APPLICATIONS OF GROUPOID AND OTHER TECHNIQUES TO GLUEING AND CO-GLUEING THEOREMS IN THE CATEGORY OF TOPOLOGICAL PAIRS

SISTER ROSALITA A. FUREY

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BY

SISTER ROSALITA FUREY ©

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INTRODUCTION

The purpose of this thesis is two-fold - first to develop a theory of pullbacks and pushouts, fibrations and cofibrations in the category Toppair of topological pairs and continuous maps - and second to develop a groupoid technique for pairs as an aid to proving the Glueing (c.f. Theorem 4.5.9) and Co-glueing (c.f. Theorem 3.3.1) Theorems for Toppair.

We remark that the theory which we devise in order to formalize the proofs of Theorems 3.3.1 and 4.5.9, sets up machinery which would allow one to specialize similar results of Kamps.

Our study of groupoids leads us to examine and partially present a manuscript of Brown [3]. As an application of the theory we develop, we show how a Mayer-Victoris type sequence of groupoids arises from a fibration of topological pairs. As a further application we show how this same type of sequence of groupoids arises from a homotopy pullback of pairs.

We would like to emphasize that many of the Toppair results given throughout the thesis restrict to similar results in the category Top and more importantly in Top_*. Regarding the latter category much of the literature completely ignores base-points and very often equates free and based results. Our theory attempts to make precise conditions which avoid this type of reasoning.

CHAPTER I - INTRODUCTION

In section one we state the basic definitions of category and functor and give examples. Sections two and three are devoted to showing how the concept of homotopy makes precise the concept of continuous deformation. This leads us to the homotopy category of topological pairs and homotopy equivalence of pairs. In section four we examine the notions of initial and final topologies and show how these can be used to define the universal constructions of pullbacks and pushouts in the category of topological spaces (Top). Section five exhibits the existence of an exponential law in the category Toppair.

§1.1 CATEGORY THEORY

Definition 1.1.1: A category C consists of

a) A class of objects

b) For every ordered pair of objects X and Y, a set $Mor(X,Y)$ of morphisms with domain X, range Y. If f belongs to $Mor(X,Y)$, we write $f : X \to Y$ or $X \xrightarrow{f} Y$ and if $Mor(X,Y) \cap Mor(X',Y') \neq \phi$, then $X = X'$, $Y = Y'$.

c) For every ordered triple of objects X, Y, Z, function associating to a pair of morphisms $f : X \to Y$, $g : Y \to Z$ their composite $gf : X \to Z$.

These satisfy the following two axioms:

(i) Associativity - If $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$ then $h(gf) = (hg)f : X \to W$.

(ii) Identity - For every object Y there is a morphism $1_Y : Y \to Y$ such that if $X \xrightarrow{f} Y$, then $1_Y f = f$ and if $h : Y \to Z$, $h1_Y = h$.

In particular if the class of objects is a set, the category is said to be SMALL.

Examples of categories include the following:

1) Set - Objects - Sets
   Morphisms - Functions

2) Top - Objects - Topological spaces
   Morphisms - Continuous functions = maps

3) Top_* - Objects - Topological spaces with base points
   Morphisms - Base-point preserving maps

4) G — Objects - Groups
Morphisms - Homomorphisms

Definition 1.1.2: Let C and $\mathcal{D}$ be categories. A covariant (contravariant) functor T from C to $\mathcal{D}$ consists of an object function which assigns to every object X of C an object $T(X)$ of $\mathcal{D}$ and a morphism function which assigns to every morphism $f : X \to Y$ of C a morphism $T(f) : T(X) \to T(Y)$ $(T(f) : T(Y) \to T(X))$ of $\mathcal{D}$ such that:

a) $T(1_X) = 1_{T(X)}$.

b) $T(gf) = T(g)T(f)$ (or $T(gf) = T(f)T(g)$).

Examples of each type of functor follow -

1) A covariant functor T : Top $\to$ Set. Here each topological space is sent via T to its underlying set, and each map $f : X \to Y$ in Top is sent to its underlying function $f : X \to Y$ in Set. This is called the FORGETFUL FUNCTOR.

2) For any category C there is a contravariant functor to its opposite category C^* which assigns to an object X of C the object X^* of C^* and to a morphism $f : X \to Y$ in C the morphism

$$f^* : Y^* \to X^*.$$

From now on unless otherwise specified, functors will be covariant.

Proposition 1.1.3: Let T be a functor from a category C to a category $\mathcal{D}$. Then T maps isomorphisms in C to isomorphisms in $\mathcal{D}$.

Proof: Assume that T is a covariant functor (the argument is similar for a contravariant functor). Let $f : X \to Y$ be an isomorphism in C. Then $ff^{-1} = 1_X$. Therefore,

$$1_{T(X)} = T(1_X) = T(f^{-1}f) = T(f^{-1})T(f).$$

So $T(f^{-1})$ is left inverse for $T(f)$. Similarly $T(f)T(f^{-1}) = 1_{T(Y)}$. Thus, $T(f^{-1})$ is a two-sided inverse of $T(f)$. Hence $T(f)$ is an isomorphism in $\mathcal{D}$. //

1.2 ELEMENTARY CONCEPTS IN HOMOTOPY THEORY

We recall the following points from HOMOTOPY THEORY.

Definition 1.2.1: A TOPOLOGICAL PAIR (X,A) consists of a topological space X and a subspace $A \hookrightarrow X$. If A is empty, i.e. if $A = \phi$, we shall not distinguish between the pair (X,ϕ) and the space X.

Definition 1.2.2: A map $f : (X,A) \to (Y,B)$ between pairs is a continuous function $f : X \to Y$ such that $f(A) \subseteq B$. We write f_0 for f restricted to A, i.e. $f_0 = f|A$, throughout the thesis.

Definition 1.2.3: Let X, Y be objects of Top and $\{h_t\}$, t belonging to I, the unit interval, be a family of maps. The family $\{h_t\}$ is said to be continuous if and only if the function $H : X \times I \to Y$ defined by $H(x,t) = h_t(x)$ for all x in X, for all t in I, is continuous. A continuous family $\{h_t\}$ or equivalently the map H is usually called a HOMOTOPY.

Let $f, g : X \to Y$ be maps. We say that f is homotopic to g, denoted $f \simeq g$, if and only if there exists a continuous map $H : X \times I \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$. We write

$H : f \simeq g$ and say that f is "continuously deformed" into g.

Proposition 1.2.4: Any two maps $f, g : X \to R^N$ [R^N is Euclidean N-space] from a space X into Euclidean N-space are homotopic.

Proof: We simply define $H(x,t) = (1-t)f(x)+g(x)$ for all x in X, and for all t in I. //

Proposition 1.2.5: The relation $\simeq$ between maps of X into Y is an equivalence relation.

Proof: We prove the proposition for a more general case in Proposition 3.1.2. //

It follows from the above proposition that the maps from X to Y are divided into disjoint equivalence classes called HOMOTOPY CLASSES.

Definition 1.2.6: Two maps $f, g : (X,A) \to (Y,B)$ are homotopic as maps of pairs if there exists a map

$$F : (X\times I,\ A\times I) \to (Y,B)$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Note that this says that in continuously deforming f into g, it is required that at each stage we map A into B.

To illustrate the difference between maps being absolutely homotopic and homotopic as maps of pairs, consider the following:

Example 1.2.7: Let $(X,A) = (I,\dot{I})$; $(Y,B) = (S^1,\{1\})$ [$\dot{I} = \{0,1\}$].

Define $f : I \to S^1$ by $f(s) = e^{2\pi is}$ for all s in I and $g : I \to S^1$ by $g(s) = 1$ for all s in I. It is obvious that f and g are both maps of pairs since $f(0) = e^0 = 1 \subseteq \{1\}$ and $f(1) = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 \subset \{1\}$ and $g(0) = g(1) = 1 \subseteq \{1\}$.

To see that f and g are absolutely homotopic we define $H : I \times I \to S^1$ by $H(s,t) = e^{(1-t)2\pi is}$ for all t in I, s in I. This is a continuous map, and $H(s,0) = e^{2\pi is} = f(s)$ and $H(s,1) = e^0 = 1 = g(s)$. However, under H, $I \times I$ need not be included in $\{1\}$. Now $\{0\} \times I$, under H is always inside $\{1\}$ but $\{1\} \times I$ under H need not be inside $\{1\}$. For example if $t = \frac{1}{2}$, $H(1,\frac{1}{2}) = e^{\pi i} = \cos \pi + i \sin \pi = -1 + 0 = -1 \not\subset \{1\}$. Hence H is not a homotopy of pairs. We say that it is an absolute homotopy between f and g but not a relative homotopy between f and g.

We remark here that by taking $A = B = \phi$, the pair concepts reduce to concepts in Top; in this sense, the concept of homotopic pairs is more general than that of homotopic spaces.

Definition 1.2.8: Given a pair (X,A), we let $(X,A) \times I$ denote the pair $(X \times I, A \times I)$. Let $X' \subset X$ and suppose that

$$f', f : (X,A) \to (Y,B) \text{ agree}$$

on X' [that is, $f'|X' = f|X'$]. Then f' is homotopic to f relative to X', denoted $f' \simeq f \text{ rel } X'$, if there exists a map

$$F : (X,A) \times I \to (Y,B) \text{ such that } F(x,0) = f'(x),$$

$F(x,1) = f(x)$, for all x in X and $F(x,t) = f'(x)$ for all x in X' and t in I. Such a map F is called a HOMOTOPY RELATIVE to X', denoted $F : f' \simeq f$ rel X'. If $X' = \phi$, we omit the phrase "relative to ϕ".

Example 1.2.9: Let $X = Y = R^N$ and define $f'(x) = x$, $f(x) = 0$ for all x in R^N, i.e. $f' = 1_{R^N}$, and f is the constant map of R^N to its origin. Then if $F : R^N \times I \to R^N$ is defined by $F(x,t) = (1-t)x$, $F : f' \simeq f$ rel 0.

Example 1.2.10: Let X be an arbitrary space and let Y be a convex subset of R^N. Let f', $f : X \to Y$ be maps which agree on some subspace $X' \subseteq X$. Then $f' \simeq f$ rel X' because the map

$$F : X \times I \to Y \text{ defined by } F(x,t) = tf_1(x) + (1-t)f'(x)$$

is a homotopy relative to X' from f' to f.

THEOREM 1.2.11: Homotopy relative to X' is an equivalence relation in the set of maps from (X,A) to (Y,B).

Proof: (Reflexivity) For $f : (X,A) \to (Y,B)$ define $F : f \simeq f$ rel X by $F(x,t) = f(x)$ for all x in X, t in I.

(Transitivity) Given $F : f' \simeq f$ rel X', $G : f \simeq f''$ rel X' define $H : f' \simeq f''$ rel X' by

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Now H is continuous since its restriction to each of the closed sets $X \times [0, \frac{1}{2}]$, $X \times [\frac{1}{2},1]$ is continuous.

(Symmetry) Given $F : f \simeq f'$ rel X. Define $H : f' \simeq f$ rel X by $H(x,t) = F(x,1-t)$. //

Definition 1.2.12: If $H : f \simeq g : (X,A) \times I \to (Y,B)$ and $p : (Y,B) \to (Z,C)$ is a map such that pH equals a constant, then we say that f is homotopic to g rel p.

§1.3 THE CATEGORY OF TOPPAIR AND HOMOTOPY EQUIVALENCES

Example 1.3.1: We now give a further example of a category - the category which we call Toppair. Its objects are pairs of topological spaces (X,X_0) such that $X_0 \subseteq X$. Its morphisms are maps between such pairs as follows - $(X,X_0) \xrightarrow{f} (Y,Y_0)$ is a Toppair morphism if $f : X \to Y$ is a map such that $f(X_0) \subseteq Y_0$, i.e. the morphisms are sub-space preserving maps.

Definition 1.3.2: A map $f : (X,X_0) \to (Y,Y_0)$ in Toppair is a HOMOTOPY EQUIVALENCE if and only if there exists a map

$$g : (Y,Y_0) \to (X,X_0) \text{ such that } gf \simeq 1$$

and $fg \simeq 1$. If f is a homotopy equivalence we say that (X,X_0) and (Y,Y_0) are of the same HOMOTOPY TYPE.

Example 1.3.3: We give the following example to show that if A is homotopically equivalent to B and X is of the same homotopy type as Y and $(X,A) \to (Y,B)$ is a map of pairs, (X,A) need not be of the same homotopy type as (Y,B). Let $X = Y = S^1 \cup \{0\}$, let $y_0 = (1,0)$ in S^1. Then X is of the same homotopy type as Y and

$A = \{0\}$ is of the same homotopy type as $\{y_0\}$; yet there is no pair homotopy equivalence $f : (X, \{0\}) \to (Y, \{y_0\})$, since any $g : (Y, \{y_0\}) \to (X, \{0\})$ must send all of S^1 to $\{0\}$.

Lemma 1.3.4: If $f : (X, X_0) \to (Y, Y_0)$ is continuous in Toppair, then $f_0 : X_0 \to Y_0$ is continuous in Top.

Proof: Consider the following commutative diagram:

$$\begin{array}{ccc} X_0 & \overset{f_0}{\dashrightarrow} & Y_0 \\ {\scriptstyle i_{X_0}}\downarrow & & \downarrow{\scriptstyle i_{Y_0}} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Now Y_0 has the subspace topology with respect to Y; hence f_0 is continuous if and only if $i_{Y_0} f_0$ is continuous since the subspace topology on Y_0 coincides with the initial topology on Y_0 with respect to i_{Y_0}. But $i_{Y_0} f_0 = f i_{X_0}$ which is continuous - (composition of continuous functions is continuous). Hence the result. (See §1.4 for initial topology). //

Lemma 1.3.5: If $f : (A, A_0) \to (B, B_0)$, $g : (B, B_0) \to (C, C_0)$ are homotopy equivalences in Toppair, then gof is a homotopy equivalence.

Proof: Let $f' : (B, B_0) \to (A, A_0)$, $g' : (C, C_0) \to (B, B_0)$ be homotopy inverses of f, g respectively, i.e. $f'f \simeq 1_{(A,A_0)}$, $ff' \simeq 1_{(B,B_0)}$, $g'g \simeq 1_{(B,B_0)}$ and $gg' \simeq 1_{(C,C_0)}$. We require a homotopy inverse h of gf such that $(gf)h \simeq 1_{(A,A_0)}$ $h(gf) \simeq 1_{(C,C_0)}$. Now $f'g'$ is the likely candidate and $f'g'gf \simeq f'1_{(B,B_0)}f = f'f \simeq 1_{(A,A_0)}$ and $gff'g' \simeq g1_{(B,B_0)}g' = gg' \simeq 1_{(C,C_0)}$, so $f'g'$ is indeed a left and right homotopy inverse of gf and hence gf is a homotopy equivalence in Toppair. //

Proposition 1.3.6: If in the following commutative diagram in Toppair f, f_1 and g are homotopy equivalences, then so also is g_1.

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{g_1} & (B,B_0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ (C,C_0) & \xrightarrow[g]{} & (D,D_0) \end{array}$$

Proof: Consider the following diagram:

$$(A,A_0) \xrightarrow{g_1} (B,B_0) \xrightarrow{f_1} (D,D_0), \qquad (A,A_0) \xrightarrow{gf} (D,D_0)$$

where gf is a homotopy equivalence by Lemma 1.3.5. Now $gf = f_1g_1$ by commutativity of given diagram. Hence $gf \simeq f_1g_1$ by constant homotopy. The map f_1 is a homotopy equivalence (given). Let h be a homotopy inverse of gf, i.e., $(gf)h \simeq 1_{(D,D_0)}$ and $h(gf) \simeq 1_{(A,A_0)}$. But $f_1g_1 \simeq gf$ implies that $hf_1g_1 \simeq hgf \simeq 1_{(A,A_0)}$ which implies that $hf_1g_1 \simeq 1_{(A,A_0)}$. But this means that hf_1 is a left homotopy inverse of g_1. We still need to show that hf_1 is a right homotopy inverse of g, i.e., that $g_1hf_1 \simeq 1_{(B,B_0)}$. Since f_1 is a homotopy equivalence, there exists f_1', a homotopy inverse of f_1 such that $f_1'f_1 \simeq 1_{(B,B_0)}$ and $f_1f_1' \simeq 1_{(D,D_0)}$. Now $f_1'f_1 \simeq 1_{(B,B_0)}$ implies that $f_1'f_1g_1hf_1 \simeq 1_{(B,B_0)}g_1hf_1$. But this implies that $f_1'f_1g_1hf_1 \simeq g_1hf_1$ or $g_1hf_1 \simeq f_1'f_1g_1hf_1$. Now $f_1'f_1g_1hf_1 \simeq f_1'gfhf_1$ (since $f_1g_1 \simeq g_0f_0$) and $f_1'gfhf_1 \simeq f_1'1_{(D,D_0)}f_1 = f_1'f_1 \simeq 1_{(B,B_0)}$. Thus we actually have $g_1hf_1 \simeq 1_{(B,B_0)}$. Hence hf_1 is a right homotopy inverse of g_1 and so g_1 is indeed a homotopy equivalence. //

Note 1.3.7: We can also prove in the same proposition that if f, g_1, f_1 are given homotopy equivalences and the diagram of the proposition commutes, then g is a homotopy equivalence. In this case we have

$$(A,A_0) \xrightarrow{f} (C,C_0) \xrightarrow{g} (D,D_0), \quad f_1g_1 = gf$$

We can show that fh is a homotopy inverse for g where h is a homotopy inverse for f_1g_1. We notice that for the proposition to hold true we simply need to have f_1g_1 homotopic to gf.

§1.4 INITIAL AND FINAL TOPOLOGIES AND UNIVERSAL CONSTRUCTIONS

DEFINITION 1.4.1: Let X be a topological space with topology τ, Y a topological space with topology τ'. A function $f : X_\tau \to Y_{\tau'}$ is said to be continuous if $f^{-1}(U)$ is open in τ for all U open in τ'.

Note: X_τ means X with topology τ.

Proposition 1.4.2: The following are equivalent for f as defined above:

(i) f is continuous,

(ii) $f^{-1}(U)$ is open for all U in β where β is a basis for τ,

(iii) $f^{-1}(U)$ is open for each sub-basic U (U an element of a sub-basis).

Proof: (i) implies (ii) Trivial from definition

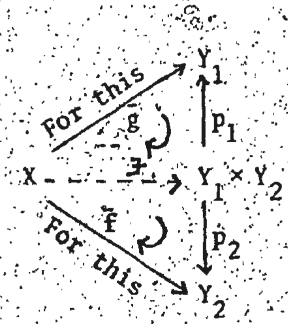
(ii) implies (i) Since $f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$

(i) if and only if (iii) Trivial from definition

(iii) implies (ii) Follows since $f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2)$. //

Definition 1.4.3: Given $\{X_\lambda\}$, where λ belongs to the indexing set Λ, an arbitrary family of topological spaces and $\{f_\lambda : X \to X_\lambda\}$, a family of functions. A topology I on X is said to be INITIAL with respect to $\{f_\lambda\}$ if it has the following property: for any topological space Y a function $Y \xrightarrow{g} X_I$ is continuous if and only if $f_\lambda g : Y \to X_\lambda$ is continuous for each λ in Λ.

Example 1.4.4: Let $\{X_\lambda\} = \{Y_1, Y_2\}$ be a family of topological spaces and let $X = Y_1 \times Y_2$, $\{f_\lambda\} = \{Y_1 \times Y_2 \xrightarrow{p_1} Y_1, Y_1 \times Y_2 \xrightarrow{p_2} Y_2\}$. Then the initial topology on $Y_1 \times Y_2$ with respect to p_i, $i = 1,2$, is the usual product topology on $Y_1 \times Y_2$



Proposition 1.4.5: If I is an initial topology on X with respect to $\{f_\lambda : X \to X_\lambda\}$, then I is the smallest topology such that each f_λ is continuous.

Proof: (i) With initial topology I on X, each f_λ is continuous because we have

$(X,I) \xrightarrow{1} (X,I) \xrightarrow{f_\lambda} X_\lambda$ i.e. $X_I \xrightarrow{1} X_I \xrightarrow{f_\lambda} X_\lambda$.

Since I is initial $f_\lambda \circ 1 = f_\lambda$ is continuous.

(ii) Let J be a topology on X such that $f_\lambda : X_J \to X_\lambda$ is continuous for all λ in Λ. We show that $1 : X_J \to X_I$ is continuous. But this is continuous if and only if $f_\lambda \circ 1 = f_\lambda$ is continuous. Hence the result. //

It follows from the above proposition that the initial topology on X with respect to $\{f_\lambda\}$ is unique.

NOTE 1.4.6: "The" initial topology on $Y_1 \times Y_2$ has as its sub-basis all things of the form $p_1^{-1}(U)$, $p_2^{-1}(V)$ where U is open in Y_1, V is open in Y_2 and $p_1^{-1}(U) = U \times Y_2$, $p_2^{-1}(V) = Y_1 \times V$.

Proposition 1.4.7: "The" initial topology exists and has as a sub-basis the sets $f_\lambda^{-1}(U)$ for U open in X_λ.

Proof: We first prove that I generated by $f_\lambda^{-1}(U)$ is initial. Let $K : Y_J \to X_I$ be continuous. We show that $f_\lambda \circ k$ is continuous for all λ in Λ with $f_\lambda \circ k : Y_J \to X_\lambda$. Let U be open in X_λ. Then $(f_\lambda k)^{-1}(U) = K^{-1}(f_\lambda^{-1}(U))$. But $f_\lambda^{-1}(U)$ is an open sub-basic set and

k is continuous; hence $(f_\lambda k)^{-1}(U)$ is open. Thus $f_\lambda \circ k$ is continuous for all λ in Λ. Hence I is initial.

We next suppose that $k : Y_J \to X_I$ is a function such that $f_\lambda \circ k$ is continuous. We show that k is then continuous. It is sufficient to show that k^{-1} (sub-basic set) is open. Let $f_\lambda^{-1}(U)$ be a sub-basic set. Then $k^{-1}(f_\lambda^{-1}(U)) = (f_\lambda k)^{-1}(U)$ is open since U is open in X_λ and $f_\lambda k$ is continuous. Hence k is continuous. //

Definition 1.4.8: Let C be a category. Consider the following commutative diagram in C.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\bar{f}]{} & D \end{array}$$

Diagram 1.4.9

The square 1.4.9 is said to be a pullback in C, if for every pair of C-morphisms $f' : Z \to C$, $g' : Z \to B$ such that $gg' = \bar{f}f'$ there exists a unique C-morphism $\langle g' : f' \rangle : Z \to A$ such that $f \circ \langle g' : f' \rangle = g'$ and $\bar{g} \circ \langle g' : f' \rangle = f'$. The latter property is called the UNIVERSAL PROPERTY OF PULLBACKS.

Lemma 1.4.10: A pullback in C is unique up to isomorphism.

Proof: Let the following diagrams

Diagram 1.4.11

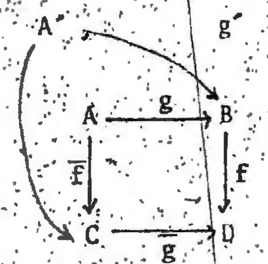
$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\bar{g}]{} & D \end{array}$$

and

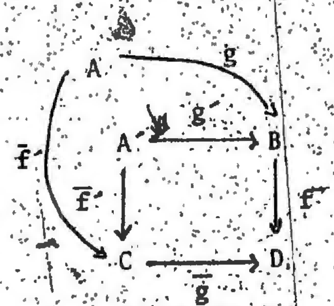
$$\begin{array}{ccc} A' & \xrightarrow{g'} & B \\ {\scriptstyle \bar{f}'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\bar{g}]{} & D \end{array}$$

Diagram 1.4.12

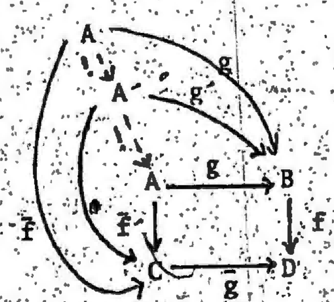
be pullback diagrams in C. Thus we have a commutative diagram as follows:



But diagram 1.4.11 is a pullback in C, so by the universal property of pullbacks there exists a unique C-morphism $\langle \bar{f}' : g' \rangle$ such that $g \langle \bar{f}' : g' \rangle = g'$ and $\bar{f} \langle \bar{f}' : g' \rangle = \bar{f}'$. Similarly we have the following commutative diagram -

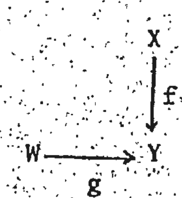


But diagram 1.4.12 is a pullback in C. Hence there exists a unique C-morphism $\langle \bar{f} : g \rangle$ such that $g' \circ \langle \bar{f} : g \rangle = g$ and $\bar{f}' \circ \langle \bar{f} : g \rangle = \bar{f}$. Putting the latter diagrams together we have



with $\langle \bar{f}' : g' \rangle \circ \langle \bar{f} : g \rangle$ a unique C-morphism making the diagram commute. But 1_A also satisfies the commutativity of the diagram. Hence $\langle \bar{f}' : g' \rangle \circ \langle \bar{f} : g \rangle = 1_A$. Similarly we can show that $\langle \bar{f} : g \rangle \circ \langle \bar{f}' : g' \rangle = 1_{A'}$. Thus $\langle \bar{f}' : g' \rangle : A' \to A$ is an isomorphism in C. //

We now show how to form a pullback in the category of Top. Consider the following diagram in Top.



We can form a set $W \sqcap X = \{(w,x) \text{ in } W \times X \text{ such that } f(x) = g(w)\}$. We can then give $W \sqcap X$ the subspace topology of $W \times X$ or the initial topology with respect to the projections, $p_1 : W \sqcap X \to X$ and $p_2 : W \sqcap X \to W$, and the composition $q = fp_1 = gp_2$, of the following diagram:

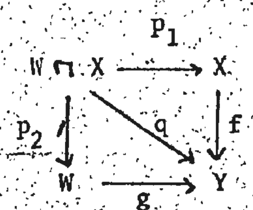


Diagram 1.4.13

Proposition 1.4.14: Either of the two topologies on $W \sqcap X$ as stated above makes diagram 1.4.13 a pullback in Top.

Proof: Let $W \sqcap X_I$ denote $W \sqcap X$ with the initial topology with

respect to p_1, p_2, and q. Let Z be a topological space and let $Z \xrightarrow{f'} X$, $Z \xrightarrow{g'} W$ be maps such that $ff' = gg'$. We require a unique map $h : Z \to W \sqcap X$ such that $p_1 h = f'$ and $p_2 h = g_1$. Define $h(z)$ to be $(g'(z), f'(z))$. This is unique by construction. We show that h is continuous. Since $W \sqcap X$ has the initial topology with respect to p_1, p_2, q then h is continuous if and only if $p_1 h$, $p_2 h$ and qh are continuous.
But $p_1 h(z) = p_1(g'(z), f'(z)) = g'(z)$ which is continuous (given) - and $p_2 h(z) = p_2(g'(z), f'(z)) = f'(z)$ which is continuous (given) - and $qh(z) = q(g'(z), f'(z)) = fp_1(g'(z), f'(z)) = fg'(z)$ which is continuous since composition of continuous functions is continuous. Hence h is continuous and Diagram 1.4.13 is a pullback in Top.

Let $W \sqcap X_\tau$ denote $W \sqcap X$ with the subspace topology of $X \times W$.

Consider $W \sqcap X_\tau \xrightarrow{i} W \times X \xrightarrow{p_X} X$ and $W \sqcap X_\tau \xrightarrow{i} W \times X \xrightarrow{p_W} W$ where $X \times W$ has the product topology. Let $p_X i = p_1'$, $p_W i = p_2'$. These are continuous by definition of product topology. Consider also diagram 1.4.15.

$$\begin{array}{ccc} W \sqcap X_\tau & \xrightarrow{p_1'} & X \\ \downarrow{\scriptstyle p_2'} & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[g]{} & Y \end{array}$$

Diagram 1.4.15

Suppose we are given a topological space Q and maps $\bar{f} : Q \to X$, $\bar{g} : Q \to W$ such that $f\bar{f} = g\bar{g}$. Define $h' : Q \to W \sqcap X$ by $h'(q) = (\bar{g}(q), \bar{f}(q))$. Again this is unique by construction. We need h' continuous. Let $U \subseteq X$ be open. Then $(p_1' h')^{-1}(U) = \bar{f}^{-1}(U)$

which is open in Q since $\overline{f}$ is continuous. But

$$(p_1 h)^{-1}(U) = h^{-1}(p_1^{-1}(U)).$$

Now $p_1^{-1}(U)$ is open in τ since p_1 is continuous so that we have h^{-1} (of an open set) open. Hence h is indeed continuous. Thus diagram 1.4.15 is a pullback in Top. Hence by Lemma 1.4.10 $I = \tau$. //

Definition 1.4.16: Let $\{f_\alpha : X_\alpha \to X\}$ be a family of maps for α belonging to an indexing set A, and X, X_α topological spaces. A topology F on X is said to be final with respect to $\{f_\alpha\}$ if for any topological space Z and any function $g : X_F \to Z$ we have g continuous if and only if $gf_\alpha : X_\alpha \to Z$ is continuous for all α in A.

Proposition 1.4.17: If F is the final topology on X with respect to $\{f_\alpha\}$ then a) each $f_\alpha : X_\alpha \to X_F$ is continuous

b) if τ is any topology on X such that $f_\alpha : X_\alpha \to X_\tau$ is continuous, then F is larger than τ.

Proof: a) The identity $1 : X_F \to X_F$ is continuous and hence $1f_\alpha : X_\alpha \to X$ is continuous for all α in A. Since $1f_\alpha = f_\alpha$ it follows that f_α is continuous for all α in A.

b) Let $1 : X_F \to X_\tau$ be the identity function. Then since $1 \circ f_\alpha = f_\alpha$ is continuous and F is final topology on X with respect to f_α, then 1 is continuous. Hence each τ-open set is F-open. Thus F is larger than τ. //

Proposition 1.4.18: The final topology on X with respect to $\{f_\alpha\}$ exists and is characterized by either of the following conditions:

a) $U \subseteq X$ is open if and only if $f_\alpha^{-1}(U)$ is open in X_α for all α in A.

b) same as (a) with "open" replaced by "closed".

Proof: We must first show that (a) does in fact define a topology. ϕ belongs to F and $f_\alpha^{-1}(X) = X_\alpha$ so X belongs to F. Let U and V be open in F. Then $f_\alpha^{-1}(U \cap V) = f_\alpha^{-1}(U) \cap f_\alpha^{-1}(V)$ which is open in X_α; so $U \cap V$ belongs to F. Let U_β be open in X, β belong to an indexing set B. Then $f_\alpha^{-1}(\bigcup_\beta U_\beta) = \bigcup_\beta f_\alpha^{-1}(U_\beta)$ is open in F. Hence (a) does define a topology. In a similar way we can show that (b) defines a topology.

We next prove that this topology is final. Let $g : X \to Z$ be a function. Suppose g is continuous. With the topology as given it is clear that f_α is continuous for all α in A. Hence $g \circ f_\alpha$ is continuous.

Conversely - assume gf_α is continuous. We need g continuous. Let W be open in Z. Then $(gf_\alpha)^{-1}(W) = f_\alpha^{-1}(g^{-1}(W))$. But gf_α is continuous - hence $(gf_\alpha)^{-1}(W)$ is open in X_α as is $f_\alpha^{-1}(g^{-1}(W))$. Since f_α^{-1} is continuous by hypothesis, $g^{-1}(W)$ is open in X_F. Hence g is continuous. //

Example 1.4.19: Let $X \sqcup Y$ denote $X \times \{0\} \cup Y \times \{1\}$. This is the disjoint union and is called the sum in the categorical sense. This can be given the final topology with respect to the inclusions

$$X \xrightarrow{i_X} X$$

This coincides with
sum topology if for maps f :
map h : $X \sqcup Y \to Z$ such that
referred to as the universal

Example 1.4.20: Let X be a
relation on X. We give $X/\sim$
with respect to the projecti
if and only if $p^{-1}(U)$ is o

Definition 1.4.21: Let C
commutative diagram in C

$$\begin{array}{c} A \longrightarrow \\ \bar{f}\big\downarrow \\ C \end{array}$$

The square above is
if for a pair of C-morphisms
there exists a unique C-morph
$\langle f' : g' \rangle \circ g = f'$, $\langle f' : g' \rangle$

Lemma 1.4.22: The pushout of
isomorphism in C.

Proof: This is exactly dual t

$X \sqcup Y \xleftarrow{i_Y} Y$

the sum topology, i.e. $X \sqcup Y$ has the
$X \to Z$, $g : Y \to Z$ there exists a unique
t $hi_Y = g$ and $hi_X = f$. This is
property of a sum.

topological space and $\sim$ an equivalence
, the quotient set, the final topology
n $p : X \to X/\sim$. Thus $U \subset X/\sim$ is open
en in X.

e a category. Consider the following

$\xrightarrow{\bar{g}}$ B

$\downarrow f$

g $\to$ D

said to be a PUSHOUT (DUAL OF PULLBACK)
C $\xrightarrow{f'}$ Z, B $\xrightarrow{g'}$ Z such that $f'\bar{f} = g'\bar{g}$
ism $\langle f' : g' \rangle : D \to Z$ such that
o f = g'.

a pair of maps is unique up to

to Lemma 1.4.10. //

We next show how to form the pushout of the following diagram in Top.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

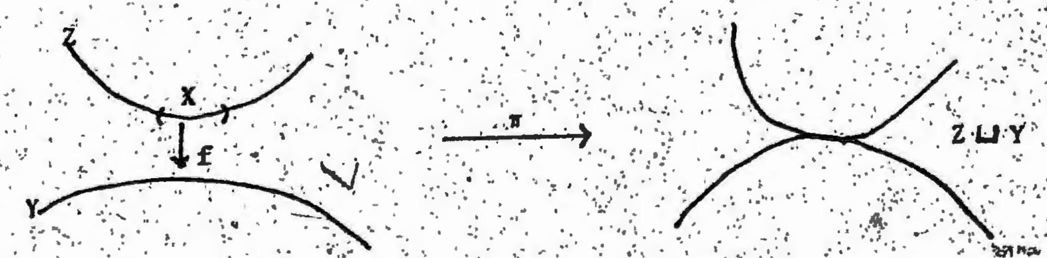
We take the set $Z \sqcup Y$, the disjoint union, and quotient it under the smallest equivalence relation generated by $f(x) \sim g(x)$ for all x in X. Then the maps $Z \xrightarrow{\bar{f}} Z \sqcup Y/\sim$ and $Y \xrightarrow{\bar{g}} Z \sqcup Y/\sim$ are the following compositions

$$Z \underset{\text{inclusion}}{\overset{i_Z}{\hookrightarrow}} Z \sqcup Y \underset{\text{projection}}{\overset{\pi}{\twoheadrightarrow}} Z \sqcup Y/\sim$$

and $$Y \overset{i_Y}{\hookrightarrow} Z \sqcup Y \overset{\pi}{\twoheadrightarrow} Z \sqcup Y/\sim .$$

The pushout in Top is the set $Z \sqcup Y/\sim$ topologized by the final topology with respect to the composites πi_Z and πi_Y. We denote the pushout space by $Z\,{}_g\!\sqcup_f Y$ or simply $Z \sqcup Y$ where no confusion arises.

In the case where X is a subspace of Z and g the inclusion, we can visualize $Z \sqcup Y$ as



§1.5 THE EXPONENTIAL LAW FOR TOPPAIR

Definition 1.5.1: Let X,Y be topological spaces and let $Y^X = \mathrm{Map}(X,Y)$ denote the set of all maps from X to Y. For any two sets $K \subset X$, $U \subset Y$ let $\mathrm{Map}(K,U)$ denote the subset of $\mathrm{Map}(X,Y)$ defined by $\mathrm{Map}(K,U) = \{f$ belong to $\mathrm{Map}(X,Y)$ such that $f(K) \subset U\}$. This set $\mathrm{Map}(K,U)$ will be called a sub-basic set of Y^X if K is compact in X and U is open in Y. The COMPACT-OPEN TOPOLOGY κ of Y^X is defined by the above sub-basis. Every set in κ is the union of a collection of finite intersections of sub-basic sets.

Note 1.5.2: The subset $(Y,Y_0)^{(X,X_0)} = \{f : X \to Y$ such that $f(X_0) \subseteq Y_0\}$ of Y^X can be given the subspace topology with respect to Y^X so that $(Y^X, (Y,Y_0)^{(X,X_0)})$ is certainly an object of Toppair. In particular if $X = X_0 = I$, the space $(Y^I, (Y,Y_0)^{(I,I)})$ can be written in a modified form as we show.

Now any map $\lambda : I \to Y$ belongs to the subspace if and only if $\lambda(I) \subseteq Y_0$. Thus the subspace $(Y,Y_0)^{(I,I)}$ can be considered as $Y_0{}^I$. We sometimes write the pair $(Y^I, Y_0{}^I)$ as $(Y,Y_0)^I$.

Definition 1.5.3: A function $\varepsilon : (Y^X \times X, (Y,Y_0)^{(X,X_0)} \times X_0) \to (Y,Y_0)$, where $(Y^X, (Y,Y_0)^{(X,X_0)}) \times (X,X_0) = (Y^X \times X, (Y,Y_0)^{(X,X_0)} \times X_0)$ which takes an element (f,x) of $Y^X \times X$ to $f(x)$ and clearly behaves well with respect to the subspaces is called the EVALUATION FUNCTION for Toppair.

Note: This is the usual evaluation function in Top if $X_0 = Y_0 = \phi$.

Definition 1.5.4: A topology τ is admissible if and only if the evaluation function $\bar{\varepsilon}$ is continuous.

Note 1.5.5: The propositions, theorems and definitions given for the Exponential Law in Top (see BROWN [2], HEATH [13], HU [17]) hold as well in Toppair - see Lemma 1.3.4.

For completeness we conclude with the following comments:

1) Let f belong to $(Y^X, (Y,Y_0)^{(X,X_0)})$ and (Z,Z_0) be an arbitrary object of Toppair. Then $f_* : (Y^X, (Y,Y_0)^{(X,X_0)}) \to (Z^X, (Z,Z_0)^{(X,X_0)})$ defined by $f_*(h) = f \circ h$ is a map of pairs (h in Y^X) and $f^* : (Z^Y, (Z,Z_0)^{(Y,Y_0)}) \to (Z^X, (Z,Z_0)^{(X,X_0)})$ defined by $f^*(k) = k \circ f$ is a map of pairs (k in Z^Y).

2) If Y is locally compact and regular then, as in Top, any map $(X,X_0) \times (Y,Y_0) \xrightarrow{f} (Z,Z_0)$ or $(X \times Y, X_0 \times Y_0) \xrightarrow{f} (Z,Z_0)$ is continuous as a map of pairs if and only if $(X,X_0) \xrightarrow[\hat{f}]{} (Z^Y, (Z,Z_0)^{(Y,Y_0)})$ is continuous.

3) Let Y be locally compact and regular. Then there is a bijection

$$\text{Toppair}\ ((Z,Z_0) \times (Y,Y_0),(X,X_0)) \cong \text{Toppair}\ ((Z,Z_0),(X^Y,(X,X_0)^{(Y,Y_0)})$$

This is known as the Toppair exponential law. We note that the Toppair exponential law does not restrict to the Top_* exponential law. In case $(Y,Y_0) = (I,I)$ we write $(Z,Z_0) \times (Y,Y_0)$ as $(Z,Z_0) \times I$. In this case the exponential law reduces to

$$\text{Toppair}\ ((Z,Z_0) \times I,\ (X,X_0)) \cong \text{Toppair}\ ((Z,Z_0),(X,X_0)^I).$$

CHAPTER II - INTRODUCTION

In section one we define the covering homotopy property (CHP) of pairs. This leads naturally to the concept of Toppair fibration and the problem of forming pullbacks in Toppair. We see that the pullback of two maps

$$(X,X_0) \xrightarrow{g} (B,B_0) \xleftarrow{f} (E,E_0)$$

is a pair of pullbacks

$$(X \sqcap B,\ X_0 \sqcap B_0)$$

in the Top sense.

The second section shows that the usual "Top" proof of the factorization lemma easily adapts to prove that any Toppair map

$$f : (E,E_0) \to (B,B_0)$$

factors as

$$(E,E_0) \xrightarrow{u} (E \sqcap B^I, E_0 \sqcap B_0^I) \xrightarrow{\theta} (B,B_0)$$

where u is a homotopy equivalence and θ is a fibration in Toppair.

Finally, in section three, we give a characterization of Toppair fibrations which essentially generalizes similar concepts due to Hurewicz [19].

§2.1 FIBRATIONS AND PULLBACKS IN TOPPAIR

Definition 2.1.1: A map $p : (E,E_0) \to (B,B_0)$ in **TOPPAIR** is said to have the COVERING HOMOTOPY PROPERTY (abbreviated CHP) with respect to the pair (X,X_0) if for every commutative diagram of the form

$$\begin{array}{ccc} (X,X_0)\times\{0\} & \xrightarrow{\quad f \quad} & (E,E_0) \\ \text{inclusion} = \; i \big\downarrow & & \big\downarrow p \\ (X,X_0)\times I & \xrightarrow[H:f\simeq g]{} & (B,B_0) \end{array}$$

there exists a homotopy $\tilde{H} : f \simeq \tilde{H}(-,1) : (X,X_0)\times I \to (E,E_0)$ such that $p\tilde{H} = H$ and $\tilde{H}i = f$. We note that $H_0 : X_0 \times I \to B_0$ is also a homotopy. The map p is said to be a FIBRATION in Toppair if p has the CHP for all pairs (X,X_0). As an example of such a fibration we have the projection

$$pr : (X\times Y,\ X_0\times Y_0) \to (X,X_0). \qquad \text{See Remark 2.1.9.}$$

This follows from the universal properties of products.

Definition 2.1.2: Two fibrations $p : (E,E_0) \to (B,B_0)$, $q : (E',E'_0) \to (B,B_0)$ in Toppair are said to be FIBRE HOMOTOPICALLY EQUIVALENT, i.e. have the same fibre homotopy type, if there exist maps

$$f : (E,E_0) \to (E',E'_0),\ g : (E',E'_0) \to (E,E_0)$$

preserving the fibres in the sense that $qf = p$, $pg = q$;

and there exist homotopies H, H' such that H is a homotopy between gf and $1_{(E,E_0)}$ rel p; and H' is a homotopy between fg and $1_{(E',E'_0)}$ rel q. Both f and g are homotopy equivalences.

Proposition 2.1.3: If $p : (E,E_0) \to (B,B_0)$ is a fibration in Toppair, then the restrictions (i) $p : E \to B$ and (ii) $p_0 : E_0 \to B_0$ are fibrations in Top.

Proof: (i) For each commutative diagram in Top of form 2.1.4, we need a homotopy $\bar{H}$ such that $p \circ \bar{H} = H$.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow p \\ X \times I & \xrightarrow[H:f\simeq g]{} & B \end{array}$$

Diagram 2.1.4

Now for $X_0 = \phi$ Diagram 2.1.4 gives us in Toppair Diagram 2.1.5 as follows:

$$\begin{array}{ccc} (X,\phi) \times \{0\} & \xrightarrow{\tilde{f}} & (E,E_0) \\ \downarrow & & \downarrow p \\ (X,\phi) \times I & \xrightarrow[H:f\simeq g]{} & (B,B_0) \end{array}$$

Diagram 2.1.5

But p is a fibration in Toppair, so there exists a homotopy $\tilde{H} : (X,\phi) \times I \to (E,E_0)$ making Diagram 2.1.5 commute and such that $\tilde{H}_0(\phi \times I) \subseteq E_0$. Thus $\tilde{H}$ gives us the required completion for Diagram 2.1.4. Hence $p : E \to B$ is a fibration in Top.

(ii) For each commutative diagram in Top, we need a completion as in (i) of the form 2.1.6

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{r}} & E_0 \\ \big\downarrow & & \big\downarrow p_0 \\ X \times I & \xrightarrow[K:r\simeq t]{} & B_0 \end{array}$$

Diagram 2.1.6

i.e. we need a homotopy $\tilde{K}$ such that $p_0 \circ \tilde{K} = K$ and $\tilde{K} \circ i = \tilde{r}$. Putting $X_0 = X$ we have the following commutative diagram in Toppair

$$\begin{array}{ccc} (X,X) \times \{0\} & \longrightarrow & (E,E_0) \\ i \big\downarrow & & \big\downarrow p \\ (X,X) \times I & \xrightarrow[K:r\simeq t]{} & (B,B_0) \end{array}$$

Diagram 2.1.7

where $K_0 : r_0 \simeq t_0 : X \times I \to B_0$. But p is a fibration in Toppair. Hence there exists a homotopy $\tilde{K} : (X,X) \times I \to (E,E_0)$ such that $p\tilde{K} = K$ and $\tilde{K}_0$ is a map from $X \times I$ to E_0. Taking this restriction of $\tilde{K}$ gives us the required completion for Diagram 2.1.6. Thus p_0 is indeed a fibration in Top. //

We next consider the problem of forming a pullback of the following diagram in Toppair

$$\begin{array}{ccc} & & (E,E_0) \\ & & \big\downarrow f \\ (X,X_0) & \xrightarrow[g]{} & (B,B_0) \end{array}$$

We can form the set $X \sqcap E = \{(x,e)$ belonging to $X \times E$ such that $f(e) = g(x)\}$

and the set $X_0 \sqcap E_0 = \{(x_0, e_0)$ belonging to $X_0 \times E_0$ such that $f(e_0) = g(x_0)\}$. As a set, $X_0 \sqcap E_0$ is then a subset of $X \sqcap E$. Now we know by Proposition 1.4.12 that $X \sqcap E$ with the subspace topology of $X \times E$ makes the outside diagram a pullback. We need to examine the possible topologies for $X_0 \sqcap E_0$. Now $X_0 \sqcap E_0$ can be considered as a subspace of $X_0 \times E_0$ and as such it will be a pullback of

$$\begin{array}{ccc} & & E_0 \\ & & \downarrow \\ X_0 & \rightarrow & B_0 \end{array}$$

or, it can be considered with the subspace topology with respect to $X \sqcap E$. The following proposition shows that both these topologies coincide, and hence that

$$\begin{array}{ccc} (X \sqcap E, X_0 \sqcap E_0) & \xrightarrow{\bar{g}} & (E, E_0) \\ \bar{f} \downarrow & & \downarrow f \\ (X, X_0) & \xrightarrow[g]{} & (B, B_0) \end{array}$$

is a pullback in Toppair.

Proposition 2.1.8: If $X_0 \sqcap E_{0_\tau}$ is a subspace of $X \sqcap E$ and $X_0 \sqcap E_{0_{\tau'}}$ is a subspace of $X_0 \times E_0$, then $\tau = \tau'$.

Proof: Consider the following diagram where i, i', j, j' are the obvious inclusions and 1 is the identity function:

$$\begin{array}{ccccc} X_0 \sqcap E_{0_\tau} & \xrightarrow{i} & X \sqcap E & \xrightarrow{i'} & X \times E \\ 1 \downarrow & & & \nearrow j' & \\ X_0 \sqcap E_{0_{\tau'}} & \xrightarrow[j]{} & X_0 \times E_0 & & \end{array}$$

The function l need not be continuous but $i'i\,(x,e) = j'j\,l\,(x,e)$.
Let U be open in τ. Then since $X_0 \sqcap E_0$ is a subspace of $X \sqcap E$ there exists V open in $X \sqcap E$ such that $U = i^{-1}(V) = V \cap (X_0 \sqcap E_0)$. But $X \sqcap E$ is a subspace of $X \times E$; thus there exists W open in $X \times E$ such that $V = i'^{-1}(W) = W \cap (X \sqcap E)$ so that

$$U = i^{-1}(V) = i^{-1}(i'^{-1}(W)) = (i'i)^{-1}(W).$$

But $\quad i'i = j'jl = j'j$

so that $\quad U = (j'j)^{-1}(W) = j^{-1}(j'^{-1}(W))$.

But $\quad j^{-1}(j'^{-1}(W))$ is open in τ' (inclusions are continuous).

Hence $\quad U$ is open in τ' i.e. $\tau \subseteq \tau'$.

In a similar manner we can show that $\tau' \subseteq \tau$.

Hence $\quad \tau = \tau'$.

//

Remark 2.1.9: The pullback of the following diagram in Toppair

$$\begin{array}{ccc} & & (E,E_0) \\ & & \downarrow \\ (X,X_0) & \longrightarrow & (*,*) \end{array}$$

is then $(X \times E, X_0 \times E_0) = (X,X_0) \times (E,E_0)$.

§2.2 TOPPAIR MAPPING TRACK AND FIBRATION CHARACTERIZATION

Definition 2.2.1: Let $(Y,Y_0)^I \xrightarrow{\theta} (Y,Y_0)$ be defined by

$$\theta(\lambda) = \lambda(0) \text{ for all } \lambda \in (Y,Y_0)^I.$$

Then the MAPPING TRACK of a map $f : (X,X_0) \to (Y,Y_0)$ in Toppair is the following pullback in Toppair:

$$\begin{array}{ccc} (X \sqcap Y^I, X_0 \sqcap Y_0^{\,I}) & \xrightarrow{\ \bar{f}\ } & (Y,Y_0)^I \\ \downarrow{\scriptstyle \bar{\theta}} & & \downarrow{\scriptstyle \theta} \\ (X,X_0) & \xrightarrow{\ f\ } & (Y,Y_0) \end{array}$$

where $\bar{f}$ and $\bar{\theta}$ are the projections and

$$X \sqcap Y^I = \{(x,\lambda) \in X \times Y^I \text{ such that } \theta(\lambda) = f(x) = \lambda(0)\}.$$

Proposition 2.2.2: Any map $p : (E,E_0) \to (B,B_0)$ can be factored through the mapping track of pairs as follows:

$$(E,E_0) \xrightarrow{u} (E \sqcap B^I, E_0 \sqcap B_0^{\,I}) \xrightarrow{\theta} (B,B_0)$$

where (i) u is a homotopy equivalence and (ii) θ, defined by $\theta(\lambda,e) = \lambda(1)$, is a fibration.

Proof: Let $(E,E_0) \to (E \sqcap B^I, E_0 \sqcap B_0^{\,I})$ be defined by $u(e) = (e,\tilde{p}(e))$ where $\tilde{p}(e)$ is the constant path at $p(e)$ and $u : E \to E \sqcap B^I$.

Let $u' : (E \sqcap B^I, E_0 \sqcap B_0^{\,I}) \to (E,E_0)$, be the projection $u'(e,\lambda) = e$, $u' : E \sqcap B^I \to E$. We then have $u'u(e) = u'(e,\tilde{p}(e)) = e = 1(e)$. Hence $u'u = 1_{(E,E_0)}$.

We define $H : (E \sqcap B^I, E_0 \sqcap B_0^{\,I}) \times I \to (E \sqcap B^I, E_0 \sqcap B_0^{\,I})$ by defining $H : (E \sqcap B^I) \times I \to E \sqcap B^I$ as $H(e,\lambda,t) = (e,\lambda^t)$ where $\lambda^t(s) = \lambda(st)$ for all s,t belonging to I. Then $H(e,\lambda,0) = (e,\lambda^0) = (e,\tilde{p}(e))$ and $uu'(e,\lambda) = u(e) = (e,\tilde{p}(e))$. Hence $H(e,\lambda,0) = uu'(e,\lambda)$. Similarly $H(e,\lambda,1) = (e,\lambda^1) = (e,\lambda) = 1_{(E \sqcap B^I, E_0 \sqcap B_0^{\,I})}$.

So we have that H is a homotopy between uu' and 1.

We need to check that H is in fact a homotopy of pairs, i.e. that $H_0[(E_0 \sqcap B_0{}^I) \times I] \subseteq E_0 \sqcap B_0{}^I$. To this purpose let (e_0, λ_0, t) belong to $(E_0 \sqcap B_0{}^I) \times I$. Then $H_0(e_0, \lambda_0, t) = (e_0, \lambda_0{}^t)$ by definition of H. But $\lambda_0{}^t$ belongs to $B_0{}^I$ since λ_0 belongs to $B_0{}^I$. We also know that e_0 belongs to E_0.

Hence H is the required homotopy and so u is a homotopy equivalence in Toppair.

(ii) For all (Z, Z_0) and every commutative diagram of form 2.2.3 in Toppair we need that there exists $\bar{H} : g \simeq r$, say, making the diagram commute.

$$\begin{array}{ccc} (Z, Z_0) \times \{0\} & \xrightarrow{\ g\ } & (E \sqcap B^I, E_0 \sqcap B_0{}^I) \\ \downarrow & & \downarrow \theta \\ (Z, Z_0) \times I & \xrightarrow[H : f \simeq q]{} & (B, B_0) \end{array}$$

Diagram 2.2.3

Let $g(z) = (e, \lambda)$; $g(z_0) = (e_0, \lambda_0)$.

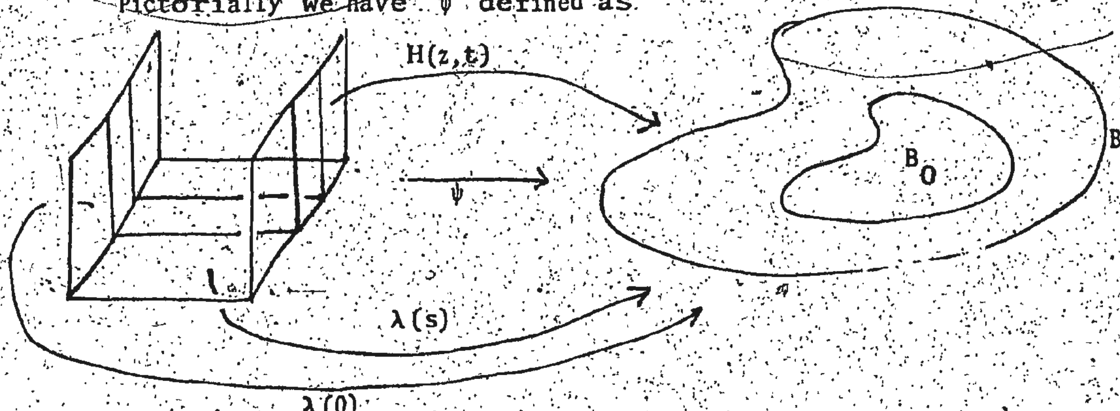
Define a map $\psi : (Z, Z_0) \times (I \times \{0\} \cup \{0\} \times I \cup \{1\} \times I) \to (B, B_0)$ as follows:

$$\psi(z, s, 0) = \lambda(s) \qquad [\psi_0(z_0, s, 0) = \lambda_0(s)]$$

$$\psi(z, 0, t) = \lambda(0) = \theta(e) \quad [\psi_0(z_0, 0, t) = \lambda_0(0) = \theta(e_0)]$$

$$\psi(z, 1, t) = H(z, t) \quad [H_0(z_0, t) \text{ if } z \in Z_0]$$

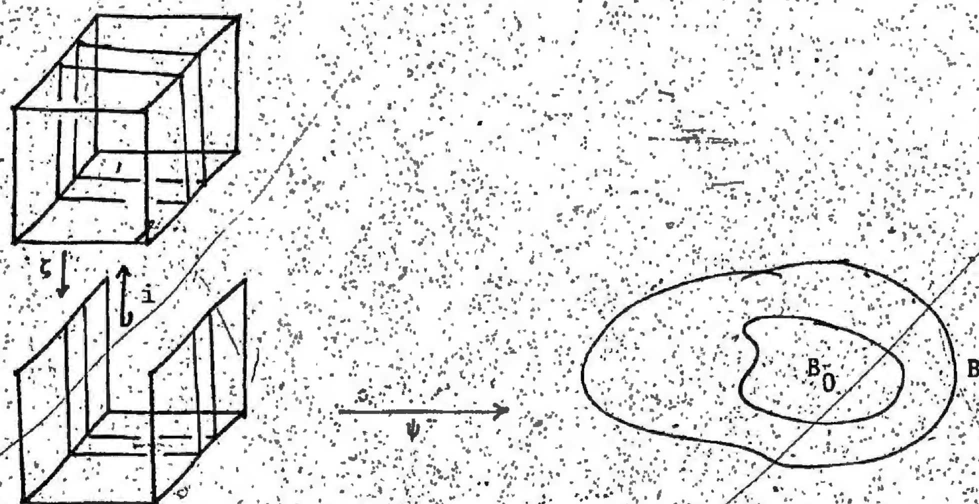
Pictorially we have ψ defined as



But we have the retraction

$$\zeta : (Z, Z_0) \times I \times I \to (Z, Z_0) \times (I \times \{0\} \cup \{0\} \times I \cup \{1\} \times I)$$

which can be pictured as



Now for each z in Z define a family of paths λ_t^z in B by setting $\lambda_t^z(s) = \psi(z, \zeta(s,t))$ [restrictions to subspaces holding].

Finally we define the required homotopy $\tilde{H}$ by letting

$$\tilde{H}(z,t) = (e, \lambda_t^z) = (\psi(z), \widetilde{\zeta(-,t)}, e).$$

Before showing that $\tilde{H}$ is the homotopy we require, we check that (e, λ_t^z) belongs to $E \sqcap B^I$ in the pair $(E \sqcap B^I, E_0 \sqcap B_0^I)$.

Now $(E \sqcap B^I) = \{(e,\lambda) \in E \times B^I \mid p(e) = \theta(\lambda) = \lambda(0)\}$,

$$\theta(\lambda_z^t) = \lambda_z^t(0) \qquad \text{[definition of } \theta],$$

and $\lambda_z^t(0) = \psi(z, \zeta(0,t)) = \psi(z,0,t) = \lambda(0)$.

But $\lambda(0) = p(e)$. It is trivial to see that the restrictions to the subspaces hold. We now show that $\tilde{H}$ is the required homotopy. The map $\tilde{H}$ evaluated at "0", i.e.

$$\tilde{H}(z,0) = (e,\lambda_0{}^z) = (\psi(z,\zeta(-,0),e)$$
$$= (\psi(z,-,0),e) = (e,\lambda) = g(z)$$

and $\theta\tilde{H}(z,t) = \theta(e,\lambda_t{}^z) = \lambda_t{}^z(1)$ [Definition of θ]

$$= \psi(z,\zeta(1,t)) = \psi(z,1,t) = H(z,t).$$

So $\tilde{H}$ is the homotopy we require. Hence θ is a fibration in Toppair.

//

Note 2.2.4: In the following diagram $\bar{\varepsilon}_1$ is a homotopy equivalence. It is in effect a homotopy inverse for u, the homotopy equivalence of the mapping track factorization of p.

$$\begin{array}{ccc} (E \sqcap B^I, E_0 \sqcap B_0{}^I) & \rightarrow & (B,B_0)^I \\ \bar{\varepsilon}_1 \downarrow & & \downarrow \varepsilon_1 \\ (E,E_0) & \xrightarrow{p} & (B,B_0) \end{array}$$

Lemma 2.2.5: If in the following pullback diagram in Toppair p is a fibration, then π_2 is a fibration

$$\begin{array}{ccc} (X \sqcap E, X_0 \sqcap E_0) & \xrightarrow{\pi_1} & (E,E_0) \\ \pi_2 \downarrow & & \downarrow p \\ (X,X_0) & \xrightarrow{f} & (B,B_0) \end{array}$$

Proof: For all (Z,Z_0) in Toppair we have the following commutative diagram

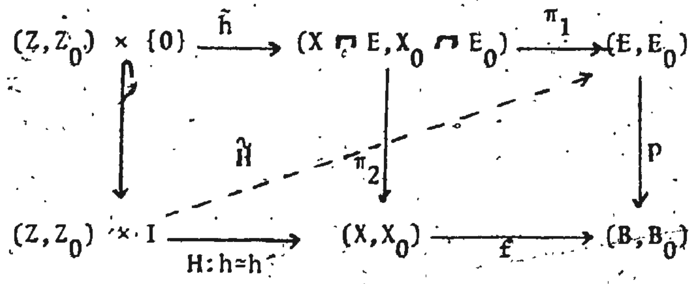
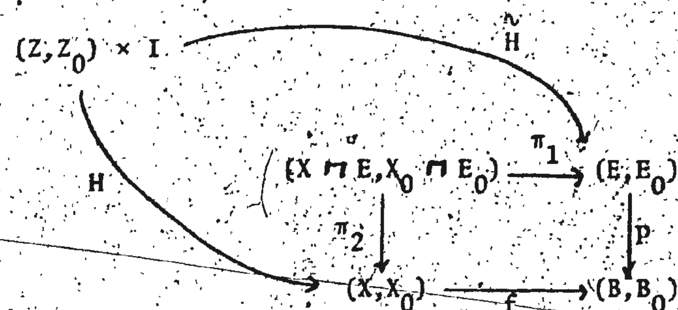


Diagram 2.2.6

But p is a fibration so there exists a homotopy

$$\tilde{H} : \pi_1\tilde{h} \simeq \tilde{H}(-,1) : (Z,Z_0) \times I \to (E\ E_0) .$$

making the outside diagram commute. We then can form the following diagram



where $p\tilde{H} = fH$ by commutativity of Diagram (2.2.6).

Therefore there exists a unique map $\psi : (Z,Z_0) \times I \to (X \sqcap E, X_0 \sqcap E_0)$ making the diagram commute, i.e. such that $\pi_2\psi = H$.

Thus π_2 is a fibration in Toppair.

//

Given a map $p : (E,E_0) \to (B,B_0)$ there is an induced map $q = q_p : (E,E_0)^I \to (E \sqcap B^I, E_0 \sqcap B_0^I)$ defined by

$$q(\lambda) = (\lambda(0), p\lambda).$$

The following characterization essentially generalizes similar concepts due to HUREWICZ [19].

Proposition 2.2.7: Let $p : (E,E_0) \to (B,B_0)$ be a map in Toppair.

Then the following are equivalent:

(i) p is a fibration

(ii) q_p has a section, i.e. a right inverse

(iii) q_p lifts over any map $(X,X_0) \to (E \sqcap B^I, E_0 \sqcap B_0^{\,I})$

(iv) p has the CHP with respect to the pair $(E \sqcap B^I, E_0 \sqcap B_0^{\,I})$.

Proof: a) (i) if and only if (ii)

(Sufficiency) Let $s : (E \sqcap B^I, E_0 \sqcap B_0^{\,I}) \to (E,E_0)^I$ be a section to q. Given any commutative diagram of form 2.2.8, we require a homotopy $\bar{F} : (Z,Z_0) \times I \to (E,E_0)$ making the diagram commute.

$$\begin{array}{ccc} (Z,Z_0) & \xrightarrow{\quad f \quad} & (E,E_0) \\ \downarrow & \circlearrowleft & \downarrow p \\ (Z,Z_0) \times I & \xrightarrow[F : g \simeq h]{} & (B,B_0) \end{array}$$

Diagram 2.2.8

Now the following diagram is a pullback in Toppair

$$\begin{array}{ccc} (E \sqcap B^I, E_0 \sqcap B_0^{\,I}) & \xrightarrow{\quad \bar{p} \quad} & (B,B_0)^I \quad \lambda \\ \bar{\varepsilon}_0 \downarrow & & \varepsilon_0 \downarrow \qquad \downarrow \\ (E,E_0) & \xrightarrow[\quad p \quad]{} & (B,B_0) \quad \lambda(0) \end{array}$$

Diagram 2.2.9

where $\bar{p}$ and $\bar{\varepsilon}_0$ are projections.

The function $(Z,Z_0) \xrightarrow{\hat{F}} (B,B_0)^I$ is continuous (Exponential law) and the function $(Z,Z_0) \overset{f}{\to} (E,E_0)$ is continuous (Diagram 2.2.8).

So we have the following diagram of continuous maps:

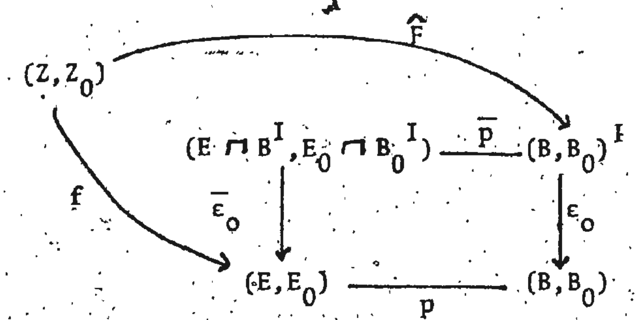


Diagram 2.2.10

and $\varepsilon_o\hat{F}(z) = \hat{F}(z)(0) = F(z,0) = g(z) = gi(z) = pf.$

Hence there exists a unique map $m : (Z,Z_0) \to (E \sqcap B^I, E_0 \sqcap B_0^I)$ making Diagram 2.2.10 commute.

Now $sm : (Z,Z_0) \to (E,E_0)^I$ is a continuous map,

$$sm(z)(t) = s(f(z),\hat{F}(z))(t) = \hat{F}(z)(t).$$

Therefore $\underline{sm} : (Z,Z_0) \times I \to (E,E_0)$ is continuous (Exponential law). This is the map we require for the completion of Diagram 2.2.8. It is easy to check that this map makes 2.2.8 commute, i.e. $p(\underline{sm})(z,t) = F(z,t)$ and $\underline{sm}i = f.$ Hence p is a fibration.

(Necessity) Given a map $q : (E,E_0)^I \to (E \sqcap B^I, E_0 \sqcap B_0^I)$ we require a map $s : (E \sqcap B^I, E_0 \sqcap B_0^I) \to (E,E_0)^I$ such that $q \circ s = 1.$ Now 2.2.9 is a pullback diagram and there exists the following commutative diagram in Toppair

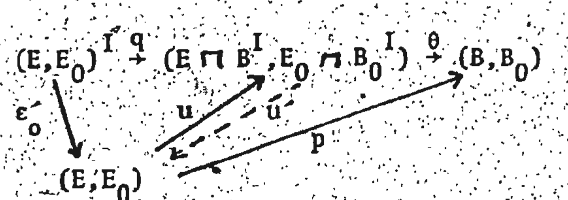


Diagram 2.2.11

where $\theta u = p$ is the factorization of p through the mapping track and u' is a homotopy inverse for u.

There exists also the following commutative diagram in Toppair

$$\begin{array}{ccc}
(E \sqcap B^I, E_0 \sqcap B_0{}^I) & \xrightarrow{u'} & (E, E_0) \\
\Big\downarrow i & & \Big\downarrow p \\
(E \sqcap B^I, E_0 \sqcap B_0{}^I) \times I & \xrightarrow[\hat{p}]{} & (B, B_0)
\end{array}$$

Diagram 2.2.12

where $\hat{p}((e,\lambda),t) = \overline{p}(e,\lambda)(t) = \lambda(t)$.

But $pu'(e,\lambda) = p(e)$ (definition of u') and

$$\hat{p}i((e,\lambda),0) = \hat{p}((e,\lambda),0) = \overline{p}(e)(\lambda(0)) = \lambda(0).$$

Now (e,λ) belongs to $E \sqcap B^I$ if and only if

$p(e) = \lambda(0)$. So Diagram 2.2.12 commutes. But p is a fibration so there exists a homotopy

$$G : (E \sqcap B^I, E_0 \sqcap B_0{}^I) \times I \to (E, E_0),$$

with $pG = \hat{p}$. Then $\hat{G} : (E \sqcap B^I, E_0 \sqcap B_0{}^I) \to (E, E_0)^I$ serves as a section to q. We check that $q\hat{G} = 1$ as follows:

$$\begin{aligned}
q\hat{G}(e,\lambda)(t) &= (\hat{G}(e,\lambda)(0), p\hat{G}(e,\lambda)(t)) && \text{[definition of } q\text{]} \\
&= (G((e,\lambda),0), pG((e,\lambda),t)) && \text{[definition of } \hat{G}\text{]} \\
&= (Gi((e,\lambda),0), pG((e,\lambda),t)) \\
&= (u'((e,\lambda),0), \hat{p}((e,\lambda),t)) && \text{[Diagram 2.2.12 commutes]} \\
&= (e, \overline{p}(e,\lambda)(t)) \\
&= (e,\lambda)(t) = 1(e,\lambda)(t).
\end{aligned}$$

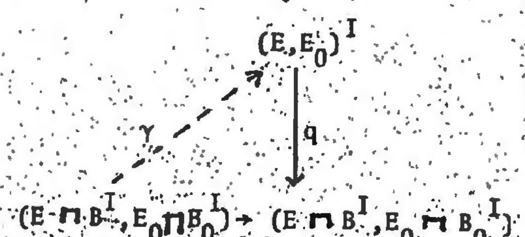
Thus $\hat{G}$ is indeed a section to q.

b) (ii) if and only (iii).

(Sufficiency) - Since q lifts over any map

$$h : (X,X_0) \to (E \sqcap B^I, E_0 \sqcap B_0^I)$$

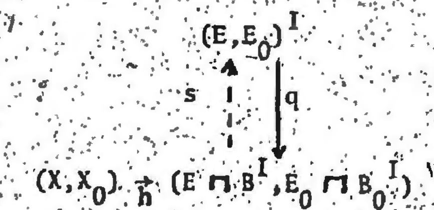
it lifts in particular over the identity, i.e. there exists a map γ as shown by dotted arrow making the diagram commute.



Thus γ is a section to q.

(Necessity)

Given the following diagram



we require a map $t : (X,X_0) \to (E,E_0)^I$ such that $qt = h$. Now q has a section s as shown by dotted arrow. Therefore $s \circ h$ lifts h. And

$$q \circ (s \circ h) = (q \circ s) \circ h = 1 \circ h = h.$$

c) (iv) if and only if (ii).

We prove only the "necessity" which we need for d).

From the pullback Diagram 2.2.9, we get the following commutative diagram in Toppair by using the exponential law -

$$\begin{array}{ccc} (E \sqcap B^I, E_0 \sqcap B_0^I) & \xrightarrow{\overline{\varepsilon}_0} & (E, E_0) \\ \downarrow & & \downarrow p \\ (E \sqcap B^I, E_0 \sqcap B_0^I) \times I & \xrightarrow[\hat{p}]{} & (B, B_0) \end{array} \qquad \text{Diagram 2.2.13}$$

But p has the CHP with respect to $(E \sqcap B^I, E_0 \sqcap B_0^I)$. Thus there exists $\hat{\tilde{p}}$ completing Diagram 2.2.13. Then

$$\hat{\tilde{p}} : (E \sqcap B^I, E_0 \sqcap B_0^I) \to (E, E_0)^I$$

is a section to q

$$\begin{aligned} \text{since} \quad q\hat{\tilde{p}}(e,\lambda)(t) &= (\hat{\tilde{p}}(e,\lambda)(0),\ p\hat{\tilde{p}}(e,\lambda)(t)) && [\text{Definition of } q] \\ &= (\hat{\tilde{p}}((e,\lambda),0),\ p\hat{\tilde{p}}((e,\lambda),t)) && [\text{Definition of } \hat{\tilde{p}}] \\ &= (\overline{\varepsilon}_0(e,\lambda),\ \hat{p}((e,\lambda),t)) && [\text{Diagram 2.2.13 commutes}] \\ &= (e,\ \overline{p}(e,\lambda)(t)) && [\text{Definition of } \overline{\varepsilon}_0, \hat{p}] \\ &= (e, \lambda(t)) = (e,\lambda)(t) = 1(e,\lambda)(t). \end{aligned}$$

So $\hat{\tilde{p}}$ is a section to q.

(i) if and only if (iv).

(Necessity) - Trivial

(Sufficiency) - Given Diagram 2.2.8 we require a completion.
Now there exists $s : (E \sqcap B^I, E_0 \sqcap B_0^I) \to (E, E_0)^I$ by (c).
and $\hat{s} : (E \sqcap B^I, E_0 \sqcap B_0^I) \times I \to (E, E_0)$
is a map completing the following diagram

$$\begin{array}{ccc} (E \sqcap B^I, E_0 \sqcap B_0^I) & \xrightarrow{\bar{\varepsilon}_0} & (E, E_0) \\ \Big\downarrow & \overset{\hat{s}}{\nearrow} & \Big\downarrow p \\ (E \sqcap B^I, E_0 \sqcap B_0^I) \times I & \xrightarrow[\hat{p}]{} & (B, B_0) \end{array}$$

Diagram 2.2.14

We also have $\gamma : (Z, Z_0) \to (E \sqcap B^I, E_0 \sqcap B_0^I)$, where

$$\gamma(z) = (f(z), \hat{F}(z)) \quad \text{from Diagram 2.2.10.}$$

Then $\hat{s}(\gamma \times 1) : (Z, Z_0) \times I \to (E, E_0)$ completes Diagram 2.2.8,

and
$$\begin{aligned} p\hat{s}(\gamma \times 1)(z,t) &= p\hat{s}((f(z), \hat{F}(z)), t) && [\text{Definition of } \gamma] \\ &= \hat{\bar{p}}((f(z), \hat{F}(z)), t) && [\text{Diagram 2.2.14 commutes}] \\ &= \bar{p}((f(z), \hat{F}(z)), t) \\ &= \hat{F}(z)(t) = F(z,t). \end{aligned}$$

So p is a fibration. This completes the proof of the proposition. //

Corollary 2.2.15: Any Top_* fibration is a Toppair fibration, and if $p : E \to B$ is a Top fibration then $p : (E, \phi) \to (B, \phi)$ is a Toppair fibration.

Proof: We prove the last statement first. Since $p : E \to B$ is a fibration in Top, $q_p : E^I \to E \sqcap B^I$ has a section in Top; hence $q_p : (E, \phi)^I \to (E \sqcap B^I, \phi)$ has a section in Toppair, i.e. p is a Toppair fibration. For the first part we have that $(E^I, *) \to (E \sqcap B^I, *)$ has a section in Top_* ;
hence $(E^I, *) \to (E \sqcap B^I, *)$ has a section in Toppair.
Thus the result. //

NOTE: We remark here that Proposition 2.1.3 can also be proved using this characterization of a Toppair fibration.

Proposition 2.2.16: If $p : (E,E_0) \to (B,B_0)$ is a Toppair map and $p : E \to B$, $p_0 : E_0 \to B_0$ are Top fibrations, then p has the Toppair CHP with respect to all pairs (X,X_0) that are closed with the HOMOTOPY EXTENSION PROPERTY in Toppair.

[Homotopy Extension Property in Toppair is defined in 4.2.1].

Proof: This involves lifting H_0 in 2.1.1 to $\tilde{H}_0$ at f_0 by p_0, then using STROM RELATIVE LIFTING THEOREM to lift $\tilde{H}$ extending $(f \times 0) \cup F_0 : X \times \{0\} \cup X_0 \times I \to E$. //

CHAPTER III - INTRODUCTION

In section one we first define the concept of groupoid and give examples. We then explain the notion of fibration of groupoids and prove that any Toppair fibration induces a fibration of groupoids.

Section two gives several propositions and lemmas which we require for the proof of the Co-glueing Theorem in Toppair. We note that key results of this section are given in Theorem 3.2.12 and Proposition 3.2.15.

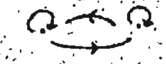
In section three we state and prove the Co-glueing Theorem for the category of Toppair and hence "a fortiori" for the category of Top.

§3.1 GROUPOIDS AND FIBRATIONS OF GROUPOIDS

Definition 3.1.1: A groupoid G is a small category in which every morphism is an isomorphism.

The following are examples of groupoids:

i)  called $\mathbb{1}$

ii)  called $\mathbb{2}$

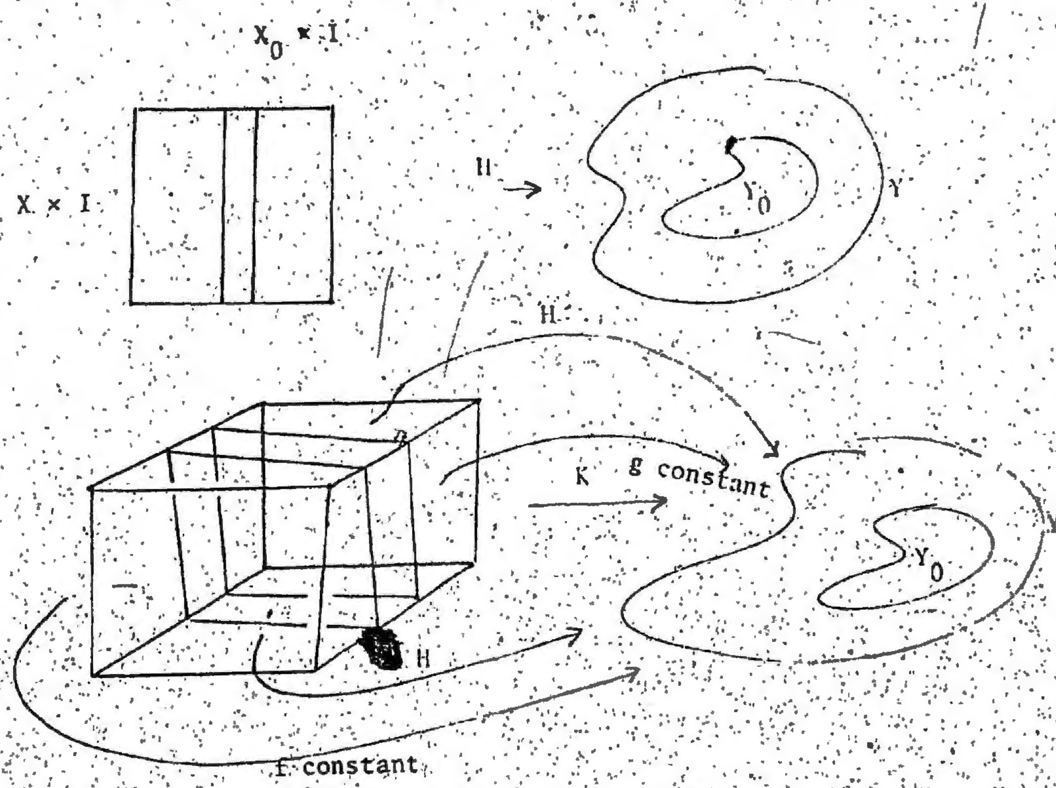
iii) any group G whose object is G itself and whose morphisms are the elements of G

iv) $\pi((X,X_0), (Y,Y_0))$ whose objects are the set of maps from $(X,X_0) \to (Y,Y_0)$ in Toppair denoted by Map $((X,X_0), (Y,Y_0))$. The set of morphisms between any two maps f and g is a quotient set of homotopies $H : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$ - this quotient may be empty.

The equivalence relation $\sim$ is given by : $H \sim H'$ if and only if there exists a homotopy $K : H \simeq H' : (X,X_0) \times I \times I \to (Y,Y_0)$ such that K is always $f(x)$ on the left hand side of the homotopy and $g(x)$ on the right hand side of the homotopy.

Note: We call the homotopy K in example (iv) a homotopy rel the end maps since the restriction to the "ends" $X \times \{0\} \times I$ and $X \times \{1\} \times I$ are f and g respectively.

Diagramatically we can view example (iv) as follows:



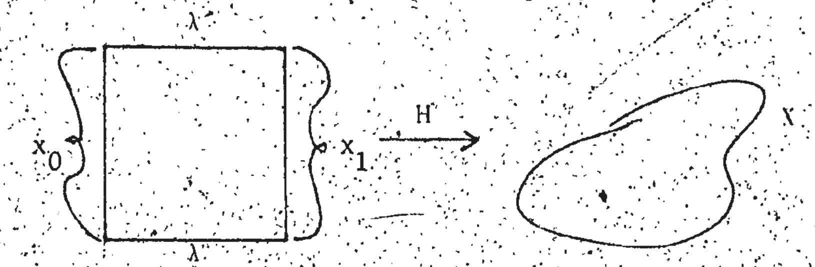
We prove that $\pi((X,X_0),(Y,Y_0))$ is in fact a groupoid in Proposition 3.1.4.

We note that this is a generalization of the definition of the groupoid $\pi(X,Y)$ given in Heath [13].

The groupoid $\pi(X,Y)$ is written as $\pi((X,\phi),(Y,\phi))$ in our notation.

Example: In particular we have that $\pi(*,X)$ is a groupoid written as $\pi(X)$ and called the FUNDAMENTAL GROUPOID OF X. Here the objects can be considered the underlying set of X and the morphisms, classes of paths between the objects, e.g. if $[\lambda]$ belongs to $Mor[\pi(X)]$ then $[\lambda]$ is a class of paths between x_0

and x_1, say, and $\lambda \simeq \lambda'$ if and only if there exists a homotopy $H : \lambda \simeq \lambda' : I \times I \to X$ such that $H(0,t) = x_0$, $H(1,t) = x_1$, i.e. we have H as follows:



SEE HEATH - QUEEN'S NOTES [13], BROWN [2] p. 217.

Proposition 3.1.2: $\simeq$ is an equivalence relation of pairs.

Proof: (i) Reflexivity - Let $f : (X,X_0) \to (Y,Y_0)$ be a map of pairs and let $H(-,t) : (X,X_0) \times I \to (Y,Y_0)$ be defined by $H(-,t) = f(x)$ for all t in I and all x in X. Then $H(-,0) = f(x)$ and $H(-,1) = f(x)$, so H is a homotopy between f and f, i.e. $f \simeq f$.

(ii) Symmetry - Let f,g be maps of pairs and let H be a homotopy between f and g, i.e.

$$H : f \simeq g : (X,X_0) \times I \to (Y,Y_0).$$

Let $H' : (X,X_0) \times I \to (Y,Y_0)$ be defined as follows: $H'(-,t) = H(-,1-t)$ for all t in I, and for all x in X. Then $H'(-,0) = H(-,1) = g(x)$ and $H'(-,1) = H(-,0) = f(x)$. Hence H' is a homotopy between g and f, i.e. $H' : g \simeq f : (X,X_0) \times I \to (Y,Y_0)$.

(iii) Transitivity - Let f,g,h be maps of pairs from (X,X_0) to (Y,Y_0);

H be a homotopy between f and g;

H' be a homotopy between g and h. Define a homotopy H'' as follows:

$$H''(-,t) = \begin{cases} H(-,2t), & 0 \le t \le \frac{1}{2} \\ H'(-,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Then $H''(-,0) = H(-,0) = f(x)$

and $H''(-,1) = H'(-,1) = h(x)$. H'' is obviously continuous.

Hence H'' is a homotopy between f and h.

//

Remark 3.1.3: If $F : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$ and $F' : g \simeq h : (X,X_0) \times I \to (Y,Y_0)$, then

$(F' + F) : f \simeq h : (X,X_0) \times I \to (Y,Y_0)$ is given by

$$(F' + F)(-,t) = \begin{cases} F(-,2t), & 0 \le t \le \frac{1}{2} \\ F'(-,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Furthermore if $F'' : h \simeq h' : (X,X_0) \times I \to (Y,Y_0)$ we can see that $(F'' + F') + F$ is equivalent to $F'' + (F' + F)$, i.e. we have associativity of composition. We prove this in Proposition 3.1.4.

We note that the addition which we use here to denote composition is not commutative.

Proposition 3.1.4: $\pi((X,X_0),(Y,Y_0))$ is a groupoid for all objects $(X,X_0),(Y,Y_0)$ in Toppair.

Proof: We have that $\simeq$ is an equivalence relation from Proposition 3.1.2 and we have composition of homotopies defined in Remark 3.1.3.

We first need to show a) the existence of identities and b) associativity of composition. This will ensure that $\pi((X,X_0),(Y,Y_0))$ is a category.

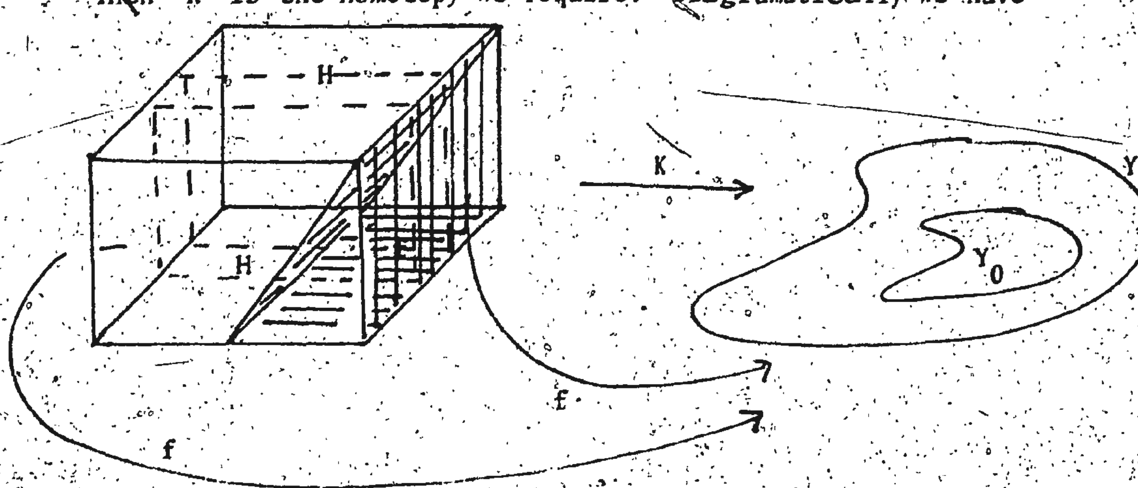
a) Let $H : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$ and $0_f : f \simeq f : (X,X_0) \times I \to (Y,Y_0)$ be homotopies where $0_f(-,t) = f(x)$ for all t in I and for all x in X.

We need that $0_f + H \sim H$, i.e. we need to define a homotopy $K : (X,X_0) \times I \times I \to (Y,Y_0)$ such that K is a homotopy between $0_f + H$ and H rel the end maps.

We define K as follows:

$$K(-,t,t') = \begin{cases} H(-,\frac{2t}{t'+1}) & 0 \leq t \leq \frac{t'+1}{2} \\ 0_f & \frac{t'+1}{2} \leq t \leq 1 \end{cases}$$

Then K is the homotopy we require. Diagramatically we have



In a similar manner we can show that

$H + O_g \sim H$. Thus the constant homotopy is the identity.

b) Let $H : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$,

$H' : g \simeq h : (X,X_0) \times I \to (Y,Y_0)$ and

$H'' : h \simeq \tau : (X,X_0) \times I \to (Y,Y_0)$. We need

that $(H'' + H') + H$ is equivalent to

$H'' + (H' + H)$, i.e.

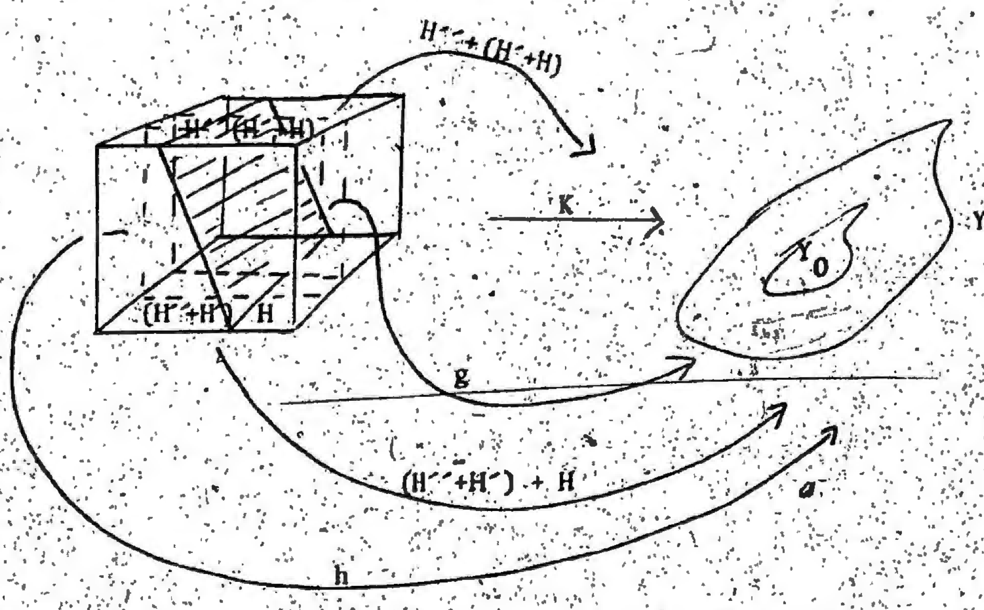
we need a homotopy

$K : (H'' + H') + H \simeq H'' + (H' + H)$ rel the end maps.

Let K be defined as follows:

$$K(-,t,t') = \begin{cases} H''(-, \frac{4t}{t'+1}), & 0 \le t \le \frac{t'+1}{4} \\ H'(-,4t - t' - 1), & \frac{t'+1}{4} \le t \le \frac{t'+2}{4} \\ H(-, \frac{4t-2-t'}{2-t'}), & \frac{t'+2}{4} \le t \le 1 \end{cases}$$

Then K is the homotopy we require. Diagramatically we have



So a) and b) hold and hence $\pi((X,X_0),(Y,Y_0))$ is a well-defined category. We still need to show that, each H in $Mor[\pi((X,X_0),(Y,Y_0))]$ has an inverse. To this purpose let $H : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$ be a morphism in $\pi((X,X_0),(Y,Y_0))$. We define $-H$ as follows:

$-H(-,t) = H(-,1-t)$ for all t in I and all x in X.

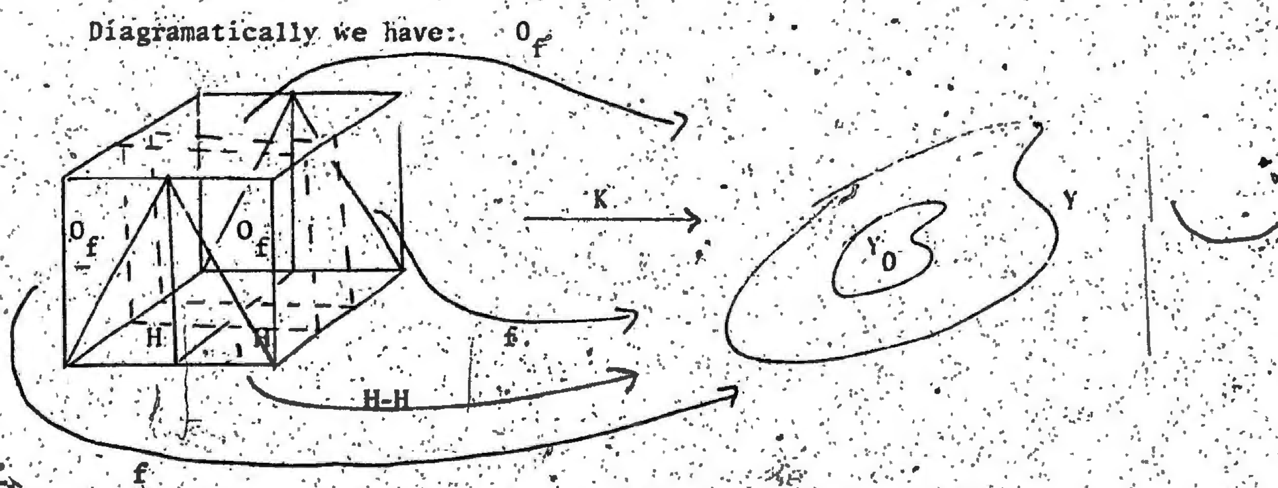
Then $-H(-,0) = H(-,1) = g(x)$ and $-H(-,1) = H(-,0) = f(x)$.

We need to check that $H - H \simeq 0_f$. Let

$K : (X,X_0) \times I \times I \to (Y,Y_0)$ be a homotopy as follows:

$$K(-,t,t') = \begin{cases} H(-,0) & , \ 0 \le t \le \frac{t'}{2} \\ H(-,2t-t') & , \ \frac{t'}{2} \le t \le \frac{1}{2} \\ H(-,2-2t-t') & , \ \frac{1}{2} \le t \le 1 - \frac{t'}{2} \\ H(-,0) & , \ 1 - \frac{t'}{2} \le t \le 1 \end{cases}$$

Diagramatically we have:



Then K is the required homotopy - a homotopy between $H - H$ and 0_f rel the end maps. See Spanier [25]. This last fact along with

a) and b) above prove that $\pi((X,X_0),(Y,Y_0))$ is indeed a groupoid.

//

Remark 3.1.5.: Let $(X,X_0) \xrightarrow{f} (Z,Z_0)$ and $(Y,Y_0) \xrightarrow{g} (W,W_0)$ be maps in Toppair. Then f induces

$$f^* : \pi((Z,Z_0),(Y,Y_0)) \to \pi((X,X_0),(Y,Y_0))$$

and g induces

$$g_* : \pi((X,X_0),(Y,Y_0)) \to \pi((X,X_0),(W,W_0))$$

as follows. For any object h in $\pi((X,X_0),(Y,Y_0))$ $f^*(h) = (h + f)$ and for any object r in $\pi((Z,Z_0),(Y,Y_0))$ $g_*(r) = (g + r)$. Furthermore, if $[H]$ belongs to $\mathrm{Mor}(\pi((Z,Z_0),(Y,Y_0)))$ and a representative H of $[H]$ is a homotopy between k and ℓ, say, then we define

$$f^*([H]) = [H + (f \times 1)] \text{ where } H + (f \times 1)$$

is a homotopy between $k + f$ and $\ell + f$. Similarly if $[H']$ belongs to $\mathrm{Mor}(\pi((X,X_0),(Y,Y_0)))$ and a representative H' of $[H']$ is a homotopy between k' and ℓ', say, then we define

$$g_*([H']) = [g + H'] \text{ where } g + H' \text{ is a}$$

homotopy between $g + k'$ and $g + \ell'$. It can be readily seen that f^* and g_* are well-defined.

Lemma 3.1.6.: f^* and g_* are functors.

Proof: That f^* is an object function assigning every object t in $\pi((Z,Z_0),(Y,Y_0))$ to an object $f^*(t)$ in $\pi((X,X_0),(Y,Y_0))$ is obvious from the definition. It is also obvious from the definition that f^* is a morphism function. We simply need to check that f^* preserves a) identities and b) composition.

a) Let $0_h : h \to h$ be the constant homotopy at h, i.e. $0_h = C_h$.

Then $f^*([0_h]) = f^*([C_h]) = [C_h + (f \times 1)]$ (definition of f^*)

$= [C_{h+f}]$ ($f \approx f \times 1$ since 1 holds I fixed).

But $[C_{h+f}] = [0_{h+f}] = [0_{f^*h}]$.

Thus f^* preserves identities.

b) Let H,K be morphisms of $\pi((Z,Z_0),(Y,Y_0))$.

Then $f^*([H] + [K]) = f^*([H + K])$

$= [(H + K) + (f \times 1)]$.

and $f^*([H]) + f^*([K]) = [H + (f \times 1)] + [K + (f \times 1)]$

$= [H + (f \times 1) + K + (f \times 1)]$

$= [(H + K) + (f \times 1)]$

Hence f^* preserves composition. Note that we are using the additive notation for composition.

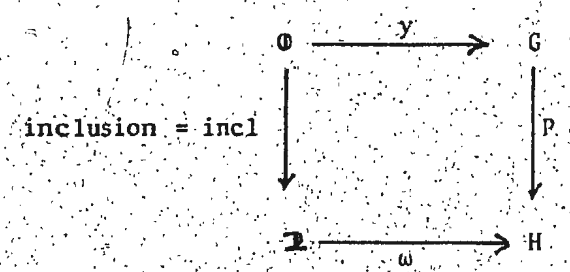
Now a) and b) together imply that f^* is a contravariant factor. In a similar manner we can show that g_* is a covariant functor.

//

Definition 3.1.7.: Let G be a groupoid. We partition the set of objects of G into equivalence classes called COMPONENTS and denoted $\pi_0 G$ as follows: Two objects x and y of G are in the same class if and only if there exists a functor $\omega : \mathbf{2} \to G$ such that

$\omega(0) = x$ and $\omega(1) = y$, i.e. there is a morphism in G between x and y.

Definition 3.1.8.: A functor $p : G \to H$ of groupoids is said to be a FIBRATION of GROUPOIDS (Heath [13], Brown [4]) if, given any commutative diagram as follows,



there exists a functor $\tilde{\omega} : \mathbf{2} \to G$ such that

$$p\tilde{\omega} = \omega \quad \text{and} \quad \tilde{\omega} i = f.$$

NOTE: $\mathbf{0}$ and $\mathbf{2}$ are examples i) and ii) page 44 and y is an object of G while ω is an invertible morphism of the groupoid H which we identify with $\omega(i)$.

Example 3.1.9.: A groupoid functor $p : G \to H$ which is surjective on both objects and morphisms is a fibration of groupoids.

Proposition 3.1.10.: If $p : (E,E_0) \to (B,B_0)$ has the CHP with respect to (Z,Z_0) then $p_* : \pi((Z,Z_0),(E,E_0)) \to \pi((Z,Z_0),(B,B_0))$ is a fibration of groupoids.

Proof: Suppose we are given any commutative diagram of groupoids and functors of the following form:

$$\begin{array}{ccc} \mathbf{0} & \xrightarrow{\tilde{f}} & \pi((Z,Z_0),(E,E_0)) \\ \Big\downarrow \text{incl} & & \Big\downarrow p_* \\ \mathbf{2} & \xrightarrow[{[H]:f\simeq g}]{} & \pi((Z,Z_0),(B,B_0)) \end{array}$$

Diagram 3.1.11

Let H belong to $[H]$. Then the above diagram gives rise to the following commutative diagram in Toppair:

$$\begin{array}{ccc} (Z,Z_0) & \xrightarrow{\tilde{f}} & (E,E_0) \\ \Big\downarrow \text{incl} & & \Big\downarrow p \\ (Z,Z_0)\times I & \xrightarrow[H:f\simeq g]{} & (B,B_0) \end{array}$$

But p has the CHP with respect to (Z,Z_0), so there exists a homotopy $\tilde{H} : \tilde{f} \simeq \tilde{H}(-,1)$ such that $p\tilde{H} = H$. Taking $[\tilde{H}] : \tilde{f} \simeq \tilde{H}(-,1) : \mathbf{2} \to \pi((Z,Z_0),(E,E_0))$, we have the required functor making Diagram 3.1.11 commute. Hence p_* is a fibration of groupoids.

//

Before giving two corollaries to the proposition, we agree to identify the groupoid $\pi((Z,\phi),(E,\phi))$ and the groupoid $\pi(Z,E)$. We explain why in the following paragraph.

Now $p : (E,\phi) \to (B,\phi)$ is a map in Toppair, i.e. a map $p : E \to B$ such that $p_0(\phi) \subseteq \phi$. But this is nothing more than a map in Top $p : E \to B$. Thus the objects of the groupoid $\pi((Z,\phi),(E,\phi))$ are maps $p : Z \to E$ such that $p_0(\phi) \subseteq \phi$, i.e. maps

$p : Z \to E$. The set of morphisms between any two maps f and g is a quotient set of homotopies $H : f \simeq g : (Z,\phi) \times I \to (E,\phi)$. The equivalence relation $\sim$ is given by: $H \sim H'$ if and only if there exists a homotopy $K : H \simeq H' : (Z,\phi) \times I \times I \to (E,\phi)$ such that $K(z,0,t) = f(z)$ and $K(z,1,t) = g(z)$.

We note that this set of morphisms is the same as the set given for $\pi(Z,E)$ in [13] since $H_0(\phi \times I) \subsetneq \phi$ and $K_0(\phi \times I \times I) \subsetneq \phi$. Now $\pi(Z,E)$ has the "same" objects and morphisms, so the two groupoids can be identified.

Similarly, $p : (E,e_o) \to (B,b_o)$ is a map in Toppair such that $p(e_o) = b_o$. But this is also a Top_* map.

Corollary 3.1.12 : If $p : E \to B$ is a fibration in Top, then for all Z, $p_* : \pi(Z,E) \to \pi(Z,B)$ is a fibration of groupoids.

Proof: Now $p : (E,\phi) \to (B,\phi)$ has the CHP with respect to (Z,ϕ), for all (Z,ϕ) in Toppair. Hence p_* is a fibration of groupoids.

Corollary 3.1.13 : If $p : (E,*) \to (B,*)$ is a fibration in Top_*, then for all $(Z,*)$, $p_* : \pi((Z,*),(E,*)) \to \pi((Z,*),(B,*))$ is a fibration of groupoids.

Proof: This is similar to Heath [13].

§3,2 GROUPOID FIBRES AND HOMOTOPY EQUIVALENCES IN THE CATEGORY OF COMMUTATIVE SQUARES IN TOPPAIR

Definition 3.2.1. Let $f \in Ob[\pi((Z,Z_0),(B,B_0))]$ and

$$p_* : \pi((Z,Z_0),(E,E_0)) \to \pi((Z,Z_0),(B,B_0))$$

be a groupoid functor. Then the GROUPOID FIBRE over f, written F_f, is the following pullback of groupoids:

$$\begin{array}{ccc} F_f & \hookrightarrow & \pi((Z,Z_0),(E,E_0)) \\ \downarrow & & \downarrow p_* \\ f & \hookrightarrow & \pi((Z,Z_0),(B,B_0)) \end{array}$$

(definition of pullback of groupoids is the obvious one).

In the diagram that follows we show an example of a groupoid fibre over f containing <u>two</u> components, i.e. $\pi_o F_f$ with two components [see Definition 3.1.7 for $\pi_o G$].

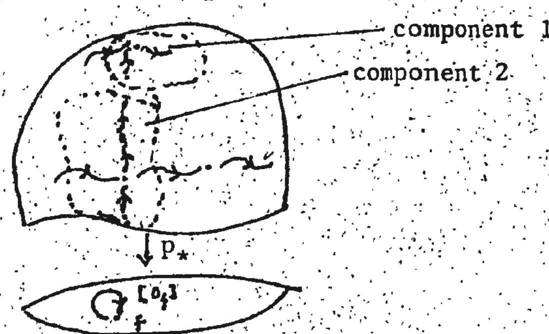


Diagram 3.2.2

THEOREM 3.2.3. - Brown [2], Heath [13]. Let $p : G \to H$ be a fibration of groupoids. Then there exists a functor $\# : H \to Set$ such that $\#(b) = \pi_o F_b$ for an object b in Ob H. For a morphism $\omega : b \to d$ in H, $\#(\omega)$, written $\omega_\#$ in Heath [13], is a bijection $\pi_o F_b \to \pi_o F_d$.

Proof: We do this proof in several steps.

a) Let $\omega : b \to d$ be a morphism in H and let g belong to $\text{Ob } F_b$.

Then we have the following commutative diagram of groupoids (taking g as an object of G):

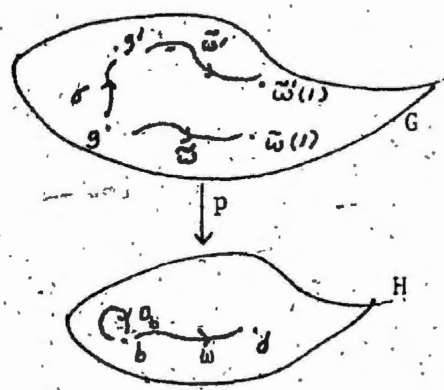
$$\begin{array}{ccc} 0 & \xrightarrow{g} & G \\ \downarrow & & \downarrow p \\ 2 & \xrightarrow[\omega]{} & H \end{array}$$

But p is a fibration of groupoids; therefore there exists $\tilde{\omega} : 2 \to G$ making the diagram commute. We define $\#(\omega)(\pi_0 g) = \pi_0(\tilde{\omega}(1))$, i.e.

$$\left(\begin{array}{ccc} \pi_0 F_b & \xrightarrow{\#(\omega)} & \pi_0 F_d \\ \pi_0 g & \longmapsto & \pi_0(\tilde{\omega}(1)) \end{array}\right)$$

b) We show that $\#(\omega)$ is well-defined.

Now we have chosen g within its component in F_b and $\tilde{\omega}$ as a lift of ω. Let $\pi_0 g = \pi_0 g'$; that is, let g, g' belong to the same component, and let $\tilde{\omega}'$ be a lift of ω at g' [we can do this since p is a fibration of groupoids]. Then there exists $\sigma : g \to g'$ in F_b such that $p(\sigma) = 0_b$. Now the morphism $\tilde{\omega}' + \sigma + \tilde{\omega}^{-1} : \tilde{\omega}(1) \to \tilde{\omega}'(1)$ is a morphism in F_b since $p(\tilde{\omega}' + \sigma + \tilde{\omega}^{-1}) = p(\tilde{\omega}) + p(\sigma) + p(\tilde{\omega}^{-1}) = 0_b$. See diagram which follows.



Hence $\#(\omega)$ is well-defined.

c) We next show that $\#(0) = 0$. This follows since, if $\omega = 0_b$ in the diagram above, we can choose $\tilde{\omega}$ to be the identity at g, i.e. 0_g. Then $\#(0_b)$ links $\pi_o F_b$ to $\pi_o F_b$ and $\#(0_b)(\pi_o g) = \pi_o g$.

Hence $\#(0_b) = 0_{\pi_o F_b} = 0_{\#(b)}$ and $\#$ preserves identities.

d) We then show that $\#(\eta + \omega) = \#(\eta) + \#(\omega)$. Consider g, ω, $\tilde{\omega}$ as in a) and let $\eta : d \to c$ be given. Choose $\tilde{\eta}$ as a lift of η at $\tilde{\omega}(1)$; then $\tilde{\eta} + \tilde{\omega}$ is a lift of $\eta + \omega$ at g, i.e. $\#$ preserves composition. Thus $\#$ is indeed a functor.

e) We finally show that $\#(\omega)$ is a bijection for each $\omega : b \to d$. Consider $\omega^{-1} : d \to b$ for all such ω. Then

$$\#(\omega) + \#(\omega^{-1}) = \#(\omega + \omega^{-1}) \quad \text{(by (d))}$$
$$= \#(0) = 0 \quad \text{(by (c))}.$$

Similarly $\#(\omega^{-1}) + \#(\omega) = 0$. Thus $\#(\omega)$ is a bijection.

//

Corollary 3.2.4.: If $p_* : \pi((X,X_0),(E,E_0)) \to \pi((X,X_0),(B,B_0))$ is a fibration of groupoids, then $\# : \pi((X,X_0),(B,B_0)) \to \text{Set}$ defined by $\#(f) = \pi_0 F_f$, the set of path components of the fibre over f, on objects; and $\#([\omega]) = [\tilde{\omega}(1)]$ on morphisms, is a functor.

Proof: It follows immediately from the Theorem above.

//

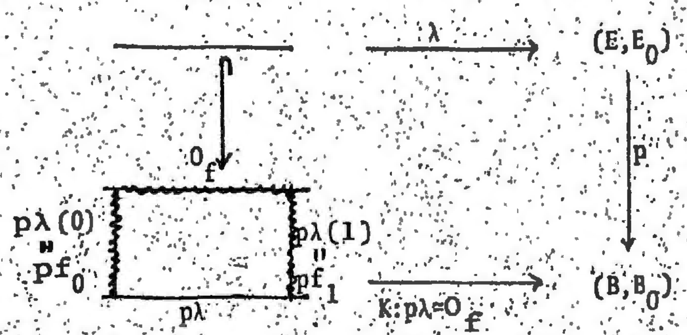
Definition 3.2.5[1]: Let $q : X \to B$, $p : E \to B$ be maps. A FIBRE MAP $f : q \to p$ is a map $f : X \to E$ such that $pf = q$. A fibre (vertical) homotopy $H : f \simeq g : q \to p$ is a homotopy $H : f \simeq g : X \times I \to E$ such that $pH = 0_q$. The map p is said to be FIBRE HOMOTOPIC EQUIVALENT to the map q if there exist maps $h : E \to X$, $r : X \to E$ with $pr = q$, $qh = p$ and homotopies $H : rh \simeq 1_E$ rel p [i.e. pH constant], $K : hr \simeq 1_X$ rel q [i.e. qK constant]). We denote the set of (vertical) homotopy classes of fibre maps from q into p by $[X,E]_{//q}$ or $[f,p]$.

Lemma 3.2.6.: If $p : (E,E_0) \to (B,B_0)$ is a fibration in Toppair and f belongs to $\text{Ob}[\pi((X,X_0),(B,B_0))]$ and $\lambda : f_0 \simeq f_1 : (X,X_0) \times I \to (E,E_0)$ such that $p\lambda \sim 0_f$, then there

[1]This definition readily adapts to the Toppair category and we write $[(X,X_0),(E,E_0)]_{//q}$ or $[f,p]$ where no confusion arises.

exists a homotopy $\lambda' : f_0 \simeq f_1 : (X,X_0) \times I \to (E,E_0)$ such that $p\lambda' = 0_f$.

Proof: Now $p\lambda \simeq 0_f$ implies that there exists a homotopy $K : p\lambda \simeq 0_f : (X,X_0) \times I \times I \to (B,B_0)$ rel end maps [see diagram which follows].



But p is a fibration. So there exists

$\tilde{K} : (X,X_0) \times I \times I \to (E,E_0)$ such that $p\tilde{K} = K$ and $\tilde{K} : \lambda \simeq \tilde{K}(-,-,1)$. By definition, $\tilde{K}(-,-,0) = \lambda$ which links f_0 and f_1. Let $\tilde{K}$, restricted to $(X,X_0) \times [(I \times \{1\}) \cup (\{0\} \times I) \cup (\{1\} \times I)]$, be called R, and let R be defined as follows:

$$R(-,s,t) = \begin{cases} \tilde{K}(-,0,3t), & 0 \leq t \leq 1/3, \quad s = 0 \\ \tilde{K}(-,3s-1,1), & 1/3 \leq s \leq 2/3, \quad t = 1 \\ \tilde{K}(-,1,3-3t), & 2/3 \leq t \leq 1, \quad s = 1 \end{cases}$$

Then $pR = p\tilde{K}| = K| = 0_f + (0_f + 0_f) \simeq 0_f$, and

$R(-,-,0) = \tilde{K}(-,0,0) = \lambda(-,0) = f_0$

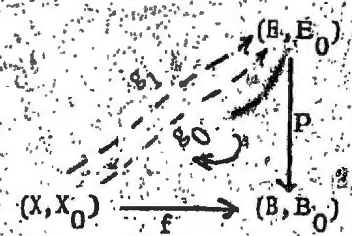
$R(-,-,1) = \tilde{K}(-,1,0) = \lambda(-,1) = f_1$

So R is the homotopy λ' we require. //

Proposition 3.2.7.: If $p : (E,E_0) \to (B,B_0)$ is a fibration in Toppair, then $[f,p]$ or $[(X,X_0),(E,E_0)]_{//f} = \pi_0 F_f$.

Proof: We want that $[f,p] \subseteq \pi_0 F_f$ and $\pi_0 F_f \subseteq [f,p]$.

Since p is a fibration in Toppair, then by Proposition 3.1.10 $p_* : \pi((X,X_0),(E,E_0)) \to \pi((X,X_0),(B,B_0))$ is a fibration of groupoids. Let g_0, g_1 be objects of $[f,p]$ (see diagram below).



Then by definition of $[f,p]$, there exists a homotopy $H : g_0 \simeq g_1 : (X,X_0) \times I \to (B,B_0)$ such that $pH = 0_f$. But this implies that $[pH] = [0_f]$. Hence $pH \sim 0_f$ (see Theorem 3.2.3) and $[f,p] \subseteq \pi_0 F_f$. Lemma 3.2.6 gives us the opposite inclusion. Thus $\pi_0 F_f = [f,p]$.

//

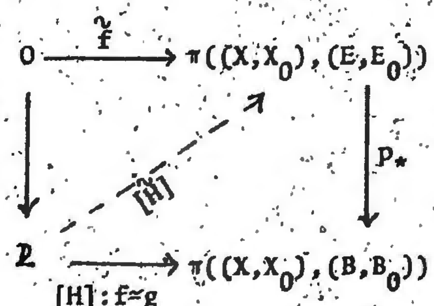
Corollary 3.2.8.: a) If $p : (E,E_0) \to (B,B_0)$ is a fibration in Toppair and if $H : f \simeq g : (X,X_0) \times I \to (B,B_0)$, then there exists a bijection

$$\#([H]) : [f,p] \to [g,p].$$

b) If $\tilde{f},\tilde{g}$ are lifts of f and g respectively, and $\#([H])[\tilde{f}] = [\tilde{g}]$, then there exists a lift H' of $0_f + H$ such that
$H' : \tilde{f} \simeq \tilde{g} : (X,X_0) \times I \to (E,E_0)$.

Proof: a) Since $\pi_0 F_f = [f,p]$, $\pi_0 F_g = [g,p]$ and since a functor preserves isomorphisms [Proposition 1.1.3] the result follows.

b) The following diagram is commutative in the category of groupoids and p_* is a fibration of groupoids since p is a fibration in Toppair.



Let $\tilde{H}$ belong to $[\tilde{H}]$. Then $\tilde{H} : \tilde{f} \simeq \tilde{H}(-,1) : (X,X_0) \times I \to (E,E_0)$.
By definition of $\#$ we have that $\#([H])[\tilde{f}] = [\tilde{H}(-,1)]$. But $\#([H])[\tilde{f}] = [\tilde{g}]$ by hypothesis. Hence
$[\tilde{H}(-,1)] = [\tilde{g}]$ in $\pi_0 F_g$, and there exists a homotopy
$K : \tilde{H}(-,1) \simeq \tilde{g} : (X,X_0) \times I \to (E,E_0)$ such that $pK = 0_f$.
Then $K + \tilde{H} : \tilde{f} \simeq \tilde{g} : (X,X_0) \times I \to (E,E_0)$ and

$$p(K + \tilde{H}) = pK + p\tilde{H} = 0_f + H \text{ so that } K + \tilde{H}$$

is the required lift H' of $0_f + H$.

//

Lemma 3.2.9.: If $H : f \simeq g : (W,W_0) \times I \to (X,X_0)$, then there exists a morphism $[\theta]$ in $\mathrm{Mor}[\pi((W,W_0),(B,B_0))]$ such that the following diagram commutes for each $k : (X,X_0) \to (B,B_0)$ and Toppair fibration $p : (E,E_0) \to (B,B_0)$.

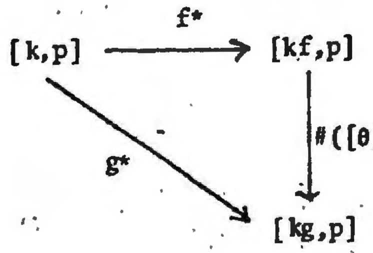
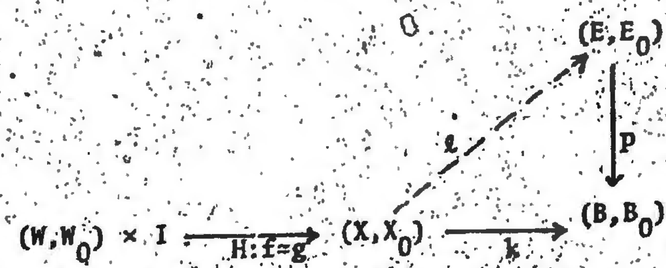


Diagram 3.2.10

Proof: Now $\#([\theta])$ is a bijection [Corollary 3.2.8(a)], and we have the following commutative diagram in Toppair:



where (ℓ) belongs to $[k,p]$, i.e. $p\ell = k$.

It is obvious that $kH : kf \simeq kg : (W,W_0) \times I \to (B,B_0)$ so that $[kH]$ is a morphism in $\pi((W,W_0),(B,B_0))$ and this is the morphism $[\theta]$ we require. Now ℓH is a lift of kH and $\#([\theta])f^*(\ell) = \#([\theta])(\ell f) = \ell g = g^*(\ell)$.

Thus the diagram commutes.

//

Proposition 3.2.11.: If $f : (W,W_0) \to (X,X_0)$ is a homotopy equivalence in Toppair and $p : (E,E_0) \to (B,B_0)$ has the CHP with respect to (X,X_0) and if $k : (X,X_0) \to (B,B_0)$ is a map, then $f^* : [k,p] \to [kf,p]$ is a bijection.

Proof: As a special case let $(W,W_0) = (X,X_0)$ in Lemma 3.2.9 and let $f = 1$. Then $f^* : [k,p] \to [kf,p]$ is a bijection since Diagram 3.2.10 commutes.

For the general case let $g : (X,X_0) \to (W,W_0)$ be a homotopy inverse of f, i.e. $gf \simeq 1_{(W,W_0)}$ and $fg \simeq 1_{(X,X_0)}$.

Consider the following diagram:

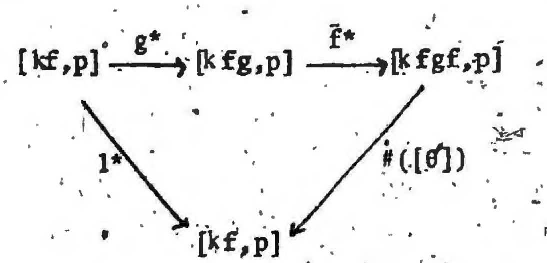
$$\begin{array}{ccccc} [(X,X_0),(E,E_0)]_{//k} & \xrightarrow{f^*} & [(W,W_0),(E,E_0)]_{//kf} & \xrightarrow{g^*} & [(X,X_0),(E,E_0)]_{//kfg} \\ & \searrow^{1^*} & & \swarrow_{\#([\theta])} & \\ & & [(X,X_0),(E,E_0)]_{//k} & & \end{array}$$

which is equivalent to

$$\begin{array}{ccccc} [k,p] & \xrightarrow{f^*} & [kf,p] & \xrightarrow{g^*} & [kfg,p] \\ & \searrow^{1^*} & & \swarrow_{\#([\theta])} & \\ & & [k,p] & & \end{array}$$

Now $\#([\theta])g^*f^* = 1^*$ [Lemma 3.2.9].

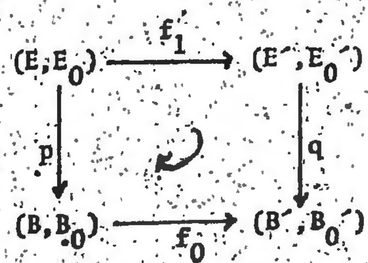
But $\#([\theta])$ is a bijection [Corollary 3.2.8(a)] and 1^* is a bijection, so g^*f^* is a bijection. This implies that g^* is epi and f^* is mono. Consider the following diagram:



As above this diagram commutes and $\#([\theta'])$ is a bijection, so $\#([\theta'])\bar{f}^*g^* = 1^*$. But this implies that $\bar{f}^*g^*$ is a bijection. Thus g^* is mono and hence a bijection. But g^*f^* is a bijection from above - so f^* is a bijection as required.

//

THEOREM 3.2.12: If f_0 and f_1 are homotopy equivalences and p and q have the CHP with respect to (E,E_0), $(E,E_0) \times I$, (E',E_0'), $(E',E_0') \times I$ in the following commutative diagram,



then (f_0,f_1) is a homotopy equivalence in the category of commutative squares in Toppair.

Proof: Let g_0 be a homotopy inverse of f_0 and consider the following diagram

$$\begin{array}{ccccc} (E,E_0) & \xrightarrow{f_1} & (E',E_0') & & (E,E_0) \\ \downarrow p & & \downarrow q & & \downarrow p \\ (B,B_0) & \xrightarrow[f_0]{} & (B',B_0') & \xrightarrow[g_0]{} & (B,B_0) \end{array}$$

Diagram 3.2.13

Consider also the sequence

$$[(E',E_0'),(E,E_0)]_{//g_0q} \xrightarrow{f_1^*} [(E,E_0),(E,E_0)]_{//g_0qf_1=g_0f_0p} \xrightarrow{\#([\theta])} [(E,E_0),(E,E_0)]_{//p}$$

which is equivalent to

$$[g_0q,p] \xrightarrow{f_1^*} [g_0qf_1,p] = [g_0f_0p,p] \xrightarrow{\#([\theta])} [p,p]$$

[equality follows from commutativity of given Diagram 3.2.13]

where $\theta = H(p \times 1)$ and $H : g_0f_0 \simeq 1 : (B,B_0) \times I \to (B,B_0)$.

Now we have the following diagram

$$\begin{array}{ccc} & & (E,E_0) \\ & \overset{1}{\nearrow} & \downarrow p \\ (E,E_0) & \xrightarrow[p]{} & (B,B_0) \end{array}$$

so that $[p,p] \neq \phi$.

But since $\#([\theta])$ and f_1^* are bijections [Corollary 3.2.8(a), Proposition 3.2.11 respectively], this implies that $[g_0q,p] \neq \phi$, i.e. that there exists $g_1 : (E',E_0') \to (E,E_0)$ such that $pg_1 = g_0q$ and further that $\#([\theta])(f_1^*)[g_1] = [1]$.

Thus, by definition of $f_1^*([g_1])$, $\#([\theta])(f_1^*)[g_1] = \#([\theta])([g_1f_1])=[1]$.

[Equation A].

We also have that $p1 = p$ and $pg_1f_1 = g_0f_0p$. Thus 1 and g_1f_1 are lifts of p and g_0f_0p respectively. Using this latter fact and equation [A] above, we are in the situation of Corollary 3.2.8(b); hence there exists a homotopy $K : g_1f_1 \simeq 1$ such that K is a lift of $0 + H$. We have actually found a left homotopy inverse of f_1 making Diagram 3.2.13 commute. We can apply a similar argument to show that there exists a map $\overline{f} : (E,E_0) \to (E',E_0')$ making the diagram commute and a homotopy $K' : \overline{f}g_1 \simeq 1$ over $0 + H'$ where $H' : f_0g_0 \simeq 1$.

Note: K is actually the homotopy derived as follows:

$$\begin{array}{ccc}
0 & \xrightarrow{\;g_1f_1\;} & \pi((E,E_0),(E,E_0)) \\
\downarrow & \overset{[\tilde{H}(p\times 1)]}{\nearrow} & \downarrow p_* \\
I & \xrightarrow[H(p\times 1):g_0f_0p\simeq p]{} & \pi((E,E_0),(B,B_0))
\end{array}$$

p_* is a fibration of groupoids since p is a fibration in Toppair for all (X,X_0) and in particular for $(X,X_0) = E,E_0)$.

Now $\#([H(p\times 1)])([g_1f_1]) = [1]$ and $\tilde{H}(p\times 1)$ belongs to $[\tilde{H}(p\times 1)]$, so $\tilde{H}(p\times 1) : g_1f_1 \simeq \tilde{H}(p\times 1)(-,1)$, and $\#([H(p\times 1)])([g_1f_1]) = [\tilde{H}(p\times 1)(-,1)]$ by definition of $\#$; therefore $[\tilde{H}(p\times 1)(-,1)] = [1]$ in $\pi_o F_p = [p,p]$.

Hence there exists a homotopy

$G : \tilde{H}(p \times 1)(-,1) \simeq 1 : E,E_0) \times I \to (E,E_0)$

such that $pG = 0_{g_0 f_0 p}$ by definition of $[p,p]$ or $\pi_0 F_p$.

Then $G + \tilde{H}(p \times 1)$ is the K we require, i.e.

$$G + \tilde{H}(p \times 1) : g_1 f_1 \simeq 1 \text{ and } p(G + \tilde{H}(p \times 1)) = pG + p\tilde{H}(p \times 1) = 0_{g_0 f_0 p} + H(p \times 1).$$

Then we have $(g_1,g_0) \circ (f_1,f_0) \simeq (1,1)$.

We require $(f_1,f_0) \circ (g_1,g_0) \simeq (1,1)$. But

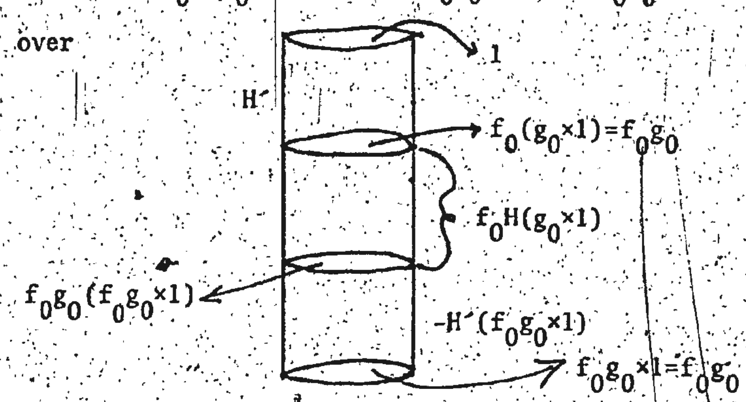
$(f_1,f_0) \circ (g_1,g_0) = (1,1) \circ (f_1,f_0) \circ (g_1,g_0)$.

$$\begin{aligned} \text{And } (1,1) \circ (f_1,f_0) \circ (g_1,g_0) &\simeq (\bar{f},f_0) \circ (g_1,g_0) \circ (f_1,f_0) \circ (g_1,g_0) \\ &\simeq (\bar{f},f_0) \circ (1,1) \circ (g_1,g_0) \\ &\simeq (\bar{f},f_0) \circ (g_1,g_0) \\ &\simeq (1,1) \end{aligned}$$

So (f_1,f_0) is indeed a homotopy equivalence in the category of commutative squares in Toppair.

//

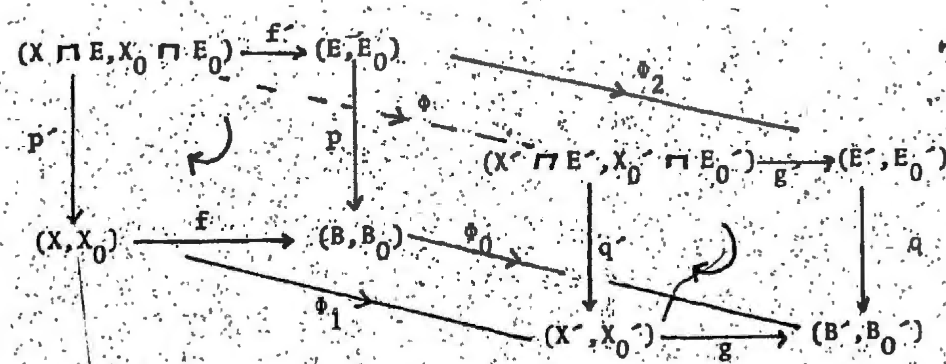
Remark 3.2.14.: $f_1 g_1$ is homotopic to 1 over a conjugate homotopy $H' + f_0 H(g_0 \times 1) - H'(f_0 g_0 \times 1) : f_0 g_0 \simeq 1$, i.e. over



where $(B',B_0') \times I \xrightarrow[g_0\times 1]{} (B,B_0) \times I \xrightarrow[H:g_0 f_0 \simeq 1]{} (B,B_0) \xrightarrow[f_0]{} (B',B_0')$

and $(B',B_0') \times I \xrightarrow[f_0 g_0 \times 1]{} (B',B_0') \times I \xrightarrow[H':f_0 g_0 \simeq 1]{} (B',B_0')$.

Proposition 3.2.15.: If in the following diagram p,q,f,g are fibrations in Toppair and Φ_0, Φ_1, Φ_2 are homotopy equivalences in Toppair, then the induced map Φ is a homotopy equivalence in Toppair.



Proof: By Theorem 3.2.12 there exists g_0, a homotopy inverse of Φ_0.

Let $K : g_0 \Phi_0 \simeq 1_{(B,B_0)}$ and $\overline{K} : \Phi_0 g_0 \simeq 1_{(B',B_0')}$. There also exists g_1, a homotopy inverse of Φ_1 with $g_1 \Phi_1 \simeq 1_{(X,X_0)}$, $\Phi_1 g_1 \simeq 1_{(X',X_0')}$, and there exists g_2, a homotopy inverse of Φ_2 with $g_2 \Phi_2 \simeq 1_{(E,E_0)}$, $\Phi_2 g_2 \simeq 1_{(E',E_0')}$ making the following diagram commute.

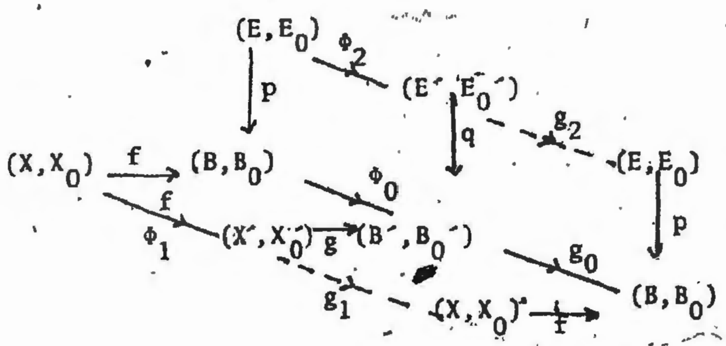


Diagram 3.2.16

Now by the same Theorem we know that there exist lifts F of $0 \to K$, $\overline{F}$ of $0 \to \overline{K}$, i.e. $F : g_2\Phi_2 \simeq 1 : (E,E_0) \times I \to (E,E_0)$ and $\overline{F} : \Phi_2 g_2 \simeq 1 : (E',E_0') \times I \to (E',E_0')$. We can also find lifts $G : g_1\Phi_1 \simeq 1 : (X,X_0) \times I \to (X,X_0)$ and $\overline{G} : \Phi_1 g_1 \simeq 1 : (X',X_0') \times I \to (X',X_0')$ where

$pF = K(p \times 1) : g_0\Phi_0 p \simeq p : (E,E_0) \times I \to (B,B_0)$ [Equation B]

$q\overline{F} = \overline{K}(q \times 1) : \Phi_0 g_0 q \simeq q : (E',E_0') \times I \to (B',B_0')$ [Equation B']

$fG = K(f \times 1) : g_0\Phi_0 f \simeq f : (X,X_0) \times I \to (B,B_0)$ [Equation C]

and

$g\overline{G} = \overline{K}(g \times 1) : \Phi_0 g_0 g \simeq g : (X',X_0') \times I \to (B',B_0')$ [Equation C'].

We also have that $F(f' \times 1)$ and $G(p' \times 1)$ are the following homotopies:

$$F(f' \times 1) : g_2\Phi_2 f' \simeq f' : (X \sqcap E, X_0 \sqcap E_0) \times I \to (E,E_0)$$

$$G(p' \times 1) : g_1\Phi_1 p' \simeq p' : (X \sqcap E, X_0 \sqcap E_0) \times I \to (X,X_0).$$

Thus we have the following commutative diagram:

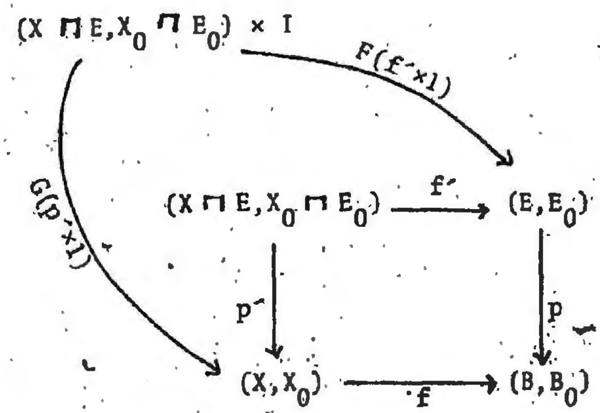


Diagram 3.2.17

Now $pF(f' \times 1) = K(p \times 1)(f' \times 1)$ by substitution in Equation B and $fG(p' \times 1) = K(f \times 1)(p' \times 1)$ by substitution in Equation C.

But $(p \times 1) \circ (f' \times 1) = (f \times 1) \circ (p' \times 1)$ since $pf' = fp'$.

Therefore there exists a unique map

$H : (X \sqcap E, X_0 \sqcap E_0) \times I \to (X \sqcap E, X_0 \sqcap E_0)$ making Diagram 3.2.17 commute by the universal property of pullbacks. But

$H(-,0) \simeq H(-,1)$ since H is a homotopy and

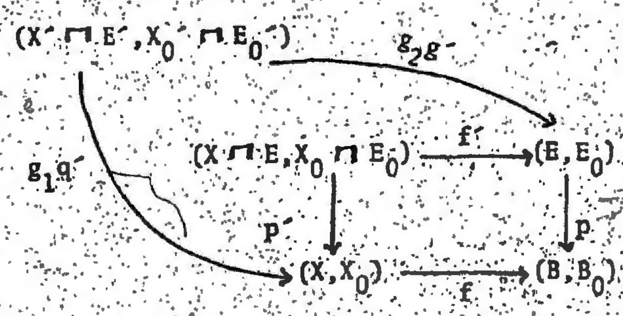
$f' \circ H(-,0) = F \circ (f' \times 1)(-,0) = g_2\phi_2 f'$,

$f' \circ H(-,1) = F \circ (f' \times 1)(-,1) = f'$.

Similarly $p'H = G(p' \times 1)$ so that $p'H(-,0) = G(p' \times 1)(-,0)$

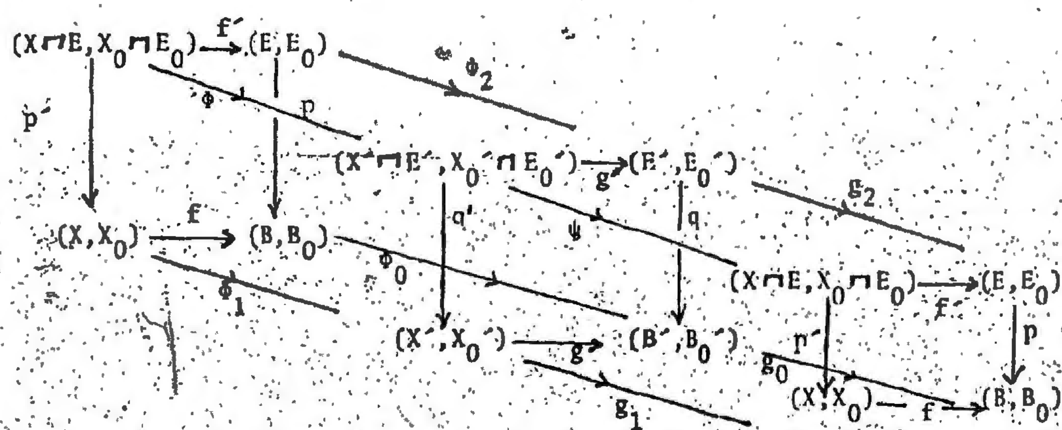
$= g_1\phi_1 p'$ and $p'H(-,1) = p'$.

Now we also have the following diagram

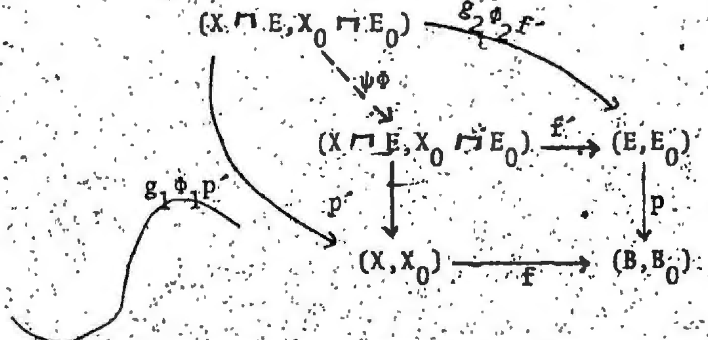


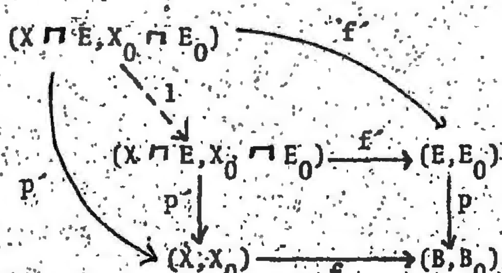
which commutes since $pg_2g'' = g_0qg' = g_0gq' = fg_1q'$ by commutativity of Diagram 3.2.16.

Again by the universal property of pullbacks there exists a unique map $\psi : (X' \sqcap E', X_0' \sqcap E_0') \to (X \sqcap E, X_0 \sqcap E_0)$ making the following diagram commute:



But this implies that the following diagrams commute:

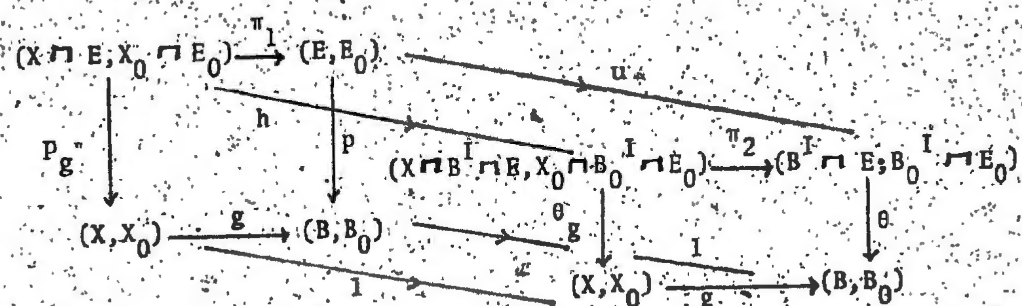




But by the uniqueness of the completion in the pullback diagrams we have that $H(-,0) = \psi\phi$ and $H(-,1) = 1$. Hence H is a homotopy between $\psi\phi$ and 1. In a similar manner we can show that ψ is a right homotopy inverse of ϕ. Thus ϕ is indeed a homotopy equivalence in Toppair.

//

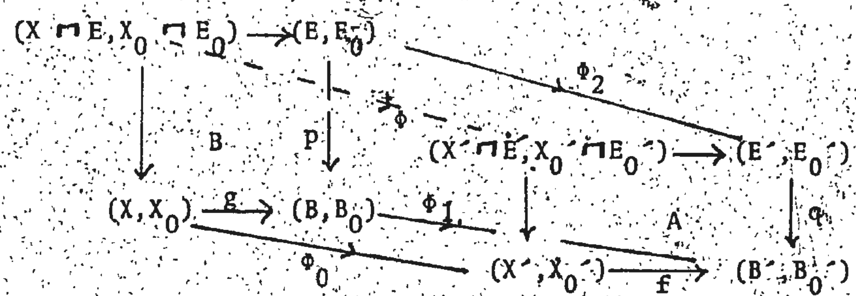
Note 3.2.18: We can follow the procedure of the above proposition to show that, if u is a fibre homotopy equivalence and p is a fibration, then the induced map $H : (X \sqcap E, X_0 \sqcap E_0) \to (X \sqcap B^I \sqcap E, X_0 \sqcap B_0^I \sqcap E_0)$ of the following diagram is a fibre homotopy equivalence even though g need not be a fibration:



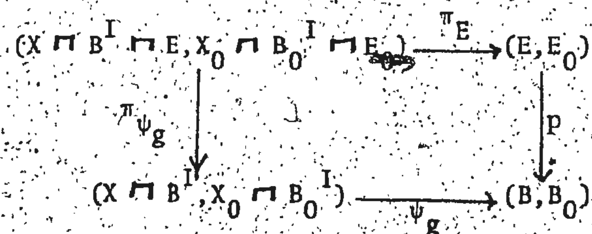
§3.3 THE RELATIVE CO-GLUEING THEOREM

We are now in a position to state and prove the main theorem of this chapter - THE RELATIVE CO-GLUEING THEOREM.

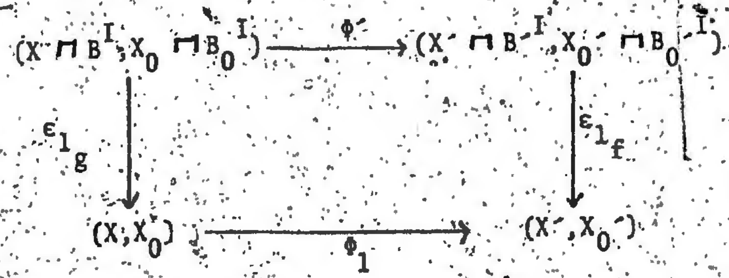
THEOREM 3.3.1: If in the following commutative diagram in Toppair A and B are pullbacks, p,q are fibrations, ϕ_0, ϕ_1, ϕ_2 are homotopy equivalences, then the induced map ϕ is a homotopy equivalence in Toppair.



Proof: We have the following pullback diagram,



where π_E, π_{ψ_g} are projections and ψ_g is the fibration of the factorization of $g : (X, X_0) \to (B, B_0)$ through the MAPPING TRACK of pairs [see Proposition 2.2.2]. And p is a fibration by hypothesis. We also have the following diagram:



where $\phi'(x,\lambda) = (\phi(x), \phi_0 \circ \lambda)$ [we define it in that way].

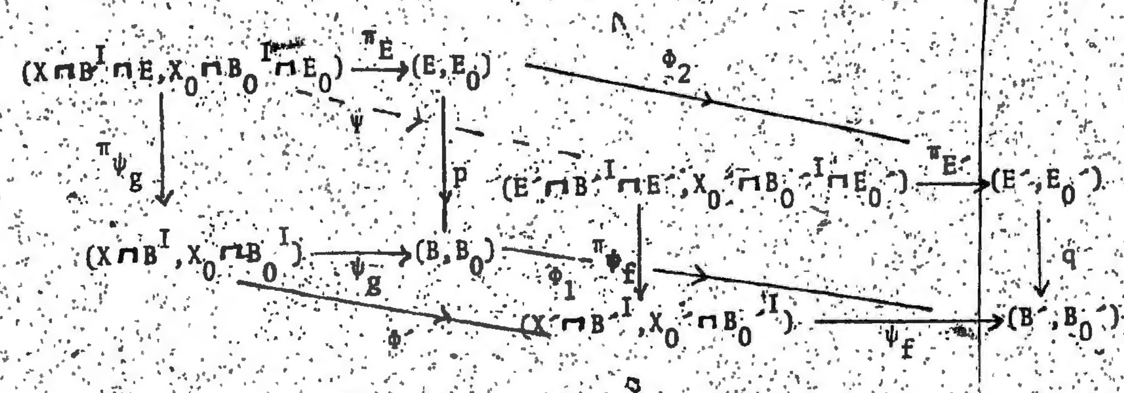
But ϕ_1 is a homotopy equivalence by hypothesis and by Note 2.2.4 ε_{1_g} and ε_{1_f} are homotopy equivalences.

Also $\varepsilon_{1_f}\phi'(x,\lambda) = \varepsilon_{1_f}(\phi_1(x), \phi_0 \circ \lambda) = \phi_1(x)$

$\phi_1\varepsilon_{1_g}(x,\lambda) = \phi_1(x)$, so the diagram commutes.

Thus by Proposition 1.3.6. ϕ' is a homotopy equivalence.

Now we also have the following commutative diagram:



in which p and q are fibrations by hypothesis, ϕ_1 and ϕ_2 are homotopy equivalences by hypothesis, ϕ' is a homotopy equivalence (see above), and ψ_g and ψ_f are the fibrations of the MAPPING TRACKS of g and f respectively.

Hence by Proposition 3.2.15, the induced map

$$\Psi : (X \sqcap B^I \sqcap E, X_0 \sqcap B_0^I \sqcap E_0) \to (X' \sqcap B'^I \sqcap E', X_0' \sqcap B_0'^I \sqcap E_0')$$

defined by $\Psi(x, \tilde{p}(e), e) = (\phi_0(x), \phi_1(\tilde{p}(e)), \phi_2(e))$

is a homotopy equivalence.

Now we have

$$\begin{array}{ccccc} (x,e) & (X \sqcap E, X_0 \sqcap E_0) & \xrightarrow{\Phi} & (X' \sqcap E', X_0' \sqcap E_0') & (x',e') \\ \big\downarrow & \big\downarrow h & & \big\downarrow h' & \big\downarrow \\ (x, \tilde{p}(e), e) & (X \sqcap B^I \sqcap E, X_0 \sqcap B_0^I \sqcap E_0) & \xrightarrow{\Psi} & (X' \sqcap B'^I \sqcap E', X_0' \sqcap B_0'^I \sqcap E_0') & (x', \tilde{q}(e'), e') \end{array}$$

Diagram 3.3.2

where Ψ is the homotopy equivalence above and h, h' the fibre homotopy equivalence of Note 3.2.18, and $\Phi(x,e)$ is defined to be the pair $(\phi_0(x), \phi_2(e))$.

Now $h'\Phi(x,e) = h'(\phi_0(x), \phi_2(e)) = (\phi_0(x), \tilde{q}(\phi_2(e)), \phi_2(e))$

$= (\phi_0(x), \phi_1\tilde{p}(e), \phi_2(e))$ $\quad [\tilde{q}\phi_2 = \phi_1\tilde{p}$ - given].

And $\Psi h(x,e) = \Psi(x, \tilde{p}(e), e) = (\phi_0(x), \phi_1\tilde{p}(e), \phi_2(e))$

[Definition of Ψ].

Hence Diagram 3.3.2 commutes.

And so by Proposition 1.3.6, Φ is a homotopy equivalence.

//

Note: If $E_0 = B_0 = \phi$ in the above Theorem, and p and q are (weak) fibrations in Top and ϕ_0, ϕ_1, ϕ_2 are homotopy equivalences in Top, then Φ is a homotopy equivalence in Top.

Corollary 3.3.3 : If

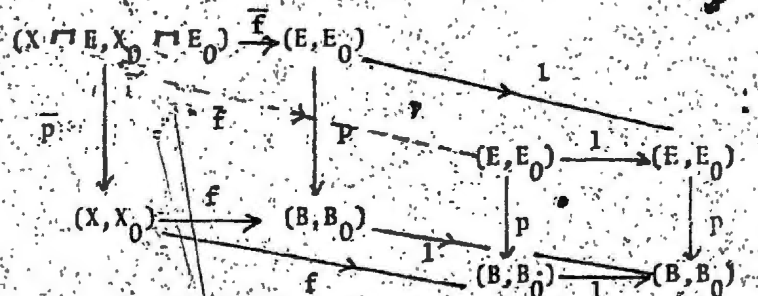
$$\begin{array}{ccc} (X \sqcap E, X_0 \sqcap E_0) & \xrightarrow{\overline{f}} & (E, E_0) \\ \Big\downarrow{\scriptstyle \overline{p}} & & \Big\downarrow{\scriptstyle p} \\ (X, X_0) & \xrightarrow{f} & (B, B_0) \end{array}$$

is a pullback in Toppair and p is a fibration and f is a homotopy equivalence, then $\overline{f}$ is a homotopy equivalence.

Proof: We have the following pullbacks in Toppair:

$$\begin{array}{ccc} (X \sqcap E, X_0 \sqcap E_0) & \longrightarrow & (E, E_0) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ (X, X_0) & \longrightarrow & (B, B_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} (E, E_0) & \xrightarrow{1} & (E, E_0) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ (B, B_0) & \xrightarrow{1} & (B, B_0) \end{array}$$

Putting the two diagrams together we have the following diagram:



By the Theorem proved above, since 1 and f are homotopy equivalences and p is a fibration, the induced map

$$\overline{f} : (X \sqcap E, X_0 \sqcap E_0) \to (E, E_0) \text{ is a homotopy equivalence.}$$

//

Corollary 3.3.4 : If in the following commutative diagram in Toppair p and q are fibrations, and f and g are homotopy equivalences, then the fibre over (b, b_0) in (B, B_0) is of the same homotopy type as the fibre over $f(b)$ in (B',B_0').

$$\begin{array}{ccc} (E,E_0) & \xrightarrow{g} & (E',E_0') \\ \downarrow p & & \downarrow q \\ (B,B_0) & \xrightarrow[f]{} & (B',B_0') \end{array}$$

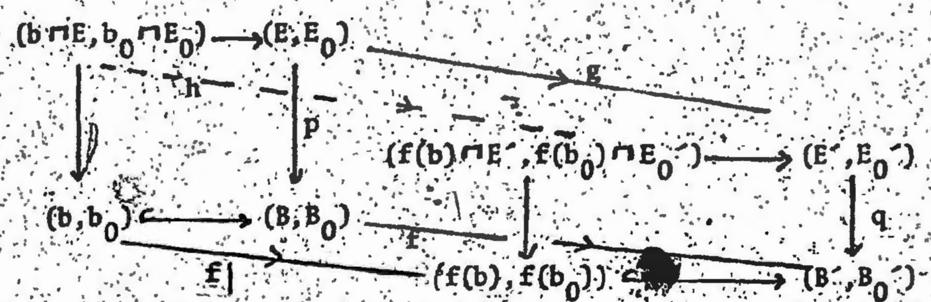
Proof: We have the following diagrams (pullback) in Toppair:

$$\begin{array}{ccc} (b \sqcap E, b_0 \sqcap E_0) & \longrightarrow & (E,E_0) \\ \downarrow & & \downarrow p \\ (b,b_0) & \hookrightarrow & (B,B_0) \end{array}$$

and

$$\begin{array}{ccc} (f(b) \sqcap E', f(b_0) \sqcap E_0') & \longrightarrow & (E',E_0') \\ \downarrow & & \downarrow q \\ (f(b),f(b_0)) & \hookrightarrow & (B',B_0') \end{array}$$

Putting the two diagrams together we have:



Now since $f|$, f and g are homotopy equivalences and p and q are fibrations, the induced map

$$h: (b \sqcap E, b_0 \sqcap E_0) \to (f(b) \sqcap E', f(b_0) \sqcap E_0')$$

is a homotopy equivalence by Theorem 3.3.1.

CHAPTER IV - INTRODUCTION

In section one we consider the problem of forming the Toppair pushout of two maps $(X,X_0) \xleftarrow{f} (A,A_0) \xrightarrow{g} (Y,Y_0)$ and we find that this is not exactly "dual" to the pullback case - i.e. a pair of pushouts in the Top sense is not always the pushout in Toppair. We see that $(X \sqcup_f Y, X_0 \sqcup_{f_0} Y_0)$ need not be an object of Toppair and that the induced function $f_0 \sqcup g_0$ (in Set) need not be injective. We solve this problem by defining an equivalence relation on $X_0 \sqcup_{f_0} Y_0$ so that it is injectively included in $X \sqcup_f Y$. We emphasize that even when $X_0 \sqcup_{f_0} Y_0$ is a subset of $X \sqcup_f Y$ the subspace and identification topologies need not coincide though the equivalence relation is trivial. This is seen in tom Dieck, Kamps, Puppe [9].

In section two we turn to a discussion of cofibrations in Toppair. We see that every map $f : (X,X_0) \to (Y,Y_0)$ factors through the Toppair mapping cylinder

$$(X,X_0) \xrightarrow{t} (M_f, M_{f_0}) \xrightarrow{u} (Y,Y_0)$$

where t is a cofibration and u is a homotopy equivalence in Toppair.

Section three gives a brief study of retractions and deformations in Toppair while section four gives a characterization of Toppair cofibrations.

In section five we state and prove the Glueing Theorem for Toppair and hence for Top and Top_*.

We note that our proof of Proposition 4.5.3 mimics the proof of Theorem 1 in Heath [11]. Furthermore, our proof of the Glueing Theorem in Toppair is a generalization of an earlier proof of Brown [2].

§4.1 PUSHOUTS IN TOPPAIR

Definition 4.1.1.: Let $\sim$ be an equivalence relation on the space Y and let $f : Y \to Z$ be a map compatible with $\sim$. The existence of a unique map $g : Y/\sim \to Z$ such that $g \circ \pi = f$ in the following diagram:

Projection = π

$$\begin{array}{ccc} Y & & \\ \downarrow & \searrow^{f} & \\ Y/\sim & \overset{g}{\dashrightarrow} & Z \end{array}$$

is called the UNIVERSAL PROPERTY of $\sim$.

Consider any diagram in Toppair of the following form:

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{f} & (Y,Y_0) \\ g\downarrow & & \\ (X,X_0) & & \end{array}$$

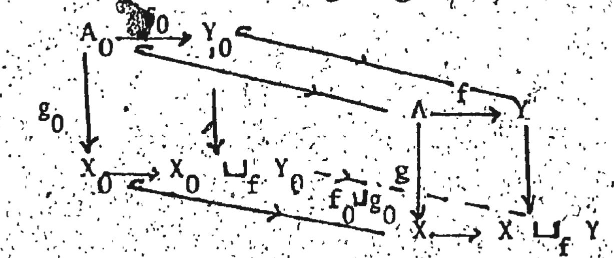
Diagram 4.1.2

We want to form a Toppair pushout of this diagram. We would expect it to be $(X \sqcup_f Y, X_0 \sqcup_f Y_0)$ but this is not always the case as the following counter-example shows.

Let $(\mathring{A},A_0) = (I,\{0\})$, $(X,X_0) = (I,I)$, $(Y,Y_0) = (*,*)$ and let f and g be the obvious maps, i.e. we have

$$\begin{array}{ccc} (I,\{0\}) & \xrightarrow{f} & (*,*) \\ g\downarrow & & \\ (I,I) & & \end{array}$$

In this case $X \sqcup_f Y = *$ and $X_0 \sqcup_{f_0} Y_0 = I$. But obviously $(*, I)$ is not an object of Toppair. There might also be a problem with the topology of $X_0 \sqcup_f Y_0$. We explain this later. Consider the following diagram in Set:



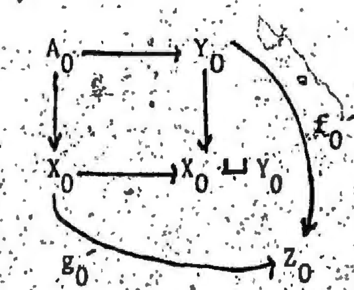
f_0 and g_0 are the restrictions of f and g respectively. The function $f_0 \sqcup g_0 = h$ exists but is not necessarily injective. Using the counter example given above we can see that

q exists but is not injective!

We define an equivalence relation $\sim$ on $X_0 \sqcup_{f_0} Y_0$ as follows: $z_1 \sim z_2$ if and only if $(f_0 \sqcup g_0)(z_1) = (f_0 \sqcup g_0)(z_2)$. Then the quotient set $X_0 \sqcup_{f_0} Y_0/\sim$ is included injectively in $X \sqcup_f Y$.

Proposition 4.1.3.: The pair $(X \sqcup_f Y, X_0 \sqcup_f Y_0/\sim)$ where $X_0 \sqcup_f Y_0/\sim$ is given the subspace topology of $X \sqcup_f Y$ is a pushout of Diagram 4.1.2 in Toppair. $X \sqcup_f Y$ is given the usual Top identification topology.

Proof: Let $f' : (Y,Y_0) \to (Z,Z_0)$ and $g' : (X,X_0) \to (Z,Z_0)$ be maps such that $g'f = f'g$. These maps define a unique Top map $h : X \sqcup_f Y \to Z$. We need that $h|(X_0 \sqcup Y_0/\sim) \subseteq Z_0$ and $h|(X_0 \sqcup Y_0/\sim)$ is continuous. Consider the following diagram in Set:



where f_0' and g_0' are the restrictions of f' and g' respectively. Now there exists a unique function $h_0 : X_0 \sqcup Y_0 \to Z_0$ in Set by definition of pushout in Set. Consider also the following diagram:

Projection = π

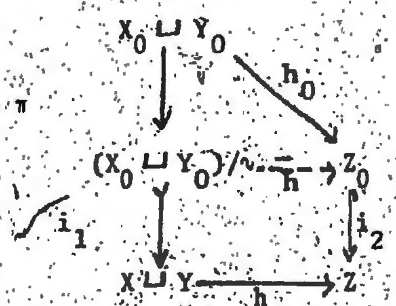


Diagram 4.1.4

Now by the universal property of $\sim$, there exists a unique function $\bar{h} : (X_0 \sqcup Y_0)/\sim \to Z_0$, the dotted arrow in diagram. This is the restriction of h to $(X_0 \sqcup Y_0)/\sim$. We still need that this restriction is continuous. Now $Z_0 \subseteq Z$ and as such has the subspace topology or the initial topology with respect to i_2. Hence $\bar{h}$ is continuous if and only if $i_2 \circ \bar{h}$ is continuous.

But $i_2\bar{h} = hi_1$ which is continuous. Hence Diagram 4.1.2 is indeed a pushout in Toppair.

Under certain conditions the pushout in Toppair is indeed the one we expect, i.e. $(X \sqcup_f Y, X_0 \sqcup_f Y_0)$. We now give some such conditions.

Proposition 4.1.5.: If both f and g are inclusions, then $(X \sqcup_f Y, X_0 \sqcup_{f_0} Y_0)$, where $X_0 \sqcup_{f_0} Y_0$ has the subspace topology, is the pushout of Diagram 4.1.2.

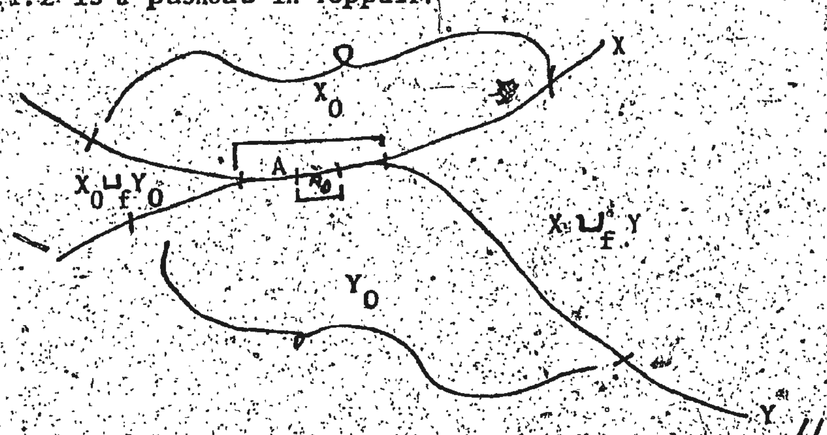
Proof: Consider Diagram 4.1.6 or the "picture" 4.1.7.

$$\begin{array}{ccc} X_0 \dot{\cup} Y_0 & \hookrightarrow & X \dot{\cup} Y \\ \text{Proj.}_1 \downarrow & & \downarrow \text{Proj.}_2 \\ X_0 \sqcup_f Y_0 & \hookrightarrow & X \sqcup_f Y \end{array}$$

Diagram 4.1.6

Any point in $X_0 \sqcup_f Y_0$ outside* the adjunction has a pre-image in either X_0 or Y_0 or both. In this case it is included in $X \dot{\cup} Y$ and so projects to itself in $X \sqcup_f Y$. If a point of $X_0 \sqcup_f Y_0$ is in the adjunction, it has a pre-image consisting of two points - one in X_0 and one in Y_0 and this pre-image is included in $X_0 \dot{\cup} Y_0$ which is included in $X \dot{\cup} Y$ and as before projects to the same point in $X \sqcup_f Y$. Hence $X_0 \sqcup_f Y_0 \to X \sqcup_f Y$ is an injection and 4.1.2 is a pushout in Toppair.

"Picture 4.1.7



*Points of $X_0 \sqcup_f Y_0$ not glued together by the identification.

We consider also, cases in which either f or g is an inclusion in the following diagram:

$$\begin{array}{ccc} (A, A_0) & \xrightarrow{\ f\ } & (Y, Y_0) \\ {\scriptstyle g}\downarrow & & \\ (X, X_0) & & \end{array}$$

Diagram 4.1.8

Without loss of generality assume that (A, A_0) is a subpair of (X, X_0), i.e. that g is an inclusion.

Now the sets $X \sqcup_f Y$ and $X_0 \sqcup_f Y_0$ can also be written as the disjoint unions $Y \,\dot{\cup}\, (X \setminus A)$ and $Y_0 \,\dot{\cup}\, (X_0 \setminus A_0)$ respectively. We then have the following proposition.

Proposition 4.1.9.: The set $X_0 \sqcup_f Y_0$ is a subset of $X \sqcup_f Y$ if $X_0 \setminus A_0$ is a subset of $X \setminus A$.

Proof: Assuming that $X_0 \setminus A_0$ is a subset of $X \setminus A$ the result is obvious.

//

In the Corollaries that follow we assume that the second space always has the subspace topology with respect to the first.

Corollary 4.1.10.: The Toppair mapping cylinder of a map $f : (X, X_0) \to (Y, Y_0)$ is the pair (M_f, M_{f_0}) where M_f is the usual Top mapping cylinder and as a set M_{f_0} is the mapping cylinder of the inclusion (f_0 is the restriction of f). We topologize M_{f_0} as the subspace of M_f.

Proof: In the notation of the proposition the map $X_0 \backslash A_0 \to X \backslash A$ is precisely $X_0 \times (0,1] \hookrightarrow X \times (0,1]$ which is clearly an inclusion, i.e. we have the following diagram

Diagram 4.1.11

$$\begin{array}{ccc} (X,X_0) \times \{0\} & \xrightarrow{f} & (Y,Y_0) \\ \downarrow & & \downarrow \\ (X,X_0) \times I & \longrightarrow & ((X \times I) \sqcup_f Y, (X_0 \times I) \sqcup_f Y_0) \end{array}$$

which by the Proposition, is a pushout in Toppair.

//

Remark 4.1.12. The Top_* and Toppair mapping cylinders of a map $f: (X,x_0) \to (Y,y_0)$ are not identical. This is clear since the Top_* mapping cylinder of f is the following pushout:

$$\begin{array}{ccc} (X,x_0) & \xrightarrow{f} & (Y,y_0) \\ \downarrow & & \downarrow \\ (X,x_0) \times I & \longrightarrow & ((X \times I)/x_0 \times I \sqcup_f Y, *) \end{array}$$

But the Toppair mapping cylinder of f is the pushout

$$\begin{array}{ccc} (X,x_0) & \longrightarrow & (Y,y_0) \\ \downarrow & & \downarrow \\ (X,x_0) \times I & \longrightarrow & ((X \times I) \sqcup_f Y, (x_0 \times I) \sqcup_{f_0} \{y_0\}) \end{array}$$

It is clear that the mapping cylinder of $f_0 : \{x_0\} \to \{y_0\}$ is not a point.

Corollary 4.1.13 : Let (X,X_0) be an object of Toppair. Then the suspension of (X,X_0) and the mapping cone of (X,X_0) are respectively (SX,SX_0) and (CX,CX_0).

Proof: We again have that $X_0 \setminus A_0 \hookrightarrow X|A$ which in the case of the suspension is $(X_0 \times I) \setminus (X_0 \times \dot{I}) \hookrightarrow (X \times I) \setminus (X \times \dot{I})$, i.e. $X_0 \times (0,1) \hookrightarrow X \times (0,1)$. Hence we have that the suspension of (X,X_0) is the pushout

$$\begin{array}{ccc} (X,X_0) \times \dot{I} & \xrightarrow{f} & (*,*) \\ \downarrow & & \downarrow \\ (X,X_0) \times I & \longrightarrow & [(X \times I) \sqcup_f *,\ (X_0 \times I) \sqcup_f *] \end{array}$$

Similarly in the case of the mapping cone of (X,X_0) we have $(X_0 \times I) \setminus (X_0 \times \{0\}) \hookrightarrow (X \times I) \setminus (X \times \{0\})$, i.e. $X_0 \times (0,1] \hookrightarrow X \times (0,1]$ is an inclusion so that

$(X_0 \times I) \sqcup *$ is a subset of $(X \times I) \sqcup *$ and

the mapping cone of (X,X_0) is the following pushout

$$\begin{array}{ccc} (X,X_0) \times \{0\} & \xrightarrow{f} & (*,*) \\ \downarrow & & \downarrow \\ (X,X_0) \times I & \longrightarrow & ((X \times I) \sqcup_f *,\ (X_0 \times I) \sqcup_f *) \end{array}$$

//

In the same manner as above we can derive the reduced mapping cone and the reduced suspension in Toppair.

Note 4.1.14 : We emphasize that the subspace topology on $X_0 \sqcup Y_0$ need not coincide with the pushout topology, i.e. the identification topology, even when the equivalence relation $\sim$ is trivial. Consider the following example.

Example 4.1.15: Let $A = (0,1]$, $X = [0,1]$ and consider the mapping cylinder of the inclusion $(X,A) \hookrightarrow (X,X)$. It is shown in [9] that the subspace topology of $X \times \{0\} \cup A \times I$ in $X \times I$ does not coincide with the identification topology.

§4.2 COFIBRATIONS IN TOPPAIR - ELEMENTARY PROPERTIES

Definition 4.2.1.: A map $(A,A_0) \overset{i}{\hookrightarrow} (X,X_0)$ is a COFIBRATION in Toppair if any commutative diagram of the following form

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{\quad i \quad} & (X,X_0) \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_X} \\ (A,A_0) \times I & \xrightarrow[i \times 1]{} & (X,X_0) \times I \end{array}$$

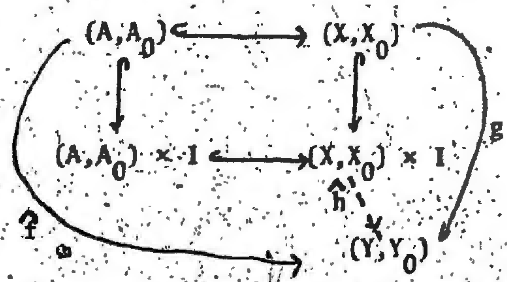
is a WEAK pushout, i.e. for any space (Y,Y_0) and any maps $f : (X,X_0) \to (Y,Y_0)$, $H : (A,A_0) \times I \to (Y,Y_0)$ such that $H(i \times 1) = fi$. there exists a map $H' : (X,X_0) \times I \to (Y,Y_0)$ such that $H' \circ (i \times 1) = H$ and $H' i_X = f$. The map i is said to have the HOMOTOPY EXTENSION PROPERTY, abbreviated HEP, if the definition holds with respect to a particular (Y,Y_0).

Alternate Definition 4.2.2.: (STRØM-LIKE Generalization): $(A,A_0) \overset{i}{\hookrightarrow} (X,X_0)$ has the HEP in Toppair if for any commutative diagram of the following form:

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{\quad f \quad} & (Y,Y_0)^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ (X,X_0) & \xrightarrow[g]{} & (Y,Y_0) \end{array}$$

Diagram 4.2.3

there exists a map $h : (X,X_0) \to (Y,Y_0)^I$ such that $\varepsilon_o h = g$, or $hi = f$. It is easy to see, using the exponential law for pairs, that this definition is equivalent to Definition 4.2.1, i.e. Diagram 4.2.3 gives rise to



and conversely the diagram of Definition 4.2.1 gives rise to Diagram 4.2.3. Hence the equivalence.

Proposition 4.2.4.: A pushout of a cofibration in Toppair is a cofibration in Toppair.

Proof: Let $(A,A_0) \hookrightarrow (X,X_0)$ be a cofibration in Toppair and let $f : (A,A_0) \to (Z,Z_0)$ be a map. Consider the following pushout in Toppair:

Diagram 4.2.5

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{f} & (Z,Z_0) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ (X,X_0) & \xrightarrow[\bar{f}]{} & (X \sqcup Z, X_0 \sqcup Z_0/\sim) \end{array}$$

We are required to prove that $\bar{i}$ is a cofibration, i.e. for each commutative diagram of the following form in Toppair

Diagram 4.2.6

$$\begin{array}{ccc} (Z,Z_0) & \xrightarrow{h} & (Y,Y_0)^I \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ (X \sqcup Z, X_0 \sqcup Z_0/\sim) & \xrightarrow[\bar{h}]{} & (Y,Y_0) \end{array}$$

we require the existence of a map $q : (X \sqcup Z, X_0 \sqcup Z_0/\sim) \to (Y,Y_0)^I$ such that $\varepsilon_0 q = \bar{h}$ and $q\bar{i} = h$. Consider the following diagram obtained by joining Diagram 4.2.5 to Diagram 4.2.6.:

$$\begin{array}{ccccc} (A,A_0) & \xrightarrow{f} & (Z,Z_0) & \xrightarrow{h} & (Y,Y_0)^I \\ \downarrow i & \circlearrowleft & \downarrow \bar{i} & \circlearrowleft & \downarrow \varepsilon_0 \\ (X,X_0) & \xrightarrow[\bar{f}]{} & (X \sqcup Z, X_0 \sqcup Z_0/\sim) & \xrightarrow[\bar{h}]{} & (Y,Y_0) \end{array}$$

Now i is a cofibration; therefore there exists $g : (X,X_0) \to (Y,Y_0)^I$ such that $\varepsilon_0 g = \bar{h}\bar{f}$ and $gi = hf$. Thus we have

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{f} & (Z,Z_0) \\ \downarrow i & \circlearrowleft & \downarrow \bar{i} \\ (X,X_0) & \xrightarrow[\bar{f}]{} & (X \sqcup Z, X_0 \sqcup Z_0/\sim) \end{array} \qquad (Z,Z_0) \xrightarrow{h} (Y,Y_0)^I, \quad (X,X_0) \xrightarrow{g} (Y,Y_0)^I$$

where $hf = gi$.

But the diagram of Definition 4.2.1 is a pushout in Toppair; hence there exists a unique map $\bar{q} : (X \sqcup Z, X_0 \sqcup Z_0/\sim) \to (Y,Y_0)^I$ such that $\bar{q}\bar{i} = h$ and $\bar{q}\bar{f} = g$. Then $\bar{q}$ completes Diagram 4.2.6 commutatively. Hence $\bar{i}$ is indeed a cofibration in Toppair.

//

Proposition 4.2.7.: If $A \hookrightarrow X$ is a cofibration in Top, then $(A,A) \hookrightarrow (X,X)$ is a cofibration in Toppair.

Proof: Given any commutative diagram of the following form:

Diagram 4.2.8

$$\begin{array}{ccc} (A,A) & \xrightarrow{K} & (Y,Y_0)^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ (X,X) & \xrightarrow[h]{} & (Y,Y_0) \end{array}$$

We require a completion $t : (X,X) \to (Y,Y_0)^I$ such that $\varepsilon_0 t = h$ and $ti = K$.

In effect we need a commutative completion for any diagram:

$$\begin{array}{ccc} A & \xrightarrow{K} & Y^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ X & \xrightarrow[h]{} & Y \end{array}$$

whose restriction commutatively completes the inside of Diagram 4.2.8. But i is a cofibration in Top; hence there exists a map $\overline{h} : X \to Y^I$ such that $\varepsilon_0 \overline{h} = h$ and $\overline{h} i = K$.

Also

$$\begin{array}{ccc} A & \xrightarrow{K_0} & Y_0^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ X & \xrightarrow[h_0]{} & Y_0 \end{array}$$

is a commutative diagram in Top; hence there exists a commutative completion $V : X \to Y_0^I$ since i is a cofibration in Top. It is easily seen that V is the restriction of $\overline{h}$. Thus $\overline{h}$ commutatively completes Diagram 4.2.8. Hence $(A,A) \hookrightarrow (X,X)$ is a cofibration in Toppair.

//

Proposition 4.2.9.: Let $(A,A_0) \overset{i}{\hookrightarrow} (X,X_0)$ be an inclusion such that $A_0 \hookrightarrow X_0$ is a cofibration in Top. Then $(A,A_0) \overset{\bar{i}}{\hookrightarrow} (X_0 \sqcup A, X_0)$ is a cofibration in Toppair.

Proof: Consider the following diagram:

$$\begin{array}{ccc} (A_0,A_0) & \xrightarrow{\ j\ } & (A,A_0) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ (X_0,X_0) & \longrightarrow & (X_0 \sqcup A, X_0) \end{array}$$

This is indeed a pushout in Toppair [see §4.1].
And i is a cofibration in Toppair by Proposition 4.2.7.
Thus by Proposition 4.2.4, $\bar{i}$ is a cofibration in Toppair.

//

Corollary 4.2.10.: Let $(Y;Y_1,Y_2)$ be a Mayer-Vietoris Triad, i.e. $Y = Y_1 \sqcup Y_2$ and Y_1, Y_2 are closed in Y (see Brown [2]). Then if $Y_0 = Y_1 \cap Y_2$, $(Y_1,Y_0) \hookrightarrow (Y,Y_2)$ and $(Y,Y_0) \hookrightarrow (Y,Y_1)$ are Toppair cofibrations.

Proof: For the first part take $A_0 = Y_0, X_0 = Y_2, A = Y_1$ in the Proposition. For the second part take $A_0 = Y_0, X_0 = Y_1, A = Y$ in the Proposition.

//

Proposition 4.2.11.: Let $i : (A,A_0) \hookrightarrow (X,X_0)$ be a map in Toppair such that (i) $A_0 \hookrightarrow X_0$ is a cofibration in Top

(ii) $X_0 \cup A \hookrightarrow X$ is a cofibration in Top

(iii) $A \cap X_0 = A_0$

(iv) $X_0 \sqcup A = X_0 \cup A$ implies $X_0 \sqcup A \hookrightarrow X$ is a cofibration in Top. Then i is a cofibration in Toppair.

Proof: For any commutative diagram of the following form:

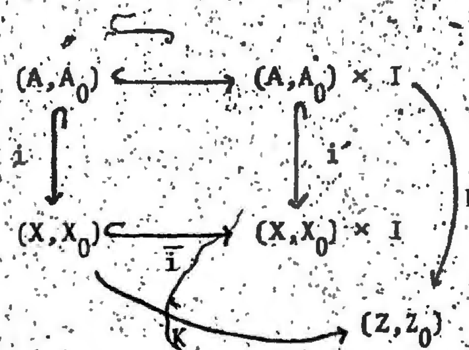
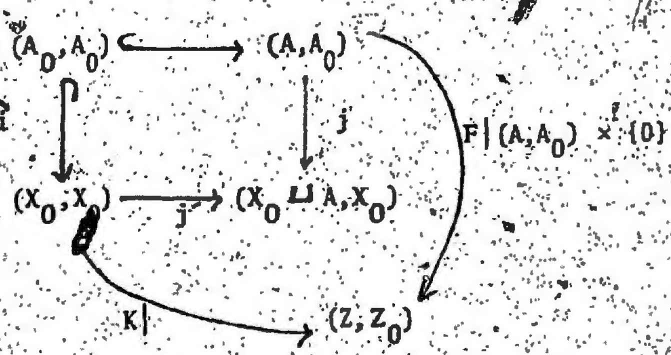


Diagram 4.2.12

we require a map $H : (X,X_0) \times I \to (Z,Z_0)$ such that $H \circ \bar{i} = K$ and $H \circ i' = F$. Now $(A,A_0) \xrightarrow{j} (X_0 \sqcup A, X_0)$ is a cofibration in Toppair by Proposition 4.2.9. Thus we have the following diagram:



But $\hat{i}$ is a cofibration in Toppair since $A_0 \hookrightarrow X_0$ is a cofibration in Top; hence there exists a map ψ such that $\psi \circ j = F|(A,A_0) \times \{0\}$ and $\psi \circ j' = K|$. We now have the following commutative diagram in Toppair.

$$\begin{array}{ccc}
(A,A_0) & \hookrightarrow & (A,A_0)\times I \\
\downarrow j & & \downarrow \bar{j} \\
(X_0 \sqcup A, X_0) & \xrightarrow{t} & (X_0 \sqcup A, X_0)\times I
\end{array}
\qquad F : (A,A_0)\times I \to (Z,Z_0), \quad \psi : (X_0 \sqcup A, X_0) \to (Z,Z_0)$$

where j is a cofibration in Toppair. Thus there exists a map $Q : (X_0 \sqcup A, X_0) \times I \to (Z, Z_0)$ such that $Q \circ \bar{j} = F$ and $Q \circ t = \psi$. We also have the following commutative diagram in Top:

$$\begin{array}{ccc}
X_0 \cup A & \hookrightarrow & (X_0 \cup A)\times I \\
\downarrow t & & \downarrow t\times 1 \\
X & \xrightarrow{i_1} & X \times I
\end{array}
\qquad Q : (X_0 \cup A)\times I \to Z, \quad K : X \to Z$$

But by hypothesis t is a cofibration in Top; hence there exists a map $Q' : X \times I \to Z$ such that $Q'(t \times 1) = Q$ and $Q' i_1 = K$. Then the maps Q' and Q_0 (the restriction of Q) commutatively complete Diagram 4.2.12. Thus i is a cofibration in Toppair.

//

Corollary 4.2.13. If $A \hookrightarrow X$ is a cofibration in Top, then $(A, A_0) \hookrightarrow (X, A)$ is a cofibration in Toppair.

Proof: Put $X_0 = A = A_0$; then $X_0 \cup A = A$ in the Proposition.

//

Corollary 4.2.14.: If $B \hookrightarrow Y$ is a cofibration in Top, then $(B,*) \hookrightarrow (Y,*)$ is a cofibration in Toppair and in particular in Top_*.

Proof: Put $X_0 = A_0 = *$ in the proposition. Then $X_0 \cup A = A \hookrightarrow X$.

//

Lemma 4.2.15.: A map $i : (A,A_0) \hookrightarrow (X,X_0)$ is a cofibration in Toppair if and only if $i^* : (Z^X,(Z,Z_0)^{(X,X_0)}) \to (Z^A,(Z,Z_0)^{(A,A_0)})$ is a fibration for all (Z,Z_0).

Proof: (Necessity) We need for each commutative diagram of the following form (where T is locally compact and regular)

Diagram 4.2.16

$$\begin{array}{ccc} (T,T_0) & \xrightarrow{\tilde{f}} & (Z^X,(Z,Z_0)^{(X,X_0)}) \\ \varepsilon_0 \downarrow & & \downarrow i^* \\ (T,T_0) \times I & \longrightarrow & (Z^A,(Z,Z_0)^{(A,A_0)}) \end{array}$$

a map $\tilde{H} : (T,T_0) \times I \to (Z^X,(Z,Z_0)^{(X,X_0)})$ making the diagram commute. Now from Diagram 4.2.16 we get (using the Exponential Law) the following commutative diagram:

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{H} & (Z^T,(Z,Z_0)^{(T,T_0)})^I \\ i \downarrow & & \downarrow \varepsilon_0^* \\ (X,X_0) & \longrightarrow & (Z^T,(Z,Z_0)^{(T,T_0)}) \end{array}$$

Diagram 4.2.17

But i is a cofibration in Toppair; hence there exists a map

$\overline{H} : (X,X_0) \to (Z^T, (Z,Z_0)^{(T,T_0)^I})$ making Diagram 4.2.17 commute.

From $\overline{H}$ we get (using the exponential law) a map

$\tilde{H} : (T,T_0) \times I \to (Z^X, (Z,Z_0)^{(X,X_0)})$, making Diagram 4.2.16 commute. Hence i^* is a fibration in Toppair.

(Sufficiency): We need for any commutative diagram of the following form

Diagram 4.2.18

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{\hat{h}} & (Y,Y_0)^I \\ i \downarrow & & \downarrow \varepsilon_0 \\ (X,X_0) & \xrightarrow{f} & (Y,Y_0) \underset{\theta}{\cong} (Y^*, (Y,Y_0)^{(*,*)}) \end{array}$$

that there exists a map $\overline{H} : (X,X_0) \times I \to (Y,Y_0)$ making the diagram commute.

Now $\theta f : (X,X_0) \to (Y^*, (Y,Y_0)^{(*,*)})$ is a map if and only if $\widehat{\theta f} : (X \times *, X_0 \times *) \to (Y,Y_0)$ is a map.

But $X \times * \cong * \times X$, so $\underset{\sim}{\theta f} : (*,*) \to (Y^X, (Y,Y_0)^{(X,X_0)})$ is a map. And $\hat{h} : (A,A_0) \to (Y,Y_0)^I$ gives a map

$$\underset{\sim}{h} : I \to (Y^A, (Y,Y_0)^{(A,A_0)})$$

Hence from Diagram 4.2.18 we get the following commutative diagram:

$$
\begin{array}{ccc}
(*,*) & \xrightarrow{\theta f} & (Y^X, (Y,Y_0)^{(X,X_0)}) \\
\downarrow & & \downarrow i^* \\
(I,I) & \xrightarrow[h]{} & (Y^A, (Y,Y_0)^{(A,A_0)})
\end{array}
$$

Diagram 4.2.19

But i^* is a fibration in Toppair, so there exists a map

$$H : (I,I) \to (Y^X, (Y,Y_0)^{(X,X_0)})$$

making Diagram 4.2.19 commute. From H using the exponential law, we get the required map $\bar{H} : (X,X_0) \to (Y,Y_0)^I$ making Diagram 4.2.18 commute. Thus i is a cofibration in Toppair. //

Proposition 4.2.20.: Let $f : (X,X_0) \to (Y,Y_0)$ be a map in Toppair. Then $q : (X,X_0) \to (M_f, M_{f_0})$ defined by $q(x) = [(x,1)]$ is a cofibration in Toppair.

Proof: We have the following sequence of maps:

$$x \longmapsto (x,1)$$

$$X \xhookrightarrow[i_x]{} X \times I \longrightarrow (X \times I) \cup Y \twoheadrightarrow (X \times I) \sqcup_f Y$$

To show that q is a cofibration we show that for any commutative diagram of the following form

$$
\begin{array}{ccccc}
(X,X_0) & \xhookrightarrow{i} & (X,X_0) \times I & & \\
\downarrow q & & \downarrow q \times 1 & \searrow F & \\
(M_f, M_{f_0}) & \xrightarrow[\bar{i}]{} & (M_f, M_{f_0}) \times I & & \\
& \searrow g & & & (Z, Z_0)
\end{array}
$$

there exists a map $H : (M_f, M_{f_0}) \times I \to (Z, Z_0)$ such that

$H(q \times 1) = F$ and $Hi = g$.

We define $\lambda : (X, X_0) \times (\{0\} \times I \cup \{1\} \times I \cup I \times \{0\}) \to (Z, Z_0)$

by sending $(x,0,t)$ to $g([f(x)])$,

$(x,1,t)$ to $F(x,t)$ and $(x,s,0)$ to $g([x,s])$.

Now we have the usual retraction $I \times I \xrightarrow{\zeta} (\{0\} \times I \cup \{1\} \times I \cup I \times \{0\})$

so that we can define a homotopy

$$\eta_t : M_f \times I \to Z \quad \text{by setting}$$

$$\eta_t([x],s) = \lambda(x, \zeta(s,t)).$$

Then this is the homotopy we require since

$$\eta_t \bar{i}([x],0) = \eta_t([x],0) = \lambda(x, \zeta(0,t))$$

$$= \lambda(x,0,t) = g([f(x)]).$$

And $\eta_t(q \times 1)(x,t) = \eta_t(q(x),t) = \eta_t([x,1],t) =$

$$\lambda(x,\zeta(1,t)) = \lambda(x,1,t) = F(x,t).$$

We still need to check that the restriction of η_t is inside Z_0.

But for $[x_0]$ in M_{f_0}, $\eta_t([x_0],s) = \lambda(x_0, \zeta(s,t))$

which belongs to Z_0 since λ is a map of pairs.

And $\eta_t \bar{i}_0([x_0],0) = \eta_t([x_0],0) = \lambda(x_0, \zeta(0,t))$

$$= \lambda(x_0,0,t) = g_0([f_0(x_0)]).$$

And $\eta_t(q_0 \times 1)(x_0,t) = \eta_t(q_0(x_0),t) = \eta_t([x_0,1],t)$

$$= \lambda(x_0,\zeta(1,t)) = \lambda(x_0,1,t) = F_0(x_0,t).$$

Hence q is indeed a Toppair cofibration. //

Proposition 4.2.21. Let $h : (M_f, M_{f_0}) \to (Y, Y_0)$ be defined by $h([(x,t)]) = f(x)$ and $h([y]) = y$. Then h is a homotopy equivalence.

Proof: Now h is well-defined and $hq(x) = h([x,1]) = f(x)$ so that $hq = f$, i.e. $(X, X_0) \xrightarrow{q} (M_f, M_{f_0}) \xrightarrow{h} (Y, Y_0)$, with composite f,

and

$$h_0 q_0(x_0) = h_0([(x_0,1)]) = f_0(x_0).$$

Define $i : (Y, Y_0) \to (M_f, M_{f_0})$ by $i(y) = [y]$ for all y in (Y, Y_0). Then $hi(y) = h([y]) = y$ and $h_0 i_0(y_0) = h_0([y_0]) = y_0$. So $hi = 1_{(Y,Y_0)}$, i.e. i is a right inverse for h and as such is thus a right homotopy inverse for h, i.e. $hi \simeq 1_{(Y,Y_0)}$. We still need to find a left homotopy inverse for h.

Now $ih([(x,t)]) = if(x) = [f(x)] = [(x,0)]$

and $i_0 h_0([(x_0,t)]) = i_0 f_0(x_0) = [f_0(x_0)]$.

And $ih([y]) = i(y) = [y]$; $i_0 h_0([y_0]) = i_0(y_0) = [y_0]$.

Define $\theta_t : M_f \to M_f$ for t belong to I, by

$$\theta_t([(x,s)]) = [(x,st)] , \; \theta_t([y]) = [y]$$

θ_t is well defined and $\theta_0([y]) = [y] = ih(y)$

and $\theta_0[(x,s)] = [(x,0)] = ih([(x,s)])$

and $\theta_1[(x,s)] = [(x,s)]$; $\theta_1([y]) = [y]$.

Hence $\theta_0 = ih$ and $\theta_1 = 1_{M_f}$; thus $\theta : ih \simeq 1_{M_f}$

We still need to check that θ_t restricted to M_{f_0} remains inside M_{f_0}. Let $[y_0]$ belong to M_{f_0}. Then $\theta_t([y_0]) = [y_0]$ which is in M_{f_0} and, if $[(x_0,t)]$ belongs to M_{f_0} then $\theta_t([(x_0,t)]) = [(x_0,st)]$.

And $\theta_0([(x_0,s)]) = [(x_0,0)] = i_0h_0([(x_0,s)])$,

$\theta_1([(x_0,s)]) = [(x_0,s)]$ and $\theta_1([y_0]) = [y_0]$.

Hence the restriction of $\theta_0 = i_0h_0$ and the restriction of $\theta_1 = 1_{M_{f_0}}$.

Thus i is a left homotopy inverse for h and h is a homotopy equivalence. //

Note 4.2.22. Propositions 4.2.20 and 4.2.21 show us that, as in Top, any Toppair map $f : (X,X_0) \to (Y,Y_0)$ can be factored through the mapping cylinder of pairs (M_f,M_{f_0}), as follows:

$$(X,X_0) \xrightarrow{q} (M_f,M_{f_0}) \xrightarrow{h} (Y,Y_0)$$

where $hq = f$ (q is a cofibration as defined in Proposition 4.2.20 and h is a homotopy equivalence).

§4.3 TOPPAIR RETRACTIONS

Definition 4.3.1.: A subspace (A,A_0) of a space (X,X_0) is called a RETRACT of (X,X_0) if the inclusion map $i : (A,A_0) \hookrightarrow (X,X_0)$ has a left inverse in the category of Toppair. Thus (A,A_0) is a retract of (X,X_0), if and only if there exists a continuous map

$$r : (X,X_0) \to (A,A_0) \quad \text{such that} \quad ri = 1_{(A,A_0)},$$

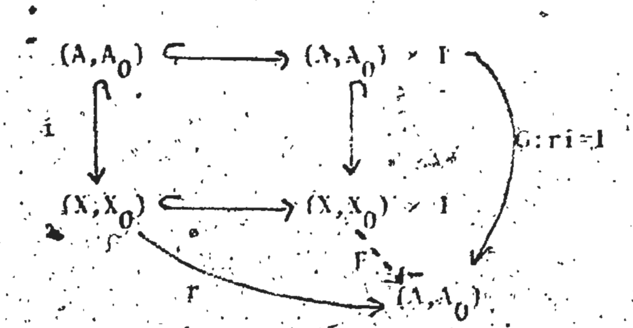
i.e. $r(x) = x$ for all x belonging to (A,A_0). The map r is called a RETRACTION of (X,X_0) onto (A,A_0).

Definition 4.3.2.: A subspace (A,A_0) of (X,X_0) is called a WEAK RETRACT of (X,X_0) if the inclusion map $i : (A,A_0) \hookrightarrow (X,X_0)$ has a left homotopy inverse. Thus (A,A_0) is a weak retract of (X,X_0) if and only if there exists a continuous map $r : (X,X_0) \to (A,A_0)$ such that $ri \simeq 1_{(A,A_0)}$. The map r is called a WEAK RETRACTION of (X,X_0) onto (A,A_0).

THEOREM 4.3.3.: If $(A,A_0) \hookrightarrow (X,X_0)$ has the HEP with respect to (A,A_0), then (A,A_0) is a weak retract of (X,X_0) if and only if (A,A_0) is a retract of (X,X_0).

Proof: (Necessity) We show that any weak retraction $r : (X,X_0) \to (A,A_0)$ is homotopic to a retraction. Let $i : (A,A_0) \hookrightarrow (X,X_0)$ - then $ri \simeq 1$. Let

$G : ri \simeq 1 : (A,A_0) \times I \to (A,A_0)$. Then $G(x,0) = ri(x) = r(x)$ for all x in (A,A_0). But $(A,A_0) \hookrightarrow (X,X_0)$ has the HEP with respect to (A,A_0). Thus we have the following diagram:



so that there exists F making the diagram commute. And $F(x,0) = G(x,0) = ri(x) = r(x)$. If $r' : (X,X_0) \to (A,A_0)$ is defined by $r'(x) = F(x,1)$ for all x in (X,X_0), then r' is a retraction of (X,X_0) onto (A,A_0).

Since $r'(x) = F(x,1) = G(x,1) = 1_{(A,A_0)} = x$ and $F : r \simeq r'$, the result follows.

(Sufficiency) Trivial.

//

We also consider inclusion maps with right homotopy inverses. This consideration leads to the following definitions.

Definition 4.3.4. Given $(A,A_0) \hookrightarrow (X,X_0)$, a DEFORMATION D of (A,A_0) in (X,X_0) is a homotopy $D : (A,A_0) \times I \to (X,X_0)$ such that $D(a,0) = a$ for all a in (A,A_0). Moreover, if $D((A,A_0) \times 1)$ is contained in a subspace (X',X_0') of (X,X_0), D is said to be a DEFORMATION of (A,A_0) into (X',X_0') and (A,A_0) is said to be deformable in (X,X_0) into (X',X_0').

Note 4.3.5.: A space (X,X_0) is said to be DEFORMABLE into a subspace (X',X_0') if it is deformable in itself into (X,X_0). Thus a space (X,X_0) is contractible if and only if it is deformable into one of its points.

Lemma 4.3.6.: A space (X,X_0) is deformable into a subspace (A,A_0) if and only if the inclusion map $i : (A,A_0) \hookrightarrow (X,X_0)$ has a right homotopy inverse.

Proof: (Sufficiency) Let $f : (X,X_0) \to (A,A_0)$ be a map such that $if \simeq 1$. Let $F : 1 \simeq if : (X,X_0) \times I \to (X,X_0)$. Then $F(x,0) = x$. So F is a deformation of (X,X_0) and $F((X,X_0) \times 1) = if(X,X_0) \subset (A,A_0)$.
Thus (X,X_0) is deformable into (A,A_0).

(Necessity) If (X,X_0) is deformable into (A,A_0) let $D : (X,X_0) \times I \to (X,X_0)$ be a deformation such that $D((X,X_0) \times 1) \subset (A,A_0)$. Let $f : (X,X_0) \to (A,A_0)$ be defined by $if(x) = D(x,1)$, for all x in (X,X_0).
Then $D : 1 \simeq if$ which shows that f is a right homotopy inverse of i.

//

Note 4.3.7.: In Top or Toppair $i : (A,A_0) \hookrightarrow (X,X_0)$ with HEP has a right homotopy inverse only in the trivial case $(A,A_0) = (X,X_0)$.

Definition 4.3.8.: A subspace (A,A_0) of (X,X_0) is called a WEAK DEFORMATION RETRACT of (X,X_0) if the inclusion map $i : (A,A_0) \hookrightarrow (X,X_0)$ is a homotopy equivalence.

Lemma 4.3.9.: (A,A_0) is a weak deformation retract of (X,X_0) if and only if (A,A_0) is a weak retract of (X,X_0) and (X,X_0) is deformable into A.

Proof: It follows trivially from the fact that left and right inverses are equal and from Lemma 4.3.6.

//

Definition 4.3.10: A subspace (A,A_0) of (X,X_0) is a STRONG DEFORMATION RETRACT if there exists a retraction $r : (X,X_0) \to (A,A_0)$ such that, if $(A,A_0) \xhookrightarrow{i} (X,X_0)$, then $1 \simeq ir$ rel (A,A_0).

Definition 4.3.11.: A subspace (A,A_0) of (X,X_0) is called a DEFORMATION RETRACT if there is a retraction r of (X,X_0) to (A,A_0) such that $1 \simeq ir$ where $i : (A,A_0) \hookrightarrow (X,X_0)$ is the inclusion map. If $F : 1 \simeq ir$, F is called a deformation retraction of (X,X_0) to (A,A_0). A homotopy $F : (X,X_0) \times I \to (X,X_0)$ is called a deformation retraction if and only if $F(x,0) = x$ for all x belonging to (X,X_0) and $F((X,X_0) \times 1) \subset (A,A_0)$ and $F(x,1) = x$ for all x in (A,A_0). It is a STRONG DEFORMATION RETRACT if and only if it also satisfies the condition $F(x,t) = x$ for all x in (A,A_0) and all t in I.

Lemma 4.3.12.: If (X,X_0) is deformable into a retract (A,A_0) then (A,A_0) is a deformation retract of (X,X_0).

Proof: Let $(A,A_0) \hookrightarrow (X,X_0)$ be the inclusion map and $r : (X,X_0) \to (A,A_0)$ a retraction. Then $ri = 1$ by definition of retract, i.e. r is a left homotopy inverse of i. But (X,X_0) is deformable into (A,A_0), so by Lemma 4.3.6, i has a right homotopy inverse. But we know that if i has a right homotopy inverse and a left homotopy inverse they are the same. Hence $1 \simeq ir$ and (A,A_0) is a deformation retract of (X,X_0).

//

Corollary 4.3.13. If $(A,A_0) \hookrightarrow (X,X_0)$ has the HEP with respect to (A,A_0), then (A,A_0) is a weak deformation retract of (X,X_0) if and only if (A,A_0) is a deformation retract of (X,X_0).

Proof: This follows from Lemma 4.3.12 and Theorem 4.3.3.

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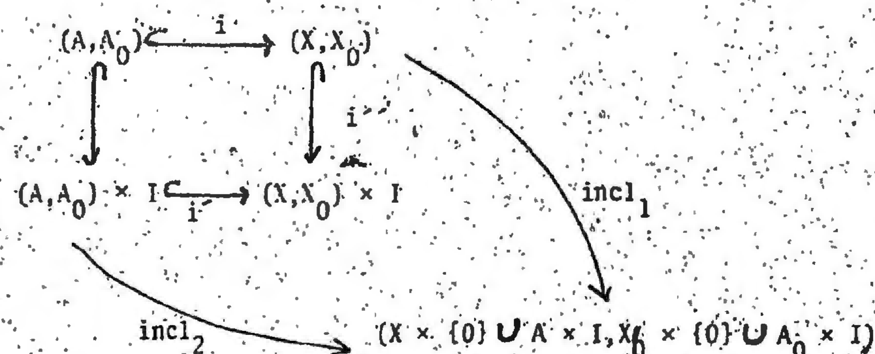
§4.4 CHARACTERIZATION OF COFIBRATIONS IN TOPPAIR

THEOREM 4.4.1.: The map $(A,A_0) \xhookrightarrow{i} (X,X_0)$ is a cofibration in Toppair if and only if $(X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I)$ is a retract of $(X \times I, X_0 \times I)$.

Proof: (Necessity) Assume i is a cofibration. We need $r : (X,X_0) \times I \to (X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I)$ such that $r\bar{i} = 1$ where $\bar{i} : (X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I) \hookrightarrow (X,X_0) \times I$.

Now we have the following diagram:



But i is a cofibration; therefore there exists a map

$$r : (X,X_0) \times I \to (X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I)$$

such that $ri'' = \text{incl}_1$ and $ri' = \text{incl}_2$. We still need that $r\bar{i} = 1$. But we have that $ri'(a,t) = r(a,t) = \text{incl}_2\,(a,t) =$

$$(a,t) = 1_{(A,A_0)\times I}(a,t)$$

And $ri''(x) = r(x) = \text{incl}_1(x) = x = 1_{(X,X_0)}(x)$.

And $\bar{i} = \text{incl}_1 \cup \text{incl}_2$. So we have the required result.

Before giving the sufficiency we prove the following Lemma of STRØM [26].

Lemma 4.4.2.: If (X,A) is a pair such that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, then a subset C of $X \times \{0\} \cup A \times I$ is open in $X \times \{0\} \cup A \times I$ if and only if $C \cap X \times \{0\}$ is open in $X \times \{0\}$ and $C \cap A \times I$ is open in $A \times I$.

Proof: (Sufficiency) - Assume that $C \subset X \times \{0\} \cup A \times I$ and $C \cap X \times \{0\}$ is open in $X \times \{0\}$, $C \cap A \times I$ is open in $A \times I$. We need that C is open in $X \times \{0\} \cup A \times I$. We first write C in the following way.

If $C \cap A \times I$ is open in $A \times I$, then $C \cap (A \times (0,1])$ is open in $C \cap A \times I$. And $C \cap (A \times (0,1])$ is open in $(C \cap A \times I) \cup (C \cap X \times \{0\})$ which is open in $A \times I \cup X \times \{0\}$.

Let $U = \{x$ in X such that $(x,0)$ belongs to $C\}$ be open in X.

Let $U_N = \cup \{V$ such that V is open in X and $(V \cap A) \times [0,1/N) \subset C\}$.

[Note: Then $A \cap U = A \cap \overset{\infty}{\underset{1}{\cup}} U_N$ and, if V is open in X such that $V \cap A \subset U_N$, then $V \subset U_N$].

Let $B = U \times \{0\} \cup \overset{\infty}{\underset{1}{\cup}}((A \cap U_N) \times [0,1/N))$; then $C = C \cap (A \times (0,1]) \cup B$. We now need to show that B is open in $X \times \{0\} \cup A \times I$. We first prove that $U \subset \overset{\infty}{\underset{1}{\cup}} U_N$. Assume that x belongs to $X - \overset{\infty}{\underset{1}{\cup}} U_N$. Then x belongs to the closure of A, i.e. to $\bar{A}$. Let t belong to $(0,1]$. Then $r(x,t)$ belongs to $r(\bar{A} \times t) = A \times t$ where $r : X \times I \to (X \times \{0\} \cup A \times I)$ is the retraction.

If $r(x,t)$ belongs to some $U_N \times I$ there must exist open neighbourhoods V of x and W of t such that
$r(V \times W) \subset U_N \times I$.
We should then have that $(V \cap A) \times t = r(V \cap A) \times t$ since x is in $\bar{A}$ and V is a neighbourhood of x, and
$r(V \cap A) \times t \subset U_N \times I$ since $r(V \times W) \subset U_N \times I$,
i.e. $(V \cap A) \times t \subset U_N \times I$, i.e. $V \cap A \subset U_N$. But this implies that $V \subset U_N$ from Note above and so x belongs to
$V \subset U_N \subset \bigcup_1^\infty U_N$ which contradicts the hypothesis - x in
$X - \bigcup_1^\infty U_N$.
Hence $r(x,t)$ belongs to $(A - \bigcup_1^\infty U_N) \times I$. But
$A \cap U = A \cap \bigcup_1^\infty U_N$, so $A - \bigcup_1^\infty U_N = A - U$. Thus $r(x,t)$
belongs to $(A - U) \times I \subset (X - U) \times I$ for t in $[0,1]$.
Now $X - U$ is closed in X since U is open in X.
And since $X - U$ is closed and r is continuous -

$$(x,0) = r(x,0) \text{ belongs to } (X - U) \times I,$$

i.e. x is in $X - U$.
But this implies that $X - \bigcup_1^\infty U_N \subset X - U$. Thus $U \subset \bigcup_1^\infty U_N$.
Let $V_N = U \cap U_N$, $N = 1, \ldots$. Then each V_N is open in X since U and U_N are open in X for all N. Also we have

$$U = \bigcup_1^\infty V_N, \text{ i.e. } U = U \cap U_1 \cup U \cap U_2 \cup U \cap U_3 \cup \ldots,$$

$$= U \cap \bigcup_1^\infty U_N = U \text{ since } U \subset \bigcup_1^\infty U_N.$$

And $A \cap V_N = A \cap (U \cap U_N)$ [definition of V_N]

$= (A \cap U) \cap U_N$

$= (A \cap \bigcup_1^\infty U_N) \cap U_N$ [$A \cap U = A \cap \bigcup_1^\infty U_N$ already seen by definition of U and $\bigcup_1^\infty U_N$]

$= A \cap (\bigcup_1^\infty U_N \cap U_N)$

$= A \cap U_N$

And $C = C \cap (A \times (0,1]) \cup B \subset X \times \{0\} \cup A \times I$

with $C \cap (A \times (0,1])$ open in $X \times \{0\} \cup A \times I$ [from above].

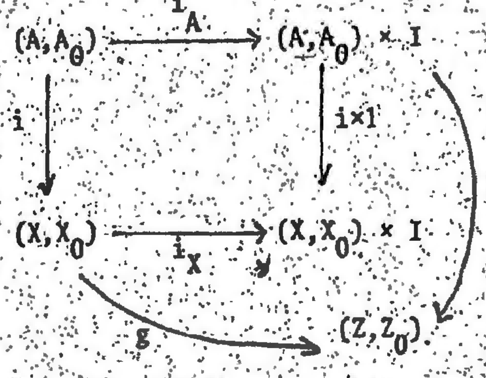
And $B = (X \times \{0\} \cup A \times I) \cap \bigcup_1^\infty (V_N \times [0,1/N))$ is open in $X \times \{0\} \cup A \times I$. Hence C is open in $X \times \{0\} \cup A \times I$ and the Lemma is proved. //

We note that the Lemma is proved in Top not Toppair and this is sufficient for our purpose.

We now prove the sufficiency of Theorem 4.4.1.

(Sufficiency) Let $r : (X \times I, X_0 \times I) \to (X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I)$ be a retraction. Let $f : (A, A_0) \times I \to (Z, Z_0)$ and $g : (X, X_0) \to (Z, Z_0)$ be maps. We need that any commutative diagram of the following form

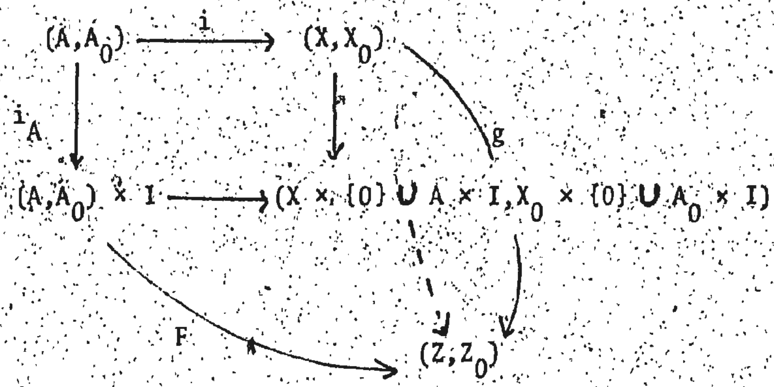
Diagram 4.4.3



is a weak pushout.

Now since r is a retraction and as a consequence of Lemma 4.4.2, we have a pushout:

Diagram 4.4.4



Therefore there exists $g \cup F = Q$, say, making Diagram 4.4.4 commute. Then $Q \circ r$ is a map that commutatively completes Diagram 4.4.3. Hence i is indeed a cofibration in Toppair

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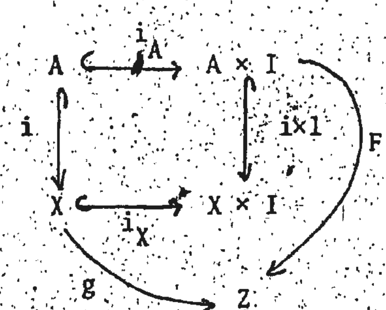
Proposition 4.4.5. If $i : (A,A_0) \hookrightarrow (X,X_0)$ is a cofibration in Toppair, then $A \hookrightarrow X$ and $A_0 \hookrightarrow X_0$ are cofibrations in Top.

Proof: For any commutative diagram of the following form

Diagram 4.4.6

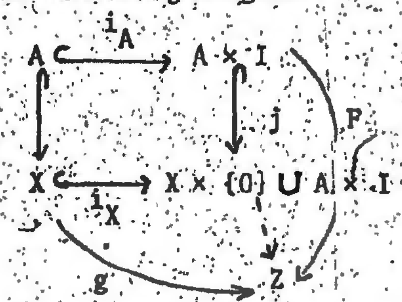


we require a map $H : X \times I \to Z$ such that $H(i \times 1) = F$, $Hi_X = g$. But since i is a cofibration in Toppair, there exists a retraction

$r : (X \times I, X_0 \times I) \to (X \times \{0\} \cup A \times I, X_0 \times \{0\} \cup A_0 \times I)$.

But $r : X \times I \to X \times \{0\} \cup A \times I$ is then a retraction in Top.

And so we have the following diagram



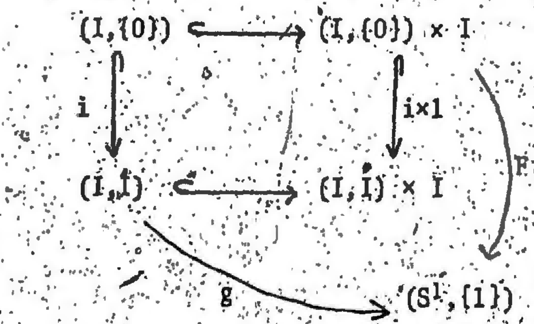
which is a pushout in Top (Lemma 4.4.2.). Hence there exists a map H' such that $H' i_X = g$ and $H' j = F$. Taking the composition $H' \circ r$ gives us the map which commutatively completes Diagram 4.4.6. In a similar way we see that $A_0 \hookrightarrow X_0$ is a cofibration in Top.

//

Remark 4.4.7.: That $i : (A, A_0) \hookrightarrow (X, X_0)$ is a Toppair inclusion and that $A \hookrightarrow X$ and $A_0 \hookrightarrow X_0$ are cofibrations in Top are necessary but not sufficient conditions for i to be a Toppair cofibration. We give the following counter-example. Let $(X, X_0) = (I, \dot{I})$, $(A, A_0) = (I, \{0\})$ and $(Z, Z_0) = (S^1, \{1\})$. We note that $X_0 \cap A = \dot{I} \cap I = \dot{I} \neq A_0$.

Consider the diagram:

Diagram 4.4.8



and maps $g : (I,\dot{I}) \to (S^1,\{1\})$ given by $g(s) = 1$ and

$$F : (I,\{0\}) \times I \to (S^1,\{1\}) \text{ given by}$$

$F(s,t) = e^{(1-t)2\pi is}$.

Obviously g is a map of pairs and F a homotopy of pairs.

Then the extension of $F : I \times I \to S^1$ is F.

But $F(\dot{I} \times I)$ does not stay inside the subspace $\{1\}$,

i.e. for the point of $X_0 \cap A$ outside of $A_0 \setminus \{1\}$,

$F(\dot{I} \times I)$ does not lie inside $\{1\}$. For example

$F(1,\frac{1}{2}) = e^{\frac{1}{2}2\pi i} = e^{\pi i} = \cos\pi + i\sin\pi = -1 \not\subset \{1\}$.

Hence Diagram 4.4.8 cannot be a weak pushout.

Thus $i : (A,A_0) \hookrightarrow (X,X_0)$ is not a cofibration in Toppair.

§4.5 INDUCED HOMOTOPY EQUIVALENCES AND UNIVERSAL CONSTRUCTIONS

THEOREM 4.5.1.: If (X,X_0), (Y,Y_0) and (Z,Z_0) are objects of Toppair, X and Z are Hausdorff and Y is locally compact, then the function

$$(Y^X,(Y,Y_0)^{(X,X_0)}) \times (Z^Y,(Z,Z_0)^{(Y,Y_0)}) \to (Z^X,(Z,Z_0)^{(X,X_0)})$$

is continuous where $T : Y^X \times Z^Y \to Z^X$ is defined by $T(f,g) = g \circ f$.

Proof: See Dugundji, p. 259 [10].

Proposition 4.5.2.: Let $f,g : (X,X_0) \to (Y,Y_0)$ be maps in Toppair and let (Z,Z_0) be an object of Toppair with X,Z Hausdorff and Y locally compact. Then f is homotopic to g if and only if

f^* and $g^* : (Z^Y,(Z,Z_0)^{(Y,Y_0)}) \to (Z^X,(Z,Z_0)^{(X,X_0)})$ are homotopic.

Proof: (Necessity) Assume $H : f \simeq g : (X,X_0) \times I \to (Y,Y_0)$, i.e. $H : f \simeq g : (X,X_0) \times (I,I) \to (Y,Y_0)$.

Now H is continuous if and only if $\hat{H} : (I,I) \to (Y^X,(Y,Y_0)^{(X,X_0)})$ is continuous if and only if

$$\hat{H} \times 1 : (I,I) \times (Z^Y,(Z,Z_0)^{(Y,Y_0)}) \to (Y^X,(Y,Y_0)^{(X,X_0)}) \times (Z^Y,(Z,Z_0)^{(Y,Y_0)})$$

is continuous (See §1.5). So we have the following sequence of continuous maps:

$$(I,I)\times(Z^Y,(Z,Z_0)^{(Y,Y_0)}) \xrightarrow[H\times 1]{} (Y^X,(Y,Y_0)^{(X,X_0)})\times(Z^Y,(Z,Z_0)^{(Y,Y_0)}) \xrightarrow{T} (Z^X,(Z,Z_0)^{(X,X_0)})$$

And $T \circ (\hat{H} \times 1)$ is continuous. But

$$(Z^Y, (Z,Z_0)^{(Y,Y_0)}) \times (I,\dot{I}) \overset{\theta}{\cong} (I,\dot{I}) \times (Z^Y, (Z,Z_0)^{(Y,Y_0)})$$

So $T \circ (\hat{H} \times 1) \circ \theta$ is the required homotopy between f^* and g^*.

(Sufficiency) Assume

$$Q : f^* \simeq g^* : (Y^Y, (Y,Y_0)^{(Y,Y_0)}) \times (I,\dot{I}) \to (Y^X, (Y,Y_0)^{(X,X_0)})$$

Let $i : (*,*) \to (Y^Y, (Y,Y_0)^{(Y,Y_0)})$ be defined by sending

$$* \text{ to } 1_{Y^Y} \text{ in } * \to Y^Y .$$

Consider the induced map

$$i^* : \mathrm{Map}[(Y^Y, (Y,Y_0)^{(Y,Y_0)}), (Y^X, (Y,Y_0)^{(X,X_0)})] \to [(Y^X)^*, ((Y,Y_0)^{(X,X_0)})^{(*,*)}]$$

and the map

$$\beta : [(Y^X)^*, ((Y,Y_0)^{(X,X_0)})^{(*,*)}] \to (Y^{**X}, (Y,Y_0)^{(**X,**X_0)}).$$

Consider also the isomorphism

$$(Y^X, (Y,Y_0)^{(X,X_0)}) \to (Y^{**X}, (Y,Y_0)^{(**X,**X_0)}).$$

We then have the following sequence:

$$\mathrm{Map}[(Y^Y, (Y,Y_0)^{(Y,Y_0)}), (Y^X, (Y,Y_0)^{(X,X_0)})] \xrightarrow{i^*} [(Y^X)^*, ((Y,Y_0)^{(X,X_0)})^{(*,*)}] \xrightarrow{\beta}$$

$$[(Y^{**X}, (Y,Y_0)^{(**X,**X_0)})] \xrightarrow[(\ell^{-1})^*]{} (Y^X, (Y,Y_0)^{(X,X_0)}).$$

Now Q belongs to $\mathrm{Map}[(Y^Y, (Y,Y_0)^{(Y,Y_0)}), (Y^X, (Y,Y_0)^{(X,X_0)})]$, and

$(\ell^{-1})^* \circ \beta \circ i^*(Q)$ is a homotopy between f and g. //

Proposition 4.5.3.: Let $f : (X,X_0) \to (Y,Y_0)$ be a map in Toppair, (Z,Z_0) an arbitrary object of Toppair, X and Z Hausdorff, and Y locally compact. Then f is a homotopy equivalence if and only if

$f^* : (Z^Y, (Z,Z_0)^{(Y,Y_0)}) \to (Z^X, (Z,Z_0)^{(X,X_0)})$ is a homotopy equivalence.

Proof: (Necessity) Let $g : (Y,Y_0) \to (X,X_0)$ be a homotopy inverse for f. Then $gf \simeq 1$ and $fg \simeq 1$. Now by Proposition 4.5.2 $r \simeq q$ if and only if $r^* \simeq q^*$. Therefore $(gf)^* \simeq 1$ and $(fg)^* \simeq 1$. By functorial properties this implies that $f^*g^* \simeq 1$ and $g^*f^* \simeq 1$. Hence f^* is indeed a homotopy equivalence.

(Sufficiency) Since f^* is a homotopy equivalence for all (Z,Z_0), it is so in particular for $(Z,Z_0) = (X,X_0)$.

Let $g : (X^X, (X,X_0)^{(X,X_0)}) \to (X^Y, (X,X_0)^{(Y,Y_0)})$ be a homotopy inverse for f^*. Let $g(1_X) = h : (Y,Y_0) \to (X,X_0)$. Let H be a homotopy between f^*g and 1, i.e.

$H : f^*g \simeq 1 : (X^X, (X,X_0)^{(X,X_0)}) \times (I,I) \to (X^X, (X,X_0)^{(X,X_0)})$.

Holding 1_X fixed in $(X^X, (X,X_0)^{(X,X_0)})$, we have a map

$$H_{1_{(X,X_0)}} = K : (I,I) \to (X^X, (X,X_0)^{(X,X_0)}) \text{ defined by}$$

$$H_{1_{(X,X_0)}}(t)(-) = K(t)(-) = H(1_{(X,X_0)}(-),t) \text{ for all } x \text{ in } X$$

and for all t in I. Hence

$K(0)(-) = H(1_{(X,X_0)}(-),0) = f^*g1_X(-) = f^*h = hf$. And

$K(1)(-) = H(1_{(X,X_0)}(-),1) = 1_{X^X}(1_{(X,X_0)}(-)) = 1_{(X,X_0)}$, i.e.

K is a path between hf and 1.

Using the exponential law, we get a homotopy

$$\hat{K} : (X,X_0) \times (I,I) \to (X,X_0).$$

And $\hat{K}(-,0) \simeq \hat{K}(-,1)$. But $\hat{K}(-,0) = K(0)(-) = H_{1_{(X,X_0)}}(0)(-) = hf$

and

$$\hat{K}(-,1) = K(1)(-) = H_{1_{(X,X_0)}}(1)(-) = 1_{(X,X_0)}.$$

So $\hat{K} : hf \simeq 1_{(X,X_0)} : (X,X_0) \times I \to (X,X_0)$.

But h does not necessarily induce g since we simply have that

$g(1_{(X,X_0)}) = h$ and $g(K)$ need not equal h^* for $K \neq 1$.

However, since $hf \simeq 1$ we can say that

$(hf)^* \simeq 1_{(Z^X,(Z,Z_0)^{(X,X_0)})}$ for all (Z,Z_0), i.e.

$$f^*h^* \simeq 1 : (Z^X,(Z,Z_0)^{(X,X_0)}) \times (I,I) \to (Z^X,(Z,Z_0)^{(X,X_0)}).$$

But f^* and 1 are homotopy equiavlances. Thus by Proposition 1.3.6 and Note 1.3.7 so too is h^*. Now we have that f is a right inverse for h. We still need a left homotopy inverse for h. Applying the argument above to

$h^* : (Y^X,(Y,Y_0)^{(X,X_0)}) \to (Y^Y,(Y,Y_0)^{(Y,Y_0)})$ we get our result.

Now h^* is a homotopy equivalence, so there exists

$r : (Y^Y, (Y,Y_0)^{(Y,Y_0)}) \to (Y^X, (Y,Y_0)^{(X,X_0)})$ such that $rh^* = 1$ and $h^*r = 1$. Let $r(1_{(Y,Y_0)}) = q : (X,X_0) \to (Y,Y_0)$ and we get

$$qh \simeq 1_{(Y,Y_0)}.$$

Now $qhfh \simeq 1fh = fh$ [If $L : m \simeq n : (A,A_0) \times I \to (B,B_0)$ and $t : (B,B_0) \to (C,C_0)$ is a map, then tL is a homotopy between tm and tn].

But $qhfh = q1_{(X,X_0)}h = qh$. Therefore $qh \simeq qhfh \simeq fh \simeq 1_{(Y,Y_0)}$, i.e. f is a left homotopy inverse for h. Hence h is a homotopy equivalence. But $hf = 1$; hence f is a homotopy equivalence.

//

Proposition 4.5.4.: If Z is Hausdorff and B is locally compact and Hausdorff then commutative Diagram 4.5.5 is a pushout in Toppair if and only if commutative Diagram 4.5.6 is a pullback in Toppair.

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{u_1} & (B,B_0) \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle i_B} \\ (C,C_0) & \xrightarrow[i_C]{} & (B \sqcup C, B_0 \sqcup C_0) \end{array}$$

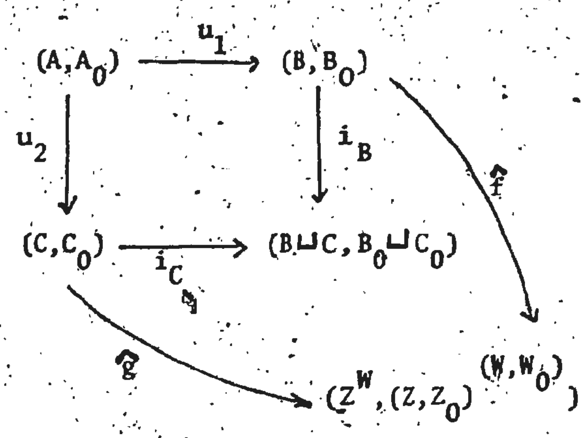
Diagram 4.5.5.

$$\begin{array}{ccc} (Z^{B \sqcup C}, (Z,Z_0)^{(B \sqcup C, B_0 \sqcup C_0)}) & \xrightarrow{t_1} & (Z^B, (Z,Z_0)^{(B,B_0)}) \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ (Z^C, (Z,Z_0)^{(C,C_0)}) & \xrightarrow[q_2]{} & (Z^A, (Z,Z_0)^{(A,A_0)}) \end{array}$$

Diagram 4.5.6.

Proof: (Necessity) Let W be locally compact,

$f : (W,W_0) \to (Z^B,(Z,Z_0)^{(B,B_0)})$, $g : (W,W_0) \to (Z^C,(Z,Z_0)^{(C,C_0)})$ be maps such that $q_1 \circ f = t_2 \circ g$. We need a map $(W,W_0) \to (Z^{B \sqcup C},(Z,Z_0)^{(B\sqcup C,B_0 \sqcup C_0)})$ making Diagram 4.5.6 commute. Using the exponential law for Toppair, Diagram 4.5.6 gives rise to the following commutative diagram



But the above diagram is a pushout in Toppair, so there exists a map $\overline{Q} : (B \sqcup C,B_0 \sqcup C_0) \to (Z^W,(Z,Z_0)^{(W,W_0)})$ making the diagram commute. Again using the exponential law $\overline{Q}$ gives rise to $Q : (W,W_0) \to (Z^{B\sqcup C},(Z,Z_0)^{(B\sqcup C,B_0\sqcup C_0)})$ which makes Diagram 4.5.6 commute. The uniqueness of Q comes from the fact that

$$\mathrm{Map}[(W,W_0),(Z^{B\sqcup C},(Z,Z_0)^{(B\sqcup C,B_0\sqcup C_0)})] \cong$$

$$\mathrm{Map}[(B\sqcup C,B_0\sqcup C_0),(Z^W,(Z,Z_0)^{(W,W_0)})].$$

Hence Diagram 4.5.6 is a pullback for all (Z,Z_0).

(Sufficiency) Let T be Hausdorff and $f : (B,B_0) \to (T,T_0)$, $g : (C,C_0) \to (T,T_0)$ be maps such that $f \circ u_1 = g \circ u_2$.

We require the existence of a unique map

$$\psi : (B \sqcup C, B_0 \sqcup C_0) \to (T,T_0)$$

making any diagram of the following form commute:

$$\begin{array}{ccccc}
(A,A_0) & \xrightarrow{u_1} & (B,B_0) & & \\
\downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle i_B} & & \\
(C,C_0) & \xrightarrow{i_C} & (B \sqcup C, B_0 \sqcup C_0) & & \searrow{\scriptstyle f} \\
& \searrow{\scriptstyle g} & & & (T,T_0)
\end{array}$$

Diagram 4.5.7.

Now from Diagram 4.5.7 we deduce the following commutative diagram

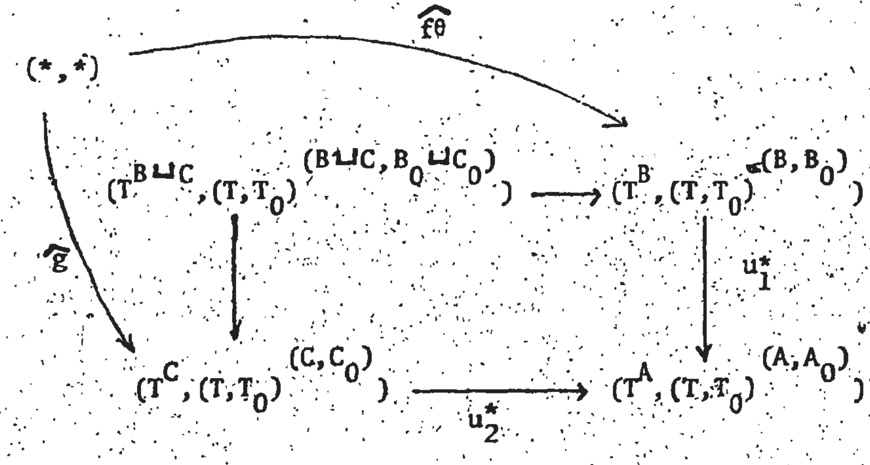
$$\begin{array}{ccc}
(T^{B \sqcup C}, (T,T_0)^{(B \sqcup C, B_0 \sqcup C_0)}) & \xrightarrow{i_B^*} & (T^B, (T,T_0)^{(B,B_0)}) \\
\downarrow{\scriptstyle i^*_C} & & \downarrow{\scriptstyle u_1^*} \\
(T^C, (T,T_0)^{(C,C_0)}) & \xrightarrow[u_2^*]{} & (T^A, (T,T_0)^{(A,A_0)})
\end{array}$$

Now $(B,B_0) \times (*,*) \overset{\theta}{\cong} (B,B_0) \xrightarrow{f} (T,T_0)$, so using the exponential law

$$\widehat{f\theta} : (*,*) \to (T^B, (T,T_0)^{(B,B_0)}) \text{ is a map.}$$

Similarly $\hat{g} : (*,*) \to (T^C, (T,T_0)^{(C,C_0)})$ is a map.

Thus we have the following commutative diagram in Toppair:



But Diagram 4.5.6 is a pullback in Toppair for all (Z,Z_0) so in particular for $(Z,Z_0) = (T,T_0)$. Therefore there exists a unique map $Q' : (*,*) \to (T^{B\sqcup C},(T,T_0)^{(B\sqcup C,B_0\sqcup C_0)})$ making the above diagram commute. Using the exponential law we get a map $\hat{Q}' : (B\sqcup C,B_0\sqcup C_0) \to (T,T_0)$ making Diagram 4.5.7 commute. Using an argument similar to the one used for the "if" part of the proof, we get the uniqueness of $\overline{Q}'$. Hence Diagram 4.5.7 is a pushout in Toppair.

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Consider the following commutative diagram in Toppair:

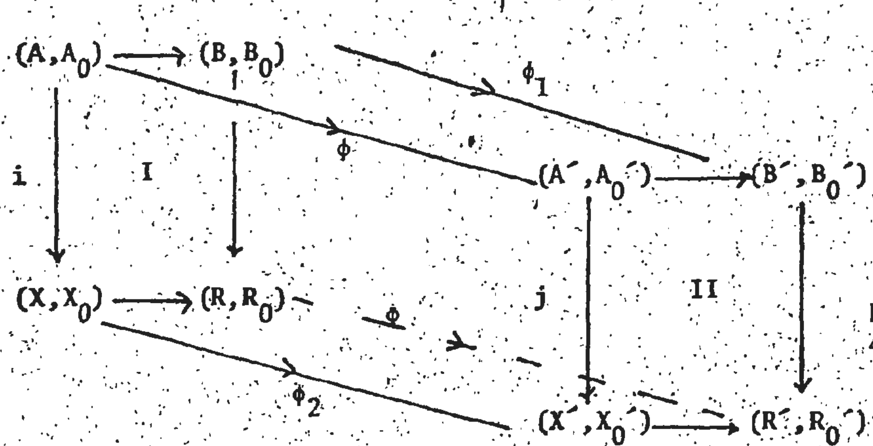


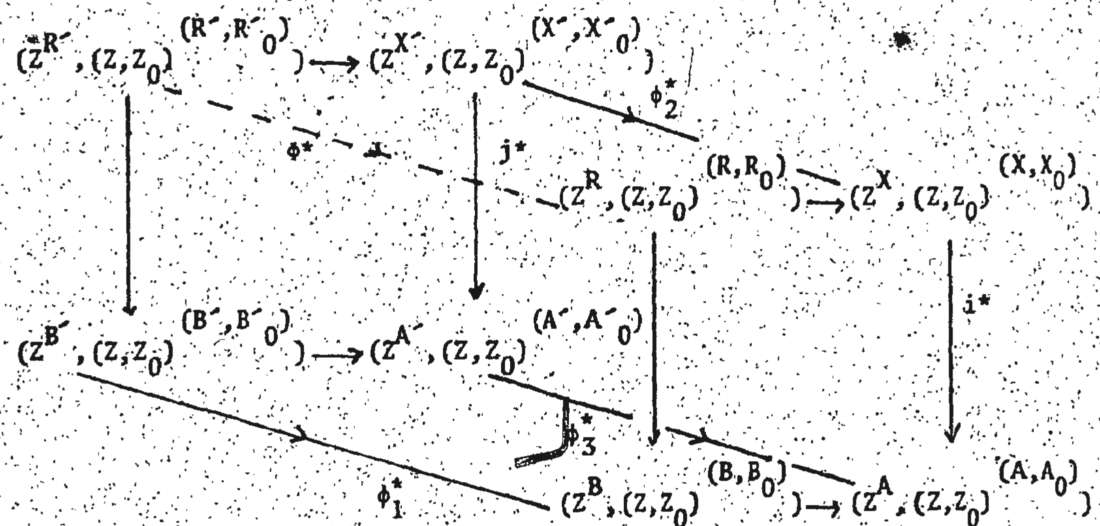
Diagram 4.5.8.

in which i, j are cofibrations, squares I and II are pushouts, and ϕ_3, ϕ_1, ϕ_2 are homotopy equivalences.

RELATIVE GLUEING THEOREM:

THEOREM 4.5.9.: The induced map ϕ in Diagram 4.5.8 is a homotopy equivalence.

Proof: For all (Z,Z_0), Diagram 4.5.8 induces the following commutative diagram in which squares I′ and II′ are pullbacks in Toppair (Proposition 4.5.4.), i^* and j^* are fibrations (Lemma 4.2.14.) and ${\phi^*}_3$, ${\phi^*}_1$, ${\phi^*}_2$ are homotopy equivalences (Proposition 4.5.3.):



By the Co-glueing Theorem (see 3.3.1) ϕ^* is a homotopy equivalence for all (Z,Z_0). Hence by Proposition 4.5.3, ϕ is a homotopy equivalence.

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Note: The above theorem is a generalization of a similar result due to Brown [2] and a specialization of similar results of Kamps.

CHAPTER V - INTRODUCTION

In section one, we present part of a manuscript by Brown [3]. The main theorem we quote is Theorem 5.1.3. We also show how any Toppair pullback in which one map is a fibration induces diagrams of groupoids which fit into a Mayer-Vietoris type sequence with certain exactness properties.

In section two we define the concept of homotopy pullbacks in Toppair and show how a Mayer-Vietoris type sequence arises from a Toppair homotopy pullback.

§5.1 MAYER-VIETORIS SEQUENCES OF GROUPOIDS

Definition 5.1.1.: Let G be a groupoid. A SUBGROUPOID of G is a subcategory H of G such that a belongs to H implies a^{-1} belongs to H; i.e. H is a subcategory which is also a groupoid.

We say that H is a FULL SUBGROUPOID if H is a full subcategory. The full subgroupoid of G on one object of G is written $G\{x\}$ - this groupoid has one zero and $a \circ b$ is defined for all a, b in $G\{x\}$. A groupoid with only one object is called a group and in particular $G\{x\}$ is called the group at x.

Note 5.1.2.: Throughout this chapter the component of an object x of a groupoid will be written $\tilde{x}$. Also if $f : A \to C$ is a functor between groupoids, f will also be used to denote the induced morphism of components, i.e. $f : \pi_0 A \to \pi_0 C$. As well throughout this chapter we will consider a pullback square of groupoids:

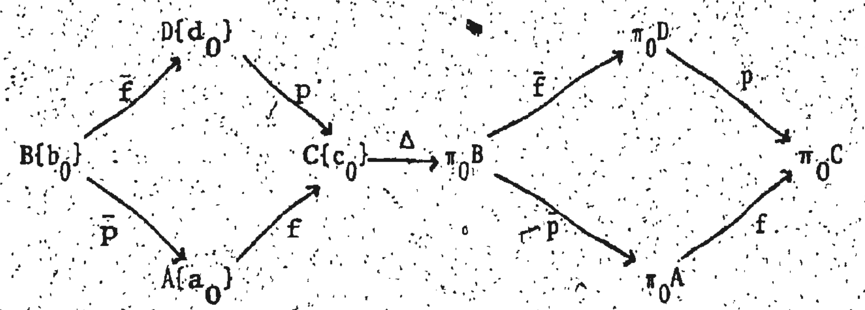
$$\begin{array}{ccc} B & \xrightarrow{\overline{f}} & D \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & C \end{array}$$

where B is the subgroupoid of $A \times D$ whose morphisms are pairs (α, δ) such that $f(\alpha) = p(\delta)$ and

$\overline{f}(\alpha, \delta) =$, $\overline{p}(\alpha, \delta) = \alpha$.

Let $b_0 = (a_0, d_0)$ belong to $\mathrm{Ob}\, B$, i.e. $f(a_0) = p(d_0)$. Let $c_0 = f(a_0)$. We give the following Theorem of Brown [3]:

THEOREM 5.1.3.: If p is a fibration of groupoids, then there is a function $\Delta : C\{c_0\} \to \pi_0 B$ which fits into a diagram



with the following exactness properties:

(i) If $\tilde{a}$ belongs to $\pi_0 A$, $\tilde{d}$ belongs to $\pi_0 D$ and satisfy $f(\tilde{a}) = p(\tilde{d})$, then there is $\tilde{b}$ belonging to $\pi_0 B$ such that $\bar{f}(\tilde{b}) = \tilde{d}$, $\bar{p}(\tilde{b}) = \tilde{a}$.

(ii) $\mathrm{Im}(\Delta) = \bar{f}^{-1}(\tilde{d}_0) \cap \bar{p}^{-1}(\tilde{a}_0)$

(iii) $\Delta(1) = \tilde{b}_0$

(iv) γ_1, γ_2 belonging to $C\{c_0\}$ satisfy $\Delta(\gamma_1) = \Delta(\gamma_2)$ if and only if there exist elements α of $A\{a_0\}$, σ of $D\{d_0\}$ such that

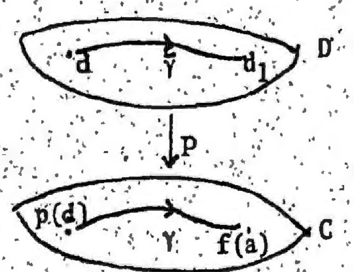
$$\gamma_1 = f(\alpha)\gamma_2 p(\delta)$$

(v) $B\{b_0\}$ is the pullback of $A\{a_0\} \to C\{c_0\} \leftarrow D\{d_0\}$.

Proof: (i) Since $f(\tilde{a}) = p(\tilde{d})$ in $\pi_0 C$, there exists γ belonging to $C(p(d), f(a))$ by definition of $\pi_0 C$,

i.e. we have the diagram

But p is a fibration of groupoids - so there exists a lift $\bar{\gamma}$ of γ in D where $\bar{\gamma}$ belongs to $D(d, d_1)$ say, and $p(\bar{\gamma}) = \gamma$:



$p(d) = p(d), p(d_1) = f(a)$

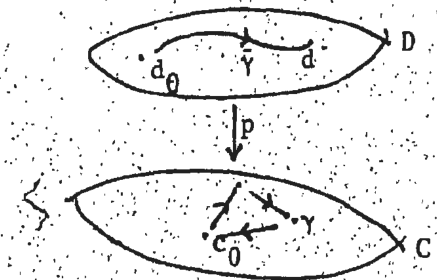
so (a, d_1) belongs to Ob B.

Taking the component of of (a, d_1), i.e. $\widetilde{(a, d_1)}$ we have $\tilde{b}$ in $\pi_0 B$ and $\bar{f}\widetilde{(a, d_1)} = \tilde{d}_1$, $\bar{p}\widetilde{(a, d_1)} = \tilde{a}$ since $\bar{f}(a, d_1) = d_1$ and $\bar{p}(a, d_1) = a$.

But $\tilde{d}_1 = \tilde{d}$ since $\bar{\gamma}$ joins d to d_1. This completes the proof of (i).

(ii) We first need to define Δ.

Let γ belong to $C(c_0)$. Since p is a fibration of groupoids, there exists a lift $\bar{\gamma}$ belonging to $D(d_0, d)$ say, i.e. we have the following diagram:

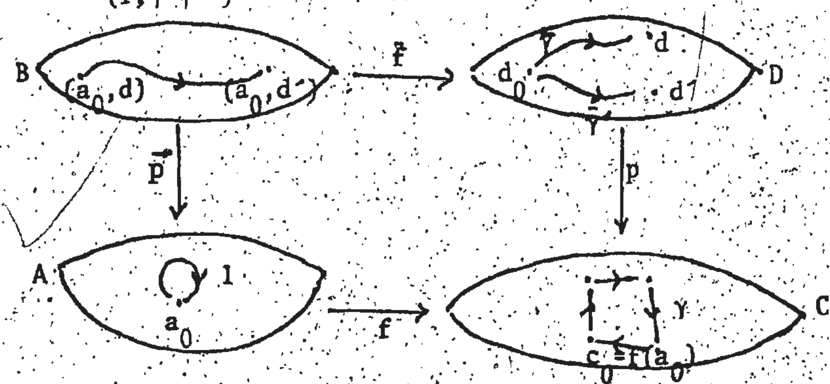


such that $p(\bar{\gamma}) = \gamma$

and $p(d_0) = c_0 = f(a_0)$

$p(d) = c_0 = f(a_0)$.

Thus (a_0,d) belongs to Ob B. We define $\Delta(\gamma) = (a_0,d)^{\sim}$, the component of (a_0,d) in B. We now need to show that Δ is well-defined. Suppose $\bar{\gamma}'$ also lifts γ, $\bar{\gamma}'$ belonging to $D(d_0,d')$ say, $[p(\bar{\gamma}') = \gamma]$ and 1 = identity at a_0:



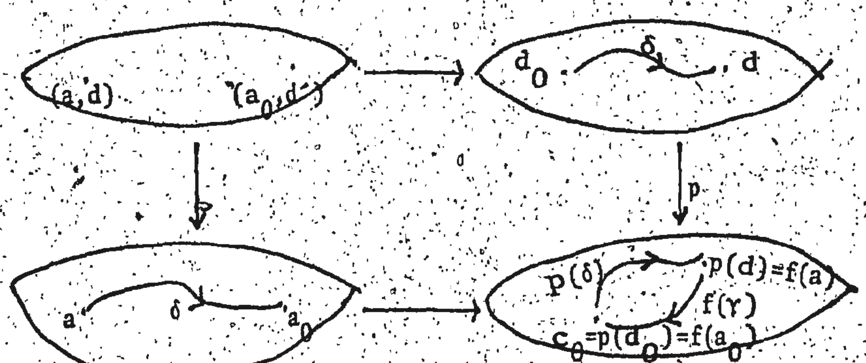
$p(d_0) = c_0 = f(a_0)$, $p(d') = c_0 = f(a_0)$ — so (a_0,d') belongs to Ob B. Now $p(\bar{\gamma}') = p(\bar{\gamma}) = \gamma$, $(1,\bar{\gamma}'\bar{\gamma}^{-1})$ joins (a_0,d) to (a_0,d'), $f(1) = 1$ (Functor preserves identities) and $p(\bar{\gamma}'\bar{\gamma}^{-1}) = p(\bar{\gamma}')p(\bar{\gamma}^{-1}) = \gamma\gamma^{-1} = 1$, i.e.

$f(1) = p(\bar{\gamma}'\bar{\gamma}^{-1})$. Thus Δ is well-defined since $\Delta(\gamma) = (a_0,d)^{\sim}$, $\Delta(\gamma) = (a_0,d')^{\sim}$ and $(a_0,d)^{\sim} = (a_0,d')^{\sim}$.

We also have that $\bar{p}(a_0,d) = a_0$, $\bar{f}(a_0,d) = d$ and $\tilde{d}_0 = \tilde{d}$ so that $\bar{p}(\mathrm{Im}\,\Delta) = \{\tilde{a}_0\}$ and $\bar{f}(\mathrm{Im}\,\Delta) = \{\tilde{d}_0\}$, i.e.

$\mathrm{Im}\,\Delta \subseteq \bar{p}^{-1}(\tilde{a}_0) \cap \bar{f}^{-1}(\tilde{d}_0)$.

To prove the remaining part of (ii), let $\tilde{b} = (a,d)^{\sim}$ in $\pi_0 B$ and suppose $\bar{p}(\tilde{b}) = \tilde{a}_0$, $\bar{f}(\tilde{b}) = \tilde{d}_0$. We show that $\tilde{b}$ belongs to $\mathrm{Im}\,\Delta$. Then $\bar{p}^{-1}(\tilde{a}_0) \cap \bar{f}^{-1}(\tilde{d}_0) \subseteq \mathrm{Im}\,\Delta$, i.e. we show that there exists γ belonging to $C\{c_0\}$ such that $\Delta(\gamma) = \tilde{b}$. Now since $\bar{p}(\tilde{b}) = \bar{p}(a,d)^{\sim} = \tilde{a}_0$ and $\bar{f}(\tilde{b}) = \tilde{d}_0$ there are elements α in $A(a,a_0)$, δ in $D(d_0,d)$ with $p(d_0) = f(a_0) = c_0$; $p(d) = f(a)$ since (a,d) belongs to Ob B, i.e. diagramatically we have



Thus $\gamma = f(\alpha)p(\delta)$ is an element of $C\{c_0\}$. And since p is a fibration there exists a lift of γ, $\bar{\gamma}$ say, where $\bar{\gamma}$ belongs to $D(d_0,d')$ and $p(\bar{\gamma}) = \gamma$, $p(d_0) = c_0 = f(a_0) = p(d')$ so that (a_0,d') belongs to Ob B. And $(\alpha, \bar{\gamma}\delta^{-1})$ joins (a,d) to (a_0,d') in B and $f(\alpha) = p(\bar{\gamma}\delta^{-1})$

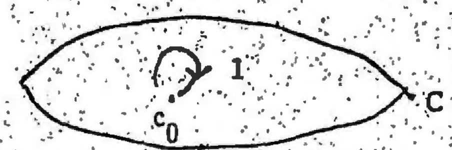
$$[p(\bar{\gamma})p(\delta^{-1}) = \gamma p(\delta^{-1}) = f(\alpha)p(\delta)p(\delta^{-1}) = f(\alpha)].$$

Hence $\Delta(\gamma) = \Delta(f(\alpha)p(\delta)) =$

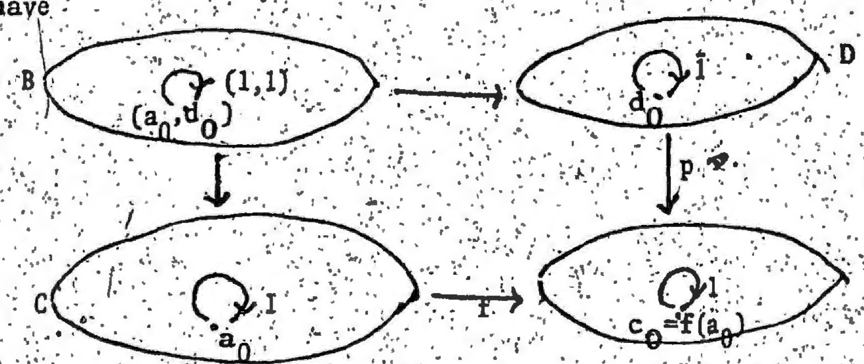
$$(a_0,d')^{\sim} = (a,d)^{\sim} = \tilde{b},$$ i.e.

$\bar{p}^{-1}(\tilde{a}_0) \cap \bar{f}^{-1}(\tilde{d}_0) \subseteq \mathrm{Im}\,\Delta$. So we have completed the proof of (ii) since both inclusions imply equality.

(iii) Let l belong to $C(c_0)$ as in the diagram:



Then since p is a fibration and $p(d_0) = c_0 = f(a_0)$ there exists a lift $\bar{l}$ of l such that $\bar{l}$ belongs to $D(d_0)$, i.e. we have

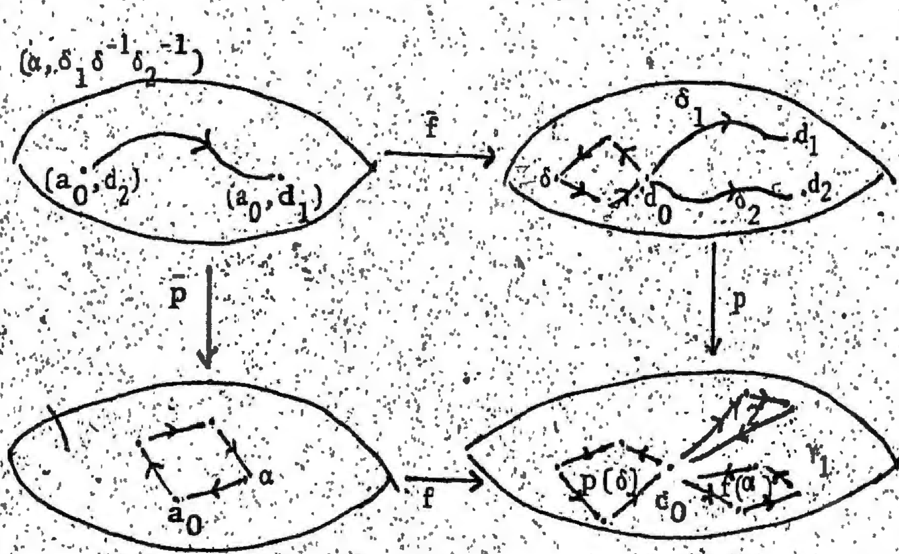


and $p(\bar{l}) = l = f(1)$, $p(d_0) = f(a_0)$. So $\Delta(l) = (a_0, d_0)^{\sim} = \tilde{b}_0$. This completes the proof of (iii)

(iv) "if" part - Suppose γ_1, γ_2 belong to $C(c_0)$. Let δ_1 belonging $D(d_0, d_1)$, δ_2 belonging to $D(d_0, d_2)$ be lifts of γ_1, γ_2 respectively.

So $\Delta(\gamma_1) = (a_0, d_1)^{\sim}$; $\Delta(\gamma_2) = (a_0, d_2)^{\sim}$ in $\pi_0 B$.

Assume also that there are elements α in $A(a_0)$, δ in $D(d_0)$ such that $\gamma_1 = f(\alpha)\gamma_2 p(\delta)$. We prove that $\Delta(\gamma_1) = \Delta(\gamma_2)$. Now we have the following diagram:



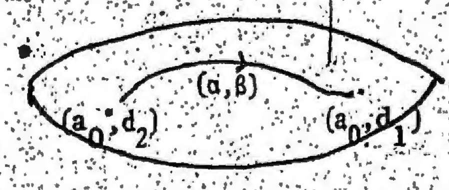
Then $p(\delta_1) = \gamma_1 = f(\alpha)\gamma_2 p(\delta) = f(\alpha)p(\delta_2)p(\delta)$. Thus $(\alpha,\delta_1\delta^{-1}\delta_2^{-1})$ joins (a_o,d_2) to (a_0,d_1) in B since $f(\alpha) = p(\delta_1\delta^{-1}\delta_2^{-1})$, i.e. $p(\delta_1\delta^{-1}\delta_2^{-1}) = p(\delta_1)p(\delta^{-1})p(\delta_2^{-1})$

$$= \gamma_1 p(\delta^{-1}\delta_2^{-1}) = f(\alpha)\gamma_2 p(\delta)p(\delta^{-1})p(\delta_2^{-1}) = f(\alpha).$$

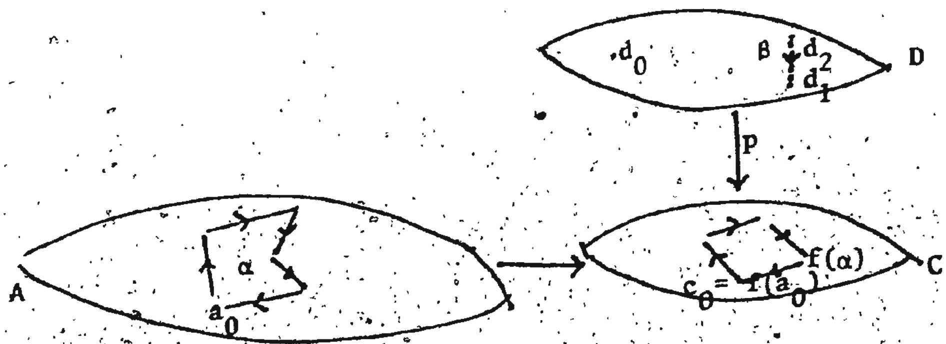
And $(a_0,d_2)^{\sim} = (a_0,d_1)^{\sim}$, i.e. $\Delta(\gamma_1) = \Delta(\gamma_2)$. This completes the sufficiency.

"only if" part - Assume γ_1,γ_2 belong to $C\{c_0\}$ such that $\Delta(\gamma_1) = \Delta(\gamma_2)$. We need elements α in $A\{a_0\}$, δ in $D\{d_0\}$ such that $\gamma_1 = f(\alpha)\gamma_2 p(\delta)$.

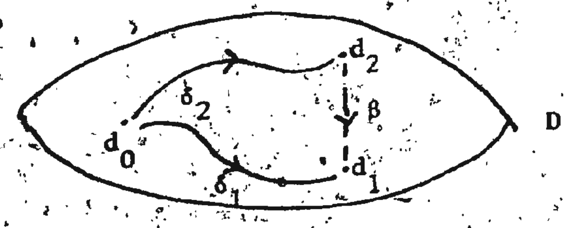
Now since $\Delta(\gamma_1) = \Delta(\gamma_2)$, i.e. $(A_0,d_1)^{\sim} = (a_0,d_2)^{\sim}$ say, there exists an element (α,β) in B joining (a_0,d_2) to (a_0,d_1).



Hence there exists α in $A\{a_0\}$ and β in $D(d_2,d_1)$ such that $f(\alpha) = p(\beta)$ as in the following diagram:



But we have δ_1 in $D(d_0,d_1)$, δ_2 in $D(d_0,d_2)$ which are lifts of γ_1, γ_2, i.e. $p(\delta_1) = \gamma_1$ and $p(\delta_2) = \gamma_2$ since p is a fibration of groupoids. So in D we actually have the following "picture":



Now let $\delta^{-1} = \delta_1^{-1}\beta\delta_2$, i.e. call the "road" above, δ^{-1}.

Then $\delta_1 = \beta\delta_2\delta$, $[\delta_1 = \beta\delta_2(\delta_1^{-1}\beta\delta_2)^{-1} = \beta\delta_2\delta_2^{-1}\beta^{-1}\delta_1 = \delta_1]$.

And $\gamma_1 = p(\delta_1) = p(\beta\delta_2\delta) = p(\beta)p(\delta_2)p(\delta)$,

i.e. $\gamma_1 = f(\alpha)\gamma_2 p(\delta)$. This completes the necessity.

(v) We need that

$$\begin{array}{ccc} B\{b_0\} & \xrightarrow{\bar{f}} & D\{d_0\} \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ A\{a_0\} & \xrightarrow{f} & C\{c_0\} \end{array}$$

is a pullback diagram, i.e. we need that

$B\{b_0\} = A\{a_o\} \sqcap D\{d_0\}$ where $b_0 = (a_0, d_0)$.

Now $A\{a_0\} \sqcap D\{d_0\} = \{(\alpha,\delta)\}$ in $A\{a_0\} \times D\{d_0\}$ such that $p(\delta) = f(\alpha)\}$.

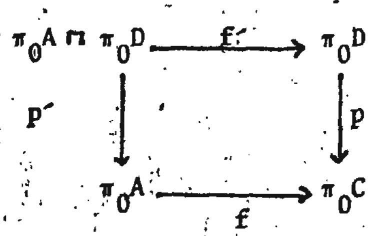
And $B\{b_0\} = B\{(a_0, d_0)\} = (A \sqcap D)\{(a_0, d_0)\}$

$= \{(\alpha,\delta)$ in $(A \times D)\{(a_0, d_0)\}$ such that $p(\delta) = f(\alpha)\}$ by definition of B.

So $B\{b_0\} = (A \sqcap D)\{b_0\} = A\{a_0\} \sqcap D\{d_0\}$ since $A\{a_0\} \times D\{d_0\} = (A \times D)\{(a_o, d_0)\}$ where $(\alpha,\delta) : b_0 \to b_0$, i.e. the diagram is indeed a pullback.

//

Consider the pullback



Now $\bar{f} : \pi_0 B \to \pi_0 D$

and $\bar{p} : \pi_0 B \to \pi_0 A$

are maps such that $p\bar{f} = f\bar{p}$. Thus by the universal property of pullbacks there exists a unique map

$\phi : \pi_0 B \to \pi_0 A \sqcap \pi_0 D$ making the diagram above commute.

Corollary 5.1.4.: ϕ is surjective.

Proof: Brown [3]. This has already been proved in (i) of the theorem. For if $\tilde{a}$ belongs to $\pi_0 A$, $\tilde{d}$ belongs to $\pi_0 D$

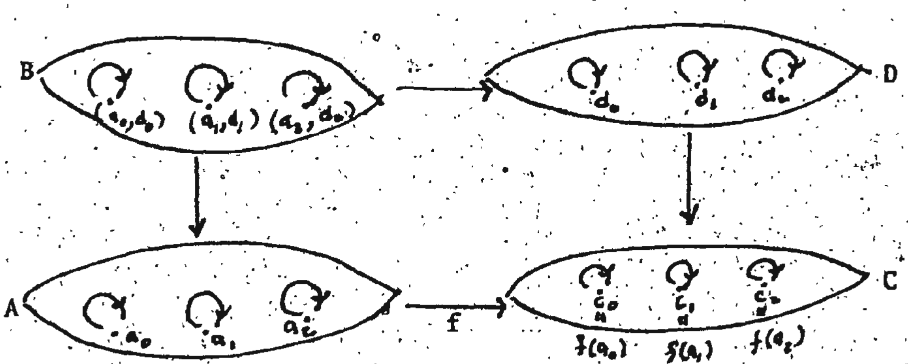
and $f(\tilde{a}) = p(\tilde{d})$, we have seen that there exists $\tilde{b}$ in $\pi_0 B$ such that $\bar{f}(\tilde{b}) = \tilde{d}$ and $\bar{p}(\tilde{b}) = \tilde{a}$, i.e. there exists $\tilde{b}$ in $\pi_0 B$ such that $\phi(\tilde{b}) = (\tilde{a},\tilde{d})$.

//

Definition 5.1.5.: A groupoid C is simply-connected if $C\{c_0\}$ consists of a single element for all c_0 belonging to Ob C.

Corollary 5.1.6.: If Δ constant for all choices of (a_0,d_0) in Ob B, then ϕ is bijective. In particular ϕ is bijective if C is simply-connected.

Proof: Brown [3] Now Δ is constant for all choices of (a_0,d_0) in B (see diagram)



which means $f(1) = p(1)$ so that $(1,1)$ joins (a_i,d_i) $i = 0,1,2$ to itself since $\text{Im}(\Delta) = \bar{p}^{-1}(\tilde{a}_0) \cap \bar{f}^{-1}(\tilde{d}_0) = \tilde{b}_0 - \Delta$ is constant, i.e. $\phi^{-1}(\tilde{a}_0,\tilde{d}_0) = \tilde{b}_0$ only. But every element of $\pi_0 A \sqcap \pi_0 D$ is of the form $(\tilde{a}_0,\tilde{d}_0)$ for some object (a_0,d_0) of B since ϕ is surjective. So if Δ is constant, ϕ is injective, i.e. ϕ is bijective.

//

Now we have already seen that, if $p : (E,E_0) \to (B,B_0)$ is a fibration in Toppair, then for all (Z,Z_0),
$p_* : \pi((Z,Z_0),(E,E_0)) \to \pi((Z,Z_0),(B,B_0))$ is a fibration of groupoids.

Consider the following pullback diagram in Toppair in which p is a fibration:

$$\begin{array}{ccc} (X \sqcap E, X_0 \sqcap E_0) & \longrightarrow & (E,E_0) \\ \downarrow & & \downarrow p \\ (X,X_0) & \xrightarrow{f} & (B,B_0) \end{array}$$

Consider also the following diagrams of groupoids - the first, a pullback diagram; the second, the induced diagram:

$$\begin{array}{ccc} \pi((Z,Z_0),(X,X_0)) \sqcap \pi((Z,Z_0),(E,E_0)) & \to & \pi((Z,Z_0),(E,E_0)) \\ \downarrow & & \downarrow p_* \\ \pi((Z,Z_0),(X,X_0)) & \xrightarrow{f_*} & \pi((Z,Z_0),(B,B_0)) \end{array}$$

$$\begin{array}{ccc} \pi((Z,Z_0),(X \sqcap E, X_0 \sqcap E_0)) & \longrightarrow & \pi((Z,Z_0),(E,E_0)) \\ \downarrow & & \downarrow p_* \\ \pi((Z,Z_0),(X,X_0)) & \xrightarrow{f_*} & \pi((Z,Z_0),(B,B_0)) \end{array}$$

Now by universal property of pullbacks there exists a unique map

$\psi' : \pi_0[\pi((Z,Z_0),(X,X_0)) \sqcap \pi((Z,Z_0),(E,E_0))] \to \pi_0[\pi((Z,Z_0),(X \sqcap E, X_0 \sqcap E_0))]$

defined as follows:

Let $[(h',h)]$ belong to $\pi_0[\pi((Z,Z_0),(X,X_0)) \sqcap \pi((Z,Z_0),(E,E_0))]$, i.e. we have

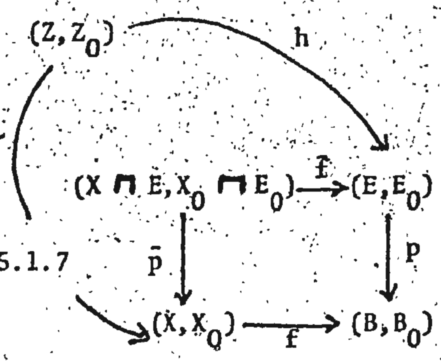


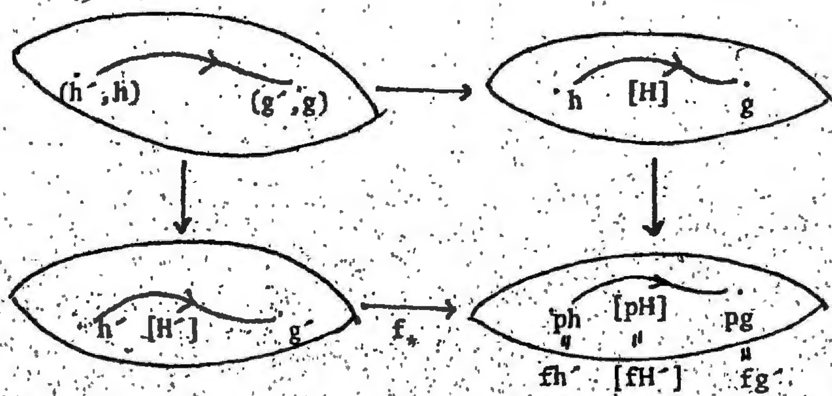
Diagram 5.1.7 $\qquad ph = fh'$

But 5.1.7 is a pullback diagram in Toppair, so there exists a unique map $Q : (Z,Z_0) \to (X \sqcap E, X_0 \sqcap E_0)$ such that $\bar{f}Q = h$, $\bar{p}Q = h'$. For $[Q]$ belonging to $\pi_0[\pi((Z,Z_0),(X \sqcap E, X_0 \sqcap E_0))]$ we define $\psi([(h,h')])$ to be $[Q]$.

Proposition 5.1.8.: ψ, as defined above, is a canonical bijection.

Proof: We prove (i) that ψ is well-defined (ii) that ψ is a bijection.

(i) Let (g',g) be a representative of $[(h',h)]$, i.e. $[(g',g)] = [(h',h)]$; and let $\psi([(h',h)]) = [Q]$, $\psi([(g',g)]) = [Q']$. Then we have the following picture

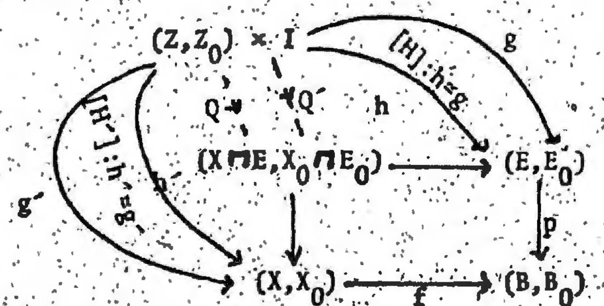


where $[H] : h \simeq g : (Z,Z_0) \times I \to (E,E_0)$ and

$[H'] : h' \simeq g' : (Z,Z_0) \times I \to (X,X_0)$ and

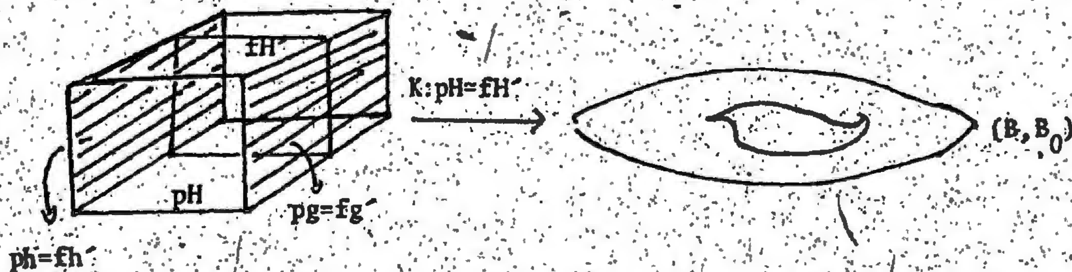
$p_*[H] = f_*[H']$. But this implies that $[pH] = [fH']$.

Thus we have the diagram

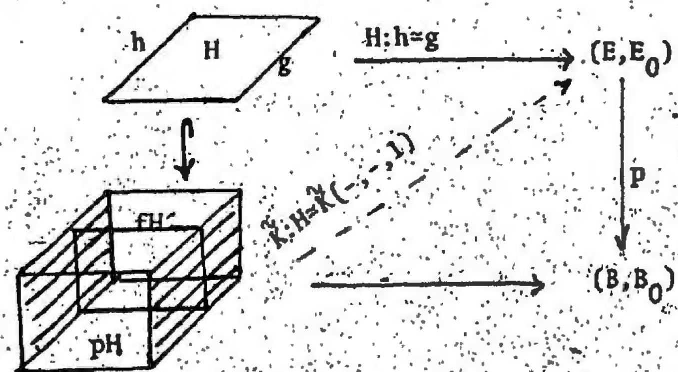


We need a morphism $[G] : Q \simeq Q'$ and this will imply that $[Q] = [Q']$.

Now $[pH] = [fH']$ means that there exists a homotopy $K : pH \simeq fH' : (Z,Z_0) \times I \times I \to (B,B_0)$ rel and maps, i.e. we have

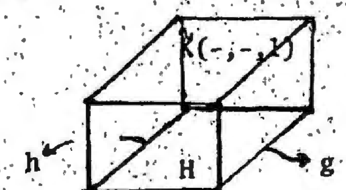


And we have



where H belongs to [H].

But p is a fibration in Toppair. Therefore there exists a lift $\tilde{K}$ of K such that $p\tilde{K} = K$ - diagramatically



Now we require a homotopy $\overline{H} : h \simeq g$ and a homotopy $\underline{H} : h' \simeq g'$ such that $p\overline{H} = f\underline{H}$; that is, we need homotopies such that the following diagram commutes:

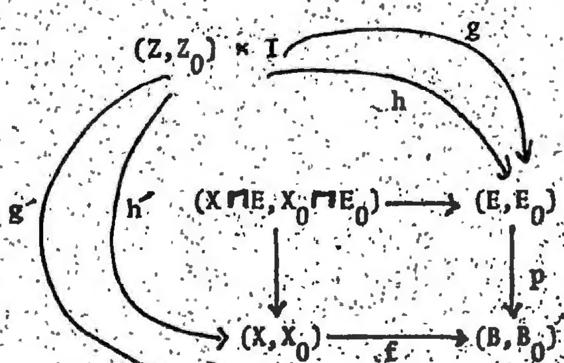
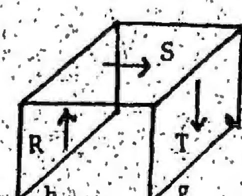


Diagram 5.1.9

Let $\overline{H}$ be the restriction of $\tilde{K}$ to



which is indeed a homotopy between h and g,

$(T + (S + R)) : h \simeq g.$

And let $\underline{H} = (0_{g'} + (H' + 0_g))$. This is a homotopy between h' and g' since H' is such a homotopy. Also we have that

$$f(0_{g'} + (H' + 0_g)) = 0_{fg'} + (fH' + 0_{fg}) = 0_{pg} + (fH' + 0_{ph})$$

$$= p(T + (S + R)) \text{ since } \tilde{K} \text{ is a lift of } K.$$

So we have homotopies which make Diagram 5.1.9 commute. Hence there exists a homotopy $G : (Z, Z_0) \times I \to (X \sqcap E, X_0 \sqcap E_0)$ making Diagram 5.1.9 commute. So the class of G, $[G]$, is a homotopy class between Q and Q' determined by the homotopies between h and g and h' and g'. Thus $[Q] = [Q']$.

Now if we have $[Q] = [Q']$ and $\psi([h',h)]) = [Q]$ and $\psi([(g',g)]) = [Q']$ we can easily show that there is a morphism between (h',h) and (g',g), i.e.

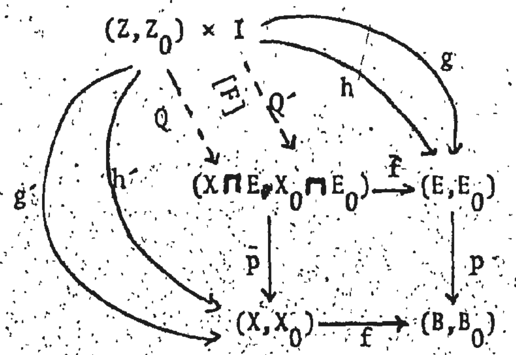
$[(h',h)] = [(g',g)]$.

Now Q and Q' are in the same class implies that there exists a homotopy class $[F] : Q \simeq Q' : (Z, Z_0) \times I \to (X \sqcap E, X_0 \sqcap E_0)$.

And $F \sim F'$ if and only if there exists

$L : F \simeq F' : (Z, Z_0) \times I \times I \to (X \sqcap E, X_0 \sqcap E_0)$ rel end maps and such that $L|Z_0 \times I \times I : F_0 \simeq F'_0 : Z_0 \times I \times I \to X_0 \sqcap E_0$ (restriction of L is a homotopy between the restriction of F and F').

So we have the following diagram in Toppair:



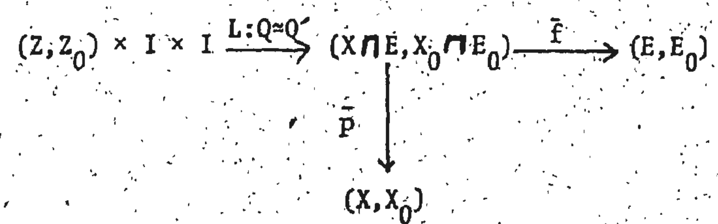
$\bar{f}Q = h$

$\bar{p}Q = h'$

$\bar{f}Q' = g$

$\bar{p}Q' = g'$

We have also the diagram



so that $\bar{f}L : \bar{f}Q \simeq \bar{f}Q' : (Z,Z_0) \times I \times I \to (E,E_0)$, i.e.

$\bar{f}L : h \simeq g$,

and $\bar{p}L : \bar{p}Q \simeq \bar{p}Q' : (Z,Z_0) \times I \times I \to (X,X_0)$, i.e.

$\bar{p}L : h' \simeq g'$,

and $p(\bar{f}L) = f(\bar{p}L)$ so that

$(\bar{f}L, \bar{p}K)$ joins (h,h') to (g,g') in

$\pi_0[\pi((Z,Z_0),(X,X_0)) \sqcap \pi((Z,Z_0),(E,E_0))]$,

i.e. ψ is well-defined.

For (ii) we see from above that ψ is injective. That ψ is surjective can be readily seen by taking

$[h]$ in $\pi_0(\pi((Z,Z_0),(X \sqcap E, X_0 \sqcap E_0)))$. Then there exists

$[(\bar{f}h, \bar{p}h)]$ in $\pi_0(\pi((Z,Z_0),(X,X_0)) \sqcap \pi((Z,Z_0),(E,E_0)))$

such that $\psi([(\bar{f}h,\bar{p}h)]) = [h]$.

Hence ψ is bijective.

//

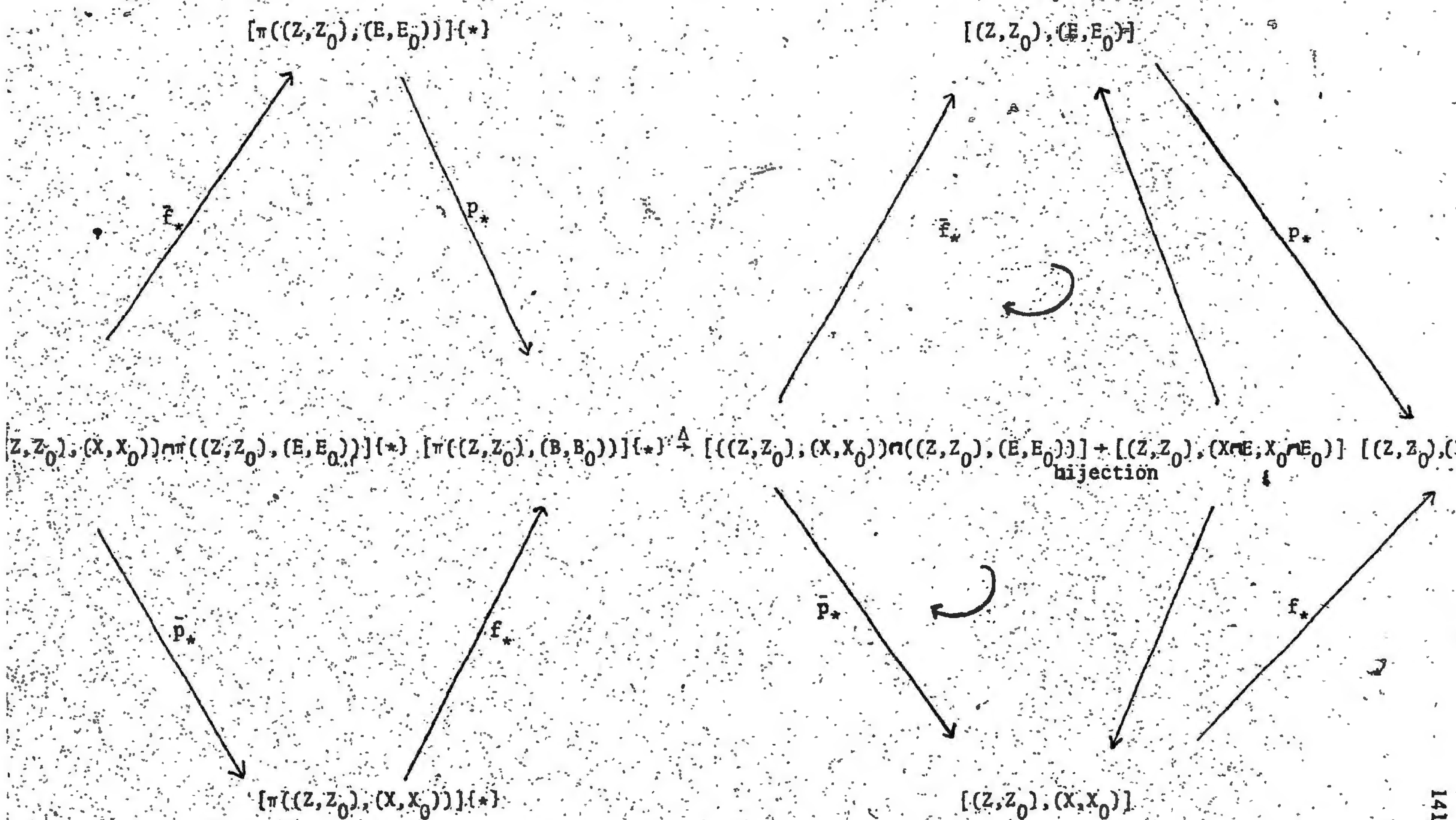
The following corollary to the above proposition ammends a stronger statement of BROWN in Proposition 4.1 of [3], the proof of which contains an error.

Corollary 5.1.10.: $\pi_0[\pi(Z, X \sqcap E)] \xrightarrow{\rho} \pi_0[\pi(Z,X) \sqcap \pi(Z,E)]$ is a canonical bijection.

Proof: This follows directly from the Proposition and the various Lemmas and Propositions that show $\pi(Z,X)$ as a special case of $\pi((Z,Z_0),(X,X_0))$.

//

We now give an example of the Theorem 5.1.3 where $\pi((Z,Z_0),(E,E_0))$, $\pi((Z,Z_0),(B,B_0))$ are groupoids and p_* is a fibration of groupoids. We recall that $\pi_0[((A,A_0),(B,B_0))]$ can be written simply as $[(A,A_0),(B,B_0)]$.


[π((Z,Z_0),(E,E_0))]{*}
[(Z,Z_0),(E,E_0)]
f̄_*
p_*
f̄_*
p_*
[(Z,Z_0),(X,X_0))⊓π((Z,Z_0),(E,E_0))]{*} [π((Z,Z_0),(B,B_0))]{*} Δ→ [((Z,Z_0),(X,X_0))⊓((Z,Z_0),(E,E_0))] → [(Z,Z_0),(X⊓E,X_0⊓E_0)] [(Z,Z_0),(B,B_0)]
bijection
p̄_*
f_*
p̄_*
f_*
[π((Z,Z_0),(X,X_0))]{*}
[(Z,Z_0),(X,X_0)]

We also have a Mayer-Vietoris type sequence arising from the fact that, if $p : E \to B$ is a fibration in Top and the diagram

$$\begin{array}{ccc} X \sqcap E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

is a pullback in Top, then we have the following diagrams of groupoids - the first, the induced diagram; the second, a pullback of groupoids

$$\begin{array}{ccc} \pi(Z,(X \sqcap E)) & \longrightarrow & \pi(Z,E) \\ \downarrow & & \downarrow p_* \\ \pi(Z,X) & \xrightarrow{f_*} & \pi(Z,B) \end{array} \qquad \begin{array}{ccc} \pi(Z,X) \sqcap \pi(Z,E) & \longrightarrow & \pi(Z,E) \\ \downarrow & & \downarrow p_* \\ \pi(Z,X) & \xrightarrow{f_*} & \pi(Z,B) \end{array}$$

where $\pi_0[\pi(Z,X \sqcap E)] \overset{\text{canonical bijection}}{=} \pi_0[\pi(Z,X) \sqcap \pi(Z,E)] = [Z, X \sqcap E]$.

Thus the M-V sequence is as follows:

$$\begin{array}{ccccc} & [\pi(Z,E)]\{*\} & & & \\ \nearrow & & \searrow & & \\ [\pi(Z,X) \sqcap \pi(Z,E)]\{*\} & & & & [\pi(Z,B)]\{*\} \\ \searrow & & \nearrow & & \\ & [\pi(Z,X)]\{*\} & & & \end{array} \quad \triangleq \quad \begin{array}{ccccc} & [Z,E] & & & \\ \nearrow & & \searrow & & \\ [Z, X \sqcap E] & & & & [Z,B] \\ \searrow & & \nearrow & & \\ & [Z,X] & & & \end{array}$$

§5.2 HOMOTOPY PULLBACKS IN TOPPAIR

Definition 5.2.1.: Consider the following square in Toppair

$$\begin{array}{ccc} (A,A_0) & \xrightarrow{f} & (B,B_0) \\ {\scriptstyle g}\downarrow & \overset{H}{\swarrow} & \downarrow{\scriptstyle h} \\ (C,C_0) & \xrightarrow[k]{} & (D,D_0) \end{array} \qquad \text{Diagram 5.2.2}$$

in which $H : hf \simeq kg : (A,A_0) \times I \to (D,D_0)$ (restrictions holding).

Now: h can be factored through the Mapping Track of pairs (Proposition 2.2.2.) as

$$(B,B_0) \overset{u}{\to} (B \sqcap D^I, B_0 \sqcap D_0{}^I) \overset{\theta}{\to} (D,D_0)$$

where u is a homotopy equivalence and θ is a fibration.

We now have the following pullback diagram in Toppair:

$$\begin{array}{ccc} (C \sqcap B \sqcap D^I, C_0 \sqcap B_0 \sqcap D_0{}^I) & \longrightarrow & (B \sqcap D^I, B_0 \sqcap D_0{}^I) \\ \downarrow & & \downarrow{\scriptstyle \theta} \\ (C,C_0) & \xrightarrow[K]{} & (D,D_0) \end{array} \qquad \text{Diagram 5.2.3.}$$

Denote $(C \sqcap B \sqcap D^I, C_0 \sqcap B_0 \sqcap D_0{}^I)$ by $(E_{h,k}, E_{0_{h,k}})$ and consider the following square:

$$\begin{array}{ccc} (E_{h,k}, E_{0_{h,k}}) & \xrightarrow{p} & (B,B_0) \\ {\scriptstyle q}\downarrow & \overset{G}{\swarrow} & \downarrow{\scriptstyle h} \\ (C,C_0) & \xrightarrow[k]{} & (D,D_0) \end{array} \qquad \text{Diagram 5.2.4.}$$

where $E_{h,k} = \{(c,b,\theta)$ in $C \times B \times D^I$ such that $k(c) = \theta(1)$, $h(b) = \theta(0)\}$ and p,q are the restrictions of the projections and $G((c,b,\theta),t)$ is defined as $\theta(t)$. There is also an associated map called the WHISKER MAP

$$w : (A,A_0) \to (E_{h,k}, E_{0_{h,k}})$$

defined by $w(a) = (g(a), f(a), H|a \times I)$. This map has the following properties:

(i) $p \circ w = f$ (ii) $q \circ w = g$ (iii) $G \circ w = H$

($G \circ w$ is actually $G \circ (w \times I)$).

If w is a homotopy equivalence in Toppair, we say that Square 5.2.2 is a HOMOTOPY PULLBACK. We call Square 5.2.4 the STANDARD HOMOTOPY PULLBACK in Toppair.

Note: The above definition can easily be seen to hold in Top.

Example 5.2.5.: Consider the following pullback in Toppair in which p is a fibration and H the static (constant) homotopy. We show that this is actually a homotopy pullback.

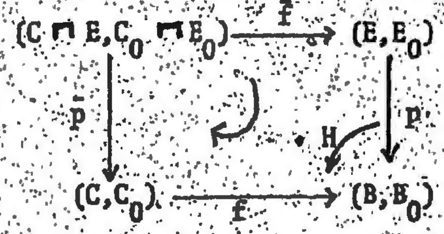


Diagram 5.2.6.

$H : p\bar{f} \simeq f\bar{p} : (C \sqcap E, C_0 \sqcap E_0) \times I \to (B,B_0)$.

Now the canonical factorization of p is as follows:

$$(E,E_0) \xrightarrow{u'} (E \sqcap B^I, E_0 \sqcap B_0^I) \xrightarrow{\theta'} (B,B_0)$$

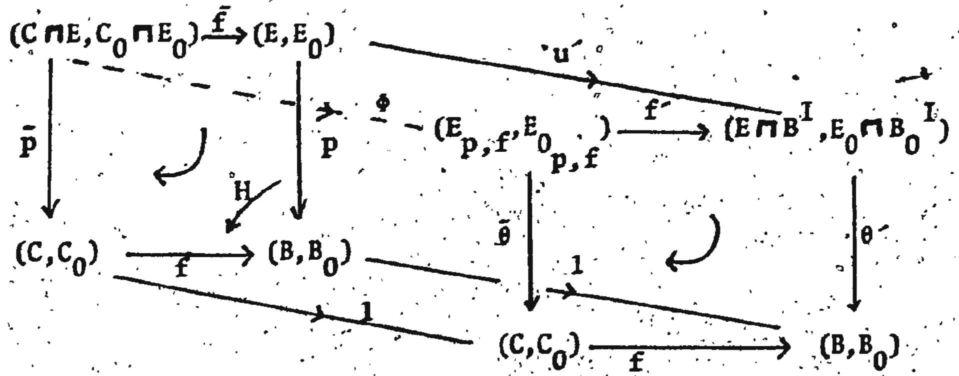
in which u' is a homotopy equivalence and θ' a fibration.

We have the following pullback in Toppair

$$\begin{array}{ccc} (E_{p,f}, E_{0_{p,f}}) & \xrightarrow{f'} & (E \sqcap B^I, E_0 \sqcap B_0^I) \\ \downarrow \tilde{\theta} & & \downarrow \theta' \\ (C,C_0) & \xrightarrow{f} & (B,B_0) \end{array}$$

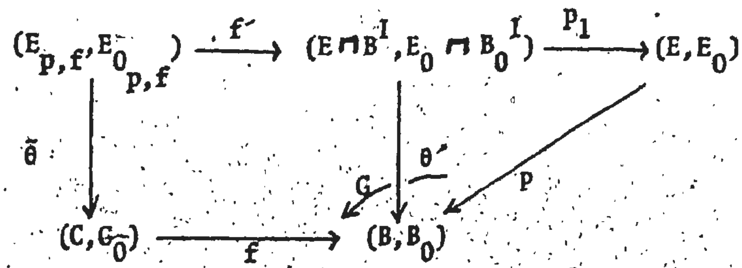
where f' and $\tilde{\theta}$ are projections.

Consider also the following diagram in Toppair:



Now p, θ' are fibrations, $u, 1$ are homotopy equivalences and the front and back squares are pullbacks in Toppair. Thus by Theorem 3.3.1, the induced map ϕ is a homotopy equivalence. We now show that ϕ has the properties of the whisker map in the definition of a homotopy pullback. We first write the standard homotopy pullback for this example

$$(c,e,\theta) \longmapsto (e,\theta) \longmapsto e$$



where $G : pp_1f' \simeq f\tilde{\theta} : (E_{p,f}, E_{0_{p,f}}) \times I \to (B,B_0)$.

Now $p_1f'\phi = p_1u'\bar{f}$ by commutativity of Diagram 5.2.7.

But we have the sequence

$$(E,E_0) \xrightarrow{u'} (E \sqcap B^I, E_0 \sqcap B_0^I) \xrightarrow{p_1} (E,E_0)$$

where $p_1u' = 1_{(E,E_0)}$. So $p_1f'\phi = p_1u'\bar{f} = 1\bar{f} = \bar{f}$.

Thus (i) of the Definition 5.2.1 holds.

Part (ii) of the definition holds by commutativity of Diagram 5.2.7.

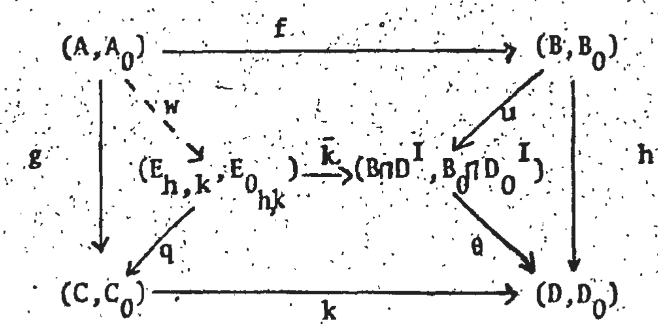
And for (iii) we have $G \circ \phi : pp_1f'\phi \simeq f\tilde{\theta}\phi : (C \sqcap E, C_0 \sqcap E_0) \times I \to (B,B_0)$.

But $pp_1f'\phi = pp_1u\bar{f} = p1\bar{f} = p\bar{f}$.

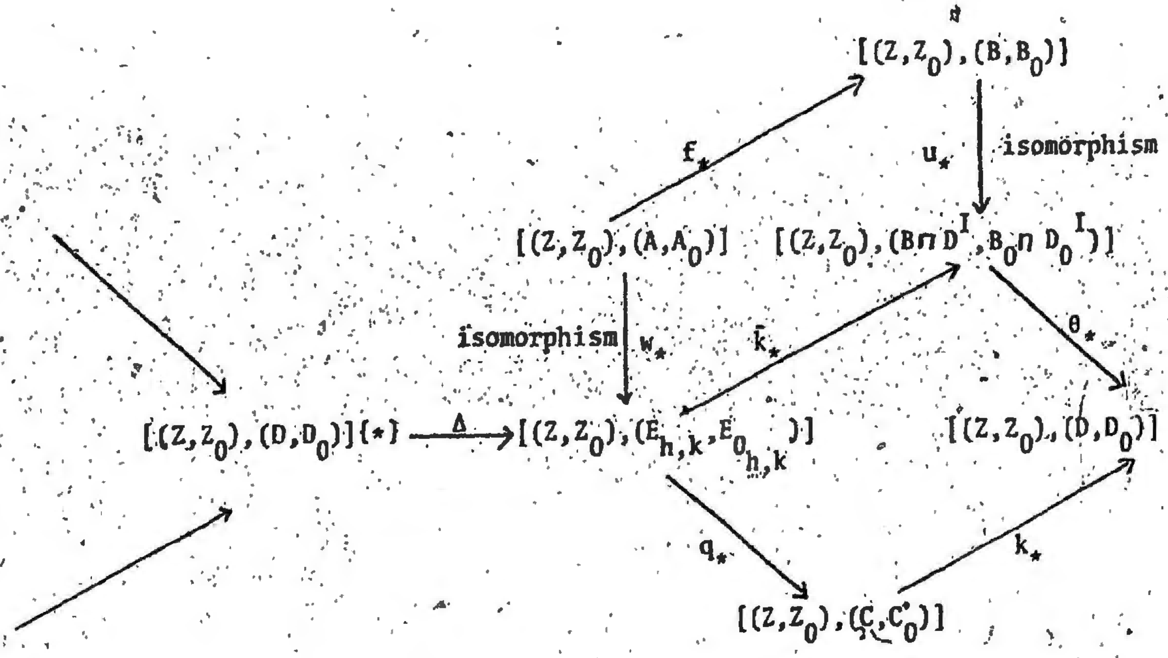
And $f\tilde{\theta}\phi = f\bar{p}$. Thus $G \circ \Phi = H$, and Φ is indeed the whisker map satisfying the necessary conditions to make Diagram 5.2.6 a homotopy pullback. We note that as a particular case the ordinary pullback in the category Top is also a homotopy pullback in Top.

We can form a Mayer-Vietoris sequence associated with a homotopy pullback as follows:

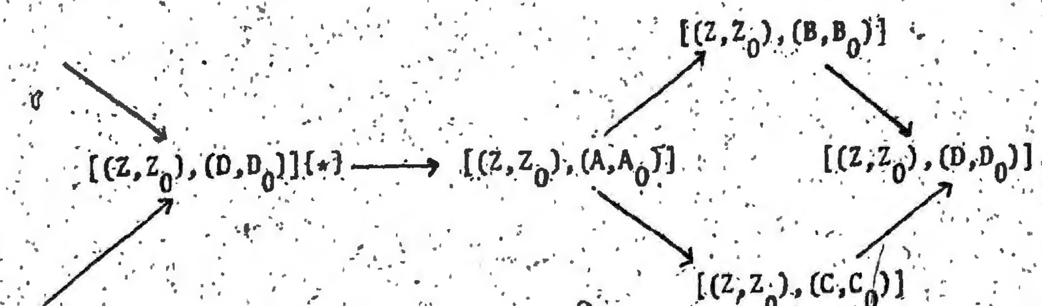
Consider Diagrams 5.2.2 and 5.2.3 and assume that 5.2.2 is a homotopy pullback. These diagrams can be written together as follows:



The front trapezoid is a pullback in Toppair and the back square is a homotopy pullback. It is easily seen that the two triangles commute, i.e. $qw = g$ and $\theta u = h$ and that the top trapezoid is homotopy commutative. Furthermore, we have that w and u are homotopy equivalences in Toppair and θ is a Toppair fibration. Hence for all (Z,Z_0) in Ob Toppair, we have the following induced diagram which commutes:



Thus for any homotopy pullback in Toppair we get the induced Mayer-Vietoris type diagram that follows:



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