-}c = {}^{-}ac$	${}^{-}5\ (2p - 7) = {}^{-}10p - 35$	1
	PSM8	${}^{-}ax\ (c - bx)$ ${}^{-}ax \cdot {}^{-}bx = {}^{-}abx^2$	${}^{-}5p\,(7 - 2p) = {}^{-}35p - 10p^2$	1
	CSA1	$a + {}^{-}b = {}^{-}(a - b)\ (b < a)$	$8 + {}^{-}3 = {}^{-}5$	2
	CSA2	$a + {}^{-}b = b - a\ (b > a)$	$10 + {}^{-}24 = 14$	2
	CSA3	${}^{-}a + {}^{-}b = a + b$	${}^{-}10 + {}^{-}13 = 23$	2
	PSA1	${}^{-}ax^n + {}^{-}bx^n = (a + b)\ x^n$	${}^{-}2x^2 + {}^{-}5x^2 = 7x^2$	2
	PSA2	$-bx^n + ax^n$ $ax^n + {}^{-}bx^n$ $= (b - a)\ x^n$ $(b > a)$	$5r + {}^{-}21r = 16r$ $-23x^2 + 12x^2 = 11x^2$	2
	PSA3	${}^{-}ax^n + bx^n = {}^{-}(b - a)\ x^n$ $(b > a)$	${}^{-}6w^2 + 15w^2 = {}^{-}9w^2$	2
	PSA4	${}^{-}ax^n + {}^{-}bx^n = (a + b)\ x^n$ (indirect)		2
	PSA5	$ax^n + {}^{-}bx^n = (b - a)\ x^n\ (b > a)$ (indirect)		3
	PSA6	${}^{-}ax^n + bx^n = {}^{-}(b - a)\,x^n\ (b > a)$ (indirect)		1
	CSS1	$a - {}^{-}b = {}^{-}(a + b)$	$13 - {}^{-}12 = {}^{-}25$	2
	CSS2	${}^{-}a - {}^{-}b = {}^{-}(b - a)\ (b > a)$	${}^{-}40 - {}^{-}73 = {}^{-}33$	2
	CSS3	${}^{-}a - b = b + a$	${}^{-}27 - 32 = 59$	2
	CSS4	$a - b = b - a\ (b > a)$	$25 - 35 = 10$	2
	CSS5	${}^{-}a - {}^{-}b = a - b\ (a > b)$	${}^{-}41 - {}^{-}21 = 20$	2

Table 18 (cont'd.)

Error Category	Code	Description	Example	Max.*
	PSS1	$ax^n - bx^n = (b - a)\ x^n\ (b > a)$	$4p^2 - 6p^2 - 2p^2$	2
	PSS2	$ax^n - bx^n = {}^-(a - b)\ x^n\ (b > a)$	$8d^2 - 7d^2 = {}^-d^2$	3
	PSS3	${}^-ax^n - bx^n = (a - b)\ x^n$	${}^-18x^2 - 15x^2 = {}^-3x^2$	1
Basic Fact Errors	CF1**	$a \cdot b = c\ (c \neq a \cdot b)$	$2 \cdot 5 = 11$	4
	CF2**	$a + b = c\ (c \neq a + b)$	$2 + 5 = 8$	16
	PF1**	$ax^n \cdot bx^n = cx^n\ (c \neq a \cdot b)$	$2x \cdot 3x = 7x^2$	16
	PF2**	$ax + bx = cx\ (c \neq a + b)$	$2x + 3x = 4x$	11
Wrong Operation	CW1**	$a \cdot b = a + b$	${}^-13 \cdot 9 = {}^-4$	4
	CW2**	$a + b = a \cdot b$	$18 + {}^-7 = {}^-126$	6
	CW3**	$a - b = a + b$	${}^-40 - {}^-73 = {}^-113$	6
	CW4**	$a - {}^-b = a - b$	$13 - {}^-12 = 1$	2
	CW5	${}^-a - b = a - b$	${}^-14 - 12 = 2$	2
	CW6	$a + {}^-b = a + b$	$18 + {}^-7 = 25$	4
	PW1**	$ax^n \cdot bx^m = (a + b)\ x^{n+m}$	$4w^2 \cdot 9w^5 = 13w^7$	4
	PW2	${}^-a\ (bx \pm c) = ({}^-a + b)\ x \pm ac$	${}^-6\ (13a + 8) = 7a + {}^-48$	4
	PW3	${}^-a\ (bx \pm c) = {}^-abx \pm ({}^-a + c)$	${}^-6\ (13a + 8) = {}^-78a + 2$	4
	PW4	${}^-ax\ (bx \pm c) = (a + b)\ x \pm acx$	${}^-5p\ (2p - 7) = {}^-3p + 35p$	4
	PW5	${}^-ax\ (bx \pm c) = abx^2 \pm (a + c)\ x$	${}^-5p\ (2p - 7) = {}^-10p^2 - 12p$	4
	PW6**	$ax \cdot b = (a + b)\ x$	$9m \cdot {}^-3 = 6m$	2
	PW7**	$ax \cdot bx = (a + b)\ x^2$	$18a \cdot 4a = 22a^2$	2
	PW8	$ax \cdot bx^n = (a + b)\ x^{1+n}$	$4w \cdot 9w = 13w^6$	4
	PW9**	$ax \pm bx = abx$	$5x + 3x = 15x$	5
	PW10**	$ax^n \pm bx^n = abx^n$	$4x^2 + 7x^2 = 28x^2$	4
	PW11**	$ax \pm bx = abx$ (indirect)		4
	PW12**	$ax^n + bx^n = abx^n$ (indirect)		2
	PW13	$(ax^n + b) - (cx^n + d)$ $= (a + c)\ x^n - (b + d)$	$(17x + 2) - (12x + 9)$ $= 29x - 11$	2
	PW14	$(ax^n - b) - (cx^n - d)$ $= (a + c)\ x^n - (b + d)$	$(8d^2 - 13) - (7d^2 - 4)$ $= 15d^2 - 17$	2
	PW15	$ax^n + {}^-bx^n = \pm (a + b)\ x^n$	$5r + {}^-3r = 8r$	7
	PW16**	$ax^n + bx^n = abx^{2n}$	$4x^2 + 7x^2 = 28x^4$	2
	PW17	$ax + x = ax^2$	$13x + x = 13x^2$	2
	PW18	$(ax^n \pm b) - (cx^n + d)$ $= acx^{2n} \pm adx^n \pm bcx^n \pm bd$	$(8d^2 - 13) - (7d^2 - 4)$ $= 56d^4 - 32d^2 + 91d^2 + 52$	4
Distribution	AD1	$a - (b + c) = a - b + c$	$169 - (153 + 189) = 169 - 153 + 189$	2
	AD3	${}^-a\ (b - c) = ab - c$	${}^-59\ (65 - 97) = {}^-59 \cdot 65 - 97$	4

Table 18 (cont'd.)

Error Category	Code	Description	Example	Max.*
	AD4	$a - (b + c) = a - b + a - c$	$169 - (153 + 189) = 169 - 153 + 169 - 189$	2
	AD5	$a - (b + c) = a \cdot b \pm a \cdot c$	$169 - (153 + 189) = 169 \cdot 153 \pm 169 \cdot 189$	2
	PD1	${}^{-}a\ (bx \pm c) = {}^{-}abx \pm c$	${}^{-}6\ (13a + 8) = {}^{-}78a + 8$	4
	PD2	${}^{-}ax\ (bx \pm c) = {}^{-}abx \pm c$	${}^{-}6x\ (3x + 4) = {}^{-}18x^2 + 4$	4
	PD3	$(ax^n - b) - (cx^n - d)$ $= ax^n - b - cx^n - d)$	$(4p^2 - 3) - (6p^2 - 5)$ $= 4p^2 - 3 - 6p^2 - 5$	2
	PD4	$(ax + b) - (cx + d)$ $= ax + b - cx + d$	$(17x + 2) - (12x + 9)$ $= 17x + 2 - 12x + 9$	2
	PD5	${}^{-}ax\ (bx \pm c) = abx^2 \pm ac$	${}^{-}7w\ (3w - 6) - {}^{-}21w^2 + 42$	4
	PD6	${}^{-}a\ (bx \pm c) = bx \pm ac$	${}^{-}6\ (13a + 8) = 13a - 48$	4
Grouping	AG1+	$ab + cb = a + c \cdot b$	$19 \cdot 107 + {}^{-}42 \cdot 107$ $= 19 \cdot 107 + {}^{-}42 \cdot 107$	8
	AG2	$ab + cb - d = (a + c) \cdot b$	$(58 \cdot 171 + (43 \cdot 171) - 516$ $= (58 + 43 - 516)\ 171$	2
	AG3	$a + {}^{-}b\ (c - d) = (a + {}^{-}b)\ (c - d)$	$139 + {}^{-}5\ (487 - 632)$ $= (139 + {}^{-}5)\ (487 - 632)$	2
	AG4	$ab + b = a\ (b + b)$	$35 \cdot 789 + 789$ $= 35\ (789 + 789)$	2
	AG5	$ab + cb - d = (a + c)\ (b + d) - d$	$(58 \cdot 171) + (43 \cdot 171) - 516$ $= (58 + 43)\ (171 + 171) - 516$	2
	AG6+	$ab + cb = (a + c)\ (b + b)$	$19 \cdot 107 + {}^{-}42 \cdot 107$ $= (19 + {}^{-}42)\ (107 + 107)$	2
	PG1	$ax + {}^{-}b\,(cx - d) = (ax + b)\ (cx - d)$	$5r + {}^{-}3\ (7r - 2)$ $= 6r + {}^{-}3)\ (7r - 2)$	
Incorrect Symbolism	AL1	writes $\cdot$ instead of +		12
	AL2	writes + instead of $\cdot$		12
	PL1	writes $\cdot$ instead of +		19
	PL2	writes + instead of $\cdot$		19
Numerical Bases	AB1	$a^n \cdot a^m = (a \cdot a)^{n+m}$	$8^2 \cdot 8^4 = 64^6$	6
Exponent Errors	AEM1	$a \cdot a = a$	$231 \cdot 231 = 231$	3
	AEM2	$a \cdot a^n = a^n$	$107 \cdot 107^3 = 107^3$	1
	AEM3	$a^n \cdot a^m = a^{n \cdot m}$	$8^4 \cdot 8^7 = 8^{28}$	2
	PEM1	$ax \cdot bx - abx$	$8a \cdot 13a = 104a$	2
	PEM2	$ax \cdot bx^n = abx^n$	$4w \cdot 9w^5 = 36w^5$	4
	PEM3°	$ax^n \cdot bx^m = abx^{n \cdot m}$	$11n^3 \cdot 2n^5 = 22n^{15}$	4
	PEM4	${}^{-}ax\ (bx \pm c) = abx \pm acx$	${}^{-}5p\ (2p - 7) = {}^{-}10p + 35p$	4
	PEM5	$ax \cdot bx^n = abx$	$4w \cdot 9w^5 = 36w$	4
	PEM6°	$ax^n \cdot bx^m = abx$	$11x^3 \cdot 2x^5 = 22x$	4
	AEA1+	$ab + cb = (a + c)\ b^2$	$19 \cdot 107 + {}^{-}42 \cdot 107 = (19 + {}^{-}42)\ 107^2$	6
	AEA2	${}^{-}ac^n + bc^n = ({}^{-}a + b)\ c^{2n}$	${}^{-}9 \cdot 18^2 + 17 \cdot 18^2 = ({}^{-}9 + 17)\ 18^4$	2

Table 18 (cont'd.)

Error Category	Code	Description	Example	Max.*
	PEA1°	$ax \pm bx = (a \pm b)\ x^2$	$16d + 31d = 47d^2$	5
	PEA2§	$ax^n \pm bx^n = (a \pm b)\ x^{2n}$	$-4x^2 + 5x^2 = x^4$	4
	PEA3**	$ax + bx = (a + b)\ x^2$ (indirect)		4
	PEA4**	$ax^n \pm bx^n = (a \pm b)\ x^{2n}$ (indirect)		2
	PES1	$ax - bx = (a - b)\ x^2$	$2x - 3x = {}^-x^2$	3
	PES2	$ax^n - bx^n = (a - b)\ x^{2n}$	$4p^2 - 6p^2 = {}^-2p^4$	2
Like Term Errors	PT1°	$ax \pm b = (a \pm b)\ x$	$27d - 10 = 17d$	14
	PT2°	$ax^n \pm b = (a \pm b)\ x^n$	$15x^2 + 3 = 18x^2$	4
	PT3°	$ax \pm b = (a \pm b)\ x$ (indirect)		10
	PT4°	$ax^n \pm b = (a \pm b)\ x^n$ (indirect)		4
	PT5°	$ax^n + bx = (a + b)\ x^{n+1}$	$15x^2 + 3x = 18x^3$	2
	PT6**	$ax^n + bx = (a + b)\ x^{n+1}$ (indirect)		6
	PT7°	$ax^n + bx = (a + b)\ x^n$	$15x^2 + 3x = 18x^2$	2
	PT8°	$ax^n + bx = (a + b)\ x^n$ (indirect)		6
	PT9	$ax + {}^-b\ (cx \pm d) = (ax + cx) + ({}^-b \pm d)$	$5r + {}^-3\ (7r + 2) = 12r + {}^-1$	2
Miscellaneous	PM1	omits variable	$2x + 3x = 5$	32
	PM2	$x^m \cdot x^n = (m + n)\ x$	$z^4 \cdot z^9 = 13z$	2
	AM3	$a - b = b - a$	$169 - (189 + 156) = (189 + 156) - 169$	4
	PM4	$x^n \cdot x^m = 2x^{n+m}$	$z^4 \cdot z^9 = 2z^{13}$	2
	PM5	$(ax \pm b) - (cx \pm d) = (a - c)\ x \pm (b + d)$	$(17x + 2) - (12x + 9) = 5x + 11$	4
	B	blanks		
	I	incomplete solution		
	D	correct, but different		
Others		$ab + b = ab^2$	$13 \cdot 789 + 789 = 13 \cdot 789^2$	2

*Max. means the maximum number of items in which the error could arise.

**a, b ξ I

+a, b ξ N, c ξ I

°a ξ N, b ξ I

§a ξ I, b ξ N

APPENDIX D

INDIVIDUAL CODING SHEET

Each individual's errors were coded on a separate coding sheet and records of all the errors were kept. Since the test and item number were indicated, as was the general error categories, numbers were used to indicate the specific error type the child made on a particular item. For example, if a student made the second sign error in subtraction on item 4 on the computation test, the number 2 was placed in the appropriate block. This example is illustrated in Figure 32.

	Computation										Arithmetic		
Error Type	1	2	3	4	5	6	7	8	9	10 . . .	1	2	3 . . .
Sign Errors													
multiplication													
addition													
subtraction				2									
Basic Fact Error													
.						.				.			
.						.				.			
.						.				.			

FIGURE 32. Coding sheets used for individual students.

APPENDIX E

SUMMARY SHEET USED
FOR EACH GROUP

Using the coding sheets available for each of the 200 students, a summary of the number of errors made in each error category was transferred to the appropriate summary sheet. A separate summary sheet was available for each grade, sex, and program combination. The abbreviations used for coding each error type correspond to those used in Appendix C.

For example, PSM2 was the code used to name the algebraic (P), sign (S) error in multiplication (M), number 2. If student 3 in a particular group made error PSM2, three times, a three was recorded in the appropriate box. The diagram in Figure 33 illustrates this.

MASTER SUMMARY

GRADE: ______________ SEX: ______________ ALGEBRA TEST

			STUDENT NUMBER									
			1	2	3	4	5	6	7	8	9	10 . . .
ERRORS		MAX*	NUMBER OF TIMES ERROR MADE									
Sign:	PSM1	2										.
	PSM2	4			3							.
	PSM3	4										.
	.	.			.							
	.	.			.							
	.	.			.							

*MAX indicates maximum number of items in which the error could occur.

FIGURE 33. Summary sheet used for each group.

APPENDIX F

SUMMARY SHEET OF NUMBER OF STUDENTS
AND FREQUENCY OF ERRORS

On this summary sheet, the number of students who made an error at a particular frequency was recorded for each group. The frequencies were divided into three groups: those less than 50% (<50%), those 50% exactly (=50%), and those greater than 50% (>50%). For example, if two males in grade 9 matriculation made error CSM2 in 50% of the possible items, then the number 2 was recorded in the second column, as shown in Figure 34.

Due to the breadth of the study and its exploratory nature, the "50% exactly" category was adopted because some skills were tested in only two items.

GRADE, PROGRAM, SEX	9M-M			9M-F			. . .
FREQUENCY	<50	=50	>50	<50	=50	>50	
ERROR TYPE							
CSM1							
CSM2		2					
.				.			.
.				.			.
.				.			.

FIGURE 34. Summary sheet of number of students and frequency of errors.

GRADE AND SEX:		9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10H-F			TOTAL		
FREQUENCY		< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50
ERRORS																												
SIGN:	CSM1		2	1		1	1					3			1											0	7	2
	CSM2		2			1						1									2			1		0	7	0
	CSA1		2			4						1														0	7	0
	CSA2		4			2			1	1		2			2						1			2		0	14	1
	CSA3		3	2		4			1	1		2			1						1					0	12	3
	CSS1		2																							0	2	0
	CSS2		5			5	2		1			1									1					0	13	2
	CSS3		2	2		3			2			2			1	1					2			1		0	13	3
	CSS4		9			3	1		3			3			1						1			1		0	21	1
	CSS5		4			5	1		2			2			1						3			1		0	18	1
WR.OP.	CW1				3						1						1									5	0	0
	CW2	4			2			2						3			3			1			3			18	0	0
	CW3	7		2	4	3	1	3			4	1	2	1					1	1						20	4	6
	CW4		5	4		3	3		1	3		3	6		5			3						2		0	22	16
	CW5		5	5		5	5			1		2	2								1			1		0	14	13
	CW6	2		1		1		1				1														3	2	1
BS.FT.	CF1	5	1		2				1		6						3			1			3			20	2	0
	CF2*	9			8			2			6			2			4			4			1			36	0	0
SIGN:	ASM1*																									0	0	0
	ASM2	5	3		5	4	1	9		1	2	3	2	3		1	1		1	4	1	1	1	1	1	30	12	8
	ASM3*	1																								1	0	0
DIST.	AD1		4	3			2		1	2		1	1					1				3			4	0	7	15
	AD3	1		1	3	1			1	1																4	2	2
	AD4		4	2		2	2		1	1		1	1		1	9		2	6		3	1		1	3	0	15	25
	AD5		1	2		2	2		2	1		2	2			1		5				1			2	0	12	11
GROUP.	AG1	3		1	2	1					1			1			1						1			9	1	1
	AG2						3			1		1	3			2			1			1		1		0	2	11
	AG3		2	2			1					2	1			1						2			1	0	4	8
	AG4		5	4		2	2		3	4		5	1			3		4	2			3		2	2	0	21	21
	AG5			2																						0	0	2
	AG6		1																							0	1	0
INCOP.	AL1*	2			2			1			4						1			1			1			12	0	0
	AL2*	1																								1	0	0
NO.BS.	AB1	5			1	2		2			2			3			2									15	2	0
EXP.	AEM1						2	1			1															2	0	2
	AEM2						1																			0	0	1
	AEM3		1										1								1			1		0	3	1
	AEA1	2	1		1	1					1			1	1					1			1			7	3	0
	AEA2		4	1		3	2											1			1			1		0	10	3
MISC.	AM3		2	4		2		2	2			3		2			1			1	2	1				6	11	5
$ab + b = ab^2$			1	2		4	3		3			3	2		1	2		3	2					1		0	16	11

(cont'd.)

TABLE 19. Summary of the frequency of errors for different groups

Table 19 (cont'd.)

GRADE AND SEX:		9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10-H-F			TOTAL		
FREQUENCY		< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50
ERRORS																												
SIGN:	PSM1					2									1											0	3	0
	PSM2	1			1	1		1			2			1			1									7	1	0
	PSM3	1			1	1		2			1												1			6	1	0
	PSM4	7		2	5	2	2	1		1	4	2		4		1	1			1						23	4	6
	PSM5	2						1																		3	0	0
	PSM6				2		1	1			2															5	0	1
	PSM7			1			1			2			3									2				0	0	9
	PSM8			3			4			1			2			1			1						1	0	0	13
	PSA1						1		1				1								1			1		0	3	2
	PSA2		11	1		7	1		4	3		4	1		1	1		2				1		3		0	32	7
	PSA3					2																				0	2	0
	PSA4																									0	0	0
	PSA5	1			1					1															1	2	0	2
	PSA6*																									0	0	0
	PSS1		8			2			2			2									1					0	15	0
	PSS2*										2											1	1			3	0	0
	PSS3*																									0	0	0
BS.FT.	PF1*	9			7			8			5			4			2			5			4			44	0	0
	PF2*	8			4			4			3			3			1			4			3			30	0	0
WROP.	PW1*	1									1															2	0	0
	PW2*	2																								2	0	0
	PW3								1																	0	1	0
	PW4	1	1					1	1																	2	2	0
	PW5*																									0	0	0
	PW6																	1								0	1	0
	PW7		1									1														0	2	0
	PW8*							1																		1	0	0
	PW9*	1																								1	0	0
	PW10*	1						1			2						1									5	0	0
	PW11*																									0	0	0
	PW12*																									0	0	0
	PW13*		3	1		2	1		1				1		1	1										0	7	4
	PW14		1	1		5				1					1		1	1								0	8	2
	PW15*	2			3			1			1															7	0	0
	PW16		10			8	1		2						5			5			7			3		0	40	1
	PW17		2	9		2	7			2		1	2		3	2		4	2		1	1			1	0	13	26
	PW18	1		3	2	2	3	1		1			3					1	1			1				4	3	12
DIST.	PD1	3	2	2	2				1		2			2												9	3	2
	PD2	2	1	4	2													1								4	2	4
	PD3		1	1		3	1		1			1	3		3	2		1	1							0	10	8
	PD4		1	1		1	4		1	2		2	1			1		1	1		1					0	7	10
	PD5*	3																								3	0	0
	PD6		2	1																						0	2	1

Table 19 (cont'd.)

GRADE AND SEX:		9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10H-F			TOTAL		
FREQUENCY		< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50	< 50	= 50	> 50
ERRORS																												
GROUP.	PG1			1		1	2																			0	1	3
INCOP.	PL1*	1																								1	0	0
	PL2*																									0	0	0
EXP.	PEM1			3					1						1	1					1	2		1	1	0	4	7
	PEM2	2		2	1	2		1		1	1			1												6	2	3
	PEM3	1	1								2			1				1		1	1					5	3	0
	PEM4	8	1	3	2	3	1	5		1	6		1	2		2	4		1	2			1			30	4	9
	PEM5*										1															0	0	0
	PEM6*																									0	0	0
	PEA1	4		2	2		2				2						1									9	0	4
	PEA2	4	3	1	5	2		1	1		3	1		3	1					1			1			18	8	1
	PEA3	3			1						2		1													6	0	1
	PEA4*																									0	0	0
	PES1*																1									1	0	0
	PES2					1	1		1																	0	2	1
LK.T.	PT1*	3			2			2																		7	0	0
	PT2	1	1		1																					2	1	0
	PT3	2			2						2		1													6	0	1
	PT4	1					1				2															3	0	1
	PT5		1									1														0	2	0
	PT6*	1									1															2	0	0
	PT7		1																							0	1	0
	PT8*	1																								1	0	0
	PT9		2	2			1																			0	2	3
MISC.	PM1*				10			6			13			5			3			4			4			45	0	0
	PM2*																									0	0	0
	PM3*																									0	0	0
	PM4					2	1					1														0	3	1
	PM5*																									0	0	0

*These errors were not made systematically by either student.

TABLE 2

Sign Errors

Grade, Program, Sex	*9M-M			9M-F			10M-M			10M-F			9H-M			9H-F			10H-M			10H-F			TOTAL		
	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>	<	=	>
**Frequency	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
***Error Type	Number of Students																										
Sign																											
CSM1		2	1		1	1					3			1											0	7	2
CSM2		2			1						1									2			1		0	7	0
CSA1		2			4						1														0	7	0
CSA2		4			2			1	1		2			2						1			2		0	14	1
CSA3		3	2		4			1	1		2			1						1					0	12	3
CSS1		2																							0	2	0
CSS2		5			5	2		1			1									1					0	13	2
CSS3		2	2		3			2			2			1	1					2			1		0	13	3
CSS4		9			3	1		3			3			1						1			1		0	21	1
CSS5		4			5	1		2			2			1						3			1		0	18	1
ASM1																									0	0	0
ASM2	5	3		5	4	1	9		1	2	3	2	3		1	1		1	4	1	1	1	1	1	30	12	8
ASM3	1																								1	0	0
PSM1					2									1											0	3	0
PSM2	1			1	1		1			2			1			1									7	1	0
PSM3	1			1	1		2			1												1			6	1	0
+PSM4	7		2	5	2	2	1		1	4	2		4		1	1									23	4	6
PSM5	2						1																		3	0	0
PSM6				2		1	1			2															5	0	1
PSM7			1			1			2			3									2				0	0	9
+PSM8			3			4			1			2				1		1						1	0	0	13
PSA1						1		1				1								1			1		0	3	2
+PSA2		11	3		7	1		4	3		4	1			1		2				1		3		0	32	7
PSA3					2																				0	2	0
PSA4																									0	0	0
PSA5	1			1					1															1	2	0	2
PSA6																									0	0	0
+PSS1		8			2			2			2									1					0	15	0
PSS2										2												1			3	0	0
PSS3																									0	0	0

*9M-M, for example, denotes grade nine matriculation males.

**"<50, =50, >50" represent the frequency at which an error occurred. "<50" denotes that the error occurred in less than 50% of the possible occasions, "=50"denotes that it occurred in exactly 50% of the possible occasions, and ">50" denotes that it occurred in more than 50% of the possible occasions.

***A description and example of each error type is present in Appendix F.

+A common algebraic error.