FINITE ELEMENT LINEAR PROGRAMMING APPROACH TO FOUNDATION SHAKEDOWN

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TO FOUNDATION SHAKEDOWN

by

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To My Parents

ABSTRACT

Shakedown load factors are evaluated for certain plane stress and plane strain problems using the kinematic shakedown theorem. The numerical procedure is based on coupling of the finite element method with linear programming.

For foundations subjected to loads varying in time in a nonproportional manner within prescribed limits, the classical limit theorems can give unsafe estimates of the collapse loads, as failure can occur at loads well below the static collapse values. Shakedown theorems, which are generalizations of the limit theorems, can provide appropriate safe bounds for complex loading programmes. A few applications of the static shakedown theorem to plane stress problems are available in the literature, but plane strain applications have not yet been reported.

The nonlinear mathematical formulations for shakedown analysis of continuum problems can be adapted to linear programming by piecewise linearization of the yield surfaces. The continuum is discretised into a finite number of constant strain triangular elements. The elastic-perfectly plastic piecewise linear constitutive law for each element is defined by the yield condition and the associated flow rule. The equilibrium and compatibility conditions are derived from the requirements of the static and the kinematic shakedown theorems respectively. The plastic dissipation energy is minimized subject to compatibility and maximum positive external work conditions.

A computer programme is developed for application of the kinematic shakedown theorem. The software is applied to (1) shakedown analysis of a square plate, with a circular central hole, subjected to biaxial, variable repeated loading for the plane stress condition, and (2) limit analysis of a strip footing subjected to uniformly distributed loading for the plane strain condition. The results are in excellent agreement with available analytical (Case 1) and numerical (Cases 1 and 2) solutions. The code is then applied to the problem of a footing subjected to inclined, eccentric and variable repeated loading for the plane strain case. It is observed that the shakedown load varies almost linearly with the uniaxial compressive strength of the underlying soil for the particular case of tension cut-off.

Although the analysis is restricted to dry soil, the work can be extended to include the effect of cyclic loading on pore pressure. Nonassociative behaviour, work hardening/softening properties, and inertia and damping effects for a prescribed loading history can also be considered. The use of the hybrid finite element model and the sparse matrix technique in linear programming will give better estimates of the shakedown loads with minimum computation time.

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NOTATION

$F_\alpha(t)$: Loading programme

$\Phi(Q_j)$: Yield surface

$\psi(Q_j)$: Limit surface

Q_j : Stress field

q_j : Strain field

q_j^e : Elastic component of strain field

q_j^p : Plastic component of strain field

C_{jk} : Matrix of elastic coefficients

λ : Plastic multiplier

Q_j^E : Fictitious elastic stress response

ρ_j : Residual stress distribution

V : Volume of the body

ξ : Fictitious elastic strain energy

u_α : Generalized displacement field

$\bar{\rho}_j(s)$: Time-independent residual stress field

$\rho_j^o(s)$: Shakedown residual stress field

D : Internal plastic energy dissipation

β : Load factor

$\dot{q}^{P}_{jo}$: Admissible plastic strain rate cycle

$\dot{u}_{\alpha o}$: Kinematically admissible velocity field

Q_{jo} : Stress field associated with $\dot{q}^{P}_{jo}$

$\dot{\rho}_{jo}$: Residual stress rate associated with $\dot{q}^{P}_{jo}$

$\bar{f}_{\alpha}(t)$: Segmental loading programme

F^{+}_{α} &

F^{-}_{α} : Limiting values of variable repeated loading programme

$\{\sigma\}$: Total stress vector

$\{\sigma^{E}\}$: Elastic stress response vector

$\{\sigma^{R}\}$: Residual stress vector

$\{\varepsilon\}$: Total strain vector

$\{\varepsilon^{e}\}$: Elastic component of strain vector

$\{\varepsilon^{P}\}$: Plastic component of strain vector

$\{\varepsilon^{E}\}$: Fictitious elastic strain response vector

$\{\varepsilon^{R}\}$: Residual strain vector

[B] : Strain-displacement matrix

[C] : Matrix of elastic coefficients

Φ_j^i : Piecewise linearized yield surface

$\{N_j^i\}$: Direction cosines of the yield plane j at the check point i

K_{oj}^i : Distance from stress origin to the yield plane j at the check point i

Y^i : Number of yield planes of the yield polyhedron at the check point i

n : Number of elements = number of check points

Y^n : Total number of yield planes of the assembled structure

$\dot{\alpha}$: Plastic multiplier rate

$\{\lambda\}$: Plastic multiplier vector

U : Strain energy density

$\{u\}$: Displacement vector

$\{u^E\}$: Elastic displacement vector associated with $\{\varepsilon^E\}$ and $\{\sigma^E\}$

$\{q\}$: Nodal displacement vector

$\{q^E\}$: Nodal displacement vector associated with $\{u^E\}$

$[T]$: Displacement interpolation function

$\{P_b\}$: Body force vector

$\{P_t\}$: Surface traction vector

$\{F\}$: Overall load vector

[K] : Overall stiffness matrix

M_j^i : Maximum with respect to time of the projection of the $\{\sigma^{Ei}\}$ on the j-th yield plane

k : Load factor

s : Shakedown load factor

[E] : Assembled equilibrium matrix

$[E]^t$: Assembled compatibility matrix

$[S^i]$: Matrix of the coordinates of the vertices of the original yield surface at the check point i

$\{\bar{U}\}$: Vector of unit enteries

d : Number of stress components

$[H^i]$: Matrix of the coordinates of the vertices of the elastic stress polyhedron at the check point i

$\{L^i\}^t$: Vector of elastic factors of the elastic stress polyhedron at the check point i

NDF : Number of degrees of freedom

$\bar{V}_o^i$: Number of vertices of the original yield surface at the check point i

$\bar{V}^i$: Number of vertices of the elastic stress polyhedron at the check point i

$\{F_o\}$: Dead load vector

c : Cohesive strength

ϕ : Angle of internal friction

σ_o : Yield stress in simple tension

f_{cu} : Uniaxial compressive strength

ν : Poission's ratio

CHAPTER I

INTRODUCTION

1.1 General Remarks

The methods of determining the safety of structures from collapse based on the theory of plasticity are characterized by the types of loading. For loads varying according to a prescribed time history, it is possible to follow the development of the displacements and strains by step-by-step integration of the elastic-plastic equations, thus tracing the stress history of the structure up to the collapse. This is the well known incremental elastoplastic analysis, for which available methods are valid under the general hypothesis of material behaviour.

For loads increasing proportionally, with the assumption of elastic-perfectly plastic structures at collapse, the theorems of limit analysis can be adapted to supply the collapse load and the mechanism. Once the loading pattern has been specified, all loads are supposed to increase in proportion until a collapse mechanism forms at a specific load factor, which can be determined either by a trial and error procedure, or by mathematical programming.

For random loads, independently varying in magnitude, sense and direction within prescribed limits, cyclic plastic deformations may be repeated as the load cycle is repeated, which can create two problems. In the first one, the cyclic sensitivity of the material itself has to be determined since the stress-strain curves for the elastic-plastic solids are altered by cyclic straining, which governs the low cycle fatigue life of a structure. The second problem concerns the response of an elastic-plastic structure when the material properties may be considered cycle-independent.

In the second problem it is convenient to distinguish between the two major categories, i.e., either plastic deformation stabilizing after a finite number of cycles, or plastic flow continuing during cycling. In the first major category, the structure is said to shake down to a state of permanent strains and corresponding residual stresses, and responds to any further cycling in a purely elastic manner, provided that the loads remain within the prescribed range; an appropriate shakedown domain can therefore be specified in the load factor space. In the second major category, the plastic deformation does not stabilize, leading to failure of the structure. This type of failure is appropriately called failure by cyclic plastic deformation or inadaptation, and it is convenient here to discern two modes of inadaptation, as the structure in this case becomes unserviceable either due to alternating plasticity or due to incremental collapse.

In the first inadaptation mode, plastic strain develops repeatedly in the opposite sense and in a bounded form in the zone at yield at a certain point in the structure; this eventually results in a low cycle fatigue leading to a local failure of the structure through the appearance of localized fractures. The second inadaptation mode is that of incremental collapse, due to accumulation of plastic strains in the same sense at the end of each cycle, leading to progressive increase of permanent displacements and failure of the global system.

In the incremental collapse mode, it is necessary to mention that, no failure mechanism can be produced due to any single combination of loads, but plastic zones can be developed adjoined by elastic zones in the structure, such that for each combination of loads, there will be a corresponding development of plastic zones. Thus, by continuous cycling, it may happen that plastic zones forming in sequence and in the same sense would constitute a collapse mechanism as if they all occurred simultaneously, and permanent displacements, during successive cycles of loading, may increase beyond all the bounds, making the structure unserviceable. Such a mode of inadaptation includes ratchetting. A classification of problems of elastoplastic analysis is given in Table 1.

The behaviour of the structures under slowly varying repeated loadings is of considerable interest. These loads are prescribed only at their range of variation, neither their sequence, nor their frequencies being prescribed. The material properties are assumed cycle-insensitive, and the effect of change in the configuration on the equilibrium equations is neglected by considering that the hypothesis of the geometrically linear theory holds. The loading assumed to be sufficiently slow to enable the neglect of inertia forces; the elastic response of the structure involves no dynamic effects. The shakedown analysis includes both the adaptation of a structure to the prescribed loading range and the two modes of inadaptation.

Two fundamental theorems for the shakedown of an ideally elastic-plastic structure, formulated by Melan and Koiter, known as the static and the kinematic shakedown theorems respectively, can bound the values of the repeatedly varying loading factors from below and above. These shakedown theorems are very difficult to understand with respect to the underlying definitions and basic assumptions. Many investigators have worked on proofs and generalizations, but there is a lack of numerical illustration mainly for continuum problems and particularly for geotechnical applications.

1.2 Statement of the Problem

In this thesis, Koiter's theorem is used to evaluate the load factor for a hypothetical foundation subjected to variable repeated loading under dry conditions using a finite element and linear programming approach based on piecewise linearization of the Mohr-Coulomb yield criterion in plane strain.

1.3 Layout

Chapter II reviews the literature on the shakedown phenomenon, theorems, proofs, assumptions, extensions, generalizations, and applications in the areas of structures and geotechnical engineering.

Chapter III presents in indicial notation the concepts, assumptions, definitions, and the two theorems of Melan and Koiter.

Chapter IV describes a finite element linear programming approach; the theorems are applied in a discretised form by using a compatible constant strain triangular element and the revised simplex algorithm.

Chapter V illustrates numerical examples to check the code developed with available solutions for plane stress shakedown analysis and plane strain limit analysis. Numerical examples are investigated regarding the shakedown of a footing in plane strain condition with different soil properties.

Chapter VI presents the concluding remarks, and lists the engineering applications of the method and areas for further research.

Appendies A, B and C illustrate a linear programming problem, linearized yield surfaces and a flow chart for the computer code respectively.

CHAPTER II

LITERATURE REVIEW

2.1 Introduction

The topics reviewed are the shakedown theorems with modifications and extensions, and their applications in the fields of structural analysis and geotechnical engineering.

2.2 Shakedown Theorems and Their Applications in Structural Analysis

The static shakedown theorem was first stated by Bleich (8), who presented a limited proof. A general proof for trusses was given by Melan (56, 57) and later by Symonds and Prager (71). Neal (62) has given a similar proof for ideal beam sections, and also for real sections (61). Proofs of the static shakedown theorem for continuous media have been given by Melan (58), Symonds (72), and Koiter (37). A special proof based on the adaptation of the Symonds proof for the particular case of one-dimensional stress distribution, such as that which exists in beams and frames under bending, was given by Hodge (30) and Horne (31).

The kinematic shakedown theorem was stated and proved for continuous media by Koiter (37, 38).

Detailed and clear descriptions of the definitions, basic assumptions, and proofs of the shakedown theorems have been given by Martin (55).

The largest number of applications of the shakedown theorems are in the field of framed structures; detailed numerical illustrations can be found in the books of Horne (31), Hodge (30), and Martin (55) who treated the problem as a linear programming one. In these applications, the solution was based on the combined mechanism procedure, formulated by Neal and Symonds (63), by considering the effect of bending moment only and neglecting the effect of the axial force. An application to a circular shaft subjected to combined axial force and twisting moment, treated as a continuous problem, and using the Von Mises yield criterion was given by Symonds (72).

The effect of combined axial force and bending moment for the case of beams and plates was considered by Grundy (26). König (40) developed an approximate method for the shakedown analysis of frames and arches, considering the combined action of the axial force and the bending moment, based on a linear programming approach and the 'elastic locus concept': "a domain in the space of the generalized stresses, within which the response at any point of the section remains perfectly elastic". Chon et al (12, 13) developed a method and a code for the design and analysis of the shakedown problems of frame by using any of the static or the kinematic formulations and ulilizing the standard simplex algorithm. Eyre and Galambos (19) carried out theoretical and experimental studies on steel bars and beams to investigate their shakedown performances in bridge systems. Deflection analysis for shakedown of beams was carried out by Eyre and Galambos (20), by considering that the yield zones are spread along the length of the beam of a strain hardening material. Eyre and Galambos (21) have given an analytical solution for the shakedown of a beam on elastic support.

The first application of the shakedown theorems for complex structures was carried out by Lecki and Penny (47, 48) by evaluating the lower bound estimates of shakedown factors for pressure and thrust and moment loadings applied through a radial flush or protruding nozzle in a spherical pressure vessel. Melan's theorem was used with available elastic solutions and standard linear programming techniques. König (39) developed the shakedown theory of plates based on Melan's theorem using the elastic locus concept; shakedown load factors were evaluated for simply supported and clamped plates by using the Tresca yield criterion. A complete survey for the shakedown theorems, including the accommodation of a structure to the prescribed loading range as well as the inadaptation and unserviceability was given by Sawczuk (69). The definitions and the basic assumptions of the shakedown theorems were illustrated by numerous applications for struts, frames, plates and cylinderical shells. Gokhfeld and Cherniarsky (25) transformed the fundamental shakedown theorems to separate a time-dependent analysis of stress and strain from the solution of the fundamental problem. The analysis of incremental collapse was reduced to the limit analysis problem for a nonhomogeneous body. This method, utilizing generalized variables, is based mainly on the concept of the fictitious yield surface: "surface in the nine-dimensional stress space bounding the region of admissible, time-independent distribution of residual stresses". It permits one to employ the classical procedures of limit analysis for shakedown analysis of plates and shells, and provides an exact and approximate method for solving the shakedown problems of cylindrical shells and circular plates.

Maier (49) used the matrix description of the mechanical behaviour based on finite element discretization and piecewise linearization of yield surfaces, in contradistinction to the traditional continuous field description. A linear programming concept was used to formulate a general matrix shakedown theory and the basis for relevant solution procedures. Melan's theorem was presented as a primal linear programming problem, and Koiter's theorem as its dual one. Koiter's theorem was extended to allow for variable dislocations (e.g. temperature cycles) for systems with associated flow laws. For systems with nonassociated flow laws, two theorems were formulated for lower and upper bounds to the shakedown load factor. Ceradini (9) extended Melan's theorem to the dynamic range by considering the effect of the inertia forces and viscous damping. Maier (50) extended Ceradini's theorem to systems which obey a fairly general hardening rule. The basis was established for shakedown theorem of hardening structures, similar to Melan's theorem, previously mentioned in Ref. 49, for perfectly plastic structures and quasi-static loading. However, lack of normality was not dealt with for the work hardening materials. By the duality relationship between the static and the kinematic shakedown theorems, Koiter's theorem was extended to allow for work hardening.

Maier (51, 52) also extended the theorems of Melan and Koiter to allow for second order geometric effects, i.e., in situations where the change in the configuration affects the equilibrium equations.

Melan's theorem was generalized to allow for work softening in the dynamic and the quasi-static ranges. Corradi and Maier (14) extended Koiter's theorem to the dynamic range, i.e., to establish sufficient and necessary conditions for inadaptation of structures under given external action history, in the presence of significant inertia and viscous damping effects, taking into account the imposed strains and or displacements (dislocations) among the external actions, as in Ref. 49 for the quasi-static case.

The first application using Maier's generalization for shakedown theorems was given by Vitiello (74); finite element and linear programming approaches were used to analyze the shakedown of a statically indeterminate beam of an elastic-perfectly plastic material with associated flow laws, subjected to variable repeated loads. Also, a method of calculation was presented for the upper bound value of the local plastic deformation in a prescribed zone of the structure at the shakedown state. König (41, 42) presented a method of bounding the shakedown deflections by using finite element and linear programming.

The first numerical solution for shakedown analysis in continuum problems was given by Belytschko (7) who formulated the problem as a nonlinear programming one by using the Von Mises yield criterion. Airy's stress function was used to express the stress field in the finite element formulation of Melan's theorem, and the method applied to analyze the shakedown of a square plate with a circular hole of an elastic-perfectly plastic material subjected to in-plane variable repeated loading under plane stress condition. This method was used successfully in limit analysis by Belytschko and Hodge (6). Corradi and Zavelani (15) employed Maier's formulation to establish a numerical tool for determining the load domain multipliers for two and three-dimensional structures subjected to variable repeated loads which can give rise to shakedown. A new procedure was proposed to reduce the number of constraints in Maier's approach to Melan's theorem while retaining the linear nature of the problem. This procedure was used successfully before in limit analysis by Zavelani (78), and is based mainly on expressing the yield condition by means of the coordinates of the vertices of the yield polyhedron instead of the direction cosines and the plastic capacity of each yield plane. This procedure was applied in Ref. 15 to analyze the shakedown of the same plane stress problem solved earlier in Ref. 7, but with a compatible finite element model and a linear programming approach by piecewise linearization of the Von Mises yield criterion. The results agreed to some extent with Belytschko's solution, Ref. 7, but the main advantages of this approach were the reduction in computation time, and the avoidance of computational difficulties associated with the nonlinear programming technique.

Prager (66), formulated a shakedown theorem for structures whose materials are idealized as rigid-plastic kinematic hardening. Ref. 66 developed a condition of adaptation for trusses similar to Melan's theorem for elastic-perfectly plastic structures. A generalization of Prager's theorem to a broader class of hardening rules and an inadaptation theorem similar to Koiter's theorem for elastic-perfectly plastic structures were given by Polizzoto

(65). König and Maier (43), extended Prager's theorem to the general case of discrete structures, similar to Melan's theorem for elastic-perfectly plastic structures, and for a broader class of hardening rules including temperature and second order geometric effects. Two methods for bounding the maximum shakedown deflection were presented.

Hung and König (34) presented a finite element nonlinear programming formulation for the shakedown analysis; based on the static approach and the 'yield criterion of the mean'. This later assumption is based on Hencky's interpretation of the Von Mises yield criterion, i.e., plastic flow occurs when the distortion energy density reaches the ultimate value. This approximate procedure reduces the number of discrete nonlinear yield constraints in Belytschko's formulation, Ref. 7, by averaging the yield function over each element. For a fine mesh, the error in this procedure becomes very small as it was successfully used in limit analysis by Hung (33). The application of this formulation in shakedown analysis was presented recently by Hung and Palgen (35). The method was applied to solve the same plane stress problem solved before by Refs. 7 and 15. The results agreed with those of Ref. 7.

2.3 Application of Shakedown Theorems in Geotechnical Engineering

Compared to structural mechanics, there is very little work on shakedown in soil mechanics. The first application in this field was by Rowe (67), who carried out experimental model studies of circular surface foundations, resting on saturated clays and subjected to inclined, eccentric static and cyclic loads to model an offshore gravity platform. Shakedown was defined as follows: "On raising the wave force, the structure settles or shakes down until equilibrium is reached". The 'amount of shakedown' was considered a function of the size of the wave force and the number of cycles. The load factor was defined as the value corresponding to 'equilibrium shakedown' of the vertical displacements of the foundation. By measuring the pore pressure it was concluded that, its equilibrium is associated with the 'equilibrium of shakedown'. Rowe (68) used the loading conditions of Ref. 67 to check the general failure solutions with modified Brinch-Hansen equations, for the special case of constant vertical stress. It was observed that when the ratio between the eccentricity and the diameter is $\leq$ 1/8 'the structure could only 'fail' by shakedown due to error in c_u, especially if λ is over-estimated', λ being the ratio between the reduction factors of the shear strength associated with bearing and shear stresses.

Zienkiewicz et al (82) studied the effect of wave action on the foundations of offshore structures. It was demonstrated that even such a relatively slow cyclic loading may lead to 'progressive deformation' at loading levels much smaller than those given by the static collapse. A finite element discretisation was used to model the coupled behaviour of the soil skeleton and the pore pressure based on a visco-plastic algorithm, but the 'true creep' phenomena were excluded and attention focused only on instant plasticity. The cyclic nature of the loads, necessitated a plastic model for absolute prediction of maximum collapse loads and reasonably accurate estimates of deformations under various loading conditions. These models were classified according to the nature of the flow rule, i.e., associated or non-associated, and strain dependent or perfectly plastic models. A comparison between different models was carried out under undrained and drained conditions for hypothetical foundations; the results showed that for drained conditions the collapse loads do not depend on the nature of the flow rule, but for undrained conditions, the associative behaviour fails to give any collapse. It was concluded that both ideal plastic, non-associated and critical state models are acceptable for the study of the foundation. The 'shakedown' or 'ratcheting' phenomenon was defined such that "at loads well below static collapse values, deformation under cyclic loads can increase without limit". Contrary to the accepted definition of shakedown in the literature (for eg. Ref. 69), it was used to describe ratcheting or 'lack of shakedown' (Zienkiewicz et al (83)). This phenomenon was illustrated by examples of a footing under plane strain condition, subjected to a vertical steady loading and a cyclic bending moment due to horizontal loads. For loads and moments lower than the static collapse loads, it was shown that after a sufficient number of moments, depending on the value of the applied moments, the displacements either stablize, i.e., shakedown occurs or continues to increase, i.e., ratcheting occurs, see Figs. 1.1 and 1.2. These examples have been repeated by the same authors (84), with different parameters, to illustrate some of the factors affecting the ratcheting phenomena. The values of moments and loads were as any likely to occur in the North Sea using the critical state/ Mohr-Coulomb model under undrained conditions. The results showed that the magnitude of settlement increased either due to increase of the vertical load, or decrease of the strain hardening parameter or the angle of internal friction.

Continuing the investigations for the ratcheting phenomena, Zienkiewicz et al (85), concluded that the computation of cyclic situations is tedious and time consuming requiring a series of non-linear solutions. They attempted to describe the practical use of the 'bounding theorems' noting that such theorems have limited applicability in non-associated or strain dependent plasticity. But, as mentioned before, these limitations were overcome by Maier. Goldscheider (24) interpreted the definitions and the basic assumptions of the shakedown theorems for dry sand bodies. It was found that, the response of a dry sand body, when subjected to cyclic stressing in the safe domain, becomes more and more elastic, without reaching the perfect elasticity for a finite number of cycles, as the total strain increases proportionally with the logarithm of the number of stress cycles, Fig. 1.3. It was concluded that Melan's theorem cannot be applied for dry sand bodies due to the insufficient statical condition restricting the residual stress fields associated with the shakedown condition. However, the second statement of Koiter's theorem can be applied for dry sand bodies without any restrictions as it is based on the existence of an elastic solution for cyclic load; Koiter's first statement, based on Drucker's postulate, is meaningless for dry sand bodies.

CHAPTER III

SHAKEDOWN ANALYSIS - STATE OF THE ART

3.1 Introduction

The definition of shakedown, incremental collapse and alternating plasticity phenomenona are presented. The two fundamental theorems of Melan and Koiter are discussed and restated as bounding theorems. The discretization concept is applied to Melan's theorem.

3.2 Cyclic Response

The response of an elastic-perfectly plastic body, subjected to cyclic loading will be discussed in terms of generalized loads, $F_\alpha(t)$, with the implication that displacements on S_u are prescribed to be zero, Fig. 3.1. A cyclic loading programme with period, T, must be such that

$$F_\alpha(t) = F_\alpha(t + T) \qquad (3.1)$$

for all time t.
However, the loads are applied sufficiently slowly that inertia effects are negligible, Ref. 38.

The material is assumed to be elastic-perfectly plastic with associated flow rule. Thus, the initial yield surface, $\Phi(Q_j)$, and the limit surface, $\psi(Q_j)$, are identical. The constitutive equations are consequently given by Ref. 55:

$$\dot{q}_j = \dot{q}_j^e + \dot{q}_j^p ,$$

$$\dot{q}_j^e = C_{jk} \dot{Q}_k ,$$

$$\dot{q}_j^p = \lambda \frac{\partial \Phi}{\partial Q_j} \quad \text{for } \Phi(Q_j)=0 \text{ and } \frac{\partial \Phi}{\partial Q_j} \dot{Q}_j = 0 ,$$

$$\dot{q}_j^p = 0 \quad \text{for } \Phi(Q_j) < 0 , \tag{3.2}$$

$$\text{or } \Phi(Q_j) = 0 \text{ and } \frac{\partial \Phi}{\partial Q_j} \dot{Q}_j < 0$$

where $\dot{q}_j^e$, $\dot{q}_j^p$ are the elastic and plastic components of strain rate, $\Phi(Q_j)$ is the yield function (plastic potential), Q_j is the stress field such that $\Phi(Q_j) > 0$ is not permitted, C_{jk} is the matrix of elastic coefficients, and λ is non-negative plastic multiplier.

The stress distribution can be expressed as the sum of the elastic solution (fictitious stress, assuming that the material has unlimited elasticity, i.e., implying purely elastic response to the applied loads) and a residual stress distribution, Ref. 72:

$$Q_j(s,t) = Q_j^E(s,t) + \rho_j(s,t) \tag{3.3}$$

It is noted that $Q_j^E(s,t)$ is a cyclic function since it depends on the instantaneous value of the external loads. Thus

$$Q_j^E(s,t) = Q_j^E(s,t + T) \tag{3.4}$$

Further, because of the cyclic nature of the loads, the stress distributions, $Q_j(s,t)$ and $Q_j(s,t + T)$, are in equilibrium with the same external loads, and the stress field $\{ Q_j(s,t) - Q_j (s,t + T)\}$ is self-equilibrating or in equilibrium with zero external loads. Therefore, by the principle of virtual velocities, Ref. 55:

$$0 = \int_V \{ Q_j(t) - Q_j(t + T)\} \{ \dot{q}_j(t) - \dot{q}_j(t + T)\} \, dV \tag{3.5}$$

where $\dot{q}_j(t)$ are the total strain rates. Dividing the total strain into elastic and plastic parts gives

$$0 = \int_V \{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^e(t) - \dot{q}_j^e(t + T)\} \, dV$$

$$+ \int_V \{Q_j(t) - Q_j(t + T)\}\{\dot{q}_j^p(t) - \dot{q}_j^p(t + T)\} \, dV \qquad (3.6)$$

By making use of the relation between the stress and the elastic component of the strain rate, Eqn. 3.2, it follows that

$$\int_V \{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^e(t) - \dot{q}_j^e(t + T)\} =$$

$$\int_V C_{jk} \{ Q_j(t) - Q_j(t + T)\}\{ \dot{Q}_k(t) - \dot{Q}_k(t + T)\} \, dV =$$

$$\frac{d}{dt} \int_V C_{jk} \{ Q_j(t) - Q_j(t + T)\}\{ Q_k(t) - Q_k(t + T)\} \, dV$$

$$(3.7)$$

Since $\Phi(Q_j(t)) \leq 0$ and $\Phi(Q_j(t + T)) \leq 0$, using Drucker's postulate, (the fundamental inequality on which convexity of the yield and limit surfaces and the normality rule depend), Ref. 38, gives:

$$(Q_j(t) - Q_j^{(a)}) \, \dot{q}_j^p(t) \geq 0, \qquad (3.8a)$$

and

$$(Q_j(t + T) - Q_j^{(a)}) \, \dot{q}_j^p(t + T) \geq 0 \qquad (3.8b)$$

where $Q_j^{(a)}$ are the allowable states of stresses corresponding to the yield and limit surfaces, i.e., the boundary of the elastic domain, increments of the plastic strains can only occur on this surface. Adding Eqs. 3.8a and 3.8b gives

$$\{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^P(t) - \dot{q}_j^P(t + T)\} \geq 0 \qquad (3.9)$$

Substituting Eqs. 3.7 and 3.9 into Eqn. 3.6 gives

$$0 = \frac{d}{dt} \int_V C_{jk} \{ Q_j(t) - Q_j(t + T)\}\{ Q_k(t) - Q_k(t + T)\} dV$$

$$+ \int_V \{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^P(t) - \dot{q}_j^P(t + T)\} dV \qquad (3.10)$$

which finally yields

$$\frac{d}{dt} \int_V C_{jk} \{ Q_j(t) - Q_j(t + T)\}\{ Q_k(t) - Q_k(t + T)\} dV$$

$$= -\int_V \{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^P(t) - \dot{q}_j^P(t + T)\} dV \leq 0 \qquad (3.11)$$

The matrix of elastic coefficients, C_{jk}, is positive definite. Consequently the expression

$$\xi(Q_j(t), Q_j(t + T)) = \int_V C_{jk} \{ Q_j(t) - Q_j(t + T)\}\{ Q_k(t) - Q_k(t + T)\} dV \qquad (3.12)$$

is a quadratic form. It is non-negative, and zero only when $Q_j(s,t) = Q_j(s,t + T)$. The function $\xi(Q_j(t), Q_j(t + T))$ may be thought of as a scalar quanitity which measures twice the elastic strain energy function for the difference between the stress fields $Q_j(t)$ and $Q_j(t + T)$, Refs. 37 and 38. Eqn. 3.11 shows that

$$\frac{d\xi}{dt} \leq 0 \qquad (3.13)$$

i.e., that the difference between the stress field can only decrease with time. From Eqs. 3.3 and 3.4,it follows that

$$Q_j(t) - Q_j(t + T) = \{ Q_j^E(t) - Q_j^E(t + T)\} + \{ \rho_j(t) - \rho_j(t + T)\}$$

$$= \rho_j(t) - \rho_j(t + T) \qquad (3.14)$$

and

$$\xi(Q_j(t), Q_j(t + T)) = \xi(\rho_j(t), \rho_j(t + T)) \qquad (3.15)$$

As the body responds to cyclic loading, it can be expected that in general ξ will decrease (not necessarily continously) whenever

$$\frac{d\xi}{dt} \neq 0 \qquad (3.16)$$

i.e., $\{ Q_j(t) - Q_j(t + T)\}\{ \dot{q}_j^P(t) - \dot{q}_j^P(t + T)\} \neq 0$ (3.17)

at any point in the body. Since ξ cannot become negative, and hence cannot continue to decrease in every cycle of loads, there will be a time, denoted say by the passage of n cycles, such that for $t > nT$

$$\{ Q_j(s,t) - Q_j(s,t + T)\}\{ \dot{q}_j^P(s,t) - \dot{q}_j^P(s,t + T)\} = 0 \qquad (3.18)$$

Thus for $t > nT$, $\dot{\xi} = 0$ and ξ becomes constant. There are three non-trivial situations in which Eqn. 3.18 can be satisfied for any point in the body. The first is that

$$Q_j(s,t) = Q_j(s,t + T) \qquad \text{for} \qquad t > nT \qquad (3.19)$$

everywhere in the body. Thus, the stress field also becomes cyclic.

The second case is not of interest but it must be pointed out, the vectors $\{ Q_j(s,t) - Q_j(s,t + T)\}$ and $\{ \dot{q}_j^P(s,t) - \dot{q}_j^P(s,t + T)\}$ are orthogonal, so that the scalar product is zero. Since the quantity given in Eqn. 3.18 is the sum of two non-negative terms (Eqs. 3.8a and 3.8b), this case can occur only when $Q_j(s,t)$ and $Q_j(s,t + T)$ both lie on the same flat region of the yield surface, i.e., $\dot{q}_j^P(s,t)$ and $\dot{q}_j^P(s,t + T)$ both have the same direction in stress space. This case is shown in Fig. 3.2, and also includes the possibility of $\dot{q}_j^P(s,t) = \dot{q}_j^P(s,t + T) \neq 0$.

The third case in which Eqn. 3.18 may be satisfied at a point if

$$\dot{q}_j^P(s,t) = \dot{q}_j^P(s,t + T) = 0 \qquad \text{for} \qquad t > nT \tag{3.20}$$

This condition implies that the body behaves elastically for $t > nT$, and $q_j^P(s,t)$ becomes constant for $t > nT$. Under these circumstances, the stress field will also be cyclic, since $\rho_j(s,t)$ is uniquely determined by $q_j^P(s,t)$, Ref. 38. If $q_j^P(s,t)$ is constant for $t > nT$, it follows that $\rho_j(s,t)$ is also constant for $t > nT$, and

$$Q_j(s,t) = Q_j^E(s,t) + \rho_j(s,t) \tag{3.21}$$

is cyclic because $Q_j^E(s,t)$ is cyclic.

It can be seen from Eqn. 3.12 that $\xi = 0$ implies that the stress field becomes cyclic, and the first and third cases discussed above are sufficient conditions for $\xi = 0$ for $t > nT$. (It does not appear at present that necessary conditions can be stated for ξ to eventually become zero, as the second case includes a possibility that ξ can reach a steady non-zero value. On the other hand, it can be expected that most structures will exhibit a cyclic response eventually, with $\xi = 0$, and that the situations in which ξ reaches a constant non-zero value and the response is never cyclic are special cases, Ref. 55.)

This discussion will be continued on the assumption that the stress field in a body subjected to cyclic loading will become cyclic after a sufficiently large number of cycles. As the range of interest is narrowed, it will be seen that no less generality is implied in this assumption.

3.3 Cyclic Plastic Deformations

The response $Q_j(s,t)$ of the body for $t > nT$ when it becomes cyclic and $\xi = 0$ can be conveniently divided into two major categories.

First, suppose that

$$Q_j(s,t) = Q_j(s,t + T)$$

but

$$\dot{q}_j^p(s,t) \neq 0 \qquad \text{for} \qquad t > nT \tag{3.22}$$

for all points in the body, considering the behaviour of the body in terms of the response to one cycle of loading such that $nT \leq t \leq (n + 1)T$. The cyclic nature of the response is made evident in the observation that $\rho_j(s,nT) = \rho_j(s,(n + 1)T)$. The elastic strain rates are given by

$$\dot{q}^e(s,t) = C_{jk}\,\dot{Q}_k^E(s,t) + C_{jk}\,\dot{\rho}_k(s,t) \tag{3.23}$$

The change in the elastic strains over a cycle is zero, since

$$\Delta q_j^e(s) = q_j^e(s,(n+1)T) - q_j^e(s,nT)$$

$$= \int_{nT}^{(n+1)T} C_{jk} \dot{Q}^E(s,t)dt + \int_{nT}^{(n+1)T} C_{jk} \dot{\rho}_k(s,t)dt = 0 \quad (3.24)$$

However, the change in the total strains,

$$\Delta q_j(s) = q_j(s,(n+1)T) - q_j(s,nT) \quad (3.25)$$

must be kinematically admissible since it represents the difference between two kinematically admissible fields (i.e., it may be derivable from a velocity field satisfy the boundary conditions). As a consequence, the change in the plastic strains over a cycle defined by,

$$\Delta q_j^P = \int_{nT}^{(n+1)T} \dot{q}_j^P(s,t)dt \quad (3.26)$$

must be kinematically admissible when the body reaches a cyclic state. The changes in the plastic strains can then be associated with changes in the generalized displacements, Δu_α. The changes, $\Delta q_j^P(s)$ and Δu_α, will recur in each cycle when the response is periodic.

It is convenient to further subdivide the response into two classes:

3.3.1 Alternating Plasticity

For the first class, consider

$$\Delta q_j^P(s) = 0 \quad (3.27)$$

From Eqs. 3.24 and 3.25, $q_j^P(s,t) = q_j^P(s,t+T)$, i.e., the plastic strain $q_j^P(s,t)$ and hence the generalized displacements $u_\alpha(t)$ are cyclic. From Eqn. 3.26, the plastic strain rates, $\dot{q}_j^P(s,t)$ (which are not zero in this class) and hence the displacement rates, $\dot{u}_\alpha(t)$, are cyclic. This condition is termed as one of "alternating plastic deformation", since it is most commonly met when points in the body undergo plastic strains which change sign in the cycle. Alternating plastic deformation can be expected to lead to failure of the body through the appearance of "localized fractures", or local failure due to low cycle fatigue, Refs. 38 and 55.

3.3.2 Incremental Collapse

The second class is that when $\Delta q_j^P(s) \neq 0$. In this case the body undergoes increments in displacement, Δu_α, over each cycle. The work done by the external loads in each cycle is positive, for

$$\int_{nT}^{(n+1)T} F_\alpha \dot{u}_\alpha dt = \int_{nT}^{(n+1)T} dt \int_V Q_j \dot{q}_j^e dV + \int_{nT}^{(n+1)T} dt \int_V Q_j \dot{q}_j^P dV$$

$$= \int_{nT}^{(n+1)T} dt \int_V Q_j \dot{q}^P dV > 0 \qquad (3.28)$$

The elastic term vanishes because elastic stresses are cyclic, and $Q\dot{q}_j^P > 0$. Thus, the body will undergo increasing deformation in each cycle. If the cycle loading continues indefinitely, the body may fail, or may no longer be able to perform its function as a result of the unbounded deformations. This type of behaviour has been termed incremental collapse, Ref. 55.

3.3.3 Shakedown

The second major category is that where the plastic strain rates are identically zero after the body attains a cyclic state, i.e.,

$$\dot{q}_j^p(s,t) = 0 \qquad \text{for} \qquad t > nT \qquad (3.29)$$

In this case, the behaviour is elastic for $t > nT$, and the generalized displacements, $u_\alpha(t)$, are cyclic. The residual stress field, $\rho_j(s,t)$, is uniquely determined by the plastic strains, $q_j^p(s,t)$, which have taken place in earlier cycles, but since $\dot{q}_j^p(s,t) = 0$, $\dot{\rho}_j(s,t) = 0$, the residual stresses become constant with time. The stresses are given by

$$Q_j(s,t) = Q_j^E(s,t) + \rho_j^o(s) \qquad (3.30)$$

This type of behaviour is termed shakedown, and $\rho_j^o(s)$ is the shakedown residual stress field.

Finally, for a body subjected to cyclic loading in which shakedown does not occur, represents a situation in which failure is almost certain. The mechanism of failure may be that of alternating plasticity, incremental collapse, or a combination of these mechanisms represented by the cases in which the response of the body does not become strictly cyclic. Whether or not shakedown will eventually occur, in a body subjected to cyclic loading, depends entirely on whether or not a yield surface (for elastic, perfectly plastic bodies) can contain the load cycle. If this does occur, the behaviour thereafter will be elastic.

If not, failure by cyclic plastic deformation will occur, Ref. 55.

3.4 Shakedown Theorems

"The shakedown theorems are in fact generalizations of the limit theorems in the sense that they provide static and kinematic approaches to the question of whether or not shakedown will occur for a given body under a given cyclic loading, whereas the limit theorems provide static and kinematic approaches to the question of whether or not flow will occur in a given body under given loads. Although they are not as easy to apply in any given situation as the limit theorems, they do permit a consideration of the problem of shakedown in terms which are independent of history. Thus, like the limit theorems, they provide a partial solution of the problem (which nevertheless contains important information) at a cost in computational time and effort which is significantly less than that required to complete a full analysis" - Martin (55).

3.4.1 The Static Shakedown Theorem (Melan's Theorem)

Consider any time-independent distribution of residual stresses $\bar{\rho}_j(s)$, such that the sum of these residual stresses and the elastic solution, $Q_j^E(s,t)$, is a "safe stress state," i.e., a state of stress 'inside' the yield surface. At every point in the body and for all possible combinations of loads within the prescribed bounds, the body will shake down to 'some' time-independent distribution of residual stresses (usually depending on the actual loading program), and the response to the subsequent load variations within the prescribed limit will be elastic. This means that, the body will shake down if, Refs.

38 and 55,

$$\Phi(Q_j^E(s,t) + \bar{\rho}_j(s)) < 0 \qquad (3.31)$$

As before, $Q_j^E(s,t)$ is the elastic solution to the given body for the given cyclic loads. If shakedown does occur for any initial residual stress distribution, $\rho_j(s,o)$, the shakedown residual stress distribution will, of course, become time-independent after a sufficiently large number of cycles. If the shakedown residual stress field is denoted by $\rho_j^o(s)$, Eqn. 3.30, then the definition, Ref. 55:

$$\Phi(Q_j^E(s,t) + \rho_j^o) \leq 0 \qquad (3.32)$$

The stress field, $\rho_j^o(s)$, in Eqn. 3.32 is not necessarily identical to $\bar{\rho}_j$ in the statement of Melan's theorem and in Eqn. 3.31. The theorem states that if any $\bar{\rho}_j(s)$ can be found satisfying Eqn. 3.31, shakedown will occur with some shakedown residual stress field $\rho_j^o(s)$.

An alternative statement of the theorem can consequently be given as follows: If shakedown occurs under any initial residual stress field (leading to a shakedown residual stress field $\bar{\rho}_j$), it will occur for all admissible initial residual stress fields. From the point of view of the designer, this is an extremely important result; a correct assessment of the initial conditions, therefore, is not necessary in determining whether or not shakedown will occur, Ref. 55.

The converse of Melan's theorem can be stated as follows: Shakedown will not be possible, if no time-independent distribution of residual stresses can be found with the property that the sum of residual stresses and the elastic solution is an "allowable state of stress", i.e., a state of stress 'on' the yield surface at every point of the body for all possible load combinations. This means that the body will not shake down, if, Ref. 38,

$$\Phi(Q_j^E(s,t) + \bar{\rho}_j) \leq 0 \tag{3.33}$$

This statement is self-evident, since by definition shakedown requires that a time-independent residual stress field, $\rho_j^o(s)$, be established in the body, such that $\Phi(Q_j^E + \rho_j^o) < 0$ as a consequence of the requirement that $\dot{q}_j^P(s,t) = 0$ for $t > nT$. If no such residual stress fields can be found, clearly shakedown can never occur, Ref. 55.

3.4.2 <u>The Kinematic Shakedown Theorem (Koiter's Theorem)</u>

The body will not shakedown, i.e., it will fail ultimately by cyclic plastic deformation, if any admissible plastic strain rate cycle, $\dot{q}_{jo}^P(s,t)$ for $0 < t < T$, and any generalized load combination, $F_\alpha(t)$, within a prescribed limit can be found for which, Refs. 37 and 38,

$$\int_0^T F_\alpha(t)\cdot\dot{u}_{\alpha o}(t)\,dt > \int_0^T dt \int_V D(\dot{q}_{jo}^P)dV \tag{3.34}$$

i.e., if $W_{ext} > W_{int}$ where W_{int} is the plastic energy dissipation rate function.

On the other hand, the body will shake down, if any number $k > 1$ can be found,with the property for all admissible plastic strain rate cycles, $\dot{q}_{jo}^P(s,t)$, and all generalized load combinations, $F_\alpha(t)$, within the prescribed limits, Refs. 37 and 38, given by

$$k\left(\int_0^T F_\alpha(t)\cdot\dot{u}_{\alpha o}(t)\,dt\right) \leq \int_0^T dt \int_V D(\dot{q}^p_{jo})\,dV \qquad (3.35)$$

The upper bound of such a number, k, is then obviously the factor of safety with respect to shakedown.

In Eqs. 3.34 and 3.35, the admissible plastic strain rate cycle is thus characterized by plastic strain rates defined over one cycle of loading; it is further limited by the requirement that, Refs. 37 and 38,

$$\Delta q^p_{jo}(s) = \int_0^T \dot{q}^p_{jo}(s,t)\,dt \qquad (3.36)$$

should constitute a kinematically admissible strain distribution. It is, therefore, required that Δq^p_{jo} can be derived from an incremental displacement field, $\Delta u_{\alpha o}$, satisfying the displacement boundary conditions. The energy dissipated in plastic work, associated with this admissible plastic strain rate cycle, is

$$W_{int} = \int_0^T dt \int_V Q_{jo}(s,t)\,\dot{q}^p_{jo}(s,t)\,dV = \int_0^T dt \int_V D(\dot{q}^p_{jo})\,dV \qquad (3.37)$$

The field $Q_{jo}(s,t)$ is the stress associated with $\dot{q}^p_{jo}(s,t)$ through the plastic part of the constitutive relation. In fact, Q_{jo}, is not uniquely determined if the yield surface has flat regions or if $\dot{q}^p_{jo} = 0$, Ref. 55, however, $D(\dot{q}^p_{jo}) = Q_{jo}\,\dot{q}^p_{jo}$ is always uniquely determined in elastic, perfectly plastic materials, Ref. 38.

In addition, the admissible plastic strain rate cycle is associated with a kinematically admissible velocity field, $\dot{u}_{\alpha o}(t)$, $0 \leq t \leq T$, Ref. 55. This can be illustrated by considering a body subjected to zero external loads and plastic strains, $\dot{q}^{p}_{jo}(s,t)$, and assuming that it behaves elastically. At each instant, the solution of this elastic problem will yield a residual stress rate distribution, $\dot{\rho}_{jo}(s,t)$, which is in equilibrium with zero loads. The condition that the elastic strain rates

$$\dot{q}^{e}_{jo} = C_{jk}\,\dot{\rho}_{ko} \tag{3.38}$$

which together with the imposed admissible plastic strain rate cycle, $\dot{q}^{p}_{jo}$, will provide total strain rates,

$$\dot{q}_{jo} = \dot{q}^{e}_{j} + \dot{q}^{p}_{jo} \tag{3.39}$$

that constitute a kinematically admissible field, will be sufficient to determine $\rho_{jo}(s,t)$ uniquely, Ref. 38. The total strain rates can thus be derived from a velocity field, $\dot{u}_{\alpha o}$, which satisfies the displacement boundary conditions.

It is noted from Eqn. 3.36 that the change in ρ_{jo} over a cycle will be zero,

$$\rho_{jo}(s,t) - \rho_{jo}(s,o) = \int_{0}^{T} \dot{\rho}_{jo}(s,t)\,dt = 0 \tag{3.40}$$

Consequently, the change in the elastic strain field over a cycle will also be zero,

$$\Delta q^{e}_{jo} = \int_{0}^{T} \dot{q}^{e}_{jo}(s,t)\,dt = \int_{0}^{T} C_{jk}\,\dot{\rho}_{ko}(s,t)\,dt = 0 \tag{3.41}$$

Also, the incremental displacement field, $\Delta u_{\alpha o}$, and the velocity field are related as follows,

$$\Delta u_{\alpha o} = \int_{0}^{T} \dot{u}_{\alpha o}(t)\,dt \tag{3.42}$$

The displacement rate, $\dot{u}_{\alpha o}$, can be used to define an expression which gives the external work done over a cycle by the loads, Ref. 55, i.e.,

$$W_{o\ ext} = \int_0^T F_\alpha(t)\ \dot{u}_{\alpha o}(t)\ dt \tag{3.43}$$

3.5 Simplified Statements of Shakedown Theorems and the Nonlinear Programming Approach

Let $f_\alpha(t)$ be given set of cycles of loads, and let the body be subjected to loads

$$F_\alpha(t) = \beta f_\alpha(t) \tag{3.44}$$

where β is a non-negative constant scalar multiplier. When β is very small, the body will behave elastically. As β increases until it reaches a certain value β_s, the body will shake down. If $\beta > \beta_s$, the body will fail by cyclic plastic deformations. β cannot increase idefinitely, since loads must remain within the limit surface $\psi(\beta f(t))$ in load space. The limiting value β_L may be defined, Ref. 55, such that

$$\psi(\beta_L f_\alpha(t)) = 0 \qquad \text{for} \qquad 0 < t < T,$$

and

$$\psi(\beta_L f_\alpha(t)) < 0 \ \ \text{otherwise} \tag{3.45}$$

Then, the shakedown limit, β_s, will be

$$\beta_s \leq \beta_L \tag{3.46}$$

If the body shakes down for a multiplier, β_1, it will eventually shake down for a multiplier, $\beta_2 < \beta_1$, provided that the yield surface is convex and contains the origin, Ref. 55. To approach such a conclusion, consider a body subjected to loads, $f_\alpha(t)$, with an elastic solution, $Q_j^{oE}(s,t)$. Then, due to the linearity of the elastic solution, if the loads become $\beta f_\alpha(t)$, the stress field will be $\beta Q_j^{oE}(s,t)$. Now from Melan's theorem, if shakedown occurs for a multiplier β_1, there will be a shakedown residual stress field, $\bar{\rho}_j(s)$, such that

$$\Phi(\beta_1 Q_j^{oE} + \bar{\rho}_j) < 0 \qquad (3.47)$$

i.e., the vector $(\beta_1 Q_j^{oE} + \bar{\rho}_j)$ in Fig. 3.3 must not exceed the yield limit at any point in the body during the cycle. Then, it will be obvious from Fig. 3.3 that, the vector

$$(\beta_2/\beta_1)(\beta_1 Q_j^{oE} + \bar{\rho}_j) = \beta_2 Q_j^{oE} + (\beta_2/\beta_1)\bar{\rho}_j \qquad (3.48)$$

does not exceed the yield limit. Shakedown can occur in this case, because due to Melan's theorem there is a time-independent distribution of residual stress $(\beta_2/\beta_1)\bar{\rho}_j$.

The static and the kinematic theorems can be restated such that the bounds of β_s can be determined. Melan's theorem can be restated as follows, Ref. 55: β_s is the largest multiplier, β, for which a time independent residual stress field, $\bar{\rho}_j$, can be found such that

$$\Phi(\beta Q_j^{oE} + \rho_j) \leq 0 \qquad (3.49)$$

at all points in the body for $0 \leq t \leq T$.

The 'less than inequality' in Eqn. 3.31 is replaced by 'less than or equal' statement, because if β is reduced by an infintesimal amount, $\delta\beta$, there will be a residual stress field, $(\beta - \delta\beta)/\beta\ \rho_j$, for which the less than inequality is satisfied (see Fig. 3.3). It is implied that shakedown will occur for all $\beta < \beta_s$ without involving the case $\beta = \beta_s$.

From a programming point of view, the shakedown load factor, β_s, according to Melan's theorem can be determined by solving the following programming problem, Ref. 7:

$$\text{Maximize } \beta$$

$$\text{subject to } \Phi(\beta\, Q_j^{oE} + \bar{\rho}_j) \le 0 \quad \text{in V for } 0 \le t \le T \qquad (3.50)$$

The determination of β_s is thus a nonlinear programming problem (NLP).

In Koiter's theorem, the shakedown load factor, β_s, can be obtained by equating external and internal work for any admissible plastic strain rate cycle, $\dot{q}^p_{jo}(s,t)$, i.e., β can be obtained as

$$\beta = \frac{\int_0^T dt \int_V D(\dot{q}^p_{jo})\, dv}{\int_0^T f_\alpha(t)\, \dot{u}_o(t)\, dt} \qquad (3.51)$$

If β is increased by an infinitesimal amount, $\delta\beta$, shakedown will not occur for $\beta + \delta\beta$, since

$$(\beta + \delta\beta) \int_0^T f_\alpha(t)\, \dot{u}_o(t)\, dt > W_{o\ int} \qquad (3.52)$$

It is obvious then that Koiter's theorem can be restated as follows, Ref. 55:

"β_s is the smallest multiplier, β, obtained by equating external and internal work, i.e., from Eqn. 3.51 for any admissible plastic strain rate cycle, $\dot{q}^{p}_{jo}(s,t)$. Thus, any value of β obtained from equating $W_{o\ ext}$ and $W_{o\ int}$ in Koiter's theorem is an upper bound on β_s. This also implies that shakedown will not occur for any $\beta > \beta_s$ without involving the case, $\beta = \beta_s$".

In the next chapter, Koiter's theorem will be formulated as a minimization problem.

3.6 Discrete Formulation of the Shakedown Theorems

This section is an introduction to the next chapter in which the problem will be discussed in more detail. Now the attention will be restricted to the discussion of the discretization concept applied to the static theorem.

In order to adapt the formulation for numerical computation, the constraints given in Eqn. 3.50 must be replaced in discrete form. Since the yield condition is applied to quanities which vary in both space and time, the discretization must make it possible to consider only a finite number of instants in the loading programme and a finite number of points in the body.

3.6.1 Discretization of the Loading Programme

This discretization can be achieved by replacing the loading programme, $f_\alpha(t)$, by $\bar{f}_\alpha(t)$ which is piecewise linear in the load space and circumscribes $f_\alpha(t)$. This can be shown from the example in Fig. 3.4 in which $\alpha = 2$.

Let the corners of the locus of the loading programme, $\bar{f}_\alpha(t)$, in the load space be denoted by $f_{\alpha}^{(k)}$, $(k = 1, 2, \ldots m)$, and let the elastic solution for the loading programme $f_{\alpha}^{(k)}$ be denoted by $Q_j^{oE(k)}(s)$. The programming problem 3.50 will be as follows:

$$\text{Maximize } \beta$$
$$\text{subject to} \quad \Phi(\beta\, Q_j^{oE(k)}(s) + \bar{\rho}_j(s)) \leq 0 \quad \text{in } V \qquad (3.53)$$
$$k = 1, 2, \ldots m$$

3.6.2 Discretization of the Stress Fields

To compute $Q_j^{oE(k)}(s)$ in continuum problems, it is necessary to discretize the stress field by using finite difference or finite element methods. In addition a finite number of 'check points' in the body are required to ensure that the yield function is satisfied everywhere in the body.

3.6.3 Piecewise Linearization of the Yield Function

Eqn. 3.53 requires that the yield condition must not be violated at the chosen check points in the discretized body. The yield surface, defining the elastic—perfectly plastic behaviour of the material, is a nonlinear function of the stress components prevailing at these points. Thus, the corresponding statement of the shakedown analysis is a nonlinear programming (NLP) problem. This approach was used in solving shakedown problems, Ref. 7, and limit problems, Ref. 6. But solution algorithms for the NLP problem (in contradistinction to those for the LP problem) are not computationally sufficient to permit analysis of most practical sized engineering problems. A reasonable and practical means to overcome these difficulties is to use a piecewise linear yield surface, Refs. 1, 2, 3, 4, 15, 54, 75 and 78, in which the yield surface will be a polyhedron in the stress space, limited by suitable number of yield planes depending on the accuracy desired for a specified problem. Therefore, the condition that the yield criterion must not be violated necessitates the satisfying of a certain number of linear inequalities; the shakedown load factor can be obtained by means of standard linear programming routines.

3.7 Variable Repeated Loads

Variable repeated loads are defined as loads assumed to vary indefinitely in time according to a law which is not known except for the variation intervals of each component, Ref. 53, i.e., they are repetitive in nature and vary within prescribed limits; neither the sequence of loading nor its frequency are prescribed, Ref. 69. When the body is subjected to this type of loading, the concept of shakedown will not be changed. Shakedown is said to occur if the response of the body becomes elastic.

In fact, the variable repeated loading programme may be considered as a special case of the discretized cyclic loading programme with respect to the time parameter shown in Fig. 3.4. Fig. 3.5 illustrates a variable repeated loading programme for two generalized loads, $F_1(t)$ and $F_2(t)$, such that

$$F_1^- \le F_1(t) \le F_1^+$$

and (3.54)

$$F_2^- \le F_2(t) \le F_2^+$$

The loading programme in this case will be a rectangle in the load space with the sides, $F_1(t) = \bar{F}_1$, $F_1(t) = \overset{+}{F}_1$, $F_2(t) = \overset{+}{F}_2$ and $F_2(t) = \bar{F}_2$. The elastic solution has to be obtained for each corner of the loading programme, i.e., it is required to determine, $Q_j^E(\bar{F}_1, \bar{F}_2)$, $Q_j^E(\bar{F}_1, \overset{+}{F}_2)$, $Q_j^E(\overset{+}{F}_1, \overset{+}{F}_2)$, and $Q_j^E(\overset{+}{F}_1, \bar{F}_2)$. $Q_j^E(t)$ in Melan's statement has to be replaced by $Q_j^E(F_\alpha)$, i.e., a linear function in the generalized loading. Melan's theorem can be restated in this situation as follows, Ref. 55: Shakedown will occur if it is possible to find any time independent residual stress field, $\bar{\rho}_j(s)$, such that

$$\Phi\,(Q_j^E(F_\alpha) + \bar{\rho}_j) < 0 \qquad (3.55)$$

for all combinations of F_α.

CHAPTER IV

FINITE ELEMENT LINEAR PROGRAMMING APPROACH TO THE SHAKEDOWN ANALYSIS

4.1 Introduction

The static shakedown theorem is restated as a linear programming problem by using the finite element discretization. The dual of this problem is shown to represent the kinematic shakedown theorem. The effect of dead loading is studied. Finally, the relation between the limit and shakedown theorems is presented.

4.2 The Finite Element Model

The suggested method requires discretizing the continuum into constant strain triangle finite elements, in which the displacements are assumed to vary linearly over an element. In elasto-plastic analysis, this particular choice avoids complications (due to partially yielded elements) in defining the yield surface in the stress space, Ref. 49. In this displacement model, displacement compatibility across the boundary between two adjacent elements is satisfied, but local equilibrium conditions will be violated and equilibrium conditions satisfied in the overall sense only. The stresses are constant within the element, and defined at the element centroids. The yield conditions will be checked only at a finite number of points (check points); hence only one check point is required for each element coinciding with its centroid.

Shakedown analysis requires an elastic analysis of the structure to be performed for any individual loading condition. In this report, the same element is used for the elastic phase, and for the subsequent shakedown phase.

4.3 Constitutive Relations

Let,

$\{\sigma\}$: the total stress vector,

$\{\sigma^E\}$: the elastic stress response vector, assuming the material has unlimited elastic behaviour under the applied loads,

$\{\sigma^R\}$: the residual stress vector,

$\{\varepsilon\}$: the total strain vector,

$\{\dot{\varepsilon}\}$: the strain rate vector,

$\{\varepsilon^e\}$: the elastic component of the strain vector,

$\{\varepsilon^P\}$: the plastic component of the strain vector,

$\{\varepsilon^E\}$: the elastic strain response vector, by assuming the material has unlimited elastic behaviour under the applied loads,

and

$\{\varepsilon^R\}$: the residual strain vector.

These vectors are connected by the following relations, Refs. 53 and 49:

$$\{\sigma\} = \{\sigma^E\} + \{\sigma^R\},$$
$$\{\varepsilon\} = \{\varepsilon^e\} + \{\varepsilon^P\},$$
$$\{\varepsilon^e\} = \{\varepsilon^E\} + \{\varepsilon^R\}, \qquad (4.1)$$

and

$$\{\varepsilon\} = \{\varepsilon^E\} + \{\varepsilon^R\} + \{\varepsilon^P\}$$

In addition, Eqn. 3.2 represents the relation between the total stresses and the elastic and plastic strain rates.

The chosen finite element model requires an interpolation function, [T], to connect the displacement field, {u}, with the nodal displacements {q}, Ref. 17,

$$\{u\} = [T]\{q\} \tag{4.2}$$

The deformations are assumed to be within the framework of the first order geometric theory, i.e., the linearized strain tensor is

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4.3}$$

i.e., in matrix form

$$\{\varepsilon\} = [B]\{q\} \tag{4.4}$$

The elastic stress response is related to the elastic strain response by

$$\{\sigma^E\} = [C]\{\varepsilon^E\} \tag{4.5}$$

where [C] is the matrix of elastic coefficients.

4.4 Linearization of the Yield Conditions

The stress state within an elastic-perfectly plastic body must satisfy the yield conditions at every point, Ref. 4, i.e.,

$$\Phi(\sigma) - K_o \leq 0 \tag{4.6}$$

where both the positive constant K_o, and the function $\Phi(\sigma)$ are material dependent. The yield condition, 4.6, will be checked only at a finite number of points 'check points' within the body.

For a constant strain triangle, one check point is required for each element. Let σ^i represent the stress state within the element i, then the yield condition 4.6 will be

$$\Phi^i(\sigma^i) - K_o^i \leq 0 \qquad (i = 1, \ldots n) \qquad (4.7)$$

The equation

$$\Phi^i(\sigma^i) - K_o^i = 0 \qquad (4.8)$$

represents, in the stress space, the yield surface at the check point i, Fig. 4.1. From what has been mentioned in Chapter III, the use of linear programming algorithms requires each of the nonlinear inequalities, 4.7, to be approximated by a set of linear inequalities, Ref. 49, as follows:

$$\Phi_j^i = \{N_j^i\}^t \{\sigma^i\} - K_{oj}^i \leq 0 \qquad (j = 1, \ldots Y^i) \qquad (4.9)$$

The inequality, 4.9, represents the elastic polyhedron of element i, in which, each of its Y^i yield planes is identified with a unit normal vector, $\{N_j^i\}$, and a positive scalar, K_{oj}^i, which can be determined from simple geometrical considerations or by solving a linear programming problem, Refs. 2 and 3. Appendix B illustrates some linearized yield functions for the plane stress and plane strain cases.

4.5 Internal Plastic Energy Dissipation Rate

From Eqn. 3.2, the stress, σ, and the plastic strain rate, $\dot{\varepsilon}^p$, for an associated flow rule are related as follows:

$$\dot{\varepsilon}^p = \dot{\alpha} \frac{\partial \Phi}{\partial \sigma} ,$$

$$\dot{\alpha} \geq 0 \quad \text{if} \quad \Phi(\sigma) - K_o = 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial \sigma} \dot{\sigma} = 0 ,$$

$$\dot{\alpha} = 0 \quad \text{if} \quad \Phi(\sigma) - K_o < 0 , \tag{4.10}$$

$$\text{or} \quad \Phi(\sigma) - K_o = 0, \quad \text{and} \quad \frac{\partial \Phi}{\partial \sigma} \dot{\sigma} < 0$$

In the region of the elastic-perfectly plastic body with volume, V^i, (i = 1, . . . n), where the yield surface is approximated by the Y^i yield planes, Eqn. 4.10 becomes, Refs. 4 and 50,

$$\{\dot{\varepsilon}^{pi}\} = \sum_{J=1}^{Y^i} \{N_j^i\} \dot{\alpha}_j^i = [N^i]\{\dot{\alpha}^i\} ,$$

$$\dot{\alpha}_j^i \geq 0 \quad \text{if} \quad \{N_j^i\}^t\{\sigma^i\} - K_{oj}^i = 0 \quad \text{and} \quad \{N_j^i\}^t\{\dot{\sigma}^i\} = 0 ,$$

$$\dot{\alpha}_j^i = 0 \quad \text{if} \quad \{N_j^i\}^t\{\sigma^i\} - K_{oj}^i < 0 \tag{4.11}$$

$$\text{or} \quad \{N_j^i\}^t\{\sigma^i\} - K_{oj}^i = 0 \quad \text{and} \quad \{N_j^i\}^t\{\dot{\sigma}^i\} < 0$$

Eqn. 4.11 states that the plastic strain rate vector $\{\dot{\varepsilon}^{pi}\}$ has to be a linear combination (with non-negative but otherwise arbitrary multipliers $\dot{\alpha}_j^i$) of the vectors, $\{N_j^i\}$, normal to the planes of the i-th yield polyhedron, which are reached by the stress vector, $\{\sigma^i\}$.

From Eqn. 3.37, the plastic energy dissipation rate within V^i (total volume $V = \sum_{i=1}^{n} V^i$) is given by

$$\dot{D}^i = \int_{V^i} \{\dot{\varepsilon}^{pi}\}\{\sigma^i\}\, dV \tag{4.12}$$

Using Eqn. 4.10,

$$\dot{D}^i = \sum_{j=1}^{Y^i} \int_{V^i} \dot{\alpha}_j^i \{N_j^i\}^t \{\sigma^i\} \, dV \quad , \tag{4.13}$$

$$\dot{D}^i = \sum_{j=1}^{Y^i} K_{oj}^i \int_{V^i} \dot{\alpha}_j^i \, dV = \sum_{j=1}^{Y^i} K_{oj}^i \dot{\lambda}_j^i \quad , \tag{4.14}$$

where the non-negative $\dot{\lambda}_j^i$ variables are defined by

$$\dot{\lambda}_j^i = \int_{V^i} \dot{\alpha}_j^i \, dV \tag{4.15}$$

The plastic energy dissipation rate for the assembled structure is given by

$$\dot{D} = \sum_{i}^{n} \dot{D}^i = \sum_{i=1}^{n} \sum_{j=1}^{Y^i} K_{oj}^i \dot{\lambda}_j^i = \{K_o\}^t \{\dot{\lambda}\} \tag{4.16}$$

By defining the plastic multiplier (affected by the integration 4.15 at each check point) as, Refs. 15 and 49,

$$\lambda = \int_0^T \dot{\lambda} \, dt \tag{4.17}$$

then, the internal plastic energy dissipated during one cycle is

$$W_{int} = \int_0^T \dot{D} \, dt = \{K_o\}^t \{\lambda\} \tag{4.18}$$

4.6 Finite Element Equations of the Elastic Phase

This fictitious elastic response will be obtained by assuming the unlimited elastic behaviour of the material under the applied loads, Ref. 72. The finite element formulation for elastic analysis is presented here only for the sake of completeness. For studying the element and the assemblage structure characteristics, the principle of minimum potential energy is used. The potential energy for a linear elastic body can be expressed as the sum of the internal work and the potential of the body forces and surface tractions, Ref. 17, i.e.,

$$\Pi = \int_V dU - \int_V (p_{bx} \cdot u + p_{by} \cdot v)\, dV - \int_{st} (p_{tx} \cdot u + p_{ty} \cdot v)\, dst \qquad (4.19)$$

where, st, is the surface of the body on which the surface tractions are prescribed, and V is the volume of the body.

The strain energy density for a linear elastic body is defined as, Ref. 17,

$$dU = \frac{1}{2} \{\varepsilon^E\}^t \{\sigma^E\}\, dV = \frac{1}{2} \{\varepsilon^E\}^t [C] \{\varepsilon^E\}\, dV \; , \qquad (4.20)$$

where, by virtue of Eqs. 4.2 and 4.4

$$\{\varepsilon^E\} = [B] \{q^E\} \qquad (4.21)$$

and

$$\{u^E\} = [T] \{q^E\} \qquad (4.22)$$

Substituting Eqs. 4.20, 4.21 and 4.22 into Eqn. 4.19, the functional in terms of the displacement models for one element i, can be written as,

$$\Pi = \frac{1}{2} \int_{V^i} (\{q^{Ei}\}^t [B^i]^t [C^i] [B^i] \{q^{Ei}\} - 2 \{q^{Ei}\}^t [T^i] \{p_b^i\})\, dV - \int_{st^i} \{q^{Ei}\}^t [T^i] \{p_t^i\}\, dst \qquad (4.23)$$

Applying the variational principle, Ref. 17, gives

$$\{\delta q^{Ei}\}^t \left(\int_{V^i} [B^i]^t [C^i] [B^i] \, dV \{q^{Ei}\} - \int_{V^i} [T^i]^t \{p_b^i\} \, dV - \int_{st^i} [T^i]^t \{p_t^i\} \, dst \right) = 0 \tag{4.24}$$

Since $\{\delta q^{Ei}\}$ are arbitrary, the expression in the parentheses must vanish. This gives the equilibrium equations for the element i, as

$$[K^i] \{q^{Ei}\} = \{F^i\} \tag{4.25}$$

where the element stiffness matrix is given by

$$[K^i] = \int_{V^i} [B^i]^t [C^i] [B^i] \, dV \tag{4.26}$$

and the element load vector is given by

$$\{F^i\} = \int_{V^i} [T^i]^t \{p_b^i\} \, dV + \int_{st^i} [T^i]^t \{p_t^i\} \, dst \tag{4.27}$$

By the same variational procedure, the principle of minimum potential energy can be applied to the structure to obtain the assembled equilibrium equations, Ref. 17, as

$$[K] \{q^E\} = \{F\} \tag{4.28}$$

where

$$[K] = \sum_{i=1}^{n} [K^i] \tag{4.29}$$

and

$$\{F\} = \sum_{i=1}^{n} \{F^i\} \tag{4.30}$$

4.7 Equilibrium and Compatibility Requirements For Shakedown Analysis

When the elastic limit of the material is exceeded, the minimization of the total potential energy procedure will no longer be applicable to the formulation of the equilibrium equations; in this case the virtual work principle, Ref. 79, has to be used.

Melan's theorem is based on the concept of a statically admissible stress field, i.e., satisfaction of the equations of equilibrium in the interior of the body and the boundary conditions on the surface. The conditions of equilibrium can be studied by applying virtual velocity, $\delta\dot{u}$, at any instant during the loading cycle. The principle of virtual displacements (velocities) states that the stresses, $\{\sigma\}$, and the loads, $\{p_b\}$, and $\{p_t\}$ are in equilibrium, if for any arbitrary virtual displacement (velocity) field, $\delta\dot{u}$, the following variational equation is satisfied, Refs. 3, 4, 76 and 79:

$$\int_V \{\delta\dot{\varepsilon}\}^t \{\sigma\}\, dV - \int_V \{\delta\dot{u}\}^t \{p_b\}\, dV - \int_{st} \{\delta\dot{u}\}^t \{p_t\}\, dst = 0 \qquad (4.31)$$

The first integral represents the rate of the internal work done by the stress, $\{\sigma\}$, and the second and third integrals represent the rate of the work done by the external loads. Substituting from Eqs. 4.2 and 4.4 into Eqn. 4.31 gives

$$\{\delta\dot{q}\}^t \left(\int_V [B]^t \{\sigma\}\, dV - \int_V [T]^t \{p_b\}\, dV - \int_{st} [T]^t \{p_t\}\, ds \right) = 0 \qquad (4.32)$$

since $\{\delta\dot{q}\}$ are arbitrary, the expression in the parentheses must vanish. This gives the equilibrium equation, Ref. 79,

$$\int_V [B]^t \{\sigma\}\, dV = \{F\} \qquad (4.33)$$

Noting that the [B] does not depend on the coordinates, and the stresses, $\{\sigma\}$, are constant within the element, then the property of definite integrals, requiring that the total to be the sum of the parts, Ref. 79, gives:

$$\int_V (\ )\, dV = \sum_{i=1}^{n} \int_{V^i} (\ )\, dV^i \tag{4.34}$$

Eqn. 4.33 can now be written as, Refs. 15 and 78,

$$[E]^t \{\sigma\} = \{F\} \tag{4.35}$$

where the global equilibrium matrix $[E]^t$ is given by

$$[E]^t = \sum_{i=1}^{n} [B^i]^t V^i = \sum_{i=1}^{n} [B^i]^t A^i h^i \tag{4.36}$$

in which A^i and h^i are the area and the thickness of the element, i. If no loads are applied to the structure, only self-equilibrated residual stresses are present, and equilibrium is ensured when, Ref. 15,

$$[E]^t \{\sigma^R\} = \{0\} \tag{4.37}$$

It must be noted that if some nodal points are constrained, the effective degrees of freedom, NDF, will be less than the number of equations, NEQ; therefore, only NDF equations will be provided by Eqs. 4.35 and 4.37, and those corresponding to constrained displacements have to be eliminated, Refs. 3 and 15.

Koiter's theorem is based on the concept of an admissible plastic strain rate cycle, $\dot{\varepsilon}^p(t)$, which is characterized by the property that the integrals, Refs. 37 and 38,

$$\Delta\varepsilon^P = \int_0^T \dot{\varepsilon}^P(t)\, dt \tag{4.38}$$

for some time interval, T, should constitute a kinematically admissible strain field. Hence the strains, $\Delta\varepsilon^P$, are obtained from a displacement field, Δu, by means of Eqn. 4.3, where the displacements, Δu, vanish on S_u, in which Δu are defined as, Refs. 37 and 38,

$$\{\Delta u\} = \int_0^T \{\dot{u}(t)\}\, dt \tag{4.39}$$

In an element i, equating the external work Eqn. 3.34 and the internal energy dissipated during one cycle T, Eqn. 4.12 gives

$$\int_0^T \{F^i\}^t \cdot \{\dot{q}^i\}\, dt = \int_0^T \left(\int_{V^i} \{\sigma^i\}^t \{\dot{\varepsilon}^{Pi}\}\, dV \right) dt \tag{4.40}$$

Substituting from Eqn. 4.35 into Eqn. 4.40, yields

$$\int_0^T \{\sigma^i\}^t [E^i] \{\dot{q}^i\}\, dt = \int_0^T \left(\int_V \{\sigma^i\}^t \{\dot{\varepsilon}^{Pi}\}\, dV^i \right) dt \tag{4.41}$$

The stress states are constant within the element, and by virtue of Eqs. 4.11, 4.15 and 4.35 it follows that

$$[E^i] \{\Delta q^i\} = \int_0^T [N^i] \{\dot{\lambda}^i\}\, dt \tag{4.42}$$

By remembering the definition of the plastic multiplier from Eqn. 4.17, and by assembling for all the elements,

$$[E] \{\Delta q\} = [N] \{\lambda\} . \tag{4.43}$$

[E] is known as the compatibility matrix of the structure, Refs. 15 and 78, and $[N] \equiv \text{diag}\ [[N^1] \ . \ . \ . \ [N]^n]$ indicates the matrix with submatrices $[N^i]$ along its main diagonal, taking into account all the yield planes of each element, $[N^i] = [N_1^i \ . \ . \ . \ N_{Y^i}^i]$, Ref. 49.

4.8 Direct Formulation of Shakedown Analysis

4.8.1 The Primal Programme, Melan's Theorem

Melan's theorem, Chapter III, basically states that, an elastic-perfectly plastic structure when subjected to loads, $\{F(t)\}$, varying within a prescribed limit with unknown history, will shake down under these loads if a time-independent residual stress, $\sigma^R(x)$, can be found such that

$$\Phi(\sigma^E(x,t) + \sigma^R(x)) \leq 0 \qquad (4.44)$$

for every x and t, $\Phi(\sigma) \leq 0$ being the yield condition, and $\sigma^E(x,t)$ the linear elastic response of the structure to F(t).

For all yield planes and all elements, the maximum, with respect to time, of the projection of the elastic response vector $\{\sigma^E(t)\}$ on the normal vector $\{N_j^i\}$, Refs. 49 and 15, is defined by

$$M_j^i = \max_t \left(\{N_j^i\}^t \ \{\sigma^{Ei}(t)\}\right) \qquad (4.45)$$

Then the elastic stress response of the element, i, will be included for every time, t, in the polyhedron defined by the Y^i inequalities, Ref. 15,

$$[N^i]^t \ \{\sigma^{Ei}(t)\} \leq \{M^i\} , \qquad (4.46)$$

where

$$\{M^i\}^t \equiv \{M_1^i \ . \ . \ . \ M_{Y^i}^i\}^t \qquad (4.47)$$

Assembling in one vector, $\{M\}$, all these maximum values gives

$$\{M\}^t = \{\{M_1^1\}^t \; . \; . \; . \; \{M_{Y^n}^n\}^t\} \tag{4.48}$$

where Y^n is the total number of yield planes in the assembled structure. Correspondingly, the vector, $\{K_o\}$, of all the constants, K_{oj}^i, is introduced, and the vector of the instantaneous values of all yield functions, Eqn. 4.44, is written as, Ref. 49,

$$\{\Phi(t)\} = k \; [N]^t \; \{\sigma(t)\} - \{K_o\} \tag{4.49}$$

where k is a common positive multiplier for all the interval limits which define the loading programme kF(t). It is usual to specify the safety factor or shakedown factor as a value s, such that for any $k \leq s$, the structure shakes down; for $k > s$ it does not. Sufficient, Eqn. 3.31, and necessary, Eqn. 3.33, shakedown conditions for the loading programme $k\{F(t)\}$, can be expressed respectively, as follows :

$$k\{M\} + [N]^t \; \{\bar{\sigma}^R\} - \{K_o\} < \{0\} \tag{4.50}$$

and

$$k\{M\} + [N]^t \; \{\sigma^R\} - \{K_o\} \leq \{0\} \tag{4.51}$$

for some residual stress distribution, $\{\sigma^R\}$, Ref. 49, such that $\{\sigma^R\}$ is defined by Eqn. 4.37, Ref. 15.

A factor k is called statically admissible, if condition 4.51 is satisfied for some $\{\sigma^R\}$; then Melan's theorem will provide s as the maximum of all statically admissible multipliers, k, Ref. 49. Then, the search for s finally reduces to maximizing a linear objective function of the real variables, k and σ^R, subjected to linear constraints for these variables, Eqs. 4.37 and 4.51, i.e., the solution of a linear programming problem. Then, s, can be obtained as the optimal value of the following linear programming problem, Refs. 49 and 15:

$$s = \max_{k,\sigma^R} k \quad ,$$

subject to

$$\text{a)} \quad [E]^t \{\sigma^R\} = \{0\} \ ,$$

$$\text{b)} \quad k\{M\} + [N]^t \{\sigma^R\} \leq \{K_o\} \ ,$$

and

$$\text{c)} \quad k \geq 0 \qquad (4.52)$$

which can be written in the form of a 'tableau' as follows:

$$\begin{array}{r|c|c|c|l}
 & k & \{\sigma^R\}^t & & \\
\hline
s = \max & 1 & \{0\}^t & & \\
\hline
 & \{0\} & [E]^t & \{0\} & \text{NDF} \\
\hline
 & \{M\} & [N]^t & \{K_o\} & Y^n \\
\hline
 & 1 & dn & 1 &
\end{array} \qquad (4.53)$$

In this tableau, the dimension of the real variable vector, $\{k,\{\sigma^R\}^t\}$, is $1 + dn$, where d is the number of stress components (3 for plane stress and plane strain), and the number of constraints is $NDF + Y^n$. In order to write the programme, 4.53, in the standard form, the unrestricted variables, σ^R, can be written in terms of non-negative variables, Refs. 49, 55, 22, 46 and 64,

$$\sigma^{R} = \sigma^{R+} - \sigma^{R-} \tag{4.54}$$

with

$$\sigma^{R+} \geq 0 \quad \text{and} \quad \sigma^{R-} \geq 0 \tag{4.55}$$

Then the tableau in this case will be

	k	$\{\sigma^{R+}\}^t$	$\{\sigma^{R-}\}^t$		
$s = \max$	1	$\{0\}^t$	$\{0\}^t$		
	$\{0\}$	$[E]^t$	$-[E]^t$	$\{0\}$	NDF
	$\{M\}$	$[N]^t$	$-[N]^t$	$\{K_o\}$	Y^n
	1	dn	dn	1	

(4.56)

In this tableau, the dimension of the real variable vector becomes 1 + 2dn, but it has the same number of constraints as 4.53. In addition, each equality constraint, 4.52a, needs an artificial variable and each inequality constraint, 4.52b, needs a slack variable (see Appendix A).

4.8.2 The Dual Programme, Koiter's Theorem

In linear programming theory, any maximization (minimization) problem can be associated with a minimization (maximization) problem having the same vectors and matrices and connected with the former by some duality properties (see Appendix A). By the dualization, the dual programme can be obtained from the primal, 4.52, Ref. 15, and will be

$$s = \min_{\Delta q, \lambda} \{K_o\}^t \{\lambda\} \quad ,$$

subject to

a) $[E]\{\Delta q\} - [N]\{\lambda\} = \{0\}$,

b) $\{M\}^t\{\lambda\} = 1$,

and

c) $\{\lambda\} \geq \{0\}$ (4.57)

By discussing the mechanical interpretation of this programme, it will be proved that it represents the kinematic shakedown theorem. If the variables, $\{\Delta q\}$, is that defined by Eqs. 4.2 and 4.39 and the variables, $\{\lambda\}$, are those defined by Eqn. 4.17, then the equality constraints, 4.57a, represent the concept of the admissible plastic strain rate cycle defined before by Eqn. 4.43; the objective function, $\{K_o\}^t \{\lambda\}$, will be the plastic energy dissipated during the cycle defined before by Eqn. 4.18. Through the virtual work principle, Eqn. 4.31, the external work during one cycle can be expressed as, Refs. 49 and 55,

$$W_{ext} = \int_0^T \{F\}^t \{\dot{q}\}\, dt = \int_0^T \int_V (\{\sigma^E\}^t \{\dot{\varepsilon}^P\}\, dV)\, dt \qquad (4.58)$$

Subsituting from Eqs. 4.11, 4.15 and 4.46 into Eqn. 4.58 gives

$$W_{ext} = \int_0^T \{\sigma^E\}^t [N]\{\dot{\lambda}\}\, dt \leq \int_0^T \{M\}^t \{\dot{\lambda}\}\, dt = \{M\}^t \{\lambda\} \qquad (4.59)$$

Then, by considering the constraint, 4.57b, and the objective function of 4.57, the equation

$$\{K_o\}^t \{\lambda\} = k\{M\}^t \{\lambda\} \qquad (4.60)$$

for a load factor, k, represents an energy balance. Then, by stating that a kinematically admissible factor, k, satisfying the energy balance, Eqn. 4.60, due to an arbitrary plastic multiplier, λ, which defines the concept of an admissible plastic strain rate cycle through Eqn. 4.43, the dual programme, 4.57, will lead to Koiter's theorem. The safety factor, s, is the minimum of all kinematically admissible load factors, k.

In order to write the programme for 4.57 in the standard form, the unrestricted variables Δq, can be written in terms of non-negative variables, Refs. 46, 41 and 55, as

$$\Delta q = \Delta q^{+} - \Delta q^{-} \tag{4.61}$$

with

$$\Delta q^{+} \geq 0 \quad \text{and} \quad \Delta q^{-} \geq 0 \tag{4.62}$$

The corresponding tableau will be

	$\{\Delta q^{+}\}$	$\{\Delta q^{-}\}$	$\{\lambda\}$		
s = min	$\{0\}^t$	$\{0\}^t$	$\{K_o\}^t$		
	$[E]$	$-[E]$	$-[N]$	$\{0\}$	dn
	$\{0\}^t$	$\{0\}^t$	$\{M\}^t$	1	1
	NDF	NDF	Y^n	1	

(4.63)

The solution of the primal problem, 4.52, automatically provides the solution to the dual problem, 4.57, and vice versa, Refs. 2 and 27. For illustration, if the kinematic formulation posed by Eqn. 4.57 is used for the solution procedure, then in addition to its own solution in terms of Δq and λ, the residual stresses, σ^R, are found to be the dual variables of the compatibility constraints given by Eqn. 4.57a. On the other hand, if the static formulation, i.e., Eqn. 4.52, is used for the solution, the dual variables corresponding to the equilibrium conditions, Eqn. 4.52a, will be Δq and λ.

Programme 4.57 has about as many real variables as 4.52 has constraints, and vice versa. It is well known that the computational effort required in solving linear programming problems increases very rapidly with the number of constraints, while the same is not the case with number of the real variables. It has been suggested, Ref. 78, that the computation time may be proportional only to the number of variables, but to the cube of the number of constraints. Therefore, formulation 4.57 appears computationally preferable. However, for two or three-dimensional continua, even this formulation implies the numerical solution of a large programme, Ref. 15. An alternative formulation, proposed in Ref. 15 with fewer constraints, for the static formulation is outlined in the following.

4.9 An Alternative Formulation

As mentioned in section 4.4, the yield condition can be defined in the d-stress space at any check point i, (i = 1, . . . n) by a polyhedron of Y^i faces. Thus, by means of Eqn. 4.9 the admissible domain for σ^i is defined by

$$\{N_j^i\}^t \{\sigma^i\} \leq \{K_o^i\}, \quad (j = 1, \ldots Y^i) \tag{4.64}$$

In fact, this polyhedron can be defined alternatively by its corners (vertices). If the polyhedron at the check point, i, is defined by $\bar{V}_o^i$ corners, which are represented by the matrix, $[S^i] \equiv [\{S_1^i\}\ldots\{S_{\bar{V}o}^i\}]$, in which $\{S_j^i\}$ is the vector defining the j-corner coordinates, the stress state can be represented by a linear combination with non-negative coefficients of the vertex vectors, Refs. 15 and 78. If $\{\zeta^i\} \equiv \{\zeta_1^i \;.\;.\;.\; \zeta_j^i\}$ is the vector of these coefficients for the stress state $\{\sigma^i\}$, then

$$\{\sigma^i\} = [S^i] \{\zeta^i\} \tag{4.65}$$

It can be shown, Refs. 15 and 78, that the feasible domain for stress vector, $\{\sigma^i\}$, defined via Eqn. 4.65, always corresponds to

$$\{\zeta^i\} \geq \{0\} \quad ,$$

and (4.66)

$$\{\bar{U}^i\}^t \{\zeta^i\} \leq 1$$

where $\{\bar{U}^i\}^t$ is a vector of unit entries. Appendix B illustrates an example of the linearized Von Mises yield criterion for a plane stress problem defined by a polyhedron of 18 vertices and 14 faces.

The representation, 4.65 and 4.66, is fully equivalent to 4.64 only if the linearized yield domain is bounded in every direction, Ref. 15. For instance, the Mohr-Coulomb yield criterion for plane strain problems, Appendix B, cannot be represented by 4.65 and 4.66.

If now the diagonal matrix of n blocks $[S] \equiv \text{diag}\,[[S^1] \ldots [S^n]]$ is introduced, Eqn. 4.65 becomes

$$\{\sigma\} = [S]\,\{\zeta\} \qquad (4.67)$$

in which $\{\zeta\}^t \equiv \{\{\zeta^1\} \ldots \{\zeta^n\}\}^t$

Then, by substituting Eqn. 4.67 into Eqn. 4.37, the residual stress state will correspond to

$$[E]^t\,[S]\,\{\zeta\} = \{0\} \qquad (4.68)$$

Now consider the set of constraints, 4.52, of the primal problem. They can be substituted at every check point, i, as follows

$$\text{a)}\quad w^i\{M^i\} + [N^i]^t\{\sigma^{Ri}\} \le \{K_o^i\}$$

$$\text{b)}\quad w^i \ge 0$$

and $\qquad (i = 1, \ldots n) \qquad (4.69)$

$$\text{c)}\quad k = w^i$$

4.69a and 4.69b represent a polyhedron in the (d+1) space, in which the vector $\{\{\sigma^{Ri}\}^t, w^i\}$ is defined. The scalar, w^i, represents an elastic factor at the check point, i, connected to the load factor k through 4.69c.

As in Eqs. 4.65 and 4.66 for the original yield surface, this polyhedron can alternatively be defined by its $\bar{V}^i$ vertices.

If $[S^i] \equiv [\{S^i_1\} \ldots \{S^i_{\bar{V}^i}\}]$ is the $(d+1) \times \bar{V}^i$ matrix of the vertices coordinates, $\{\sigma^i\}$, and w^i can be obtained by

$$\begin{Bmatrix} \{\sigma^{Ri}\} \\ w^i \end{Bmatrix} = [S^i]\,\{\zeta^i\}, \text{ with } \{\zeta^i\} \geq \{0\} \ , \qquad (4.70)$$

and the $Y^n + 1$ inequalities, 4.52a and 4.52b, can be replaced, at every check point i, by a single constraint,

$$\{\bar{U}^i\}^t\,\{\zeta^i\} \leq 1 \qquad (4.71)$$

By partitioning the matrix, $[S^i]$, as

$$[S^i] = \begin{bmatrix} [H^i] \\ \{L^i\}^t \end{bmatrix} \ , \qquad (4.72)$$

Eqn. 4.70 gives

$$\{\sigma^{Ri}\} = [H^i]\,\{\zeta^i\} \ ,$$

and

$$w^i = \{L^i\}^t\,\{\zeta^i\} \ . \qquad (4.73)$$

If $[H] \equiv \text{diag}\,[[H^1] \ldots [H^n]]$, $[L] \equiv \text{diag}\,[\{L^1\}^t \ldots \{L^n\}^t]$, $[\bar{U}] = \text{diag}[\{\bar{U}^1\}^t \ldots \{\bar{U}^n\}^t]$, and $\{\zeta\}^t = \{\{\zeta^1\}^t \ldots \{\zeta^n\}^t\}$, the LP problem, 4.52, can be reformulated as follows, Ref. 15,

$$s = \max k \ ,$$

subject to

a) $[E]^t\,[H]\,\{\zeta\} = \{0\}$,

b) $k\{1\} = [L]\,\{\zeta\}$,

$[\bar{U}]\,\{\zeta\} \leq \{1\}$,

and

c) $k \geq 0$, $\{\zeta\} \geq \{0\}$ (4.74) which can be represented in a form of a tableau as follows

$$
\begin{array}{r|c|c|c|l}
 & \{\zeta\}^t & k & & \\
\hline
s = \max & \{0\}^t & 1 & & \\
\hline
 & [E]^t[H] & \{0\} & \{0\} & NDF \\
\hline
 & -[L]^t & \{1\} & \{0\} & n \\
\hline
 & [\bar{U}]^t & \{0\} & \{1\} & n \\
\hline
 & \bar{V}^n & 1 & 1 & \\
\end{array}
\qquad (4.75)
$$

where $\bar{V}^n$ is the total number of vertices for the assembled structure.

4.10 Evaluation of Vertices Coordinates

Before using the programme 4.74, it is necessary to evaluate the matrix $[S^i]$ for each check point i.

It can be shown, Ref. 15, that the total number of vertices at each check point, i, is at most $\bar{V}^i = \bar{V}^i_o + Y^i - d$, $\bar{V}^i_o$ being the number of vertices (corners) of the original yield condition 4.65 and 4.66. Then since $w^i \geq 0$, the coordinates of $\bar{V}^i_o$ of the $\bar{V}^i$ vertices of the polyhedron, 4.69a-b, can be obtained by assigning $w^i = 0$ to the (d+1) th coordinates of these $\bar{V}^i_o$ vertices. Therefore, these vertices coincide with the corners of the original linearized yield surface, and are the same for each check point, i, for homogenous bodies.

The number of the remaining vertices, for every check point, i, are $(Y^i - d)$. These vertices cannot be determined directly as they depend on the elastic polyhedron, $\{M^i\}$, in Eqn. 4.46. They have to be determined independently for each check point, i, (i = 1, . . . n), and for the jth yield planes, $\{N_j^i\}$, (j = 1, . . . $(Y^i - d)$), as the optimal vectors of the following $(Y^i - d)$ programming problems, Ref. 15:

$$\max w^i ,$$

subject to

$$w^i M_j^i + \{N_j^i\}^t \{\sigma^{Ri}\} = K_{oj}^i \quad , \qquad (4.76)$$

and

$$w^i M_m^i + \{N_m^i\}^t \{\sigma^{Ri}\} \le K_{om}^i \, , \quad m \neq j$$

i.e., for each vertix, there will be one LP problem similar to 4.76. Each individual problem is associated with Y^i constraints and $\{w^i, \{\sigma^{Ri}\}^t\}$ as a real variable vector. $\{\sigma^{Ri}\}$ will be the coordinates of the vertix required to compute the matrix, $[H^i]$, and the corresponding w^i will be the elastic multiplier required to compute $\{L^i\}^t$.

It is important to note that if w_o^i is the maximum elastic multiplier at the check point, i, among all the vertices (j = 1, . . . $(Y^i - d)$), then w_o^i represents the maximum magnification of the elastic stress polyhedron at the check point, i, under the condition that it is inside the original yield polyhedron, Fig. 4.1, then the smallest of these values

$$\bar{w} = \min_i w_o^i \qquad (4.77)$$

will be the factor of safety against alternating plasticity, Ref. 15, i.e., for a load factor $k > \bar{w}$, the lack of shakedown will occur due to the repeated development of the plastic strain in the opposite sense, and in a bounded form at one point in the body leading to a local failure through the appearance of localized fractures.

Then, if it is known in advance that the lack of shakedown will be due to alternating plasticity, Eqn. 4.76 can provide directly the shakedown load factor without any need to solve the programme 4.74.

4.11 Comparison Between the Kinematic and the Vertices Formulations

It was mentioned before that the computation time required in solving linear programming problems may be proportional to the cube of the number of constraints and varies linearly with the number of variables. This comparison will be based on these numbers.

	Kinematic Formulation	Vertices Formulation
No. of Variables	Y^n + 2 NDF	$\overline{V}^n$
No. of Constraints	dn + 1	NDF + 2n

In plane problems, where d = 3, the comparison between the number of elements, n, and the number of degrees of freedom, NDF, depends on how the elements connect with each other and with the boundary. But approximately, n is slightly smaller than the NDF, thus the kinematic formulation has slightly fewer constraints than the vertices formulation. In comparing the number of variables, consider for example the Von Mises yield criterion, Appendix B, in plane stress shakedown analysis. Each element requires 14 yield planes in the kinematic formulation and 29 vertices in the vertices formulation. This means that the number of real variables in the alternative formulation is approximately twice that in the kinematic formulation.

From a linear programming point of view, the alternative formulation, i.e., programme 4.74, has an advantage in the sense that the origin of the variables $\{\{\zeta\}^t, k\}$ is 'feasible', Ref. 15. Thus, the search for the 'basic feasible solution', i.e., the use of 'phase I' in the 'two-phase technique' is not required, and the analysis can be started directly with 'phase II', Refs. 15 and 78.

But the disadvantage of the vertices formulation is that it requires the evaluation of the coordinates of the vertices, programme 4.76, before solving programme 4.74. As mentioned before, the number of constraints in programme 4.76 are just Y^i, and the real variables are d+1. For instance, when the Von Mises yield criterion is applied, the number of constraints will be 14 and the number of real variables will be 4 (by considering the non-negativity restrictions, i.e., $\sigma^R = \sigma^{R+} - \sigma^{R-}$, $\sigma^{R+} \geq 0$, $\sigma^{R-} \geq 0$, the number of real variables will be 7). For each element, programme 4.76 has to be solved 11 times. Thus, if n is large, much time will be spent in evaluating the coordinates of the vertices.

Corradi and Zavelani (15) presented a method for reducing the number of vertices required at each check point, i, by bounding w^i in 4.69b from below and above, i.e.,

$$s_u \geq w^i \geq s_\ell \tag{4.78}$$

where s_u and s_ℓ are values certainly greater and smaller than s respectively. If s_u is taken as the limit load factor, and s_ℓ as the elastic safety factor, inequality 4.78, will hold without any approximation.

4.12 Dead Loads

In practical engineering problems, the structure may be subjected to non-repeated dead loads $\{F_o\}$, below the carrying capacity of the structure, in addition to the variable repeated loads $k\{F(t)\}$. The loading programme in this situation will be, Ref. 7,

$$\{\bar{F}(t)\} = k\{F(t)\} + \{F_o\} \tag{4.79}$$

and the energy balance, Eqn. 4.60, will be

$$\{K_o\}^t \{\lambda\} = k\{M\}^t \{\lambda\} + \int_0^T \{F_o\}^t \{\dot{q}\}\, dt$$

i.e.,

$$\{K_o\}^t \{\lambda\} - \{F_o\}^t \{\Delta q\} = k\{M\}^t \{\lambda\} \tag{4.80}$$

In addition, by specifying the factor, k, to correspond only to the variable repeated applied surface tractions, the self weight will be included in $\{F_o\}$, and $\{M\}$ will be due to the applied surface tractions only. The programming problem, 4.63, will be

	$\{\Delta q^+\}$	$\{\Delta q^-\}$	$\{\lambda\}^t$		
s = min	$-\{F_o\}^t$	$\{F_o\}^t$	$\{K_o\}^t$		
	[E]	−[E]	−[N]	{0}	dn
	$\{0\}^t$	$\{0\}^t$	$\{M\}^t$	1	1
	NDF	NDF	Y^n	1	

(4.81)

4.13 Relation Between Limit and Shakedown Analysis

It was mentioned in Sec. 3.4, that the shakedown theorems are generalizations of the limit theorems to more complex loading. If the loads are kept constant with time, Eqn. 3.54 becomes

$$F_1^+ = F_1 = F_1^- \quad ,$$

and (4.82)

$$F_2^+ = F_2 = F_2^-$$

and the external work, Eqn. 4.58, will be $\{F\}^t \{\dot{q}\}$. The internal plastic energy dissipation rate, Eqn. 4.16, is $\{K_o\}^t \{\dot{\lambda}\}$, thus the energy balance given by Eqn. 4.60 becomes

$$\{K_o\}^t \{\dot{\lambda}\} = k\{F\}^t \{\dot{q}\} = k\{\sigma^E\}^t [N] \{\dot{\lambda}\} \tag{4.83}$$

Koiter's theorem becomes the upper bound theorem of limit analysis, Ref. 55, and the limit load factor, s, can be obtained as follows, Refs. 53, 4 and 2,

$$s = \min_{\dot{\lambda},\dot{q}} \{K_o\}^t \{\dot{\lambda}\}$$

subject to

$$\text{a)}\quad [E]\,\{\dot{q}\} = [N]\,\{\dot{\lambda}\}, \qquad (4.84)$$

$$\text{b)}\quad \{F\}^t\{\dot{q}\} = 1 ,$$

and

$$\text{c)}\quad \{\dot{\lambda}\} \geq \{0\}$$

In this situation, Melan's theorem can be shown to be the lower bound theorem of limit analysis, Ref. 55, the limit load factor, s, can be obtained from Refs. 53, 4 and 2 as follows:

$$s = \max_{k,\sigma} k$$

subject to

$$\text{a)}\quad [N]^t\,\{\sigma\} \leq \{K_o\} , \qquad (4.85)$$

$$\text{b)}\quad [E]^t\,\{\sigma\} = \{F\} ,$$

and

$$\text{c)}\quad k \geq 0$$

A noticeable simplification can be seen in the limit analysis formulation as the preliminary calculations of the elastic solution are no longer required. As an illustration of the generalization of Koiter's theorem to the upper bound theorem of limit analysis, the constraint, 4.84b, can be replaced, from Eqn. 4.83, by $\{\sigma^E\}^t\,[N] = 1$, where $\{\sigma^E\}$ is obtained from the 'constant load' condition.

CHAPTER V

NUMERICAL APPLICATIONS TO GEOTECHNICAL PROBLEMS

5.1 Introduction

Based on the kinematic formulation, 4.57, a computer code (Appendix C) was developed and checked on a IBM-370 system. The programme can handle both shakedown and limit analyses for plane stress and plane strain cases. In the first part of this chapter, the output is checked with the only available solutions for plane stress shakedown and plane strain limit analysis. The programme is then used for the analysis of footing shakedown in plane strain with different soil properties.

5.2 Plane Stress Shakedown Analysis

The only available numerical solutions for shakedown analysis in continuum problems are those for plane stress by Belytschko (7), Corradi and Zavelani (15), and Hung and Palgen (35). All these investigations considered the same problem: a thin square plate with a central circular hole subjected to biaxial loading as shown in Fig. 5.1. The ratio between the diameter of the hole and the plate length was 0.2. The loading consisted of two uniform surface tractions, T_x and T_y, in the x and y directions respectively. Both T_x and T_y were allowed to vary between zero and the maximum tensile loads,and the dead load was neglected. The loading programme was square in the load space as shown in Fig. 5.2.

Belytschko (7) employed the Von Mises yield criterion, and used an equilibrated finite element model and nonlinear programming techniques, formulation 3.55, to determine the lower bound of the shakedown load under different loading programmes. The structure was discretized into 26 elements, and for the square loading programme Fig. 5.2, the shakedown load was found to be $0.431\sigma_o$, σ_o being the yield stress in tension.

Corradi and Zavelani (15) used a piecewise linearized Von Mises yield criterion, Appendix B, and a constant stress constant strain finite element model in formulation 4.74 for different loading programmes. The structure was discretized into 66 elements, and for the square loading programme, Fig. 5.2, the shakedown load was found to be $0.504\sigma_o$.

Hung and Palgen (35) employed an equilibrated finite element model and nonlinear programming technique by using the yield criterion of the mean. This assumption is based on Hencky's interpretation of the Von Mises yield criterion, i.e., plastic flow occurs when the distorion energy density reaches the ultimate value. This procedure can be achieved by averaging the yield function over each element which leads to a reduction in the number of the nonlinear constraints in Belytschko's formulation, Ref. 7. The structure is discretized into 26 elements, and for the square loading programme, Fig. 5.2, the shakedown load was found to be $.431\sigma_o$.

In this report, the kinematic formulation, 4.57, was adopted by employing the linearlized Von Mises yield criterion, Appendix

B, and constant strain triangular elements Ref. 17. The solution algorithm for the linear programming problem was the revised simplex method given by Ref. 36. For the square loading programme, with different meshes, the shakedown load is as shown in Table 2. Another check was performed on the computer programme, based on the fact that the inadaptation mode in this problem is alternating plasticity, Ref. 15. Some subroutines were deleted and others added to permit the evaluation of coordinates of vertices given by the LP problem, 4.76. s was estimated by the definition of the shakedown load factor against alternating plasticity given by Eqn. 4.77. The output in this case was the same as in the previous one, but with substantial saving in computation time as shown in Table 3.

The analytical value of the shakedown load was given by Belytschko (7) as $0.47\sigma_o$. The values of Refs. 7 and 35 are 9% lower than the analytical one,while that of Ref. 15 is 7% higher. The present solution gives a value 9% higher than the analytical one.

It is worth emphasizing that, the analytical solution of the shakedown load was estimated by a previous knowledge of the exact elastic stress distribution at point A in Fig. 5.1, where the stresses are uniaxial and the lack of shakedown is due to alternating plasticity Ref. 15. The elastic stress values, $\sigma^E_{x\ max}$ and $\sigma^E_{x\ min}$, at the point A due to individual applications of unit loads, T_x and T_y, obtained by Howland (32) are 3.14 and -1.11 respectively. The shakedown load against alternating plasticity in the uniaxial stress state can be calculated as follows, Refs. 26 and 31:

$$s\sigma_o = 2\sigma_o/(\sigma^E_{max} - \sigma^E_{min}) = 0.47\sigma_o \qquad (5.1)$$

It appears from this equation that when the inadaptation mode is due to alternating plasticity, s does not depend on the residual stress distribution in the plate, nor the yield criterion, but only on the elastic solution obtained. Thus the values, 0.504 and 0.51, indicate that the elastic stress concentration near the hole is underestimated by the compatible finite element model used. This is in contrast to the value of 0.431 obtained by Refs. 7 and 35, based on the equilibrated finite element model which represents an overestimate of elastic stress distribution.

The above remarks seem to indicate that the use of compatible models produces an upper bound to s, not only because the inadaptation mechanism for the discretized structure corresponds to kinematic solution for the continuous system as in limit analysis Ref. 4, but also because the local values of the elastic stress distribution seem to be underestimated. Similarly, a lower bound is expected if an equilibrated model is used. The use of hybrid finite element models, in which equilibrium and compatibility are imposed, leads to solutions which may give values close to the analytical ones but upper or lower bounds cannot be obtained.

The values 0.504 and 0.51 are obtained by the same finite element model, the same number of elements, and the same piece-wise linearized yield criterion. However, the 0.504 value was obtained by Ref. 15 using formulation 4.74, eliminating a limited number of vertices, but more than that permitted by Eqn. 4.78. On the other hand, the 0.51 value was obtained without any approximation, i.e., all yield planes were considered. On this basis, the 0.51 value seems to be a more realistic upper bound than the 0.504, and the error due to the elimination of vertices is on the safe side. But as far as the computation time is compared, the 0.504 value was obtained in 291 sec. on a UNIVAC 1106 system by Ref. 15. However, as mentioned before, formulation 4.74 cannot be used with the Mohr-Coulomb yield criterion which is the basis of the solution of the plane strain footing problem.

Finally, it is necessary to mention that the values 0.431 and 0.51 are not 'true' lower and upper bounds on s; the bounding nature of the Melan and Koiter theorems is lost because of the use of an approximate elastic solution and the approximation of the circular hole by straight line segments.

The meshes for 16, 42 and 66 elements are shown in Figs. 5.3, 5.4 and 5.5. In these meshes, the regions of the structure subjected to highest stresses are discretised into smaller elements, in contradistinction to larger elements for low stress areas. The 66-element mesh is very similar to that used by Ref. 15. Fig. 5.6 illustrates the convergence of the shakedown load as the mesh becomes finer.

5.3 Plane Strain Limit Analysis

The limit load can be estimated either by a complete elastoplastic analysis throughout the loading history up to collapse, Refs. 11, 77, 80 and 81, or by the limit theorems which are loading history-independent, Refs. 1, 2, 3, 4, 6, 22, 23, 33, 75 and 78. There is a substantial saving in using the limit theorems; it was suggested by Ref. 6, that the time taken is one-third that complete elastoplastic analysis. However, the elastoplastic formulation can give additional information such as load-deflection histories,which are often desired.

The problem considered is a uniformly loaded strip footing underlined by a shallow stratum of undrained clay (Von Mises material) as shown in Fig. 5.7. There are three available solutions to this problem.

The problem was solved at first by Hoeg et al (30), who assumed elastic-perfectly plastic behaviour for the soil. The shallow stratum of the clay was analyzed by the finite difference technique. The Tresca yield criterion and its associated flow rule were used with a cohesive strength of 17.5 psi. The limit load was found to be 90 psi.

Chen (10) used the incremental method for elastic-plastic behaviour in which the clay stratum was discretized into 98 finite elements. The Von Mises yield criterion was used, which in this case (plane strain) should give the same value of the limit load in terms of the yield stress in pure shear, c, (See Appendix B). The limit load in this solution was found to be 92 psi.

Valliappan (73) solved the same problem by discretizing the soil stratum into 150 finite elements and using the Von Mises yield criterion. The initial stress method (Zienkiewicz et al (80)), was used to integrate the equations and the last value at which the iterations converged was 78 psi.

In this presentation, the same problem is solved by using the kinematic formulation, 4.57, and the linearized Von Mises yield criterion (Appendix B). The soil stratum is discretized into 46 finite elements. The coarse mesh was necessitated by constraints on computer time. The limit load factor is found to be 2.4 corresponding to a limit load of 85 psi which lies inside the range of the three previous solutions.

5.4 Plastic Model For Soil

The soil is modelled as an elastic-perfectly plastic material obeying the Mohr-Coulomb criterion with its associated flow rule. The assumption of elastic-perfectly plastic behaviour will not give a very good representation of the strains but the collapse load is well approximated, Ref. 84. The assumption of associative behaviour is considered to simplify the analysis. However, in Refs. 82 and 83 comparison of the following three plastic models has been presented for hypothetical foundations:

I) elastic-perfectly plastic with Mohr-Coulomb yield surface and associated flow rule.

II) as above, with a nonassociated flow rule.

III) strain dependent (critical state) with an associated flow rule.

For the drained condition, the soil was treated as a dry one; all the three different assumptions gave similar results indicating that collapse loads are practically identical, Fig. 5.8.

The analysis presented herein will be restricted to the dry case, i.e., the effect of cyclic loading on the pore pressure is not considered.

5.5 Plane Strain Shakedown Analysis

After checking the computer programme with available solutions, a plane strain shakedown analysis of a footing, underlined by dry soil, subjected to a variable repeated loading was carried out, Fig. 5.9.

The loading programme was as shown in Fig. 5.10 which represents an eccentric, inclined variable repeated loading. The inclination angle, θ, was maintained constant 45°. The eccentricity of the load from the footing centre, e, was assumed to vary from -b/12 to b/12 where b is the footing width. Thus, the variable repeated loading programme can be expressed as

$$F^{-} \leq F(t) \leq F^{+} \tag{5.2}$$

In addition, a fixed body force, F_o, was considered to account for the specific weight of the soil, $1.7 t/m^3$. The piecewise linearized Mohr-Coulomb yield criterion, Appendix B, with its associated flow rule, was considered. The soil stratum was discretized into 62 finite elements and the adopted mesh is as shown in Fig. 5.11, with a rigid rough boundary at the base of the soil stratum and vertical smooth rigid boundaries on the sides, Ref. 82. The cohesive strength, c, was 12.5 t/m^2 and the angle of internal friction, ϕ, was 30^o. A shakedown load factor of 4.884 was obtained involving a CPU time of 71 min.. 99% of this time was spent in solving the linear programming problem (187 constraints and 508 variables). The sparse matrix technique in linear programming proposed by Reid (87), used in limit analysis by Best and McFall (86) may be most economical.

A parametric study was performed by adopting a coarser mesh of 34 elements as shown in Fig. 5.12. The same loading programme was used and the cohesive strength maintained constant at 12.5 t/m^2. The angle of internal friction, ϕ, varied as shown in Table 4. It can be noted that the shakedown load factor for $\phi = 30^o$ is 6.506, indicating an error of 25% in the results due to the coarser mesh.

A similar parametric study was performed for the same soil properties and the same loading programme, but neglecting the tensile strength of the soil. The results are shown in Table 5.

Comparing the results in Tables 4 and 5, it can be seen that the shakedown load factor for zero tensile strength is about 60% of that for the case of full tensile strength.

The uniaxial compressive strength, f_{cu}, can be expressed as, Ref. 5,

$$f_{cu} = 2c \cos\phi/(1 - \sin\phi) \qquad (5.3)$$

The shakedown load factors in Tables 4 and 5 are plotted against $f_{cu}/2c$ in Fig. 5.13. It can be noted that the shakedown load factor, s, varies almost linearly with $\cos\phi/(1 - \sin\phi)$ for the particular case of zero tensile strength.

It is worth emphasizing that the shakedown load factor was computed for unit k_o in formulation 4.57; $k_o = 2c$ from Eqn. B.5. This shows that the shakedown load varies almost linearly with the uniaxial compressive strength, for the particular case of tension cut-off.

CHAPTER VI

CONCLUSIONS

6.1 Introduction

The application of the shakedown theorems in geotechnical engineering was illustrated for a footing subjected to variable repeated loading under plane strain condition. As there was no available solution for shakedown analysis in plane strain, the formulation of the kinematic shakedown theorem was checked with the only available solution for plane stress (square plate with a central circular hole, subject to biaxial variable repeated loading). By making use of the fact that the shakedown theorems are generalizations of the limit theorems for complex loading programme, the formulation was checked with available solutions of plane strain limit analysis. In both cases,the results agreed well with available solutions, indicating the correctness of the concepts and the computer code.

6.2 Concluding Remarks

1) The discrete formulation of the kinematic shakedown theorem was chosen, with the revised simplex algorithm as a solution procedure, after comparing the following three LP problems with regard to the computational efficiency and the applicability of different yield criteria:

i) The static shakedown theorem,restated as a LP problem, in which the load factor is maximized subject to equilibrium and yield conditions: This formulation was not considered because the number of constraints is much larger than the number of variables, i.e., it appears to be computationally inefficient to the point of impracticality.

ii) The dual of the former problem, which was shown to represent the kinematic shakedown theorem in which the internal plastic energy dissipation is minimized, subject to compatibility and maximum positive external work conditions: The number of constraints is much smaller than the number of variables,which varies linearly with the number of yield planes of the linearized yield surface. But as it is difficult in shakedown analysis to decide in advance which planes can be eliminated, all the yield planes were considered in the variable vector.

iii) An alternative formulation to reduce the number of constraints in the static shakedown theorem: This formulation has few more constraints and many more variables than the kinematic formulation. The advantages of this formulation are a) the number of variables, which varies linearly with the number of vertices of the elastic polyhedra, can be reduced very much by using Eqn. 4.77, and b) the origin of the variables is feasible, i.e., the analysis can start directly from 'phase II'. The disadvantages of formulation 4.74, are a) it needs a preliminary step to calculate the coordinates of the vertices,and b) it cannot be used for the Mohr-Coulomb yield criterion, as the hydrostatic stress can increase without limit, i.e., the yield surface is unbounded. This formulation was used successfully in plane stress shakedown analysis by Ref. 15 with elimination of more vertices than that permitted by Eqn. 4.77 to reduce the computation time.

2) The numerical results for shakedown analysis in plane stress showed that the alternative formulation with the vertices elimination, used in Ref. 15, is less time consuming than the kinematic formulation. The error in the shakedown load factor due to the elimination of large number of vertices is very small and on the safe side, as the shakedown load factor is less than the upper bound obtained by the present solution.

3) The numerical results of the plane strain shakedown analysis of a footing on dry soil, subjected to variable repeated loading, showed that the shakedown load varies almost linearly with the uniaxial compressive strength, for the particular case of tension cut-off; the shakedown load factor for zero tensile strength is approximately 60% that for full tensile strength. While unfortunately, there are no available experimental investigations to check this conclusion, the analysis is a useful starting point for the application of the shakedown theorems in geotechnical engineering.

4) The shakedown load factor depends only on the choice of the finite element model (equilibrated or compatible) and not on the shakedown theorem ,(since the static and kinematic formulations are dual).

6.3 Applications of the Method

1) Study of the effect of cyclic loading on the foundations of onshore structures such as machine foundations.

2) An approximate study of the cyclic loading effects on the foundations of offshore structures by neglecting drainage; the problems are solved assuming undrained soil properties and considering soil behaviour under total stress conditions.

6.4 Recommendations for Further Research

1) Effect of cyclic loading on pore pressure,

2) Applications of the nonassociated flow rule for Coulomb materials,

3) Extension to work hardening and work softening behaviour,

4) Consideration of second order geometric effects,

5) Inclusion of inertia and damping in the elastic response analysis for a prescribed loading history,

6) Use of the sparse matrix technique in linear programming to reduce the computation time.

Table 1: Classification of Problems of Elastic-Plastic Analysis (Ref. 69)

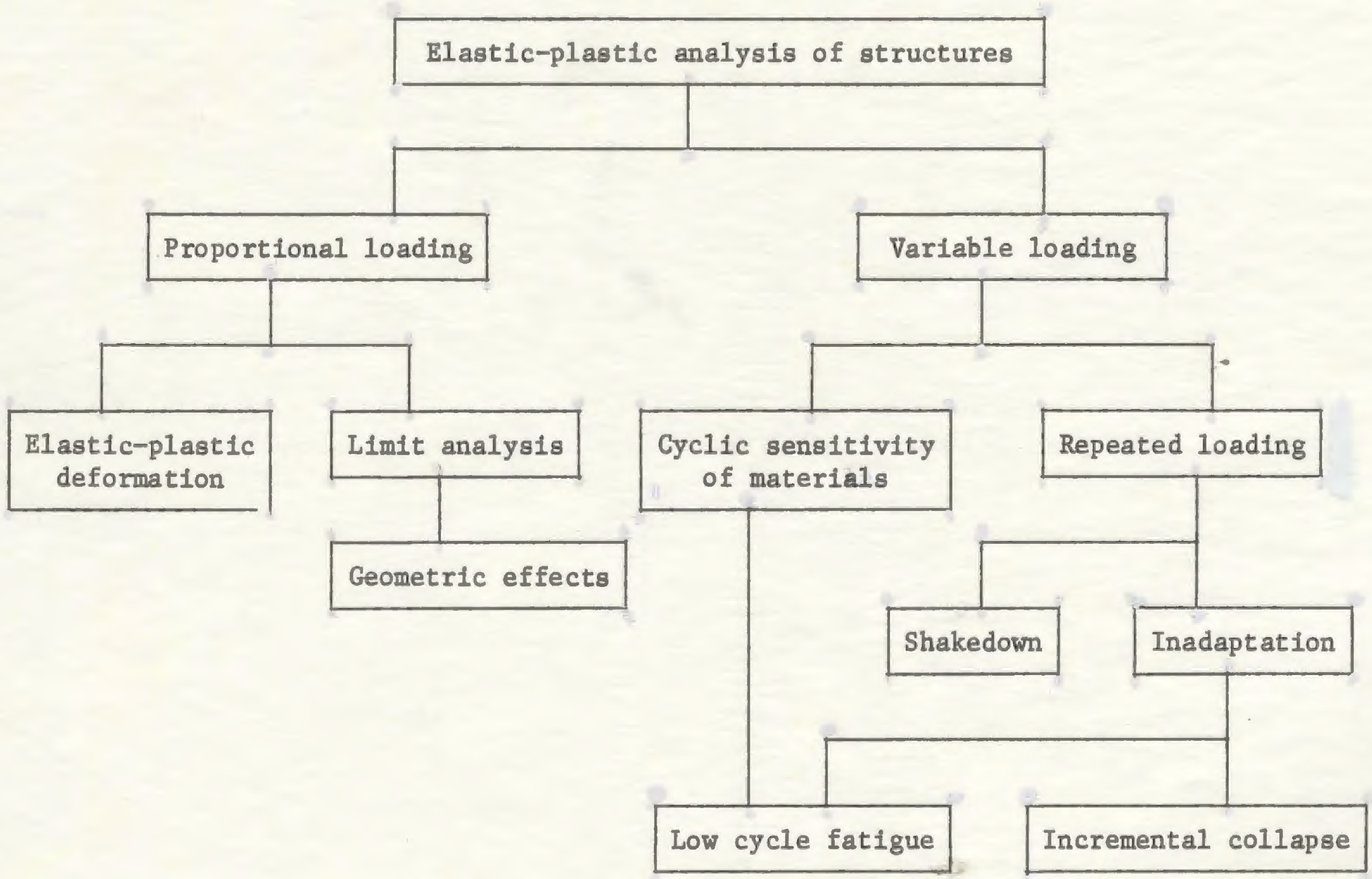

Table 2: Shakedown Load Based on the Kinematic Formulation 4.57

No. of Elements	No. of Variables	No. of Constraints	Shakedown Load	CPU Time (sec)
16	272	49	$0.654\ \sigma_o$	93
42	696	127	$0.556\ \sigma_o$	990

Table 3: Shakedown Load Calculated considering Alternating Plasticity as an Inadaptation Mode by Using Eqn. 4.77

No. of Elements	Shakedown Load	CPU Time (sec)
16	0.654 σ_o	42
42	0.556 σ_o	68
66	0.510 σ_o	96

Table 4: Shakedown Load Factors For Different Angles of Internal Friction For Full Tensile Strength

ϕ (degrees)	0	5	10	15	20	25	30
s	3.990	4.347	4.883	5.443	5.881	6.197	6.506
CPU (minutes)	9.00	9.08	9.04	11.03	9.44	10.24	12.45

Table 5: Shakedown Load Factors For Different Angles of Internal Friction For Zero Tensile Strength

ϕ (degrees)	0	5	10	15	20	25	30
s	2.362	2.648	2.965	3.315	3.684	4.011	4.328
CPU (minutes)	11.13	11.50	11.17	12.02	14.00	15.48	19.29

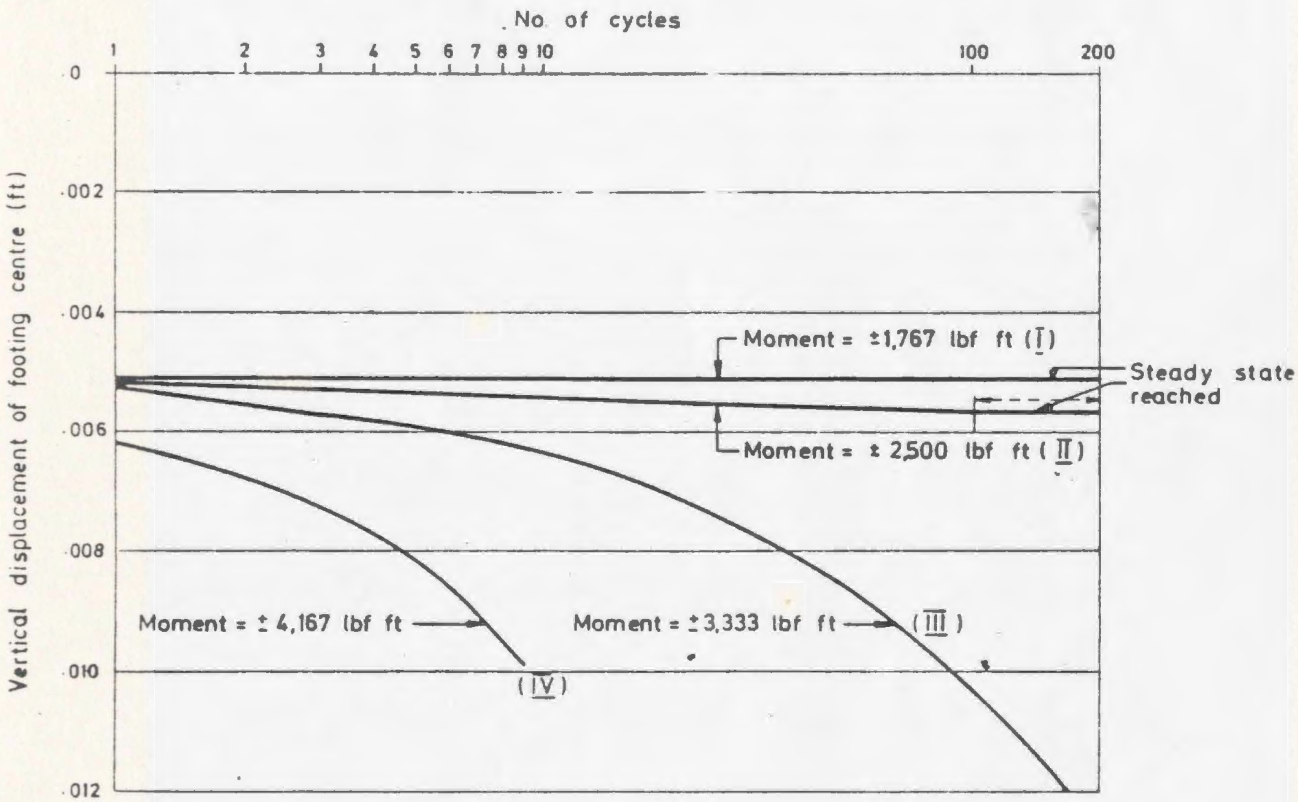

Fig. 2.1: Total Stress Analysis for Cyclic Moment Superimposed on a Vertical Loading of 250 lb/ft (Ref. 82)

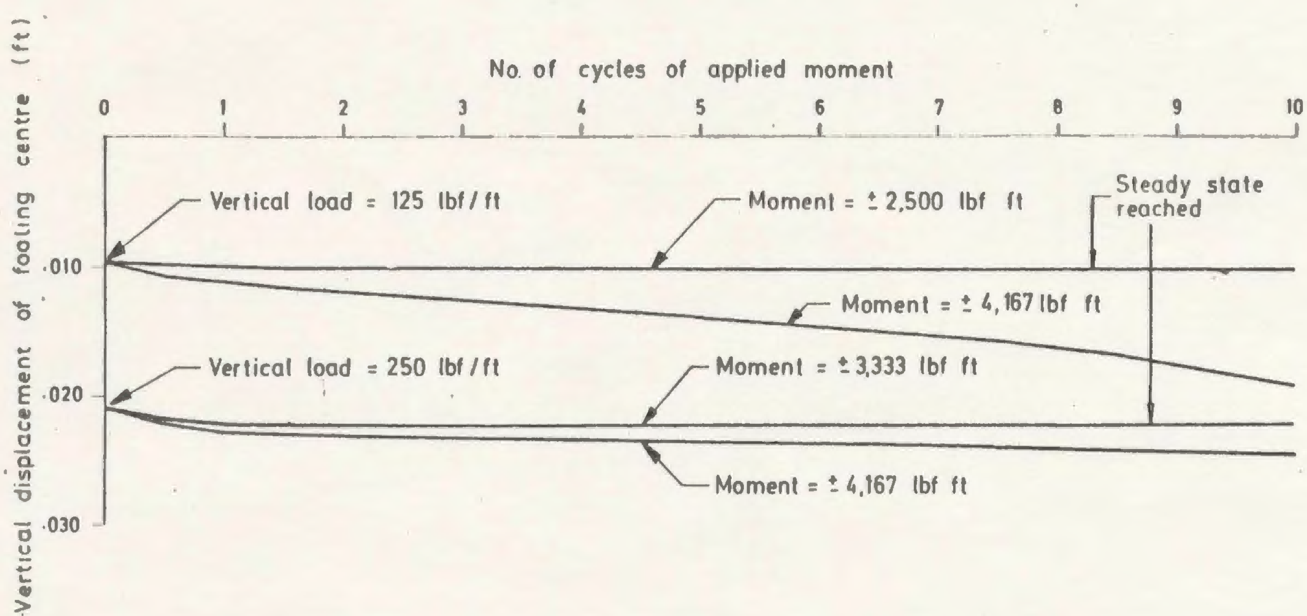

Fig. 2.2: Critical State/Mohr-Coulomb Analysis for Cyclic Moment Applied Under Undrained Conditions to a Rigid Footing Loaded Vertically Under Drained Conditions (Ref. 82)

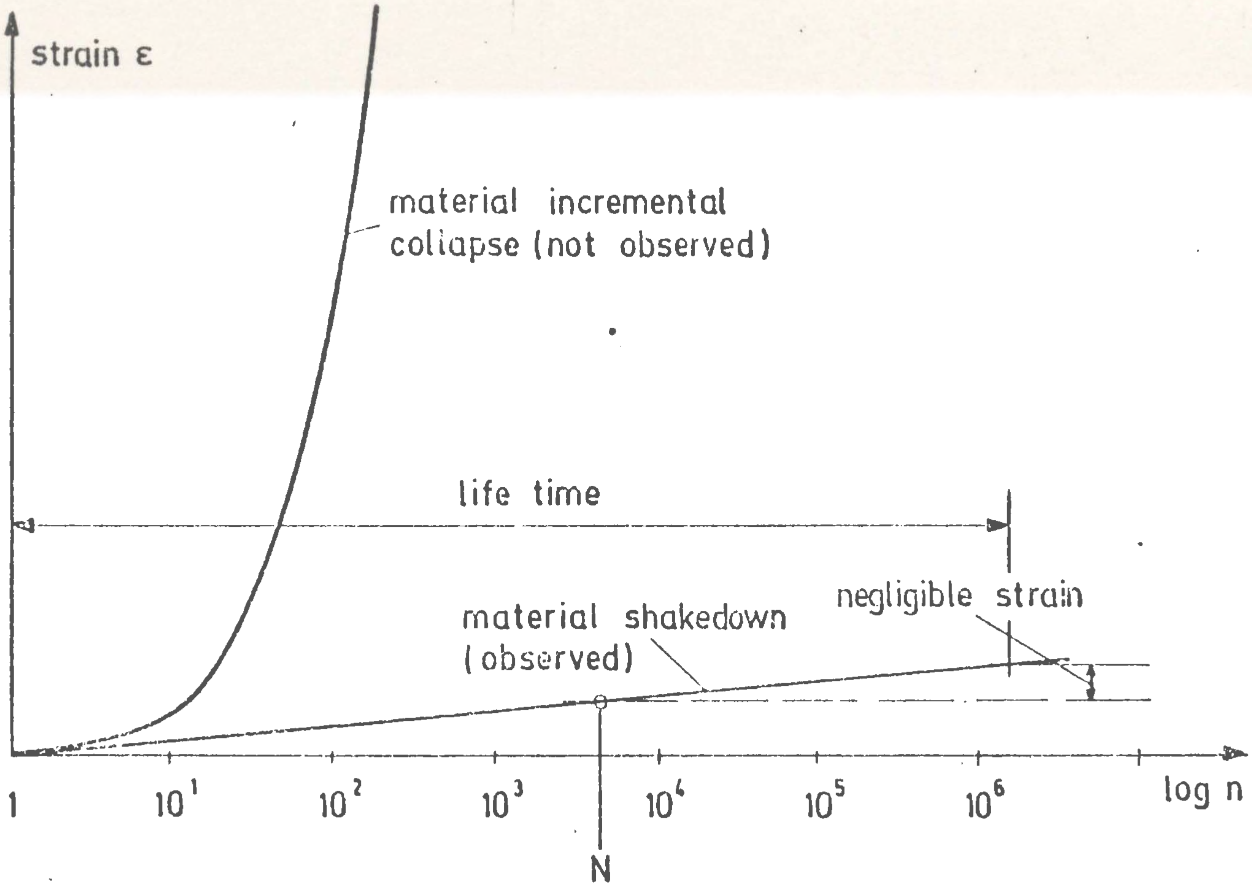

Fig. 2.3: Material Shakedown and Incremental Collapse (Ref. 24)

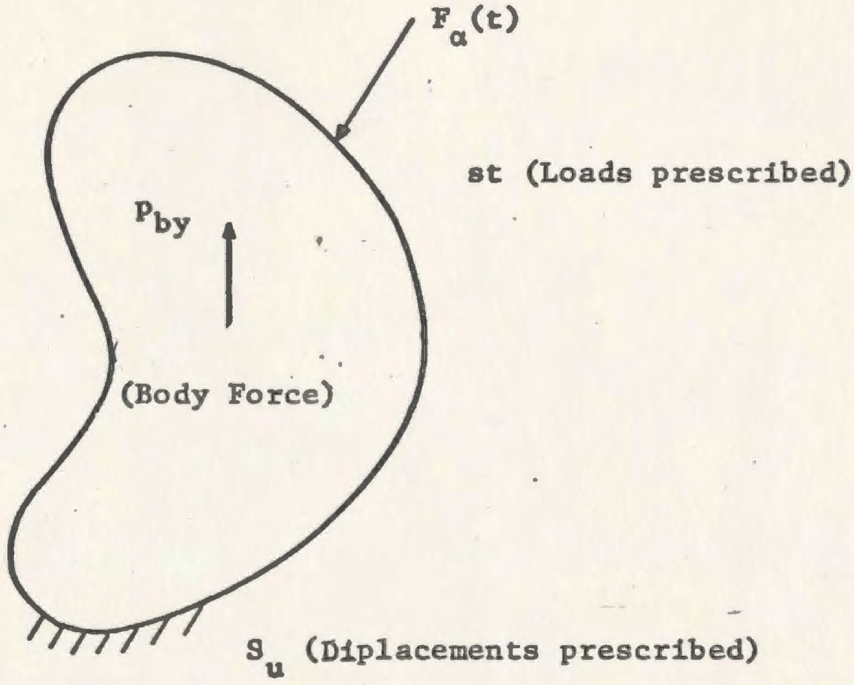

Fig. 3.1: Cyclic Loading and Boundary Conditions

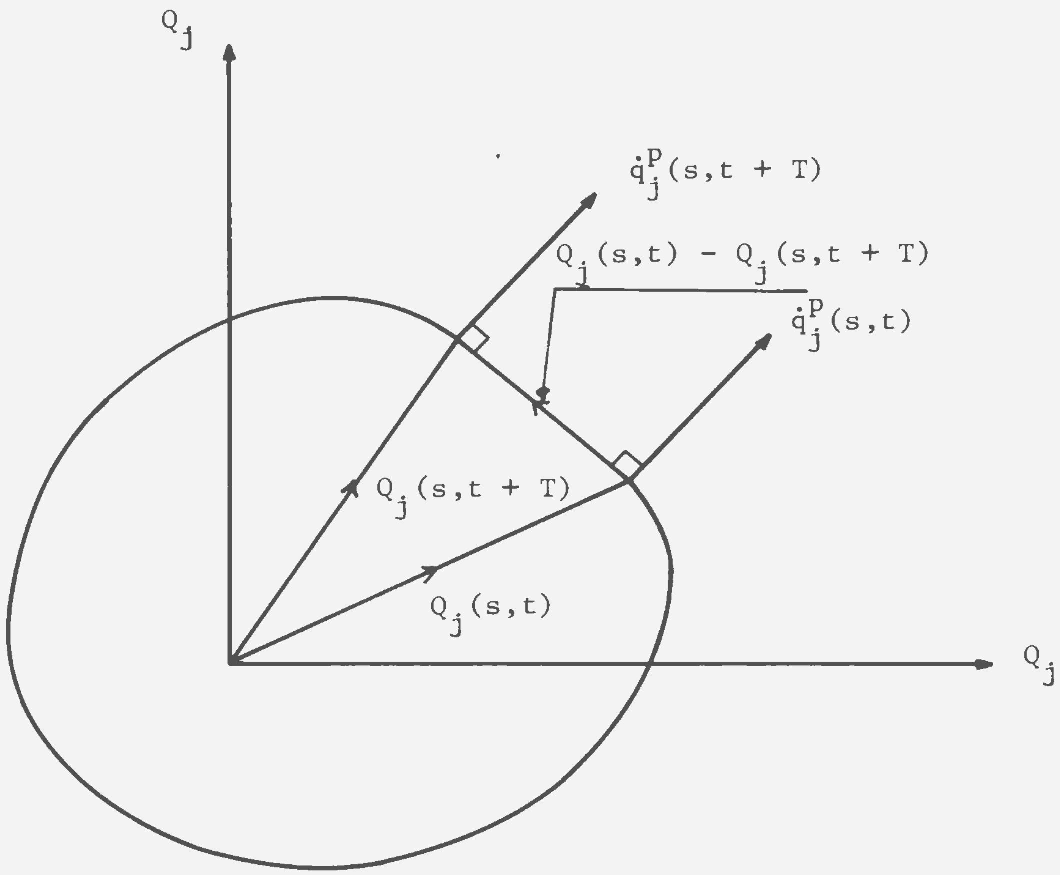

Fig. 3.2: Flat Region on the Yield Surface (Ref. 55)

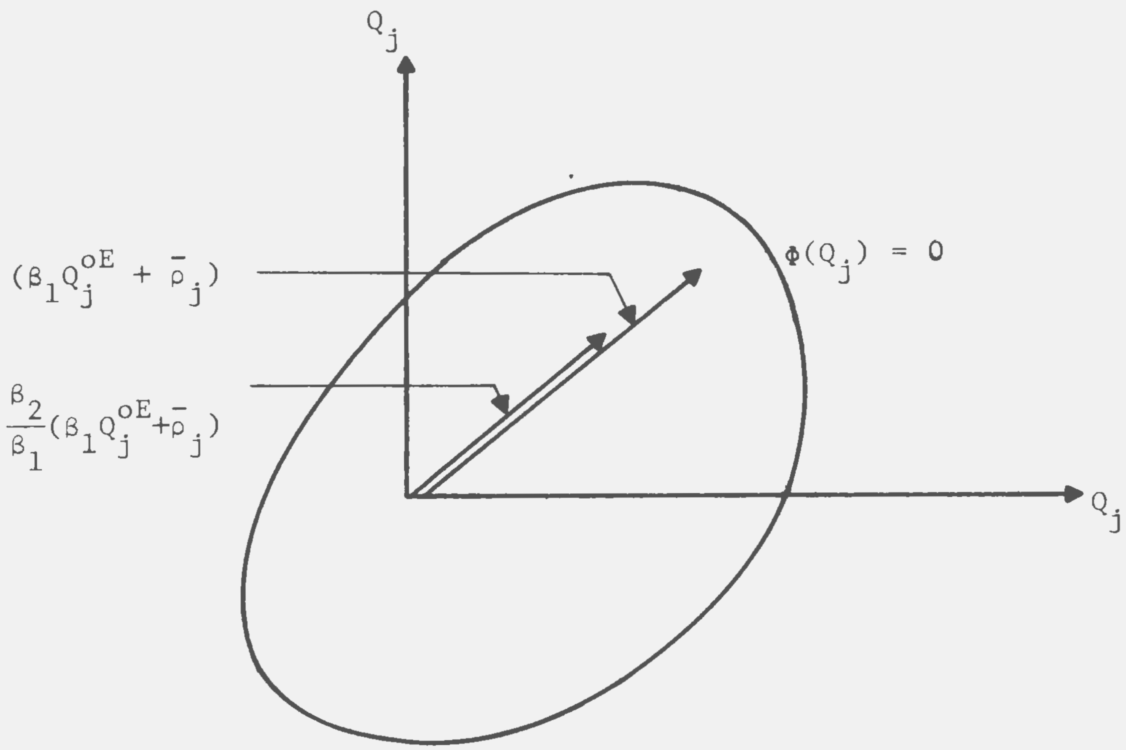

Fig. 3.3: Stress Space (Ref. 55)

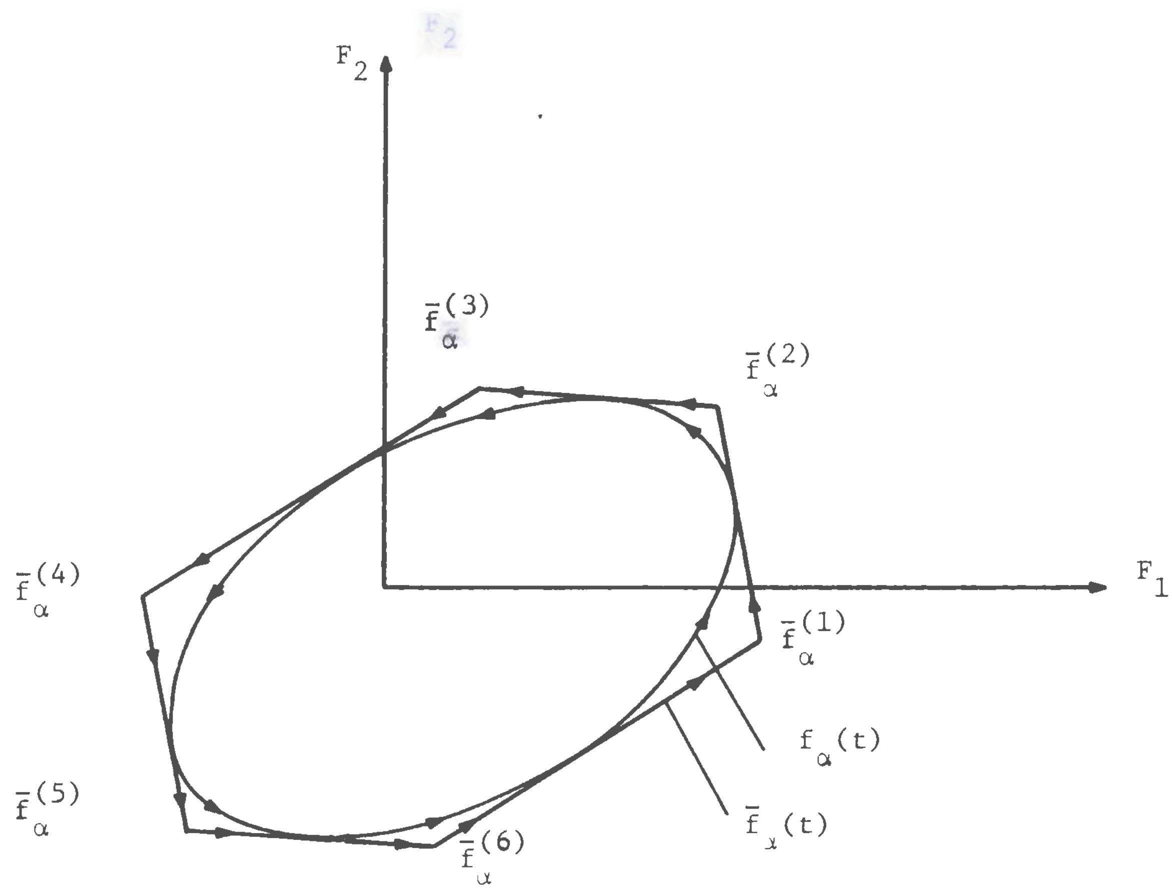

Fig. 3.4: Segmental Loading Programme (Ref. 55)

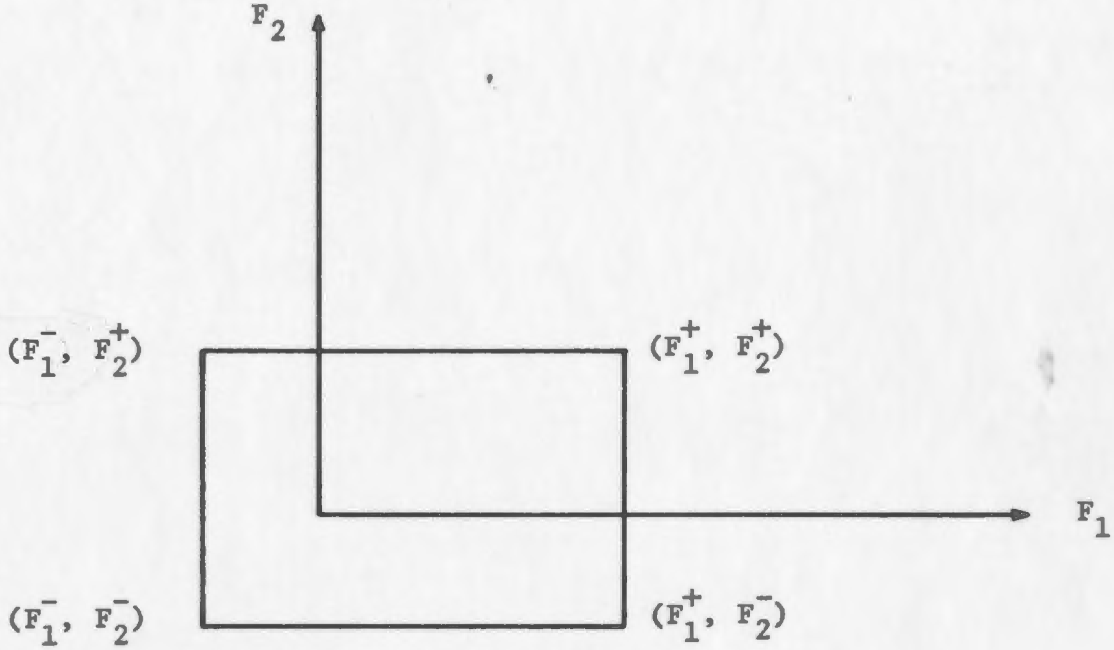

Fig. 3.5: Variable Repeated Loading within Prescribed Limits (Ref. 55)

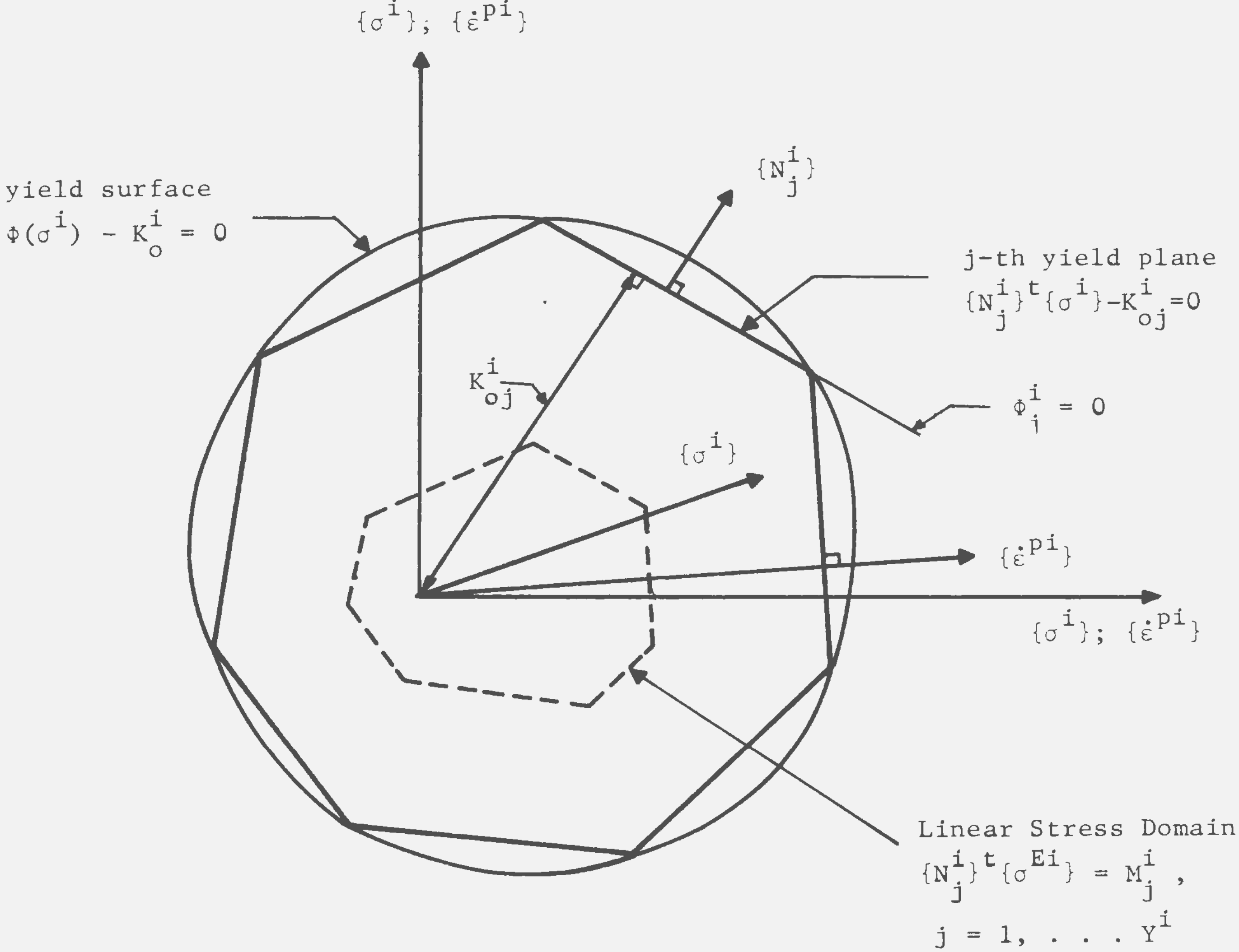

Fig. 4.1: The Linearized Yield Surface and the Linear Stress Domain at a Check Point i

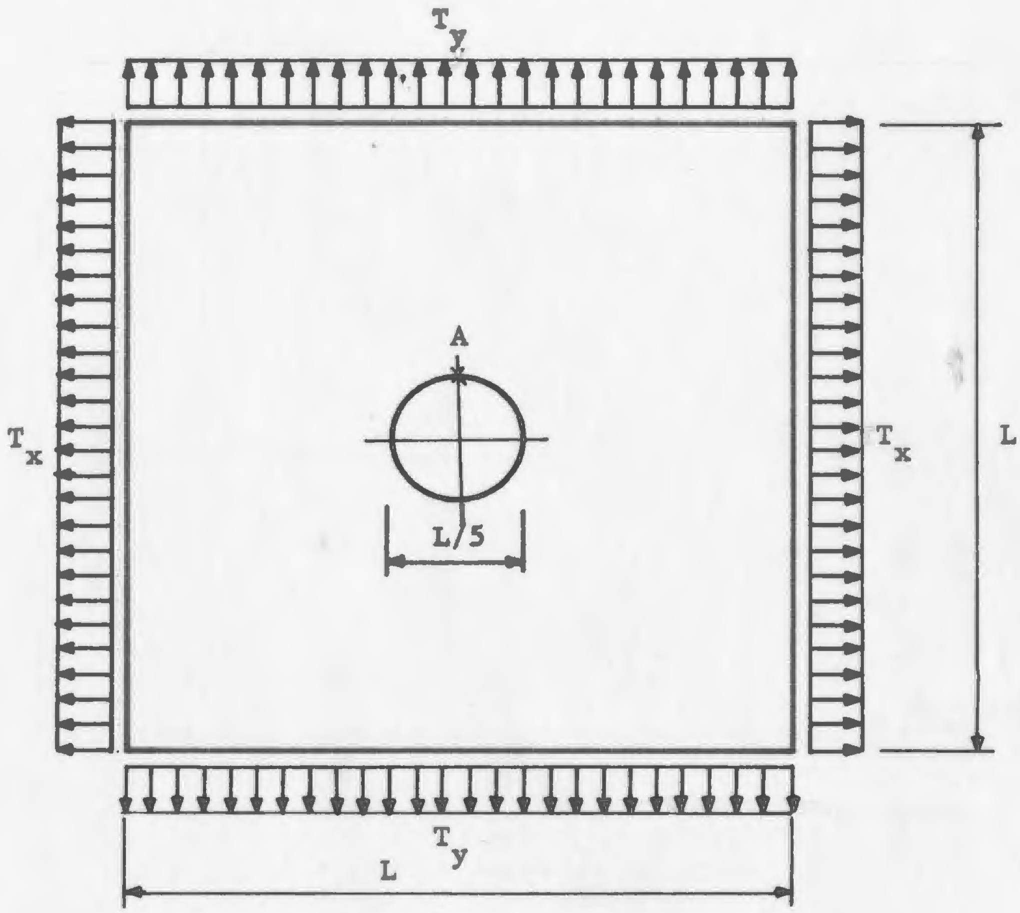

Fig. 5.1: Thin Plate with Central Circular Hole

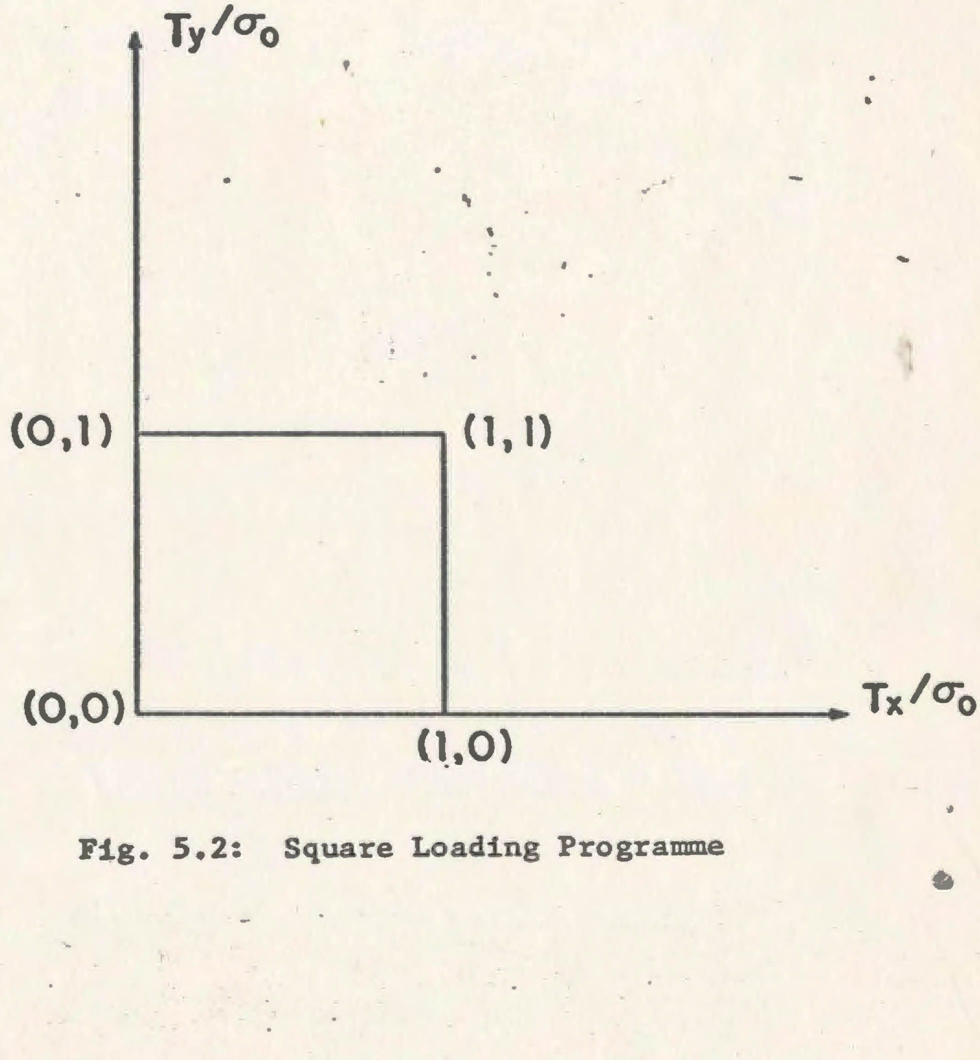

Fig. 5.2: Square Loading Programme

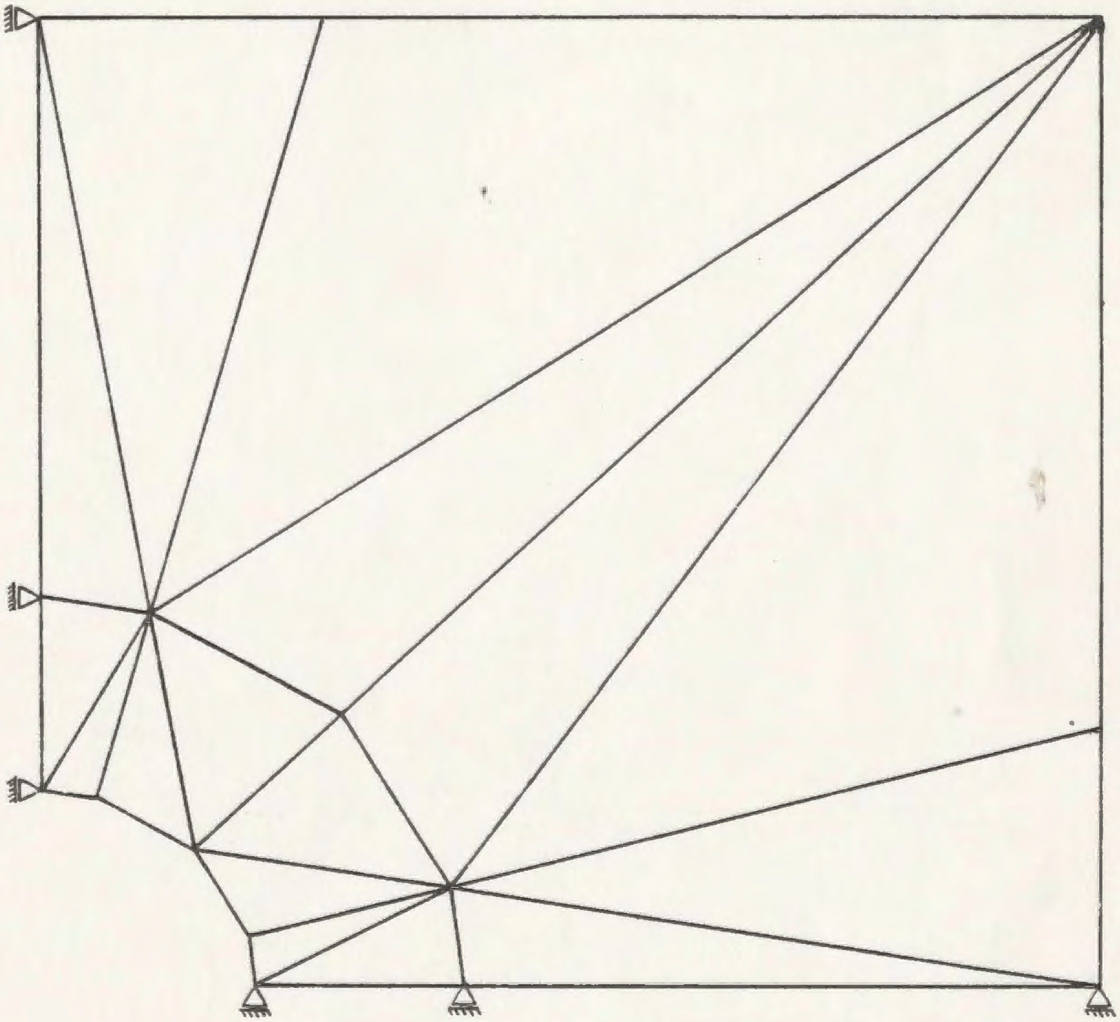

Fig. 5.3: Coarse Mesh (16 elements) for the Thin Plate

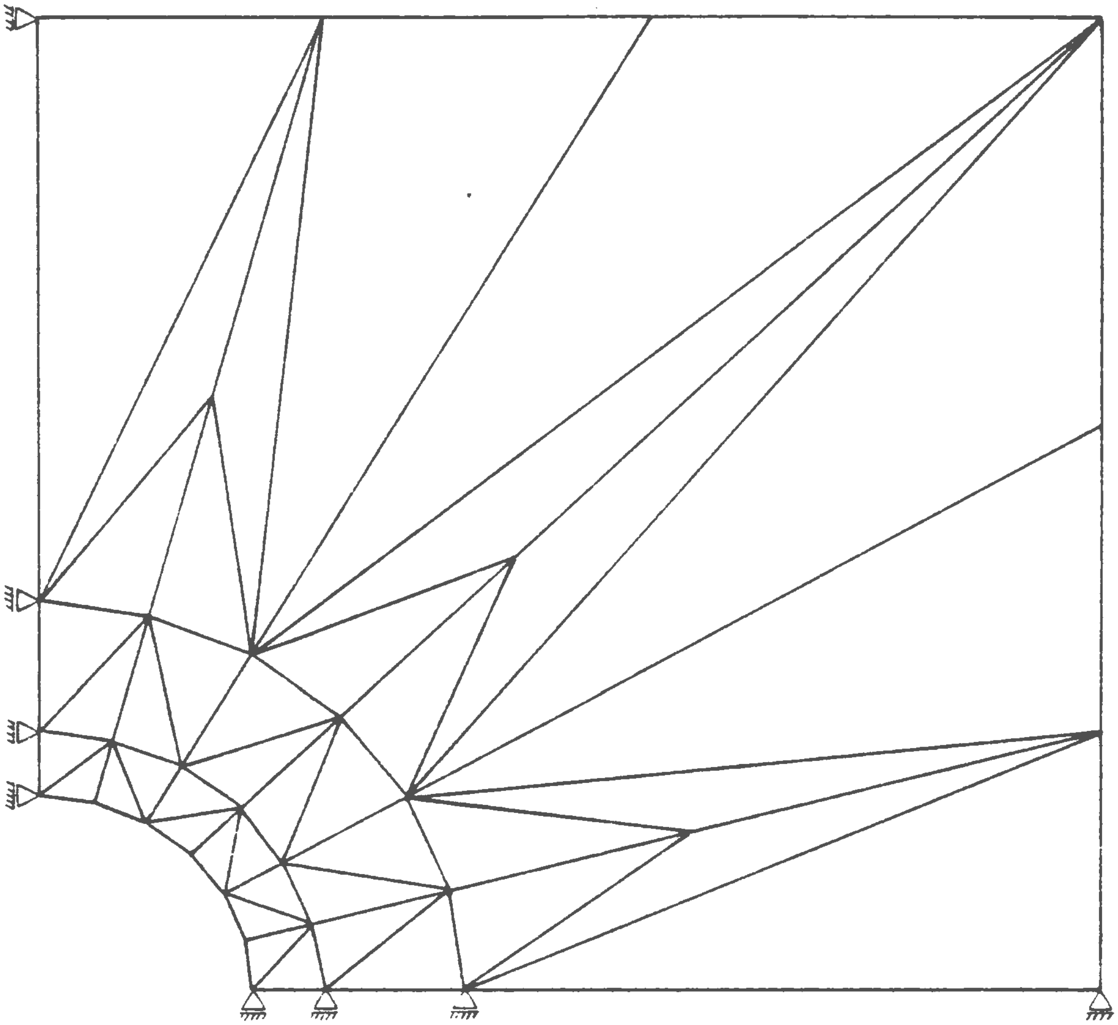

Fig. 5.4: Intermediate Mesh (42 elements) for the Thin Plate

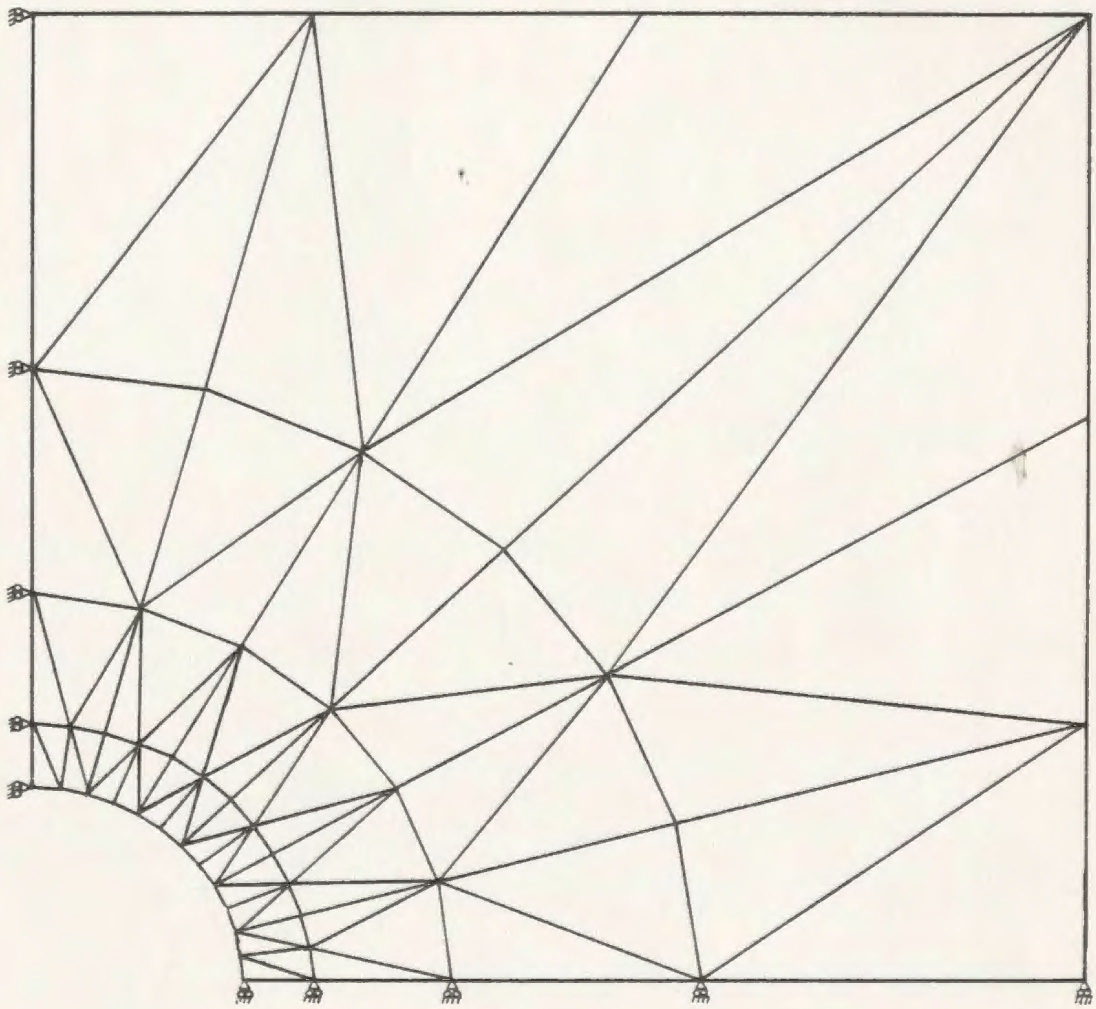

Fig. 5.5: Fine Mesh (66 elements) for the Thin Plate

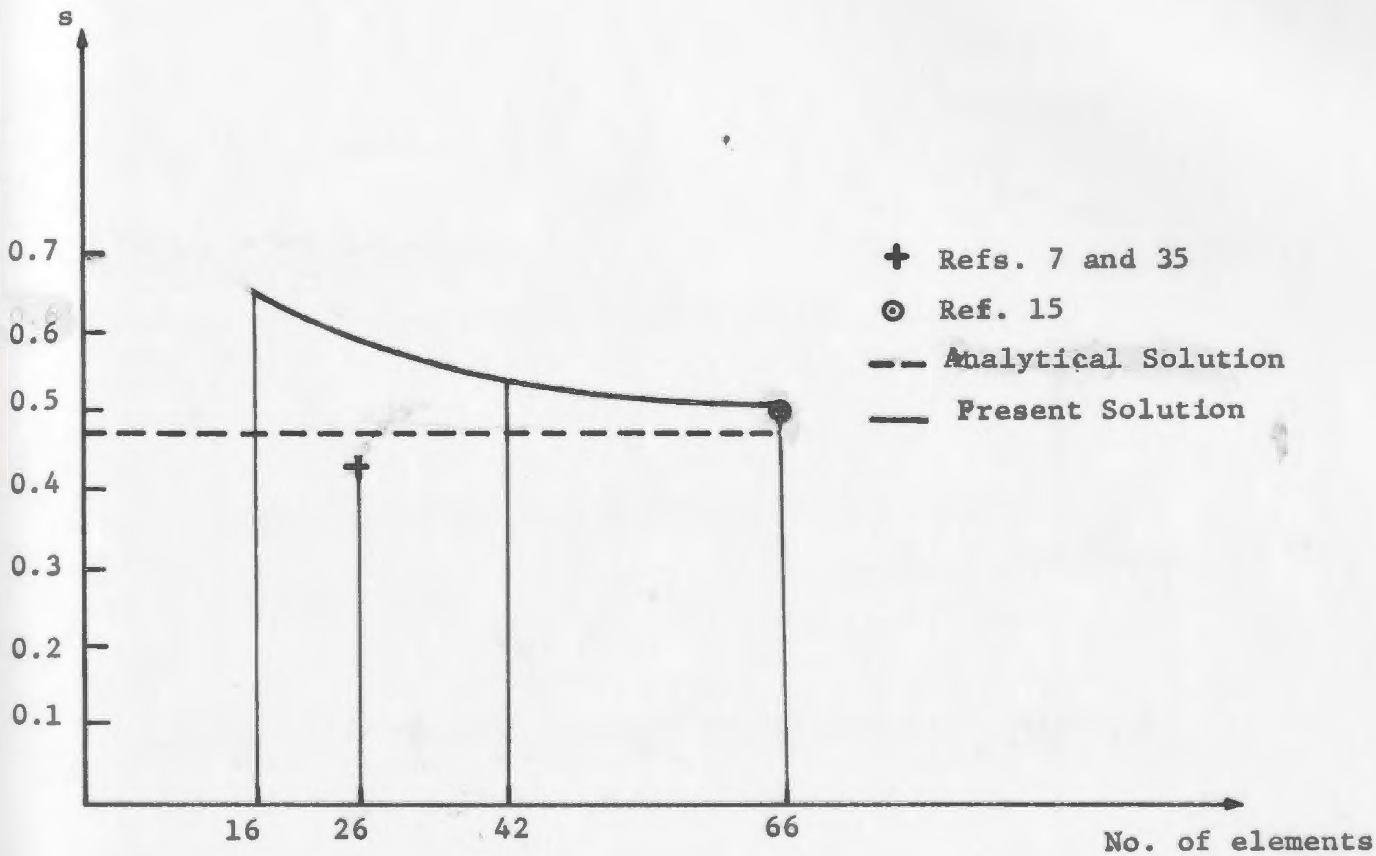

Fig. 5.6: Relation between the Number of Elements and the Shakedown Load Factors Compared to Available Solutions

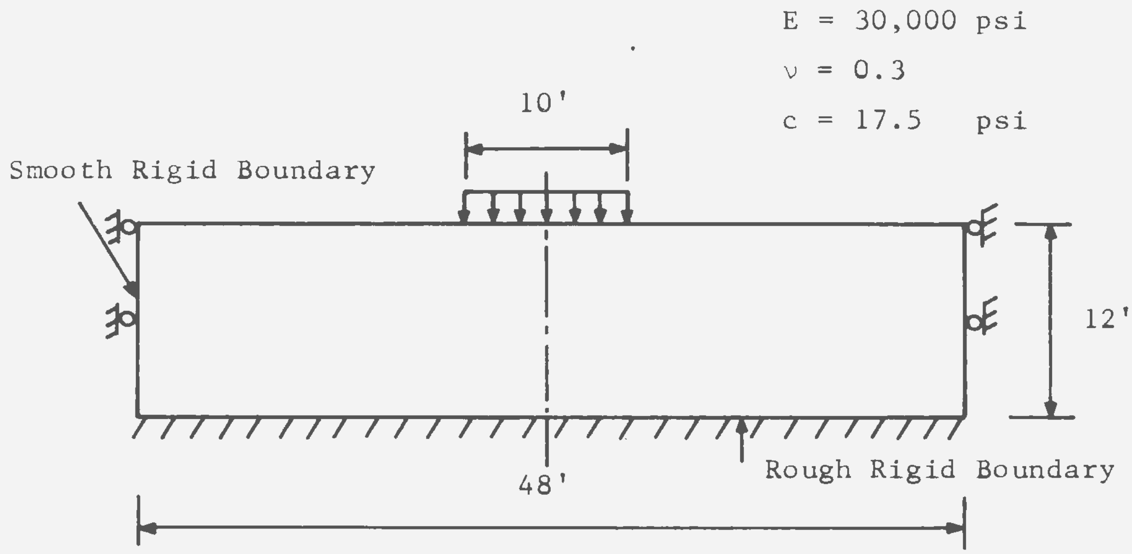

Fig. 5.7: Shallow Stratum of Undrained Clay (Ref. 10)

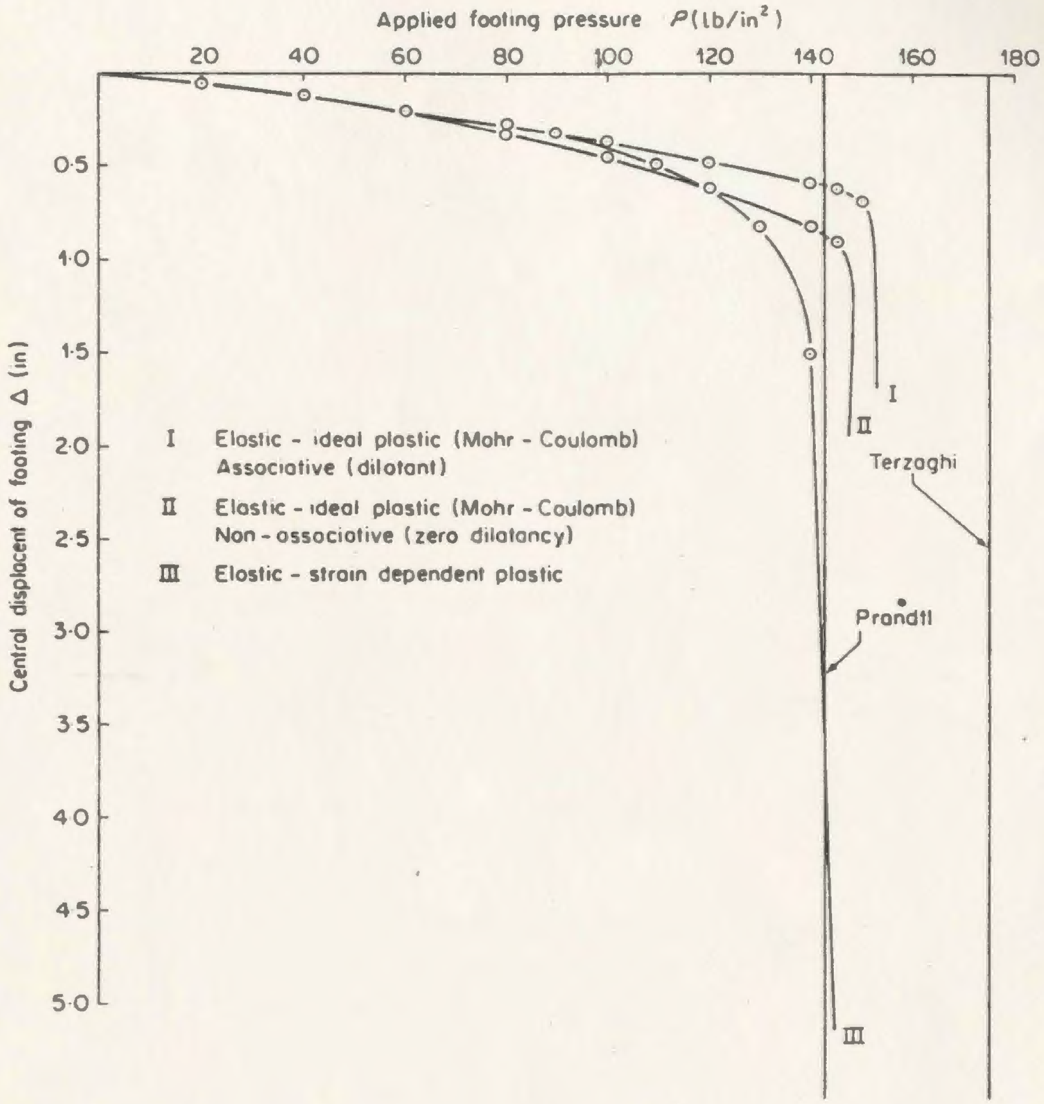

Fig. 5.8: Load Deformation Characteristics for Drained Behaviour Plane Footing (Ref. 83)

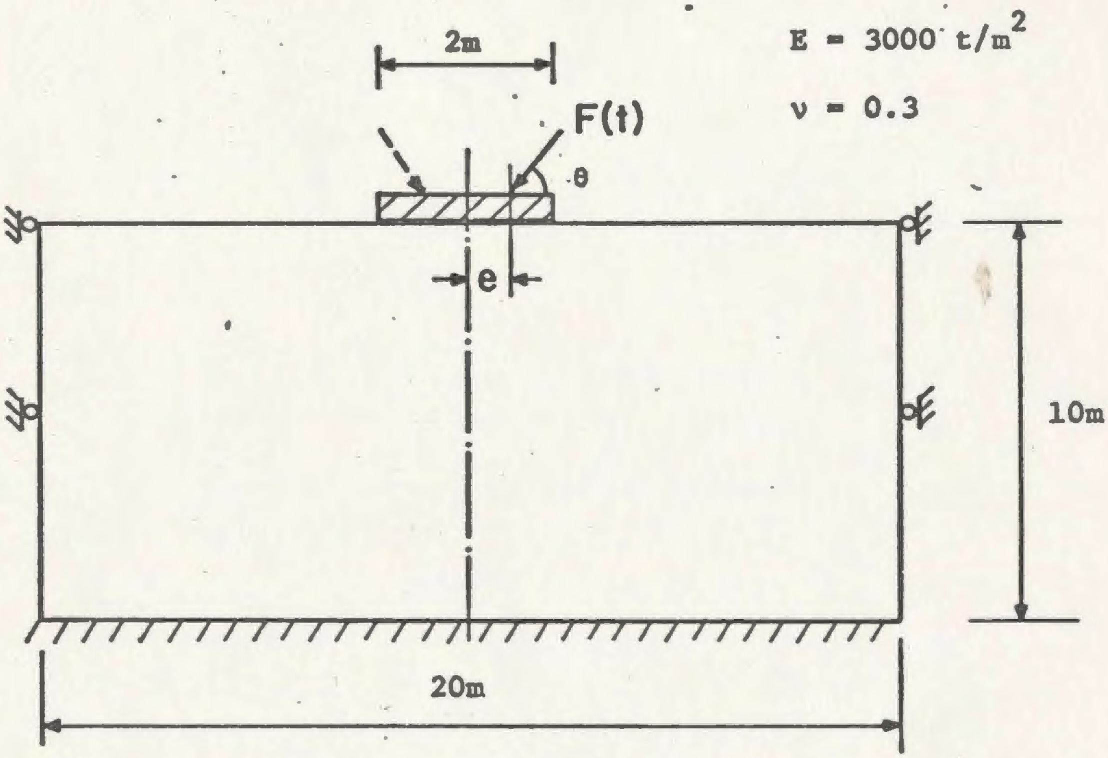

Fig. 5.9: Plane Strain Footing Subjected to Variable Repeated Loading

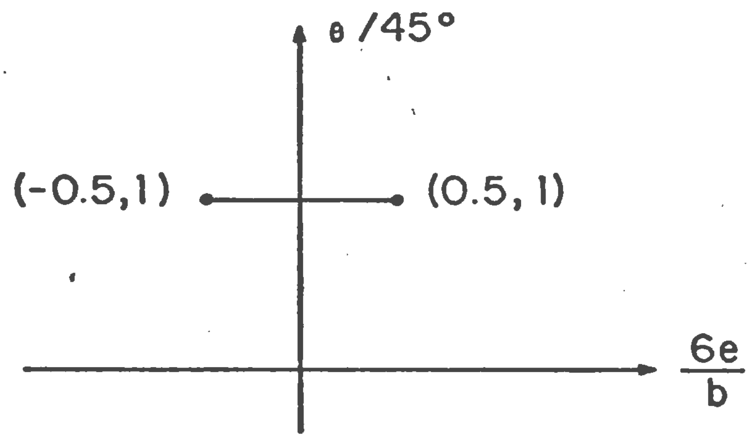

Fig. 5.10: Loading Programme for the Footing Problem

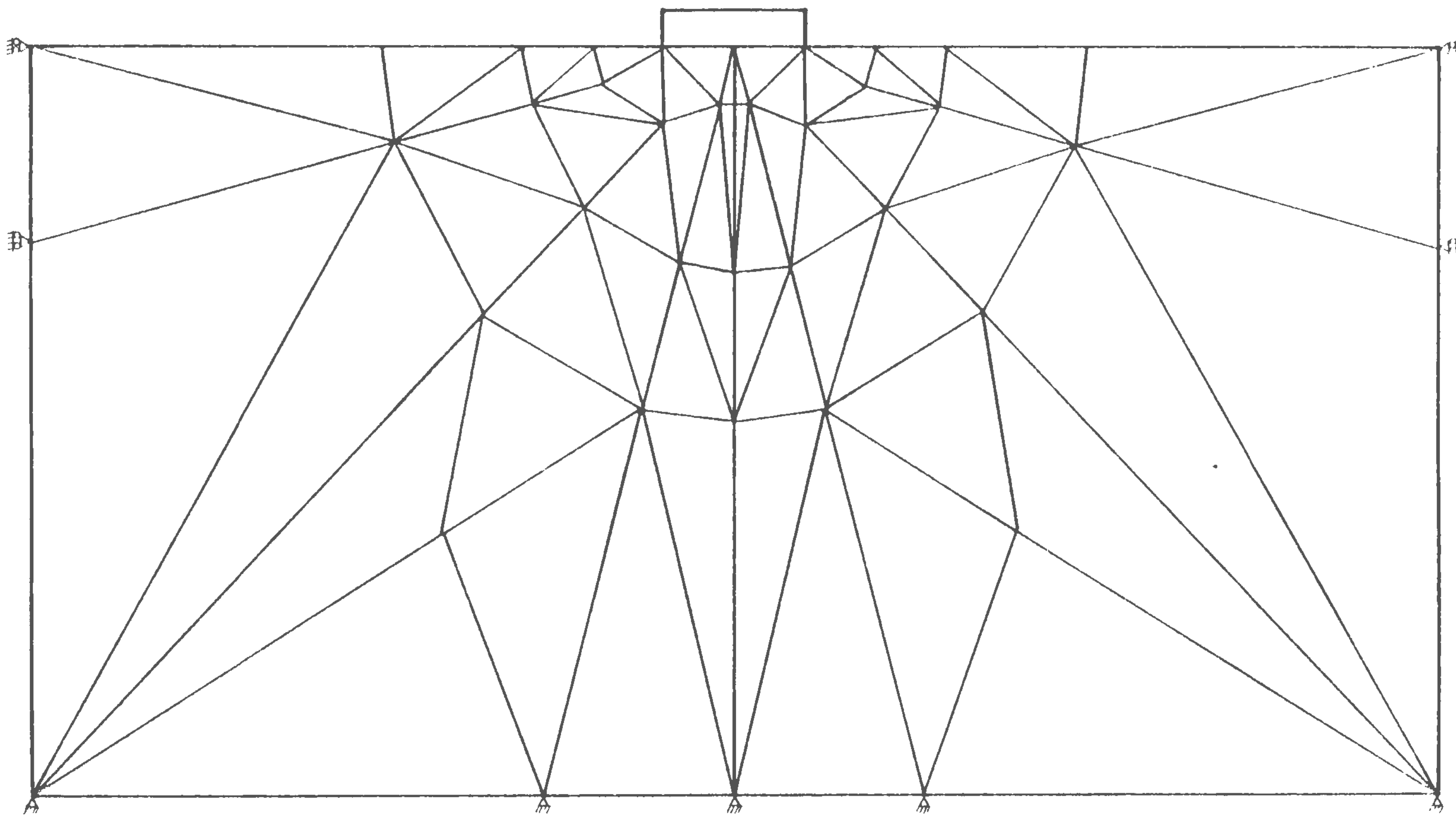

Fig. 5.11: Finite Element Mesh of 62 Elements for Footing Shakedown

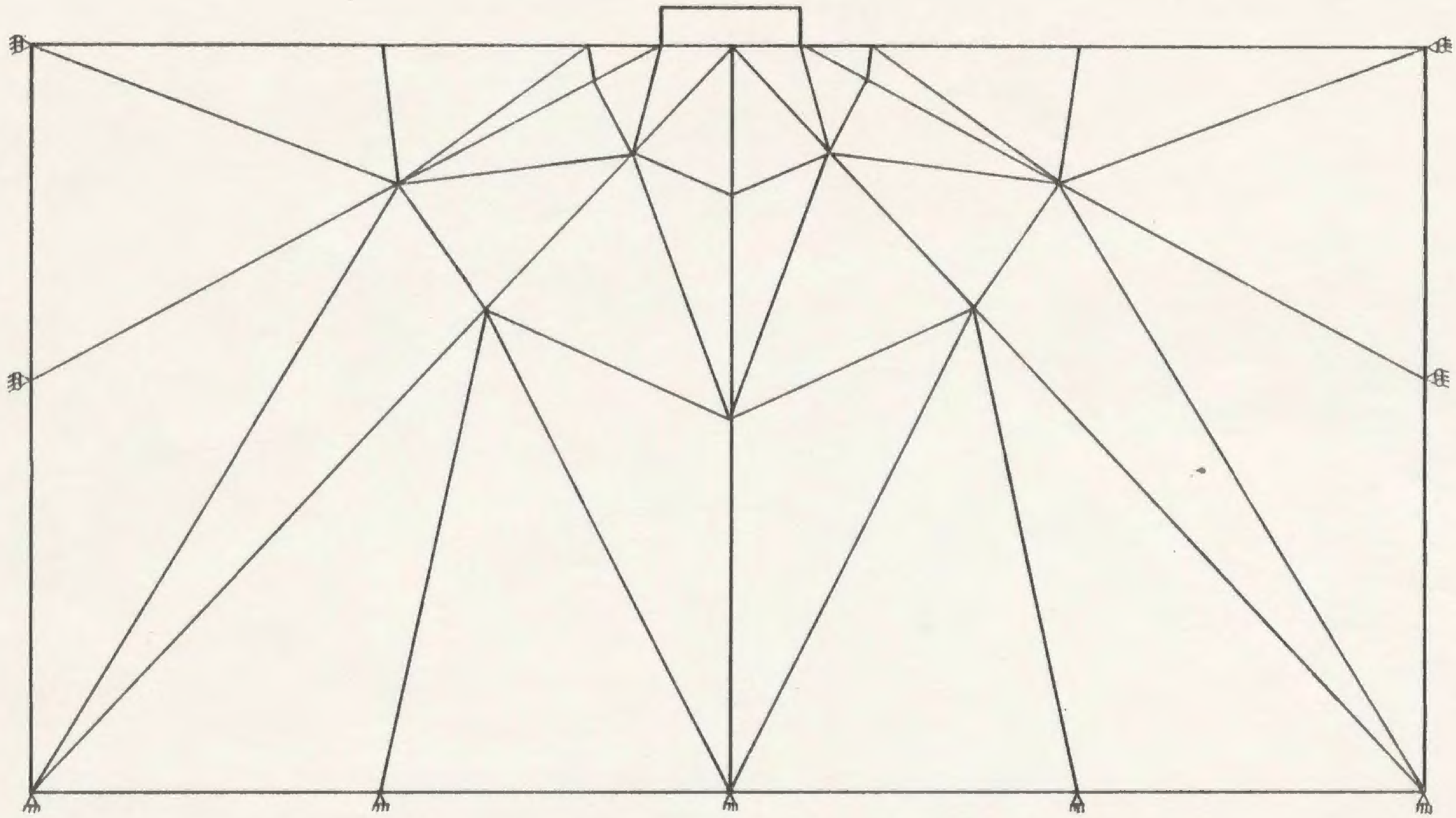

Fig. 5.12: Finite Element Mesh of 34 Elements for Footing Shakedown

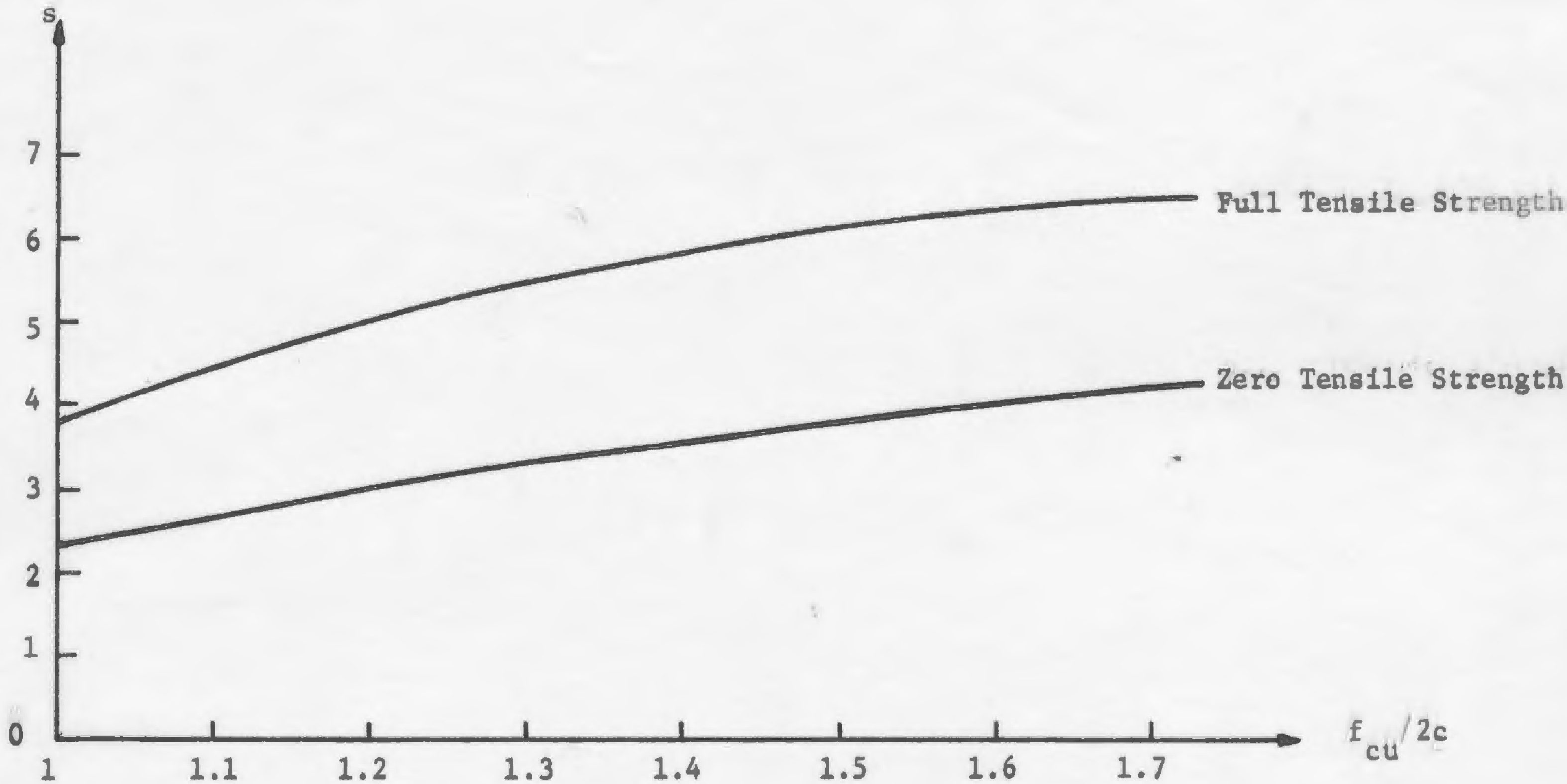

Fig. 5.13: Shakedown Load Factor Vs. $f_{cu}/2c$ for Zero and Full Tensile Strength

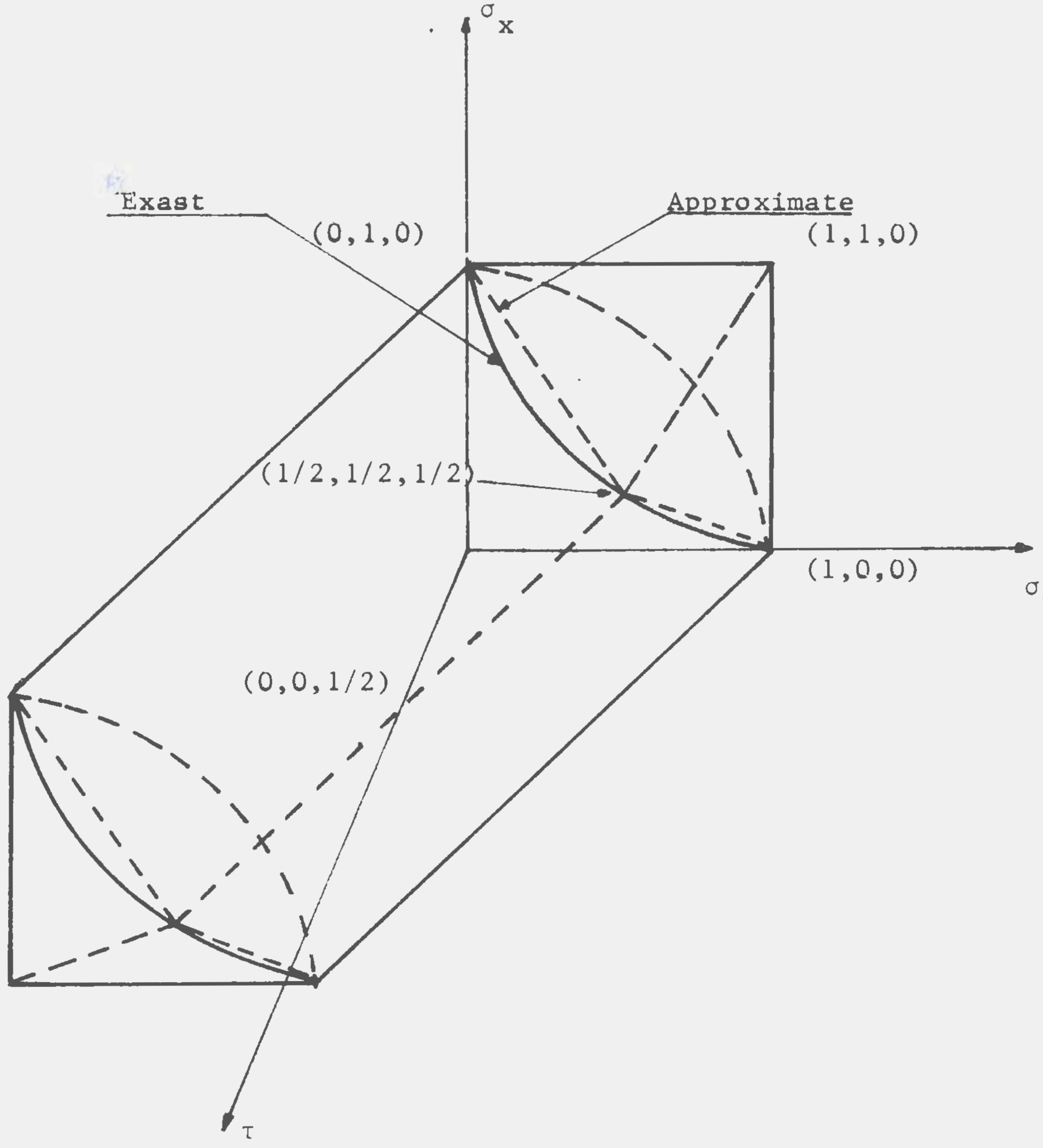

Fig. B.1 : Linearized Tresca Yield Criterion
in Plane Stress (Ref. 44)

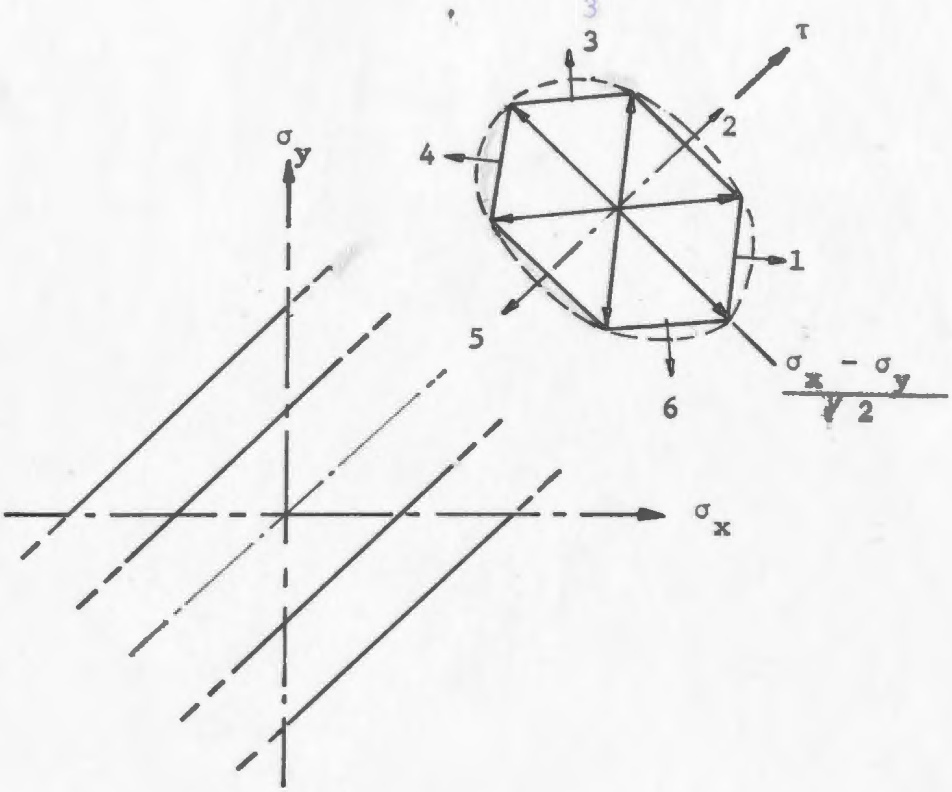

Fig. B.2: Linearized Tresca Yield Criterion in Plane Strain (Ref. 3)

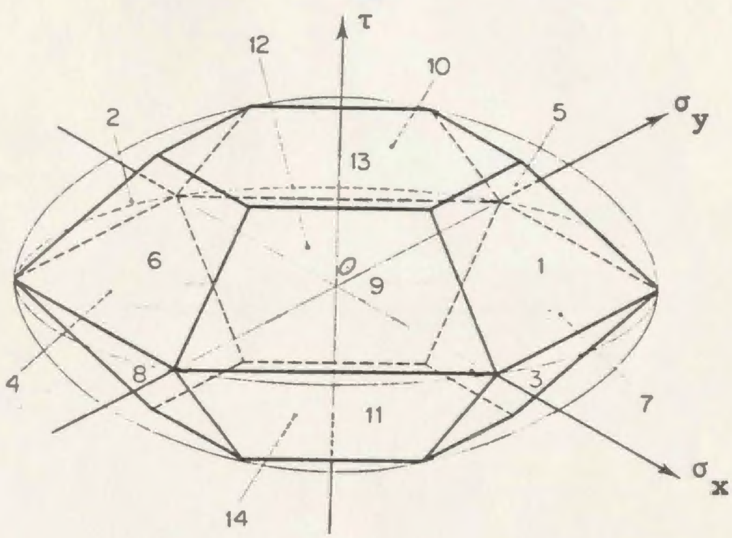

Fig. B.3: Linearized Von Mises Yield
Criterion in PlanecStress (Ref. 54)

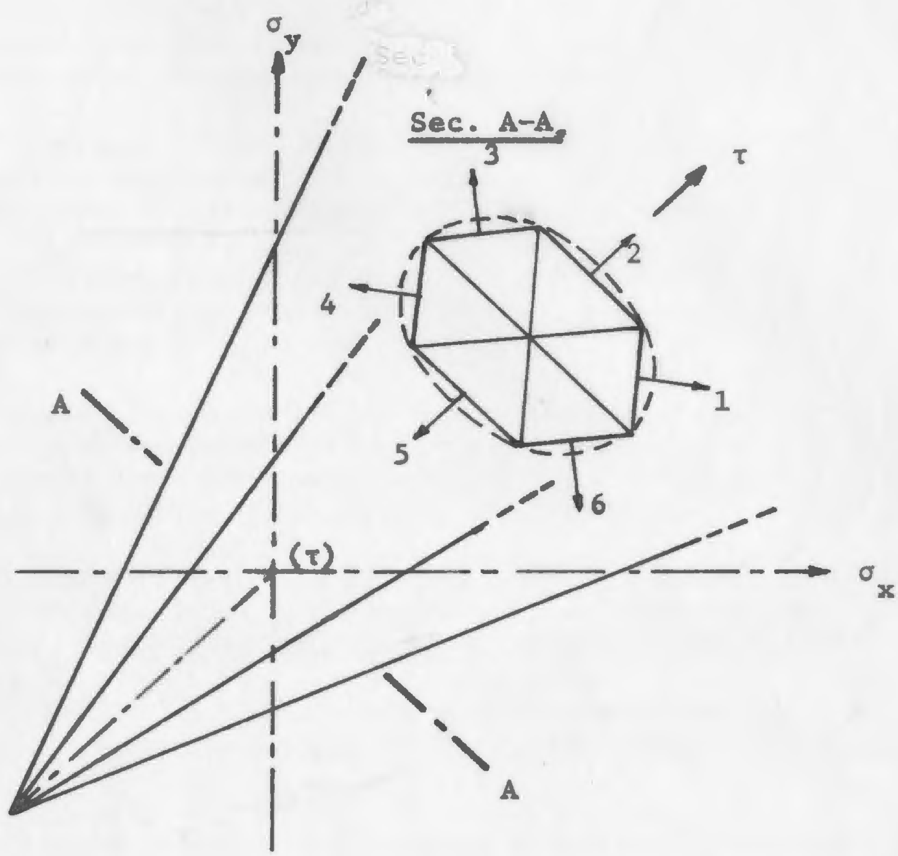

Fig. B.4: Linearized Mohr-Coulomb Yield Criterion in Plane Strain (Ref. 3)

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APPENDIX A

LINEAR PROGRAMMING (LP) PROBLEM

Introduction

The following are some definitions and properties associated with a LP problem. More details such as computation procedures can be found in Refs. 27, 45, 46, 60 and 64.

1) Standard Form of the LP Problem

Minimize the objective function

$$z = \{C\}^t_{1xn} \{x\}_{1xn} \tag{A.1}$$

subject to m constraints of the form,

$$[A]_{mxn} \{x\}_{1xn} = \{b\}_{1xm} , \tag{A.2}$$

and subject to the non-negativity conditions,

$$x_j \geq 0 \quad (j = 1, . . . n) , \tag{A.3}$$

in which C_j, (j = 1, . . . n) are termed the cost coefficients and x_j, (j = 1, . . . n) the real or structural variables.

2) Slack Variables

If a given constraint is an inequality

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i , \tag{A.4}$$

then, defining a slack variable, $x_{n+i} \geq 0$, such that

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \tag{A.5}$$

makes the inequality an equality. Similarly, if the inequality is

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad , \tag{A.6}$$

then

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i \tag{A.7}$$

3) Artificial Variables

If the constraints are in the form of equalities, it is necessary to add an artificial variable to each equation without a slack variable.

$$\sum_{j=1}^{n} a_{ij} x_j + v_{n+i} = b_i \tag{A.8}$$

4) Solution Algorithms

After adding either the slack variables or the artificial variables, the constraints are transformed into a canonical form. Having obtained the canonical form, there still remains the problem of finding an initial basic feasible solution by eliminating the artificial variables; then the search is carried out for the optimal solution.

5) Definitions

a) A feasible solution to the LP problem is a vector {x} which satisfies Eqn. A.2 and the non-negativities, Eqn. A.3.

b) A basic matrix is an (mxn) non-singular matrix formed from some m columns of the constraint matrix [A].

c) A basic solution to a LP problem is the unique vector determined by choosing a basic matrix, setting the n-m variables associated with the columns of A, not in the basic matrix, equal to zero, and solving the resulting square, non-singular system of equations for the remaining m variables.

d) A basic feasible solution is a basic solution in which all variables have non-negative values.

e) An optimal solution is a feasible solution which also minimizes z in Eqn. A.1.

6) Simplex Algorithm

The simplex method is a two-phase procedure for finding an optimal solution to LP problems. Phase I finds an initial basic feasible solution if it exists, or, gives the information that it does not exist (in this case the constraints are inconsistent and the problem has no solution). Phase II uses this solution as a starting point and either finds a minimizing solution or yields the information that the minimum is unbounded.

7) The Revised Simplex Algorithm

This algorithm updates the inverse of the basic matrix. The input tableau remains unaltered, in contrast to the standard simplex algorithm which updates the tableau itself.

8) Duality

Each LP maximization problem has its corresponding dual, a minimization problem, and vice versa. It is possible to solve either the original problem (called the primal), or the dual, to obtain the desired answer.

If the primal problem is given as

$$\begin{aligned} &\text{maximize} && z = \{C\}^t_{1xn}\ \{x\}_{1xn} \ , \\ &\text{subject to} && [A]_{mxn}\ \{x\}_{1xn} \leq \{b\}_{1xm} \ , \\ &\text{and} && x_i \geq 0, \quad (i = 1, \ldots n) \end{aligned} \qquad (A.9)$$

the primal problem can be written as

$$\begin{aligned} &\text{minimize} && s = \{b\}^t_{1xm}\ \{y\}_{1xm} \ , \\ &\text{subject to} && [A]^t_{nxm}\ \{y\}_{1xm} \leq \{C\}_{1xn} \ , \\ &\text{and} && y_j \geq 0, \quad (j = 1, \ldots m) \end{aligned} \qquad (A.10)$$

There are several interesting relationships between the optimal solutions for both the primal and the dual problems:

(i) Minimum value of s = Maximum value of z

(ii) If a slack variable occurs in the kth constraint of either system of equations, then the kth variable of its dual vanishes.

(iii) If the kth constraint in the primal is an equality then, the kth variable of its dual is unrestricted in sign.

(iv) The coefficients of the kth slack variables in the objective function row of the optimal tableau in either system correspond to the optimal values of the kth variable of its dual, i.e., the solution of either the dual or the primal problem, gives both the dual and the primal variables.

APPENDIX B

LINEARIZED YIELD SURFACES

Introduction

Different yield criteria for plane stress and plane strain are presented. The yield surface is piecewise linearized into a polyhedron in the stress space limited by a suitable number of yield planes depending upon the accuracy desired for a specified problem. The following linearized yield surfaces were used in limit analysis, plastic design, and shakedown analysis.

1) Tresca Yield Criterion

Yielding occurs when the maximum value of the extremum shear stress attains a critical value, i.e.,

$$\tau_{max} = c \ , \tag{B.1}$$

and

$$c = \sigma_o/2 \ , \tag{B.2}$$

where c is the yield stress in pure shear, and σ_o is the yield stress in simple tension, Ref. 70.

a) Plane Stress

The yield surface in this case is an elliptical cylinder bounded by two cones. The elliptical cylinder is governed by the equation, Ref. 3,

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = 4c^2 \tag{B.3}$$

Its axis is the line $\sigma_x = \sigma_y$ in the plane $\tau = 0$. The two cones have the equations

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = [4c \pm (\sigma_x + \sigma_y)]^2 \tag{B.4}$$

The apexes are on the lines $\sigma_x = \sigma_y = \pm 2c$.

This yield surface is piecewise linearized, Fig. B.1, into a polyhedron of 12 yield planes and 10 corners. The equation of each yield plane is expressed in terms of the direction cosines of each yield plane $\{N_j^i\}$, and the normal distance from the stress origin to each yield plane $\{K_{oj}^i\}$. For the 12 yield plane polyhedron, the matrix $[N^i]$ and the vector $\{K_o^i\}$ will be as follows, Refs. 3, 22 and 44:

$$[N^i] = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 2 & 2 & -2 & -2 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}_{3x12} \tag{B.5}$$

and

$$\{K_o^i\} = \{2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\ 2c\}^t_{1x12} \tag{B.6}$$

b) <u>Plane Strain</u>

The yield surface in this case is an elliptical cylinder with the equation, Refs. 3 and 70,

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = 4c^2 \tag{B.7}$$

In Fig. B.2, this yield surface is piecewise linearized

$$[N^i] = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 2/\sqrt{3} & 4/\sqrt{3} & 2/\sqrt{3} & -2/\sqrt{3} & -4/\sqrt{3} & -2/\sqrt{3} \end{bmatrix}_{3x6} \quad (B.8)$$

and

$$\{K_o^i\} = \{2c \quad 2c \quad 2c \quad 2c \quad 2c \quad 2c\}^t_{1x6} \quad (B.9)$$

2) Von-Mises Yield Criterion

This yield criterion assumes that yielding begins when the distortional energy equals that for yield in simple tension. The relation between the yield stress in pure shear and in simple tension, Ref. 59, is

$$c = \sigma_o/\sqrt{3} \quad (B.10)$$

a) Plane Stress

The yield surface in this case is an ellipsoid with the equation, Ref. 70,

$$\sigma_x^2 + \sigma_y^2 - \sigma_x\,\sigma_y + 3\tau^2 = \sigma_o^2 \quad (B.11)$$

In Fig. B.3,this yield surface is piecewise linearized by a polyhedron of 14 yield planes and 18 corners, Ref. 54, as

$$[N^i] = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}_{3x14}$$

(B.12)

and

$$\{K_o^i\} = \sigma_o\{1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0.5 \ 0.5\}^t_{1x14}$$

(B.13)

b) Plane Strain

The yield surface in this case is an elliptic cylinder with the equation

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = 4c^2 \qquad (B.14)$$

This equation is the same as that for the Tresca yield criterion in terms of the value c; the linearized yield surface will be the same as that defined by Eqs. B.8 and B.9.

3) Mohr-Coulomb Yield Criterion

This criterion is defined by

$$\tau = c + \sigma \tan\phi \qquad (B.15)$$

where τ and σ are the shear, and normal stresses, ϕ is the angle of internal friction, and c is the cohesive strength. The Tresca yield crition is a particular case of the Mohr-Coulomb yield criterion for $\phi = 0$, and the yield stress in pure shear is equal to the cohesive strength; the purely cohesive soil is called the Tresca Material, Ref. 10. For the plane strain case, the yield surface will be an elliptic cone with the equation, Refs. 3 and 16,

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = [2c \,\mathrm{Cos}\phi + (\sigma_x + \sigma_y)\,\mathrm{Sin}\phi]^2 \qquad \text{(B.16)}$$

It is worth noting that this equation represents the Drucker-Prager yield criterion in plane strain, Ref. 18, and for $\phi = 0$ material will particularize to the Tresca yield criterion and also the Von Mises yield criterion in terms of c. In Fig. B.4, this yield surface is approximated by 6 yield planes, Ref. 3, as follows:

$$[N^i] = \begin{bmatrix} 1-s & -s & -1-s & -1-s & -s & 1-s \\ -1-s & -s & 1-s & 1-s & -s & -1-s \\ 2/\sqrt{3} & 4/\sqrt{3} & 2/\sqrt{3} & -1/\sqrt{3} & -4/\sqrt{3} & -2/\sqrt{3} \end{bmatrix} \frac{1}{\cos\phi} ,$$

(B.17)

and

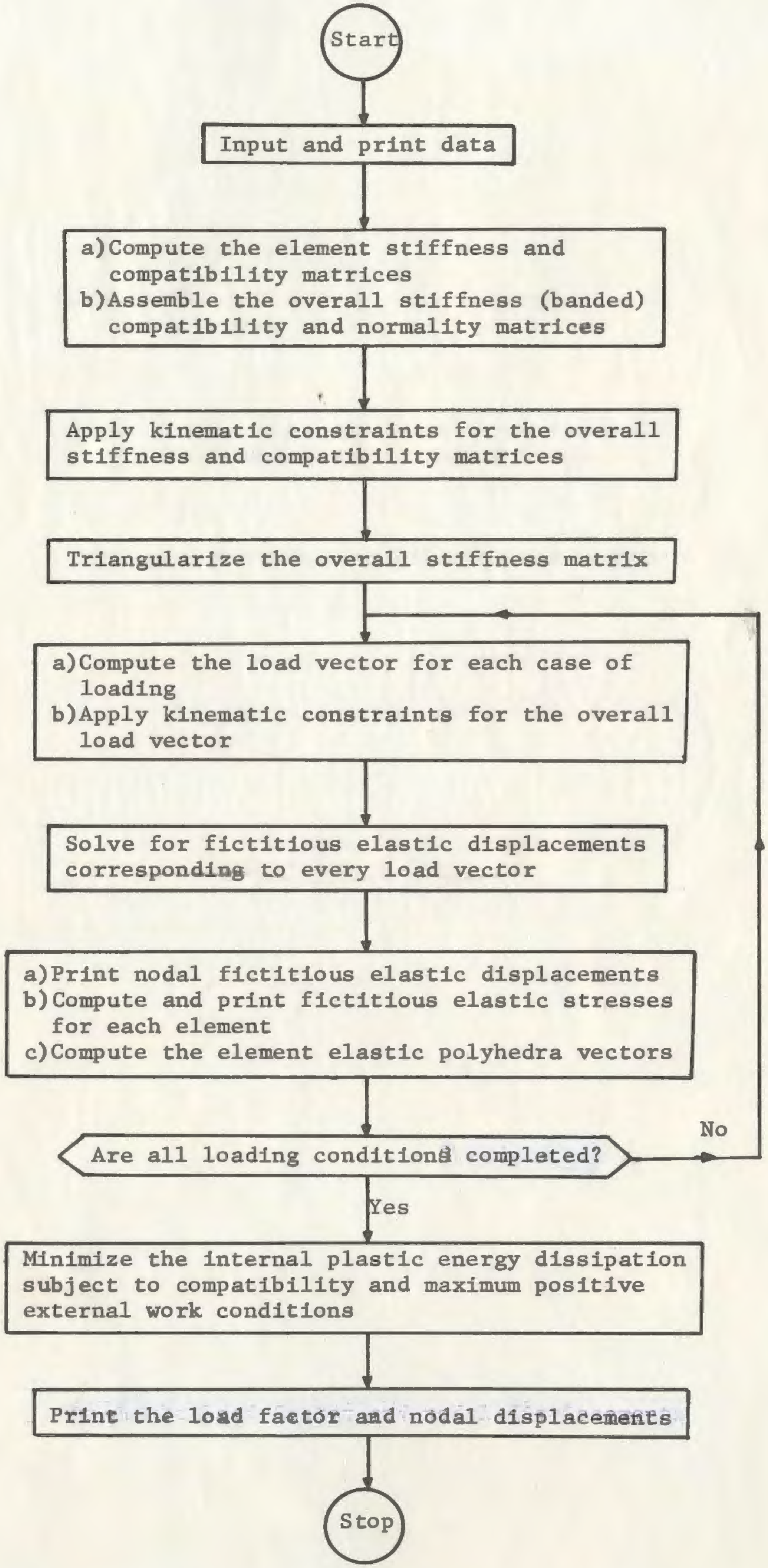
$$\{K_o^i\} = \{2c \quad 2c \quad 2c \quad 2c \quad 2c \quad 2c\}^t , \qquad \text{(B.18)}$$

where $s = \sin\phi$

Tension cut-off can be applied by adding two planes $\sigma_x = 0$ and $\sigma_y = 0$, Ref. 84.

APPENDIX C

FLOW CHART FOR THE COMPUTER PROGRAMME


Start
Input and print data
a)Compute the element stiffness and compatibility matrices
b)Assemble the overall stiffness (banded) compatibility and normality matrices
Apply kinematic constraints for the overall stiffness and compatibility matrices
Triangularize the overall stiffness matrix
a)Compute the load vector for each case of loading
b)Apply kinematic constraints for the overall load vector
Solve for fictitious elastic displacements corresponding to every load vector
a)Print nodal fictitious elastic displacements
b)Compute and print fictitious elastic stresses for each element
c)Compute the element elastic polyhedra vectors
Are all loading conditions completed?
No
Yes
Minimize the internal plastic energy dissipation subject to compatibility and maximum positive external work conditions
Print the load factor and nodal displacements
Stop