EQUILIBRIUM TIDAL RESPONSE OF A NON-GLOBAL SELF-GRAVITATING OCEAN ON A YIELDING EARTH

CENTRE FOR NEWFOUNDLAND STUDIES

TOTAL OF 10 PAGES ONLY MAY BE XEROXED

(Without Author's Permission)

JAMES B. MERRIAM
EQUILIBRIUM TIDAL RESPONSE OF A NON-GLOBAL SELF-GRAVITATING OCEAN ON A YIELDING EARTH

by

JAMES B. MERRIAM, B.Sc.

"Submitted in partial fulfilment of the requirements for the degree of Master of Science, Memorial University of Newfoundland
August, 1973"
### CONTENTS

<table>
<thead>
<tr>
<th>ABSTRACT</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHAPTER 1. EQUILIBRIUM TIDES; FIRST ORDER THEORY</td>
<td>1</td>
</tr>
<tr>
<td>1.i. Earth Deformation, Love Numbers</td>
<td>1</td>
</tr>
<tr>
<td>1.ii. The Tide Reducing Factor</td>
<td>3</td>
</tr>
<tr>
<td>1.iii. The Gravimetric Factor</td>
<td>6</td>
</tr>
<tr>
<td>1.iv. The Indirect Effect</td>
<td>9</td>
</tr>
<tr>
<td>CHAPTER 2. EQUILIBRIUM TIDES; SECOND ORDER THEORY</td>
<td>13</td>
</tr>
<tr>
<td>2.i. Introduction</td>
<td>13</td>
</tr>
<tr>
<td>2.ii. The Effect of Ocean Self-Attraction</td>
<td>13</td>
</tr>
<tr>
<td>2.iii. The Effect of Load Yielding</td>
<td>17</td>
</tr>
<tr>
<td>2.iv. The Effect of the Non-Globality of the Oceans</td>
<td>23</td>
</tr>
<tr>
<td>2.v. Equilibrium Tides on a Real Earth</td>
<td>25</td>
</tr>
<tr>
<td>2.vi. The Solution of the Equations</td>
<td>26</td>
</tr>
<tr>
<td>CHAPTER 3. THE PERTURBED POTENTIAL OF AN EARTH WITH NON-GLOBAL OCEANS</td>
<td>56</td>
</tr>
<tr>
<td>3.i. Introduction</td>
<td>56</td>
</tr>
<tr>
<td>3.ii. The Perturbed Potential in the Matrix Method</td>
<td>56</td>
</tr>
<tr>
<td>CHAPTER 4. THE INFLUENCE OF THE OCEANS ON THE CHANDLER WOBBLE</td>
<td>65</td>
</tr>
<tr>
<td>4.i. Introduction</td>
<td>65</td>
</tr>
<tr>
<td>4.ii. The Liouville Equations</td>
<td>67</td>
</tr>
</tbody>
</table>
4.iii. The Pole Tide in Global Oceans 64
4.iv. The Pole Tide in Non-Global Oceans 72
4.v. The Complete solution of the Pole Tide 75
4.vi. The Ellipticity of the Pole Path 79

CHAPTER 5
THE GRAVIMETRIC FACTOR 81
5.i. Introduction 81
5.ii. The Gravimetric Factor for a Load Potential 84
5.iii. Gravity Tides on an Earth with Non-Global Oceans 84

CHAPTER 6
SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH 87
6.i. Summary 87
6.ii. Suggestions for future Research 89

APPENDIX A. Computing the Tide Height Matrices 95
APPENDIX B. The Ocean Function 111
APPENDIX C. Surface Harmonic Synthesis 117
APPENDIX D. The Derivation of the Equation for the Potential of a Surface Layer 120
APPENDIX E. The Liouville Equations 122
APPENDIX F. The Pole Tide Potential 125
ABSTRACT

It is commonly stated that the equilibrium ocean tide, measured with respect to the solid surface of the Earth, has amplitude

\[ \xi_2 = (1 + k_2 - h_2) \frac{U_2}{g} \]

where \( U_2 \) is the tide potential of degree-2, \( g \) is the acceleration of gravity at the surface and \( k_2, h_2 \), the second degree Love numbers, are scalar quantities which characterize the elastic response of the solid Earth. In fact this equation applies strictly to the "tides" in the equipotential surfaces of oceanless, spherically symmetric planets. It is found that the mass of the oceans is sufficient to disturb the potential field to which they are responding thus creating a feedback mechanism which increases the predicted tide height by about 63%. The loading and subsequent deformation of the solid Earth by the ocean tide creates another feedback loop which decreases the tide by about 40%. The combined effect of loading and self-attraction is a 23% increase in the predicted tide height. In addition the irregular distribution of the oceans introduces complications in the response of the oceans to a prescribed potential and a single scalar equation no longer suffices. A set of matrices is developed to replace the scalar Love parameters and these are used to construct maps of several tides which are thought to be equilibrium. Matrices which give the perturbed potential and the gravity tide on a nonspherically symmetric Earth are also given, and the correct expression for
the load gravimetric factor, about which there has been some confusion, is derived. Previous estimates of the influence of equilibrium ocean tides on certain aspects of the rotation of the Earth are re-worked using the results of the matrix method.
CHAPTER 1

EQUILIBRIUM TIDES; FIRST ORDER THEORY

i) Earth Deformation, Love Numbers

Love (1909) introduced a method of characterizing the response of an elastic, spherically symmetric, gravitating sphere, initially hydrostatically stressed, to a prescribed body force potential. The fundamental assumptions in the theory are that the density and elastic parameters vary only with geocentric radius and that all displacements be small. Only in such a case can the deformation of the body have the same spherical harmonic form as the prescribed potential.

As given by Love the two parameters in the theory, called Love numbers, may be defined by the following two equations. If $U_n$ is the $n$th degree solid harmonic in the expansion of the prescribed body force potential $U$ the radial deflection of the surface of the body is

$$\xi_n = h_n U_n / g \quad (1.1)$$

where $g$ is the acceleration of gravity at the surface and $\xi_n$ is of course a surface harmonic of degree $n$. This defines the Love number $h_n$. As a result of the deformation the potential of the body at the surface will be changed by

$$k_n U_n \quad (1.2)$$

which defines the Love number $k_n$. Since all displacements must be small
the Love number method is a first order theory and in common with all
other first order theories obeys the principle of linear superposition.
This is simply because any deformation which satisfies the criterion of
being small does not alter the distribution of density and elastic
parameters sufficiently to yield a model with appreciably different Love
numbers. Love numbers of degree zero and 1 do not exist because the
associated body forces leave the Earth undeformed (a body force derived
from a degree-zero potential vanishes identically, that derived from a
degree-1 potential simply accelerates every point in the Earth by the
same amount).

Most of the easily observable periodic Earth deformations
are of degree-2 in a solid harmonic expansion and consequently prior to
about 1960 most of the interest was in evaluating \( h_2 \) and \( k_2 \) and Love
numbers of other degree were seldom mentioned in the literature. Since
1962, however, Love numbers up to degree 25 have been available from
calculations made on Earth models (Longman, 1963 a, b, 1966; Kaula,
1963; Takeuchi et al, 1962) and the problem has since been to use these
numbers to investigate more fully the phenomena which cause the deforma-
tions or to deduce how the Earth departs from the basic assumption, that
of spherical symmetry.

The response of the Earth to prescribed loads, such as
snow, may also be described in a Love number format. The defining equations
for load Love numbers are similar to those for non-load numbers. Denoting
load Love numbers here and elsewhere with a superscript prime, the radial
deflection of the solid surface in response to a load potential \( U \) is

\[
\xi_n = h_n U_n'/g
\]

(1.3)
and the change in potential at the surface is

\[ k_n' U_n' \]  \hspace{1cm} (1.4)

In direct analogy with isostasy, where yielding reduces the free air anomalies, the potential of a load is reduced by yielding. The load Love numbers are, therefore, negative (Table 1.1). Load Love numbers of degree 0 and 1 are possible: a degree-zero load potential corresponds to adding mass at the Earth's surface, and a degree-1 load is possible because a shift in surface mass distribution does not imply a shift in the centre of mass of the whole system but only of the centre of mass of the solid Earth.

ii) The Tide Reducing Factor

The tide-raising potential of the Moon, for example, is

\[ U = \sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \frac{GM}{r} \left( \frac{r}{R} \right)^n P_n(\cos \alpha) \]  \hspace{1cm} (1.5)

(Platzman 1972) where

\[ G = \text{constant of gravitation} \]
\[ M = \text{mass of the Moon} \]
\[ R = \text{radius to the Moon} \]
\[ r = \text{radius of the observation point} \]
\[ \alpha = \text{angle subtended at the centre of the Earth by radii to the observation point and the Moon} \]
\[ P_n(\cos \alpha) = \text{Legendre polynomial of degree } n \]
In response to this potential, the equipotential surfaces of a rigid Earth would exhibit a 'tide' of height

$$\sum \frac{U_n}{g}$$

(1.6)

Since the ocean will attempt to adjust itself so that its surface is an equipotential this is normally taken to be the height of the ocean tide that would be observed on the surface of a rigid Earth. This is true only to the extent that the dynamics of the ocean may be neglected.

There are in addition some other complications which will be discussed in 1.iv. If the solid part of the body is deformable the potential at the surface will be

$$\sum \frac{(1 + k_n)}{n} U_n$$

(1.7)

instead of simply

$$\sum U_n$$

(1.8)

and this is the potential to which the oceans will be responding. The equipotential surfaces of the solid body will accordingly be deflected radially by

$$\sum \frac{(1 + k_n)}{n} U_n/g$$

(1.9)

with respect to the Earth's centre. Neglecting other influences this is also the height above the centre of the Earth to which the ocean surface will be raised. Tide gauges are, however, fixed to the solid Earth surface and since this has also been deflected it must be accounted for.
The solid Earth surface will be raised by

\[ \Sigma h_n \mathcal{U}_n \frac{\gamma}{g} \tag{1.10} \]

and subtracting this from 1.9 gives the tide height as measured by a tide gauge, namely

\[ \Sigma (1 + \kappa_n - h_n) \mathcal{U}_n \frac{\gamma}{g} \tag{1.11} \]

The factor

\[ \gamma_n = 1 + \kappa_n - h_n \tag{1.12} \]

is called the tide reducing factor and it is the ratio of the tide height on an Earth with Love numbers \( h_n, \kappa_n \) to the tide height on a rigid Earth. (Mel'chior, 1966). Due to the distance of the Sun and Moon from the Earth they raise primarily degree-2 tides. Although degree-3 tides have been observed they are much smaller than the tides of degree-2 (Mel'chior and Venedikov, 1968).

In tidal measurements in lakes it is really the slope of the equipotential surfaces which is measured (Moulton, 1919; Proudman, 1925) and this is determined by the slope of the local vertical which is described by the same factor. The deflection of the local vertical is given by

\[ \frac{1}{g} \sum \left(1 - \kappa_n - h_n\right) \frac{\mathcal{U}_n}{\alpha \theta} \tag{1.13} \]
where \( a \) is the radius of the Earth, and \( \phi \) is an angle measured along the surface of the Earth in any given direction.

Darwin (1883) and Schweydar (1908) were the first to measure the tide reducing factor. Both used the monthly and fortnightly tides which they assumed to be equilibrium tides (Proudman 1960) subsequently found this to be an unjustifiable assumption) and obtained the following results for \( \alpha \) :

<table>
<thead>
<tr>
<th></th>
<th>Fortnightly</th>
<th>Monthly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Darwin</td>
<td>0.675 ± 0.084</td>
<td>0.680 ± 0.387</td>
</tr>
<tr>
<td>Schweydar</td>
<td>0.625 ± 0.043</td>
<td>0.605 ± 0.012</td>
</tr>
</tbody>
</table>

Table 1.1 The tide reducing factors \( \alpha \) (Darwin, Schweydar).

The value computed from Earth models is 0.690 using Longman's (1963) results. This small discrepancy is surprising in view of the number of errors and omissions in 1.11. The errors are discussed more fully in 1.iv.

iii) The Gravimetric Factor

The second factor which has been extensively measured is the gravimetric factor conventionally donated by \( \delta_n \). When the Earth experiences the tidal potential of the Moon, for example, there is a periodic fluctuation in the acceleration of gravity observed on the surface due to

1) The direct tidal acceleration by the Moon which would
be observed on a rigid Earth. If \( U_n \) is the \( n \)th degree component of the tide raising potential (equation 1.5) the direct contribution to the acceleration of gravity with \( g \) measured positively upwards is

\[
\Delta g = \frac{2}{2\pi} U_n
\]

\[
= \frac{n}{a} U_n
\]  

(1.14)

2) The gravitational acceleration of the deformed part of the Earth. The deformation potential of the Earth has the value

\[
k_n U_n(a)
\]

(1.15)

at the surface and falls off as

\[
k_n U_n\left(\frac{a}{r}\right)^{n+1}
\]

(1.16)

so the acceleration of gravity due to the deformation is

\[
\Delta g = \frac{2}{2\pi} \left( k_n U_n\left(\frac{a}{r}\right)^{n+1}\right)_{r=a}
\]

\[
= -k_n U_n \frac{(n+1)}{a}
\]  

(1.17)

3) The variation of \( g \) produced by moving the gravimeter in the undeformed gravity field of the Earth, \( \nabla v \). If \( \xi \) is the height the surface is raised this contribution to the gravity tide is

\[
\Delta g = \frac{2}{2\pi} \left( \nabla v \right) \xi n
\]

\[
= \frac{2}{\pi} g - \xi n
\]

\[
= \frac{2}{\pi} k_n U_n
\]  

(1.18)
<table>
<thead>
<tr>
<th>DEGREE</th>
<th>NON-LOAD LOVE NO.</th>
<th>LOVE NO.</th>
<th>TIDAL HEIGHT FACTOR</th>
<th>GRAVIMETRIC FACTOR</th>
<th>LOVE NO.</th>
<th>TIDAL HEIGHT FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>H</td>
<td>K</td>
<td>1 + K - H</td>
<td>1 + 2H / N - (N+1)K / N</td>
<td>LOAD H</td>
<td>LOAD K</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.000</td>
<td>0.0</td>
<td>0.0</td>
<td>1.134</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.0</td>
<td>1.000</td>
<td>1.000</td>
<td>-0.134</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.612</td>
<td>0.302</td>
<td>0.690</td>
<td>1.159</td>
<td>-0.290</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.290</td>
<td>0.093</td>
<td>0.803</td>
<td>1.069</td>
<td>-1.059</td>
<td>-0.310</td>
</tr>
<tr>
<td>4</td>
<td>0.175</td>
<td>0.842</td>
<td>0.867</td>
<td>1.035</td>
<td>-1.059</td>
<td>-0.197</td>
</tr>
<tr>
<td>5</td>
<td>0.129</td>
<td>0.025</td>
<td>0.896</td>
<td>1.022</td>
<td>-1.059</td>
<td>-0.133</td>
</tr>
<tr>
<td>6</td>
<td>0.107</td>
<td>0.017</td>
<td>0.910</td>
<td>1.016</td>
<td>-0.893</td>
<td>-0.090</td>
</tr>
<tr>
<td>7</td>
<td>0.095</td>
<td>0.013</td>
<td>0.918</td>
<td>1.012</td>
<td>-0.152</td>
<td>-0.082</td>
</tr>
<tr>
<td>8</td>
<td>0.087</td>
<td>0.010</td>
<td>0.923</td>
<td>1.011</td>
<td>-0.123</td>
<td>-0.076</td>
</tr>
<tr>
<td>9</td>
<td>0.081</td>
<td>0.008</td>
<td>0.927</td>
<td>1.009</td>
<td>-0.123</td>
<td>-0.069</td>
</tr>
<tr>
<td>10</td>
<td>0.076</td>
<td>0.007</td>
<td>0.931</td>
<td>1.008</td>
<td>-0.123</td>
<td>-0.066</td>
</tr>
<tr>
<td>11</td>
<td>0.072</td>
<td>0.006</td>
<td>0.934</td>
<td>1.007</td>
<td>-0.123</td>
<td>-0.066</td>
</tr>
<tr>
<td>12</td>
<td>0.069</td>
<td>0.005</td>
<td>0.936</td>
<td>1.006</td>
<td>-1.152</td>
<td>-0.064</td>
</tr>
<tr>
<td>13</td>
<td>0.066</td>
<td>0.005</td>
<td>0.939</td>
<td>1.005</td>
<td>-0.118</td>
<td>-0.062</td>
</tr>
<tr>
<td>14</td>
<td>0.064</td>
<td>0.004</td>
<td>0.940</td>
<td>1.005</td>
<td>-0.118</td>
<td>-0.060</td>
</tr>
<tr>
<td>15</td>
<td>0.062</td>
<td>0.004</td>
<td>0.942</td>
<td>1.004</td>
<td>-0.118</td>
<td>-0.058</td>
</tr>
<tr>
<td>16</td>
<td>0.060</td>
<td>0.003</td>
<td>0.943</td>
<td>1.004</td>
<td>-0.118</td>
<td>-0.056</td>
</tr>
<tr>
<td>17</td>
<td>0.058</td>
<td>0.003</td>
<td>0.945</td>
<td>1.004</td>
<td>-0.118</td>
<td>-0.055</td>
</tr>
<tr>
<td>18</td>
<td>0.056</td>
<td>0.003</td>
<td>0.947</td>
<td>1.003</td>
<td>-0.118</td>
<td>-0.054</td>
</tr>
<tr>
<td>19</td>
<td>0.055</td>
<td>0.003</td>
<td>0.948</td>
<td>1.003</td>
<td>-0.118</td>
<td>-0.052</td>
</tr>
<tr>
<td>20</td>
<td>0.053</td>
<td>0.002</td>
<td>0.949</td>
<td>1.003</td>
<td>-0.118</td>
<td>-0.051</td>
</tr>
<tr>
<td>21</td>
<td>0.052</td>
<td>0.002</td>
<td>0.950</td>
<td>1.003</td>
<td>-0.118</td>
<td>-0.050</td>
</tr>
<tr>
<td>22</td>
<td>0.051</td>
<td>0.002</td>
<td>0.951</td>
<td>1.003</td>
<td>-0.118</td>
<td>-0.049</td>
</tr>
<tr>
<td>23</td>
<td>0.050</td>
<td>0.002</td>
<td>0.952</td>
<td>1.002</td>
<td>-0.118</td>
<td>-0.048</td>
</tr>
<tr>
<td>24</td>
<td>0.048</td>
<td>0.002</td>
<td>0.954</td>
<td>1.002</td>
<td>-0.118</td>
<td>-0.047</td>
</tr>
<tr>
<td>25</td>
<td>0.047</td>
<td>0.002</td>
<td>0.955</td>
<td>1.002</td>
<td>-0.118</td>
<td>-0.046</td>
</tr>
</tbody>
</table>

**TABLE 1.1**  
LOAD AND NON-LOAD LOVE NUMBERS TO DEGREE N=25 AND SOME LOVE NUMBER PARAMETERS.  
By summing together the three contributions the total gravity tide is

$$\nabla \Delta g = (n + 2 h_n - (n+1) k_n) \frac{U}{a}$$

(1.19)

$$= \left(1 + 2 \frac{h_n}{n} - (n+1) \frac{k_n}{n}\right) n \frac{U}{a}$$

The factor

$$\delta_n = \left(1 + 2 \frac{h_n}{n} - (n+1) \frac{k_n}{n}\right)$$

(1.20)

is called the gravimetric factor and it is the ratio of the gravity tide on an Earth with Love numbers $h_n, k_n$ to the gravity tide on a rigid Earth.

In much the same way a gravimetric factor for load potentials is derived in chapter 5.

iv) The Indirect Effect

When measurements of the tide reducing factor were first made it was found that there was a wide scatter in the numerical values obtained. Values of $\delta_2$ ranged from 1.056 to 1.285 and values of $\gamma_2$ from 0.330 to 1.129 (Melchior, 1966). Hecker (1907) recognized the source of the poor results and attributed it to the indirect effect of the ocean. Equation 1.11 applies strictly to the 'tide' in the equipotential surfaces of spherically symmetric oceanless planets. The indirect effect has the following influence on the tide height:

1) The ocean itself having appreciable mass alters the potential field to which it is responding.

2) The loading of the crust by the ocean tide creates an
additional potential which tends to reduce the feedback from (1).

The simple explanation for (1) and (2) is that when a tide is produced by the prescribed potential the gravitational force of the water mass attracts more water to it and the result is a tide of increased amplitude. When the ocean bottom deforms under the load of the tide the gravitational pull of the tide is reduced in the same way that isostatic yielding reduces free air gravity anomalies.

Because the Earth is loaded by non-global oceans it does not satisfy the Love number requirement of spherical symmetry. In fact, due to the large amplitude of the ocean tide, the ocean cannot even be treated as a perturbation on an otherwise spherically symmetric Earth. The ocean must be accommodated separately. The non-globality of the oceans means that when forced by a degree-2 potential the ocean may respond with tide coefficients of any degree and order depending on the exact shape of the ocean. Any physical tide will have at least the modulation of the ocean function (Chapter 2) and as a result the coefficients of the tide will not reduce to zero with increasing degree any faster than the coefficients in the ocean function (Table B.1). To adequately describe a tide therefore it should be expanded to degree and order 8, which is the extent to which the ocean function coefficients are available (Munk and MacDonald, 1960; Balmino et al, 1973).

Kelvin (1863) first pointed out the necessity of allowing for load yielding and self-attraction. Street (1925) introduced a correction for loading and Rosenhead (1929) solved the problem of self-attraction. Proudman (1925) suggested measuring the tide in lakes which were large
enough to exhibit a tide but small enough so that the tide height would be small and hence loading and self-attraction could be neglected. For the same reasons the tide in buried pipes filled with water was measured (Michelson, 1914; Michelson and Gale, 1919). Although these methods avoided the error of using non-equilibrium tides (care being taken to assure that the resonance periods of the water bodies were far removed from the diurnal period) and eliminated loading and self-attraction problems, complete allowance for the indirect effect was still not achieved. This was because, although unknown and unsuspected by these authors, tides in the open ocean can affect the deflection of the vertical and especially the gravity tide at mid-continent stations (Pertsev, 1966). This effect of the distant oceans on the deflection of the vertical is impossible to correct for because without a knowledge of the tide in the open sea the correction is impossible to estimate. As yet the tide in the global ocean at the important diurnal and semi-diurnal frequencies is imperfectly known although Farrell (1972) has appreciably reduced the gravity tide residual at mid-continent stations (Kuo et al, 1970) by using the co-tidal chart of Hendershott (1972). Although Pertsev handles the problem correctly there still exists some confusion in the west about how the problem should be handled in a Love number format particularly as regards the load gravimetric factor (Farrell, 1970, p 25; Slichter, 1972, p 316). The correct expression is derived in 5.11.

Rosenhead (1929) attempted to apply a correction to the reducing factor which would allow for ocean self-attraction. Although substantially correct his method somehow escaped further mention in the literature.
A more systematic approach was needed to account for the indirect effect in the tide height problem. Munk and MacDonald (1960) outlined a method of predicting tide heights allowing for feedback and continentality. Their result although substantially correct was derived in an unnecessarily circuitous manner which tended to obscure the physics involved, and as originally given contains two typographical errors. Their equation 5.12.8 should read

\[ \xi = G \left[ \xi \left( J'_r U_r + J'_r U'_r + 3 q \rho_o J'_r \xi_r \right) + \text{const.} \right] \]  

(1.21)

where \( \xi \) is the tide height, \( J_r \) and \( J'_r \) are the tide reducing factors for non-load and load potentials respectively, \( U_r \) and \( U'_r \) are the prescribed non-load and load potentials respectively, \( \rho \) is the density of sea water, \( f \) the mean density of the Earth, \( G \) is the ocean function, and the constant is required to satisfy the requirement of conservation of ocean mass.

A more straightforward derivation is presented in Chapter 2.
CHAPTER 2

EQUILIBRIUM TIDES; SECOND ORDER THEORY

i) Introduction

Chapter 1 described some of the inadequacies of theoretical studies of ocean/Earth tide interactions. In this chapter the problem of the indirect effect of the oceans on the tide height and the consequences of the non-globality of the ocean for the Love number concept will be explored in detail. Also the equations for the coefficients in the surface harmonic expansion of ocean tides will be solved so that these coefficients may be readily computed for any tidal potential in terms of the coefficients of the prescribed potential.

In order to see how each effect separately alters the simplistic I.11 I shall consider each effect in turn by examining various idealized models.

ii) The Effect of Ocean Self-Attraction

To model this effect consider a rigid Earth with global oceans and suppose the height of the tide due to the prescribed potential \( U \) is \( \xi_n \) with coefficients \( U_n \) and \( \xi_n \) in a surface harmonic expansion. By treating the tide as a layer of thickness \( \xi \) and density \( \rho_0 \) equal to the density of sea water the potential of the tide may easily be found to be
\[
\sum_{n} \frac{4\pi G \rho \xi_n}{2n+1} = \sum_{n} \frac{3g \rho \xi_n}{(2n+1)\rho} 
\] (2.1)

(Munk and MacDonald, 1960, p29), where \( \rho \) is the density of sea water and \( \rho \) the mean density of the Earth. The derivation of (2.1) is given in Appendix D. The total potential at a point on the fluid surface becomes

\[
V = \sum_{n} \left( \frac{1}{n} U_n + \frac{3g \rho}{(2n+1)\rho} \xi_n \right) + \frac{2}{\sqrt{n}} (V_0) \xi + V_0 
\] (2.2)

where \( V_0 \) is the undisturbed potential of the Earth. Since \( V \) is to be constant on the fluid surface the part of (2.2) which depends on latitude and longitude must be constant, which yields

\[
g \xi = \sum_{n} \left( \frac{1}{n} U_n + \frac{3g \rho}{(2n+1)\rho} \xi_n \right) + \text{const.} 
\] (2.3)

and consequently

\[
\xi_n = \left( \frac{1}{1 - \frac{3\rho}{(2n+1)\rho}} \right) \frac{U_n}{\partial} 
\] (2.4)

The last step can only be performed because the ocean is global and as a result \( U \) and \( \xi \) have the same surface harmonic form.

Although the solid part of this Earth model is rigid and therefore has the Love number \( h_n = 0 \) the response of its fluid surface to a prescribed potential can be described by the 'Love number'.

- 14 -
\[
\tilde{f}_n = \frac{d \omega_n}{\sum_n \frac{3g}{(2n+1)\rho} \xi_n}
\]

Since the equilibrium tide in a global ocean on a rigid Earth is given simply by \(U_0/g\) when self-attraction is neglected, \(f_n\) represents an amplification factor. The theoretical tide heights are increased about 10\% by self-attraction in the case of a degree-2 potential.

Now the perturbed potential of this model is

\[
\sum_n \omega_n = \sum_n \frac{3g}{(2n+1)\rho} \xi_n = \sum_n \frac{3g}{(2n+1)\rho} \xi_n = \sum_n g_n U_n
\]

where

\[
g_n = \frac{3g}{(2n+1)\rho} \left(1 - \frac{3g}{(2n+1)\rho}\right) \quad \text{(2.7)}
\]

and \(g_n\) is the 'Love number' \(k_n\) for this model.

As long as the global ocean on this model is deeper than any possible tide the model is equivalent to a homogeneous, incompressible fluid Earth. As a check on the validity of 2.5 and 2.7 compare these solutions with those found by Kelvin (1863) and given by Takeuchi (1950, p 668) for a homogeneous incompressible Earth of rigidity \(M\). For degree-2 equations 2.5, 2.7 become

\[
f_2 = 5/2, \quad g_2 = 3/2 \quad \text{(2.8)}
\]
with \( l = \rho \). Takeuchi obtains

\[
h_2 = \frac{s}{2} m \quad k_2 = \frac{3}{2} m
\]

\[
m = \left( 1 + \frac{19 \mu}{2 \rho g a} \right)
\]

(2.9)

and for a fluid model \( \mu = 0, m = 1 \) and

\[
h_2 = \frac{s}{2} \quad k_2 = \frac{3}{2}
\]

(2.10)

<table>
<thead>
<tr>
<th>SELF ATTRACTION</th>
<th>SELF ATTRACTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEGREE EFFECTIVE</td>
<td>EFFECTIVE</td>
</tr>
<tr>
<td>( N )</td>
<td>( H )</td>
</tr>
<tr>
<td>0</td>
<td>2.200</td>
</tr>
<tr>
<td>1</td>
<td>1.222</td>
</tr>
<tr>
<td>2</td>
<td>1.122</td>
</tr>
<tr>
<td>3</td>
<td>1.085</td>
</tr>
<tr>
<td>4</td>
<td>1.065</td>
</tr>
<tr>
<td>5</td>
<td>1.052</td>
</tr>
<tr>
<td>6</td>
<td>1.044</td>
</tr>
<tr>
<td>7</td>
<td>1.038</td>
</tr>
<tr>
<td>8</td>
<td>1.033</td>
</tr>
<tr>
<td>9</td>
<td>1.030</td>
</tr>
<tr>
<td>10</td>
<td>1.027</td>
</tr>
<tr>
<td>11</td>
<td>1.024</td>
</tr>
<tr>
<td>12</td>
<td>1.022</td>
</tr>
<tr>
<td>13</td>
<td>1.021</td>
</tr>
<tr>
<td>14</td>
<td>1.019</td>
</tr>
<tr>
<td>15</td>
<td>1.018</td>
</tr>
<tr>
<td>16</td>
<td>1.017</td>
</tr>
<tr>
<td>17</td>
<td>1.016</td>
</tr>
<tr>
<td>18</td>
<td>1.015</td>
</tr>
<tr>
<td>19</td>
<td>1.014</td>
</tr>
<tr>
<td>20</td>
<td>1.013</td>
</tr>
<tr>
<td>21</td>
<td>1.013</td>
</tr>
<tr>
<td>22</td>
<td>1.012</td>
</tr>
<tr>
<td>23</td>
<td>1.012</td>
</tr>
<tr>
<td>24</td>
<td>1.011</td>
</tr>
<tr>
<td>25</td>
<td>1.011</td>
</tr>
</tbody>
</table>

Table 2.1: The effective Love numbers of a rigid earth model with global oceans taking into account ocean self attraction.
Equations 2.5 and 2.7 are consistent with Takeuchi's more general equations. The values of \( f_n \) and \( g_n \) are given in Table 2.1.

iii) The Effect of Load Yielding

The model used in this section will be a global ocean on an otherwise realistic Earth. The problem will be handled in two steps. First, the ocean surface will be considered to be acted on by all the potentials it would experience because of the feedback and the deflection with respect to the Earth's centre will be obtained. Second, the ocean bottom will be deformed by all the potentials including the ocean tide and the deflection of the solid surface with respect to the Earth's centre will be obtained. The measured tide will be obtained by subtracting one from the other.

The potentials which act on the ocean surface are:

1) the potential of the undeformed Earth:

\[
V_0(a + \xi') = V_0(a) + \frac{\partial V_0}{\partial r} \xi' = V_0 - g \xi' \tag{2.10}
\]

where \( \xi' \) is the amount the sea surface is raised above the centre of the Earth.

2) the prescribed potentials which do not/do load the Earth:

\[
\sum U_n, \sum U'_n \tag{2.11}
\]
3) the deformation potentials of the Earth as a result of the prescribed potentials:

\[ \sum_k k_n U_n + \sum_k k'_n U'_n \]  

(2.12)

4) the potential of the tide itself:

\[ \sum_n \frac{3g \rho_0 \xi_n}{(2n+1)\ell} \]  

(2.13)

where \( \xi_n \) is measured with respect to the sea floor.

5) the deformation potential of the solid Earth as a result of the ocean tide:

\[ \sum_k k'_n \frac{3g \rho_0 \xi_n}{(2n+1)\ell} \]  

(2.14)

These five potentials can now be summed together to obtain the potential at a point on the ocean surface:

\[ V = \sum_n \left( U_n + U'_n + k_n U_n + k'_n U'_n + \frac{3g \rho_0 \xi_n}{(2n+1)\ell} + k'_n \frac{3g \rho_0 \xi_n}{(2n+1)\ell} \right) - g \xi' + V_0 \]  

(2.15)

If this surface is to be an equipotential the variable part of (2.15) must be a constant on the sea surface, which gives

\[ g \xi' = \sum_n \left( U_n + U'_n + k_n U_n + k'_n U'_n + \frac{3g \rho_0 \xi_n}{(2n+1)\ell} + k'_n \frac{3g \rho_0 \xi_n}{(2n+1)\ell} \right) + \text{const.} \]  

(2.16)
Now considering the deflections of the solid Earth surface as a result of:

1) the prescribed potentials:
\[ \sum \frac{h_n U_n}{g}, \sum \frac{h_n' U_n'}{g} \]  \hspace{1cm} (2.17)

2) the ocean tides:
\[ \frac{\sum h_n' 3g \rho_0 \xi_n}{(\alpha n + 1) R} \] \hspace{1cm} (2.18)

Summing these two together gives for the total deflection of the solid Earth surface with respect to the Earth's centre,

\[ g \xi'' = \sum_{n=1} \left( h_n U_n + h_n' U_n' + h_n' 3g \rho_0 \xi_n \right) \] \hspace{1cm} (2.19)

Subtracting this from 2.16, we note that the tide measured with respect to the Earth's crust is the difference between the tide measured with respect to the Earth's centre and the solid Earth tide, i.e.

\[ \xi = \xi' - \xi'' \] \hspace{1cm} (2.20)

This results in
\[ g \xi = \sum_{n=1} \left[ (1 + k_n - h_n) U_n + (1 + k_n' - h_n') U_n' \right. \\
\left. + (1 + k_n' - h_n) 3g \rho_0 \xi_n \right] + \text{const} \] \hspace{1cm} (2.21)

or, in keeping with Munk and MacDonald's notation,
\[ \sum_n \xi_n = g^{-1} \left[ \sum_n \left( J_n U_n + J'_n U'_n + \frac{J_n 3 g \rho \xi_n}{(2n+1) \rho} \right) \right] + \text{const} \]  

where

\[ J_n = 1 + \kappa_n - h_n \]
\[ J'_n = 1 + \kappa'_n - h'_n \]  

The constant in 2.22 is determined by requiring conservation of water, i.e.

\[ \int_S \rho \xi \, ds = 0 \]  

where S is the Earth's surface. In the case of global oceans the constant is zero (except for a load which adds mass to the Earth), in non-global oceans the constant is non-zero and represents an adjustment in level referred to as the Darwin correction (M & M 1960 p. 100). We return to this point in 2.iv. Equation 2.22 gives for the tide heights \((n \geq 1)\)

\[ \xi_n = \frac{J_n}{1 - \frac{J_n 3 \rho}{(2n+1) \rho}} \frac{U_n}{g} + \frac{J'_n}{1 - \frac{J'_n 3 \rho}{(2n+1) \rho}} \frac{U'_n}{g} \]  

Again the last step can only be performed because \(\xi\) and \(U\) have the same harmonic form. The numerical values of the two Love number parameters are given in Table 2.2. For a degree-2 potential which does not load the Earth

\[ J_{\text{eff}} = \frac{J_{a}}{1 - \frac{J_{a} 3 \rho}{5 \cdot \rho}} = 0.847 \]  

so that the tide reducing factor is increased by about 20% when load yielding and self-attraction are considered.
<table>
<thead>
<tr>
<th>DEGREE K</th>
<th>GLOBAL EFFECTIVE K</th>
<th>GLOBAL EFFECTIVE TIDE HEIGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.430</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>0.238</td>
<td>1.306</td>
</tr>
<tr>
<td>2</td>
<td>0.368</td>
<td>0.847</td>
</tr>
<tr>
<td>3</td>
<td>0.152</td>
<td>0.939</td>
</tr>
<tr>
<td>4</td>
<td>0.094</td>
<td>0.982</td>
</tr>
<tr>
<td>5</td>
<td>0.069</td>
<td>0.994</td>
</tr>
<tr>
<td>6</td>
<td>0.055</td>
<td>0.996</td>
</tr>
<tr>
<td>7</td>
<td>0.046</td>
<td>0.996</td>
</tr>
<tr>
<td>8</td>
<td>0.039</td>
<td>0.994</td>
</tr>
<tr>
<td>9</td>
<td>0.034</td>
<td>0.992</td>
</tr>
<tr>
<td>10</td>
<td>0.031</td>
<td>0.992</td>
</tr>
<tr>
<td>11</td>
<td>0.028</td>
<td>0.991</td>
</tr>
<tr>
<td>12</td>
<td>0.025</td>
<td>0.990</td>
</tr>
<tr>
<td>13</td>
<td>0.024</td>
<td>0.990</td>
</tr>
<tr>
<td>14</td>
<td>0.021</td>
<td>0.989</td>
</tr>
<tr>
<td>15</td>
<td>0.020</td>
<td>0.989</td>
</tr>
<tr>
<td>16</td>
<td>0.018</td>
<td>0.988</td>
</tr>
<tr>
<td>17</td>
<td>0.018</td>
<td>0.988</td>
</tr>
<tr>
<td>18</td>
<td>0.017</td>
<td>0.989</td>
</tr>
<tr>
<td>19</td>
<td>0.016</td>
<td>0.988</td>
</tr>
<tr>
<td>20</td>
<td>0.014</td>
<td>0.988</td>
</tr>
<tr>
<td>21</td>
<td>0.014</td>
<td>0.987</td>
</tr>
<tr>
<td>22</td>
<td>0.013</td>
<td>0.987</td>
</tr>
<tr>
<td>23</td>
<td>0.013</td>
<td>0.987</td>
</tr>
<tr>
<td>24</td>
<td>0.012</td>
<td>0.988</td>
</tr>
<tr>
<td>25</td>
<td>0.012</td>
<td>0.988</td>
</tr>
</tbody>
</table>

**Table 2.2**

The effective Love number K and the effective tide height of an Earth model whose elastic response is described by the Love numbers H,K, and whose surface is covered with global oceans. The Love numbers of the elastic part are taken from Longman (1963).

The deformation potential of this model due to a non-

load potential is
\[
\sum_{n} k_n U_n + \frac{k_n^2}{(2n+1)P} \frac{3g \mu \xi_n}{(2n+1)P} + \frac{3g \mu \xi_n^2}{(2n+1)P}
\]

\[
= \sum_{n} \left[ k_n + \frac{(1+k_n^2)3\mu}{(2n+1)P} \frac{J_n}{1 - \frac{J_n^3}{(2n+1)P}} \right] U_n
\]

using 2.25.

A satellite orbiting this model Earth, or in fact the celestial body which produces the prescribed potential, will see this potential as the deformation potential of the model. Its response may be characterized by the effective Love number

\[
k_{n \text{eff}} = k_n + \frac{(1+k_n^2)3\mu}{(2n+1)P} \frac{J_n}{1 - \frac{J_n^3}{(2n+1)P}}
\]

For a degree-2 potential \( k_{2\text{eff}} = 0.366 \) which is an increase of 15% over the value Longman (1963) finds for the solid Earth. The exact value of the deformation potential of the Earth is important for studies of satellite orbit perturbations.

Smith et al. (1972) have used laser tracking of satellites in the hope of detecting polar motion. They find

\[
k_{2\text{eff}} = 0.25
\]

best fits their data. The theory of tidally-influenced satellite orbits was developed in a series of papers by Kozai (1965, 1967, 1968) who obtained

\[
k_{2\text{eff}} = 0.29
\]
Newton (1968) investigated the problem with the view particularly of finding the phase lag of the solid-Earth tide so that the tidally-influenced secular acceleration of the Earth could be deduced (Dicke, 1966, Munk, 1966). He found

\[ k_{2\text{eff}} = 0.336 \]  

(Lambeck and Cazenave, 1973), recognizing that the oceans influenced the value of \( k_{2\text{eff}} \) as seen by satellites and that the non-globality of the oceans required new approaches to Love number theory, explored the relationship between the whole Earth tide (atmospheric, oceanic, and direct and indirect solid-Earth) and a low satellite orbit. Under the assumption that the solid Earth response to both load and non-load potentials was well understood, they asserted that the satellite perturbations could be used to study and map the ocean tides. They found that for low inclination orbits \( k_{2\text{eff}} \) is 17% lower than the elastic \( k_2 \) and for high inclination orbits \( k_{2\text{eff}} \) is 5% lower. Jeffreys and Vicente (1957) in a study of the Chandler wobble deduce that \( k_{2\text{eff}} \) is increased by 12% by the oceans. In a similar study Molodenskii, (1961) obtains a 16% increase. The differences between these several observational and theoretical results, and 2.28, may be attributed to (a) shortcomings in the satellite orbit interpretation, (b) failure (except by Jeffreys and Vicente, and Molodenskii) to take into account the dynamical response of the liquid core, (c) non-equilibrium character of the ocean tides.
iv) The Effect of the Non-Globality of the Oceans

The simplest way of treating the non-globality of the oceans is to solve for the global ocean tide and simply truncate the solution to zero on land. The function which accomplishes this is the ocean function, defined by Munk and MacDonald (1960):

\[ G = \begin{cases} 0 & \text{where there is land} \\ 1 & \text{where there are oceans} \end{cases} \quad (2.32) \]

Thus the equation for tide heights on a rigid earth, for example, becomes

\[ \xi = G \sum_n U_n / g \quad (2.33) \]

and \( \xi \) will have exactly the same values where there are oceans as it would for a global ocean, and zero height where there is land. If \( G \) and \( \xi \) are expanded in surface harmonics

\[ G(\theta, \lambda) = \sum_n \sum_{m=0}^{n} P_n^m(\cos \theta) (a_n^m \cos m\lambda + b_n^m \sin m\lambda) \]

\[ \xi(\theta, \lambda) = \sum_n \xi_n = \sum_n \sum_{m=0}^{n} P_n^m(\cos \theta) (y_n^m \cos m\lambda + z_n^m \sin m\lambda) \quad (2.34) \]

where \( \theta \) is the co-latitude and \( \lambda \) the longitude, it can be seen that \( \xi \) will not in general have the same harmonic form as the forcing function \( U \). Tides of degree- \( n \) and order \( m \) will depend on the potentials of degrees other than \( n \), or more properly potentials of degree-2 order-zero will
raise tides of numerous degrees and orders depending on the ocean function. This suggests that the equations for the tide height coefficients may be cast in a matrix form such as:

\[
\begin{bmatrix}
\gamma^m_n \\
\end{bmatrix} = \begin{bmatrix}
\ddots \\
\end{bmatrix} \begin{bmatrix}
\ddots \\
\end{bmatrix} U_r
\]

A square matrix whose elements are determined by the coefficients of the ocean function and the load and non-load Love numbers (section 2.vi).

(2.35)

Here \( \gamma^m_n \) are the coefficients of the tide and \( U_r \) are the coefficients of the potential. It will be found in 2.vi that the solution for the tide coefficients can be given in this manner.

There is one additional complication introduced by non-globality, that is, an apparent non-conservation of water mass. This is overcome by requiring that the total mass of the oceans be constant, i.e.

\[
\int_S \rho \xi dS = 0 \Rightarrow \int_S \xi dS = 0
\]

(2.36)
where $S$ is the surface of the Earth. Using the orthonormality properties of the surface harmonics it is clear that 2.36 is equivalent to stating

$$\gamma_0 = 0 \quad (2.37)$$

In the next section allowance will be made for explicit mass conservation. The so-far undetermined constant in 2.21 is determined by 2.36 (or, equivalently, by 2.37).

\(\nu\) Equilibrium Tides on a Real Earth

The method of simply truncating the tide to zero on land is not consistent with the feedback of the oceans on themselves either directly or indirectly through the loading. This is because in doing so one is allowing for the feedback from land areas to sea areas as if there were ocean tides in the land areas, and then making the contradictory statement that there are no ocean tides in land areas. Clearly the condition that there be no ocean tides in land areas must be applied before the feedback is considered. Returning to equations 2.16 and 2.19, and applying the ocean function after subtracting, we find

$$g \sum R \xi_n = G \left[ \sum r \left( J_r U_r + J'_r U'_r + J_r \frac{3g \rho \hat{\xi}_r}{(2r+1)\rho} \right) \text{const} \right] \quad (2.38)$$

so that it is explicit that there is no feedback from land areas. Apart from the constant required for mass conservation, these are the equations found by Munk and MacDonald (1960) and given in a similar form by
vi) The Solution of the Equations

Hendershott's method of obtaining the solution of 2.38 is through a Green's function approach. Recognizing that the last variable in 2.38 is a feedback term it may be transformed into an integral

$$
\int \mathcal{E}(\theta', \lambda') \mathcal{C}(\theta', \lambda') \mathcal{G}(\theta', \lambda' | \theta, \lambda) \, ds',
$$

(2.39)

where

$$
\mathcal{G}(\theta', \lambda' | \theta, \lambda) = \sum_n \mathcal{T}_n \alpha_n \sum_m \cos m \alpha \cos m \lambda',
$$

$$
\mathcal{P}_n^m(\cos \phi) \mathcal{P}_n^m(\cos \phi) + \sin m \alpha \sin m \lambda' \mathcal{P}_n^m(\cos \phi) \mathcal{P}_n^m(\cos \phi),
$$

(2.40)

$$
\left[ \int_s \left[ \mathcal{P}_n^m(\cos \phi) \right] \left\{ \cos m \lambda' \right\} ds' \right]^{-1}
$$

and

$$
\alpha_n = \frac{3 \mathcal{P}_n}{(2n+1) \mathcal{P}}
$$

(2.41)

and the equation for tide heights becomes

$$
\xi(\theta, \lambda) = \mathcal{G}(\theta, \lambda) \left[ \sum_n \mathcal{T}_n \frac{U_n}{q} + \text{const} \right] + \mathcal{G}(\theta, \lambda) \int_\mathcal{E}(\theta, \lambda) \mathcal{G}(\theta, \lambda' | \theta, \lambda) ds',
$$

(2.42)

Here for brevity we consider only a non-load potential.
Although his interest was not in equilibrium tides his suggestion of an iterative solution may be used here effectively. That is, the standard formula for tide heights may be inserted in the left-hand side of 2.42 and a better approximation may be found. Thus

\[
\hat{z}(\phi, \lambda) = G(\phi, \lambda) \left[ \sum_n \frac{J_n}{q} U_n + \text{const.} \right] + \sum_n J_n \int_S U_n(\phi', \lambda') \hat{C}(\phi, \lambda | \phi', \lambda') \, ds
\]

would be the first step in the iteration. It may also be worthwhile to approach the ocean tide problem in the same way as lake tides are approached, that is through the slope of the equipotential surfaces with respect to the crust. Differentiating by \( \phi \) and \( \lambda \) gives

\[
\frac{\partial \hat{z}}{\partial \phi} = \sum_n \frac{J_n}{q} \frac{\partial U_n}{\partial \phi} + \int_S \hat{z}(\phi', \lambda') \frac{\partial G(\phi, \lambda | \phi', \lambda')}{\partial \phi} \, ds'
\]

\[
\frac{\partial \hat{z}}{\partial \lambda} = \sum_n \frac{J_n}{q} \frac{\partial U_n}{\partial \lambda} + \int_S \hat{z}(\phi', \lambda') \frac{\partial G(\phi, \lambda | \phi', \lambda')}{\partial \lambda} \, ds'
\]

(valid in the oceans) and these equations could then be solved together with the additional constraint 2.37 imposed by mass conservation.

The method suggested by Munk and MacDonald (1960) yielded simultaneous equations in the various coefficients and they gave only the degree-2 order-zero coefficient in terms of several others. The solution of the equations is attempted here in matrix form, the final result being matrices which are to the non-global oceans what the scalar Love numbers are to a spherically symmetric earth.
Starting with

$$\sum_n \xi_n = g^{-1} G \left[ \sum_r J_r U_r + \sum_r U_r + \sum_r \bar{\mathcal{E}}_g \xi_r + \text{const.} \right] (2.45)$$

and expanding all the variables in surface harmonics

$$\xi = \sum_{l=0}^n P_l^{(b)}(\cos \theta) \left[ Y_l^b \cos \alpha + Z_l^b \sin \alpha \right]$$

$$G = \sum_{m} P_m^{(m)}(\cos \theta) \left[ a_m \cos m \alpha + b_m \sin m \alpha \right]$$

$$U = \sum_{l} P_l^{(e)}(\cos \theta) \left[ U_l^e \cos \lambda + V_l^e \sin \lambda \right]$$

$$U^l = \sum_{r} P_r^{(e)}(\cos \theta) \left[ U_r^l \cos \delta + V_r^l \sin \delta \right]$$

results, after substitution in 2.45, in the following.

$$\sum_{a,b} P_a^{(b)}(\cos \theta) \left[ Y_a^b \cos b \alpha + Z_a^b \sin b \alpha \right]$$

$$= G \text{ const.}$$

$$+ \sum_{m} \left[ a_m J_r U_r^e P_m^{(m)} P_r^{(e)} \cos m \alpha \cos \lambda \right.$$}

$$+ a_m J_r V_r^e P_m^{(m)} P_r^{(e)} \cos m \alpha \sin \lambda$$

$$+ b_m J_r U_r^e P_m^{(m)} P_r^{(e)} \sin m \alpha \cos \lambda$$

$$+ b_m J_r V_r^e P_m^{(m)} P_r^{(e)} \sin m \alpha \sin \lambda$$

$$+ a_m J_r U_r^e P_m^{(m)} P_r^{(e)} \cos m \alpha \cos \lambda$$

$$= (2.47)$$
In order to facilitate computation the coefficients $\gamma^e_r$ and $\gamma^e_z$ are now not the tide height coefficients but the tide height coefficients multiplied by $g$. Multiplying both sides by $p^g_s \sin \lambda$ and integrating over the surface one obtains, after collecting terms and rearranging:

$$Z_p^q = \sum_{n \neq r} \left[ a_n^m \left( J_r U_r^e + J_r \dot{U}_r^e + J_r \gamma^e_r \right) \right]$$

$$\times \int_S p_n^m p_r^e p_p^g \cos \lambda \cos \alpha \sin \phi \, dS$$

$$+ a_n^m \left( J_r V_r^e + J_r \dot{V}_r^e + J_r \gamma^e_z \right)$$
\[
\begin{align*}
\times \int_{S} P_{n}^{m} P_{r}^{p} P_{p}^{q} \cos m \alpha \cos \alpha \sin \lambda \, ds \\
+ b_{n}^{m} \left( J_{r} U_{r}^{g} + J_{r} U_{r}^{l} + J_{r} \alpha_{r} Y_{r}^{h} \right) \\
\times \int_{S} P_{n}^{m} P_{r}^{p} P_{p}^{q} \sin m \alpha \cos \alpha \sin \lambda \, ds
\end{align*}
\]

(2.48)

by the orthonormality of the associated Legendre polynomials. Likewise, multiplying both sides of 2.47 by \( P_{p}^{q} \cos \lambda \) and integrating over the surface one obtains

\[
\begin{align*}
Y_{p}^{q} = \sum_{n,m} \left[ a_{n}^{m} \left( J_{r} U_{r}^{g} + J_{r} U_{r}^{l} + J_{r} \alpha_{r} Y_{r}^{h} \right) \\
\times \int_{S} P_{n}^{m} P_{r}^{p} P_{p}^{q} \cos m \alpha \cos \alpha \cos \lambda \, ds \\
+ a_{n}^{m} \left( J_{r} V_{r}^{e} + J_{r} V_{r}^{l} + J_{r} \alpha_{r} Z_{r}^{e} \right) \\
\times \int_{S} P_{n}^{m} P_{r}^{p} P_{p}^{q} \cos m \alpha \sin \alpha \cos \lambda \, ds
\end{align*}
\]
In order to reduce the algebra we drop explicit reference to the Darwin correction (the terms $\alpha^q \text{const.} b^q_{\text{const.}}$) and satisfy conservation of mass requirements by using 2.36.

The equations 2.48 and 2.49 may be cast in matrix form but they can be simplified somewhat by looking at the integrals over the longitude, $\lambda$. There are four types of these integrals:

1) integrals of three sine functions
2) integrals of three cosine functions
3) integrals of two sine and one cosine
4) integrals of two cosine and one sine

The product of three odd functions is odd and accordingly integrals of type 1 vanish. The product of two even and one odd functions is odd and integrals of type 4 also vanish. This leaves only integrals of types 2 and
in equations 2.48 and 2.49. The reduced equations are

\[ Y_p^q = \sum_{n,m} a_n^m \left( J_r U_r^i + J_r U_r^j + J_r \alpha_r Y_r^k \right) \]
\[ \times \int_S p_n^m p_r^e p_p^q \cos \mu \cos \alpha \cos \varphi \, ds \]
\[ + b_n^m \left( J_r V_r^i + J_r V_r^j + J_r \alpha_r Z_r^k \right) \]
\[ \times \int_S p_n^m p_r^e p_p^q \sin \mu \cos \alpha \sin \varphi \, ds \]  

(2.50)

\[ Z_p^q = \sum_{n,m} a_n^m \left( J_r V_r^i + J_r V_r^j + J_r \alpha_r Z_r^k \right) \]
\[ \times \int_S p_n^m p_r^e p_p^q \cos \mu \cos \alpha \sin \varphi \, ds \]
\[ + b_n^m \left( J_r U_r^i + J_r U_r^j + J_r \alpha_r Y_r^k \right) \]
\[ \times \int_S p_n^m p_r^e p_p^q \sin \mu \cos \alpha \sin \varphi \, ds \]

For convenience in writing define the following arrays:

\[ \sum_n a_n^m \int_S p_n^m p_r^e p_p^q \cos \mu \cos \alpha \cos \varphi \, ds \]
\[ = A1(r,e,p,q) \]

\[ \sum_n b_n^m \int_S p_n^m p_r^e p_p^q \sin \mu \cos \alpha \sin \varphi \, ds \]
Substituting these into 2.48 and 2.49 results in the equations

\[
\gamma^q_p = \sum_{rE} \left[ \left( J_r U_r^q + J_r U_r^l + J_r \alpha_r \gamma_r \right) A1(r, e, \rho_q) \right.
\]
\[
+ \left( J_r V_r^q + J_r V_r^l + J_r \alpha_r Z_r^q \right) B1(r, e, \rho_q) \right] \]  

(2.52)

\[
Z_p^q = \sum_{rE} \left[ \left( J_r V_r^q + J_r V_r^l + J_r \alpha_r Z_r^q \right) A2(r, e, \rho_q) \right.
\]
\[
+ \left( J_r U_r^q + J_r U_r^l + J_r \alpha_r \gamma_r \right) B2(r, e, \rho_q) \right] \]
These equations look very similar to matrix equations written out in subscript form except for the fact that $A_1$, $A_2$, $B_1$, and $B_2$ are 4-dimensional arrays instead of 2-dimensional matrices and $y^i_r$, $z^i_r$, $u^i_r$, $v^p_r$, $u^i_r$, and $v^i_r$ are 2-dimensional arrays instead of 1-dimensional column matrices. This problem can be circumvented if a mapping can be found that will map the 4-dimensional arrays into 2-dimensional arrays and the 2-dimensional arrays into 1-dimensional column matrices in such a way that the form of equations 2.51 is preserved. The mapping must of course be one to one and onto (by which is meant each element of $S_i$ corresponds to one and only one element in $S_o$). The following mapping satisfies these criteria.

The $r, l$ -element of $y^i_r$, etc. is mapped to the

$$\left[\frac{r(r+1)}{2} + (l+1)\right]^{st} \text{ row}$$

of the column matrix which, without introducing any confusion, is labeled with the same name. Likewise the $r, l, p, q$ -element of $A_1$, $A_2$, $B_1$, $B_2$ is mapped to the

$$\left[\frac{p(p+1)}{2} + (q+1)\right]^{st} \text{ column}$$

and the

$$\left[\frac{p(p+1)}{2} + (q+1)\right]^{st} \text{ row}$$

of the 2-dimensional matrix which again is given the same name as its image. Thus for example
Some further notation is needed before proceeding. Define

\[
\begin{align*}
\mathcal{I}_r \, A_1(r, \ell, \rho, \Omega) &= \mathcal{A}_1 \mathcal{J} \\
\mathcal{I}_r \, A_1(r, \ell, \rho, \Omega) &= \mathcal{A}_1 \mathcal{J} \\
\mathcal{I}_r \, \alpha_r \, A_1(r, \ell, \rho, \Omega) &= \mathcal{A}_1 \mathcal{J} \mathcal{A} \\
\mathcal{I}_r \, B_1(r, \ell, \rho, \Omega) &= \mathcal{B}_1 \mathcal{J} \\
\mathcal{I}_r \, B_1(r, \ell, \rho, \Omega) &= \mathcal{B}_1 \mathcal{J} \\
\mathcal{I}_r \, \alpha_r \, B_1(r, \ell, \rho, \Omega) &= \mathcal{B}_1 \mathcal{J} \mathcal{A} \\
\mathcal{I}_r \, A_2(r, \ell, \rho, \Omega) &= \mathcal{A}_2 \mathcal{J} \\
\mathcal{I}_r \, \alpha_r \, A_2(r, \ell, \rho, \Omega) &= \mathcal{A}_2 \mathcal{J} \mathcal{A} \\
\mathcal{I}_r \, B_2(r, \ell, \rho, \Omega) &= \mathcal{B}_2 \mathcal{J} \\
\mathcal{I}_r \, \alpha_r \, B_2(r, \ell, \rho, \Omega) &= \mathcal{B}_2 \mathcal{J} \mathcal{A}
\end{align*}
\]  
(2.56)
It is implicit in these identities that the R.H.S. is the image, under
the mapping 2.54, of the arrays on the L.H.S. The equations may then be
written

\[
\begin{align*}
[Y] &= [A_{13}] [U] + [A_{13}] [V] + [A_{13}] [Y] + [B_{13}] [V] + [B_{13}] [V] + [B_{13}] [Z] \\
[Z] &= [A_{23}] [V] + [A_{23}] [V] + [A_{23}] [Z] + [B_{23}] [V] + [B_{23}] [V] + [B_{23}] [Y]
\end{align*}
\]

(2.57)

and the form of equations 2.52 is preserved. The constant term in the
preceding equations is implicit in equation 2.56 in that \( Y^0 = 0 \). Re-
arranging slightly

\[
\begin{align*}
[Y] &= \left[ I - A_{13} \right]^{-1} \left\{ [A_{13}] [U] + [A_{13}] [V] + [B_{13}] [V] + [B_{13}] [V] + [B_{13}] [Z] \right\} \\
[Z] &= \left[ I - A_{23} \right]^{-1} \left\{ [A_{23}] [V] + [A_{23}] [V] + [B_{23}] [V] + [B_{23}] [V] + [B_{23}] [Y] \right\}
\end{align*}
\]

(2.58)

which may for convenience be written as

\[
\begin{align*}
[Y] &= [\alpha] + [\beta] [Z] \\
[Z] &= [\gamma] + [\lambda] [Y]
\end{align*}
\]

(2.59)

where

\[
\begin{align*}
[\alpha] &= \left[ I - A_{13} \right]^{-1} \left\{ [A_{13}] [U] + [A_{13}] [V] + [B_{13}] [V] + [B_{13}] [V] \right\} \\
[\beta] &= \left[ I - A_{13} \right]^{-1} [B_{13}]
\end{align*}
\]

(2.60)

\[
\begin{align*}
[\gamma] &= \left[ I - A_{23} \right]^{-1} \left\{ [A_{23}] [V] + [A_{23}] [V] + [B_{23}] [V] + [B_{23}] [V] \right\} \\
[\lambda] &= \left[ I - A_{23} \right]^{-1} [B_{23}]
\end{align*}
\]
Equations 2.60 are coupled matrix equations which must now be uncoupled. Substituting the first in the second and the second in the first gives

\[
\]

\[
[Z] - [\lambda][\beta][Z] = [\gamma] + [\lambda][\alpha]
\]

\[
[Y] = \left[ I - [\beta][\lambda] \right] \left\{ [\alpha] + [\beta][Y] \right\} \\
[Z] = \left[ I - [\lambda][\beta] \right] \left\{ [\gamma] + [\lambda][Z] \right\}
\]

(2.63)

The equations are now completely uncoupled. The forcing terms in 2.63 are in the matrices $\alpha$ and $\gamma$ and will now be separated out:

\[
[Y] = \left[ I - [\beta][\lambda] \right] \left[ I - [A11][A] \right] [A15] [U] \\
+ \left[ I - [\beta][\lambda] \right] \left[ I - [A15][A] \right] [A15] [U] \\
+ \left[ I - [A23][A] \right] \left[ I - [A15][A] \right] [B13] [V] \\
+ \left[ I - [A23][A] \right] \left[ I - [A15][A] \right] [B13] [V] \\
+ \left[ I - [\beta][\lambda] \right] \left[ I - [A23][A] \right] [A15] [U] \\
+ \left[ I - [\beta][\lambda] \right] \left[ I - [A23][A] \right] [B13] [V]
\]

(2.64)
\[
[Z] = \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_2][a] \right)^{-1} [a_2][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_2][a] \right)^{-1} [a_2][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_2][a] \right)^{-1} [b_2][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_2][a] \right)^{-1} [b_2][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [a_1][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [a_1][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [b_1][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [b_1][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [b_1][v] \\
+ \left( 1 - [A][b] \right)^{-1} \left( 1 - [A_1][a] \right)^{-1} [b_1][v]
\]

(2.65)

Collecting terms, equations 2.64 and 2.65 may be written as

\[
[Y] = [L_0][v] + [L_2][v] + [L_3][v] + [L_4][v]
\]

(2.66)

\[
[Z] = [L_5][v] + [L_6][v] + [L_7][v] + [L_8][v]
\]

with the L matrices defined as
Due to the nature of the forcing potential, generally only the L1 and L5 matrices will be needed. All of the tidal prediction matrices have, however, been calculated. The programming of the calculations is discussed in Chapter 3.

Among the tide height matrices L1 and L7 represent direct coupling, i.e. \( \mathbf{U}_T \) exciting \( \mathbf{Y}_T \). L3 and L5 are cross-coupling matrices. In a global ocean model L1 and L7 should be of the form

\[
\mathbf{J}_{\text{eff}} \cdot \mathbf{S}_{\text{Ar}} \cdot \mathbf{S}_{\text{me}} \cdot \mathbf{U}_T
\]

(2.68)

where \( \mathbf{J}_{\text{eff}} \) is the global effective tide reducing factor (equation 2.26) and L3 and L5 would be null matrices.
The operations of chapter 2 have been programmed and the tide height matrices are given in table 2.3a for the case of no feedback and in table 2.3b for the case of feedback. Note that in matrices L3, L4, L7 and L8 columns 1, 2, 4 and 7 contain only zeros. This is because these particular columns represent excitation from terms in the prescribed potential which do not exist i.e. the zonal coefficients with a sine function longitude dependence.
Table 2.3a The tide height matrices with the oceanic feedback neglected

| MATRIX A<sub>ij</sub> (L<sub>1 WITHOUT FEEDBACK</sub>) |  
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| -0.2180 | 0.0434 | -0.0310 | -0.0391 | -0.0219 | 0.0424 | -0.0387 | 0.0007 | 0.0639 | 0.0509 |
| -0.1871 | 0.0310 | 0.7548 | 0.0427 | -0.0345 | -0.0978 | 0.0237 | 0.0066 | -0.0486 | 0.0492 |
| -0.1340 | 0.0943 | 0.1031 | 0.4403 | 0.0092 | 0.0290 | 0.0061 | 0.0343 | 0.0289 | 0.0014 |
| -0.1033 | 0.0709 | 0.0123 | 0.0521 | -0.0855 | -0.0472 | 0.0050 | -0.0784 | 0.0642 |
| -0.0516 | 0.0342 | 0.0787 | 0.0097 | -0.0214 | 0.5131 | 0.0086 | 0.0120 | -0.0605 | 0.1270 |
| -0.1192 | 0.1214 | 0.0689 | 0.0073 | 0.0426 | -0.0312 | 0.0507 | 0.0095 | -0.0299 | 0.0091 |
| -0.1426 | 0.0032 | 0.0287 | -0.0618 | -0.0879 | 0.0650 | 0.0142 | 0.0028 | 0.0053 | 0.0273 |
| -0.1435 | 0.1115 | 0.0848 | 0.0289 | -0.0425 | 0.1742 | -0.0180 | 0.0095 | 0.5515 | 0.0380 |
| -0.0134 | -0.0148 | 0.0143 | -0.0002 | -0.0058 | 0.0456 | 0.0099 | 0.0015 | 0.0063 | 0.5897 |

| MATRIX A<sub>ij</sub> (L<sub>2 WITHOUT FEEDBACK</sub>) |  
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| -0.2472 | 0.8300 | -0.0420 | -0.0961 | -0.0538 | -0.1044 | -0.0897 | -0.0017 | 0.1482 | 0.1179 |
| -0.2122 | 0.0400 | 0.9736 | 0.1050 | -0.0848 | -0.2405 | 0.0550 | 0.0153 | 0.1126 | 0.1141 |
| -0.1520 | 0.1217 | 0.1330 | 1.0829 | 0.0226 | 0.0713 | 0.0141 | 0.0794 | 0.0671 | 0.0032 |
| -0.1171 | 0.0900 | 0.1432 | 0.0302 | 1.1087 | -0.2112 | 0.1095 | 0.1507 | -0.1819 | 0.1489 |
| -0.0535 | 0.0441 | 0.1016 | -0.0238 | -0.0526 | 1.2620 | -0.0290 | 0.0278 | -0.1666 | 0.2944 |
| -0.1399 | 0.1450 | 0.0589 | 0.0180 | 0.1048 | -0.0767 | 1.1726 | 0.0220 | 0.0694 | 0.2297 |
| -0.1617 | 0.0041 | 0.0371 | -0.1520 | 0.2163 | 0.1598 | 0.0330 | 1.3977 | 0.0122 | 0.0034 |
| -0.1627 | 0.1430 | 0.1694 | 0.0514 | 0.1044 | 0.4285 | 0.0416 | 0.0221 | 1.2789 | 0.0882 |
| -0.0152 | -0.0191 | 0.0184 | -0.0004 | -0.0142 | -0.1127 | 0.0230 | 0.0034 | 0.0147 | 1.5675 |

| MATRIX B<sub>ij</sub> (L<sub>3 WITHOUT FEEDBACK</sub>) |  
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
Table 2.3a continued

<table>
<thead>
<tr>
<th>MATRIX BIJI</th>
<th>(L4 WITHOUT FEEDBACK)</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0510</td>
<td>0.0</td>
<td>-0.0796</td>
<td>-0.1327</td>
<td>0.0</td>
<td>-0.0160</td>
<td>0.0229</td>
<td>0.0112</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0028</td>
<td>0.0</td>
<td>-0.063</td>
<td>-0.2462</td>
<td>0.0</td>
<td>-0.0129</td>
<td>0.0575</td>
<td>0.2991</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0035</td>
<td>0.0</td>
<td>0.0105</td>
<td>0.0379</td>
<td>0.0</td>
<td>-0.0586</td>
<td>0.0397</td>
<td>0.2945</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.1120</td>
<td>0.0</td>
<td>-0.0289</td>
<td>-0.1086</td>
<td>0.0</td>
<td>-0.0274</td>
<td>0.2582</td>
<td>0.1853</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0333</td>
<td>0.0</td>
<td>-0.032</td>
<td>-0.1448</td>
<td>0.0</td>
<td>0.0216</td>
<td>0.0277</td>
<td>0.2192</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0917</td>
<td>0.0</td>
<td>0.0522</td>
<td>0.2571</td>
<td>0.0</td>
<td>-0.0083</td>
<td>0.1088</td>
<td>0.2017</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0666</td>
<td>0.0</td>
<td>-0.0056</td>
<td>0.0636</td>
<td>0.0</td>
<td>-0.0011</td>
<td>0.0184</td>
<td>0.0778</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0544</td>
<td>0.0</td>
<td>-0.039</td>
<td>0.0101</td>
<td>0.0</td>
<td>0.0304</td>
<td>0.0412</td>
<td>0.0221</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX B2J</th>
<th>(LT WITHOUT FEEDBACK)</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX B2JI</th>
<th>(LS WITHOUT FEEDBACK)</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>
Table 2.3a continued

**MATRIX A2J**

(L7 WITHOUT FEEDBACK)

<table>
<thead>
<tr>
<th></th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.6928</td>
<td>0.0</td>
<td>0.0769</td>
<td>0.0740</td>
<td>0.0</td>
<td>0.0325</td>
<td>0.0022</td>
<td>0.0112</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.2472</td>
<td>0.0</td>
<td>-0.4223</td>
<td>0.0019</td>
<td>0.0</td>
<td>0.1402</td>
<td>0.0253</td>
<td>0.2085</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0596</td>
<td>0.0</td>
<td>0.0005</td>
<td>0.4968</td>
<td>0.0</td>
<td>0.0176</td>
<td>0.0123</td>
<td>0.1248</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.1415</td>
<td>0.0</td>
<td>-0.1897</td>
<td>0.0952</td>
<td>0.0</td>
<td>0.4923</td>
<td>0.0236</td>
<td>0.1357</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0039</td>
<td>0.0</td>
<td>-0.0137</td>
<td>-0.0266</td>
<td>0.0</td>
<td>0.0022</td>
<td>0.5835</td>
<td>0.0908</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0032</td>
<td>0.0</td>
<td>0.0188</td>
<td>-0.0450</td>
<td>0.0</td>
<td>0.0094</td>
<td>0.0151</td>
<td>0.5923</td>
</tr>
</tbody>
</table>

**MATRIX A2JI**

(L8 WITHOUT FEEDBACK)

<table>
<thead>
<tr>
<th></th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.8938</td>
<td>0.0</td>
<td>-0.1891</td>
<td>-0.1820</td>
<td>0.0</td>
<td>0.0753</td>
<td>0.0051</td>
<td>0.0259</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.3195</td>
<td>0.0</td>
<td>1.0385</td>
<td>0.0048</td>
<td>0.0</td>
<td>0.3250</td>
<td>0.0587</td>
<td>0.4035</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0769</td>
<td>0.0</td>
<td>0.0012</td>
<td>1.2219</td>
<td>0.0</td>
<td>0.0408</td>
<td>0.0285</td>
<td>0.2893</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.1826</td>
<td>0.0</td>
<td>-0.4665</td>
<td>0.2342</td>
<td>0.0</td>
<td>1.1415</td>
<td>0.0546</td>
<td>0.3146</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0050</td>
<td>0.0</td>
<td>-0.0317</td>
<td>-0.0653</td>
<td>0.0</td>
<td>0.0052</td>
<td>1.3530</td>
<td>0.2105</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0042</td>
<td>0.0</td>
<td>0.0463</td>
<td>-0.1107</td>
<td>0.0</td>
<td>0.0218</td>
<td>0.0351</td>
<td>1.3735</td>
</tr>
</tbody>
</table>
Table 2.3b: The tide height matrices with ocean feedback

<table>
<thead>
<tr>
<th>MATRIX L1</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4521</td>
<td>0.7948</td>
<td>-0.0249</td>
<td>-0.0466</td>
<td>-0.0276</td>
<td>0.0517</td>
<td>0.0076</td>
<td>0.0034</td>
<td>0.0752</td>
<td>0.0671</td>
<td></td>
</tr>
<tr>
<td>0.4114</td>
<td>0.0249</td>
<td>0.9577</td>
<td>0.0643</td>
<td>0.0434</td>
<td>0.1405</td>
<td>0.0276</td>
<td>0.0056</td>
<td>0.0784</td>
<td>0.0778</td>
<td></td>
</tr>
<tr>
<td>0.2822</td>
<td>0.1172</td>
<td>0.1552</td>
<td>0.5095</td>
<td>0.0137</td>
<td>0.0255</td>
<td>0.0067</td>
<td>0.0459</td>
<td>0.0262</td>
<td>0.0027</td>
<td></td>
</tr>
<tr>
<td>0.1921</td>
<td>0.0889</td>
<td>-0.1399</td>
<td>0.0182</td>
<td>0.6330</td>
<td>0.1123</td>
<td>0.0584</td>
<td>0.0874</td>
<td>0.0186</td>
<td>0.0763</td>
<td></td>
</tr>
<tr>
<td>0.1126</td>
<td>0.0416</td>
<td>-0.1131</td>
<td>0.0085</td>
<td>-0.0281</td>
<td>0.6124</td>
<td>-0.0122</td>
<td>0.0167</td>
<td>0.0997</td>
<td>0.1651</td>
<td></td>
</tr>
<tr>
<td>0.2353</td>
<td>0.1672</td>
<td>-0.0081</td>
<td>0.0527</td>
<td>-0.0439</td>
<td>0.5656</td>
<td>0.0124</td>
<td>0.0352</td>
<td>0.1227</td>
<td>0.0019</td>
<td></td>
</tr>
<tr>
<td>0.3067</td>
<td>0.0142</td>
<td>-0.0239</td>
<td>-0.0829</td>
<td>0.1185</td>
<td>0.0892</td>
<td>0.0186</td>
<td>0.6850</td>
<td>0.0114</td>
<td>0.0294</td>
<td></td>
</tr>
<tr>
<td>0.2704</td>
<td>0.1311</td>
<td>-0.1228</td>
<td>0.0191</td>
<td>-0.0563</td>
<td>-0.2161</td>
<td>-0.0212</td>
<td>-0.0101</td>
<td>0.6312</td>
<td>-0.0480</td>
<td></td>
</tr>
<tr>
<td>0.0286</td>
<td>0.0195</td>
<td>0.0226</td>
<td>0.0003</td>
<td>-0.0069</td>
<td>-0.0595</td>
<td>0.0123</td>
<td>0.0015</td>
<td>0.0080</td>
<td>0.6781</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX L2</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5127</td>
<td>0.0252</td>
<td>0.0321</td>
<td>0.1147</td>
<td>0.0679</td>
<td>0.1272</td>
<td>0.1335</td>
<td>0.0079</td>
<td>0.1744</td>
<td>0.1556</td>
<td></td>
</tr>
<tr>
<td>0.4666</td>
<td>0.0321</td>
<td>1.2355</td>
<td>0.1580</td>
<td>0.1068</td>
<td>0.3456</td>
<td>0.0639</td>
<td>0.0129</td>
<td>0.1632</td>
<td>0.1803</td>
<td></td>
</tr>
<tr>
<td>0.3176</td>
<td>-0.1454</td>
<td>0.2002</td>
<td>1.2531</td>
<td>0.0336</td>
<td>0.0627</td>
<td>0.1555</td>
<td>0.1865</td>
<td>0.0606</td>
<td>0.0063</td>
<td></td>
</tr>
<tr>
<td>0.2719</td>
<td>0.1146</td>
<td>0.1804</td>
<td>0.0448</td>
<td>1.5587</td>
<td>-0.0762</td>
<td>0.1354</td>
<td>0.2027</td>
<td>0.2425</td>
<td>-0.1769</td>
<td></td>
</tr>
<tr>
<td>0.1276</td>
<td>0.0537</td>
<td>0.1459</td>
<td>0.0289</td>
<td>0.0691</td>
<td>1.5060</td>
<td>-0.0282</td>
<td>0.0387</td>
<td>0.2312</td>
<td>-0.3828</td>
<td></td>
</tr>
<tr>
<td>0.2660</td>
<td>0.2157</td>
<td>0.1034</td>
<td>0.0198</td>
<td>0.1296</td>
<td>-0.1079</td>
<td>1.3116</td>
<td>0.0289</td>
<td>0.0817</td>
<td>0.2844</td>
<td></td>
</tr>
<tr>
<td>0.3478</td>
<td>0.0183</td>
<td>0.0308</td>
<td>0.2040</td>
<td>0.2915</td>
<td>0.2194</td>
<td>0.0432</td>
<td>1.5884</td>
<td>0.0265</td>
<td>0.0681</td>
<td></td>
</tr>
<tr>
<td>0.3157</td>
<td>0.1692</td>
<td>0.1584</td>
<td>0.0469</td>
<td>0.1386</td>
<td>0.5314</td>
<td>-0.0491</td>
<td>0.0235</td>
<td>0.4636</td>
<td>-0.1114</td>
<td></td>
</tr>
<tr>
<td>0.0325</td>
<td>0.0252</td>
<td>0.0291</td>
<td>0.0087</td>
<td>0.0169</td>
<td>0.1465</td>
<td>0.0265</td>
<td>0.0035</td>
<td>0.0166</td>
<td>1.5537</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX L3</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0397</td>
<td>0.0</td>
<td>0.0</td>
<td>0.381</td>
<td>0.0748</td>
<td>0.0</td>
<td>0.0046</td>
<td>0.0006</td>
<td>0.0324</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0196</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0340</td>
<td>0.1385</td>
<td>0.0</td>
<td>0.0054</td>
<td>0.0212</td>
<td>0.1419</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0128</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0999</td>
<td>0.0265</td>
<td>0.0</td>
<td>0.0304</td>
<td>0.0279</td>
<td>0.1724</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.1132</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0447</td>
<td>0.0490</td>
<td>0.0</td>
<td>0.0136</td>
<td>0.1400</td>
<td>0.0827</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0368</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0160</td>
<td>0.0752</td>
<td>0.0</td>
<td>0.0100</td>
<td>0.0140</td>
<td>0.0120</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0828</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0227</td>
<td>0.1332</td>
<td>0.0</td>
<td>0.0120</td>
<td>0.0015</td>
<td>0.0713</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0362</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0213</td>
<td>0.0328</td>
<td>0.0</td>
<td>0.0032</td>
<td>0.0631</td>
<td>0.0857</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0644</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0082</td>
<td>0.0384</td>
<td>0.0</td>
<td>0.0030</td>
<td>0.0448</td>
<td>0.0480</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.0572</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0207</td>
<td>0.0028</td>
<td>0.0</td>
<td>0.0159</td>
<td>0.0228</td>
<td>0.0079</td>
</tr>
</tbody>
</table>
Table 2.3b continued

<table>
<thead>
<tr>
<th>MATRIX L4</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0512</td>
<td>0.0</td>
<td>-0.0938</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0106</td>
<td>0.0</td>
<td>0.014</td>
<td>0.0</td>
<td>0.0751</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0253</td>
<td>0.0</td>
<td>-0.0836</td>
<td>0.0</td>
<td>-0.0131</td>
<td>0.0</td>
<td>-0.0125</td>
<td>0.0</td>
<td>0.0491</td>
<td>0.0</td>
<td>0.3289</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0165</td>
<td>0.0</td>
<td>0.0244</td>
<td>0.0</td>
<td>0.0705</td>
<td>0.0</td>
<td>0.0674</td>
<td>0.0</td>
<td>0.3997</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>-0.1460</td>
<td>0.0</td>
<td>-0.0115</td>
<td>0.0</td>
<td>-0.0315</td>
<td>0.0</td>
<td>-0.0326</td>
<td>0.0</td>
<td>0.1917</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0500</td>
<td>0.0</td>
<td>0.0395</td>
<td>0.0</td>
<td>0.0233</td>
<td>0.0</td>
<td>0.0325</td>
<td>0.0</td>
<td>0.2796</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.1068</td>
<td>0.0</td>
<td>0.0559</td>
<td>0.0</td>
<td>0.0278</td>
<td>0.0</td>
<td>0.0034</td>
<td>0.0</td>
<td>0.1653</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>-0.0467</td>
<td>0.0</td>
<td>0.0524</td>
<td>0.0</td>
<td>-0.0073</td>
<td>0.0</td>
<td>-0.1463</td>
<td>0.0</td>
<td>0.1987</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0830</td>
<td>0.0</td>
<td>0.0261</td>
<td>0.0</td>
<td>-0.0069</td>
<td>0.0</td>
<td>0.0111</td>
<td>0.0</td>
<td>0.1113</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0738</td>
<td>0.0</td>
<td>0.0509</td>
<td>0.0</td>
<td>0.0368</td>
<td>0.0</td>
<td>0.0528</td>
<td>0.0</td>
<td>0.0183</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX L5</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0196</th>
<th>0.0</th>
<th>0.0853</th>
<th>0.0</th>
<th>0.0285</th>
<th>0.0</th>
<th>0.0084</th>
<th>0.0</th>
<th>0.369</th>
<th>0.0</th>
<th>0.1967</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0531</td>
<td>0.0</td>
<td>0.0481</td>
<td>0.0</td>
<td>0.084</td>
<td>0.0</td>
<td>0.369</td>
<td>0.0</td>
<td>0.1967</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.047</td>
<td>0.0</td>
<td>0.0642</td>
<td>0.0</td>
<td>0.0156</td>
<td>0.0</td>
<td>0.151</td>
<td>0.0</td>
<td>0.2309</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0122</td>
<td>0.0</td>
<td>0.0752</td>
<td>0.0</td>
<td>0.0178</td>
<td>0.0</td>
<td>0.079</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.055</td>
<td>0.0</td>
<td>0.0548</td>
<td>0.0</td>
<td>0.0179</td>
<td>0.0</td>
<td>0.109</td>
<td>0.0</td>
<td>0.1887</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0759</td>
<td>0.0</td>
<td>0.0305</td>
<td>0.0</td>
<td>0.0239</td>
<td>0.0</td>
<td>0.046</td>
<td>0.0</td>
<td>0.1371</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATRIX L6</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table 2.3b continued

<table>
<thead>
<tr>
<th>Matrix L7</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.8522</td>
<td>0.0</td>
<td>-0.1017</td>
<td>-0.1022</td>
<td>0.0</td>
<td>-0.0386</td>
<td>0.0096</td>
<td>0.0247</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.3275</td>
<td>0.0</td>
<td>0.4978</td>
<td>0.0070</td>
<td>0.0</td>
<td>-0.1695</td>
<td>0.0242</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0018</td>
<td>0.5857</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.1679</td>
<td>0.0</td>
<td>0.2294</td>
<td>0.1240</td>
<td>0.0</td>
<td>0.5559</td>
<td>0.0226</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Matrix L8</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.2166</td>
<td>0.0</td>
<td>-0.5843</td>
<td>0.3059</td>
<td>0.0</td>
<td>1.2891</td>
<td>0.0524</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Several tide maps have been produced using the tide height matrices. Unfortunately, the maps show no distinct correlation with the ocean function. Figure 2.2 a,b is the tide range of the fortnightly tide. For comparison the tide range of the fortnightly tide using 1.11 and truncating to zero on land is shown in figure 2.3 a,b. The tide heights in these maps are given in decimeters and the negative sign in the polar regions is due to the phase. For a degree-2 order-zero tide potential the oceans in high latitudes oscillate 180 degrees out of phase with the low latitude oceans. The tide range of the pole tide is shown in figure 2.4 a,b. The amplitude of the polar motion was taken arbitrarily to be 2.25 seconds of arc to obtain a prescribed pole tide potential. The phase relationship of the tide is shown in that opposite quadrants of the Earth have the same phase.
Figure 2.2a. The map of the fortnightly tide range in the eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmūno et al. (1973) with \( E \leq 0.5 \).
Figure 2.2b The map of the fortnightly tide range: western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with $C < 0.5$.

<table>
<thead>
<tr>
<th></th>
<th>180</th>
<th>190</th>
<th>200</th>
<th>210</th>
<th>220</th>
<th>230</th>
<th>240</th>
<th>250</th>
<th>260</th>
<th>270</th>
<th>280</th>
<th>290</th>
<th>300</th>
<th>310</th>
<th>320</th>
<th>330</th>
<th>340</th>
<th>350</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>10</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>20</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>30</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>40</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>50</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>60</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>70</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>80</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>90</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>100</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>110</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>120</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>130</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>140</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
</tr>
<tr>
<td>150</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
</tr>
<tr>
<td>160</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>170</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>180</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
<td>1.6</td>
</tr>
<tr>
<td>190</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>200</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
<td>1.8</td>
</tr>
<tr>
<td>210</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
<td>1.9</td>
</tr>
</tbody>
</table>
Figure 2.3a The map of the tide range of the fortnightly tide, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with $C < 0.5$ (no feedback).
Figure 2.3b The map of the tide range of the fortnightly tide, western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al. (1973) with $C < 0.5$ (no feedback).
Figure 2.4a The map of the tide range of the equilibrium pole tide, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al. (1973). $C < 0.5$. 

```
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180
0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
10 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
20 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
30 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
40 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
50 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
60 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
70 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
80 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
90 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
100 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
110 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
120 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
130 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
140 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
150 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
160 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
170 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
180 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0 -0.0
```
Figure 2.4b The map of the equilibrium pole tide, western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with $B \leq 0.5$.
The lack of correlation of these tides with the maps of the ocean function (Appendix B) is due to a combination of two errors. First the ocean function is accurate to at best 30% and we are attempting to find a surface harmonic representation of the product (the tide) of two functions given only the representations of the two separate functions (the ocean function and the prescribed potential) any error in the representations of these two functions will be compounded in the representation of their product. The second error arises from a limitation imposed on the matrix method by the selection rules of the integrals in 2.48 and 2.49. For example assuming that the ocean function is available to degree-8 and that the prescribed potential is a degree-2 order-zero then the $Y^0_0$ term in the tide is given by 2.49

$$
Y^0_0 = \sum_{n, q} a^n_0 \int \frac{P^0_0}{P^0_0} \cos \lambda \cos \phi d\lambda d\phi
$$

What is the maximum degree in the tide height that can be accurately obtained? The important selection rule is

$$|\lambda - n| < \rho \leq n + 2$$

It is found that for $Y^0_8$ we need a term $a^0_0$ which is not available. $Y^0_0$ is the last term which can be evaluated exactly. Figure 2.5 shows the cutoff of terms in the matrix which can be evaluated exactly. Terms which are above the curved line can be evaluated exactly, terms which fall on or below the curved line cannot because their evaluation requires one or more ocean function coefficients which are not available. The solution of
The feedback problem requires that the matrices be square. Unfortunately, the largest matrix, all of whose elements are correct, is only about 12×12, which would give a tide to degree-4 order-2. Since this is not much an improvement over a degree-3 order-3 expansion, it was decided to use 10×10 matrices. The ocean function expanded to degree-3 order-3 is barely adequate, the position and size of only the largest land masses are apparent, and the values predicted by the ocean function are in places over 100% off. It cannot be expected that the even more variable tide will be represented well by a degree-2 order-3 expansion.

Figure 2.5 error analysis for a matrix in the matrix method.
CHAPTER 3

THE PERTURBED POTENTIAL OF AN EARTH WITH NON-GLOBAL OCEANS

i) Introduction

The results of Chapter 2 suggest that the perturbed potential of a non-spherically-symmetric Earth may be expressed in much the same way as the tide height in non-global oceans, that is with matrices. With that end in view an attempt is made in section 3.ii to write the perturbed potential of an Earth model, whose oceans respond in equilibrium fashion to a prescribed potential, in a matrix formulation. In section 3.iii the matrix method is compared with the method of Kaula (1969) and Kaula's equations are transformed into matrix equations, so that the parameters of the matrix method can be given in terms of the parameters of Kaula's method.

ii) The Perturbed Potential in the Matrix Method

In response to a prescribed body force potential

\[ U = \sum_{l,m} P_k^m (u_l^m \cos \alpha + v_l^m \sin \alpha) \]  \hspace{1cm} (3.1)

the potential due to the deformation is

\[ V = \sum_{l,m} P_k^m (q_l^m \cos \alpha + p_l^m \sin \alpha) \]  \hspace{1cm} (3.2)
with the coefficient given in subscript form by
\[ q^k_j = \sum_{j'k'} K1(l_m,j,k) U^k_{j'} + K3(l_m,j,k) V^k_{j'} \]  
(3.3)
\[ r^m_j = \sum_{j'k'} KS(p,m,j,k) U^k_{j'} + K7(p,m,j,k) V^k_{j'} \]

in direct analogy with 2.66. Using 3.3 and 3.2 the perturbed potential can be written
\[
V = \sum_{j'k'} P^m_j \left[ (K1(l_m,j,k) \cos \alpha + K5(l_m,j,k) \sin \alpha) U^k_{j'} + K3(l_m,j,k) \cos \alpha + K7(l_m,j,k) \sin \alpha) V^k_{j'} \right] \]  
(3.4)

The matrices $K1$, $K3$, $K5$, $K7$ may be evaluated using the matrices $L1$, $L3$, $L5$, $L7$ of equation 2.67, as I now show.

The potential of the ocean tide is
\[
\sum_{l=1}^{b} \frac{3g P^m_f}{(2l+1)r} \left( f_p^{m} \cos \alpha + 2 \phi^{m} \sin \alpha \right) \]  
(3.5)

Now, from 2.66 we have
\[
g f_p^{m} = \sum_{j'k'} L1(p,m,j,k) U^k_{j'} + L3(p,m,j,k) V^k_{j'} \]  
(3.6)
\[
g \phi^{m} = \sum_{j'k'} L5(p,m,j,k) U^k_{j'} + L7(p,m,j,k) V^k_{j'} \]

Therefore the potential of the equilibrium ocean tide and associated Earth deformation is
\[
T_e = \sum_{l=1}^{b} \frac{(1+K^2)}{(2l+1)r} P^m_f \left[ L1(p,m,j,k) U^k_{j'} + L3(p,m,j,k) V^k_{j'} \right] \cos \alpha \]
\[+ \left[ L5(p,m,j,k) U^k_{j'} + L7(p,m,j,k) V^k_{j'} \right] \sin \alpha \]  
(3.7)
To this must be added the potential of the solid Earth as a direct result of the prescribed potential:

\[
W = \sum_{l,m} k_p P_l^m \left( U_l^m \cos m\lambda + V_l^m \sin m\lambda \right)
\]  \hspace{1cm} (3.8)

which may be re-written using the Kronecker delta symbol

\[
W = \sum_{l,m} P_l^m \left[ (k_p \delta_{l,j} \delta_{m,k}) U_j^k \cos m\lambda \\
+ (k_p \delta_{l,j} \delta_{m,k}) V_j^k \sin m\lambda \right]
\]  \hspace{1cm} (3.9)

Adding 3.9 and 3.7 we have for the complete Earth response

\[
V = T + W = \sum_{l,m} P_l^m \left\{ \\
\left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L1(l,m,i,k) + k_p \delta_{l,j} \delta_{m,k} \right] U_j^k \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L3(l,m,i,k) \right] V_j^k \cos m\lambda \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L5(l,m,i,k) \right] U_j^k \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L7(l,m,i,k) + k_p \delta_{l,j} \delta_{m,k} \right] V_j^k \sin m\lambda \right\}
\]  \hspace{1cm} (3.10)

This may be re-arranged to give

\[
V = \sum_{l,m} P_l^m \left\{ \\
\left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L1(l,m,i,k) + k_p \delta_{l,j} \delta_{m,k} \right] \cos m\lambda \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L3(l,m,i,k) \right] \cos m\lambda \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L5(l,m,i,k) \right] \sin m\lambda \\
+ \left[ (1 + k_p^i) \frac{3 P_l^i}{(2l+1)P} L7(l,m,i,k) + k_p \delta_{l,j} \delta_{m,k} \right] \sin m\lambda \right\}
\]  \hspace{1cm} (3.11)
Comparing 3.4 and 3.13 we have

\[ k_l(e, m, j, k) = \frac{(1 + k_2) 3 \rho}{(2 \ell + 1) \rho} L_l(e, m, j, k) + k_2 s_{j \ell} s_{m k} \quad (3.12) \]

\[ k_3(e, m, j, k) = \frac{(1 + k_2) 3 \rho}{(2 \ell + 1) \rho} L_3(e, m, j, k) \]

\[ k_5(e, m, j, k) = \frac{(1 + k_2) 3 \rho}{(2 \ell + 1) \rho} L_5(e, m, j, k) \]

\[ k_7(e, m, j, k) = \frac{(1 + k_2) 3 \rho}{(2 \ell + 1) \rho} L_7(e, m, j, k) + k_2 s_{j \ell} s_{m k} \]

iii) The Method of Kaula

In a study of the effect of asymmetry in the Earth's tidal response and its consequences for the tidal evolution of the Moon's orbit, Kaula (1969) postulated that in response to a prescribed potential of the form 3.1 the deformation potential of the Earth would be given by

\[ V = \sum_{j, k} k_j(\theta, \lambda) P_j^k (u_j^k \cos k \lambda + v_j^k \sin k \lambda) \quad (3.13) \]

with the "Love number" \( k_j(\theta, \lambda) \) a function of latitude and longitude, expanded in surface harmonics as

\[ k_j(\theta, \lambda) = \sum_{\rho, q} P_{\rho q} (k_{j \rho} \cos q \lambda + l_{j \rho} \sin q \lambda) \quad (3.14) \]

This method has the disadvantage that the function \( k_j(\theta, \lambda) \) conceals the steps in the feedback process which leads to the resultant tide, whereas in the matrix method the feedback processes have been taken into account explicitly.

In any surface harmonic analysis it is necessary to truncate the series at some value of the degree. Now the effect of asymmetry will be for the perturbed potential to exhibit harmonics of both
higher and lower degree than the prescribed potential. Thus in Kaula's
method the expansion of $k_j(\theta, \phi)$ must go to higher terms than the expansion
of the prescribed potential. In the matrix method this route is not
available since (as a result of the matrix format) the highest term in
the perturbed potential is of the same degree and order as the highest
term in the prescribed potential. The low-degree terms in the tide height
expansion are easily handled by the matrix but for the high-degree terms
in the prescribed potential the matrix method truncates the expansion of
the tide height prematurely. This difficulty can be overcome in practice
by simply expanding the prescribed potential to enough terms so that the
last several are too weak to excite a measurable tide. This was the procedure
followed in Chapter 2 where allowance was being made for body-force
potential harmonics up to degree-8 order-8 even though the luni-solar
tide potential has negligible terms above degree-2. The basic incompat-
ibility of the two methods is still present although equal accuracy can be
obtained with either. This will lead to some difficulties in expressing
Kaula's parameters in terms of the matrices.

Substituting 3.14 into 3.13 we have

$$V = \sum_{j=1}^{\infty} k^q_j \left( \rho^q_j \rho^k_j \cos q \phi \cos \kappa \phi \right) U^k_j + \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} k^q_p \left( \rho^q_p \rho^k_p \sin q \phi \cos \kappa \phi \right) U^k_j$$

$$+ \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} k^q_p \left( \rho^q_p \rho^k_p \cos q \phi \sin \kappa \phi \right) V^k_j + \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} k^q_p \left( \rho^q_p \rho^k_p \sin q \phi \sin \kappa \phi \right) V^k_j \quad (3.15)$$
Each of the bracketed terms may be expanded in surface harmonics:

\[ P_p^a P_j^b \cos q \phi \cos k \lambda = \sum_{\ell m} P_{\ell}^m \left( A(p,q,j,k,\ell,m) \cos \lambda + B(p,q,j,k,\ell,m) \sin \lambda \right) \]

\[ P_p^a P_j^b \sin q \phi \cos k \lambda = \sum_{\ell m} P_{\ell}^m \left( C(p,q,j,k,\ell,m) \cos \lambda + D(p,q,j,k,\ell,m) \sin \lambda \right) \]

\[ P_p^a P_j^b \sin q \phi \sin k \lambda = \sum_{\ell m} P_{\ell}^m \left( E(p,q,j,k,\ell,m) \cos \lambda + F(p,q,j,k,\ell,m) \sin \lambda \right) \]

The coefficients \( A(p,q,j,k,1,m) \), etc. are triple product surface harmonic integrals, and defined by

\[
\begin{align*}
B(p,q,j,k,1,m) & = 0 \\
C(p,q,j,k,1,m) & = 0 \\
E(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \cos q \phi \cos k \lambda \cos \lambda \cos \theta \, d \Sigma \\
H(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \sin q \phi \cos k \lambda \sin \lambda \cos \theta \, d \Sigma \\
A(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \cos q \phi \cos k \lambda \cos \lambda \sin \theta \, d \Sigma \\
D(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \sin q \phi \cos k \lambda \sin \lambda \sin \theta \, d \Sigma \\
F(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \cos q \phi \sin k \lambda \sin \lambda \cos \theta \, d \Sigma \\
G(p,q,j,k,1,m) & = \int p_p^a p_j^b p_k^m \sin q \phi \sin k \lambda \sin \lambda \sin \theta \, d \Sigma
\end{align*}
\]
The perturbed potential is then
\[
\mathbf{V} = \sum_{j,k} \frac{P^j_k}{L^j_k} \left( k^j_p [ A(p,q,j,k,m) \cos m] U^k_j + L^j_k [ B(p,q,j,k,m) \sin m] U^j_k + k^j_p [ F(p,q,j,k,m) \sin m] V^k_j + L^j_k [ G(p,q,j,k,m) \cos m] V^j_k \right)
\] (3.18)

Comparing equations 3.18 and 3.4 we have
\[
\begin{align*}
\mathbf{k1}(l,m,j,k) &= \sum_{p,q} k^j_p \, A(p,q,l,m,j,k) \\
\mathbf{k3}(l,m,j,k) &= \sum_{p,q} L^j_p \, C(p,q,l,m,j,k) \\
\mathbf{k5}(l,m,j,k) &= \sum_{p,q} L^j_p \, D(p,q,l,m,j,k) \\
\mathbf{k7}(l,m,j,k) &= \sum_{p,q} k^j_p \, F(p,q,l,m,j,k)
\end{align*}
\] (3.19)

which gives the matrix coefficients in terms of Kaula's parameters.

The map of \( k^2 (l,m,j,k) \) (figure 3.1 a, b) was produced by using the perturbed potential matrices to construct a tide map and then dividing the value obtained at each grid point by

\[
P_0^2 (\cos \theta)
\] (3.20)

since the prescribed potential is of form \( U^0_2 \). The same comments on the accuracy of representation apply to these maps as to those of Chapter 2.
**Figure 3.1a** The map of $k_i(\theta, \phi)$, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al. with $\theta \leq 0.5$.
Figure 3.1b The map of $k_{\alpha}(\theta, \lambda)$, western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al. with $C < 0.5$. 

<table>
<thead>
<tr>
<th>180</th>
<th>190</th>
<th>200</th>
<th>210</th>
<th>220</th>
<th>230</th>
<th>240</th>
<th>250</th>
<th>260</th>
<th>270</th>
<th>280</th>
<th>290</th>
<th>300</th>
<th>310</th>
<th>320</th>
<th>330</th>
<th>340</th>
<th>350</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
<tr>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
</tr>
</tbody>
</table>
CHAPTER 4

THE INFLUENCE OF THE OCEANS ON THE CHANDLER WOBBLE

i) Introduction

The possibility of a torque-free wobble of the Earth - a motion of the geographical axis of the solid Earth about the spin axis such that any point on the Earth undergoes a periodic change in latitude - was first suggested by Euler in 1765. However, Euler's results were based on a rigid Earth model and predicted a period of 10 months rather than the eventually observed 14 months or 437 days. Most of the lengthening of the period can be attributed to elastic yielding of the solid Earth but the oceans are responsible for about 40 days. The ocean influence can be explained by noting that the period is determined by the gyroscopic restoring torque provided by the Earth's equatorial bulge. For a rigid Earth, none of the bulge can adjust to a new geographical position of the spin axis as the Earth wobbles, and the effect of the oceans on the period is limited to their contribution to the difference in moments of inertia which constitutes the bulge. But when mobility of the oceans is taken into account, together with elastic yielding of the solid Earth, the major axis of the inertia tensor of the oceans will tend to be more closely aligned with the moving spin axis, thus reducing the effective bulge and lengthening the period. In addition to influencing the period, the ocean introduces a slight ellipticity in the geographical path of the
pole because their irregular distribution causes the wobble about one axis to be excited slightly more strongly than that about the other.

The potential induced by the wobble excites an ocean tide known as the pole tide. The feedback of the ocean pole tide on the Chandler wobble has been investigated previously by Larmor (1915) and Haubrich and Munk (1959), but with simplifying assumptions about tide heights and the effect of continentality. After examining the problem in the case of global oceans (section 4.iii) and accommodating continentality (section 4.iv) a complete solution for the effect of an equilibrium pole tide is attempted in section 4.v.

Larmor (1915), the first to attempt the problem, obtained a lengthening of the period of about 28 days. He allowed for yielding of the solid Earth to the prescribed tidal potential of the wobble but did not take into account ocean loading or self-attraction. He also attempted to allow for the effect on the period, of non-globality of the oceans by reducing his tide height by 2/3 (which is approximately the fraction of the Earth covered by oceans). Although he gave no clear argument for the introduction of this factor it will be seen in section 4.iv that the leading term in the expansion of the ocean function in surface harmonics is also the leading term in the correction for continentality.

Haubrich and Munk (1959) accounted correctly for continentality but did not allow for the effect of load yielding and self-attraction on the tide height, and made an incorrect estimate (section 4.v) of the effect of load yielding on the products of inertia of a tide. Their error was attributable mostly to incorrect numerical values of the load Love numbers but, as well, their use of the Love numbers was inconsistent with
the Love number method. Their method is essentially that of section 4.14 although I have used the more recent ocean function of Balmino et al (1973) in the correction for continentality.

Here, as in all previous work, it has been assumed that the pole tide is an equilibrium tide. Although the solar annual tide is thought to be equilibrium this does not mean that all longer period tides will be equilibrium. The reason for this is that the solar annual tide potential is of degree-2, order-zero whereas the pole tide is of degree-2, order-1. This is sufficient reason to rule out equating their dynamics (Munk and MacDonald, 1960, p. 99). In fact both Haubrich and Munk (1959), and Miller (1973) in a more complete analysis, report pole tides an order of magnitude above equilibrium. However the areas in which an especially large pole tide is found are quite localized and Miller (1973) finds that the excitation from the non-equilibrium tides in the Baltic and North seas is quite insufficient to excite an appreciable wobble. On the basis of this result an investigation of the effect of the oceans on the Chandler wobble, assuming an equilibrium tidal response, may be a good approximation.

ii) The Liouville Equations

The Liouville equations which describe the rotation of a yielding Earth are derived in Appendix E. They can be written in the form

\[ \ddot{m}_1 + \dot{m}_2 = \varphi \]

\[ \ddot{m}_2 - m_1 = -\varphi \]
where $\phi_r$ is the Eulerian frequency of the wobble for a rigid Earth and $m_1$ and $m_2$ are the components of the angular velocity of the reference frame (which is attached to the solid Earth in some prescribed manner) about the axes $0^\circ, 90^\circ$ longitude in the equatorial plane, normalized by dividing by the diurnal frequency. The excitation functions, for the purposes needed here, are

$$\phi_1 = \phi_{13} / (c - a) + \phi_1'$$

$$\phi_2 = \phi_{23} / (c - a) + \phi_2'$$

(4.2)

where $c, a$ are the Earth's axial and equatorial moments of inertia, and $c_{13}, c_{23}$ are the small products of inertia generated by the pole tide. The first terms on the right hand side are the ocean excitations and $\phi_1'$ and $\phi_2'$ are the excitations of all other sources which have an effect on the period of the wobble (fluid core, elastic mantle, atmosphere). Using 4.2 equations 4.1 may be written

$$m_1 + \sigma_r m_a = \sigma_r c_{23} / (c - a) + \sigma_r \phi_2'$$

$$m_2 - \sigma_r m_1 = -\sigma_r c_{13} / (c - a) - \sigma_r \phi_1'$$

(4.3)

The most exhaustive study of the wobble of an oceanless, but otherwise real Earth is by Jeffreys and Vicente (1957). They have shown that for an oceanless Earth equations 4.3 can be rewritten in the approximate form

$$m_1 + \sigma e m_a = 0$$

$$m_2 - \sigma e m_1 = 0$$

(4.4)
Re-introducing the ocean excitation gives

\[ m_1 + \delta e m_a = \delta r \frac{C_{23}}{(c^2 - A)} \]

\[ m_2 = \delta e m_1 = -\delta r \frac{C_{13}}{(c^2 - A)} \]  \hspace{1cm} (4.5)

which may be solved for the observed frequency \( \delta \) in terms of the 'oceanless' frequency \( \delta_0 \) and the Eulerian frequency \( \delta r \). Subtraction of the 'oceanless' period from the observed period gives the lengthening of the period due to the oceans.

iii) The Pole Tide in Global Oceans

This is essentially the model of Larmor (1915). Although he did not use the perturbed Liouville equations and hence could not solve for the wobble period of an oceanless Earth, he was able to obtain an approximate solution of the problem by calculating the wobble period of a rigid Earth whose pole tide had the same height it would have on a real Earth. The lengthening calculated on this basis is about 28 days which is actually closer to the true lengthening by oceans which respond in equilibrium fashion than the 43 days we obtain here by approaching the problem correctly. Using 1.11 the height of the equilibrium tide is

\[ z_a = -\frac{\alpha^2 \alpha}{3} \sum_l \left( m_1 \cos \lambda + m_2 \sin \lambda \right) \]  \hspace{1cm} (4.7)

The wobble potential in 4.47 is derived in appendix F.
Using 2.1 for the potential of a surface layer of known height and density we have for the potential of the pole tide:

\[ V_2 = -\frac{3\rho_2}{5\rho} \frac{n^2 a^2}{3} J_{2} P_2^1 (m_2 \cos \lambda + m_2 \sin \lambda) \quad (4.8) \]

Using MacCullagh's formula for the terms in the gravitational potential due to the products of inertia \( c_{13}, c_{23} \) we have

\[ V_2 = -\frac{3c_{13} a^2}{3} P_2^1 (c_{13} \cos \lambda + c_{23} \sin \lambda) \quad (4.9) \]

and comparing with 4.8 we have

\[ c_{13} = \frac{P_2}{5\rho} \frac{n^2 a^2}{G} J_2 m_2 \quad (4.10) \]

\[ c_{23} = \frac{P_2}{5\rho} \frac{n^2 a^2}{G} J_2 m_3 \]

The wobble equations then become

\[ m_1 + \Delta e m_2 = -c_{13} \frac{P_2}{5\rho} \frac{n^2 a^2}{G(c-A)} J_2 m_3 \]

\[ m_2 - \Delta e m_1 = -c_{23} \frac{P_2}{5\rho} \frac{n^2 a^2}{G(c-A)} J_2 m_1 \quad (4.11) \]

When solved for the observed frequency these yield

\[ \Delta = \Delta e - \Delta c_{13} \frac{P_2}{5\rho} \frac{n^2 a^2}{G(c-A)} \]

\[ = \Delta e - 0.080 \Delta r \quad (4.12) \]

When the Euler and observed frequencies are used in 4.12, the increase in
the wobble period due to the oceans is 43 days, and the wobble period of an oceanless Earth is

\[ T_e = 394 \text{ days} \]  

(4.13)

and the oceans accordingly increase the period of the wobble by 43 days.

As a result of the globality of the oceans there is no preference in the excitation of either \( m_1 \) or \( m_2 \) and the path of the pole is circular.

iv) The Pole Tide in Non-Global Oceans

If the tide, as predicted by 4.7, is integrated over the oceans it is found that the integral does not reduce to zero, i.e. water is not conserved. To correct this we use the Darwin correction (section 2.iv). The corrected tide in the ocean is

\[ \xi_2 = -\frac{a^2 \Delta \zeta_2}{3g} \int \left[ m_2 \left( \frac{P_2^1 \cos \lambda - 3b^1}{40a^6} \right) + m_2 \left( \frac{P_2 \sin \lambda - 3b_2^1}{40a^6} \right) \right] \]  

(4.14)

When calculating the products of inertia due to this tide it must be truncated to zero on land:

\[ c_{13} = -a^4 P_0 \int \xi_2 \frac{G}{3} \frac{P_2^1 \cos \lambda}{3} ds \]

\[ c_{23} = -a^4 P_0 \int \xi_2 \frac{G}{3} \frac{P_2^1 \sin \lambda}{3} ds \]  

(4.15)
Substituting the surface harmonic expansions of \( G \) and \( \xi \) into 4.15 we have

\[
\begin{align*}
C_{13} &= H_2 \sum_{nm} \left[ \int_S P_n^m P_n^l P_{n,a}^m a_n \cos \alpha \cos \lambda \, ds \\
&\quad \quad \quad + \int_S P_n^m P_n^l P_{n,a}^m b_n \sin \alpha \sin \lambda \cos \lambda \, ds \\
&\quad \quad \quad - \int_S P_n^m P_n^l a_n m_1 3 a_2^2 \cos \alpha \cos \lambda \, ds \\
&\quad \quad \quad - \int_S P_n^m P_n^l a_n m_2 3 b_2^2 \cos \alpha \cos \lambda \, ds \quad (4.16)
\end{align*}
\]

\[
\begin{align*}
C_{23} &= H_2 \sum_{nm} \left[ \int_S P_n^m P_n^l P_{n,a}^m a_n \cos \alpha \sin^2 \lambda \, ds \\
&\quad \quad \quad + \int_S P_n^m P_n^l P_{n,a}^m b_n \sin \alpha \sin \lambda \cos \lambda \, ds \\
&\quad \quad \quad - \int_S P_n^m P_n^l b_n m_1 \sin \alpha \sin \lambda \, ds \\
&\quad \quad \quad - \int_S P_n^m P_n^l b_n m_2 3 b_2^2 \sin \alpha \sin \lambda \, ds \quad \text{where}
\end{align*}
\]

\[
H_2 = \frac{\Omega^4}{a^6} \frac{L_6}{q} J_2
\]

The number of non-zero integrals in 4.16 is limited by the fact that one of the Legendre functions has been specified to be \( P_2^l \). In fact the only non-zero integrals are those in which the ocean coefficients of degree 0, 2, 4, and order 0, 2 are present.
Following the notation of Haubrich and Munk (1959) these may be written

\[
C_{13} = \frac{H_2}{A} A (T_1 m_1 + R m_2)
\]

\[
C_{23} = \frac{H_2}{A} A (T_2 m_2 + R m_1)
\]

(4.17)

where

\[
T_1 = \left[ a_o^0 6000 \pi + a_2^0 4286 \frac{\pi}{2} + a_4^0 1114 \frac{\pi}{2} \right. + \left. \left( a_2^3 \cdot 5143 \frac{\pi}{2} + a_4^3 \cdot 6429 \frac{\pi}{2} \right) - \frac{9 \pi}{200} \left( \frac{b_2^1}{a_2^0} \right)^2 \right]
\]

(4.18)

\[
T_2 = \left[ a_o^0 6000 \pi + a_2^0 4286 \frac{\pi}{2} + a_4^0 1114 \frac{\pi}{2} \right. + \left. \left( a_2^3 \cdot 5143 \frac{\pi}{2} + a_4^3 \cdot 6429 \frac{\pi}{2} \right) - \frac{9 \pi}{200} \left( \frac{b_2^1}{a_2^0} \right)^2 \right]
\]

\[
R = \left[ b_2^3 \cdot 5143 \frac{\pi}{2} + b_4^3 \cdot 6429 \frac{\pi}{2} - \frac{9 \pi}{200} \frac{a_2^1 b_2^1}{a_2^0} \right]
\]

or

\[
\frac{H_2}{A} T_1 = 1.4871 \times 10^{-4}
\]

\[
\frac{H_2}{A} T_2 = 0.9159 \times 10^{-4}
\]

\[
R = 0.0672 \times 10^{-4}
\]

(4.19)

\[
\frac{H_2}{A} = 1.394 \times 10^{-4}
\]
The wobble equations become

\[
\ddot{m}_t + (\dot{\alpha}_e - \frac{T_a}{H} \dot{\alpha}_r) \dot{m}_d = -\frac{\alpha_r}{H} R \dot{m}_t,
\]

\[
\ddot{m}_d - (\dot{\alpha}_e - \frac{T_d}{H} \dot{\alpha}_r) \dot{m}_t = -\frac{\alpha_r}{H} R \dot{m}_d
\]

where

\[
H = \frac{\zeta - \beta}{\alpha} = \frac{\zeta - \beta}{\zeta}
\]

Equations 4.20 may be uncoupled to yield

\[
\ddot{m}_t + \left( \dot{\alpha}_e - \frac{T_a}{H} \dot{\alpha}_r \right) \left( \dot{\alpha}_e - \frac{T_t}{H} \dot{\alpha}_r \right) - \frac{\alpha_r}{H^3} R^3 \dot{m}_t = 0
\]

\[
\ddot{m}_d + \left( \dot{\alpha}_e - \frac{T_d}{H} \dot{\alpha}_r \right) \left( \dot{\alpha}_e - \frac{T_d}{H} \dot{\alpha}_r \right) - \frac{\alpha_r}{H^3} R^3 \dot{m}_d = 0
\]

and the frequency is

\[
\omega = \alpha_e \left[ 1 - \frac{\Omega (T_1 + T_3)}{\alpha_e} - \frac{R^3}{\alpha_e} (R^3 - T_1 T_3) \right]^{1/2}
\]

or to a first approximation

\[
\omega = \alpha_e - \left( \frac{T_1 + T_3}{\omega} \right) \Omega
\]

which gives for the wobble period of an oceanless Earth

\[
T_e = 407 \text{ days}
\]

reducing the increase in period to 30 days. The three-day difference between this and Haubrich and Munk's (1959) result is attributable to the different ocean functions used. Since the result is appreciably different from that obtained using global oceans, the importance of continentality is readily apparent.
Because $T_1$ is greater than $T_2$ the excitation is not symmetric and an elliptical pole path results.

v) The Complete Solution of the Pole Tide

The treatment in section 4.14 was an over-simplification in that the tide heights did not allow for self-attraction and loading and the consequent diminishing of the products of inertia of the ocean tide was not accounted for. It is not satisfactory to allow for load yielding, as it affects the product of inertia, simply by calculating the product of inertia of an Earth tide of amplitude

$$h_a \frac{3P_0}{5F} \Xi_a$$

and density contrast $\rho_c$, as Haubrich and Munk (1959) suggest. This is a misuse of the Love number method, as can be easily seen by noting that using this argument the potential of the solid Earth as a result of the ocean tide would be

$$h_a \frac{3P_0}{5F} \frac{3\rho_0}{5P} \Xi_a$$

But the deformation potential is actually

$$k_a \frac{3\rho_0}{5F} g \Xi_a$$

and certainly

$$k_a \neq h_a \frac{3(\rho - \rho_0)}{5P}$$
It was shown in Chapter 3 that the deformation potential of the Earth in response to a prescribed potential can be written in a manner analogous to the equations 2.74 for the tide heights provided the response of the oceans is equilibrium. The deformation potential is

\[ U = \sum_{jk} P^k_j (u^k_j \cos k\lambda + v^k_j \sin k\lambda) \]  

(4.30)

In this chapter, the K matrices will account for only the perturbed potential due to the oceans. In contrast to chapter 3, the direct perturbed potential of the solid Earth in response to the prescribed potential is not included in the K matrices because here the K matrices are to give the perturbed potential due to the oceans only. The only terms in 4.31 which are the result of the inertia products, \( c_{13} \), \( c_{23} \) are the \( u^1_j \) terms so that 4.31 may be written

\[ V = \sum_{jm} P^m_j \left[ K_1(m,j,k) u^k_j \cos m\lambda + K_3(m,j,k) u^k_j \sin m\lambda \right. \]

\[ \left. + K_{57}(m,j,k) u^k_j \cos m\lambda + K_{77}(m,j,k) u^k_j \sin m\lambda \right] \]  

(4.32)

From equations 4.9 for the potential of the products of inertia \( c_{13}, c_{23} \) we have

\[ \sum_{jk} K_1(2,1,j,k) u^k_j + K_3(2,1,j,k) v^k_j = -\frac{2G}{a^2} c_{13} \]  

(4.33)

\[ \sum_{jk} K_{57}(2,1,j,k) u^k_j + K_{77}(2,1,j,k) v^k_j = -\frac{2G}{a^2} c_{23} \]
The pole tide has only degree-2, order-1 terms

\[ U_2' = -\frac{\Omega^2 \alpha^2}{3} \ m_1, \quad V_2' = -\frac{\Omega^2 \alpha^2}{3} \ m_2 \]  

(4.34)

so we have

\[ C_{13} = \frac{\Omega^2 \alpha^5}{3G} \left[ k_1(2,1,2) \ m_1 + k_3(2,1,2) \ m_2 \right] \]  

(4.35)

\[ C_{23} = \frac{\Omega^2 \alpha^5}{3G} \left[ k_5(2,1,2) \ m_1 + k_7(2,1,2) \ m_2 \right] \]

i.e.

\[ C_{13} = A(T_1 \ m_1 + R \ m_2) \]

\[ C_{23} = A(T_2 \ m_2 + R \ m_1) \]

(4.36)

with

\[ T_1 = k_1(2,1,2) \ \frac{\Omega^2 \alpha^5}{3G} \]

\[ T_2 = k_5(2,1,2) \ \frac{\Omega^2 \alpha^5}{3G} \]

\[ R = k_3(2,1,2) \ \frac{\Omega^2 \alpha^5}{3G} \]

(4.37)

Following the previous section this leads to

\[ T_e = 411 \text{ days} \]

(4.38)

so that the oceans increase the period by 26 days.

The results and weaknesses of the three solutions are presented in Table 4.2. The non-globality of the oceans has the greatest influence, reducing the lengthening by 13 days. The slight adjustment of 4 days as a result of allowing for load yielding is the resultant of (a) an increase in the pole tide amplitude by loading and particularly self-
attraction by a factor of about .84/.69 and (b) a reduction in this tide's ability to raise products of inertia by a factor of about .69.

As reported earlier, the actual period for an oceanless Earth is about 392 days, according to Jeffreys and Vicente (1957). Presumably the discrepancy between 392 and 411 is to be accounted for by taking into account the departure of the pole tide from equilibrium, i.e. by a fully dynamical treatment.

<table>
<thead>
<tr>
<th>Method</th>
<th>Lengthening of period (days)</th>
<th>Approximations</th>
</tr>
</thead>
<tbody>
<tr>
<td>section 4.iii</td>
<td>43</td>
<td>does not allow for loading, self-attraction or continentality</td>
</tr>
<tr>
<td>section 4.iv</td>
<td>30</td>
<td>does not allow for self-attraction or loading</td>
</tr>
<tr>
<td>section 4.v</td>
<td>26</td>
<td>does not allow for non-equilibrium response of oceans</td>
</tr>
</tbody>
</table>

Table A.2 The results of the three successive approximations to the pole tide / Chandler wobble problem.
vi) The Ellipticity of the Pole Path

The path of the pole at the Chandler frequency is not circular due in part to the distribution of the oceans, and in part to the inertia ellipsoid of the solid Earth being slightly triaxial.

When a triaxial body is rotating nearly about its axis of greatest moment and wobbling slightly, the major axis of the elliptical pole path points toward the principal axis of intermediate moment of inertia. This is simply because the wobble energy is split equally between the two equatorial axes and the axis of least moment of inertia must have a greater amplitude of rotation. The axis of least moment of inertia for the Earth is in the Indian ocean which puts the intermediate axis at about 10 degrees East longitude. Haubrich and Munk (1959) place the major axis of the pole path at 107 degrees East longitude apparently by confusing the greatest equatorial radius and the greatest equatorial axis of inertia.

If $M_1$ and $M_2$ are the amplitudes of the polar motions then their magnitudes will be in the ratio of their excitations

$$\frac{M_2}{M_1} = \frac{\Delta e - T_1 \Delta \omega}{\Delta e - T_2 \Delta \omega}$$

$$\epsilon = 1 - \frac{M_2}{M_1}$$

(4.39)

$$\epsilon = 0.4$$
Writing the equations of polar motion as

\[ m_1 = M_1 \cos(\alpha t) \]
\[ m_2 = M_2 \sin(\alpha t + \beta) \]  \hspace{1cm} (4.40)

where \( \beta \) is a phase angle and \( M_1, M_2 \) are the amplitudes of the polar motion. By substituting in 4.20 we have

\[ \tan \beta = -\frac{P_i}{R_{ij}} = 0.04 \ll 1 \]  \hspace{1cm} (4.41)

Eliminating the time in 4.40 and using 4.41 we have

\[ \left( \frac{m_1}{M_1} \right)^2 + \frac{\alpha \beta}{M_1 M_2} \left( \frac{m_2}{M_2} \right)^2 = 1 \]  \hspace{1cm} (4.42)

which is an ellipse oriented at an angle \( \lambda \) such that

\[ \tan 2\lambda = \frac{\alpha \beta}{(\frac{M_2}{M_1} - \frac{M_1}{M_2})} \]  \hspace{1cm} (4.43)
\[ \lambda = 0.05 \]

The oceans thus contribute to the ellipticity of the pole path, whose major axis is aligned closely to the Greenwich meridian.
THE GRAVIMETRIC FACTOR

i) Introduction

The gravimetric factor for a prescribed non-load potential was given in section 1.iii. Farrell (1970) suggests that the gravimetric factor for a load potential is the same as that for a non-load potential with load Love numbers replacing non-load Love numbers. Slichter (1972) makes a similar statement and compounds his error by using load Love numbers of correct absolute value but wrong sign. In section 5.ii the proper expression for the load gravimetric factor is derived. This is then used, in conjunction with eq. 2.26 for the global effective tide height, to arrive at a gravimetric factor for a model with global oceans. In section 5.iii the matrices which govern the fluctuations in gravity will be derived and used to examine the gravity tides associated with several long period tides. Slichter (1972) has shown that south polar stations are very little influenced by the gravity tide from the long period ocean tides. By examining various tidal patterns it may be possible to find other areas in the Earth for which the gravity tide correction for a given tide is small.

ii) The Gravimetric Factor For a Load Potential

After yielding, the potential due to a prescribed load
potential $U'$ is

$$
\sum_n (1 + k_n^2) U_n' \tag{5.1}
$$

at the surface and

$$
\sum_n \left( \frac{a}{r} \right)^{n+l} (1 + k_n^2) U_n' \tag{5.2}
$$

outside the surface. By differentiating with respect to $r$ we obtain the perturbation in gravity at the surface:

$$
- \sum_n \frac{n+l}{a} (1 + k_n^2) U_n' \tag{5.3}
$$

There is an additional contribution to the gravity tide which arises from the vertical deflections of the Earth's surface:

$$
\frac{\partial}{\partial r} (V_0) \xi'' \tag{5.4}
$$

where $V_0$ is the undisturbed potential of the Earth and

$$
\xi'' = \sum_n h_n U_n' / g \tag{5.5}
$$

$$
\frac{\partial V_0}{\partial r} = \frac{2 g}{a}.
$$

So the total fluctuation in gravity is

$$
\left( \frac{2 h_n}{a} - \frac{n+l(1+k_n^2)}{a} \right) U_n' + \frac{\delta_n (n+l)}{a} U_n' \tag{5.6}
$$
where the load gravimetric factor is:

\[ \delta_n = 1 + \kappa_n - \frac{2}{n+1} \]  

(5.7)

In the case of equilibrium tides in global oceans we have from 2.26

\[ \zeta_n = J_n \text{eff} \frac{U_n}{a} \]  

(5.8)

and the load potential of this tide is

\[ U_n' = \frac{3\rho_a}{(2n+1)\eta} J_n \text{eff} U_n \]  

(5.9)

so that the total gravity tide is

\[ \Delta g = -\delta_n \text{eff} \frac{nU_n}{a} \]  

(5.10)

and the global effective gravimetric factor is

\[ \delta_n \text{eff} = \delta_n + \delta_n(n+1) \frac{3\rho_a}{n(2n+1)\eta} J_n \text{eff} \]  

(5.11)

Actually these gravimetric factors are to be used with measurements made on the solid Earth surface, so equation 5.11 could not, strictly speaking, apply to global oceans but only to oceans which were so extensive the tidal pattern was not greatly altered by contintality.

Numerical values of the load gravimetric factor and the global effective gravimetric factor are given in Table 5.1.
.iii) Gravity Tides on an Earth With Non-Global Oceans

As in the earlier chapters, we would like to express the gravity tide over the surface of the Earth in a surface harmonic expansion whose coefficients would be given by

\[
g_{l}^{m} = G_{l}(\ell, m, j, k) \frac{U_{j}^{k}}{a} + G_{3}(\ell, m, j, k) \frac{V_{j}^{k}}{a}
\]

\[
g_{a}^{m} = G_{5}(\ell, m, j, k) \frac{U_{j}^{k}}{a} + G_{7}(\ell, m, j, k) \frac{V_{j}^{k}}{a}
\]  (5.12)

Considering only non-load potentials and using 2.66 for the tide height and 2.1 for the load potential of this tide we have for the radial deflection of the solid Earth due only to the ocean tide

\[
g_{l}^{m} = h_{l} \frac{3\beta_{1}}{(2\pi)^{2}} \left[ L_{1}(\ell, m, j, k) U_{j}^{k} + L_{3}(\ell, m, j, k) V_{j}^{k} \right]
\]

\[
g_{a}^{m} = h_{l} \frac{3\beta_{1}}{(2\pi)^{2}} \left[ L_{3}(\ell, m, j, k) U_{j}^{k} + L_{7}(\ell, m, j, k) V_{j}^{k} \right]
\]  (5.13)

The perturbation in gravity as a result of this radial deflection of the solid surface is

\[
g_{l}^{m} = \frac{2a}{a} g_{l}^{m} \quad \Rightarrow \quad g_{a}^{m} = \frac{2a}{a} g_{a}^{m}
\]  (5.14)

and similarly for the other coefficients.

The perturbed potential of the Earth due to the ocean tide is given by 3.13 and 3.14 with the direct solid Earth perturbed potential removed:

\[
q_{l}^{m} = \sum_{j, k} k_{l}(\ell, m, j, k) U_{j}^{k} + k_{3}(\ell, m, j, k) V_{j}^{k}
\]

\[
r_{l}^{m} = \sum_{j, k} k_{5}(\ell, m, j, k) U_{j}^{k} + k_{7}(\ell, m, j, k) V_{j}^{k}
\]  (5.15)
and the perturbation in gravity is

\[ g_{1e} = -\frac{(e+1)}{a} g_{e}^m, \quad g_{2e} = -\frac{(e+1)}{a} r_{e}^m \]  \hspace{1cm} (5.16)

and similarly for the other coefficients. The total contribution of the ocean tide to the gravity tide is

\[ g_{i}^{m} = \left[ \frac{2}{a} \frac{h_{e}^{3} L_{i}(l_{e}, m, j_{i}) + (e+1) k_{i}(l_{e}, m, j_{i})}{a} \right] + \left[ \frac{2}{a} \frac{h_{e}^{3} L_{3}(l_{e}, m, j_{i}) + (e+1) k_{3}(l_{e}, m, j_{i})}{a} \right] V_{j}^{k} \]  \hspace{1cm} (5.17)

The G matrices are

\[ G_{1}(l, m, j_{i}) = \left[ 2 \frac{h_{e}^{3} L_{1}(l_{e}, m, j_{i}) + (e+1) k_{1}(l_{e}, m, j_{i})}{a} \right] \] 
\[ G_{3}(l, m, j_{i}) = \left[ 2 \frac{h_{e}^{3} L_{3}(l_{e}, m, j_{i}) + (e+1) k_{3}(l_{e}, m, j_{i})}{a} \right] \] 
\[ G_{5}(l, m, j_{i}) = \left[ 2 \frac{h_{e}^{3} L_{5}(l_{e}, m, j_{i}) + (e+1) k_{5}(l_{e}, m, j_{i})}{a} \right] \] 
\[ G_{7}(l, m, j_{i}) = \left[ 2 \frac{h_{e}^{3} L_{7}(l_{e}, m, j_{i}) + (e+1) k_{7}(l_{e}, m, j_{i})}{a} \right] \]
<table>
<thead>
<tr>
<th>DEGREE</th>
<th>LOAD 1+K-2H/(N+1)</th>
<th>GRAVIMETRIC FACTOR</th>
<th>GLOBAL EFFECTIVE FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.290</td>
<td>1.618</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.361</td>
<td>1.349</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.333</td>
<td>1.200</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.291</td>
<td>1.132</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.260</td>
<td>1.097</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.239</td>
<td>1.077</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.224</td>
<td>1.063</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.212</td>
<td>1.054</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.202</td>
<td>1.047</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.193</td>
<td>1.042</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.185</td>
<td>1.037</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1.178</td>
<td>1.034</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1.171</td>
<td>1.030</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>1.165</td>
<td>1.028</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>1.160</td>
<td>1.026</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1.156</td>
<td>1.025</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>1.151</td>
<td>1.023</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>1.146</td>
<td>1.021</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>1.143</td>
<td>1.019</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.139</td>
<td>1.019</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>1.135</td>
<td>1.018</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>1.132</td>
<td>1.017</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>1.128</td>
<td>1.016</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>1.125</td>
<td>1.015</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1.123</td>
<td>1.014</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5.1**

The Gravimetric factor for a load potential and the global effective gravimetric factor, the Love numbers are taken from Longman (1963).
SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

i) Summary

In most textbooks and published research papers on the subject, the representation of equilibrium tide heights is by the equation

\[ \bar{z} = g \left( 1 + k_2 - h_2 \right) \frac{U_0}{g} \]  \hspace{1cm} (6.1)

Munk and MacDonald (1960, p. 36) gave a formula modifying 6.1 so that the ocean feedback on itself, directly and through load yielding, was allowed for. The derivation presented by Munk and MacDonald was, however, circuitous and the actual feedback processes occurring were not transparent in the derivation. Their equation was therefore suspect and the initial purpose of this research was to either verify or refute their equation and in addition to solve for the tide height, something which Munk and MacDonald did not attempt.

Their equation was verified, (apart from a necessary correction for mass conservation which was not clearly incorporated by them), but obtained by a more comprehensible derivation, and solved so that the coefficients in the surface harmonic expansion of a tide due to any prescribed potential could be quickly obtained (Section 2.v).
the axial spin acceleration.

Much of the phase lag and amplitude discrepancy of the gravity tide observed at continental stations is due to the non-equilibrium tides in the global ocean. When a more complete knowledge of the tide in the open ocean is available it will be possible to remove the oceanic contribution to the observed gravity tide and thus obtain a solid Earth residual which will yield an accurate value, particularly of the solid-Earth phase lag, so that the dissipativeness of the solid-Earth at tidal frequencies may be obtained. This has already been attempted by Farrell (1972b). Although he does not mention the solid-Earth phase lag, his results indicate that the specific dissipation function \( Q \) is about 100 whereas without the ocean correction the midcontinental data yield a \( Q \) of about 30. Here \( Q \) is given by

\[
\frac{1}{Q} = \frac{1}{2 \pi} \int \frac{dE}{dt} \, dt \tag{6.2}
\]

where \( E \) is the peak energy stored in one cycle of the vibration under study, and \( \frac{dE}{dt} \) the rate at which energy is dissipated.

O'Hora (1973) discusses the systematic error in P.Z.T.* observations arising from the neglect of the local deflection of the vertical caused by neighbouring oceans. Although the global ocean tide has a relatively small influence on the local deflection of the vertical it may be useful to separate this effect from the P.Z.T. observations, once reliable cotidal charts are available.

* photographic zenith tube
It was found that the matrix method developed in Chapter 2 could equally well describe the perturbed potential and gravity tide on a non-spherically-symmetric Earth. Explicit formulae were given to convert the tide height matrices into the perturbed potential and gravimetric matrices (Sections 3.i1, 5.iii). Following Kaula (1969) the Love number $k_2$ was mapped as a function of position on an Earth whose oceans responded in an equilibrium fashion to a prescribed potential (Section 3.iii).

The influence of the oceans on the period of the Chandler wobble has not before been so completely investigated, in particular loading and self-attraction have not previously fully been considered. This was done in Chapter 4.

Some confusion about the load gravimetric factor was pointed out and the correct expression derived (Section 5.ii).

ii) Suggestions for Future Research

In calculating the excitation function (Munk and MacDonald, 1960, p. 39) for changes in axial rotation and wobble due to seasonal shifts in air mass and ground water, Munk and MacDonald (1960) and Van Hylckama (1970) have not considered the efficacy of a land-based load in exciting an ocean tide. The appropriate elements of $L_2$ and $L_6$ indicate that a sizable tide may be generated. The effect of neglecting this tide is to overestimate the sectorial shift in mass and underestimate the zonal, and consequently to overestimate the wobble excitation and underestimate
As is well known the sea floor is covered with a layer of sediment which may reach a kilometer in thickness. To my knowledge no one has investigated the effect of a surface layer of very low rigidity on the load Love numbers, which are very sensitive to the surface conditions.

In this thesis it has been demonstrated that self-attraction and loading have significant effects on equilibrium tide theory. It would be reasonable to expect at least as important a consequence to dynamical tide theory as the preliminary results of Hendershott (1972) suggest. In particular the dynamical interaction between the loaded sea floor and the ocean requires investigation. At present no co-tidal chart has been verified and any result obtained using one is subject to question. The secular acceleration of the Earth as a result of the ocean tides has been investigated by Pariyskiy et al (1972) and Kuznetsov (1972) using co-tidal charts derived without allowing for the ocean feedback, and these studies should be repeated with a more reliable chart.
ACKNOWLEDGEMENTS

The author wishes to acknowledge with thanks the assistance received from the following members of the Physics Department of the Memorial University of Newfoundland:

Dr. M. G. Rochester, who suggested the topic and guided the author throughout this research;

Dr. R. D. Murphy, Dr. J. A. Wright and Mr. G. P. H. Pedersen, for much helpful advice in the computer programming;

Mrs. C. Power, for typing the thesis.

Finally I am indebted to the National Research Council of Canada for financial support of this research through Dr. Rochester's operating grant, and to Memorial University of Newfoundland for a University Fellowship.
REFERENCES


Darwin, G. H. Scientific Papers 1, 340-346 (1883; 1907)


Farrell, W. E. Nature 238, 43 (1972b)


Hecker, O. V. K. Preuss. Geod. Inst. NF, No. 32 (1907)

Hendershot, M. C. Geophys. J. 29, 389-402 (1972)

Hopkins, J. J. Geophys. Res. 78, 476-477 (1973)


Kaula, W. M. J. Geophys. Res. 68, 4967-4978 (1963)


Kozai, Y. Bull. Geod. 89, 355-377 (1968)
Kozai, Y. in Trajectories of Artificial Celestial Bodies As Determined From Observations J. Kovalevsky ed; (Springer Verlag), (1965)
Michelson, A. A. Ap. J. 30, 105-138 (1914)
Munk, W. H., and G. J. F. MacDonald. *The Rotation of the Earth*
(Cambridge Univ. Press, 1960)

Newton, R. R. *Geophys. J.* 14, 505-539 (1968)


Pariyskiy, N. N. *Phys. Sol. Earth, No. 2*, 3-12 (1972)


Platzman, G. W. *Mathematical Problems in the Geophysical Sciences*

Proudman, J. *Observatory* 48, 386-388 (1925)

Proudman, J. *Geophys. J.* 3, 244-249 (1960)


Slichter, L. B. in *Nature of the Solid Earth*, ed. E. C. Robertson
(McGraw-Hill), 285-320 (1972)

Smith, D. E., R. Kolenkiewicz and P. J. Dunn, in *The Use of*
*Artificial Satellites for Geodesy*, eds. S. Henriksen, A.

(1925)


Appendix A

Computing the Tide Height Matrices

A computer program has been written to give the matrices $L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8$ of Chapter 2. The matrices have been computed to give the tide height expansion to degree and order 8 in terms of the prescribed potentials. The large size of the matrices prevents, their being entirely displayed here. However, columns 4 and 5 (which determine the tide height due to prescribed potentials of degree-2 and order-0, and degree-2 and order-1, respectively) of $L_1$ and $L_5$ are displayed in Table A.1.

Throughout the computations the associated Legendre functions used were those defined by Jeffreys and Jeffreys (1966, p. 633):

$$P_n^m(\mu) = (-1)^m (n-m)! (1-\mu^2)^{\frac{m}{2}} \frac{d^n}{d\mu^n} (\mu^2-1)^m, \mu = \cos \theta$$

These normalize as

$$\int_0^\pi (P_n^m)^2 \sin \theta d\theta = \frac{2}{2n+1} \frac{(n-m)!(n+m)!}{(n!)^2}.$$  \hspace{1cm} (A.1.1)

The triple product integrals of the associated Legendre functions,

$$\int_0^\pi P_n^m P_r^s P_q^t \sin \theta d\theta,$$

were computed using the formula of Infeld and Hull (1951). They use the fully normalized polynomials

$$P_n^m(\mu) = \left(\frac{2n+1}{\alpha^2}\right) \frac{(n-m)!}{(n+m)!} \frac{1}{2^n n!} (1-\mu^2)^{\frac{m}{2}} \frac{d^n}{d\mu^n} (\mu^2-1)^m$$
so that a normalization factor must be included in equation 2.47.

As the ocean function of Balmino et al (1973) proved to be a better representation than that of Munk and MacDonald (1960) it was decided to use the former in the programming.

The ocean function coefficients of Balmino et al were computed using Legendre functions normalized in yet a third way so that it was necessary to convert these coefficients to the normalization A1.1 by multiplying the degree-\(n\), order-\(m\) coefficient by the factor

\[
\frac{2^n}{(n+m)!(n-m)!} \frac{1}{2^n} n!
\]

The coefficients in both normalizations are displayed in Table 3.1.

The computer program with sufficient comments to clarify the operation is given below.
LEVEL 21 MAIN DATE = 73242 13/06/57

C PROGRAM TO EVALUATE THE TIDE PREDICTION MATRICES

IMPLICIT REAL*8(A-H,O-Z)
REAL*8 INTEG
INTEGER R,P,Q,RR,PP,QQ,EXPO1,EXPO2,EXP,EXP1,EXP2,EXPO
DIMENSION A(90)
DIMENSION X(4,26),AIJ(10,10)
1, AIJI(10,10), AIIJA(10,10), BIJI(10,10), BIJI(10,10), BIJA(10,10),
2, AIJ1(10,10), AIIJA(10,10), BIIJ(10,10), BIIJ(10,10), BIIJ1(10,10),
3, BIIJ1(10,10)
DIMENSION AIJIAI(10,10), AIIJA(10,10)
DIMENSION GIM(10,10), PER(10,10), PAR(10,10), PAR(10,10),
1, PAR(10,10), PAR(10,10), PAR(10,10), PAR(10,10), PAR(10,10)
DIMENSION BET(10,10), GAM(10,10), BETGAM(10,10), GAM(10,10),
C THE LOVE NUMBERS ARE FROM LONGMAN (1963) AND ARE PUNCHED ON 26 CARDS
C 10/CARD, THE FIRST IS H THE SECOND IS THE THIRD IS THE FOURTH K,
READ(5,440)((X(I,J),I=1,4),J=1,26)
440 FORMAT(4F6.3)
C THE OCEAN FUNCTION COEFFICIENTS ARE FROM BALMINO ET AL. (1973) WITH
C THE NORMALIZATION CHANGED TO THAT OF MUNK & MACDONALD (1960) AND
C ARE PUNCHED ON 9 CARDS 10/CARD
READ(5,10)((A(N),N=1,90)
10 FORMAT(10F8.4)
C SET EVERY ELEMENT IN A1, A2, B1, B2 = 0
REAL*8 A1(10,10)/100*0.00/
REAL*8 A2(10,10)/100*0.00/
REAL*8 B1(10,10)/100*0.00/
REAL*8 B2(10,10)/100*0.00/
DO 150, PP=1,4
DO 160, QQ=1,PP
DO 170, RR=1,4
DO 180, LL=1,RR
DO 190, NN=1,9
DO 200, MM=1,NN
P=PP-1
Q=QQ-1
R=RR-1
L=LL-1
N=NN-1
M=MM-1
C CARD 30 TESTS THE REQUIREMENT ON THE AZIMUTHAL INDECES
IF (NOT,(M.EQ.L+QQ OR Q.EQ.L+M OR L.EQ.Q+M)) GO TO 200
C TEST FOR SUM OF ORDERS EQUAL EVEN INTEGER
20, NRP=(N+R+P)/2
N1=N+R+P-2*NRP
IF(N1.NE.0) GO TO 200
C CARD 36 TESTS IF ORDERS SATISFY TRIANGULARITY
IF (NOT,(IABS(R-N).LE.P AND P.LE.R+N)) GO TO 200
C SUBROUTINE ORDER ORDERS THE LEGENDRE FUNCTIONS SO THAT THE
C CRITERIA OF INFELD AND HULL (1951) ARE SATISFIED
C
CALL ORDER(N,M,R,L,P,Q)
C
C SUBROUTINE VAL EVALUATES THE TRIPLE PRODUCT INTEGRALS USING THE
C FORMULA OF INFELD & HULL (1951)
C
CALL VAL(N,M,R,L,P,Q,INTEGR)
C
C SUBROUTINE AZM EVALUATES THE INTEGRAL OVER THE AZIMUTH
C
CALL AZM(M,LL,QQ,ASMCC,ASMSSC,ASMCS)
D=2*NN-1
E=2*RR-1
F=2*PP-1
C
C TERM3 IS A CONVERSION FACTOR FROM THE NORMALIZATION OF INFELD AND
C HULL TO THE NORMALIZATION OF MUNK AND MACDONALD (1960)
C
TERM3=SQRT(2.0*FACT1(NN-NN)*FACT1(NN+MM-2)*FACT1(RR-LL)
1*FACT1(RR+LL-2)*FACT1(RR+LL-2)*FACT1(RR+LL-2)*FACT1(RR+LL-2))
2*FACT1(PP-1)/(FACT1(NN-1)*FACT1(PP-1))
C
C CARDS 97-101 FORM THE 45x45 MATRICES A1, A2, B1, B2
B1(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=B1(RR*(RR-1)/2+LL,PP*(PP-1)/2+
100)+A(NN*(NN-1)/2+MM+45)*INTEG*ASMCC*TERM3
B2(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=B2(RR*(RR-1)/2+LL,PP*(PP-1)/2+
100)+A(NN*(NN-1)/2+MM+45)*INTEG*ASMCC*TERM3
A1(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=A1(RR*(RR-1)/2+LL,PP*(PP-1)/2+
100)+A(NN*(NN-1)/2+MM)*INTEG*ASMCC*TERM3
A2(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=A2(RR*(RR-1)/2+LL,PP*(PP-1)/2+
100)+A(NN*(NN-1)/2+MM)*INTEG*ASMCC*TERM3
C
C CONVERT THE MATRICES A1, B1, A2, B2 TO THE TIDE HEIGHT MATRICES WITH
C
DO 500 N=1, 4
DO 501 M=1, N
DO 502 J=1, 10
I=N*(N-1)/2+M
AIJ(J,I)=A1(I,J)*(1.0+X(2,N)-X(1,N))
BIJ(J,I)=B1(I,J)*(1.0+X(2,N)-X(1,N))
A2IJ(J,I)=A2(I,J)*(1.0+X(2,N)-X(1,N))
B2IJ(J,I)=B2(I,J)*(1.0+X(2,N)-X(1,N))
C
C NO FEEDBACK
DO 500 N=1, 4
DO 501 M=1, N
DO 502 J=1, 10
I=N*(N-1)/2+M
AIJ(J,I)=A1(I,J)*(1.0+X(2,N)-X(1,N))
BIJ(J,I)=B1(I,J)*(1.0+X(2,N)-X(1,N))
A2IJ(J,I)=A2(I,J)*(1.0+X(2,N)-X(1,N))
B2IJ(J,I)=B2(I,J)*(1.0+X(2,N)-X(1,N))
LEVEL 21       MAIN       DATE = 73242       13/06/57

B2JIA(J,I)=B2(I,J)*(1.0+X(4,N)-X(3,N))*0.5585D.00/(2*N-1)

502 CONTINUE
501 CONTINUE
500 CONTINUE

C PERFORM THE MATRIX ALGEBRA OF THE THESIS AND EVALUATE THE
C TIDE HEIGHT MATRICES
C SUBROUTINE DIV INVERTS THE FIRST ARGUMENT AND CALLS IT THE SECOND
C SUBROUTINE PRUD MULTIPLIES THE FIRST TWO AND CALLS IT THE THIRD
C SUBROUTINE TERM CALCULATES THE TIDE PREDICTION MATRICES

CALL DIV(AIJIA,AIJIAI)
CALL DIV(A2JIA,A2JIAI)
CALL PRUD(AIJIAI,BIJIA,BET)
CALL PRUD(A2JIAI,B2JIA,GAM)
CALL PRUD(BET,GAM,BETGAM)
CALL PRUD(GAM,BET,GAMBET)
CALL DIV(BETGAM,BETGIM)
CALL DIV(GAMBET,GIMBET)
CALL TERM(BETGIM,AIJIAI,AIJ,BET,A2JIAI,B2J,BETGAM)
CALL TERM(BETGIM,AIJIAI,AIJ1,BET,A2JIA1,B2JI,BETGAM)
CALL TERM(BETGIM,AIJIAI,BIJ,BET,A2JIAI,A2JI,BETGAM)
CALL TERM(BETGIM,AIJIAI,BIJ1,BET,A2JIA1,A2JI1,BETGAM)
CALL TERM(GIMBET,A2JIAI,B2J,GAM,AIJIAI,BETGAM)
CALL TERM(GIMBET,A2JIAI,B2JI,GAM,AIJIAI,BETGAM)
CALL TERM(GAMBET,A2JIAI,B2J,GAM,AIJIAI,BET)
CALL TERM(GAMBET,A2JIAI,B2JI,GAM,AIJIAI,BET)

DO 509 I=1,10
AIJ(1,I)=0.00
AIJI(1,I)=0.00
BIJ(1,I)=0.00
BIJ1(1,I)=0.00
A2J(1,I)=0.00
A2JI(1,I)=0.00
B2J(1,I)=0.00
B2JI(1,I)=0.00
PAR1(1,I)=0.00
PAR2(1,I)=0.00
PAR3(1,I)=0.00
PAR4(1,I)=0.00
PAR5(1,I)=0.00
PAR6(1,I)=0.00
PAR7(1,I)=0.00
PAR8(1,I)=0.00

509 CONTINUE
WRITE(6,890)
WRITE(6,510)
510 FORMAT('     MATRIX A1(N,M) ')
WRITE(6,520)((AI(I,J),J=1,10),I=1,10)
520 FORMAT(10F8.4)
WRITE(6,530)
530 FORMAT(' MATRIX A2(N,M) ') WRITE(6,520)((A2(I,J),J=1,10),I=1,10)
WRITE(6,540)
540 FORMAT(' MATRIX B1(N,M) ') WRITE(6,520)((B1(I,J),J=1,10),I=1,10)
WRITE(6,550)
550 FORMAT(' MATRIX B2(N,M) ') WRITE(6,520)((B2(I,J),J=1,10),I=1,10)
WRITE(6,560)
620 FORMAT(' MATRIX A1IA') WRITE(6,310)((A1IA(I,J),J=1,10),I=1,10)
WRITE(6,560)
650 FORMAT(' MATRIX B1IA') WRITE(6,310)((B1IA(I,J),J=1,10),I=1,10)
WRITE(6,560)
680 FORMAT(' MATRIX A2IA') WRITE(6,310)((A2IA(I,J),J=1,10),I=1,10)
WRITE(6,560)
710 FORMAT(' MATRIX B2IA') WRITE(6,310)((B2IA(I,J),J=1,10),I=1,10)
WRITE(6,560)
720 FORMAT(' INVERSE MATRIX I-A1IA') WRITE(6,310)((I-A1IA(I,J),J=1,10),I=1,10)
WRITE(6,570)
730 FORMAT(' INVERSE MATRIX I-A2IA') WRITE(6,310)((I-A2IA(I,J),J=1,10),I=1,10)
WRITE(6,570)
750 FORMAT(' MATRIX BET') WRITE(6,310)((BET(I,J),J=1,10),I=1,10)
WRITE(6,570)
760 FORMAT(' MATRIX GAM') WRITE(6,310)((GAM(I,J),J=1,10),I=1,10)
WRITE(6,570)
770 FORMAT(' MATRIX BETGAM') WRITE(6,310)((BETGAM(I,J),J=1,10),I=1,10)
WRITE(6,570)
780 FORMAT(' MATRIX GAMET') WRITE(6,310)((GAMET(I,J),J=1,10),I=1,10)
WRITE(6,570)
790 FORMAT(' INVERSE MATRIX BETGIM') WRITE(6,310)((I-BETGIM(I,J),J=1,10),I=1,10)
WRITE(6,570)
800 FORMAT(' INVERSE MATRIX GIMBET') WRITE(6,310)((I-GIMBET(I,J),J=1,10),I=1,10)
LEVEL 21

WRITE(6, RAG)
WRITE(6, 560)

560 FORMAT(' MATRIX AIJ (L1 WITHOUT FEEDBACK)'))
WRITE(6, 311)((AIJ(I,J), J=1,10), I=1,10)
WRITE(6, 610)

610 FORMAT(' MATRIX AIJI (L2 WITHOUT FEEDBACK)'))
WRITE(6, 311)((AIJI(I,J), J=1,10), I=1,10)
WRITE(6, 670)

630 FORMAT(' MATRIX BIJ (L3 WITHOUT FEEDBACK)'))
WRITE(6, 311)((BIJ(I,J), J=1,10), I=1,10)
WRITE(6, 630)

640 FORMAT(' MATRIX BIJI (L4 WITHOUT FEEDBACK)'))
WRITE(6, 311)((BIJI(I,J), J=1,10), I=1,10)
WRITE(6, 640)

690 FORMAT(' MATRIX B2J (L5 WITHOUT FEEDBACK)'))
WRITE(6, 311)((B2J(I,J), J=1,10), I=1,10)
WRITE(6, 700)

700 FORMAT(' MATRIX B2JI (L6 WITHOUT FEEDBACK)'))
WRITE(6, 311)((B2JI(I,J), J=1,10), I=1,10)
WRITE(6, 890)
WRITE(6, 660)

660 FORMAT(' MATRIX A2J (L7 WITHOUT FEEDBACK)'))
WRITE(6, 311)((A2J(I,J), J=1,10), I=1,10)
WRITE(6, 660)

670 FORMAT(' MATRIX A2JI (L8 WITHOUT FEEDBACK)'))
WRITE(6, 311)((A2JI(I,J), J=1,10), I=1,10)
WRITE(6, 890)
WRITE(6, 660)

810 FORMAT(' MATRIX L1'))
WRITE(6, 311)((PAR1(I,J), J=1,10), I=1,10)
WRITE(6, 820)

820 FORMAT(' MATRIX L2'))
WRITE(6, 311)((PAR2(I,J), J=1,10), I=1,10)
WRITE(6, 830)

830 FORMAT(' MATRIX L3'))
WRITE(6, 311)((PAR3(I,J), J=1,10), I=1,10)
WRITE(6, 890)
WRITE(6, 840)

840 FORMAT(' MATRIX L4'))
WRITE(6, 311)((PAR4(I,J), J=1,10), I=1,10)
WRITE(6, 850)

850 FORMAT(' MATRIX L5'))
WRITE(6, 311)((PAR5(I,J), J=1,10), I=1,10)
WRITE(6, 860)

860 FORMAT(' MATRIX L6'))
WRITE(6, 311)((PAR6(I,J), J=1,10), I=1,10)
WRITE(6, 860)
SUBROUTINE ORDER(N, M, R, L, P, Q)
SUBROUTINE ORDER ORDERS THE LEGENDRE FUNCTIONS SO THAT THE
CRITERIA OF INFELD AND HULL (1951) ARE SATISFIED
INTEGER R, L, P, Q
IF(N .EQ. 0) GO TO 7
IF(R .EQ. 0) GO TO 11
IF(P .EQ. 0) GO TO 12
GO TO 13
11 IR = R
IL = L
R = N
L = M
N = IR
M = IL
GO TO 7
12 IP = P
IQ = Q
Q = M
P = N
M = IQ
N = IP
GO TO 7
13 CONTINUE
IF(P .GT. R) GO TO 20
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
20 CONTINUE
IF(R .GT. N) GO TO 30
IR=R
IL=L
R=N
L=M
N=IR
M=IL
30 CONTINUE
IF(P .GT. R) GO TO 40
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
40 CONTINUE
IF(M .EQ. L+Q .AND. N .LE. R) GO TO 51
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
IF(M .EQ. L+Q .AND. N .LE. R) GO TO 51
IN=N
IM=H
N=R
M=L
R=IN
L=IM
IF(M .EQ. L+Q .AND. N .LE. R) GO TO 51
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
IF(M .EQ. L+Q .AND. N .LE. R) GO TO 51
IN=N
IM=H
N=R
M=L
R=IN
L=IM
IF(M .EQ. L+Q .AND. N .LE. R) GO TO 51
IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
43 CONTINUE
IF(P.GT.R) GO TO 5
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
5 CONTINUE
IF(R.GT.N) GO TO 6
IR=R
IL=L
R=N
L=M
N=IR
M=IL
6 CONTINUE
IF(P.GT.R) GO TO 7
IP=P
IQ=Q
P=R
Q=L
R=IP
L=IQ
7 CONTINUE
IF(M.EQ.L+Q) GO TO 51
IF(L.EQ.Q+M) GO TO 49
IF(Q.EQ.L+M) GO TO 50
49 Q=Q
GO TO 51
50 L=L
51 CONTINUE
RETURN
END
SUBROUTINE VAL(N,M,R,L,P,Q,S)

C SUBROUTINE VAL CALCULATES THE TRIPLE PRODUCT INTEGRALS USING THE
C FORMULA OF INFIELD & HULL (1951) AND RETURNS IT AS S.

IMPLICIT REAL*8(A-H,O-Z)
INTEGER P,Q,R,EXP,EXP1,EXP2,EXPO1,EXPO2

C CARDS 45-79 EVALUATE THE INTEGRAL.

KK=N=M+1
IF(KK=1)30,30,60
30 EXP=(P+R+N)/2+(IABS(Q))
EXP1=EXP/2
EXP2=EXP-EXP1*2
IF(EXP2)80,40,80
40 SUM=(FACT1(P+Q)*FACT1(R+N-Q))/(FACT1(N-M)*FACT1(P-Q))
1 FACT1(R-N+Q))
GO TO 110
50 SUM=-FACT1(P+Q)*FACT1(R+N-Q))/(FACT1(N-M)*FACT1(P-Q))
1 FACT1(R-N+Q))
GO TO 110
60 SUM=0.0
DO 100 I=1,KK
J=I-1
EXP0=(P+R+N)/2+(IABS(Q)+J)
EXP01=EXP0/2
EXP02=EXP0=EXP1*2
IF(EXP02)80,70,80
70 TERM4=(FACT1(P+Q+J)*FACT1(R+N-Q-J))/(FACT1(N-M-J)*FACT1(J))
1 FACT1(P-Q-J)*FACT1(R-N+Q+J))
GO TO 90
80 TERM4=(FACT1(P+Q+J)*FACT1(R+N-Q-J))/(FACT1(N-M-J)*FACT1(J))
1 FACT1(P-Q-J)*FACT1(R-N+Q+J))
90 SUM=SUM+TERM4
100 CONTINUE
110 D=2*N+1
E=2*R+1
F=2*P+1
TERM1=FACT2(P+R-N-1)*DSQRT(D*E*F))/(FACT2(P+N-R)
1 FACT2(R+N-P)*FACT2(N+P+1))
TERM2=DSQRT((FACT1(N-M)*FACT1(N-M)*FACT1(R-L)*FACT1(P-Q))/
1 (FACT1(R+L)*FACT1(P+Q)*2,F))
120 S=TERM1*TERM2*SUM

C CARDS 35-52 TEST IF A CHANGE IN SIGN OF ONE OF THE AZIMUTHAL INDECES
C WAS MADE AND CORRECTS S.

IF(Q.LE.0) GO TO 62.
IF(L.LE.0) GO TO 63.
GO TO 61

62 EXP=IABS(Q)
EXP1=EXP/2
EXP2=EXP-EXP1*2
LEVEL 21                VAL                           DATE = 73242            13/06/57

IF(EXP2,=0) GO TO 51
S=S
51 CONTINUE
RETURN,
63 EXP=IABS(L)
EXP1=EXP/2
EXP2=EXP*EXP1*2
IF(EXP2,=0) GO TO 61
S=S
61 CONTINUE
RETURN
END

LEVEL 21                AZM                           DATE = 73242            13/06/57

SUBROUTINE AZM (MM,LL,QQ,AZMCC,AZMSSC,AZMCSS,AZMSCS)
SUBROUTINE AZM EVALUATES THE INTEGRAL OVER THE AZIMUTH, THESE
WERE FOUND ANALYTICALLY FOR ALL 8 POSSIBLE CASES CARD S 5-19 TEST
WHICH CASE THE AZIMUTHAL INDICES FIT AND TRANSFERS THE COMPUTATION,
THE VALUE OF THE AZIMUTHAL INTEGRAL RETURNED IS NORMALIZED
IMPLICIT REAL*8(A-H,O-Z)
INTEGER QQ
150 IF(MM+LL+QQ,=3) GO TO 230
IF(MM+LL,=2,OR,MM+QQ,=2,OR,LL+QQ,=2) GO TO 240
IF(QQ,=1) GO TO 250
IF(LL,=1) GO TO 260
IF(MM,=1) GO TO 270
IF(QQ,=1,OR,LL+MM,=1) GO TO 280
IF(LL,=1,OR,QQ+MM,=1) GO TO 290
IF(MM,=1,OR,QQ+LL,=1) GO TO 300
230 AZMCC=1,D0
AZMSSC=0,D0
AZMCSS=0,D0
AZMSCS=0,D0
GO TO 156
240 AZMCC=0,D0
AZMSSC=0,D0
AZMCSS=0,D0
AZMSCS=0,D0
GO TO 156.
FUNCTION FACT1(MM)
SUBPROGRAM FACT1 EVALUATES THE FACTORIAL OF MM
IMPLICIT REAL*8(A-H,O-Z)
FACT1=1.DO
IF(MM.GT.1) GO TO 31
RETURN
31 CONTINUE
DO 32 K=1,MM
AK=K
FACT1=FACT1*AK
32 CONTINUE
RETURN
END
FUNCTION SUBPROGRAM FOR FACTORIAL2
FUNCTION FACT2(NN)
SUBPROGRAM FACT2 EVALUATES THE DOUBLE FACTORIAL OF NN
IMPLICIT REAL*8(A-H,O-Z)
FACT2=1.0D0
IF(NN.LE.1)GO TO 20
IF(NN.LE.2)GO TO 21
NON=NN/2
NON0=2*NON
IF(NON-NN)25,22,25
25 CONTINUE
   DO 26 K=1,NN,2
   AK=K
   FACT2=FACT2*AK
26 CONTINUE
   RETURN
22 CONTINUE
   DO 23 K=2,NN,2
   AK=K
   FACT2=FACT2*AK
23 CONTINUE
   RETURN
20 FACT2=1.0D0
   RETURN
21 FACT2=2.0D0
   RETURN
   END

SUBROUTINE PROD(AA, BB, CC)
SUBROUTINE PROD Multiplies AA and BB and returns CC
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION AA(10,10), BB(10,10), CC(10,10)
DO 1 I=1,10
   DO 2 J=1,10
      CC(I,J)=0.0D0
   2 CONTINUE
   DO 3 K=1,10
      CC(I,J)=CC(I,J)+AA(I,K)*BB(K,J)
3 CONTINUE
2 CONTINUE
1 CONTINUE
RETURN
END
SUBROUTINE DIV(B,D)

SUBROUTINE DIV USES GAUSSIAN ELIMINATION WITH MAXIMUM PIVOT TO INVERT
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION B(10,10),D(10,10),A(10,20),C(10,20)

FORM THE AUGMENTED MATRIX
DO 10 J=1,20
DO 20 I=1,10
IF(J.LE.10) GO TO 30
IF(I+10.EQ.J) GO TO 40
A(I,J)=0,0,0
GO TO 20
40 A(I,J)=1,0,0
GO TO 20.
30 IF(I.EQ.J) GO TO 31
A(I,J)=-B(I,J)
GO TO 20
31 A(I,J)=1,0,0=B(I,J)
10 CONTINUE

SEARCH FOR THE LARGEST ELEMENT IN ROW L
DO 50 L=1,9
K=1
DO 60 J=1,10
IF(DABS(A(L,K)),GT,DABS(A(L,J))) GO TO 60
K=J
60 CONTINUE
IF(K.EQ.L) GO TO 70
DO 80 J=1,10
C(J,K)=A(J,K)
C(J,K+10)=A(J,K+10)
A(J,K)=A(J,L)
A(J,K+10)=A(J,L+10)
A(J,L)=C(J,K)
A(J,L+10)=C(J,K+10)
80 CONTINUE
70 L1=L+1
DO 90 K=L1,10
S=A(K,L)
DO 100 J=L,20
A(K,J)=A(K,J)-(S/A(L,L))*A(L,J)
100 CONTINUE
90 CONTINUE
50 CONTINUE
DO 110 J=1,10
D(10,J)=A(10,10+J)/A(10,10)
DO 120 I=1,9
SUM=0,0,0,0
DO 130 K1=1,1
SUBROUTINE TERM(A,B,C,D,E,F,G)

SUBROUTINE TERM DOES THE MATRIX ALGEBRA AND RETURNS THE TIDE PREDICTION MATRICES

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(10,10),B(10,10),C(10,10),D(10,10),E(10,10),F(10,10),G(10,10),T1(10,10),S1(10,10),T2(10,10),T3(10,10),S2(10,10)

CALL PROD(A,B,T1)
CALL PROD(T1,C,S1)
CALL PROD(A,D,T2)
CALL PROD(T2,E,T3)
CALL PROD(T3,F,S2)
DO 1 I=1,10
DO 2 J=1,10
G(I,J)=S1(I,J)+S2(I,J)
2 CONTINUE
1 CONTINUE
RETURN
END

K=K1-1
SUM=SUM+D(10-K,J)*A(10-I,10-K)
130 CONTINUE
D(10-I,J)=(A(10-I,10+J)-SUM)/A(10+I,10-I)
120 CONTINUE
110 CONTINUE
RETURN
END
The Ocean Function

The ocean function has been computed by several authors (Munk and MacDonald, 1960; Lee and Kaula, 1967; Balmino et al, 1973). The expansion of Lee and Kaula (1967) has been shown to yield unsatisfactory results (Balmino et al, 1973) and will not be discussed further here.

Due to the fact that most land masses are either greater than 45 degrees or less than a few degrees in extent, an expansion to degree and order 3 will represent the ocean continent distribution almost as well as an expansion to degree and order 8. A higher order expansion does have the advantage of describing the coastlines a little better, and when synthesized it gives land value of the ocean function which are closer to zero and ocean values which are closer to 1. For the purposes of this thesis the position and shape of the continents need not be known to very great accuracy but land values must be as close to zero as possible and ocean values as close to 1. The effect of a variation from these is to introduce a spurious tide.

The expansions of Munk and MacDonald (1960) (Fig. B2 a,b) and Balmino et al, 1973) (Fig. B1 a,b) represent the ocean-continent distribution equally well, but the expansion of Munk and MacDonald gives some anomalously low values in land areas and high values in ocean areas, particularly in low latitudes. It is not difficult to trace this to values of the coefficients $b_7^7$, $b_8^8$ which are too great. A glance at Table B.1 confirms this.
<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$a_n^m$</th>
<th>$b_n^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.697</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-0.126</td>
<td>-0.056</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-0.060</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>-0.040</td>
<td>0.051</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.040</td>
<td>0.002</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.045</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.044</td>
<td>-0.032</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>-0.016</td>
<td>-0.069</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-0.024</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>0.036</td>
<td>0.038</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.090</td>
<td>-0.021</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.053</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.014</td>
<td>-0.101</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.109</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.005</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.049</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.028</td>
<td>-0.016</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>0.035</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.011</td>
<td>-0.049</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.022</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.013</td>
<td>0.023</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.019</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.002</td>
<td>-0.033</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.031</td>
<td>0.023</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.021</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.002</td>
<td>-0.013</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.000</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.004</td>
<td>-0.036</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.021</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.009</td>
<td>0.019</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.028</td>
<td>-0.010</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.005</td>
<td>0.022</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.004</td>
<td>0.030</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.001</td>
<td>0.032</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.015</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.006</td>
<td>0.027</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.002</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.008</td>
<td>0.019</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.002</td>
<td>0.018</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.012</td>
<td>-0.002</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0.014</td>
<td>-0.016</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.040</td>
<td>0.020</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.010</td>
<td>-0.021</td>
<td></td>
</tr>
</tbody>
</table>

Table 81 Surface harmonic coefficients of the ocean function.
Figure B.1 a The ocean function of Balmino et al (1973), eastern hemisphere. The boundaries are defined by
Figure B.1b The ocean function of Balmino et al (1973), western hemisphere. The boundaries are defined by $C < 0.5$. 

\[ \begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Figure B.2a The ocean function of Munk & MacDonald (1960), eastern hemisphere. The boundaries are defined by the ocean function of Balmino et al with $C < 0.5$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
<th>160</th>
<th>170</th>
<th>180</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>10</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>20</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>30</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>40</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>60</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>70</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>80</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>90</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>100</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
</tr>
<tr>
<td>110</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
</tr>
<tr>
<td>120</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>130</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>140</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>150</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>160</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>170</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>180</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Figure B.2b The ocean function of Munk & MacDonald (1960), western hemisphere. The boundaries are defined by the ocean function of Balmino et al with $C < 0.5$.
Appendix C

Surface Harmonic Synthesis

A program has been written to compute the tide height expansion due to a prescribed potential.

Following the suggestion of Hopkins, (1973) the values of the associated Legendre functions at grid points separated by 10 degrees of latitude and longitude were computed by recursion relations. For the normalization A 1.1 the recursion relations are

\[ P_{n-1}^{m}(\cos \theta) = \cos \theta P_{n}^{m}(\cos \theta) \]

\[ P_{n+1}^{m}(\cos \theta) = \frac{2n+1}{n+1} \sin \theta P_{n}^{m}(\cos \theta) \]

\[ P_{n+1}^{m}(\cos \theta) = \frac{2n+1}{n+1} \cos \theta P_{n}^{m}(\cos \theta) - \frac{(n+m)(n-m)}{n(n+1)} P_{n-1}^{m}(\cos \theta) \]

The mapping defined by 2.52 is useful in identifying a Legendre function by one index instead of the normal two.

The program, with sufficient comments to clarify the operation, is given below. The program for the ocean function is given, the programs for the tide maps being identical in operation except for a change in data cards.
C PROGRAM TO PLOT THE OCEAN FUNCTION AT EACH GRID POINT OF A 10*10 DEGREE
C PROJECTION
IMPLICIT REAL*8(A,H,O=Z)
REAL*8 OCEAN(36,19)/684*0.D0/
DIMENSION P(45),A(90),MAP(36,19),LONG1(19),LONG2(19),LAT(19)
REAL*8 C/0.17450.01/
C READ THE COEFFICIENTS OF THE OCEAN FUNCTION
READ(5,1)(A(I),I=1,45)
READ(5,1)(A(I),I=46,90)
1 FORMAT(16F7.3)
C COMPUTE THE LEGENDRE FUNCTIONS USING RECURRENCE RELATIONS
DO 30 K=1,19
F=(K-1)*C*10
DO 10 I=1,8
N=I-1
P((N+1)*(N+2)/2+N+2)=P(N*(N+1)/2+N+1)*((2*N+1)*DSIN(F)/(N+1))
P((N+1)*(N+2)/2+N+1)=P(N*(N+1)/2+N+1)*((2*N+1)*DCOS(F)/(N+1))
IF(N. LT. 1) GO TO 10
DO 20 J=1,N
M=J-1
P((N+1)*(N+2)/2+M+1)=P(N*(N+1)/2+M+1)*((2*N+1)*DCOS(F)/(N+1))
I=P(N*(N-1)/2+M+1)*(N+M)*(N=M)/(N*(N+1))
20 CONTINUE
10 CONTINUE
C COMPUTE THE VALUE OF THE OCEAN FUNCTION AT THE GRID POINTS
DO 60 L=1,36
DO 70 I=1,19
X=(L-1)*C*10
M=J-1
INM=(I*(I-1)/2+J)
61 OCEAN(L,K)=OCEAN(L,K)+P(INM)*(A(INM)*DCOS(M*X))
1+A(INM+45)*DSIN(M*X))
60 CONTINUE
70 CONTINUE
60 CONTINUE
30 CONTINUE
DO 180 I=1,19
LONG1(I)=10*(I-1)
LONG2(I)=10*(I+17)
LAT(I)=10*(I-1)
180 CONTINUE
DO 190 I=1,36
DO 200 J=1,19
IF(OCEAN(I,J).GT..5D 00) GO TO 210
MAP(I,J)=11
GO TO 200

LEVEL 21

MAIN

DATE = 73242
13/05/55

210 MAP(I,J)=22
200 CONTINUE
190 CONTINUE
WRITE(6,400)
400 FORMAT('1')
WRITE(6,260)
WRITE(6,260)
WRITE(6,260)
WRITE(6,260)
WRITE(6,220)
260 FORMAT('0')
220 FORMAT(' MAP OF THE OCEAN FUNCTION, 0 DEGREES EAST ')
1' LONGITUDE = 180 DEGREES EAST LONGITUDE ')
WRITE(6,240)(LONG1(I),I=1,19)
240 FORMAT(5X,1915)
DO 250 J=1,19
250 WRITE(6,270) LAT(J),(OCEAN(I,J),I=1,19),LAT(J)
270 FORMAT('0',15,19F5,1,15)
WRITE(6,290)(LONG1(I),I=1,19)
290 FORMAT(5X,1915)
WRITE(6,400)
WRITE(6,260)
WRITE(6,260)
WRITE(6,260)
WRITE(6,260)
WRITE(6,300)
300 FORMAT(' MAP OF THE OCEAN FUNCTION 180 DEGREES EAST ')
1' LONGITUDE = 360 DEGREES EAST LONGITUDE ')
WRITE(6,310)(LONG2(I),I=1,19)
310 FORMAT(5X,1915)
DO 320 J=1,19
320 WRITE(6,330) LAT(J),(OCEAN(I,J),I=19,36),OCEAN(1,J),LAT(J)
330 FORMAT('0',15,19F5,1,15)
WRITE(6,350)(LONG2(I),I=1,19)
350 FORMAT(5X,1915)
WRITE(6,360)
360 FORMAT('1', 'WORLD MAP ')
WRITE(6,370)((MAP(I,J),I=1,36),J=1,19)
370 FORMAT(1X,36I2)
WRITE(6,260)
STOP
END
Appendix D

Derivation of the Equation for the Potential of a Surface Layer

Consider a layer of mass of density $\rho$ and variable thickness $\xi$ on the surface of a sphere of radius $R$. Let the mass point be at $(r, \theta, \lambda)$ and the field point at $(r', \theta', \lambda')$ (figure D.1), separated by a distance $\rho$:

![Diagram](image)

Figure D.1

The potential of the mass in the layer is

$$V(r, \theta, \lambda) = \frac{G}{\rho} \int dm$$

$$= G \rho \int_s \xi(\theta', \lambda') \frac{dS'}{\rho}$$

But

$$\rho^{-1} = \frac{1}{\rho} \sum_{n=0}^{\infty} p_n(\cos \psi) \left( \frac{R}{r} \right)^n$$

(D.2)

and

$$\xi(\theta', \lambda') = \sum_{k, \ell} p_k^\ell \left( \frac{\ell}{k} \cos \lambda' + 2 \frac{\ell}{k} \sin \lambda' \right)$$

(D.3)
Substituting equations D.2 and D.3 in D.1, we have

\[ V = G_p \frac{R^2}{r} \sum_{k \leq n} \sum_{m} \int P^k_m \left( y^k_m \cos \xi + z^k_m \sin \xi \right) P_n(\cos \psi) (R/r)^m \sin \theta' \, d\theta' \, d\lambda' \]  

(D.3)

By the addition theorem we have

\[ P_n(\cos \psi) = P_n(\cos \theta) P_n(\cos \theta') \]

\[ + \sum_{m=1}^{n} \frac{n!}{(n+m)(n-m)!} P^m_n(\cos \theta) P^m_n(\cos \theta') \cos^m(\theta-\theta') \]

(D.4)

Substituting equation D.4 in D.3 and using the orthonormality properties of the Legendre functions, we have given in Appendix A, we get

\[ V(r, \theta, \lambda) = 4\pi G_p \frac{R}{r} \sum_{n} P_n^m \left( y^m_n \cos m \lambda + z^m_n \sin m \lambda \right)/(2n+1) \]

On the surface of the Earth,

\[ V(R, \theta, \lambda) = 4\pi G_p R \sum_{n} P_n^m \left( y^m_n \cos m \lambda + z^m_n \sin m \lambda \right)/(2n+1) \]

\[ = 4\pi G_p R \Sigma_n \xi_n/(2n+1) \]

\[ = \frac{3 G_p \Sigma_n \xi_n}{\rho} \]

where \( \rho \) is now the mean density of the Earth.
Appendix E

The Liouville Equations

In a coordinate frame $x_i$ ($i = 1, 2, 3$) rotating with angular velocity $\boldsymbol{\omega}$ relative to a frame fixed in space, the equations of motion of a body are

$$L_i = \frac{d}{dt} H_i + \epsilon_{ijk} \omega_j H_k$$  \hspace{1cm} (E.1)

where

$L_i$ = $i$'th component of the net external torque on the body about its centre of mass,

$H_i$ = $i$'th component of the angular momentum of the body about its centre of mass,

$\epsilon_{ijk}$ = the Levi-Civita alternating symbol;

and the summation convention over repeated subscripts is used.

For convenience take the rotating frame to be fixed to the surface of the solid Earth in some prescribed way. Then it is called a geographic frame.

$$H_i = C_{ij} \omega_j + h_i$$  \hspace{1cm} (E.2)

where $C_{ij}$ are the components of the inertia tensor in the chosen frame and $h_i$ is the $i$'th component of the body's angular momentum relative to that frame. Substituting E.2 in E.1 we have

$$L_i = \frac{d}{dt} \left( C_{ij} \omega_j + h_i \right) + \epsilon_{ijk} \omega_j (C_{k\ell} \omega_\ell + h_\ell)$$  \hspace{1cm} (E.3)
For small departures from a state in which the body is uniaxial and rotates about the axis \( x_3 \) with uniform angular speed \( \omega \), equation E.3 can be put in a perturbation form. Define

\[
\begin{aligned}
C_{11} &= A + c_1, \\
C_{22} &= A + c_2, \\
C_{33} &= C + c_3, \\
C_{12} &= c_{12}, \\
C_{13} &= c_{13}, \\
C_{23} &= c_{23}, \\
\omega_1 &= \omega m_1, \\
\omega_2 &= \omega m_2, \\
\omega_3 &= \omega (1 + m_3)
\end{aligned}
\]

(E.4)

are perturbations from the diagonal inertia tensor whose components are \( A, A, C \). The variables \( m_1, m_2 \) measure the angular departure of the rotation axis from the axis \( x_3 \), and \( m_3 \) measures changes in the length of day. Substituting E.4 into E.3 and neglecting products and squares of small quantities we have, finally,

\[
\begin{aligned}
\frac{\dot{m}_1}{\omega} + m_2 &= \phi_1, \\
\frac{\dot{m}_2}{\omega} - m_1 &= -\phi_1, \\
\dot{m}_3 &= \phi_3
\end{aligned}
\]

where

\[
\delta \tau = \left( \frac{C - A}{A} \right) \omega
\]

(E.6)

and

\[
\begin{aligned}
\lambda^2 (C - A) \phi_1 &= \lambda^2 c_{13} + \lambda^2 c_{23} + \lambda^2 h_2 + h_3 - L_2, \\
\lambda^2 (C - A) \phi_2 &= \lambda^2 c_{23} - \lambda^2 c_{13} + \lambda^2 h_2 - h_3 - L_1, \\
\lambda^2 C \phi_3 &= -\lambda^2 c_{33} - \lambda^2 h_3 + \lambda \int_0^t L_3 \, dt
\end{aligned}
\]

(E.7)
The functions $\Phi_i$ are called excitation functions because they involve the various ways in which changes in the body's rotation can be effected—changes in the mass distribution itself, changes in the internal distribution of angular momentum of the body, or external torques. The set $E.5$ are called the Liouville equations. (Munk and MacDonald, 1960, Chapters 3, 6).
Appendix F

The Pole Tide Potential

The centrifugal force experienced by a particle at distance \( r \) from the axis of rotation, about which the Earth spins with angular speed \( \vec{\omega} \), is \( \vec{a} = \vec{\omega} \times \vec{r} \). This is derived from a potential

\[
\frac{1}{2} \omega^2 r^2 = \frac{1}{2} |\vec{\omega} \times \vec{r}|^2
\]

where

\[
\vec{r} = a(\hat{e}_1 \sin \phi \cos \lambda + \hat{e}_2 \sin \phi \sin \lambda + \hat{e}_3 \cos \phi)
\]

is the distance, from the Earth's centre, of the particle whose longitude is \( \lambda \) and whose colatitude is \( \phi \). To the first order in the small quantities \( m_1, m_2 \):

\[
\vec{\omega} = \Omega (\hat{e}_1 m_1 + \hat{e}_2 m_2 + \hat{e}_3)
\]

Therefore the centrifugal force potential reduces to

\[
\frac{1}{2} m^2 a^2 \left[ \sin^2 \phi - \sin \phi \cos \phi \Omega (m_1 \cos \lambda + m_2 \sin \lambda) \right]
\]

The part which creates wobble is

\[
U_2 = -\frac{2}{3} m^2 a^2 P_2(\cos \phi)(m_1 \cos \lambda + m_2 \sin \lambda)
\]