EQUILIBRIUM TIDAL RESPONSE OF A NON-GLOBAL SELF-GRAVITATING OCEAN ON A YIELDING EARTH

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by

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ABSTRACT

It is commonly stated that the equilibrium ocean tide, measured with respect to the solid surface of the Earth, has amplitude

$$\xi_2 = (1 + k_2 - h_2)\, U_2 / g$$

where U_2 is the tide potential of degree-2, g is the acceleration of gravity at the surface and k_2, h_2, the second degree Love numbers, are scalar quantities which characterize the elastic response of the solid Earth. In fact this equation applies strictly to the "tides" in the equipotential surfaces of oceanless, spherically symmetric planets. It is found that the mass of the oceans is sufficient to disturb the potential field to which they are responding thus creating a feedback mechanism which increases the predicted tide height by about 63%. The loading and subsequent deformation of the solid Earth by the ocean tide creates another feedback loop which decreases the tide by about 40%. The combined effect of loading and self-attraction is a 23% increase in the predicted tide height. In addition the irregular distribution of the oceans introduces complications in the response of the oceans to a prescribed potential and a single scalar equation no longer suffices. A set of matrices is developed to replace the scalar Love parameters and these are used to construct maps of several tides which are thought to be equilibrium. Matrices which give the perturbed potential and the gravity tide on a non-sphericallysymmetric Earth are also given, and the correct expression for

the load gravimetric factor, about which there has been some confusion, is derived. Previous estimates of the influence of equilibrium ocean tides on certain aspects of the rotation of the Earth are re-worked using the results of the matrix method.

CHAPTER 1

EQUILIBRIUM TIDES; FIRST ORDER THEORY

i) Earth Deformation, Love Numbers

Love (1909) introduced a method of characterizing the response of an elastic, spherically symmetric, gravitating sphere, initially hydrostatically stressed, to a prescribed body force potential. The fundamental assumptions in the theory are that the density and elastic parameters vary only with geocentric radius and that all displacements be small. Only in such a case can the deformation of the body have the same spherical harmonic form as the prescribed potential.

As given by Love the two parameters in the theory, called Love numbers, may be defined by the following two equations. If U_n is the n^{th} degree solid harmonic in the expansion of the prescribed body force potential U the radial deflection of the surface of the body is

$$\xi_n = h_n U_n/g \tag{1.1}$$

where g is the acceleration of gravity at the surface and ξ_n is of course a surface harmonic of degree-n. This defines the Love number h_n. As a result of the deformation the potential of the body at the surface will be changed by

$$k_n U_n \tag{1.2}$$

which defines the Love number k_n. Since all displacements must be small the Love number method is a first order theory and in common with all other first order theories obeys the principle of linear superposition. This is simply because any deformation which satisfies the criterion of being small does not alter the distribution of density and elastic parameters sufficantly to yield a model with appreciably different Love numbers. Love numbers of degree zero and 1 do not exist because the associated body forces leave the Earth undeformed (a body force derived from a degree-zero potential vanishes identically, that derived from a degree-1 potential simply accelerates every point in the Earth by the same amount).

Most of the easily observable periodic Earth deformations are of degree-2 in a solid harmonic expansion and consequently prior to about 1960 most of the interest was in evaluating h_2 and k_2 and Love numbers of other degree were seldom mentioned in the literature. Since 1962, however, Love numbers up to degree 25 have been available from calculations made on Earth models (Longman, 1963 a, b, 1966; Kaula, 1963; Takeuchi et al, 1962) and the problem has since been to use these numbers to investigate more fully the phenomena which cause the deformations or to deduce how the Earth departs from the basic assumption, that of spherical symmetry.

The response of the Earth to prescribed loads, such as snow, may also be described in a Love number format. The defining equations for load Love numbers are similar to those for non-load numbers. Denoting load Love numbers here and elsewhere with a superscript prime, the radial deflection of the solid surface in response to a load potential U' is

$$\xi_n = h_n' U_n' / g \qquad (1.3)$$

and the change in potential at the surface is

$$k_n' U_n' \tag{1.4}$$

In direct analogy with isostasy, where yielding reduces the free air anomalies, the potential of a load is reduced by yielding. The load Love numbers are, therefore, negative (Table 1.1). Load Love numbers of degree 0 and 1 are possible: a degree-zero load potential corresponds to adding mass at the Earth's surface, and a degree-1 load is possible because a shift in surface mass distribution does not imply a shift in the centre of mass of the whole system but only of the centre of mass of the solid Earth.

ii) The Tide Reducing Factor

The tide raising potential of the Moon, for example, is

$$U = \sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \frac{GM}{R}\left(\frac{r}{R}\right)^n P_n(\cos\alpha) \tag{1.5}$$

(Platzman 1972) where

G = constant of gravitation

M = mass of the Moon

R = radius to the Moon } from the Earth's centre of mass

r = radius of the observation point } from the Earth's centre of mass

α = angle subtended at the centre of the Earth by radii to the observation point and the Moon

$P_n(\cos\alpha)$ = Legendre polynomial of degree-n

In response to this potential the equipotential surfaces of a rigid Earth would exhibit a 'tide' of height

$$\sum_n U_n/g \tag{1.6}$$

Since the ocean will attempt to adjust itself so that its surface is an equipotential this is normally taken to be the height of the ocean tide that would be observed on the surface of a rigid Earth. This is true only to the extent that the dynamics of the ocean may be neglected. There are in addition some other complications which will be discussed in 1.iv. If the solid part of the body is deformable the potential at the surface will be

$$\sum_n (1+k_n)\, U_n \tag{1.7}$$

instead of simply

$$\sum_n U_n \tag{1.8}$$

and this is the potential to which the oceans will be responding. The equipotential surfaces of the solid body will accordingly be deflected radially by

$$\sum_n (1+k_n)\, U_n/g \tag{1.9}$$

with respect to the Earth's centre. Neglecting other influences this is also the height above the centre of the Earth to which the ocean surface will be raised. Tide gauges are, however, fixed to the solid Earth surface and since this has also been deflected it must be accounted for.

The solid Earth surface will be raised by

$$\sum_n h_n U_n / g \tag{1.10}$$

and subtracting this from 1.9 gives the tide height as measured by a tide gauge, namely

$$\sum_n (1 + k_n - h_n) U_n / g \tag{1.11}$$

The factor

$$\gamma_n = 1 + k_n - h_n \tag{1.12}$$

is called the tide reducing factor and it is the ratio of the tide height on an Earth with Love numbers h_n, k_n to the tide height on a rigid Earth. (Melchior, 1966). Due to the distance of the Sun and Moon from the Earth they raise primarily degree-2 tides. Although degree-3 tides have been observed they are much smaller than the tides of degree-2 (Melchior and Venedikov, 1968).

In tidal measurements in lakes it is really the slope of the equipotential surfaces which is measured (Moulton, 1919; Proudman, 1925) and this is determined by the slope of the local vertical which is described by the same factor. The deflection of the local vertical is given by

$$\frac{1}{g} \sum_n (1 + k_n - h_n) \frac{\partial}{a \partial \theta} U_n \tag{1.13}$$

where a is the radius of the Earth, and θ is an angle measured along the surface of the Earth in any given direction.

Darwin (1883) and Schweydar (1908) were the first to measure the tide reducing factor. Both used the monthly and fortnightly tides which they assumed to be equilibrium tides (Proudman (1960) subsequently found this to be an unjustifiable assumption) and obtained the following results for γ_2 :

	fortnightly	monthly
Darwin	0.675 ± 0.084	0.680 ± 0.387
Schweydar	0.625 ± 0.043	0.605 ± 0.012

Table 1.1 The tide reducing factors γ_2 (Darwin, Schweydar).

The value computed from Earth models is 0.690 using Longman's (1963) results. This small discrepancy is surprising in view of the number of errors and omissions in 1.11. The errors are discussed more fully in 1.iv.

iii) <u>The Gravimetric Factor</u>

The second factor which has been extensively measured is the gravimetric factor conventionally donated by δ_n. When the Earth experiences the tidal potential of the Moon, for example, there is a periodic fluctuation in the acceleration of gravity observed on the surface due to

1) The direct tidal acceleration by the Moon which would

be observed on a rigid Earth. If U_n is the n^{th} degree component of the tide raising potential (equation 1.5) the direct contribution to the acceleration of gravity with g measured positively upwards is

$$\Delta g = \frac{\partial}{\partial r} U_n = \frac{n}{a} U_n \tag{1.14}$$

2) The gravitational acceleration of the deformed part of the Earth. The deformation potential of the Earth has the value

$$k_n U_n(a) \tag{1.15}$$

at the surface and falls off as

$$k_n U_n(a)\left(\frac{a}{r}\right)^{n+1} \tag{1.16}$$

so the acceleration of gravity due to the deformation is

$$\Delta g = \frac{\partial}{\partial r}\left(k_n U_n\left(\frac{a}{r}\right)^{n+1}\right)\Big|_{r=a} = -k_n U_n \frac{(n+1)}{a} \tag{1.17}$$

3) The variation of g produced by moving the gravimeter in the undeformed gravity field of the Earth, V_0. If ξ is the height the surface is raised this contribution to the gravity tide is

$$\Delta g = \frac{\partial^2}{\partial r^2}(V_0)\, \xi_n = \frac{2g}{a} \xi_n = \frac{2h_n}{a} U_n \tag{1.18}$$

DEGREE N	LOVE NO. NON-LOAD H	LOVE NO. NON-LOAD K	TIDAL HEIGHT FACTOR NON-LOAD 1+K-H	GRAVIMETRIC FACTOR NON-LOAD 1+2H/N-(N+1)K/N	LOVE NO. LOAD H	LOVE NO. LOAD K	TIDAL HEIGHT FACTOR LOAD 1+K-H
0	0.0	0.0	1.000	0.0	-0.134	0.0	1.134
1	0.0	0.0	1.000	1.000	-0.290	0.0	1.290
2	0.612	0.302	0.690	1.159	-1.007	-0.310	1.697
3	0.290	0.093	0.803	1.069	-1.059	-0.197	1.862
4	0.175	0.042	0.867	1.035	-1.059	-0.133	1.926
5	0.129	0.025	0.896	1.022	-1.093	-0.104	1.989
6	0.107	0.017	0.910	1.016	-1.152	-0.090	2.062
7	0.095	0.013	0.918	1.012	-1.223	-0.082	2.141
8	0.087	0.010	0.923	1.011	-1.296	-0.076	2.220
9	0.081	0.008	0.927	1.009	-1.369	-0.072	2.297
10	0.076	0.007	0.931	1.008	-1.439	-0.069	2.370
11	0.072	0.006	0.934	1.007	-1.506	-0.066	2.440
12	0.069	0.005	0.936	1.006	-1.572	-0.064	2.508
13	0.066	0.005	0.939	1.005	-1.631	-0.062	2.569
14	0.064	0.004	0.940	1.005	-1.691	-0.060	2.631
15	0.062	0.004	0.942	1.004	-1.747	-0.058	2.689
16	0.060	0.003	0.943	1.004	-1.798	-0.056	2.742
17	0.058	0.003	0.945	1.004	-1.852	-0.055	2.797
18	0.056	0.003	0.947	1.003	-1.902	-0.054	2.848
19	0.055	0.003	0.948	1.003	-1.949	-0.052	2.897
20	0.053	0.002	0.949	1.003	-1.994	-0.051	2.943
21	0.052	0.002	0.950	1.003	-2.037	-0.050	2.987
22	0.051	0.002	0.951	1.003	-2.078	-0.049	3.029
23	0.050	0.002	0.952	1.002	-2.117	-0.048	3.069
24	0.048	0.002	0.954	1.002	-2.156	-0.047	3.109
25	0.047	0.002	0.955	1.002	-2.194	-0.046	3.148

TABLE 1.1 LOAD AND NON-LOAD LOVE NUMBERS TO DEGREE N=25 AND SOME LOVE NUMBER PARAMETERS. THE NUMERICAL VALUES OF THE LOVE NUMBERS ARE FROM LONGMAN (1963).

By summing together the three contributions the total gravity tide is

$$\Delta g = (n + 2h_n - (n+1)k_n)\frac{U_n}{a}$$

$$= \left(1 + \frac{2h_n}{n} - \frac{(n+1)}{n}k_n\right)\frac{nU_n}{a} \qquad (1.19)$$

The factor

$$\delta_n = \left(1 + \frac{2h_n}{n} - \frac{(n+1)}{n}k_n\right) \qquad (1.20)$$

is called the gravimetric factor and it is the ratio of the gravity tide on an Earth with Love numbers h_n, k_n to the gravity tide on a rigid Earth.

In much the same way a gravimetric factor for load potentials is derived in chapter 5.

iv) The Indirect Effect

When measurements of the tide reducing factor were first made it was found that there was a wide scatter in the numerical values obtained. Values of δ_2 ranged from 1.056 to 1.285 and values of γ_2 from 0.330 to 1.129 (Melchior, 1966). Hecker (1907) recognized the source of the poor results and attributed it to the indirect effect of the ocean. Equation 1.11 applies strictly to the 'tide' in the equipotential surfaces of spherically symmetric oceanless planets. The indirect effect has the following influence on the tide height:

1) The ocean itself having appreciable mass alters the potential field to which it is responding.

2) The loading of the crust by the ocean tide creates an

additional potential which tends to reduce the feedback from (1).

The simple explanation for (1) and (2) is that when a tide is produced by the prescribed potential the gravitational force of the water mass attracts more water to it and the result is a tide of increased amplitude. When the ocean bottom deforms under the load of the tide the gravitational pull of the tide is reduced in the same way that isostatic yielding reduces free air gravity anomalies.

Because the Earth is loaded by non-global oceans it does not satisfy the Love number requirement of spherical symmetry. In fact, due to the large amplitude of the ocean tide, the ocean cannot even be treated as a perturbation on an otherwise spherically symmetric Earth. The ocean must be accommodated separately. The non-globality of the oceans means that when forced by a degree-2 potential the ocean may respond with tide coefficients of any degree and order depending on the exact shape of the ocean. Any physical tide will have at least the modulation of the ocean function (Chapter 2) and as a result the coefficients of the tide will not reduce to zero with increasing degree any faster than the coefficients in the ocean function (Table B.1). To adequately describe a tide therefore it should be expanded to degree and order 8, which is the extent to which the ocean function coefficients are available (Munk and MacDonald, 1960; Balmino _et al_ 1973).

Kelvin (1863) first pointed out the necessity of allowing for load yielding and self-attraction. Street (1925) introduced a correction for loading and Rosenhead (1929) solved the problem of self-attraction. Proudman (1925) suggested measuring the tide in lakes which were large enough to exhibit a tide but small enough so that the tide height would be small and hence loading and self-attraction could be neglected. For the same reasons the tide in buried pipes filled with water was measured (Michelson, 1914; Michelson and Gale, 1919). Although these methods avoided the error of using non-equilibrium tides (care being taken to assure that the resonance periods of the water bodies were far removed from the diurnal period) and eliminated loading and self-attraction problems, complete allowance for the indirect effect was still not achieved. This was because, although unknown and unsuspected by these authors, tides in the open ocean can affect the deflection of the vertical and especially the gravity tide at mid-continent stations (Pertsev, 1966). This effect of the distant oceans on the deflection of the vertical is impossible to correct for because without a knowledge of the tide in the open sea the correction is impossible to estimate. As yet the tide in the global ocean at the important diurnal and semi-diurnal frequencies is imperfectly known although Farrell (1972) has appreciably reduced the gravity tide residual at mid-continent stations (Kuo et al, 1970) by using the co-tidal chart of Hendershott (1972). Although Pertsev handles the problem correctly there still exists some confusion in the west about how the problem should be handled in a Love number format particularly as regards the load gravimetric factor (Farrell, 1970, p 25; Slichter, 1972, p 316). The correct expression is derived in 5.ii.

Rosenhead (1929) attempted to apply a correction to the reducing factor which would allow for ocean self-attraction. Although substantially correct his method somehow escaped further mention in the literature.

A more systematic approach was needed to account for the indirect effect in the tide height problem. Munk and MacDonald (1960) outlined a method of predicting tide heights allowing for feedback and continentality. Their result although substantially correct was derived in an unnecessarily circuitous manner which tended to obscure the physics involved, and as originally given contains two typographical errors. Their equation 5.12.8 should read

$$\xi = \mathcal{C}\left[\sum_r \left(J_r U_r + J_r' U_r' + \frac{3g\rho_0 J_r' \xi_r}{(2r+1)\rho}\right) + \text{const.}\right] \qquad (1.21)$$

where ξ is the tide height, J_r and J_r' are the tide reducing factors for non-load and load potentials respectively, U_r and U_r' are the prescribed non-load and load potentials respectively, ρ_0 is the density of sea water, ρ the mean density of the Earth, $\mathcal{C}$ is the ocean function, and the constant is required to satisfy the requirement of conservation of ocean mass.

A more straightforward derivation is presented in Chapter 2.

CHAPTER 2

EQUILIBRIUM TIDES; SECOND ORDER THEORY

i) Introduction

Chapter 1 described some of the inadequacies of theoretical studies of ocean/Earth tide interactions. In this chapter the problem of the indirect effect of the oceans on the tide height and the consequences of the non-globality of the ocean for the Love number concept will be explored in detail. Also the equations for the coefficients in the surface harmonic expansion of ocean tides will be solved so that these coefficients may be readily computed for any tidal potential in terms of the coefficients of the prescribed potential.

In order to see how each effect separately alters the simplistic 1.11 I shall consider each effect in turn by examining various idealized models.

ii) The Effect of Ocean Self-Attraction

To model this effect consider a rigid Earth with global oceans and suppose the height of the tide due to the prescribed potential U is ξ, with coefficients U_n and ξ_n in a surface harmonic expansion. By treating the tide as a layer of thickness ξ and density ρ_o equal to the density of sea water the potential of the tide may easily be found to be

$$\sum_n \frac{4\pi G}{2n+1} \rho_0 \xi_n = \sum_n \frac{3g\rho_0 \xi_n}{(2n+1)\rho} \tag{2.1}$$

(Munk and MacDonald, 1960, p29), where ρ_0 is the density of sea water and ρ the mean density of the Earth. The derivation of (2.1) is given in Appendix D. The total potential at a point on the fluid surface becomes

$$V = \sum_n \left(U_n + \frac{3g\rho_0 \xi_n}{(2n+1)\rho} \right) + \frac{\partial}{\partial r}(V_0)\,\xi + V_0 \tag{2.2}$$

where V_0 is the undisturbed potential of the Earth. Since V is to be constant on the fluid surface the part of 2.2 which depends on latitude and longitude must be constant, which yields

$$g\xi = \sum_n \left(U_n + \frac{3g\rho_0}{(2n+1)\rho} \xi_n \right) + \text{const.} \tag{2.3}$$

and consequently

$$\xi_n = \left(\frac{1}{1 - \frac{3\rho_0}{(2n+1)\rho}} \right) \frac{U_n}{g} \tag{2.4}$$

The last step can only be performed because the ocean is global and as a result U and ξ have the same surface harmonic form.

Although the solid part of this Earth model is rigid and therefore has the Love number $h_n = 0$ the response of its fluid surface to a prescribed potential can be described by the 'Love number'

$$f_n = \frac{1}{1 - \frac{3\rho_0}{(2n+1)\rho}} \qquad (2.5)$$

Since the equilibrium tide in a global ocean on a rigid Earth is given simbly by U_n/g when self-attraction is neglected, f_n represents an amplification factor. The theoretical tide heights are increased about 10% by self-attraction in the case of a degree-2 potential.

Now the perturbed potential of this model is

$$\begin{aligned} \sum_n W_n &= \sum_n \frac{3g\rho_0}{(2n+1)\rho} \xi_n \\ &= \sum_n \frac{3g\rho_0}{(2n+1)\rho} \xi_n \\ &= \sum_n g_n U_n \end{aligned} \qquad (2.6)$$

where

$$g_n = \frac{3\rho_0}{(2n+1)\rho} \cdot \frac{1}{1 - \frac{3\rho_0}{(2n+1)\rho}} \qquad (2.7)$$

and g_n is the 'Love number' k_n for this model.

As long as the global ocean on this model is deeper than any possible tide the model is equivalent to a homogeneous, incompressible fluid Earth. As a check on the validity of 2.5 and 2.7 compare these solutions with those found by Kelvin (1863) and given by Takeuchi (1950, p 668) for a homogeneous incompressible Earth of rigidity μ. For degree-2 equations 2.5, 2.7 become

$$f_2 = 5/2 \qquad g_2 = 3/2 \qquad (2.8)$$

with $\rho_o = \rho$. Takeuchi obtains

$$h_2 = 5/2\, m \qquad k_2 = 3/2\, m$$

$$m = \left(1 + \frac{19\mu}{2\rho g a}\right) \tag{2.9}$$

and for a fluid model $\mu = 0,\ m = 1$ and

$$h_2 = 5/2 \qquad k_2 = 3/2 \, . \tag{2.10}$$

DEGREE N	SELF ATTRACTION EFFECTIVE H	SELF ATTRACTION EFFECTIVE K
0	2.200	1.200
1	1.222	0.222
2	1.122	0.122
3	1.085	0.085
4	1.065	0.065
5	1.052	0.052
6	1.044	0.044
7	1.038	0.038
8	1.033	0.033
9	1.030	0.030
10	1.027	0.027
11	1.024	0.024
12	1.022	0.022
13	1.021	0.021
14	1.019	0.019
15	1.018	0.018
16	1.017	0.017
17	1.016	0.016
18	1.015	0.015
19	1.014	0.014
20	1.013	0.013
21	1.013	0.013
22	1.012	0.012
23	1.012	0.012
24	1.011	0.011
25	1.011	0.011

TABLE 2.1 THE EFFECTIVE LOVE NUMBERS OF A RIGID EARTH MODEL WITH GLOBAL OCEANS TAKING INTO ACCOUNT OCEAN SELF ATTRACTION

Equations 2.5 and 2.7 are consistent with Takeuchi's more general equations. The values of f_n and g_n are given in Table 2.1.

iii) The Effect of Load Yielding

The model used in this section will be a global ocean on an otherwise realistic Earth. The problem will be handled in two steps. First the ocean surface will be considered to be acted on by all the potentials it would experience because of the feedback and the deflection with respect to the Earth's centre will be obtained. Second the ocean bottom will be deformed by all the potentials including the ocean tide and the deflection of the solid surface with respect to the Earth's centre will be obtained. The measured tide will be obtained by subtracting one from the other.

The potentials which act on the ocean surface are:

1) the potential of the undeformed Earth:

$$V_0(a+\xi') = V_0(a) + \frac{\partial V_0}{\partial r}\xi' = V_0 - g\xi' \qquad (2.10)$$

where ξ' is the amount the sea surface is raised above the centre of the Earth.

2) the prescribed potentials which do not/do load the Earth:

$$\sum_n U_n \quad , \quad \sum_n U_n' \quad ; \qquad (2.11)$$

3) the deformation potentials of the Earth as a result of the prescribed potentials:

$$\sum_n k_n U_n \quad , \sum_n k_n' U_n' \quad ; \tag{2.12}$$

4) the potential of the tide itself:

$$\sum_n \frac{3g\rho_0 \xi_n}{(2n+1)\rho} \tag{2.13}$$

where ξ_n is measured with respect to the sea floor ;

5) the deformation potential of the solid Earth as a result of the ocean tide:

$$\sum_n \frac{k_n' 3g\rho_0 \xi_n}{(2n+1)\rho} . \tag{2.14}$$

These five potentials can now be summed together to obtain the potential at a point on the ocean surface:

$$V = \sum_n \left(U_n + U_n' + k_n U_n + k_n' U_n' + \frac{3g\rho_0 \xi_n}{(2n+1)\rho} + \frac{k_n' 3g\rho_0 \xi_n}{(2n+1)\rho} \right) - g\xi' + V_0 \tag{2.15}$$

If this surface is to be an equipotential the variable part of 2.15 must be a constant on the sea surface, which gives

$$g\xi' = \sum_n \left(U_n + U_n' + k_n U_n + k_n' U_n' + \frac{3g\rho_0 \xi_n}{(2n+1)\rho} + \frac{k_n' 3g\rho_0 \xi_n}{2n+1\rho} \right) + \text{const.} \tag{2.16}$$

Now considering the deflections of the solid Earth surface as a result of:

1) the prescribed potentials:

$$\sum_n h_n \frac{U_n}{g} , \sum_n h_n' \frac{U_n'}{g} \tag{2.17}$$

2) the ocean tides:

$$\sum_n \frac{h_n' 3 g \rho_o \xi_n}{(2n+1)\rho} \tag{2.18}$$

Summing these two together gives for the total deflection of the solid Earth surface with respect to the Earth's centre,

$$g\xi'' = \sum_n \left(h_n U_n + h_n' U_n' + \frac{h_n' 3 g \rho_o}{(2n+1)\rho} \xi_n \right) \tag{2.19}$$

Subtracting this from 2.16, we note that the tide measured with respect to the Earth's crust is the difference between the tide measured with respect to the Earth's centre and the solid Earth tide, i.e.

$$\xi = \xi' - \xi'' \tag{2.20}$$

This results in

$$g\xi = \sum_n \Big[(1+k_n-h_n) U_n + (1+k_n'-h_n') U_n' + (1+k_n'-h_n') \frac{3 g \rho_o \xi_n}{(2n+1)\rho} \Big] + const. \tag{2.21}$$

or, in keeping with Munk and MacDonald's notation,

$$\sum_n \xi_n = g^{-1}\left[\sum_n\left(J_n U_n + J_n' U_n' + J_n' \frac{3g\rho_0}{(2n+1)\rho}\xi_n\right)\right] + \text{const.} \tag{2.22}$$

where

$$\begin{aligned} J_n &= 1 + k_n - h_n \\ J_n' &= 1 + k_n' - h_n' \end{aligned} \tag{2.23}$$

The constant in 2.22 is determined by requiring conservation of water: i.e.

$$\int_S \rho_0 \xi_s \, ds = 0 \tag{2.24}$$

where S is the Earth's surface. In the case of global oceans the constant is zero (except for a load which adds mass to the Earth), in non-global oceans the constant is non-zero and represents an adjustment in level referred to as the Darwin correction (M & M 1960 p. 100). We return to this point in 2.iv. Equation 2.22 gives for the tide heights ($n \geq 1$)

$$\xi_n = \frac{J_n}{1 - \frac{J_n' 3\rho_0}{(2n+1)\rho}} \frac{U_n}{g} + \frac{J_n'}{1 - \frac{J_n' 3\rho_0}{(2n+1)\rho}} \frac{U_n'}{g} \tag{2.25}$$

Again the last step can only be performed because ξ and U have the same harmonic form. The numerical values of the two Love number parameters are given in Table 2.2. For a degree-2 potential which does not load the Earth

$$J_{2\,eff} = \frac{J_2}{1 - \frac{J_2' 3\rho_0}{5\rho}} = .847 \tag{2.26}$$

so that the tide reducing factor is increased by about 20% when load yielding and self-attraction are considered.

DEGREE N	GLOBAL EFFECTIVE K	GLOBAL EFFECTIVE TIDE HEIGHT
0	1.430	0.0
1	0.238	1.306
2	0.368	0.847
3	0.152	0.939
4	0.094	0.982
5	0.069	0.994
6	0.055	0.996
7	0.046	0.996
8	0.039	0.994
9	0.034	0.992
10	0.031	0.992
11	0.028	0.991
12	0.025	0.990
13	0.024	0.990
14	0.021	0.989
15	0.020	0.989
16	0.018	0.988
17	0.018	0.988
18	0.017	0.989
19	0.016	0.988
20	0.014	0.988
21	0.014	0.987
22	0.013	0.987
23	0.013	0.987
24	0.012	0.988
25	0.012	0.988

TABLE 2.2 THE EFFECTIVE LOVE NUMBER K AND THE EFFECTIVE TIDE HEIGHT OF AN EARTH MODEL WHOSE ELASTIC RESPONSE IS DESCRIBED BY THE LOVE NUMBERS H,K AND WHOSE SURFACE IS COVERED WITH GLOBAL OCEANS. THE LOVE NUMBERS OF THE ELASTIC PART ARE TAKEN FROM LONGMAN (1963).

The deformation potential of this model due to a non-load potential is

$$\sum_n k_n U_n + k_n' \frac{3g\rho_0}{(2n+1)\rho}\xi_n + \frac{3g\rho_0}{(2n+1)\rho}\xi_n$$

$$= \sum_n \left[k_n + (1+k_n')\frac{3\rho_0}{(2n+1)\rho} \cdot \frac{J_n}{1 - J_n' \frac{3\rho_0}{(2n+1)\rho}} \right] U_n \tag{2.27}$$

using 2.25.

A satellite orbiting this model Earth, or in fact the celestial body which produces the prescribed potential, will see this potential as the deformation potential of the model. Its response may be characterized by the effective Love number

$$k_{n\,eff} = k_n + (1+k_n')\frac{3\rho_0}{(2n+1)\rho} \cdot \frac{J_n}{1 - J_n' \frac{3\rho_0}{(2n+1)\rho}} \tag{2.28}$$

For a degree-2 potential $k_{2eff} = 0.366$ which is an increase of 15% over the value Longman (1963) finds for the solid Earth. The exact value of the deformation potential of the Earth is important for studies of satellite orbit perturbations.

Smith et al. (1972) have used laser tracking of satellites in the hope of detecting polar motion. They find

$$k_{2eff} = 0.25 \tag{2.29}$$

best fits their data. The theory of tidally-influenced satellite orbits was developed in a series of papers by Kozai (1965, 1967, 1968) who obtained

$$k_{2eff} = 0.29 \tag{2.30}$$

Newton (1968) investigated the problem with the view particularly of finding the phase lag of the solid-Earth tide so that the tidally-influenced secular acceleration of the Earth could be deduced (Dicke, 1966, Munk, 1966). He found

$$k_{2eff} = 0.336 \tag{2.31}$$

Lambeck and Cazenave (1973), recognizing that the oceans influenced the value of k_{2eff} as seen by satellites and that the non-globality of the oceans required new approaches to Love number theory, explored the relationship between the whole Earth tide (atmospheric, oceanic, and direct and indirect solid-Earth) and a low satellite orbit. Under the assumption that the solid Earth response to both load and non-load potentials was well understood, they asserted that the satellite perturbations could be used to study and map the ocean tides. They found that for low inclination orbits k_{2eff} is 17% lower than the elastic k_2 and for high inclination orbits k_{2eff} is 5% lower. Jeffreys and Vicente (1957) in a study of the Chandler wobble deduce that k_{2eff} is increased by 12% by the oceans. In a similar study Molodenskii (1961) obtains a 16% increase. The differences between these several observational and theoretical results, and 2.28, may be attributed to (a) shortcomings in the satellite orbit interpretation, (b) failure (except by Jeffreys and Vicente, and Molodenskii) to take into account the dynamical response of the liquid core, (c) non-equilibrium character of the ocean tides.

iv) The Effect of the Non-Globality of the Oceans

The simplest way of treating the non-globality of the oceans is to solve for the global ocean tide and simply truncate the solution to zero on land. The function which accomplishes this is the ocean function, defined by Munk and MacDonald (1960):

$$\mathcal{C} = \begin{cases} 0 & \text{where there is land} \\ 1 & \text{where there are oceans} \end{cases} \tag{2.32}$$

Thus the equation for tide heights on a rigid earth, for example, becomes

$$\xi = \mathcal{C} \cdot \sum_n U_n / g \tag{2.33}$$

and ξ will have exactly the same values where there are oceans as it would for a global ocean, and zero height where there is land. If $\mathcal{C}$ and ξ are expanded in surface harmonics

$$\mathcal{C}(\theta,\lambda) = \sum_n^{\infty} \sum_{m=0}^{n} P_n^m(\cos\theta)\left(a_n^m \cos m\lambda + b_n^m \sin m\lambda\right)$$

$$\xi(\theta,\lambda) = \sum_n \xi_n = \sum_n^{\infty} \sum_{m=0}^{n} P_n^m(\cos\theta)\left(y_n^m \cos m\lambda + z_n^m \sin m\lambda\right) \tag{2.34}$$

where θ is the co-latitude and λ the longitude, it can be seen that ξ will not in general have the same harmonic form as the forcing function U. Tides of degree-n and order m will depend on the potentials of degrees other than n, or more properly potentials of degree-2 order-zero will raise tides of numerous degrees and orders depending on the ocean function. This suggests that the equations for the tide height coefficients may be cast in a matrix form such as:

$$\begin{vmatrix} \vdots \\ Y_n^m \\ \vdots \end{vmatrix} = \begin{vmatrix} \text{A square matrix whose elements} \\ \text{are determined by the coeffi-} \\ \text{cients of the ocean function} \\ \text{and the load and non-load} \\ \text{Love numbers (section 2.vi).} \end{vmatrix} \begin{vmatrix} \vdots \\ U_r^\ell \\ \vdots \end{vmatrix} \qquad (2.35)$$

Here Y_n^m are the coefficients of the tide and U_r^ℓ are the coefficients of the potential. It will be found in 2.vi that the solution for the tide coefficients can be given in this manner.

There is one additional complication introduced by non-globality, that is, an apparent non-conservation of water mass. This is overcome by requiring that the total mass of the oceans be constant, i.e.

$$\int_S \rho_0 \xi \, dS = 0 \Rightarrow \int_S \xi \, dS = 0 \qquad (2.36)$$

where S is the surface of the Earth. Using the orthonormality properties of the surface harmonics it is clear that 2.36 is equivalent to stating

$$Y_0^0 = 0 \tag{2.37}$$

In the next section allowance will be made for explicit mass conservation. The so-far undetermined constant in 2.21 is determined by 2.36 (or, equivalently, by 2.37).

v) Equilibrium Tides on a Real Earth

The method of simply truncating the tide to zero on land is not consistent with the feedback of the oceans on themselves either directly or indirectly through the loading. This is because in doing so one is allowing for the feedback from land areas to sea areas as if there were ocean tides in the land areas, and then making the contradictory statement that there are no ocean tides in land areas. Clearly the condition that there be no ocean tides in land areas must be applied before the feedback is considered. Returning to equations 2.16 and 2.19, and applying the ocean function after subtracting, we find

$$g \sum_n \xi_n = C\left[\sum_r \left(J_r U_r + J_r' U_r' + \frac{J_r' 3g\rho_0 \xi_r}{(2r+1)\rho}\right) + \text{const.}\right] \tag{2.38}$$

so that it is explicit that there is no feedback from land areas. Apart from the constant required for mass conservation, these are the equations found by Munk and MacDonald (1960) and given in a similar form by

Hendershott (1972).

vi) The Solution of the Equations

Hendershott's method of obtaining the solution of 2.38 is through a Green's function approach. Recognizing that the last variable in 2.38 is a feedback term it may be transformed into an integral

$$\int_{S'} \xi(\theta',\lambda')\, \mathcal{C}(\theta',\lambda')\, G(\theta',\lambda' | \theta,\lambda)\, ds' \tag{2.39}$$

where

$$G(\theta',\lambda'|\theta,\lambda) = \sum_n J_n' \alpha_n \sum_m \cos m\lambda \cos m\lambda'$$
$$P_n^m(\cos\theta)\, P_n^m(\cos\theta') + \sin m\lambda \sin m\lambda'\, P_n^m(\cos\theta)\, P_n^m(\cos\theta') \tag{2.40}$$
$$\left[\int_{S'} \left[P_n^m(\cos\theta')\begin{Bmatrix}\cos m\lambda' \\ \sin m\lambda'\end{Bmatrix}\right]^2 dS'\right]^{-1}$$

and

$$\alpha_n = \frac{3\rho_0}{(2n+1)\rho} \tag{2.41}$$

and the equation for tide heights becomes

$$\xi(\theta,\lambda) = \mathcal{C}(\theta,\lambda)\left[\sum_n J_n \frac{U_n}{g} + \text{const.}\right] + \int \mathcal{C}(\theta',\lambda')\, \xi\, G(\theta,\lambda|\theta',\lambda')\, ds' \tag{2.42}$$

Here for brevity we consider only a non-load potential.

Although his interest was not in equilibrium tides his suggestion of an iterative solution may be used here effectively. That is the standard formula for tide heights may be inserted in the left hand side of 2.42 and a better approximation may be found. Thus

$$\xi(\theta,\lambda) = \mathcal{C}(\theta,\lambda)\left[\sum_n J_n \frac{U_n}{g} + \text{const.}\right] + \sum_n J_n \int_S U_n(\theta',\lambda')\, \mathcal{C}\, G(\theta,\lambda|\theta',\lambda')\, dS' \tag{2.43}$$

would be the first step in the iteration. It may also be worthwhile to approach the ocean tide problem in the same way as lake tides are approached, that is through the slope of the equipotential surfaces with respect to the crust. Differentiating by θ and λ gives

$$\frac{\partial}{\partial\theta}\xi = \sum_n \frac{J_n}{g}\frac{\partial}{\partial\theta}U_n + \int_{S'} \xi(\theta',\lambda')\,\mathcal{C}(\theta',\lambda')\frac{\partial}{\partial\theta} G(\theta,\lambda|\theta',\lambda')\, dS'$$

$$\frac{\partial}{\partial\lambda}\xi = \sum_n \frac{J_n}{g}\frac{\partial}{\partial\theta}U_n + \int_{S'} \xi(\theta',\lambda')\,\mathcal{C}(\theta',\lambda')\frac{\partial}{\partial\lambda} G(\theta\lambda|\theta',\lambda')\, dS' \tag{2.44}$$

(valid in the oceans) and these equations could then be solved together with the additional constraint 2.37 imposed by mass conservation.

The method suggested by Munk and MacDonald (1960) yielded simultaneous equations in the various coefficients and they gave only the degree-2 order-zero coefficient in terms of several others. The solution of the equations is attempted here in matrix form, the final result being matrices which are to the non-global oceans what the scalar Love numbers are to a spherically symmetric earth.

Starting with

$$\sum_{n} \xi_n = g^{-1} \mathcal{C} \left[\sum_{r} J_r U_r + J_r' U_r' + \frac{J_r' 3 g \rho_0}{(2r+1)\rho} \xi_r + \text{const.} \right] \qquad (2.45)$$

and expanding all the variables in surface harmonics

$$\begin{aligned}
\xi &= \sum_{pq} P_p^q(\cos\theta) \left[Y_p^q \cos q\lambda + Z_p^q \sin q\lambda \right] \\
\mathcal{C} &= \sum_{nm} P_n^m(\cos\theta) \left[a_n^m \cos m\lambda + b_n^m \sin m\lambda \right] \\
U &= \sum_{r\ell} P_r^\ell(\cos\theta) \left[U_r^\ell \cos \ell\lambda + V_r^\ell \sin \ell\lambda \right] \\
U' &= \sum_{r\ell} P_r^\ell(\cos\theta) \left[U_r'^\ell \cos \ell\lambda + V_r'^\ell \sin \ell\lambda \right]
\end{aligned} \qquad (2.46)$$

results, after substitution in 2.45, in the following.

$$\begin{aligned}
&\sum_{ab} P_a^b(\cos\theta) \left[Y_a^b \cos b\lambda + Z_a^b \sin b\lambda \right] \\
&= \mathcal{C} \,\text{const.} \\
&+ \sum_{\substack{nm \\ r\ell}} \Big[a_n^m J_r U_r^\ell P_n^m P_r^\ell \cos m\lambda \cos \ell\lambda \\
&\quad + a_n^m J_r V_r^\ell P_n^m P_r^\ell \cos m\lambda \sin \ell\lambda \\
&\quad + b_n^m J_r U_r^\ell P_n^m P_r^\ell \sin m\lambda \cos \ell\lambda \\
&\quad + b_n^m J_r V_r^\ell P_n^m P_r^\ell \sin m\lambda \sin \ell\lambda \\
&\quad + a_n^m J_r' U_r'^\ell P_n^m P_r^\ell \cos m\lambda \cos \ell\lambda
\end{aligned} \qquad (2.47)$$

$$+ a_n^m J_r' V_r'^{\ell} P_n^m P_r^{\ell} \cos m\lambda \sin \ell\lambda$$

$$+ b_n^m J_r' U_r'^{\ell} P_n^m P_r^{\ell} \sin m\lambda \cos \ell\lambda$$

$$+ b_n^m J_r' V_r'^{\ell} P_n^m P_r^{\ell} \sin m\lambda \sin \ell\lambda$$

$$+ a_n^m J_r' \alpha_r Y_r^{\ell} P_n^m P_r^{\ell} \cos m\lambda \cos \ell\lambda$$

$$+ a_n^m J_r' \alpha_r Z_r^{\ell} P_n^m P_r^{\ell} \cos m\lambda \sin \ell\lambda$$

$$+ b_n^m J_r' \alpha_r Y_r^{\ell} P_n^m P_r^{\ell} \sin m\lambda \cos \ell\lambda$$

$$+ b_n^m J_r' \alpha_r Z_r^{\ell} P_n^m P_r^{\ell} \sin m\lambda \sin \ell\lambda \Big]$$

In order to facilitate computation the coefficients Y_r^{ℓ} and Z_r^{ℓ} are now not the tide height coefficients but the tide height coefficients multiplied by g. Multiplying both sides by $P_p^q \sin q\lambda$ and integrating over the surface one obtains, after collecting terms and rearranging.

$$Z_p^q = \sum_{\substack{nm \\ r\ell}} \Big[a_n^m \left(J_r U_r^{\ell} + J_r' U_r'^{\ell} + J_r' \alpha_r Y_r^{\ell} \right)$$

$$\times \int_S P_n^m P_r^{\ell} P_p^q \cos m\lambda \cos \ell\lambda \sin q \, dS$$

$$+ a_n^m \left(J_r V_r^{\ell} + J_r' V_r'^{\ell} + J_r' \alpha_r Z_r^{\ell} \right)$$

$$
\begin{aligned}
&\times\int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \sin q\lambda \, ds \\
&+ b_n^m \left(J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r Y_r^\ell \right) \\
&\times\int_S P_n^m P_r^\ell P_p^q \sin m\lambda \cos \ell\lambda \sin q\lambda \, ds \\
&+ b_n^m \left(J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r Z_r^\ell \right) \\
&\times\int_S P_n^m P_r^\ell P_p^q \sin m\lambda \sin \ell\lambda \sin q\lambda \, ds \Big] \\
&+ b_p^q \,\mathrm{const.}
\end{aligned}
\tag{2.48}
$$

by the orthonormality of the associated Legendre polynomials. Likewise multiplying both sides of 2.47 by $P_p^q \cos q\lambda$ and integrating over the surface one obtains

$$
\begin{aligned}
y_p^q = \sum_{\substack{nm \\ r\ell}} \Big[& a_n^m \left(J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r Y_r^\ell \right) \\
&\times\int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \cos q\lambda \, ds \\
&+ a_n^m \left(J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r Z_r^\ell \right) \\
&\times\int_S P_n^m P_r^\ell P_p^q \cos m\lambda \sin \ell\lambda \cos q\lambda \, ds
\end{aligned}
$$

$$+ b_n^m (J_r \dot{U}_r^l + J_r' U_r'^l + J_r' \alpha_r Y_r^l)$$

$$\times \int_s P_n^m P_r^l P_p^q \sin m\lambda \cos l\lambda \cos q\lambda \, ds$$

$$b_n^m (J_r V_r^l + J_r' V_r'^l + J_r' \alpha_r Z_r^l)$$

$$\times \int_s P_n^m P_r^l P_p^q \sin m\lambda \sin l\lambda \cos q\lambda \, ds \qquad (2.49)$$

$$+ a_p^q \text{const.}$$

In order to reduce the algebra we drop explicit reference to the Darwin correction (the terms a_p^qconst. b_p^qconst.) and satisfy conservation of mass requirements by using 2.36.

The equations 2.48 and 2.49 may be cast in matrix form but they can be simplified somewhat by looking at the integrals over the longitude, λ. There are four types of these integrals:

1) integrals of three sine functions
2) integrals of three cosine functions
3) integrals of two sine and one cosine
4) integrals of two cosine and one sine

The product of three odd functions is odd and accordingly integrals of type 1 vanish. The product of two even and one odd functions is odd and integrals of type 4 also vanish. This leaves only integrals of types 2 and

3 in equations 2.48 and 2.49. The reduced equations are

$$Y_p^q = \sum_{\substack{nm \\ r\ell}} \Big[a_n^m (J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r Y_r^\ell)$$

$$\times \int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \cos q\lambda \, dS$$

$$+ b_n^m (J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r Z_r^\ell)$$

$$\times \int_S P_n^m P_r^\ell P_p^q \sin m\lambda \sin \ell\lambda \cos q\lambda \, dS \Big] \tag{2.50}$$

$$Z_p^q = \sum_{\substack{nm \\ r\ell}} \Big[a_n^m (J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r Z_r^\ell)$$

$$\times \int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \sin q\lambda \, dS$$

$$+ b_n^m (J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r Y_r^\ell)$$

$$\times \int_S P_n^m P_r^\ell P_p^q \sin m\lambda \cos \ell\lambda \sin q\lambda \, dS \Big]$$

For convenience in writing define the following arrays:

$$\sum_{nm} a_n^m \int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \cos q\lambda \, dS$$

$$= AI(r, \ell, p, q)$$

$$\sum_{nm} b_n^m \int_S P_n^m P_r^\ell P_p^q \sin m\lambda \sin \ell\lambda \cos q\lambda \, dS,$$

$$= B1(r,\ell,p,q) \tag{2.51}$$

$$\sum_{nm} a_n^m \int_S P_n^m P_r^\ell P_p^q \cos m\lambda \sin \ell\lambda \sin q\lambda \, ds$$

$$= A2(r,\ell,p,q)$$

$$\sum_{nm} b_n^m \int_S P_n^m P_r^\ell P_p^q \sin m\lambda \cos \ell\lambda \sin q\lambda \, ds$$

$$= B2(r,\ell,p,q)$$

Substituting these into 2.48 and 2.49 results in the equations

$$y_p^q = \sum_{r\ell}\Big[\left(J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r y_r^\ell\right) A1(r,\ell,p,q)$$

$$+ \left(J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r z_r^\ell\right) B1(r,\ell,p,q)\Big]$$

$$z_p^q = \sum_{r\ell}\Big[\left(J_r V_r^\ell + J_r' V_r'^\ell + J_r' \alpha_r z_r^\ell\right) A2(r,\ell,p,q)$$

$$+ \left(J_r U_r^\ell + J_r' U_r'^\ell + J_r' \alpha_r y_r^\ell\right) B2(r,\ell,p,q)\Big] \tag{2.52}$$

These equations look very similar to matrix equations written out in subscript form except for the fact that A1, A2, B1, and B2 are 4-dimensional arrays instead of 2-dimensional matrices and y_r^ℓ, z_r^ℓ, U_r^ℓ, V_r^ℓ, $U_r'^\ell$ and $V_r'^\ell$ are 2-dimensional arrays instead of 1-dimensional column matrices. This problem can be circumvented if a mapping can be found that will map the 4-dimensional arrays into 2-dimensional arrays and the 2-dimensional arrays into 1-dimensional column matrices in such a way that the form of equations 2.51 is preserved. The mapping must of course be one to one and onto, (by which is meant each element of S_1 corresponds to one and only one element in S_2). The following mapping satisfies these criteria.

The r, ℓ -element of y_r^ℓ, etc. is mapped to the

$$\left[r\left(\frac{r+1}{2}\right) + (\ell + 1)\right]^{th} \text{ row} \tag{2.53}$$

of the column matrix which, without introducing any confusion, is labeled with the same name. Likewise the r, ℓ, p, q, element of A1, A2, B1, B2 is mapped to the

$$\left[r\left(\frac{r+1}{2}\right) + (\ell + 1)\right]^{th} \text{ column}$$

and the (2.54)

$$\left[p\left(\frac{p+1}{2}\right) + (q+1)\right]^{th} \text{ row}$$

of the 2-dimensional matrix which again is given the same name as its image. Thus for example

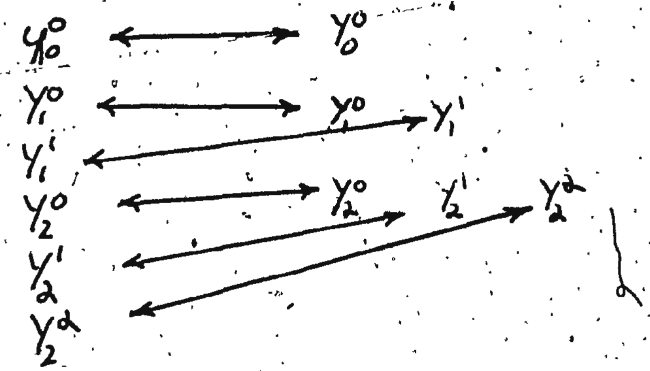

Figure 2.1 The mapping of equation 2.53.

Some further notation is needed before proceeding. Define

$$
\begin{aligned}
J_r\, A1(r,\ell,p,q) &\equiv [A1J] \\
J_r'\, A1(r,\ell,p,q) &\equiv [A1J1] \\
J_r'\, \alpha_r\, A1(r,\ell,p,q) &\equiv [A1J1A] \\
J_r\, B1(r,\ell,p,q) &\equiv [B1J] \\
J_r'\, B1(r,\ell,p,q) &\equiv [B1J1] \\
J_r'\, \alpha_r\, B1(r,\ell,p,q) &\equiv [B1J1A] \\
J_r\, A2(r,\ell,p,q) &\equiv [A2J] \\
J_r'\, \alpha_r\, A2(r,\ell,p,q) &\equiv [A2J1A] \\
J_r'\, A2(r,\ell,p,q) &\equiv [A2J1] \\
J_r\, B2(r,\ell,p,q) &\equiv [B2J] \\
J_r'\, B2(r,\ell,p,q) &\equiv [B2J1] \\
J_r'\, \alpha_r\, B2(r,\ell,p,q) &\equiv [B2J1A]
\end{aligned}
\tag{2.56}
$$

It is implicit in these identities that the R.H.S. is the image, under the mapping 2.54, of the arrays on the L.H.S. The equations may then be written

$$[Y] = [A1J][U] + [A1J1][U1] + [A1J1A][Y] + [B1J][V] + [B1J1][V1] + [B1J1A][Z]$$

$$[Z] = [A2J][V] + [A2J1][V1] + [A2J1A][Z] + [B2J][U] + [B2J1][U1] + [B2J1A][Y] \tag{2.57}$$

and the form of equations 2.52 is preserved. The constant term in the preceding equations is implicit in equation 2.56 in that $Y_0^0 = 0$. Rearranging slightly

$$[Y] = [I - A1J1A]^{-1}\{[A1J][U] + [A1J1][U1] + [B1J][V] + [B1J1][V1] + [B1J1A][Z]\}$$

$$[Z] = [I - A2J1A]^{-1}\{[A2J][V] + [A2J1][V1] + [B2J][U] + [B2J1][U1] + [B2J1A][Y]\} \tag{2.58}$$

which may for convenience be written as

$$[Y] = [\alpha] + [\beta][Z]$$

$$[Z] = [\gamma] + [\lambda][Y] \tag{2.59}$$

where

$$[\alpha] = [I - A1J1A]^{-1}\{[A1J][U] + [A1J1][U1] + [B1J][V] + [B1J1][V1]\}$$

$$[\beta] = [I - A1J1A]^{-1}[B1J1A]$$

$$[\gamma] = [I - A2J1A]^{-1}\{[A2J][V] + [A2J1][V1] + [B2J][U] + [B2J1][U1]\} \tag{2.60}$$

$$[\lambda] = [I - A2J1A]^{-1}[B2J1A]$$

Equations 2.60 are coupled matrix equations which must now be uncoupled. Substituting the first in the second and the second in the first gives

$$[y] = [\alpha] + [\beta]\{[\gamma] + [\lambda][y]\}$$
$$[z] = [\gamma] + [\lambda]\{[\alpha] + [\beta][z]\} \tag{2.61}$$

$$[y] - [\beta][\lambda][y] = [\alpha] + [\beta][\gamma]$$
$$[z] - [\lambda][\beta][z] = [\gamma] + [\lambda][\alpha] \tag{2.62}$$

$$[y] = [1 - [\beta][\lambda]]^{-1}\{[\alpha] + [\beta][\gamma]\}$$
$$[z] = [1 - [\lambda][\beta]]^{-1}\{[\gamma] + [\lambda][\alpha]\} \tag{2.63}$$

The equations are now completely uncoupled. The forcing terms in 2.63 are in the matrices α and γ and will now be separated out:

$$\begin{aligned}
[y] = {} & [1 - [\beta][\lambda]]^{-1}[1 - [A1][A]]^{-1}[A1][U] \\
& + [1 - [\beta][\lambda]]^{-1}[1 - [A1][A]]^{-1}[A1'][U'] \\
& + [1 - [\beta][\lambda]]^{-1}[1 - [A1][A]]^{-1}[B1][V] \\
& + [1 - [\beta][\lambda]]^{-1}[1 - [A1][A]]^{-1}[B1'][V'] \\
& + [1 - [\beta][\lambda]]^{-1}[\beta][1 - [A2][A]]^{-1}[A2][V] \\
& + [1 - [\beta][\lambda]]^{-1}[\beta][1 - [A2][A]]^{-1}[B2][U] \\
& + [1 - [\beta][\lambda]]^{-1}[\beta][1 - [A2][A]]^{-1}[B2'][U'] \\
& + [1 - [\beta][\lambda]]^{-1}[\beta][1 - [A2][A]]^{-1}[A2'][V']
\end{aligned} \tag{2.64}$$

$$
\begin{aligned}
[Z] = {} & [I-[\lambda][\beta]]^{-1}\,[I-[A2][A]]^{-1}\,[A2]\,[V] \\
& + [I-[\lambda][\beta]]^{-1}\,[I-[A2][A]]^{-1}\,[A2']\,[V'] \\
& + [I-[\lambda][\beta]]^{-1}\,[I-[A2][A]]^{-1}\,[B2]\,[U] \\
& + [I-[\lambda][\beta]]^{-1}\,[I-[A2][A]]^{-1}\,[B2']\,[U'] \\
& + [I-[\lambda][\beta]]^{-1}\,[\lambda]\,[I-[A1][A]]^{-1}\,[A1]\,[U] \\
& \quad [I-[\lambda][\beta]]^{-1}\,[\lambda]\,[I-[A1][A]]^{-1}\,[A1']\,[U'] \\
& \quad [I-[\lambda][\beta]]^{-1}\,[\lambda]\,[I-[A1][A]]^{-1}\,[B1]\,[V] \\
& \quad [I-[\lambda][\beta]]^{-1}\,[\lambda]\,[I-[A1][A]]^{-1}\,[B1']\,[V']
\end{aligned}
\qquad (2.65)
$$

Collecting terms, equations 2.64 and 2.65 may be written as

$$
\begin{aligned}
[Y] &= [L1][U] + [L2][U'] + [L3][V] + [L4][V'] \\
[Z] &= [L5][U] + [L6][U'] + [L7][V] + [L8][V']
\end{aligned}
\qquad (2.66)
$$

with the L matrices defined as

$$[L1] = [I-[\beta][\lambda]]^{-1}[I-[A1][A]]^{-1}[A1] + [I-[\beta][\lambda]]^{-1}[\beta][I-[A2][A]]^{-1}[\beta 2]$$

$$[L2] = [I-[\beta][\lambda]]^{-1}[I-[A1][A]]^{-1}[A1] + [I-[\beta][\lambda]]^{-1}[\beta][I-[A2][A]]^{-1}[A1]$$

$$[L3] = [I-[\beta][\lambda]]^{-1}[I-[A1][A]]^{-1}[B1] + [I-[\beta][\lambda]]^{-1}[\beta][I-[A2][A]]^{-1}[A2]$$

$$[L4] = [I-[\beta][\lambda]]^{-1}[I-[A1][A]]^{-1}[B1] + [I-[\beta][\lambda]]^{-1}[\beta][I-[A2][A]]^{-1}[A2] \qquad (2.67)$$

$$[L5] = [I-[\lambda][\beta]]^{-1}[I-[A2][A]]^{-1}[B2] + [I-[\lambda][\beta]]^{-1}[\lambda][I-[A1][A]]^{-1}[A1]$$

$$[L6] = [I-[\lambda][\beta]]^{-1}[I-[A2][A]]^{-1}[B2] + [I-[\lambda][\beta]]^{-1}[\lambda][I-[A1][A]]^{-1}[B2]$$

$$[L7] = [I-[\lambda][\beta]]^{-1}[I-[A2][A]]^{-1}[A2] + [I-[\lambda][\beta]]^{-1}[\lambda][I-[A1][A]]^{-1}[B1]$$

$$[L8] = [I-[\lambda][\beta]]^{-1}[I-[A2][A]]^{-1}[A2] + [I-[\lambda][\beta]]^{-1}[\lambda][I-[A1][A]]^{-1}[B1]$$

Due to the nature of the forcing potential, generally only the L1 and L5 atrices will be needed. All of the tidal prediction matrices have, nowever, been calculated. The programming of the calculations is discussed in Chapter 3.

Among the tide height matrices L1 and L7 represent direct coupling, i.e. U_r^ℓ exciting Y_r^ℓ. L3 and L5 are cross-coupling matrices. In a global ocean model L1 and L7 should be of the form

$$J_{\eta\, eff} \cdot \delta_{nr}\delta_{m\ell}\, U_r^\ell \qquad (2.68)$$

where $J_{\eta\, eff}$ is the global effective tide reducing factor (equation 2.26) and L3 and L5 would be null matrices.

The operations of chapter 2 have been programmed and the tide height matrices are given in table 2.3a for the case of no feedback and in table 2.3b for the case of feedback. Note that in matrices L3, L4, L7 and L8 columns 1,2,4 and 7 contain only zeros. This is because these particular columns represent excitation from terms in the prescribed potential which do not exist i.e. the zonal coefficients with a sine function longitude dependence.

Table 2.3a The tide height matrices with the oceanic feedback neglected

MATRIX AIJ (L1 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,2180	0,6434	-0,0310	-0,0391	-0,0219	0,0424	-0,0387	-0,0007	0,0639	-0,0509
-0,1871	-0,0310	0,7548	0,0427	-0,0345	-0,0978	0,0237	0,0066	-0,0486	0,0492
-0,1340	-0,0943	0,1031	0,4403	0,0092	0,0290	0,0061	-0,0343	0,0289	-0,0014
-0,1033	-0,0704	-0,1110	0,0123	0,5321	-0,0855	0,0472	-0,0650	-0,0784	-0,0642
0,0516	0,0342	-0,0787	0,0097	-0,0214	0,5131	-0,0086	0,0120	-0,0805	-0,1270
0,1190	-0,1124	0,0689	0,0073	0,0426	-0,0312	0,5057	0,0095	-0,0299	0,0991
0,1426	-0,0032	0,0287	-0,0618	-0,0879	0,0650	0,0142	0,6028	0,0053	0,0273
0,1435	0,1115	-0,0848	0,0209	-0,0425	-0,1742	-0,0180	-0,0095	0,5515	-0,0380
-0,0134	-0,0148	0,0143	-0,0002	-0,0058	-0,0458	0,0099	0,0015	-0,0063	0,5897

MATRIX AIJI (L2 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,2472	0,8300	-0,0400	-0,0961	-0,0538	0,1044	-0,0897	-0,0017	0,1482	-0,1179
-0,2122	-0,0400	0,9736	0,1050	-0,0848	-0,2405	0,0550	0,0153	-0,1128	0,1141
-0,1520	-0,1217	0,1330	1,0829	0,0226	0,0713	0,0141	-0,0794	0,0671	-0,0032
-0,1171	-0,0908	-0,1432	0,0302	1,3087	-0,2102	0,1095	-0,1507	-0,1819	-0,1489
0,0585	0,0441	-0,1016	0,0238	-0,0526	1,2620	-0,0200	0,0278	-0,1866	-0,2944
0,1349	-0,1450	0,0889	0,0180	0,1048	-0,0767	1,1726	0,0220	-0,0694	0,2297
0,1617	-0,0041	0,0371	-0,1520	-0,2163	0,1598	0,0330	1,3977	0,0122	0,0634
0,1627	0,1438	-0,1094	0,0514	-0,1044	-0,4285	-0,0416	-0,0221	1,2789	-0,0882
-0,0152	-0,0191	0,0184	-0,0004	-0,0142	-0,1127	0,0230	0,0034	-0,0147	1,3675

MATRIX BIJ (L3 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	-0,0395	0,0	-0,0323	-0,0539	0,0	-0,0069	-0,0099	0,0048
0,0	0,0	0,0016	0,0	-0,0270	-0,1001	0,0	-0,0056	-0,0248	-0,1290
0,0	0,0	0,0027	0,0	0,0043	-0,0154	0,0	-0,0253	0,0171	0,1270
0,0	0,0	-0,0869	0,0	-0,0085	-0,0442	0,0	-0,0118	-0,1113	-0,0799
0,0	0,0	-0,0258	0,0	0,0131	-0,0589	0,0	0,0093	0,0119	-0,0945
0,0	0,0	0,0711	0,0	0,0212	0,1045	0,0	0,0117	0,0053	-0,0415
0,0	0,0	-0,0242	0,0	-0,0160	0,0245	0,0	-0,0036	-0,0469	0,0870
0,0	0,0	0,0516	0,0	-0,0023	0,0258	0,0	-0,0005	0,0079	-0,0336
0,0	0,0	-0,0422	0,0	0,0158	-0,0041	0,0	0,0131	0,0178	-0,0095

Table 2.3a continued

MATRIX B1JI (L4 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	-0,0510	0,0	-0,0796	-0,1327	0,0	-0,0160	-0,0229	0,0112
0,0	0,0	0,0020	0,0	-0,0663	-0,2462	0,0	-0,0129	-0,0575	-0,2991
0,0	0,0	0,0035	0,0	0,0105	-0,0379	0,0	-0,0586	0,0397	0,2945
0,0	0,0	-0,1120	0,0	-0,0209	-0,1086	0,0	-0,0274	-0,2582	-0,1853
0,0	0,0	-0,0333	0,0	0,0322	-0,1448	0,0	0,0216	0,0277	-0,2192
0,0	0,0	0,0917	0,0	0,0522	0,2571	0,0	0,0272	0,0123	-0,0962
0,0	0,0	-0,0313	0,0	-0,0394	0,0603	0,0	-0,0083	-0,1088	0,2017
0,0	0,0	0,0666	0,0	-0,0056	0,0636	0,0	-0,0011	0,0184	-0,0778
0,0	0,0	-0,0544	0,0	0,0389	-0,0101	0,0	0,0304	0,0412	-0,0221

MATRIX B2J (L5 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,0970	-0,0395	0,0016	0,0011	-0,0270	-0,0321	0,0245	-0,0056	0,0296	-0,1451
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1517	-0,1042	-0,0869	0,0057	-0,0085	0,0524	0,0235	-0,0118	-0,0042	0,1754
0,0026	-0,0434	-0,0806	-0,0051	-0,0110	-0,0589	0,0290	0,0045	0,0119	-0,0113
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1037	-0,0300	-0,0242	-0,0456	-0,0160	0,0505	0,0176	-0,0036	-0,0039	0,1577
-0,1824	-0,0172	-0,0432	0,0124	-0,0603	0,0258	0,0032	-0,0177	0,0079	0,1066
-0,0745	0,0014	-0,0375	0,0153	-0,0072	-0,0341	-0,0041	0,0084	-0,0056	-0,0095

MATRIX B2JI (L6 WITHOUT FEEDBACK)

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1100	-0,0510	0,0020	0,0028	-0,0663	-0,0789	0,0568	-0,0129	0,0686	-0,3365
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1493	-0,1344	-0,1120	0,0141	-0,0209	0,1290	0,0545	-0,0274	-0,0097	0,4067
0,0029	-0,0560	-0,1040	-0,0126	-0,0271	-0,1448	0,0672	0,0105	0,0277	-0,0263
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1176	-0,0387	-0,0313	-0,1122	-0,0394	0,1241	0,0408	-0,0083	-0,0091	0,3657
-0,2068	-0,0222	-0,0557	0,0304	-0,1482	0,0636	0,0074	-0,0410	0,0184	0,2472
-0,0845	0,0018	-0,0483	0,0376	-0,0177	-0,0839	-0,0096	0,0195	-0,0130	-0,0221

Table 2.3a continued

MATRIX A2J (L7 WITHOUT FEEDBACK)

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.6928	0.0	-0.0769	-0.0740	0.0	-0.0325	0.0022	0.0112
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.2477	0.0	-0.4223	0.0019	0.0	-0.1402	-0.0253	0.2085
0.0	0.0	-0.0596	0.0	0.0005	0.4968	0.0	0.0176	-0.0123	-0.1248
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.1415	0.0	-0.1897	0.0952	0.0	0.4923	-0.0236	0.1357
0.0	0.0	0.0039	0.0	-0.0137	-0.0266	0.0	0.0022	0.5835	-0.0908
0.0	0.0	0.0032	0.0	0.0188	-0.0450	0.0	0.0094	-0.0151	0.5923

MATRIX A2JI (L8 WITHOUT FEEDBACK)

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.8938	0.0	-0.1891	-0.1820	0.0	-0.0753	0.0051	0.0259
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.3195	0.0	1.0385	0.0048	0.0	-0.3250	-0.0587	0.4835
0.0	0.0	-0.0769	0.0	0.0012	1.2219	0.0	0.0408	-0.0285	-0.2893
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.1826	0.0	-0.4665	0.2342	0.0	1.1415	-0.0546	0.3146
0.0	0.0	0.0050	0.0	-0.0337	-0.0653	0.0	0.0052	1.3530	-0.2105
0.0	0.0	0.0042	0.0	0.0463	-0.1107	0.0	0.0218	-0.0351	1.3735

Table 2.3b The tide height matrices with ocean feedback

MATRIX L1

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,4521	0,7948	-0,0249	-0,0466	-0,0276	0,0517	-0,0576	-0,0034	0,0752	-0,0671
-0,4114	-0,0249	0,9577	0,0643	-0,0434	-0,1405	0,0276	0,0056	-0,0704	0,0778
-0,2022	-0,1127	0,1552	0,5095	0,0137	0,0255	0,0067	-0,0459	0,0262	0,0027
-0,1921	-0,0889	-0,1399	0,0182	0,6338	-0,1123	0,0584	-0,0874	-0,1046	-0,0763
0,1126	0,0416	-0,1131	0,0085	-0,0281	0,6124	-0,0122	0,0167	-0,0997	-0,1651
0,2353	-0,1672	0,0801	0,0081	0,0527	-0,0439	0,5656	0,0124	-0,0352	0,1227
0,3067	-0,0142	0,0239	-0,0829	-0,1185	0,0892	0,0186	0,6850	0,0114	0,0294
0,2784	0,1311	-0,1228	0,0191	-0,0563	-0,2161	-0,0212	-0,0101	0,6312	-0,0480
-0,0286	-0,0195	0,0226	0,0003	-0,0069	-0,0595	0,0123	0,0015	-0,0080	0,6701

MATRIX L2

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,5127	1,0252	-0,0321	-0,1147	-0,0679	0,1272	-0,1335	-0,0079	0,1744	-0,1556
-0,4666	-0,0321	1,2355	0,1580	-0,1068	-0,3456	0,0639	0,0129	-0,1632	0,1803
-0,3201	-0,1454	0,2002	1,2531	0,0336	0,0627	0,0155	-0,1065	0,0606	0,0063
-0,2179	-0,1146	-0,1804	0,0448	1,5587	-0,2762	0,1354	-0,2027	-0,2425	-0,1769
0,1276	0,0537	-0,1459	0,0209	-0,0691	1,5060	-0,0282	0,0387	-0,2312	-0,3828
0,2668	-0,2157	0,1034	0,0198	0,1296	-0,1079	1,3116	0,0289	-0,0817	0,2844
0,3478	-0,0183	0,0308	-0,2040	-0,2915	0,2194	0,0432	1,5884	0,0265	0,0681
0,3157	0,1692	-0,1584	0,0469	-0,1386	-0,5314	-0,0491	-0,0235	1,4636	-0,1114
-0,0325	-0,0252	0,0291	0,0007	-0,0169	-0,1465	0,0285	0,0035	-0,0186	1,5537

MATRIX L3

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	-0,0397	0,0	-0,0381	-0,0748	0,0	-0,0046	-0,0006	0,0324
0,0	0,0	0,0196	0,0	-0,0340	-0,1385	0,0	-0,0054	-0,0212	-0,1419
0,0	0,0	0,0128	0,0	0,0099	-0,0265	0,0	-0,0304	0,0279	0,1724
0,0	0,0	-0,1132	0,0	-0,0047	-0,0490	0,0	-0,0136	-0,1400	-0,0827
0,0	0,0	-0,0388	0,0	0,0160	-0,0752	0,0	0,0100	0,0140	-0,1206
0,0	0,0	0,0828	0,0	0,0227	0,1332	0,0	0,0120	-0,0015	-0,0713
0,0	0,0	-0,0362	0,0	-0,0213	0,0328	0,0	-0,0032	-0,0631	0,0857
0,0	0,0	0,0644	0,0	-0,0082	0,0384	0,0	-0,0030	0,0048	-0,0480
0,0	0,0	-0,0572	0,0	0,0207	-0,0028	0,0	0,0159	0,0228	-0,0079

Table 2.3b continued

MATRIX L4

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	-0,0512	0,0	-0,0938	-0,1841	0,0	-0,0106	-0,0014	0,0751
0,0	0,0	0,0253	0,0	-0,0836	-0,3406	0,0	-0,0125	-0,0491	-0,3289
0,0	0,0	0,0165	0,0	0,0244	-0,0652	0,0	-0,0705	0,0647	0,3997
0,0	0,0	-0,1460	0,0	-0,0115	-0,1204	0,0	-0,0315	-0,3246	-0,1917
0,0	0,0	-0,0500	0,0	0,0395	-0,1849	0,0	0,0233	0,0325	-0,2796
0,0	0,0	0,1068	0,0	0,0559	0,3277	0,0	0,0278	-0,0034	-0,1653
0,0	0,0	-0,0467	0,0	-0,0524	0,0807	0,0	-0,0073	-0,1463	0,1987
0,0	0,0	0,0830	0,0	-0,0201	0,0944	0,0	-0,0069	0,0111	-0,1113
0,0	0,0	-0,0738	0,0	0,0509	-0,0068	0,0	0,0368	0,0528	-0,0183

MATRIX L5

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1969	-0,0397	0,0196	0,0053	-0,0351	-0,0481	0,0285	-0,0084	0,0369	-0,1967
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,2341	-0,1228	-0,1094	0,0132	-0,0047	0,0642	0,0252	-0,0156	-0,0151	0,2309
0,0293	-0,0603	-0,1115	-0,0088	-0,0122	-0,0752	0,0369	0,0059	0,0178	-0,0079
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1763	-0,0199	-0,0234	-0,0550	-0,0179	0,0548	0,0179	-0,0030	-0,0109	0,1887
-0,3598	-0,0011	-0,0370	0,0200	-0,0759	0,0305	-0,0008	-0,0239	0,0046	0,1371
-0,1515	0,0094	-0,0412	0,0207	-0,0075	-0,0435	-0,0071	0,0090	-0,0080	-0,0079

MATRIX L6

0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,2233	-0,0512	0,0253	0,0131	-0,0864	-0,1184	0,0660	-0,0194	0,0855	-0,4561
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,2654	-0,1584	-0,1412	0,0325	-0,0116	0,1579	0,0584	-0,0361	-0,0350	0,5354
0,0332	-0,0777	-0,1438	-0,0217	-0,0301	-0,1849	0,0856	0,0138	0,0412	-0,0182
0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,1999	-0,0257	-0,0302	-0,1354	-0,0439	0,1348	0,0415	-0,0068	-0,0252	0,4375
-0,4081	-0,0014	-0,0477	0,0492	-0,1866	0,0750	-0,0019	-0,0555	0,0107	0,3180
-0,1718	0,0121	-0,0532	0,0509	-0,0185	-0,1069	-0,0165	0,0210	-0,0185	-0,0184

Table 2.3b continued

MATRIX L7

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.8522	0.0	-0.1017	-0.1022	0.0	-0.0386	0.0096	0.0247
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.3275	0.0	0.4978	0.0070	0.0	-0.1695	-0.0242	0.2764
0.0	0.0	-0.0822	0.0	0.0018	0.5857	0.0	0.0229	-0.0154	-0.1580
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.1679	0.0	-0.2294	0.1240	0.0	0.5559	-0.0226	0.1561
0.0	0.0	0.0164	0.0	-0.0137	-0.0330	0.0	0.0055	0.6673	-0.0897
0.0	0.0	0.0071	0.0	0.0249	-0.0570	0.0	0.0108	-0.0151	0.6829

MATRIX L8

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0993	0.0	-0.2501	-0.2512	0.0	-0.0895	0.0223	0.0573
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.4225	0.0	1.2243	0.0172	0.0	-0.3929	-0.0562	0.6410
0.0	0.0	-0.1061	0.0	0.0043	1.4406	0.0	0.0530	-0.0357	-0.3664
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.2166	0.0	-0.5643	0.3050	0.0	1.2891	-0.0524	0.3620
0.0	0.0	0.0211	0.0	-0.0336	-0.0812	0.0	0.0127	1.5474	-0.2081
0.0	0.0	0.0092	0.0	0.0613	-0.1402	0.0	0.0251	-0.0349	1.5834

Several tide maps have been produced using the tide height matrices. Unfortunately the maps show no distinct correlation with the ocean function. Figure 2.2 a,b is the tide range of the fortnightly tide. For comparison the tide range of the fortnightly tide using 1.11 and truncating to zero on land is shown in figure 2.3 a,b. The tide heights in these maps are given in decimeters and the negative sign in the polar regions is due to the phase. For a degree-2 order-zero tide potential the oceans in high latitudes oscilate 180 degrees out of phase with the low latitude oceans. The tide range of the pole tide is shown in figure 2.4 a,b. The amplitude of the polar motion was taken arbitrarily to be .25 seconds of arc to obtain a prescribed pole tide potential. The phase relationship of the tide is shown in that opposite quadrants of the Earth have the same phase.

Figure 2.2a The map of the fortnightly tide range , eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al (1973) with $e \leq 0.5$

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	0
10	-0.9	-0.9	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	10
20	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.8	-0.8	-0.8	-0.8	20
30	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.6	-0.6	-0.6	-0.6	30
40	-0.4	-0.4	-0.4	-0.4	-0.3	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	40
50	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	-0.1	-0.0	0.0	0.1	0.1	0.1	0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.0	50
60	-0.0	-0.0	-0.0	-0.0	-0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2	60
70	0.1	0.1	0.1	0.1	0.1	0.2	0.3	0.3	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.5	70
80	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.6	0.6	80
90	0.3	0.2	0.2	0.2	0.2	0.3	0.4	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.6	0.7	90
100	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.5	0.5	0.6	100
110	0.2	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.4	0.4	110
120	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.2	120
130	-0.2	-0.2	-0.2	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	130
140	-0.4	-0.4	-0.4	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.4	-0.4	-0.4	-0.4	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	140
150	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.8	-0.8	150
160	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-1.0	160
170	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.1	-1.1	170
180	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure 2.2b The map of the fortnightly tide range. western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with C ≤ 0.5

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360
0	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	0.0
10	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	0.0
20	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.7	-0.7	-0.7	-0.7	-0.7	0.0
30	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.5	-0.5	-0.5	-0.5	-0.5	0.0
40	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	0.0
50	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	0.0
60	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.0
70	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.0
80	0.6	0.7	0.7	0.8	0.7	0.7	0.7	0.6	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.0
90	0.7	0.7	0.8	0.8	0.8	0.8	0.7	0.6	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.0
100	0.6	0.7	0.7	0.7	0.7	0.7	0.6	0.5	0.4	0.4	0.4	0.3	0.4	0.4	0.4	0.4	0.3	0.3	0.0
110	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.0
120	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	0.0	0.0	0.0	0.0
130	-0.2	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	0.0
140	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.6	-0.6	-0.6	-0.6	-0.5	-0.5	-0.5	-0.4	-0.4	-0.4	-0.0
150	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.7	-0.7	-0.7	-0.7	-0.6	0.0
160	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-0.9	-0.9	-0.9	-0.9	-0.9	-0.8	0.0
170	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	0.0
180	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1	0.0
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360

Figure 2.3a The map of the tide range of the fortnightly tide, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with $C \leq 0.5$ (no feedback)

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	0
10	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.8	-0.8	-0.8	-0.8	-0.8	10
20	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.7	-0.7	-0.7	-0.7	20
30	-0.5	-0.5	-0.5	-0.5	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.5	-0.5	-0.5	-0.5	-0.5	30
40	-0.3	-0.3	-0.3	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.3	-0.3	-0.3	-0.3	-0.3	40
50	-0.1	-0.1	-0.2	-0.1	-0.1	-0.1	-0.0	0.0	0.0	0.1	0.1	0.1	0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.1	50
60	0.0	-0.0	-0.0	-0.0	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.1	0.2	60
70	0.1	0.1	0.1	0.1	0.1	0.2	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.4	70
80	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.5	0.5	80
90	0.3	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.5	0.5	0.5	90
100	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.5	100
110	0.2	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.3	0.3	0.3	110
120	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	120
130	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	130
140	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	140
150	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	-0.6	150
160	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	160
170	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	170
180	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure 2.3b The map of the tide range of the fortnightly tide, western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al (1973) with $e \leq 0.5$ (no feedback)

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360
0	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	0.0
10	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.7	0.0
20	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.6	-0.6	-0.6	-0.6	0.0
30	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	0.0
40	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	0.0
50	-0.1	-0.1	-0.1	-0.0	-0.0	-0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	0.0
60	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.0
70	0.4	0.4	0.4	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.0
80	0.5	0.5	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.0
90	0.5	0.6	0.6	0.7	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.0
100	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.0
110	0.3	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.0
120	0.1	0.2	0.2	0.2	0.2	0.1	0.1	0.0	0.0	-0.0	-0.0	-0.0	-0.0	0.0	0.0	0.0	0.1	0.0	0.0
130	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	-0.1	0.0
140	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.4	-0.4	-0.4	-0.4	-0.3	0.0
150	-0.6	-0.6	-0.6	-0.6	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.6	-0.6	-0.6	-0.6	-0.6	-0.5	0.0
160	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.8	-0.7	-0.7	0.0
170	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	0.0
180	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	-0.9	0.0
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	3 0	340		

Figure 2.4a The map of the tide range of the equilibrium pole tide, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al (1973). C < 0.5

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0
10	-0.1	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	-0.1	-0.1	-0.1	-0.1	-0.0	-0.0	0.0	0.0	0.0	0.1	0.1	0.1	10
20	[illegible]	-0.2	-0.2	-0.2	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	-0.1	-0.1	-0.0	0.0	0.1	0.2	0.2	0.2	0.3	20
30	-0.2	-0.3	-0.3	-0.3	-0.3	-0.4	-0.3	-0.3	-0.3	-0.2	-0.2	-0.1	-0.0	0.1	0.2	0.2	0.3	0.4	0.4	30
40	-0.2	-0.3	-0.3	-0.4	-0.4	-0.4	-0.4	-0.4	-0.4	-0.3	-0.2	-0.1	-0.0	0.1	0.2	0.3	0.4	0.5	0.6	40
50	-0.2	-0.3	-0.3	-0.4	-0.5	-0.5	-0.5	-0.5	-0.4	-0.4	-0.3	-0.2	-0.1	0.0	0.2	0.3	0.4	0.5	0.6	50
60	-0.1	-0.2	-0.3	-0.4	-0.5	-0.5	-0.5	-0.5	-0.4	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.4	0.5	0.6	60
70	-0.0	-0.2	-0.3	-0.4	-0.5	-0.5	-0.5	-0.5	-0.4	-0.3	-0.2	-0.2	-0.1	-0.0	0.0	0.1	0.3	0.4	0.5	70
80	0.0	-0.1	-0.2	-0.3	-0.4	-0.4	-0.4	-0.3	-0.3	-0.2	-0.2	-0.1	-0.1	-0.1	-0.0	0.0	0.1	0.2	0.3	80
90	0.1	0.0	-0.1	-0.2	-0.2	-0.2	-0.2	-0.1	-0.1	-0.0	-0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.0	0.0	90
100	0.2	0.2	0.1	0.0	0.0	0.0	0.1	0.1	0.2	0.2	0.2	0.1	0.0	-0.1	-0.2	-0.3	-0.3	-0.3	-0.3	100
110	0.4	0.3	0.3	0.3	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.2	0.1	-0.1	-0.2	-0.4	-0.5	-0.5	-0.5	110
120	0.5	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.6	0.6	0.5	0.4	0.2	-0.0	-0.2	-0.4	-0.6	-0.7	-0.7	120
130	0.6	0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.7	0.6	0.4	0.2	0.0	-0.2	-0.4	-0.6	-0.8	-0.9	130
140	0.6	0.7	0.7	0.8	0.8	0.9	0.9	0.9	0.8	0.8	0.6	0.5	0.3	0.1	-0.2	-0.4	-0.6	-0.7	-0.9	140
150	0.6	0.7	0.7	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.6	0.4	0.2	0.1	-0.1	-0.3	-0.5	-0.6	-0.7	150
160	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.6	0.6	0.5	0.4	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.4	-0.5	160
170	0.3	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	-0.0	-0.1	-0.2	-0.2	-0.3	170
180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure 2.4b The map of the equilibrium pole tide, western hemisphere. The oceanic boundaries are defined by the ocean function of Balmino *et al* with $e \leq 0.5$

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360
0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	0.0
10	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	-0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	0.0
20	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	0.0
30	0.4	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.1	0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2	-0.2	0.0
40	0.6	0.6	0.6	0.6	0.5	0.4	0.3	0.2	0.1	0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2	0.0
50	0.6	0.7	0.7	0.7	0.6	0.5	0.4	0.3	0.2	0.1	-0.0	-0.0	-0.1	-0.0	-0.0	-0.0	-0.1	-0.1	0.0
60	0.6	0.7	0.7	0.7	0.7	0.6	0.4	0.3	0.2	0.1	-0.0	-0.0	-0.0	0.0	0.0	0.0	0.0	-0.0	0.0
70	0.5	0.6	0.6	0.7	0.6	0.5	0.4	0.3	0.1	0.0	-0.0	-0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.0
80	0.3	0.4	0.4	0.5	0.4	0.4	0.3	0.2	0.1	-0.0	-0.0	-0.0	0.0	0.1	0.2	0.2	0.2	0.1	0.0
90	0.0	0.1	0.2	0.2	0.2	0.1	0.1	-0.0	-0.1	-0.1	-0.1	-0.0	0.0	0.1	0.2	0.3	0.3	0.2	0.0
100	-0.3	-0.2	-0.2	-0.1	-0.1	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	0.0	0.1	0.2	0.3	0.3	0.3	0.0
110	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.4	-0.4	-0.4	-0.3	-0.2	-0.0	0.1	0.2	0.3	0.4	0.4	0.0
120	-0.7	-0.8	-0.8	-0.8	-0.8	-0.7	-0.7	-0.7	-0.6	-0.5	-0.4	-0.2	-0.1	0.1	0.2	0.3	0.4	0.5	0.0
130	-0.9	-0.9	-1.0	-1.0	-1.0	-0.9	-0.9	-0.8	-0.7	-0.6	-0.5	-0.3	-0.1	0.0	0.2	0.3	0.4	0.5	0.0
140	-0.9	-0.9	-1.0	-1.0	-1.0	-1.0	-0.9	-0.9	-0.8	-0.6	-0.5	-0.3	-0.2	-0.0	0.1	0.3	0.4	0.5	0.0
150	-0.7	-0.8	-0.9	-0.9	-0.9	-0.9	-0.9	-0.8	-0.7	-0.6	-0.5	-0.3	-0.2	-0.0	0.1	0.2	0.4	0.5	0.0
160	-0.5	-0.6	-0.7	-0.7	-0.7	-0.7	-0.7	-0.6	-0.5	-0.5	-0.4	-0.3	-0.1	-0.0	0.1	0.2	0.3	0.4	0.0
170	-0.3	-0.3	-0.3	-0.4	-0.4	-0.4	-0.3	-0.3	-0.3	-0.2	-0.2	-0.1	-0.1	-0.0	0.0	0.1	0.2	0.2	0.0
180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360

The lack of correlation of these tides with the maps of the ocean function (Appendix B) is due to a combination of two errors. First the ocean function is accurate to at best 30% and we are attempting to find a surface harmonic representation of the product (the tide) of two functions given only the representations of the two separate functions (the ocean function and the prescribed potential) any error in the representations of these two functions will be compounded in the representation of their product. The second error arises from a limitation imposed on the matrix method by the selection rules of the integrals in 2.48 and 2.49. For example assuming that the ocean function is available to degree-8 and that the prescribed potential is a degree-2 order-zero then the Y_p^q term in the tide is given by 2.49

$$Y_p^q = \sum_{\substack{nm \\ r\ell}} a_n^m J_2 U_2^0 \int_S P_n^m P_r^\ell P_p^q \cos m\lambda \cos \ell\lambda \cos q\lambda \, dS \tag{2.69}$$

What is the maximum degree in the tide height that can be accurately obtained? The important selection rule is

$$|2-n| \leq p \leq n+2 \tag{2.70}$$

It is found that for Y_8^0 we need a term a_{10}^0 which is not available. Y_6^0 is the last term which can be evaluated exactly. Figure 2.5 shows the cutoff of terms in the matrix which can be evaluated exactly. Terms which are above the curved line can be evaluated exactly, terms which fall on or below the curved line cannot because their evaluation requires one or more ocean function coefficients which are not available. The solution of the feedback problem requires that the matrices be square. Unfortunately the largest matrix, all of whose elements are correct, is only about 12×12 which would give a tide to degree-4 order-2. Since this is not much an improvement over a degree-3 order-3 expansion it was decided to use 10×10 matrices. The ocean function expanded to degree-3 order-3 is barely adequate, the position and size of only the largest land masses are apparent, and the values predicted by the ocean function are in places over 100% off. It cannot be expected that the even more variable tide will be represented well by a degree-2 order-3 expansion.

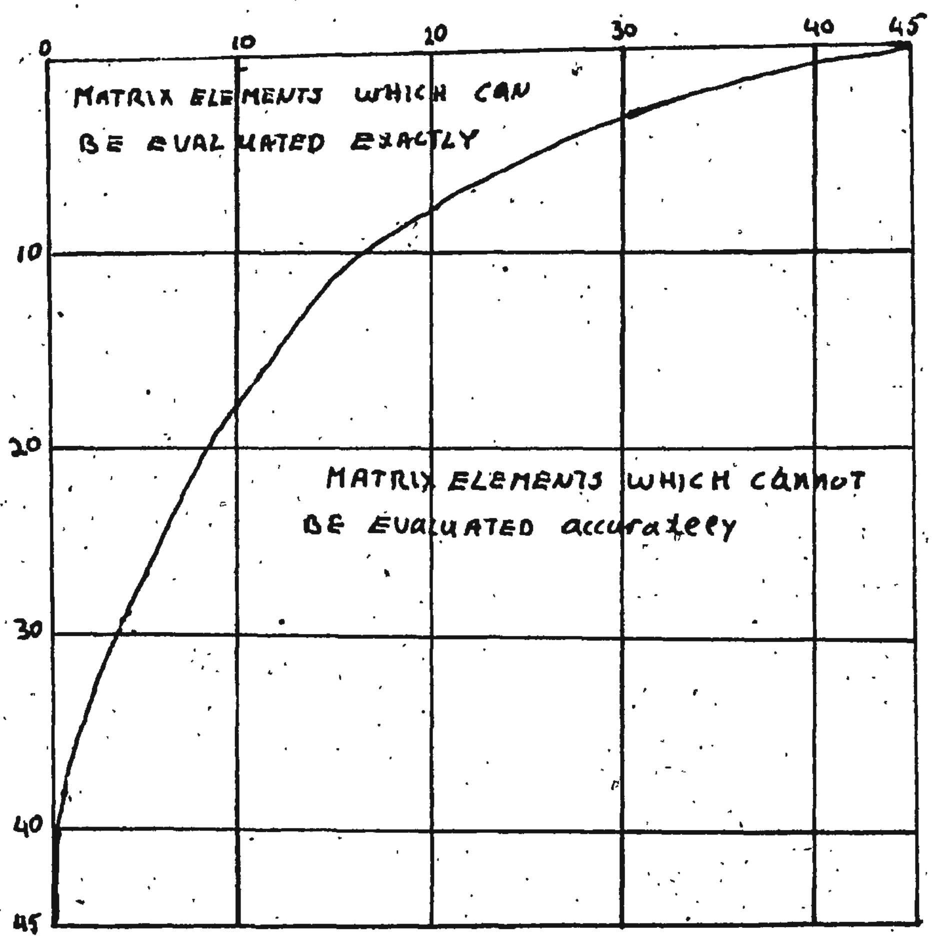

Figure 2.5 error analysis for a matrix in the matrix method

CHAPTER 3

THE PERTURBED POTENTIAL OF AN EARTH WITH NON-GLOBAL OCEANS

i) Introduction

The results of Chapter 2 suggest that the perturbed potential of a non-spherically-symmetric Earth may be expressed in much the same way as the tide height in non-global oceans, that is with matrices. With that end in view an attempt is made in section 3.ii to write the perturbed potential of an Earth model, whose oceans respond in equilibrium fashion to a prescribed potential, in a matrix formulation. In section 3.iii the matrix method is compared with the method of Kaula (1969) and Kaula's equations are transformed into matrix equations, so that the parameters of the matrix method can be given in terms of the parameters of Kaula's method.

ii) The Perturbed Potential in the Matrix Method

In response to a prescribed body force potential

$$U = \sum_{\ell m} P_\ell^m (u_\ell^m \cos m\lambda + v_\ell^m \sin m\lambda) \qquad (3.1)$$

the potential due to the deformation is

$$V = \sum_{\ell m} P_\ell^m (q_\ell^m \cos m\lambda + r_\ell^m \sin m\lambda) \qquad (3.2)$$

with the coefficient given in subscript form by

$$q_\ell^m = \sum_{jk} K1(\ell,m,j,k)\, U_j^k + K3(\ell,m,j,k)\, V_j^k$$
$$r_\ell^m = \sum_{jk} K5(\ell,m,j,k)\, U_j^k + K7(\ell,m,j,k)\, V_j^k \tag{3.3}$$

in direct anology with 2.66. Using 3.3 and 3.2 the perturbed potential can be written

$$V = \sum_{\substack{\ell m\\ jk}} P_\ell^m \Big[\big(K1(\ell,m,j,k) \cos m\lambda + K5(\ell,m,j,k) \sin m\lambda \big) U_j^k$$
$$+ K3(\ell,m,j,k) \cos m\lambda + K7(\ell,m,j,k) \sin m\lambda \big) V_j^k \tag{3.4}$$

The matrices K1, K3, K5, K7 may be evaluated using the matrices L1, L3, L5, L7 of equation 2.67 , as I now show.

The potential of the ocean tide is

$$\sum_{\ell m} \frac{3 g \rho_0}{(2\ell+1)\rho} P_\ell^m \left(y_\ell^m \cos m\lambda + z_\ell^m \sin m\lambda \right) \tag{3.5}$$

Now from 2.66 we have

$$g\, y_\ell^m = \sum_{jk} L1(\ell,m,j,k)\, U_j^k + L3(\ell,m,j,k)\, V_j^k$$
$$g\, z_\ell^m = \sum_{jk} L5(\ell,m,j,k)\, U_j^k + L7(\ell,m,j,k)\, V_j^k \tag{3.6}$$

Therefore the potential of the equilibrium ocean tide and associated Earth deformation is

$$T = \sum_{\substack{\ell m\\ jk}} (1+k_\ell') \frac{3\rho_0}{(2\ell+1)\rho} P_\ell^m \Big\{ \big[L1(\ell,m,j,k)\, U_j^k + L3(\ell,m,j,k)\, V_j^k \big] \cos m\lambda$$
$$+ \big[L5(\ell,m,j,k)\, U_j^k + L7(\ell,m,j,k)\, V_j^k \big] \sin m\lambda \tag{3.7}$$

To this must be added the potential of the solid Earth as a direct result of the prescribed potential :

$$W = \sum_{\ell m} k_\ell P_\ell^m \left(U_\ell^m \cos m\lambda + V_\ell^m \sin m\lambda \right) \tag{3.8}$$

which may be re-written using the Kronecker delta symbol

$$W = \sum_{\substack{jk \\ \ell m}} P_\ell^m \Big[(k_\ell \delta_{\ell j} \delta_{mk}) U_j^k \cos m\lambda \\ + (k_\ell \delta_{\ell j} \delta_{mk}) V_j^k \sin m\lambda \Big] \tag{3.9}$$

Adding 3.9 and 3.7 we have for the complete Earth response

$$V = T + W = \sum_{\substack{jk \\ \ell m}} P_\ell^m \Big\{ \\ \Big(\Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L1(\ell,m,j,k) + k_\ell \delta_{\ell j} \delta_{mk} \Big] U_j^k \\ + \Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L3(\ell,m,j,k) \Big] V_j^k \Big) \cos m\lambda \\ + \Big(\Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L5(\ell,m,j,k) \Big] U_j^k \\ + \Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L7(\ell,m,j,k) + k_\ell' \delta_{\ell j} \delta_{mk} \Big] V_j^k \Big) \sin m\lambda \Big\} \tag{3.10}$$

This may be re-arranged to give

$$V = \sum_{\substack{\ell m \\ jk}} P_\ell^m \Big\{ \\ \Big(\Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L1(\ell,m,j,k) + k_\ell \delta_{\ell j} \delta_{mk} \Big] \cos m\lambda \\ + \Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L5(\ell,m,j,k) \Big] \sin m\lambda \Big) U_j^k \\ + \Big(\Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L3(\ell,m,j,k) \Big] \cos m\lambda \\ + \Big[(1+k_\ell') \frac{3\rho_a}{(2\ell+1)\rho} L7(\ell,m,j,k) + k_\ell \delta_{j\ell} \delta_{mk} \Big] \sin m\lambda \Big) V_j^k \Big\} \tag{3.11}$$

Comparing 3,4 and 3.13 we have

$$\begin{aligned}
K1(\ell,m,j,k) &= \frac{(1+k_\ell')\,3\rho_0}{(2\ell+1)\,\rho}\,L1(\ell,m,j,k) + k_\ell\,\delta_{j\ell}\,\delta_{mk} \\
K3(\ell,m,j,k) &= \frac{(1+k_\ell')\,3\rho_0}{(2\ell+1)\,\rho}\,L3(\ell,m,j,k) \\
K5(\ell,m,j,k) &= \frac{(1+k_\ell')\,3\rho_0}{(2\ell+1)\,\rho}\,L5(\ell,m,j,k) \\
K7(\ell,m,j,k) &= \frac{(1+k_\ell')\,3\rho_0}{(2\ell+1)\,\rho}\,L7(\ell,m,j,k) + k_\ell\,\delta_{j\ell}\,\delta_{mk}
\end{aligned} \qquad (3.12)$$

iii) The Method of Kaula

In a study of the effect of asymmetry in the Earth's tidal response and its consequences for the tidal evolution of the Moon's orbit, Kaula (1969) postulated that in response to a prescribed potential of the form 3.1 the deformation potential of the Earth would be given by

$$V = \sum_{jk} k_j(\theta,\lambda)\, P_j^k (U_j^k \cos k\lambda + V_j^k \sin k\lambda) \qquad (3.13)$$

with the "Love number" $k_j(\theta,\lambda)$ a function of latitude and longitude, expanded in surface harmonics as

$$k_j(\theta,\lambda) = \sum_{pq} P_p^q (k_{jp}^q \cos\lambda + \ell_{jp}^q \sin q\lambda) \qquad (3.14)$$

This method has the disadvantage that the function $k_j(\theta,\lambda)$ conceals the steps in the feedback process which leads to the resultant tide, whereas in the matrix method the feedback processes have been taken into account explicitly.

In any surface harmonic analysis it is necessary to truncate the series at some value of the degree. Now the effect of asymmetry will be for the perturbed potential to exhibit harmonics of both higher and lower degree than the prescribed potential. Thus in Kaula's method the expansion of $k_j(\theta,\lambda)$ must go to higher terms than the expansion of the prescribed potential. In the matrix method this route is not available since (as a result of the matrix format) the highest term in the perturbed potential is of the same degree and order as the highest term in the prescribed potential. The low-degree terms in the tide height expansion are easily handled by the matrix but for the high-degree terms in the prescribed potential the matrix method truncates the expansion of the tide height prematurely. This difficulty can be overcome in practice by simply expanding the prescribed potential to enough terms so that the last several are too weak to excite a measurable tide. This was the procedure followed in Chapter 2 where allowance was being made for body-force potential harmonics up to degree-8 order-8 even though the luni-solar tide potential has negligible terms above degree-2. The basic incompatibility of the two methods is still present although equal accuracy can be obtained with either. This will lead to some difficulties in expressing Kaula's parameters in terms of the matrices.

Substituting 3.14 into 3.13 we have

$$
\begin{aligned}
V = \sum_{\substack{jk\\pq}} & k_{jp}^{q} \left(P_p^q P_j^k \cos q\lambda \cos k\lambda \right) U_j^k \\
& + \ell_{jp}^{q} \left(P_p^q P_j^k \sin q\lambda \cos k\lambda \right) U_j^k \\
& + k_{jp}^{q} \left(P_p^q P_j^k \cos q\lambda \sin k\lambda \right) V_j^k \\
& + \ell_{jp}^{q} \left(P_p^q P_j^k \sin q\lambda \sin k\lambda \; V_j^k \right.
\end{aligned}
\qquad (3.15)
$$

Each of the bracketed terms may be expanded in surface harmonics;

$$
\begin{aligned}
&P_p^q P_j^k \cos q\lambda \cos k\lambda \\
&\quad = \sum_{\ell m} P_\ell^m \left(A(p,q,j,k,\ell,m) \cos m\lambda + B(p,q,j,k,\ell,m) \sin m\lambda \right) \\
&P_p^q P_j^k \sin q\lambda \cos k\lambda \\
&\quad = \sum_{\ell m} P_\ell^m \left(C(p,q,j,k,\ell,m) \cos m\lambda + D(p,q,j,k,\ell,m) \sin m\lambda \right) \qquad (3.16) \\
&P_p^q P_j^k \cos q\lambda \sin k\lambda \\
&\quad = \sum_{\ell m} P_\ell^m \left(E(p,q,j,k,\ell,m) \cos m\lambda + F(p,q,j,k,\ell,m) \sin m\lambda \right) \\
&P_p^q P_j^k \sin m\lambda \sin k\lambda \\
&\quad = \sum_{\ell m} P_\ell^m \left(G(p,q,j,k,\ell,m) \cos m\lambda + H(p,q,j,k,\ell,m) \sin m\lambda \right)
\end{aligned}
$$

The coefficients $A(p,q,j,k,l,m)$, etc. are triple product surface harmonic integrals and defined by

$$
\left.
\begin{aligned}
&B(p,q,j,k,l,m) \\
&C(p,q,j,k,l,m) \\
&E(p,q,j,k,l,m) \\
&H(p,q,j,k,l,m)
\end{aligned}
\right\} = 0
$$

$$
\begin{aligned}
A(p,q,j,k,l,m) &= \int_S P_p^q P_j^k P_\ell^m \cos q\lambda \cos k\lambda \cos m\lambda \, dS \qquad (3.17) \\
D(p,q,j,k,l,m) &= \int_S P_p^q P_j^k P_\ell^m \sin q\lambda \cos k\lambda \sin m\lambda \, dS \\
F(p,q,j,k,l,m) &= \int_S P_p^q P_j^k P_\ell^m \cos q\lambda \sin k\lambda \sin m\lambda \, dS \\
G(p,q,j,k,l,m) &= \int_S P_p^q P_j^k P_\ell^m \sin q\lambda \sin k\lambda \cos m\lambda \, dS
\end{aligned}
$$

The perturbed potential is then

$$V = \sum_{\substack{jk \\ lm}} P_l^m \Big\{ k_{jp}^{q} \big[A(p,q,j,k,l,m) \cos m\lambda \big] U_j^k$$
$$l_{jp}^{q} \big[D(p,q,j,k,l,m) \sin m\lambda \big] U_j^k$$
$$k_{jp}^{q} \big[F(p,q,j,k,l,m) \sin m\lambda \big] V_j^k \qquad (3.18)$$
$$l_{jp}^{q} \big[G(p,q,j,k,l,m) \cos m\lambda \big] V_j^k$$

Comparing equations 3.18 and 3.4 we have

$$k1(l,m,j,k) = \sum_{pq} k_{jp}^{q} A(p,q,l,m,j,k)$$
$$k3(l,m,j,k) = \sum_{pq} l_{jp}^{q} G(p,q,l,m,j,k) \qquad (3.19)$$
$$k5(l,m,j,k) = \sum_{pq} l_{jp}^{q} D(p,q,l,m,j,k)$$
$$k7(l,m,j,k) = \sum_{pq} k_{jp}^{q} F(p,q,l,m,j,k).$$

which gives the matrix coefficients in terms of Kaula's parameters.

The map of $k_2(\theta,\lambda)$ (figure 3.1 a,b) was produced by using the perturbed potential matrices to construct a tide map and then dividing the value obtained at each grid point by

$$P_2^0(\cos\theta) \qquad (3.20)$$

since the prescribed potential is of form U_2^0. The same comments on the accuracy of representation apply to these maps as to those of Chapter 2.

Figure 3.1a The map of $k_i(\theta,\lambda)$, eastern hemisphere. The oceanic boundaries are defined by the ocean function of Balmino et al with $e \leq 0.5$

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	0
10	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	10
20	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	20
30	3.4	3.4	3.4	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.2	3.2	3.2	3.2	3.2	3.2	3.3	3.3	3.3	30
40	3.4	3.4	3.4	3.4	3.4	3.3	3.3	3.2	3.2	3.2	3.1	3.1	3.1	3.1	3.1	3.1	3.1	3.1	3.1	40
50	3.8	3.8	3.8	3.8	3.6	3.5	3.2	3.0	2.8	2.7	2.6	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.4	50
60	2.8	2.7	2.7	2.7	2.9	3.1	3.3	3.6	3.8	4.0	4.1	4.2	4.2	4.2	4.2	4.3	4.3	4.4	4.5	60
70	3.1	3.0	3.0	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.6	3.7	3.7	3.7	3.7	3.7	3.8	3.8	3.9	70
80	3.1	3.1	3.1	3.1	3.1	3.2	3.3	3.4	3.4	3.5	3.5	3.6	3.6	3.6	3.6	3.6	3.6	3.7	3.7	80
90	3.1	3.1	3.1	3.1	3.1	3.2	3.2	3.3	3.4	3.4	3.5	3.5	3.5	3.5	3.5	3.5	3.6	3.6	3.7	90
100	3.1	3.0	3.0	3.0	3.1	3.1	3.2	3.3	3.4	3.4	3.4	3.5	3.5	3.5	3.5	3.5	3.6	3.6	3.7	100
110	3.0	2.9	2.9	3.0	3.0	3.1	3.2	3.2	3.3	3.4	3.4	3.4	3.4	3.4	3.4	3.5	3.5	3.6	3.7	110
120	2.4	2.4	2.4	2.4	2.5	2.7	2.8	3.0	3.1	3.2	3.2	3.3	3.3	3.3	3.3	3.4	3.5	3.6	3.8	120
130	4.3	4.3	4.3	4.2	4.2	4.0	3.9	3.8	3.7	3.7	3.7	3.7	3.7	3.7	3.7	3.6	3.5	3.4	3.3	130
140	3.7	3.6	3.6	3.6	3.6	3.6	3.6	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	140
150	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	150
160	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	160
170	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	170
180	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure 3.1b The map of $k_2(\theta,\lambda)$, western hemisphere. The oceanic boundaries are dfined by the ocean function of Balmino et al with $C \leq 0.5$

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360
0	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	0.0
10	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	0.0
20	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	0.0
30	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.3	3.4	3.4	0.0
40	3.1	3.1	3.1	3.1	3.1	3.2	3.2	3.2	3.2	3.2	3.3	3.3	3.3	3.3	3.3	3.4	3.4	3.4	0.0
50	2.4	2.4	2.3	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.1	3.2	3.3	3.4	3.5	3.7	0.0
60	4.5	4.6	4.6	4.6	4.6	4.5	4.3	4.2	4.0	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	2.9	0.0
70	3.9	3.9	3.9	3.9	3.9	3.9	3.8	3.7	3.6	3.5	3.5	3.4	3.4	3.4	3.3	3.3	3.2	3.1	0.0
80	3.7	3.8	3.8	3.8	3.8	3.7	3.7	3.6	3.5	3.5	3.4	3.4	3.4	3.3	3.3	3.3	3.2	3.1	0.0
90	3.7	3.7	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.4	3.4	3.3	3.3	3.3	3.3	3.2	3.2	3.1	0.0
100	3.7	3.7	3.8	3.8	3.7	3.7	3.6	3.5	3.4	3.4	3.3	3.3	3.2	3.2	3.2	3.2	3.1	3.1	0.0
110	3.7	3.7	3.8	3.8	3.7	3.7	3.6	3.5	3.4	3.3	3.2	3.1	3.1	3.1	3.1	3.1	3.0	3.0	0.0
120	3.8	3.9	4.0	4.0	3.9	3.7	3.5	3.3	3.0	2.8	2.7	2.6	2.5	2.5	2.5	2.5	2.5	2.4	0.0
130	3.3	3.3	3.2	3.2	3.3	3.5	3.6	3.8	4.0	4.2	4.3	4.4	4.4	4.4	4.4	4.4	4.4	4.4	0.0
140	3.5	3.5	3.5	3.5	3.5	3.5	3.6	3.6	3.7	3.7	3.7	3.7	3.7	3.7	3.7	3.7	3.7	3.7	0.0
150	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	0.0
160	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	0.0
170	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	0.0
180	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5	0.0
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360

CHAPTER 4

THE INFLUENCE OF THE OCEANS ON THE CHANDLER WOBBLE

i) Introduction

The possibility of a torque-free wobble of the Earth - a motion of the geographical axis of the solid Earth about the spin axis such that any point on the Earth undergoes a periodic change in latitude - was first suggested by Euler in 1765. However, Euler's results were based on a rigid Earth model and predicted a period of 10 months rather than the eventually observed 14 months or 437 days. Most of the lengthening of the period can be attributed to elastic yielding of the solid Earth but the oceans are responsible for about 40 days. The ocean influence can be explained by noting that the period is determined by the gyroscopic restoring torque provided by the Earth's equatorial bulge. For a rigid Earth, none of the bulge can adjust to a new geographical position of the spin axis as the Earth wobbles, and the effect of the oceans on the period is limited to their contribution to the difference in moments of inertia which constitutes the bulge. But when mobility of the oceans is taken into account, together with elastic yielding of the solid Earth, the major axis of the inertia tensor of the oceans will tend to be more closely aligned with the moving spin axis, thus reducing the effective bulge and lengthening the period. In addition to influencing the period the ocean introduces a slight ellipticity in the geographical path of the pole because their irregular distribution causes the wobble about one axis to be excited slightly more strongly than that about the other.

The potential induced by the wobble excites an ocean tide known as the pole tide. The feedback of the ocean pole tide on the Chandler wobble has been investigated previously by Larmor (1915) and Haubrich and Munk (1959), but with simplifying assumptions about tide heights and the effect of continentality. After examining the problem in the case of global oceans (section 4.iii) and accommodating continentality (section 4.iv) a complete solution for the effect of an equilibrium pole tide is attempted in section 4.v.

Larmor (1915), the first to attempt the problem, obtained a lengthening of the period of about 28 days. He allowed for yielding of the solid Earth to the prescribed tidal potential of the wobble but did not take into account ocean loading or self-attraction. He also attempted to allow for the effect, on the period, of non-globality of the oceans by reducing his tide height by 2/3 (which is approximately the fraction of the Earth covered by oceans). Although he gave no clear argument for the introduction of this factor it will be seen in section 4.iv that the leading term in the expansion of the ocean function in surface harmonics is also the leading term in the correction for continentality.

Haubrich and Munk (1959) accounted correctly for continentality but did not allow for the effect of load yielding and self-attraction on the tide height, and made an incorrect estimate (section 4.v) of the effect of load yielding on the products of inertia of a tide. Their error was attributable mostly to incorrect numerical values of the load Love numbers but, as well, their use of the Love numbers was inconsistent with the Love number method. Their method is essentially that of section 4.iv although I have used the more recent ocean function of Balmino et al (1973) in the correction for continentality.

Here, as in all previous work, it has been assumed that the pole tide is an equilibrium tide. Although the solar annual tide is thought to be equilibrium this does not mean that all longer period tides will be equilibrium. The reason for this is that the solar annual tide potential is of degree-2, order-zero whereas the pole tide is of degree-2, order-1. This is sufficient reason to rule out equating their dynamics (Munk and MacDonald, 1960, p. 99). In fact both Haubrich and Munk (1959), and Miller (1973) in a more complete analysis, report pole tides an order of magnitude above equilibrium. However the areas in which an especially large pole tide is found are quite localized and Miller (1973) finds that the excitation from the non-equilibrium tides in the Baltic and North seas is quite insufficient to excite an appreciable wobble. On the basis of this result an investigation of the effect of the oceans on the Chandler wobble, assuming an equilibrium tidal response, may be a good approximation.

ii) The Liouville Equations

The Liouville equations which describe the rotation of a yielding Earth are derived in Appendix E. They can be written in the form

$$\begin{aligned} \dot{m}_1/\sigma_r + m_2 &= \phi_2 \\ \dot{m}_2/\sigma_r - m_1 &= -\phi_1 \end{aligned} \qquad (4.1)$$

where σ_r is the Eulerian frequency of the wobble for a rigid Earth and m_1 and m_2 are the components of the angular velocity of the reference frame (which is attached to the solid Earth in some prescribed manner) about the axes 0^o, 90^oE longitude in the equatorial plane, normalized by dividing by the diurnal frequency. The excitation functions, for the purposes needed here, are

$$\begin{aligned} \phi_1 &= c_{13}/(C-A) + \phi_1' \\ \phi_2 &= c_{23}/(C-A) + \phi_2' \end{aligned} \qquad (4.2)$$

where C,A are the Earth's axial and equatorial moments of inertia, and c_{13}, c_{23} are the small products of inertia generated by the pole tide. The first terms on the right hand side are the ocean excitations and ϕ_1' and ϕ_2' are the excitations of all other sources which have an effect on the period of the wobble (fluid core, elastic mantle, atmosphere). Using 4.2 equations 4.1 may be written

$$\begin{aligned} \dot{m}_1 + \sigma_r m_2 &= \sigma_r c_{23}/(C-A) + \sigma_r \phi_2' \\ \dot{m}_2 - \sigma_r m_1 &= -\sigma_r c_{13}/(C-A) - \sigma_r \phi_1' \end{aligned} \qquad (4.3)$$

The most exhaustive study of the wobble of an oceanless, but otherwise real Earth is by Jeffreys and Vicente (1957). They have shown that for an oceanless Earth equations 4.3 can be rewritten in the approximate form

$$\begin{aligned} \dot{m}_1 + \sigma_e m_2 &= 0 \\ \dot{m}_2 - \sigma_e m_1 &= 0 \end{aligned} \qquad (4.4)$$

Re-introducing the ocean excitation gives

$$\dot{m}_1 + \sigma_e m_2 = \sigma_r c_{23}/(C-A)$$
$$\dot{m}_2 - \sigma_e m_1 = -\sigma_r c_{13}/(C-A) \qquad (4.5)$$

which may be solved for the observed frequency σ in terms of the 'oceanless' frequency σ_e and the Eulerian frequency σ_r. Subtraction of the 'oceanless' period from the observed period gives the lengthening of the period due to the oceans.

iii) The Pole Tide in Global Oceans

This is essentially the model of Larmor (1915). Although he did not use the perturbed Liouville equations and hence could not solve for the wobble period of an oceanless Earth, he was able to obtain an approximate solution of the problem by calculating the wobble period of a rigid Earth whose pole tide had the same height it would have on a real Earth. The lengthening calculated on this basis is about 28 days which is actually closer to the true lengthening by oceans which respond in equilibrium fashion than the 43 days we obtain here by approaching the problem correctly. Using 1.11 the height of the equilibrium tide is

$$\xi_2 = -\frac{\Omega^2 a^2}{g}\frac{2}{3}J_2 P_2^1(m_1 \cos\lambda + m_2 \sin\lambda) \qquad (4.7)$$

The wobble potential in 4.47 is derived in appendix F.

Using 2.1 for the potential of a surface layer of known height and density we have for the potential of the pole tide:

$$V_2 = -\frac{3\rho_0}{5\rho}\Omega^2 a^2 \frac{2}{3} J_2 P_2^1 (m_1 \cos\lambda + m_2 \sin\lambda) \quad (4.8)$$

Using MacCullagh's formula for the terms in the gravitational potential due to the products of inertia c_{13}, c_{23} we have

$$V_2 = -\frac{3G}{a^3}\frac{2}{3} P_2^1 (c_{13}\cos\lambda + c_{23}\sin\lambda) \quad (4.9)$$

and comparing with 4.8 we have

$$\begin{aligned} c_{13} &= \frac{\rho_0}{5\rho}\frac{\Omega^2 a^5}{G} J_2 m_1 \\ c_{23} &= \frac{\rho_0}{5\rho}\frac{\Omega^2 a^5}{G} J_2 m_2 \end{aligned} \quad (4.10)$$

The wobble equations then become

$$\begin{aligned} \dot{m}_1 + \sigma_e m_2 &= \frac{\sigma_r \rho_0}{5\rho}\frac{\Omega^2 a^5}{G(C-A)} J_2 m_2 \\ \dot{m}_2 - \sigma_e m_1 &= -\frac{\sigma_r \rho_0}{5\rho}\frac{\Omega^2 a^5}{G(C-A)} J_2 m_1 \end{aligned} \quad (4.11)$$

When solved for the observed frequency these yield

$$\begin{aligned} \sigma &= \sigma_e - \sigma_r \frac{\rho_0}{5\rho}\frac{J_2 \Omega^2 a^5}{G(C-A)} \\ &= \sigma_e - .080\,\sigma_r \end{aligned} \quad (4.12)$$

When the Euler and observed frequencies are used in 4.12, the increase in the wobble period due to the oceans is 43 days, and the wobble period of an oceanless Earth is

$$T_e = 394 \text{ days} \tag{4.13}$$

and the oceans accordingly increase the period of the wobble by 43 days.

As a result of the globality of the oceans there is no preference in the excitation of either m_1 or m_2 and the path of the pole is circular.

iv) <u>The Pole Tide in Non-Global Oceans</u>

If the tide, as predicted by 4.7, is integrated over the oceans it is found that the integral does not reduce to zero, i.e. water is not conserved. To correct this we use the Darwin correction (section 2.iv). The corrected tide in the ocean is

$$\xi_2 = -\frac{\Omega^2 a^2}{3g} J_2\left[m_1\left(P_2^1\cos\lambda - \frac{3a_2^1}{40a_0^0}\right) + m_2\left(P_2^1\sin\lambda - \frac{3b_2^1}{40a_0^0}\right)\right] \tag{4.14}$$

When calculating the products of inertia due to this tide it must be truncated to zero on land:

$$\begin{aligned} C_{13} &= -a^4\rho_0\int_S \xi_2\, \mathcal{C}\,\frac{2}{3} P_2^1\cos\lambda\, dS \\ C_{23} &= -a^4\rho_0\int_S \xi_2\, \mathcal{C}\,\frac{2}{3} P_2^1\sin\lambda\, dS \end{aligned} \tag{4.15}$$

Substituting the surface harmonic expansions of $\mathcal{C}$ and ξ into 4.15 we have

$$C_{13} = H_2 \sum_{nm} \Big[\Big[\int_S P_n^m P_2^1 P_2^1 a_n^m \cos m\lambda \cos^2\lambda \; ds$$

$$+ \int_S P_n^m P_2^1 P_2^1 b_n^m \sin m\lambda \sin\lambda \cos\lambda \; ds$$

$$- \int_S P_n^m P_2^1 a_n^m m_1 \frac{3a_2^1}{40a_0^0} \cos m\lambda \cos\lambda \; ds$$

$$- \int_S P_n^m P_2^1 a_n^m m_2 \frac{3b_2^1}{40a_0^0} \cos m\lambda \cos\lambda \; ds \qquad (4.16)$$

$$C_{23} = H_2 \sum_{nm} \Big[\Big[\int_S P_n^m P_2^1 P_2^1 a_n^m m_2 \cos m\lambda \sin^2\lambda \; ds$$

$$+ \int_S P_n^m P_2^1 P_2^1 b_n^m m_1 \sin m\lambda \sin\lambda \cos\lambda \; ds$$

$$- \int_S P_n^m P_2^1 b_n^m m_1 \sin m\lambda \sin\lambda \; ds$$

$$- \int_S P_n^m P_2^1 b_n^m m_2 \frac{3b_2^1}{40a_0^0} \sin m\lambda \sin\lambda \; ds$$

where

$$H_2 = \frac{\Omega^2 a^6 4 \rho_0 J_2}{9}$$

The number of non-zero integrals in 4.16 is limited by the fact that one of the Legendre functions has been specified to be P_2^1. In fact the only non-zero integrals are those in which the ocean coefficients of degree 0,2,4, and order 0,2 are present.

Following the notation of Haubrich and Munk (1959) these may be written

$$
\begin{aligned}
C_{13} &= \frac{H_2}{A}\, A\,(T_1 m_1 + R m_2) \\
C_{23} &= \frac{H_2}{A}\, A\,(T_2 m_2 + R m_1)
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
T_1 = \Big[& a_0^0 \,.6000\pi + a_2^0 \,.4286\frac{\pi}{2} + a_4^0 \,.1714\frac{\pi}{2} \\
& + \left(a_2^2 \,.5143\frac{\pi}{2} + a_4^2 \,.6429\frac{\pi}{2}\right) - \frac{9\pi}{200}\frac{(a_2^1)^2}{a_0^0} \Big] \\
T_2 = \Big[& a_0^0 \,.6000\pi + a_2^0 \,.4286\frac{\pi}{2} + a_4^0 \,.1714\frac{\pi}{2} \\
& - \left(a_2^2 \,.5143\frac{\pi}{2} + a_4^2 \,.6429\frac{\pi}{2}\right) - \frac{9\pi}{200}\frac{(b_2^1)^2}{a_0^0} \Big] \\
R = \Big[& b_2^2 \,.5143\frac{\pi}{2} + b_4^2 \,.6429\frac{\pi}{2} - \frac{9\pi}{200}\frac{a_2^1 b_2^1}{a_0^0} \Big]
\end{aligned}
\tag{4.18}
$$

or

$$
\begin{aligned}
\frac{H_2}{A} T_1 &= 1.4871 \times 10^{-4} \\
\frac{H_2}{A} T_2 &= 0.9159 \times 10^{-4} \\
R &= 0.0672 \times 10^{-4} \\
\frac{H_2}{A} &= 1.394 \times 10^{-4}
\end{aligned}
\tag{4.19}
$$

The wobble equations become

$$\dot{m}_1 + \left(\sigma_e - \frac{T_2}{H}\sigma_r\right) m_2 = \frac{\sigma_r R}{H} m_1$$
$$\dot{m}_2 - \left(\sigma_e - \frac{T_1}{H}\sigma_r\right) m_1 = -\frac{\sigma_r R m_2}{H} \qquad (4.20)$$

where

$$H = \frac{C-A}{A} \doteq \frac{C-A}{C} \qquad (4.21)$$

Equations 4.20 may be uncoupled to yield

$$\ddot{m}_1 + \left(\left(\sigma_e - \frac{T_2\sigma_r}{H}\right)\left(\sigma_e - \frac{T_1\sigma_r}{H}\right) - \frac{\sigma_r^2 R^2}{H^2}\right) m_1 = 0$$
$$\ddot{m}_2 + \left(\left(\sigma_e - \frac{T_1\sigma_r}{H}\right)\left(\sigma_e - \frac{T_2\sigma_r}{H}\right) - \frac{\sigma_r^2 R^2}{H^2}\right) m_2 = 0 \qquad (4.22)$$

and the frequency is

$$\sigma = \sigma_e\left[1 - \frac{\Omega(T_1+T_2)}{\sigma_e} - \frac{\Omega^2}{\sigma_e^2}\left(R^2 - T_1T_2\right)\right]^{1/2} \qquad (4.23)$$

or to a first approximation

$$\sigma = \sigma_e - \frac{(T_1+T_2)}{2}\Omega \qquad (4.24)$$

which gives for the wobble period of an oceanless Earth

$$T_e = 407 \text{ days}, \qquad (4.25)$$

reducing the increase in period to 30 days. The three-day difference between this and Haubrich and Munk's (1959) result is attributable to the different ocean functions used. Since the result is appreciably different from that obtained using global oceans, the importance of continentality is readily apparent.

Because T1 is greater than T2 the excitation is not symmetric and an elliptical pole path results.

v) The Complete Solution of the Pole Tide

The treatment in section 4.iv was an over-simplification in that the tide heights did not allow for self-attraction and loading and the consequent diminishing of the products of inertia of the ocean tide was not accounted for. It is not satisfactory to allow for load yielding, as it affects the product of inertia, simply by calculating the product of inertia of an Earth tide of amplitude

$$h_2' \frac{3\rho_0}{5\rho} \xi_2 \qquad (4.26)$$

and density contrast $\rho-\rho_0$, as Haubrich and Munk (1959) suggest. This is a misuse of the Love number method, as can be easily seen by noting that using this argument the potential of the solid Earth as a result of the ocean tide would be

$$h_2' \frac{3(\rho-\rho_0)}{5\rho} g \frac{3\rho_0}{5\rho} \xi_2 \qquad (4.27)$$

But the deformation potential is actually

$$k_2' \frac{3\rho_0}{5\rho} g \xi_2 \qquad (4.28)$$

and certainly

$$k_2' \neq h_2' \frac{3(\rho-\rho_0)}{5\rho} \qquad (4.29)$$

It was shown in Chapter 3 that the deformation potential of the Earth in response to a prescribed potential

$$U = \sum_{jk} P_j^k (U_j^k \cos k\lambda + V_j^k \sin k\lambda) \tag{4.30}$$

can be written in a manner analogous to the equations 2.74 for the tide heights provided the response of the oceans is equilibrium. The deformation potential is

$$V = \sum_{\substack{lm \\ jk}} P_l^m [K1(l,m,j,k)\, U_j^k \cos m\lambda + K5(l,m,j,k)\, U_j^k \sin m\lambda$$
$$+ K3(l,m,j,k)\, V_j^k \cos m\lambda + K7(l,m,j,k)\, V_j^k \sin m\lambda] \tag{4.31}$$

In this chapter the K matrices will account for only the perturbed potential due to the oceans. In contrast to chapter 3, the direct perturbed potential of the solid Earth in response to the prescribed potential is not included in the K matrices because here the K matrices are to give the perturbed potential due to the oceans only. The only terms in 4.31 which are the result of the inertia products c_{13}, c_{23} are the u_2^1 terms so that 4.31 may be written

$$V = \sum_{jk} P_2^1 [(K1(2,1,j,k)\, U_j^k + K3(2,1,j,k)\, V_j^k) \cos\lambda$$
$$+ (K5(2,1,j,k)\, U_j^k + K7(2,1,j,k)\, V_j^k) \sin\lambda] \tag{4.32}$$

From equations 4.9 for the potential of the products of inertia c_{13}, c_{23} we have

$$\sum_{jk} K1(2,1,j,k)\, U_j^k + K3(2,1,j,k)\, V_j^k = -\frac{2G}{a^3} C_{13}$$
$$\sum_{jk} K5(2,1,j,k)\, U_j^k + K7(2,1,j,k)\, V_j^k = -\frac{2G}{a^3} C_{23} \tag{4.33}$$

The pole tide has only degree-2, order-1 terms

$$U_2^1 = -\Omega^2 a^2 \frac{2}{3} m_1 \quad , \quad V_2^1 = -\Omega^2 a^2 \frac{2}{3} m_2 \tag{4.34}$$

so we have

$$\begin{aligned} C_{13} &= \frac{\Omega^2 a^5}{3G}\left[K1(2,1,2,1)\, m_1 + K3(2,1,2,1)\, m_2 \right] \\ C_{23} &= \frac{\Omega^2 a^5}{3G}\left[K5(2,1,2,1)\, m_1 + K7(2,1,2,1)\, m_2 \right] \end{aligned} \tag{4.35}$$

i.e.

$$\begin{aligned} C_{13} &= A(T_1 m_1 + R m_2) \\ C_{23} &= A(T_2 m_2 + R m_1) \end{aligned} \tag{4.36}$$

with

$$\begin{aligned} T_1 &= K1(2,1,2,1)\, \Omega^2 a^5 / 3GA \\ T_2 &= K7(2,1,2,1)\, \Omega^2 a^5 / 3GA \\ R &= K3(2,1,2,1)\, \Omega^2 a^5 / 3GA \end{aligned} \tag{4.37}$$

Following the previous section this leads to

$$T_e = 411 \text{ days} \tag{4.38}$$

so that the oceans increase the period by 26 days.

The results and weaknesses of the three solutions are presented in Table 4.2. The non-globality of the oceans has the greatest influence, reducing the lengthening by 13 days. The slight adjustment of 4 days as a result of allowing for load yielding is the resultant of (a) an increase in the pole tide amplitude by loading and particularly self-attraction by a factor of about .84/.69 and (b) a reduction in this tide's ability to raise products of inertia by a factor of about .69.

As reported earlier, the actual period for an oceanless Earth is about 392 days, according to Jeffreys and Vicente (1957). Presumably the discrepancy between 392 and 411 is to be accounted for by taking into account the departure of the pole tide from equilibrium, i.e. by a fully dynamical treatment.

Method	Lengthening of period (days)	Approximations
section 4.iii	43	does not allow for loading, self-attraction or continentality
section 4.iv	30	does not allow for self-attraction or loading
section 4.v	26	does not allow for non-equilibrium response of oceans

Table 4.2 The results of the three successive approximations to the pole tide / Chandler wobble problem.

vi) The Ellipticity of the Pole Path

The path of the pole at the Chandler frequency is not circular due in part to the distribution of the oceans, and in part to the inertia ellipsoid of the solid Earth being slightly triaxial.

When a triaxial body is rotating nearly about its axis of greatest moment and wobbling slightly, the major axis of the elliptical pole path points toward the principal axis of intermediate moment of inertia. This is simply because the wobble energy is split equally between the two equatorial axes and the axis of least moment of inertia must have a greater amplitude of rotation. The axis of least moment of inertia for the Earth is in the Indian ocean which puts the intermediate axis at about 10 degrees East longitude. Haubrich and Munk (1959) place the major axis of the pole path at 107 degrees East longitude apparently by confusing the greatest equatorial radius and the greatest equatorial axis of inertia.

If M_1 and M_2 are the amplitudes of the polar motions then their magnitudes will be in the ratio of their excitations

$$\frac{M_2}{M_1} = \frac{\sigma_e - T_1 \Omega}{\sigma_e - T_2 \Omega} \qquad (4.39)$$

$$\therefore \; \epsilon = 1 - \frac{M_2}{M_1}$$

$$= .04$$

Writing the equations of polar motion as

$$m_1 = M_1 \cos(\sigma t)$$
$$m_2 = M_2 \sin(\sigma t + \beta) \qquad (4.40)$$

where β is a phase angle and M_1 M_2 are the amplitudes of the polar motion. By substituting in 4.20 we have

$$\tan\beta = -\frac{\Omega R}{\sigma} = .004 \ll 1 \qquad (4.41)$$

Eliminating the time in 4.40 and using 4.41 we have

$$\left(\frac{m_1}{M_1}\right)^2 - \frac{2\beta m_1 m_2}{M_1 M_2} + \left(\frac{m_2}{M_2}\right)^2 = 1 \qquad (4.42)$$

which is an ellipse oriented at an angle λ such that

$$\tan 2\lambda = 2\beta \Big/ \left(\frac{M_1}{M_2} - \frac{M_2}{M_1}\right)$$
$$\lambda = .05 \qquad (4.43)$$

The oceans thus contribute to the ellipticity of the pole path, whose major axis is aligned closely to the Greenwich meridian.

CHAPTER 5

THE GRAVIMETRIC FACTOR

i) Introduction

The gravimetric factor for a prescribed non-load potential was given in section 1.iii. Farrell (1970) suggests that the gravimetric factor for a load potential is the same as that for a non-load potential with load Love numbers replacing non-load Love numbers. Slichter (1972) makes a similar statement and compounds his error by using load Love numbers of correct absolute value but wrong sign. In section 5.ii the proper expression for the load gravimetric factor is derived. This is then used, in conjunction with eq. 2.26 for the global effective tide height, to arrive at a gravimetric factor for a model with global oceans. In section 5.iii the matrices which govern the fluctuations in gravity will be derived and used to examine the gravity tides associated with several long period tides. Slichter (1972) has shown that south polar stations are very little influenced by the gravity tide from the long period ocean tides. By examining various tidal patterns it may be possible to find other areas in the Earth for which the gravity tide correction for a given tide is small.

ii) The Gravimetric Factor For a Load Potential

After yielding, the potential due to a prescribed load potential U' is

$$\sum_n (1+k_n') U_n' \tag{5.1}$$

at the surface and

$$\sum_n \left(\frac{a}{r}\right)^{n+1} (1+k_n') \frac{U_n'}{r} \tag{5.2}$$

outside the surface. By differentiating with respect to r we obtain the perturbation in gravity at the surface;

$$-\sum_n \frac{n+1}{a} (1+k_n') U_n' \tag{5.3}$$

There is an additional contribution to the gravity tide which arises from the vertical deflections of the Earth's surface :

$$\frac{\partial^2}{\partial r^2}(V_0)\, \xi'' \tag{5.4}$$

where V_0 is the undisturbed potential of the Earth and

$$\xi'' = \sum_n h_n' U_n'/g \,, \qquad \frac{\partial^2 V_0}{\partial r^2} = \frac{2g}{a} . \tag{5.5}$$

So the total fluctuation in gravity is

$$\left(\frac{2h_n'}{a} - \frac{n+1}{a}(1+k_n')\right) U_n' = -\delta_n'\left(\frac{n+1}{a}\right) U_n' \tag{5.6}$$

where the load gravimetric factor is

$$\delta_n' = 1 + k_n' - \frac{2h_n'}{n+1} \tag{5.7}$$

In the case of equilibrium tides in global oceans we have from 2.26

$$\xi_n = J_{n\,eff} \frac{U_n}{g} \tag{5.8}$$

and the load potential of this tide is

$$U_n' = \frac{3\rho_o}{(2n+1)\rho} J_{n\,eff} U_n \tag{5.9}$$

so that the total gravity tide is

$$\Delta g = -\delta_{n\,eff} \frac{n U_n}{a} \tag{5.10}$$

and the global effective gravimetric factor is

$$\delta_{n\,eff} = \delta_n + \delta_n' \frac{(n+1)}{n} \frac{3\rho_o}{(2n+1)\rho} J_{n\,eff} \tag{5.11}$$

Actually these gravimetric factors are to be used with measurements made on the solid Earth surface, so equation 5.11 could not, strictly speaking, apply to global oceans but only to oceans which were so extensive the tidal pattern was not greatly altered by continentality.

Numerical values of the load gravimetric factor and the global effective gravimetric factor are given in Table 5.1.

iii) Gravity Tides on an Earth With Non-Global Oceans

As in the earlier chapters, we would like to express the gravity tide over the surface of the Earth in a surface harmonic expansion whose coefficients would be given by

$$
\begin{aligned}
g_{1\ell}^{m} &= G1(\ell,m,j,k)\,\frac{U_j^k}{a} + G3(\ell,m,j,k)\,\frac{V_j^k}{a} \\
g_{2\ell}^{m} &= G5(\ell,m,j,k)\,\frac{U_j^k}{a} + G7(\ell,m,j,k)\,\frac{V_j^k}{a}
\end{aligned}
\tag{5.12}
$$

Considering only non-load potentials and using 2.68 for the tide height and 2.1 for the load potential of this tide we have for the radial deflection of the solid Earth due only to the ocean tide

$$
\begin{aligned}
gY_\ell^m &= h_\ell' \frac{3\rho_o}{(2\ell+1)\rho}\left[L1(\ell,m,j,k)\,U_j^k + L3(\ell,m,j,k)\,V_j^k\right] \\
gZ_\ell^m &= h_\ell' \frac{3\rho_o}{(2\ell+1)\rho}\left[L3(\ell,m,j,k)\,U_j^k + L7(\ell,m,j,k)\,V_j^k\right]
\end{aligned}
\tag{5.13}
$$

The perturbation in gravity as a result of this radial deflection of the solid surface is

$$
g_{1\ell}^m = \frac{2g}{a}\,Y_\ell^m \quad , \quad g_{2\ell}^m = \frac{2g}{a}\,Z_\ell^m \tag{5.14}
$$

and similarly for the other coefficients.

The perturbed potential of the Earth due to the ocean tide is given by 3.13 and 3.14 with the direct solid Earth perturbed potential removed:

$$
\begin{aligned}
q_\ell^m &= \sum_{jk} K1(\ell,m,j,k)\,U_j^k + K3(\ell,m,j,k)\,V_j^k \\
r_\ell^m &= \sum_{jk} K5(\ell,m,j,k)\,U_j^k + K7(\ell,m,j,k)\,V_j^k
\end{aligned}
\tag{5.15}
$$

and the perturbation in gravity is

$$g_{1\ell}^{m} = -\frac{(\ell+1)}{a} q_\ell^m \;, \quad g_{2\ell}^{m} = -\frac{(\ell+1)}{a} r_\ell^m \tag{5.16}$$

and similarly for the other coefficients. The total contribution of the ocean tide to the gravity tide is

$$\begin{aligned} g_{1\ell}^{m} &= \left[\frac{2}{a}\frac{h_\ell' 3\rho_0}{(2\ell+1)\rho} L1(\ell,m,j,k) - \frac{(\ell+1)}{a} K1(\ell,m,j,k)\right] U_j^k \\ &+ \left[\frac{2}{a}\frac{h_\ell' 3\rho_0}{(2\ell+1)\rho} L3(\ell,m,j,k) - \frac{(\ell+1)}{a} K3(\ell,m,j,k)\right] V_j^k \\ g_{2\ell}^{m} &\left[\frac{2}{a}\frac{h_\ell' 3\rho_0}{(2\ell+1)\rho} L5(\ell,m,j,k) - \frac{(\ell+1)}{a} K5(\ell,m,j,k)\right] U_j^k \\ &+ \left[\frac{2}{a}\frac{h_\ell' 3\rho_0}{(2\ell+1)\rho} L7(\ell,m,j,k) - \frac{(\ell+1)}{a} K7(\ell,m,j,k)\right] V_j^k \end{aligned} \tag{5.17}$$

The G matrices are

$$G1(\ell,m,j,k) = \left[2h_\ell' - (\ell+1)(1+k_\ell')\right]\frac{3\rho_0}{(2\ell+1)\rho} L1(\ell,m,j,k)$$

$$G3(\ell,m,j,k) = \left[2h_\ell' - (\ell+1)(1+k_\ell')\right]\frac{3\rho_0}{(2\ell+1)\rho} L3(\ell,m,j,k)$$

$$G5(\ell,m,j,k) = \left[2h_\ell' - (\ell+1)(1+k_\ell')\right]\frac{3\rho_0}{(2\ell+1)\rho} L5(\ell,m,j,k)$$

$$G7(\ell,m,j,k) = \left[2h_\ell' - (\ell+1)(1+k_\ell')\right]\frac{3\rho_0}{(2\ell+1)\rho} L7(\ell,m,j,k)$$

DEGREE N	GRAVIMETRIC FACTOR LOAD 1+K-2H/(N+1)	GLOBAL EFFECTIVE GRAVIMETRIC FACTOR
0	0.0	0.0
1	1.290	1.618
2	1.361	1.349
3	1.333	1.200
4	1.291	1.132
5	1.260	1.097
6	1.239	1.077
7	1.224	1.063
8	1.212	1.054
9	1.202	1.047
10	1.193	1.042
11	1.185	1.037
12	1.178	1.034
13	1.171	1.030
14	1.165	1.028
15	1.160	1.026
16	1.156	1.025
17	1.151	1.023
18	1.146	1.021
19	1.143	1.019
20	1.139	1.019
21	1.135	1.018
22	1.132	1.017
23	1.128	1.016
24	1.125	1.015
25	1.123	1.014

TABLE 5.1 THE GRAVIMETRIC FACTOR FOR A LOAD POTENTIAL AND THE GLOBAL EFFECTIVE GRAVIMETRIC FACTOR. THE LOVE NUMBERS ARE TAKEN FROM LONGMAN (1963)

CHAPTER 6

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

i) Summary

In most textbooks and published research papers on the subject, the representation of equilibrium tide heights is by the equation

$$\xi = G(1+k_2-h_2)\frac{U_2}{g} \qquad (6.1)$$

Munk and MacDonald (1960, p. 36) gave a formula modifying 6.1 so that the ocean feedback on itself, directly and through load yielding, was allowed for. The derivation presented by Munk and MacDonald was, however, circuitous and the actual feedback processes occurring were not transparent in the derivation. Their equation was therefore suspect and the initial purpose of this research was to either verify or refute their equation and in addition to solve for the tide height, something which Munk and MacDonald did not attempt.

Their equation was verified, (apart from a necessary correction for mass conservation which was not clearly incorporated by them), but obtained by a more comprehensible derivation, and solved so that the coefficients in the surface harmonic expansion of a tide due to any prescribed potential could be quickly obtained (Section 2.v).

the axial spin acceleration.

Much of the phase lag and amplitude discrepancy of the gravity tide observed at continental stations is due to the non-equilibrium tides in the global ocean. When a more complete knowledge of the tide in the open ocean is available it will be possible to remove the oceanic contribution to the observed gravity tide and thus obtain a solid Earth residual which will yield an accurate value, particularly of the solid-Earth phase lag, so that the dissipativeness of the solid-Earth at tidal frequencies may be obtained. This has already been attempted by Farrell (1972b). Although he does not mention the solid-Earth phase lag, his results indicate that the specific dissipation function Q is about 100 whereas without the ocean correction the midcontinental data yield a Q of about 30. Here Q is given by

$$1/Q = (1/2\pi E)\oint \frac{dE}{d\tau}\, d\tau \qquad (6.2)$$

where E is the peak energy stored in one cycle of the vibration under study, and $\frac{dE}{d\tau}$ the rate at which energy is dissipated.

O'Hora (1973) discusses the systematic error in P.Z.T.* observations arising from the neglect of the local deflection of the vertical caused by neighbouring oceans. Although the global ocean tide has a relatively small influence on the local deflection of the vertical it may be useful to separate this effect from the P.Z.T. observations, once reliable cotidal charts are available.

* photographic zenith tube

It was found that the matrix method developed in Chapter 2 could equally well describe the perturbed potential and gravity tide on a non-spherically-symmetric Earth. Explicit formulae were given to convert the tide height matrices into the perturbed potential and gravimetric matrices (Sections 3.ii, 5.iii). Following Kaula (1969) the Love number k_2 was mapped as a function of position on an Earth whose oceans responded in an equilibrium fashion to a prescribed potential (Section 3.iii).

The influence of the oceans on the period of the Chandler wobble has not before been so completely investigated, in particular loading and self-attraction have not previously fully been considered. This was done in Chapter 4.

Some confusion about the load gravimetric factor was pointed out and the correct expression derived (Section 5.ii).

ii) Suggestions for Future Research

In calculating the excitation function (Munk and MacDonald, 1960, p. 39) for changes in axial rotation and wobble due to seasonal shifts in air mass and ground water, Munk and MacDonald (1960) and Van Hylckama (1970) have not considered the efficacy of a land based load in excitating an ocean tide. The appropriate elements of L2 and L6 indicate that a sizable tide may be generated. The effect of neglecting this tide is to overestimate the sectorial shift in mass and underestimate the zonal, and consequently to overestimate the wobble excitation and underestimate

As is well known the sea floor is covered with a layer of sediment which may reach a kilometer in thickness. To my knowledge no one has investigated the effect of a surface layer of very low rigidity on the load Love numbers, which are very sensitive to the surface conditions.

In this thesis it has been demonstrated that self-attraction and loading have significant effects on equilibrium tide theory. It would be reasonable to expect at least as important a consequence to dynamical tide theory, as the preliminary results of Hendershott (1972) suggest. In particular the dynamical interaction between the loaded sea floor and the ocean requires investigation. At present no co-tidal chart has been verified and any result obtained using one is subject to question. The secular acceleration of the Earth as a result of the ocean tides has been investigated by Pariyskiy <u>et al</u> (1972) and Kuznetsov (1972) using co-tidal charts derived without allowing for the ocean feedback, and these studies should be repeated with a more reliable chart.

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REFERENCES

Balmino, G. K. Lambeck and W. M. Kaula. J. Geophys. Res. 78, 478-481 (1973)

Darwin, G. H. Scientific Papers 1, 340-346 (1883 ; 1907)

Dicke, R. H. in The Earth Moon System B. G. Marsden and A. G. W. Cameron eds., (Plenum Press), 98-164 (1966)

Farrell, W. E. Ph.D. Thesis, University of California (1970)

Farrell, W. E. Revs. Geophys. Space Phys. 10, 761-797 (1972a)

Farrell, W. E. Nature 238, 43 (1972b)

Haubrich, R., and W. H. Munk. J. Geophys. Res. 64, 2373-2387 (1959)

Hecker, O. V. K. Preuss. Geod. Inst. NF, No. 32 (1907)

Hendershott, M. C. Geophys. J. 29, 389-402 (1972)

Hopkins, J. J. Geophys. Res. 78, 476-477 (1973)

Hylckama, T.E.A. Proceedings of the Reading Symposium World Water Balance, (International Association of Scientific Hydrology), 434-444 (1970)

Infeld, L., and T. E. Hull, Revs. Mod. Phys. 32, 21-68 (1951)

Jeffreys, H., and R. O. Vincente. Mon. Not. Roy. Astron. Soc. 47, 162-173 (1957)

Kaula, W. M. J. Geophys. Res. 68, 4967-4978 (1963)

Kaula, W. M. Astron. J. 74, 1108-1114 (1969)

Kelvin, Lord., Phil. Trans. Roy. Soc. Lond. 153, 573 (1863)

Kuo, J. T., R. C. Jachens, M. Ewing, G. White . Science 168, 968-971 (1970)

Kozai, Y. Phil. Trans. Roy. Soc. Lond. A 262, 135-136 (1967)

Kozai, Y. Bull. Geod. 89, 355-377 (1968)

Kozai, Y. in Trajectories of Artificial Celestial Bodies As Determined From Observations J. Kovalevsky ed; (Springer Verlag), (1965)

Kuznetsov, M. V. Phys. Sol. Earth No. 12, 779-784 (1972)

Lambeck, K., and A. Cazenave. Groupe de Recherches de Geodesie Spatiale Bull. No. 7 (1973)

Larmor, J. Proc. Lond. Math. Soc. 14, Ser. 2, 489-499 (1915)

Lee, W. H. K. and W. M. Kaula. J. Geophys. Res. 72 No. 2, 753-758 (1967)

Longman, I. M. J. Geophys. Res. 68, 485-497 (1963a)

Longman, I. M. J. Geophys. Res. 68, 845-851 (1963b)

Longman, I. M. Geophys. J. 11, 133-137 (1966)

Love, A. E. H. Proc. Roy. Soc. Lond. A82, 73-88 (1909)

Melchior, P. Earth Tides (Pergamon Press, 1966)

Melchior, P., and A. Venedikov. Phys. Earth Planet. Interiors 1, 363-372 (1968)

Michelson, A. A. Ap. J. 30, 105-138 (1914)

Michelson, A. A., and H. G. Gale. Ap. J. 50, 330-345 (1919)

Miller, S. M.Sc. Thesis, Mass. Inst. Tech. (1973)

Molodenskii, M. S. Comm. Obs. Roy. Belgique 188, 25-56 (1961)

Moulton, F. R. Ap. J. 50, 346-355 (1919)

Munk, W. H. in The Earth Moon System B. G. Marsden and A. G. W. Cameron eds., (Plenum Press), 52-69 (1966)

Munk, W. H., and G. J. F. MacDonald. The Rotation of the Earth (Cambridge Univ. Press, 1960)

Newton, R. R. Geophys. J. 14, 505-539 (1968)

O'Hora, N. P. J. Phys. Earth. Planet. Interiors 7, No. 1, 92-96 (1973)

Pariyskiy, N. N. Phys. Sol. Earth, No. 2, 3-12 (1972)

Pertsev, B. P. Phys. Sol. Earth, No. 10, 636-639 (1966)

Platzman, G. W. Mathematical Problems in the Geophysical Sciences W. H. Reid, ed. (American Math. Soc.) 239-292 (1972)

Proudman, J. Observatory 48, 386-388 (1925)

Proudman, J. Geophys. J. 3, 244-249 (1960)

Rosenhead, L. Mon. Nat. Roy. Astron. Soc. Geophys. Suppl. 2, 171-196 (1929)

Scheweydar, W. Gerl. Beitr. Geoph. 9, 41-77 (1908)

Slichter, L. B. in Nature of the Solid Earth, ed. E. C. Robertson (McGraw-Hill), 285-320 (1972)

Smith, D. E., R. Kolenkiewicz and P. J. Dunn, in The Use of Artificial Satellites for Geodesy, eds. S. Henriksen, A. Mancini and B. Chovitz, (Amer. Geophys. Un.), 187-196 (1972)

Street, R. C. Mon. Not. Roy. Astron. Soc. Geophys. Suppl. 1, 292-306 (1925)

Takeuchi, H. Trans. Am. Geophys. Un. 31, 651-689 (1950)

Takeuchi, H., M. Saito and N. Kobayashi. J. Geophys. Res. 67, 1141-1154 (1962)

Appendix A

Computing the Tide Height Matrices

A computer program has been written to give the matrices L1, L2, L3, L4, L5, L6, L7, L8 of chapter 2. The matrices have been computed to give the tide height expansion to degree and order 8 in terms of the prescribed potentials. The large size of the matrices prevents their being entirely displayed here. However, columns 4 and 5 (which determine the tide height due to prescribed potentials of degree-2 and order-0, and degree-2 and order-1, respectively) of L1 and L5 are displayed in Table A.1.

Throughout the computations the associated Legendre functions used were those defined by Jeffreys and Jeffreys (1966, p. 633):

$$P_n^m(\mu) = \frac{(n-m)!}{2^n n!^2}(1-\mu^2)^{\frac{m}{2}}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n;\quad \mu = \cos\theta$$

These normalize as

$$\int_0^\pi (P_n^m)^2 \sin\theta\, d\theta = \frac{2}{2n+1}\,\frac{(n-m)!(n+m)!}{(n!)^2} \qquad \text{(A.1.1)}$$

The triple product integrals of the associated Legendre functions,

$$\int_0^\pi P_n^m P_r^\ell P_p^q \sin\theta\, d\theta,$$

were computed using the formula of Infeld and Hull (1951). They use the fully normalized polynomials

$$P_n^m(\mu) = \left(\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right)^{\frac{1}{2}}\frac{1}{2^n n!}(1-\mu^2)^{\frac{m}{2}}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n$$

so that a normalization factor must be included in equation 2.47.

As the ocean function of Balmino et al (1973) proved to be a better representation than that of Munk and MacDonald (1960) it was decided to use the former in the programming.

The ocean function coefficients of Balmino et al were computed using Legendre functions normalized in yet a third way so that it was necessary to convert these coefficients to the normalization A1.1 by multiplying the degree-n, order-m coefficient by the factor

$$\left(\frac{2n+1}{(n+m)!(n-m)!}\right)^{\frac{1}{2}} n!$$

The coefficients in both normalizations are displayed in Table 3.1.

The computer program with sufficient comments to clarify the operation is given below.

```
LEVEL  21                    MAIN                    DATE = 73242              13/06/57
C  PROGRAM TO EVALUATE THE TIDE PREDICTION MATRICES
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 INTEG
      INTEGER R,P,Q,RR,PP,QQ,EXPO1,EXPO2,EXP,EXP1,EXP2,EXPO
      DIMENSION A(90)
       DIMENSION X(4,26),AIJ(10,10)
     1,AIJI(10,10),AIJIA(10,10),BIJ(10,10),BIJI(10,10),BIJIA(10,10),
     2A2J(10,10),A2JI(10,10),A2JIA(10,10),B2J(10,10),B2JI(10,10),
     3B2JIA(10,10)
      DIMENSION AIJIAI(10,10),A2JIAI(10,10)
      DIMENSION GIMBET(10,10),BETGIM(10,10),PAR1(10,10),PAR2(10,10),
     1PAR3(10,10),PAR4(10,10),PAR5(10,10),PAR6(10,10),PAR7(10,10),
     1PAR8(10,10)
      DIMENSION BET(10,10),GAM(10,10),BETGAM(10,10),GAMBET(10,10)
C  THE LOVE NUMBERS ARE FROM LONGMAN (1963) AND ARE PUNCHED ON 26 CARDS
C  10/CARD,THE FIRST.IS H THE SECOND K THE THIRD H' THE FOURTH K'.
      READ(5,440)((X(I,J),I=1,4),J=1,26)
 440  FORMAT(4F6.3)
C  THE OCEAN FUNCTION COEFFICIENTS ARE FROM BALMINO ET AL (1973) WITH
C  THE NORMALIZATION CHANGED TO THAT OF MUNK & MACDONALD (1960) AND
C  ARE PUNCHED ON 9 CARDS 10/CARD
      READ(5,10)(A(N),N=1,90)
  10  FORMAT(10F8.4)
C     SET EVERY ELEMENT IN A1,A2,B1,B2=0
      REAL*8 A2(10,10)/100*0.D0/
      REAL*8 A1(10,10)/100*0.D0/
      REAL*8 B1(10,10)/100*0.D0/
      REAL*8 B2(10,10)/100*0.D0/
      DO 150 PP=1,4
      DO 160 QQ=1,PP
      DO 170 RR=1,4
      DO 180 LL=1,RR
      DO 190 NN=1,9
      DO 200 MM=1,NN
      P=PP-1
      Q=QQ-1
      R=RR-1
      L=LL-1
      N=NN-1
      M=MM-1
C  CARD 30 TESTS THE REQUIREMENT ON THE AZIMUTHAL INDECES
      IF(.NOT.(M.EQ.L+Q.OR.Q.EQ.L+M.OR.L.EQ.Q+M))GO TO 200
C    TEST FOR SUM OF ORDERS EQUAL EVEN INTEGER
  20  NRP=(N+R+P)/2
      N1=N+R+P-2*NRP
      IF(N1.NE.0) GO TO 200
C  CARD 36 TESTS IF ORDERS SATISFY TRIANGULARITY
      IF(.NOT.(IABS(R-N).LE.P.AND.P.LE.R+N)) GO TO 200
```

```
C     SUBROUTINE ORDER ORDERS THE LEGENDRE FUNCTIONS SO THAT THE
C     CRITERIA OF INFELD AND HULL (1951) ARE SATISFIED
         CALL ORDER(N,M,R,L,P,Q)
C    SUBROUTINE VAL EVALUATES THE TRIPLE PRODUCT INTEGRALS USING THE
C    FORMULA OF INFELD & HULL (1951)
     4   CALL VAL (N,M,R,L,P,Q,INTEG)
C    SUBROUTINE AZM EVALUATES THE INTEGRAL OVER THE AZIMUTH
         CALL AZM (MM,LL,QQ,ASMCCC,ASMSSC,ASMCSS,ASMSCS)
         D=2*NN-1
         E=2*RR-1
         F=2*PP-1
C    TERM3 IS A CONVERSION FACTOR FROM THE NORMALIZATION OF INFELD AND
C    HULL TO THE NORMALIZATION OF MUNK AND MACDONALD (1960)
         TERM3=DSQRT(2.D0*FACT1(NN-MM)*FACT1(NN+MM-2)*FACT1(RR-LL)
        1*FACT1(RR+LL-2)*F/(D*E*FACT1(PP-QQ)*FACT1(PP+QQ-2)))
        2*FACT1(PP-1)/(FACT1(NN-1)*FACT1(RR-1))
C    CARDS 97-101 FORM THE 45$45 MATRICES A1,A2,B1,B2
         B1(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=B1(RR*(RR-1)/2+LL,PP*(PP-1)/2+
        1QQ)+A(NN*(NN-1)/2+MM+45)*INTEG*ASMSSC*TERM3
         B2(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=B2(RR*(RR-1)/2+LL,PP*(PP-1)/2+
        1QQ)+A(NN*(NN-1)/2+MM+45)*INTEG*ASMSCS*TERM3
         A1(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=A1(RR*(RR-1)/2+LL,PP*(PP-1)/2+
        1QQ)+A(NN*(NN-1)/2+MM)*INTEG*ASMCCC*TERM3
         A2(RR*(RR-1)/2+LL,PP*(PP-1)/2+QQ)=A2(RR*(RR-1)/2+LL,PP*(PP-1)/2+
        1QQ)+A(NN*(NN-1)/2+MM)*INTEG*ASMCSS*TERM3
   200   CONTINUE
   190   CONTINUE
   180   CONTINUE
   170   CONTINUE
   160   CONTINUE
   150   CONTINUE
C     CONVERT THE MATRICES A1,B1,A2,B2 TO THE TIDE HEIGHT MATRICES WITH
C     NO FEEDBACK
         DO 500 N=1,4
         DO 501 M=1,N
         DO 502 J=1,10
         I=N*(N-1)/2+M
         AIJ(J,I)=A1(I,J)*(1.0+X(2,N)-X(1,N))
         AIJI(J,I)=A1(I,J)*(1.0+X(4,N)-X(3,N))
         AIJIA(J,I)=A1(I,J)*(1.0+X(4,N)-X(3,N))*0.5585D 00/(2*N-1)
         BIJ(J,I)=B1(I,J)*(1.0+X(2,N)-X(1,N))
         BIJI(J,I)=B1(I,J)*(1.0+X(4,N)-X(3,N))
         BIJIA(J,I)=B1(I,J)*(1.0+X(4,N)-X(3,N))*0.5585D 00/(2*N-1)
         A2J(J,I)=A2(I,J)*(1.0+X(2,N)-X(1,N))
         A2JI(J,I)=A2(I,J)*(1.0+X(4,N)-X(3,N))
         A2JIA(J,I)=A2(I,J)*(1.0+X(4,N)-X(3,N))*0.5585D 00/(2*N-1)
         B2J(J,I)=B2(I,J)*(1.0+X(2,N)-X(1,N))
         B2JI(J,I)=B2(I,J)*(1.0+X(4,N)-X(3,N))
```

```
      B2JIA(J,I)=B2(I,J)*(1,0+X(4,N)-X(3,N))*0.5585D 00/(2*N-1)
  502 CONTINUE
  501 CONTINUE
  500 CONTINUE
C     PERFORM THE MATRIX ALGEBRA OF THE THESIS AND EVALUATE THE
C     TIDE HEIGHT MATRICES
C     SUBROUTINE DIV INVERTS THE FIRST ARGUMENT AND CALLS IT THE SECOND
C     SUBROUTINE PROD MULTIPLIES THE FIRST TWO AND CALLS IT THE THIRD
C     SUBROUTINE TERM CALCULATES THE TIDE PREDICTION MATRICES
      CALL DIV(AIJIA,AIJIAI)
      CALL DIV(A2JIA,A2JIAI)
      CALL PROD(AIJIAI,BIJIA,BET)
      CALL PROD(A2JIAI,B2JIA,GAM)
      CALL PROD(BET,GAM,BETGAM)
      CALL PROD(GAM,BET,GAMBET)
      CALL DIV(BETGAM,BETGIM)
      CALL DIV(GAMBET,GIMBET)
      CALL TERM(BETGIM,AIJIAI,AIJ,BET,A2JIAI,B2J,PAR1)
      CALL TERM(BETGIM,AIJIAI,AIJI,BET,A2JIAI,B2JI,PAR2)
      CALL TERM(BETGIM,AIJIAI,BIJ,BET,A2JIAI,A2J,PAR3)
      CALL TERM(BETGIM,AIJIAI,BIJI,BET,A2JIAI,A2JI,PAR4)
      CALL TERM(GIMBET,A2JIAI,B2J,GAM,AIJIAI,AIJ,PAR5)
      CALL TERM(GIMBET,A2JIAI,B2JI,GAM,AIJIAI,AIJI,PAR6)
      CALL TERM(GIMBET,A2JIAI,A2J,GAM,AIJIAI,BIJ,PAR7)
      CALL TERM(GIMBET,A2JIAI,A2JI,GAM,AIJIAI,BIJI,PAR8)
      DO 509 I=1,10
      AIJ(1,I)=0.D0
      AIJI(1,I)=0.D0
      BIJ(1,I)=0.D0
      BIJI(1,I)=0.D0
      A2J(1,I)=0.D0
      A2JI(1,I)=0.D0
      B2J(1,I)=0.D0
      B2JI(1,I)=0.D0
      PAR1(1,I)=0.D0
      PAR2(1,I)=0.D0
      PAR3(1,I)=0.D0
      PAR4(1,I)=0.D0
      PAR5(1,I)=0.D0
      PAR6(1,I)=0.D0
      PAR7(1,I)=0.D0
      PAR8(1,I)=0.D0
  509 CONTINUE
      WRITE(6,890)
      WRITE(6,510)
  510 FORMAT('             MATRIX A1(N,M) ')
      WRITE(6,520)((A1(I,J),J=1,10),I=1,10)
  520 FORMAT(10F8.4)
```

```
      WRITE(6,530)
  530 FORMAT('                MATRIX A2(N,M) ')
      WRITE(6,520)((A2(I,J),J=1,10),I=1,10)
      WRITE(6,540)
  540 FORMAT('                MATRIX B1(N,M) ')
      WRITE(6,520)((B2(I,J),J=1,10),I=1,10)
      WRITE(6,550)
  550 FORMAT('                MATRIX B2(N,M) ')
      WRITE(6,520)((B2(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,620)
  620 FORMAT('     MATRIX AIJIA')
      WRITE(6,31)((AIJIA(I,J),J=1,10),I=1,10)
      WRITE(6,650)
  650 FORMAT('      MATRIX BIJIA')
      WRITE(6,31)((BIJIA(I,J),J=1,10),I=1,10)
      WRITE(6,680)
  680 FORMAT('      MATRIX A2JIA')
      WRITE(6,31)((A2JIA(I,J),J=1,10),I=1,10)
      WRITE(6,710)
  710 FORMAT('      MATRIX B2JIA')
      WRITE(6,31)((B2JIA(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,720)
  720 FORMAT('      INVERSE MATRIX I-AIJIA')
      WRITE(6,31)((AIJIAI(I,J),J=1,10),I=1,10)
      WRITE(6,730)
  730 FORMAT('       INVERSE MATRIX I-A2JIA')
      WRITE(6,31)((A2JIAI(I,J),J=1,10),I=1,10)
      WRITE(6,750)
  750 FORMAT('      MATRIX BET')
      WRITE(6,31)((BET(I,J),J=1,10),I=1,10)
      WRITE(6,760)
  760 FORMAT('      MATRIX GAM')
      WRITE(6,31)((GAM(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,770)
  770 FORMAT('      MATRIX BETGAM')
      WRITE(6,31)((BETGAM(I,J),J=1,10),I=1,10)
      WRITE(6,780)
  780 FORMAT('      MATRIX GAMBET')
      WRITE(6,31)((GAMBET(I,J),J=1,10),I=1,10)
      WRITE(6,790)
  790 FORMAT('      INVERSE MATRIX BETGIM')
      WRITE(6,31)((BETGIM(I,J),J=1,10),I=1,10)
      WRITE(6,800)
  800 FORMAT('      INVERSE MATRIX GIMBET')
      WRITE(6,31)((GIMBET(I,J),J=1,10),I=1,10)
```

```
      WRITE(6,890)
      WRITE(6,560)
  560 FORMAT('      MATRIX AIJ  (L1 WITHOUT FEEDBACK)')
      WRITE(6,31)((AIJ(I,J),J=1,10),I=1,10)
      WRITE(6,610)
  610 FORMAT('      MATRIX AIJI  (L2 WITHOUT FEEDBACK)')
      WRITE(6,31)((AIJI(I,J),J=1,10),I=1,10)
      WRITE(6,630)
  630 FORMAT('      MATRIX BIJ  (L3 WITHOUT FEEDBACK)')
      WRITE(6,31)((BIJ(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,640)
  640 FORMAT('      MATRIX BIJI  (L4 WITHOUT FEEDBACK)')
      WRITE(6,31)((BIJI(I,J),J=1,10),I=1,10)
      WRITE(6,690)
  690 FORMAT('      MATRIX B2J  (L5 WITHOUT FEEDBACK)')
      WRITE(6,31)((B2J(I,J),J=1,10),I=1,10)
      WRITE(6,700)
  700 FORMAT('      MATRIX B2JI  (L6 WITHOUT FEEDBACK)')
      WRITE(6,31)((B2JI(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,660)
  660 FORMAT('      MATRIX A2J  (L7 WITHOUT FEEDBACK)')
      WRITE(6,31)((A2J(I,J),J=1,10),I=1,10)
      WRITE(6,670)
  670 FORMAT('      MATRIX A2JI  (L8 WITHOUT FEEDBACK)')
      WRITE(6,31)((A2JI(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,810)
  810 FORMAT('      MATRIX L1')
      WRITE(6,31)((PAR1(I,J),J=1,10),I=1,10)
      WRITE(6,820)
  820 FORMAT('      MATRIX L2')
      WRITE(6,31)((PAR2(I,J),J=1,10),I=1,10)
      WRITE(6,830)
  830 FORMAT('      MATRIX L3')
      WRITE(6,31)((PAR3(I,J),J=1,10),I=1,10)
      WRITE(6,890)
      WRITE(6,840)
  840 FORMAT('      MATRIX L4')
      WRITE(6,31)((PAR4(I,J),J=1,10),I=1,10)
      WRITE(6,850)
  850 FORMAT('      MATRIX L5')
      WRITE(6,31)((PAR5(I,J),J=1,10),I=1,10)
      WRITE(6,860)
  860 FORMAT('      MATRIX L6')
      WRITE(6,31)((PAR6(I,J),J=1,10),I=1,10)
      WRITE(6,890)
```

```
LEVEL  21                    MAIN                 DATE = 73242           13/06/57

      WRITE(6,870)
  870 FORMAT('        MATRIX L7')
      WRITE(6,31)((PAR7(I,J),J=1,10),I=1,10)
      WRITE(6,880)
  880 FORMAT('        MATRIX L8')
      WRITE(6,31)((PAR8(I,J),J=1,10),I=1,10)
  890 FORMAT('1')
   31 FORMAT(10F8.4)
      STOP
      END
```

```
LEVEL  21                    ORDER                DATE = 73242           13/06/57

      SUBROUTINE ORDER(N,M,R,L,P,Q)
C     SUBROUTINE ORDER ORDERS THE LEGENDRE FUNCTIONS SO THAT THE
C     CRITERIA OF INFELD AND HULL (1951) ARE SATISFIED
      INTEGER R,L,P,Q
      IF(N.EQ.0) GO TO 7
      IF(R.EQ.0) GO TO 11
      IF(P.EQ.0) GO TO 12
      GO TO 13
   11 IR=R
      IL=L
      R=N
      L=M
      N=IR
      M=IL
      GO TO 7
   12 IP=P
      IQ=Q
      Q=M
      P=N
      M=IQ
      N=IP
      GO TO 7
   13 CONTINUE
```

```
      IF(P.GT.R) GO TO 20
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
   20 CONTINUE
      IF(R.GT.N) GO TO 30
      IR=R
      IL=L
      R=N
      L=M
      N=IR
      M=IL
   30 CONTINUE
      IF(P.GT.R) GO TO 40
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
   40 CONTINUE
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51


      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
      IN=N
      IM=M
      N=R
      M=L
      R=IN
      L=IM
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
      IN=N
      IM=M
      N=R
      M=L
      R=IN
      L=IM
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
```

```
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
      IF(M.EQ.L+Q.AND.N.LE.R) GO TO 51
   43 CONTINUE
      IF(P.GT.R) GO TO 5
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
    5 CONTINUE
      IF(R.GT.N) GO TO 6
      IR=R
      IL=L
      R=N

      L=M
      N=IR
      M=IL
    6 CONTINUE
      IF(P.GT.R) GO TO 7
      IP=P
      IQ=Q
      P=R
      Q=L
      R=IP
      L=IQ
    7 CONTINUE
      IF(M.EQ.L+Q) GO TO 51
      IF(L.EQ.Q+M) GO TO 49
      IF(Q.EQ.L+M) GO TO 50
   49 Q=-Q
      GO TO 51
   50 L=-L
   51 CONTINUE
      RETURN
      END
```

```
LEVEL  21                    VAL.                 DATE = 73242          13/06/57
      SUBROUTINE VAL (N,M,R,L,P,Q,S)
C   SUBROUTINE VAL CALCULATES THE TRIPLE PRODUCT INTEGRALS USING THE
C   FORMULA OF INFELD & HULL (1951) AND RETURNS IT AS S
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER P,Q,R,EXP,EXP1,EXP2,EXPO,EXPO1,EXPO2
C   CARDS 45-79 EVALUATE THE INTEGRAL
      KK=N-M+1
      IF(KK-1)30,30,60
   30 EXP=(P-R+N)/2+(IABS(Q))
      EXP1=EXP/2
      EXP2=EXP-EXP1*2
      IF(EXP2)50,40,50
   40 SUM= (FACT1(P+Q)*FACT1(R+N-Q))/(FACT1(N-M)*FACT1(P-Q)
     1 *FACT1(R-N+Q))
      GO TO 110
   50 SUM=-(FACT1(P+Q)*FACT1(R+N-Q))/(FACT1(N-M)*FACT1(P-Q)
     1 *FACT1(R-N+Q))
      GO TO 110
   60 SUM=0.D0
      DO 100 I=1,KK
      J=I-1
      EXPO=(P-R+N)/2+(IABS(Q)+J)
      EXPO1=EXPO/2
      EXPO2=EXPO-2*EXPO1
      IF(EXPO2)80,70,80
   70 TERM4= (FACT1(P+Q+J)*FACT1(R+N-Q-J))/(FACT1(N-M-J)*FACT1(J)
     1  *FACT1(P-Q-J)*FACT1(R-N+Q+J))
      GO TO 90
   80 TERM4=-(FACT1(P+Q+J)*FACT1(R+N-Q-J))/(FACT1(N-M-J)*FACT1(J)
     1 *FACT1(P-Q-J)*FACT1(R-N+Q+J))
   90 SUM=SUM+TERM4
  100 CONTINUE
  110 D=2*N+1
      E=2*R+1
      F=2*P+1
      TERM1=(FACT2(P+R-N-1)*DSQRT(D*E*F))/(FACT2(P+N-R)
     1  *FACT2(R+N-P)*FACT2(N+R+P+1))
      TERM2=DSQRT((FACT1(N+M)*FACT1(N-M)*FACT1(R-L)*FACT1(P-Q))/
     1 (FACT1(R+L)*FACT1(P+Q)*2.0))
  120    S=TERM1*TERM2*SUM
C   CARDS 35-52 TEST IF A CHANGE IN SIGN OF ONE OF THE AZIMUTHAL INDECES
C   WAS MADE AND CORRECTS S
      IF(Q.LE.0) GO TO 62
      IF(L.LE.0) GO TO 63
      GO TO 61
   62 EXP=IABS(Q)
      EXP1=EXP/2
      EXP2=EXP-EXP1*2
```

```
LEVEL  21                   VAL                   DATE = 73242            13/06/57
      IF(EXP2,EQ.0) GO TO 51
      S=-S
   51 CONTINUE
      RETURN
   63 EXP=IABS(L)
      EXP1=EXP/2
      EXP2=EXP-EXP1*2
      IF(EXP2,EQ.0) GO TO 61
      S=-S
   61 CONTINUE
      RETURN
      END
```

```
LEVEL  21                   AZM                   DATE = 73242            13/06/57
      SUBROUTINE AZM (MM,LL,QQ,AZMCCC,AZMSSC,AZMCSS,AZMSCS)
C  SUBROUTINE AZM EVALUATES THE INTEGRAL OVER THE AZIMUTH. THESE
C  WERE FOUND ANALYTICALLY FOR ALL 8 POSSIBLE CASESC  CARDS 5-19 TEST
C  WHICH CASE THE AZIMUTHAL INDECES FIT AND TRANSFERS THE COMPUTATION.
C  THE VALUE OF THE AZIMUTHAL INTEGRAL RETURNED IS NORMALIZED
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER QQ
  150 IF(MM+LL+QQ.EQ.3) GO TO 230
      IF(MM+LL.EQ.2.OR.MM+QQ.EQ.2.OR.LL+QQ.EQ.2) GO TO 240
      IF(QQ.EQ.1) GO TO 250
      IF(LL.EQ.1) GO TO 260
      IF(MM.EQ.1) GO TO 270
      IF(QQ.EQ.LL+MM-1) GO TO 280
      IF(LL.EQ.QQ+MM-1) GO TO 290
      IF(MM.EQ.QQ+LL-1) GO TO 300
  230 AZMCCC=1.D0
      AZMSSC=0.D0
      AZMCSS=0.D0
      AZMSCS=0.D0
      GO TO 156
  240 AZMCCC=0.D0
      AZMSSC=0.D0
      AZMCSS=0.D0
      AZMSCS=0.D0
      GO TO 156
```

```
250   AZMCCC=0.50D 00
      AZMSSC=0.50D 00
      AZMCSS=0.D0
      AZMSCS=0.D0
      GO TO 156
260   AZMCCC=1.D0
      AZMSSC=0.D0
      AZMCSS=0.D0
      AZMSCS=1.D0
      GO TO 156
270   AZMCCC=1.D0
      AZMSSC=0.D0
      AZMCSS=1.D0
      AZMSCS=0.D0
      GO TO 156
280   AZMCCC=0.50D 00
      AZMSSC=-0.50D 00
      AZMCSS=0.50D 00
      AZMSCS=0.50D 00
      GO TO 156
290   AZMCCC=0.50D 00
      AZMSSC=0.50D 00
      AZMCSS=0.50D 00
      AZMSCS=-0.50D 00
      GO TO 156
300   AZMCCC=0.50D 00
      AZMSSC=0.50D 00
      AZMCSS=-0.50D 00
      AZMSCS=0.50D 00
156   CONTINUE
      RETURN
      END
```

```
.EVEL  21                    FACT1               DATE = 73242          13/06/57

      FUNCTION FACT1(MM)
C  SUBPROGRAM FACT1 EVALUATES THE FACTORIAL OF MM
      IMPLICIT REAL*8(A-H,O-Z)
      FACT1=1.D0
      IF(MM.GT.1) GO TO 31
      RETURN
   31 CONTINUE
      DO 32 K=1,MM
      AK=K
      FACT1=FACT1*AK
   32 CONTINUE
      RETURN
      END
```

```
.EVEL  21                    MAIN                    DATE = 73242          13/06/57
C      FUNCTION SUBPROGRAM FOR FACTORIAL2
      FUNCTION FACT2(NN)
C  SUBPROGRAM FACT2 EVALUATES THE DOUBLE FACTOTIAL OF NN
      IMPLICIT REAL*8(A-H,O-Z)
      FACT2=1.D0
      IF(NN.LE.1)GO TO 20
      IF(NN.EQ.2)GO TO 21
      NON=NN/2
      NONO=2*NON
      IF(NONO-NN)25,22,25
   25 CONTINUE
      DO 26 K=1,NN,2
      AK=K
      FACT2=FACT2*AK
   26 CONTINUE
      RETURN
   22 CONTINUE
      DO 23 K=2,NN,2
      AK=K
      FACT2=FACT2*AK
   23 CONTINUE
      RETURN
   20 FACT2=1.D0
      RETURN
   21 FACT2=2.D0
      RETURN
      END
```

```
.EVEL  21                    PROD                    DATE = 73242          13/06/57
      SUBROUTINE PROD(AA,BB,CC)
C  SUBROUTINE PROD MULTIPLIES AA ANDBB AND RETURNS CC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AA(10,10),BB(10,10),CC(10,10)
      DO 1 I=1,10
      DO 2 J=1,10
      CC(I,J)=0.D0
      DO 3 K=1,10
      CC(I,J)=CC(I,J)+AA(I,K)*BB(K,J)
    3 CONTINUE
    2 CONTINUE
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DIV(B,D)
C  SUBROUTINE DIV USES GAUSSIAN ELIMINATION WITH MAXIMUM PIVOT TO INVERT
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(10,10),D(10,10),A(10,20),C(10,20)
C  FORM THE AUGMENTED MATRIX
      DO 10 J=1,20
      DO 20 I=1,10
      IF(J.LE.10) GO TO 30
      IF(I+10.EQ.J) GO TO 40
      A(I,J)=0.D0
      GO TO 20
   40 A(I,J)=1.D0
      GO TO 20
   30 IF(I.EQ.J) GO TO 31
      A(I,J)=-B(I,J)
      GO TO 20
   31 A(I,J)=1.D0-B(I,J)
   20 CONTINUE
   10 CONTINUE
C  SEARCH FOR THE LARGEST ELEMENT IN ROW L
      DO 50 L=1,9
      K=1
      DO 60 J=1,10
      IF(DABS(A(L,K)).GT.DABS(A(L,J))) GO TO 60
      K=J
   60 CONTINUE
      IF(K.EQ.L) GO TO 70
      DO 80 J=1,10
      C(J,K)=A(J,K)
      C(J,K+10)=A(J,K+10)
      A(J,K)=A(J,L)
      A(J,K+10)=A(J,L+10)
      A(J,L)=C(J,K)
      A(J,L+10)=C(J,K+10)
   80 CONTINUE
   70 L1=L+1
      DO 90 K=L1,10
      S=A(K,L)
      DO 100 J=L,20
      A(K,J)=A(K,J)-(S/A(L,L))*A(L,J)
  100 CONTINUE
   90 CONTINUE
   50 CONTINUE
      DO 110 J=1,10
      D(10,J)=A(10,10+J)/A(10,10)
      DO 120 I=1,9
      SUM=0.D0
      DO 130 K1=1,I
```

```
.EVEL  21                  DIV                DATE = 73242          13/06/57
      K=K1-1
      SUM=SUM+D(10-K,J)*A(10-I,10-K)
 130  CONTINUE
      D(10-I,J)=(A(10-I,10+J)-SUM)/A(10-I,10-I)
 120  CONTINUE
 110  CONTINUE
      RETURN
      END
```

```
.EVEL  21                  TERM               DATE = 73242          13/06/57
      SUBROUTINE TERM(A,B,C,D,E,F,G)
C     SUBROUTINE TERM DOES THE MATRIX ALGEBRA AND RETURNS THE TIDE
C     PREDICTION MATRICES
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(10,10),B(10,10),C(10,10),D(10,10),E(10,10),F(10,10),
     1G(10,10),T1(10,10),S1(10,10),T2(10,10),T3(10,10),S2(10,10)
      CALL PROD(A,B,T1)
      CALL PROD(T1,C,S1)
      CALL PROD(A,D,T2)
      CALL PROD(T2,E,T3)
      CALL PROD(T3,F,S2)
      DO 1 I=1,10
      DO 2 J=1,10
      G(I,J)=S1(I,J)+S2(I,J)
    2 CONTINUE
    1 CONTINUE
      RETURN
      END
```

Appendix B

The Ocean Function

The ocean function has been computed by several authors (Munk and MacDonald, 1960; Lee and Kaula, 1967; Balmino et al 1973). The expansion of Lee and Kaula (1967) has been shown to yield unsatisfactory results (Balmino et al, 1973) and will not be discussed further here.

Due to the fact that most land masses are either greater than 45 degrees or less than a few degrees in extent, an expansion to degree and order 3 will represent the ocean continent distribution almost as well as an expansion to degree and order 8. A higher order expansion does have the advantage of describing the coastlines a little better, and when synthesized it gives land value of the ocean function which are closer to zero and ocean values which are closer to 1. For the purposes of this thesis the position and shape of the continents need not be known to very great accuracy but land values must be as close to zero as possible and ocean values as close to 1. The effect of a variation from these is to introduce a spurious tide.

The expansions of Munk and MacDonald (1960) (Fig. B2 a,b) and Balmino et al, 1973) (Fig. B1 a,b) represent the ocean-continent distribution equally well, but the expansion of Munk and MacDonald gives some anomalously low values in land areas and high values in ocean areas, particularly in low latitudes. It is not difficult to trace this to values of the coefficients b_7^7, b_8^8 which are too great. A glance at Table B.1 confirms this.

n	m	Balmino et al (1973) a_n^m	b_n^m	Balmino et al M&M norm a_n^m	b_n^m	Munk and MacDonald (1960) a_n^m	b_n^m
0	0	0.697	0.0	0.697	0.0	0.714	0.0
1	0	-0.126	0.0	-0.218	0.0	-0.213	0.0
1	1	-0.108	-0.056	-0.187	-0.097	-0.188	-0.094
2	0	-0.060	0.0	-0.134	0.0	-0.130	0.0
2	1	-0.040	-0.051	-0.103	-0.132	-0.101	-0.158
2	2	0.040	0.002	0.052	0.003	0.099	-0.007
3	0	0.045	0.0	0.119	0.0	0.116	0.0
3	1	0.044	-0.032	0.143	-0.104	0.149	-0.128
3	2	0.070	-0.089	0.143	-0.182	0.257	-0.367
3	3	-0.016	-0.089	-0.013	-0.074	-0.014	-0.211
4	0	-0.024	0.0	-0.072	0.0	-0.078	0.0
4	1	0.036	0.030	0.137	0.114	0.153	0.095
4	2	0.090	-0.021	0.241	-0.056	0.471	-0.115
4	3	-0.053	0.005	-0.076	0.007	-0.207	0.010
4	4	0.014	-0.101	0.007	-0.051	0.035	-0.206
5	0	0.109	0.0	0.362	0.0	0.325	0.0
5	1	-0.005	0.011	-0.021	0.047	-0.034	0.079
5	2	0.049	0.029	0.159	0.094	0.314	0.167
5	3	-0.028	-0.016	-0.055	-0.032	-0.212	-0.071
5	4	-0.100	0.035	-0.093	0.033	-0.340	0.099
5	5	-0.001	-0.049	-0.000	-0.014	0.0	-0.076
6	0	-0.022	0.0	-0.079	0.0	-0.118	0.0
6	1	0.013	0.023	0.061	0.109	0.045	0.095
6	2	0.019	0.005	0.071	0.019	0.124	-0.024
6	3	0.002	-0.033	0.005	-0.082	-0.006	-0.186
6	4	-0.031	0.023	-0.042	0.031	-0.150	0.124
6	5	0.021	0.029	0.012	0.017	0.063	0.066
6	6	-0.002	-0.013	-0.000	-0.002	-0.002	-0.013
6	6	-0.002	-0.013	-0.000	-0.002	-0.002	-0.013
7	0	0.060	0.0	0.232	0.0	0.197	0.0
7	1	-0.004	-0.036	-0.020	-0.184	-0.031	-0.177
7	2	-0.021	-0.006	-0.088	-0.025	-0.203	-0.009
7	3	0.009	-0.019	0.027	-0.056	0.128	-0.095
7	4	0.028	-0.010	0.050	-0.018	0.194	-0.046
7	5	-0.005	0.022	-0.004	0.020	-0.019	0.106
7	6	0.004	0.030	0.001	0.010	0.011	0.057
7	7	0.001	0.032	0.000	0.003	0.004	0.021
8	0	0.015	0.0	0.062	0.0	0.040	0.0
8	1	0.006	0.027	0.033	0.148	-0.009	0.149
8	2	-0.002	0.003	-0.009	0.014	-0.094	0.112
8	3	-0.008	-0.019	-0.027	-0.064	-0.156	-0.092
8	4	-0.002	-0.018	-0.004	-0.039	0.0	-0.113
8	5	0.012	-0.002	0.015	-0.002	0.141	-0.002
8	6	-0.014	-0.016	-0.008	-0.009	-0.041	-0.065
8	7	-0.040	-0.020	-0.008	-0.004	-0.050	-0.025
8	8	0.010	-0.021	0.000	-0.001	0.007	0.009

Table B1 Surface harmonic coefficients of the ocean function.

Figure B.1 a The ocean function of Balmino et al (1973), eastern hemisphere. The boundaries are defined by $e \leq 0.5$

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0
10	0.9	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	10
20	0.7	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.4	0.4	0.5	0.6	0.6	20
30	0.6	0.4	0.3	0.2	0.1	0.1	0.1	0.1	0.0	-0.0	-0.1	-0.1	-0.0	0.0	0.2	0.3	0.5	0.6	0.7	30
40	0.5	0.4	0.2	0.1	0.0	-0.0	-0.0	-0.0	-0.1	-0.1	-0.2	-0.2	-0.0	0.2	0.4	0.6	0.7	0.8	0.9	40
50	0.4	0.3	0.3	0.3	0.2	0.1	0.0	0.0	0.0	-0.1	-0.1	-0.0	0.2	0.5	0.8	0.9	1.0	1.1	1.1	50
60	0.1	0.1	0.3	0.4	0.3	0.2	0.2	0.2	0.2	0.1	0.0	0.2	0.5	0.9	1.0	1.1	1.1	1.1	1.1	60
70	-0.0	-0.1	0.0	0.2	0.3	0.3	0.4	0.5	0.5	0.3	0.2	0.4	0.7	1.0	1.1	1.0	0.9	1.0	1.0	70
80	0.2	-0.1	-0.1	0.0	0.2	0.5	0.8	0.9	0.9	0.6	0.4	0.6	0.9	1.1	1.0	0.9	0.9	1.0	1.0	80
90	0.6	0.2	-0.0	-0.0	0.3	0.7	1.1	1.2	1.1	0.9	0.7	0.8	0.9	0.9	0.8	0.8	1.0	1.1	1.1	90
100	1.0	0.5	0.1	0.1	0.4	0.8	1.2	1.2	1.1	1.0	0.9	0.9	0.7	0.5	0.4	0.7	1.0	1.2	1.1	100
110	1.0	0.6	0.2	0.2	0.5	0.8	1.0	1.0	0.9	0.9	1.0	0.8	0.5	0.2	0.1	0.5	0.9	1.1	1.0	110
120	0.9	0.6	0.4	0.5	0.7	0.9	0.9	0.9	0.9	1.0	1.0	0.9	0.5	0.2	0.2	0.5	0.8	1.0	0.9	120
130	0.9	0.8	0.8	0.9	1.0	1.1	1.1	1.0	1.0	1.1	1.1	1.0	0.8	0.6	0.6	0.7	0.9	1.0	0.9	130
140	1.1	1.0	1.1	1.1	1.2	1.2	1.2	1.1	1.1	1.1	1.1	1.1	1.0	1.0	1.0	1.0	1.1	1.1	1.0	140
150	1.0	1.0	1.0	1.0	1.0	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.9	0.9	1.0	1.0	1.0	1.0	1.0	150
160	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.6	0.6	0.7	0.7	160
170	0.1	0.1	0.0	-0.0	-0.1	-0.1	-0.1	-0.2	-0.2	-0.2	-0.1	-0.1	-0.1	-0.0	0.0	0.1	0.2	0.2	0.3	170
180	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure B.1b The ocean function of Balmino et al (1973), western hemisphere. Thr boundaries are defined by e<0.5

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360	
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0
10	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.8	0.8	0.8	0.9	0.9	0.9	0.9	0.9	10
20	0.6	0.6	0.6	0.6	0.5	0.4	0.3	0.2	0.2	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.8	0.7	20
30	0.7	0.7	0.7	0.7	0.5	0.3	0.1	-0.1	-0.2	-0.2	-0.1	0.1	0.4	0.6	0.7	0.8	0.8	0.7	0.6	30
40	0.9	0.9	0.9	0.9	0.7	0.5	0.2	-0.1	-0.3	-0.2	-0.0	0.2	0.5	0.8	0.9	1.0	0.9	0.7	0.5	40
50	1.1	1.1	1.1	1.0	1.0	0.7	0.4	0.1	-0.1	0.0	0.3	0.6	0.9	1.0	1.1	1.1	0.9	0.6	0.4	50
60	1.1	1.0	1.0	1.1	1.1	0.9	0.6	0.3	0.2	0.3	0.6	0.9	1.1	1.2	1.2	1.0	0.7	0.4	0.1	60
70	1.0	0.9	0.9	1.0	1.1	1.0	0.8	0.6	0.5	0.6	0.7	0.9	1.0	1.1	1.1	1.0	0.7	0.3	-0.0	70
80	1.0	0.9	0.9	1.0	1.1	1.1	0.9	0.9	0.9	0.8	0.6	0.5	0.6	0.8	0.9	1.0	0.8	0.5	0.2	80
90	1.1	0.9	0.9	1.0	1.1	1.0	0.9	1.0	1.1	0.9	0.5	0.2	0.1	0.4	0.7	0.9	1.0	1.0	0.6	90
100	1.1	1.0	1.0	1.1	1.1	0.9	0.9	1.1	1.3	1.1	0.5	-0.0	-0.1	0.1	0.5	0.8	1.1	1.2	1.0	100
110	1.0	0.9	1.0	1.1	1.1	0.9	0.8	1.0	1.3	1.1	0.6	0.1	-0.0	0.2	0.5	0.8	1.1	1.2	1.0	110
120	0.9	0.9	1.0	1.1	1.1	0.9	0.9	1.0	1.1	1.1	0.7	0.4	0.3	0.5	0.7	0.9	1.1	1.1	0.9	120
130	0.9	0.9	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8	0.7	0.7	0.8	1.0	1.1	1.1	1.1	0.9	130
140	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0	1.0	0.9	0.8	0.8	0.8	0.9	1.1	1.2	1.2	1.1	1.1	140
150	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0	1.0	0.9	0.9	0.8	0.9	0.9	1.0	1.0	1.0	1.0	1.0	150
160	0.7	0.8	0.8	0.8	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	160
170	0.3	0.3	0.4	0.4	0.4	0.4	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.1	170
180	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	180
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360	

Figure B.2a The ocean function of Munk& macDonald (1960), eastern hemisphere. The boundaries are defined by the ocean function of Balmino et al with $C \leq 0.5$

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	
0	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0
10	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.8	0.8	10
20	~~0.8~~	0.6	0.5	0.4	0.4	0.4	0.5	0.5	0.4	0.3	0.2	0.1	0.1	0.2	0.3	0.5	0.6	0.7	0.8	20
30	0.8	0.5	0.2	0.1	-0.0	0.0	0.1	0.2	0.1	-0.0	-0.3	-0.5	-0.4	-0.1	0.2	0.6	0.8	1.0	1.0	30
40	0.8	0.5	0.2	0.0	-0.2	-0.2	-0.1	0.1	0.0	-0.3	-0.7	-0.9	-0.6	0.0	0.7	1.1	1.3	1.3	1.2	40
50	0.2	0.2	0.3	0.3	0.1	-0.1	-0.1	0.1	0.1	-0.3	-0.8	-0.9	-0.3	0.7	1.5	1.7	1.5	1.3	1.2	50
60	-0.7	-0.4	0.1	0.4	0.4	0.3	0.3	0.4	0.3	-0.2	-0.8	-0.8	0.2	1.4	2.0	1.6	1.1	1.0	1.1	60
70	-1.0	-0.8	-0.5	-0.2	0.3	0.9	1.2	1.1	0.6	-0.0	-0.5	-0.4	0.7	1.9	2.1	1.2	0.4	0.6	1.2	70
80	-0.1	-0.4	-1.2	-1.5	-0.4	1.3	2.2	1.8	0.9	0.3	-0.0	0.2	0.9	1.7	1.8	0.9	0.3	0.6	1.4	80
90	1.7	0.5	-1.6	-2.6	-1.0	1.7	2.9	2.2	1.0	0.6	0.8	0.8	0.7	0.9	1.0	0.9	0.7	1.0	1.4	90
100	2.7	1.1	-1.6	-2.8	-1.0	1.8	3.0	1.9	0.8	1.0	1.7	1.4	0.2	-0.5	-0.2	0.7	1.4	1.5	1.3	100
110	2.4	0.9	-1.2	-1.9	-0.3	1.9	2.4	1.2	0.5	1.3	2.3	1.7	-0.3	-1.7	-1.3	0.4	1.7	1.8	1.2	110
120	-1.4	0.4	-0.6	-0.5	0.8	1.9	1.7	0.7	0.4	1.4	2.4	1.8	-0.2	-1.7	-1.4	0.2	1.6	1.8	1.2	120
130	0.8	0.4	0.2	0.7	1.5	1.8	1.3	0.5	0.5	1.3	2.1	1.7	0.5	-0.6	-0.5	0.6	1.5	1.7	1.2	130
140	0.9	0.8	0.9	1.3	1.5	1.5	1.0	0.6	0.6	1.1	1.6	1.6	1.1	0.6	0.6	1.0	1.4	1.5	1.2	140
150	1.0	1.0	1.0	1.1	1.1	1.0	0.7	0.5	0.5	0.7	0.9	1.1	1.0	1.0	1.0	1.1	1.2	1.1	1.0	150
160	0.7	0.6	0.6	0.5	0.5	0.3	0.2	0.2	0.2	0.2	0.3	0.4	0.5	0.5	0.6	0.6	0.7	0.7	0.7	160
170	0.2	0.1	0.1	0.0	-0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1	-0.0	0.0	0.1	0.1	0.2	0.2	0.3	170
180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	180
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	

Figure B.2b The ocean function of Munk & MacDonald (1960), western hemisphere. The boundaries are defined by the ocean function of Balmino *et al* with $e \leq 0.5$

	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360	
0	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0
10	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.7	0.7	0.8	0.8	0.8	0.8	0.8	0.8	10
20	0.8	0.9	0.9	0.8	0.7	0.5	0.3	0.1	0.0	0.0	0.2	0.3	0.5	0.7	0.9	0.9	0.9	0.9	0.8	20
30	1.0	1.1	1.1	1.1	0.8	0.4	-0.1	-0.6	-0.8	-0.7	-0.3	0.2	0.6	1.0	1.2	1.3	1.3	1.1	0.8	30
40	1.2	1.2	1.3	1.4	1.3	0.7	-0.3	-1.2	-1.5	-1.1	-0.3	0.5	1.0	1.4	1.6	1.7	1.5	1.2	0.8	40
50	1.2	1.1	1.2	1.5	1.7	1.3	0.0	-1.3	-1.8	-1.1	0.1	1.0	1.5	1.8	2.0	1.9	1.4	0.7	0.2	50
60	1.1	1.0	0.8	1.2	1.8	1.9	0.7	-0.9	-1.5	-0.6	0.6	1.3	1.5	1.9	2.3	2.0	1.0	-0.2	-0.7	60
70	1.2	1.1	0.5	0.6	1.5	2.1	1.2	-0.1	-0.5	0.2	0.9	0.8	0.7	1.5	2.5	2.4	1.0	-0.5	-1.0	70
80	1.4	1.2	0.4	0.2	1.1	1.8	1.4	0.6	0.8	1.3	0.8	-0.4	-0.8	0.6	2.4	2.7	1.5	0.3	-0.1	80
90	1.4	1.2	0.6	0.5	1.0	1.3	1.0	1.1	1.9	2.3	0.8	-1.6	-2.3	-0.6	1.7	2.5	2.0	1.7	1.7	90
100	1.3	1.0	0.8	1.1	1.2	0.9	0.5	1.1	2.5	2.9	0.9	-1.9	-2.9	-1.4	0.9	2.0	2.2	2.6	2.7	100
110	1.2	0.7	0.9	1.5	1.5	0.8	0.2	0.9	2.4	2.8	1.1	-1.4	-2.3	-1.2	0.6	1.6	2.0	2.4	2.4	110
120	1.2	0.7	0.9	1.4	1.4	0.8	0.4	0.9	1.9	2.1	1.0	-0.6	-1.1	-0.3	0.9	1.6	1.7	1.7	1.4	120
130	1.2	0.7	0.8	1.1	1.2	1.0	0.8	1.0	1.4	1.4	0.7	-0.0	-0.1	0.5	1.3	1.7	1.6	1.3	0.8	130
140	1.2	0.8	0.8	0.9	1.1	1.2	1.2	1.2	1.1	0.9	0.5	0.3	0.4	0.9	1.4	1.6	1.5	1.2	0.9	140
150	1.0	0.9	0.9	1.0	1.1	1.3	1.3	1.2	1.1	0.8	0.6	0.5	0.6	0.8	1.1	1.2	1.2	1.1	1.0	150
160	0.7	0.7	0.8	0.8	0.9	1.0	1.0	1.0	0.9	0.7	0.6	0.6	0.5	0.6	0.7	0.7	0.7	0.7	0.7	160
170	0.3	0.3	0.4	0.4	0.4	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	170
180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	180
	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360	

Appendix C

Surface Harmonic Synthesis

A program has been written to compute the tide height expansion due to a prescribed potential.

Following the suggestion of Hopkins, (1973) the values of the associated Legendre functions at grid points separated by 10 degrees of latitude and longitude were computed by recursion relations. For the normalization A 1.1 the recursion relations are

$$P_n^{n-1}(\cos\theta) = \cot\theta \, P_n^n(\cos\theta)$$

$$P_{n+1}^{n+1}(\cos\theta) = \frac{2n+1}{n+1} \sin\theta \, P_n^n(\cos\theta)$$

$$P_{n+1}^m(\cos\theta) = \frac{2n+1}{n+1} \cos\theta \, P_n^m(\cos\theta) - \frac{(n+m)(n-m)}{n(n+1)} P_{n-1}^m(\cos\theta)$$

The mapping defined by 2.52 is useful in identifying a Legendre function by one index instead of the normal two.

The program, with sufficient comments to clarify the operation, is given below. The program for the ocean function is given , the programs for the tide maps being identical in operation except for a change in data cards.

```
LEVEL  21                    MAIN                    DATE = 73242            13/05/55

C  PROGRAM TO PLOT THE OCEAN FUNCTION AT EACH GRID POINT OF A 10*10 DEGREE
C  PROJECTION
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 OCEAN(36,19)/684*0.D0/
      DIMENSION P(45),A(90),MAP(36,19),LONG1(19),LONG2(19),LAT(19)
      REAL*8 C/0.1745D-01/
C   READ THE COEFFICIENTS OF THE OCEAN FUNCTION
      READ(5,1)(A(I),I=1,45)
      READ(5,1)(A(I),I=46,90)
    1 FORMAT(10F7.3)
C   COMPUTE THE LEGENDRE FUNCTIONS USING RECURRENCE RELATIONS
      P(1)=1.D0
      DO 30 K=1,19
      F=(K-1)*C*10
      DO 10 I=1,8
      N=I-1
      P((N+1)*(N+2)/2+N+2)=P(N*(N+1)/2+N+1)*(2*N+1)*DSIN(F)/(N+1)
      P((N+1)*(N+2)/2+N+1)=P(N*(N+1)/2+N+1)*(2*N+1)*DCOS(F)/(N+1)
      IF(N.LT.1) GO TO 10
      DO 20 J=1,N
      M=J-1
      P((N+1)*(N+2)/2+M+1)=P(N*(N+1)/2+M+1)*(2*N+1)*DCOS(F)/(N+1)
     1-P(N*(N-1)/2+M+1)*(N+M)*(N-M)/(N*(N+1))
   20 CONTINUE
   10 CONTINUE
C   COMPUTE THE VALUE OF THE OCEAN FUNCTION AT THE GRID POINTS
      DO 60 L=1,36
      DO 70 I=1,9
      DO 80 J=1,I
      X=(L-1)*C*10
      M=J-1
      INM=(I*(I-1)/2+J)
   81 OCEAN(L,K)=OCEAN(L,K)+P(INM)*(A(INM)*DCOS(M*X)
     1+A(INM+45)*DSIN(M*X))
   80 CONTINUE
   70 CONTINUE
   60 CONTINUE
   30 CONTINUE
      DO 180 I=1,19
      LONG1(I)=10*(I-1)
      LONG2(I)=10*(I+17)
      LAT(I)=10*(I-1)
  180 CONTINUE
      DO 190 I=1,36
      DO 200 J=1,19
      IF(OCEAN(I,J).GT..5D 00) GO TO 210
      MAP(I,J)=11
      GO TO 200
```

```
LEVEL  21                    MAIN                DATE = 73242            13/05/55
 210   MAP(I,J)=22
 200   CONTINUE
 190   CONTINUE
       WRITE(6,400)
 400   FORMAT('1')
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,220)
 260   FORMAT('0')
 220   FORMAT('           MAP OF THE OCEAN FUNCTION,0 DEGREES EAST ',
      1'LONGITUDE -180 DEGREES EAST LONGITUDE')
       WRITE(6,240)(LONG1(I),I=1,19)
 240   FORMAT(5X,19I5)
       DO 250 J=1,19
 250   WRITE(6,270) LAT(J),(OCEAN(I,J),I=1,19),LAT(J)
 270   FORMAT('0',I5,19F5.1,I5)
       WRITE(6,290)(LONG1(I),I=1,19)
 290   FORMAT(5X,19I5)
       WRITE(6,400)
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,260)
       WRITE(6,300)
 300   FORMAT('           MAP OF THE OCEAN FUNCTION 180 DEGREES EAST',
      1' LONGITUDE - 360 DEGREES EAST LONGITUDE')
       WRITE(6,310)(LONG2(I),I=1,19)
 310   FORMAT(5X,19I5)
       DO 320 J=1,19
 320   WRITE(6,330) LAT(J),(OCEAN(I,J),I=19,36),OCEAN(1,J),LAT(J)
 330   FORMAT('0',I5,19F5.1,I5)
       WRITE(6,350)(LONG2(I),I=1,19)
 350   FORMAT(5X,19I5)
       WRITE(6,360)
 360   FORMAT('1','              WORLD MAP')
       WRITE(6,370)((MAP(I,J),I=1,36),J=1,19)
 370   FORMAT(1X,36I2)
       WRITE(6,260)
       STOP
       END
```

Appendix D

Derivation of the Equation for the Potential of a Surface Layer

Consider a ~~layer~~ of mass of density ρ_o and variable thickness ξ on the surface of a sphere of radius R. Let the mass point be at (R, θ', λ') and the field point at (r, θ, λ) (figure D.1), separated by a distance ρ:

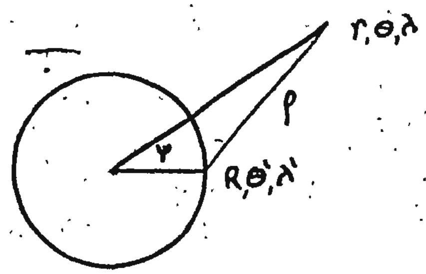

Figure D.1

The potential of the mass in the layer is

$$V(r,\theta,\lambda) = G\int \frac{dm}{\rho}$$

$$= G\rho_o \int_{S'} \frac{\xi(\theta'\lambda')}{\rho}\, dS' \qquad \text{(D.1)}$$

But

$$\rho^{-1} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\psi)\left(\frac{R}{r}\right)^n \qquad \text{(D.2)}$$

and

$$\xi(\theta'\lambda') = \sum_{k\ell} P_k^\ell \left(y_k^\ell \cos\ell\lambda' + z_k^\ell \sin\ell\lambda'\right) \qquad \text{(D.3)}$$

Substituting equations D.2 and D.3 in D.1 we have

$$V = G\rho_0 \frac{R^2}{r} \sum_{kl} \sum_{n} \iint P_k^l (y_k^l \cos l\lambda' + z_k^l \sin l\lambda') P_n(\cos\psi) \cdot (R/r)^n \sin\theta' d\theta' d\lambda' \tag{D.3}$$

By the addition theorem we have

$$P_n(\cos\psi) = P_n(\cos\theta) P_n(\cos\theta') + 2\sum_{m=1}^{n} \frac{n!^2}{(n+m)!(n-m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\lambda-\lambda') \tag{D.4}$$

Substituting equation D.4 in D.3 and using the orthonormality properties of the Legendre functions we have given in Appendix A, we get

$$V(r,\theta,\lambda) = 4\pi G\rho_0 R\left(\frac{R}{r}\right)^{n+1} \times \sum_{nm} P_n^m (y_n^m \cos m\lambda + z_n^m \sin m\lambda)/(2n+1)$$

On the surface of the Earth,

$$V(R,\theta,\lambda) = 4\pi G\rho_0 R \times \sum_{nm} P_n^m(\cos\theta) (y_n^m \cos m\lambda + z_n^m \sin m\lambda)/(2n+1)$$

$$= 4\pi G\rho_0 R \sum_n \Xi_n/(2n+1)$$

$$= 3g\frac{\rho_0}{\rho} \sum_n \Xi_n/(2n+1)$$

where ρ is now the mean density of the Earth.

Appendix E

The Liouville Equations

In a coordinate frame x_i ($i = 1, 2, 3$) rotating with angular velocity $\vec{\omega}$ relative to a frame fixed in space, the equations of motion of a body are

$$L_i = \frac{d}{dT} H_i + \epsilon_{ijk} \omega_j H_k \qquad \text{(E.1)}$$

where

L_i = i' th component of the net external torque on the body about its centre of mass,

H_i = i'th component of the angular momentum of the body about its centre of mass,

ϵ_{ijk} = the Levi-Civita alternating symbol;

and the summation convention over repeated subscripts is used. For convenience take the rotating frame to be fixed to the surface of the solid Earth in some prescribed way. Then it is called a geographic frame.

$$H_i = C_{ij} \omega_j + h_i \qquad \text{(E.2)}$$

where C_{ij} are the components of the inertia tensor in the chosen frame and h_i is the i'th component of the body's angular memehtum relative to that frame. Substituting E.2 in E.1 we have

$$L_i = \frac{d}{d\tau} (C_{ij} \omega_j + h_i) + \epsilon_{ijk} \omega_j (C_{k\ell} \omega_\ell + h_k) \qquad \text{(E.3)}$$

For small departures from a state in which the body is uniaxial and rotates about the axis x_3 with uniform angular speed Ω, equation E.3 can be put in a perturbation form. Define

$$\begin{array}{lll} C_{11} = A + c_{11} & C_{22} = A + c_{22} & C_{33} = C + c_{33} \\ C_{12} = c_{12} & C_{13} = c_{13} & C_{23} = c_{23} \\ \omega_1 = \Omega m_1 & \omega_2 = \Omega m_2 & \omega_3 = \Omega(1+m_3) \end{array} \tag{E.4}$$

c_{ij} are perturbations from the diagonal inertia tensor whose components are A, A, C. The variables m_1, m_2 measure the angular departure of the rotation axis from the axis x_3, and m_3 measures changes in the length of day. Substituting E.4 into E.3 and neglecting products and squares of small quantities we have, finally,

$$\begin{aligned} \frac{\dot{m}_1}{\sigma_r} + m_2 &= \phi_2 \\ \frac{\dot{m}_2}{\sigma_r} - m_1 &= -\phi_1 \\ \dot{m}_3 &= \dot{\phi}_3 \end{aligned} \tag{E.5}$$

where

$$\sigma_r = \frac{(C-A)}{A}\Omega \tag{E.6}$$

and

$$\begin{aligned} \Omega^2(C-A)\phi_1 &= \Omega^2 c_{13} + \Omega \dot{c}_{23} + \Omega h_1 + \dot{h}_2 - L_2 \\ \Omega^2(C-A)\phi_2 &= \Omega^2 c_{23} - \Omega \dot{c}_{13} + \Omega h_2 - \dot{h}_1 - L_1 \\ \Omega^2 C \phi_3 &= -\Omega^2 c_{33} - \Omega h_3 + \Omega\int_0^t L_3\, dt \end{aligned} \tag{E.7}$$

The functions Φ_i are called excitation functions because they involve the various ways in which changes in the body's rotation can be effected - changes in the mass distribution itself, changes in the internal distribution of angular momentum of the body, or external torques. The set E.5 are called the Liouville equations. (Munk and MacDonald, 1960, Chapters 3,6).

Appendix F

The Pole Tide Potential

The centrifugal force experienced by a particle at distance $\vec{d}$ from the axis of rotation, about which the Earth spins with angular speed $\vec{\omega}$, is $\omega^2 \vec{d}$. This is derived from a potential

$$\frac{1}{2}\omega^2 d^2 = \frac{1}{2}|\vec{\omega} \times \vec{r}|^2$$

where

$$\vec{r} = a(\hat{e}_1 \sin\theta\cos\lambda + \hat{e}_2 \sin\theta\sin\lambda + \hat{e}_3 \cos\theta$$

is the distance, from the Earth's centre, of the particle whose longitude is λ and whose colatitude is θ. To the first order in the small quantities m_1, m_2:

$$\vec{\omega} = \Omega(\hat{e}_1 m_1 + \hat{e}_2 m_2 + \hat{e}_3)$$

Therefore the centrifugal force potential reduces to

$$\frac{1}{2}\Omega^2 a^2 [\sin^2\theta - \sin 2\theta (m_1 \cos\lambda + m_2 \sin\lambda)]$$

The part which creates wobble is

$$U_1 = -\frac{2}{3}\Omega^2 a^2 P_2^1(\cos\theta)(m_1\cos\lambda + m_2\sin\lambda)$$