QUASIBOUNDED MAPPINGS, FIXED POINT THEOREMS AND APPLICATIONS

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BY

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ABSTRACT

Chapter I contains some necessary preliminaries which may be found in most functional analysis texts. Also some fixed point theorems including those of Schauder [46], Furi & Vignoli [22], Swaminathan & Thompson [51], Nussbaum [36] and Petryshyn [37] are given in this chapter.

Chapter II deals with the study of quasibounded mappings and their fixed points. A systematic and up to date summary of known results is given in this chapter. Also some of the known results have been extended.

In Chapter III some applications of the fixed point theorems are illustrated by taking suitable examples.

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INTRODUCTION

Existence theorems in analysis first appeared in the ninteenth century and since then have received much attention. These theorems were considered for some time by mathematicians such as Cauchy, Picard, Birkoff and Kellogg. Then in 1922, S. Banach [2] formulated his classical theorem, commonly called the Banach Contraction Principle, which is based on a geometric interpretation of Picard's method of successive approximations, it reads as follows:

"A contraction mapping of a complete metric space into itself has a unqiue fixed point".

Because of its usefulness, the contraction mapping principle has motivated a great deal of research in the existence and uniqueness theorems of differential equations, partial differential equations, integral equations, random differential equations, etc.

In Chapter I, we have given a brief survey of the fixed point theorems proven for contraction, contractive and nonexpansive mappings in metric spaces. In the later portion of the chapter we have given some fixed point theorems for k-set-contraction, densifying and 1-set-contraction mappings. Some of these results are necessary in many proofs of the thesis.

In Chapter II, we have studied quasibounded mappings and their fixed points. This mapping was first introduced by Granas [24] and we have tried to give here a systematic and up to date summary of known results.

First section of this chapter is devoted to the existence of solutions of nonlinear equations while some results for p-quasibounded mappings, introduced by Cain & Nashed [8] are given in the second section. In the same section we have also obtained solutions of equations for p-quasibounded mappings which generalize the results due to Granas [24], Vignoli [52] and Nashed & Wong [34]. Some intersection theorems for quasibounded mappings have been given in the third section while in the fourth section we have added some further results for these mappings.

In Chapter III, we have given some selected applications of fixed point theorems established in the previous two chapters.

CHAPTER I

Preliminaries on Some Fixed Point Theorems

Our purpose in this chapter is to discuss some preliminary definitions and some of the well-known fixed point theorems in metric and linear spaces.

1.1. Metric Spaces

Definition 1.1.1. Let X be a set and d be a function from $X \times X$ into $\mathbb{R}^+$ such that for every $x,y,z \in X$ we have:

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0 \iff x = y$,

(iii) $d(x,y) = d(y,x)$ (symmetry),

(iv) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

Then d is called a metric (or distance function) for X, and the pair (X,d) is called a metric space.

When no confusion seems possible, we will refer to X as a metric space.

Definition 1.1.2. A sequence $\{x_n\}$ of points of a metric space X is said to converge to a point x_o if given $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that $d(x_n, x_o) < \varepsilon$ whenever $n \geq N(\varepsilon)$, or $\lim_{n\to\infty} d(x_n, x_o) = 0$. We denote this by $x_n \to x_o$.

It can be easily verified that if $x_n \to x_o$ and $x_n \to y_o$ then $x_o = y_o$, i.e., a convergent sequence has a unique limit in a metric space.

Definition 1.1.3. A sequence $\{x_n\}$ of points of a metric space X is said to be a Cauchy sequence, if for arbitrary $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for every $n,m \geq N(\varepsilon)$.

It follows directly from the triangle inequality that every convergent sequence is Cauchy.

Definition 1.1.4. A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.1.5. Let $T : X \to Y$ be a mapping of a metric space X into a metric space Y. Then T is said to be continuous at a point $x_o \in X$ if given any $\varepsilon > 0$ there exists $\delta > 0$ such that $d'(Tx, Tx_o) < \varepsilon$ whenever $d(x, x_o) < \delta$, where d and d' are metrics in X and Y respectively.

The mapping T is said to be continuous on X if it is continuous at every point $x \in X$.

Definition 1.1.6. Let A be a subset of a metric space X. Then A is said to be bounded if there exists a positive number M such that $d(x,y) \leq M$ for every $x,y \in A$.

If A is bounded, we define the diameter of A as

$$\text{diam. } (A) = \sup\{d(x,y) \mid x,y \in A\} .$$

If A is not bounded, we write

$$\text{diam. } (A) = \infty .$$

Definition 1.1.7. A subset A of a metric space X is said to be totally bounded if given $\varepsilon > 0$ there exists a finite number of subsets $A_1, A_2, \ldots, A_n$ of X such that diam. $(A_i) < \varepsilon$ $(i = 1,2, \ldots, n)$ and $A \subset \bigcup_{i=1}^{n} A_i$.

Clearly, if a subset A of a metric space X is totally bounded then it is bounded but the converse is not true. However, in $\mathcal{R}$, bounded and totally bounded sets are equivalent.

An important and useful property of totally bounded sets is the following. (e.g. see Goldberg [23]).

Theorem 1.1.8. A subset A of a metric space X is totally bounded if and only if every sequence of points of A contains a Cauchy subsequence.

Definition 1.1.9. A metric space X is said to be compact if every open covering of X has a finite subcovering.

Definition 1.1.10. Let T be a mapping of a set X into itself. A point $x \in X$ is called a fixed point of T if $Tx = x$, i.e., the fixed point is a point that remains invariant under a mapping.

Definition 1.1.11. A mapping T of a metric space X into itself is said to satisfy Lipschitz condition if there exists a real number k such that $d(Tx, Ty) \leq kd(x,y)$ for every $x,y \in X$.

In particular, if $0 \leq k < 1$, T is said to be a contraction mapping.

A contraction mapping is always continuous.

Now we give the well-known "Principle of Contraction Mappings" formulated by a famous Polish Mathematician S. Banach (1892-1945)[2] which is perhaps the most elementary and by far the most fruitful method for mapping theorems on existence and uniqueness of solutions of equations of various types.

Theorem 1.1.12. (Banach Contraction Principle): Every contraction mapping T defined on a complete metric space X into itself has a unique fixed point.

<u>Proof</u>. Let $x_o \in X$ be an arbitrary point and let

$$\left.\begin{aligned} x_1 &= Tx_o \\ x_2 &= Tx_1 = T^2x_o \\ &\cdots\cdots\cdots \\ x_n &= Tx_{n-1} = \ldots = T^n x_o \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (1)$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. From the definition of a contraction mapping, we have

$$d(Tx, Ty) \leq kd(x,y) \quad \text{for every } x,y \in X \text{ and } 0 \leq k < 1.$$

Therefore,
$$\begin{aligned} d(x_n, x_m) &= d(Tx_{n-1}, Tx_{m-1}) \\ &\leq kd(x_{n-1}, x_{m-1}) \\ &\leq k^2 d(x_{n-2}, x_{m-2}) \\ &\vdots \\ &\leq k^n d(x_o, x_{m-n}), \quad m > n \\ &\leq k^n [d(x_o, x_1) + d(x_1, x_2) + \ldots + d(x_{m-n-1}, x_{m-n})] \\ &\leq k^n d(x_o, x_1)[1 + k + k^2 + \ldots + k^{m-n-1}] \\ &\leq k^n d(x_o, x_1) \left(\frac{1}{1-k}\right) \\ &\longrightarrow 0 \text{ as } n \to \infty, \text{ since } 0 \leq k < 1. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, therefore $\lim_{n\to\infty} x_n$ exists.

We set $\lim_{n\to\infty} x_n = x$.

Then by the continuity of T we get,

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Thus the existence of a fixed point is proved.

We now show that this fixed point is unique. Let x and y be two distinct fixed points of T,

i.e., $Tx = x$ and $Ty = y$. $(x \neq y)$.

Then $d(Tx, Ty) = d(x,y)$.

But $d(Tx, Ty) \leq kd(x,y)$.

Hence $d(x,y) \leq kd(x,y)$.

i.e., $1 \leq k$, which is a contradiction to the fact that $k < 1$.

This contradiction implies that $x = y$.

This proves that the fixed point is unique.

Remark 1.1.13. Besides showing that an equation of the form $Tx = x$ has a unique solution, the above theorem also gives a practical method for finding the solution, i.e., calculation of the "successive approximations" (1). In fact, as shown in the proof, the approximations (1) actually converge to the solution of the equation $Tx = x$. For this reason, this fixed point theorem is often called the method of successive approximations.

Remark 1.1.14. Both conditions of the above theorem are necessary:

(a) The mapping $T : (0,1] \to (0,1]$ defined by $Tx = \frac{x}{2}$ is a contraction but has no fixed point. We note that the condition of completeness of the space is violated in this case.

(b) The mapping $T : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, defined by $Tx = x + 1$ is not a contraction and has no fixed point although $\mathbb{R}$ is a complete metric space.

The following two worth mentioning theorems have been given by Chu and Diaz [10].

Theorem 1.1.15. If T maps a complete metric space X into itself and if T^n (n is a positive integer) is a contraction mapping in X, then T has a unique fixed point.

Theorem 1.1.16. Let E be any nonempty set of elements and T be a map of E into itself. If for some positive integer n, T^n has a unique fixed point, then T also has a unique fixed point.

The above theorem has been improved, under different conditions, by Chu and Diaz [11] as follows:

Theorem 1.1.17. Let T be a mapping defined on a nonempty set E into itself, K be another function defined on X mapping it into itself such that $KK^{-1} = I$, where I is the identity function of X. Then T has a unique fixed point if and only if $K^{-1}TK$ has a unique fixed point.

The following is an immediate corollary to the above theorem:

Corollary 1.1.18. Let X be a complete metric space, $T : X \to X$ and $K : X \to X$ be such that $KK^{-1} = I$, the identity function. If $K^{-1}TK$ is a contraction in X, then T has a unique fixed point.

The proof of this corollary follows directly from the Theorem 1.1.17. and Banach's fixed point theorem.

Definition 1.1.19. A mapping T of a metric space X into itself is said to be contractive if $d(Tx, Ty) < d(x,y)$ for every $x,y \in X$, $x \neq y$.

Clearly a contractive map is continuous and if such a mapping has a fixed point, then this fixed point is unique. However, a contractive mapping of a complete metric space into itself need not have a fixed point, which can be seen from the following example:

Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = x + \frac{\pi}{2} - \arctan x$. Since $\arctan x < \frac{\pi}{2}$ for every x, the mapping T has no fixed point although it is a contractive map, for $T'x = 1 - \frac{1}{1 + x^2} < 1$.

The following theorem due to Edelstein [16] states the sufficient conditions for the existence of a fixed point for a contractive mapping.

<u>Theorem 1.1.20</u>. Let T be a contractive mapping of a metric space X into itself and $x_o \in X$ be such that the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to a point $z \in X$, then z is a unique fixed point of T.

A simple proof of the above theorem based on the same lines as due to Cheney & Goldstein [9] is given here.

<u>Proof</u>. Since T is contractive, hence continuous and we may write

$$\begin{aligned} d(T^n x_o, T^{n+1} x_o) &= d(T \cdot T^{n-1} x_o, T \cdot T^n x_o) \\ &< d(T^{n-1} x_o, T^n x_o) \\ &\quad \cdots\cdots\cdots\cdots \\ &< d(x_o, Tx_o). \end{aligned}$$

Thus, $\{d(T^n x_o, T^{n+1} x_o)\}$ is a decreasing sequence of real numbers bounded below by zero, and therefore has a limit.

Since $\{T^{n_i}(x_o)\}$ converges to $z \in X$, therefore, the sequence $\{T^{n_i+1}(x_o)\}$ converges to Tz and $\{T^{n_i+2}(x_o)\}$ converges to $T^2 z$.

Now if $z \neq Tz$, $d(z, Tz) = \lim_{k\to\infty} d(T^{n_k}(x_o), T^{n_k+1}(x_o))$

$$= \lim_{k\to\infty} d(T^{n_k}(x_o), T \cdot T^{n_k}(x_o))$$

By the continuity of T, we have

$$
\begin{aligned}
d(z, Tz) &= \lim_{k\to\infty} d(T^{n_k+1}(x_o), T^{n_k+2}(x_o)) \\
&= \lim_{k\to\infty} d(T \cdot T^{n_k}(x_o), T^2 \cdot T^{n_k}(x_o)) \\
&= d(Tz, T^2 z),
\end{aligned}
$$

a contradiction to the fact that T is contractive.

Hence $Tz = z$.

For uniqueness of z, let $y \neq z$ be another fixed point of T. Then $d(y,z) = d(Ty, Tz) < d(y,z)$, a contradiction.

Hence z is a unique fixed point of T.

The following corollary is due to Edelstein [16].

Corollary 1.1.21. If T is a contractive mapping of a metric space X into a compact metric space $Y \subset X$, then T has a unique fixed point.

Definition 1.1.22. A mapping T of a metric space X into itself is said to be nonexpansive if $d(Tx, Ty) \leq d(x,y)$ for every $x,y \in X$.

Cheney and Goldstein [9] proved the following theorem.

Theorem 1.1.23. Let T be a mapping of a metric space X into itself such that

(i) $d(Tx, Ty) \leq d(x,y)$

(ii) if $x \neq Tx$, then $d(Tx, T^2x) < d(x, Tx)$

(iii) for each x, the sequence $\{T^n(x)\}$ has a cluster point.

Then for each x, the sequence $\{T^n(x)\}$ converges to a fixed point of T.

K.L. Singh [49] proved the above theorem by relaxing conditions (ii) and (iii) and obtained a unique fixed point. We give this theorem below.

Theorem 1.1.24. Let T be a mapping of a compact metric space X into itself such that $d(Tx, Ty) \leq d(x,y)$, equality holds only when $x = y$. Then T has a unique fixed point.

Proof. The compactness of X and the condition $d(Tx, Ty) \leq d(x,y)$ imply that each x in X^T generates an isometric sequence (Edelstein [17], Theorem 1'). Therefore, by the definition of isometric sequence, $d(x, Tx) = d(Tx, T^2x)$; but from the given condition we have $d(Tx, T^2x) \leq d(x, Tx)$. This shows $d(x, Tx) = 0$, which implies $x = Tx$, i.e., x is a fixed point of T.

To prove the uniqueness, let us assume that y is another point such that $y \neq x$ and $y = Ty$. Then $d(Tx, Ty) = d(x,y)$ contradicting the condition $d(Tx, Ty) \leq d(x,y)$ unless $x = y$. Thus x is a unique fixed point.

1.2. Linear Spaces

Definition 1.2.1. Let X be a nonempty set, K a field (of real or complex numbers). A structure of vector space (or linear space) on X is defined by two maps:

(1) a map $(x,y) \mapsto x + y$ from $X \times X$ into X, called addition,

(2) a map $(\alpha,x) \mapsto \alpha x$ from $K \times X$ into X, called scalar multiplication.

These maps must satisfy the following axioms for every $x,y,z \in X$ and for every $\alpha,\beta \in K$:

(i) $(x + y) + z = x + (y + z)$ (commutativity)

(ii) $x + y = y + x$ (associativity)

(iii) There exists an element $0 \in X$, called zero element, such that $x + 0 = x$

(iv) For each $x \in X$ there exists $-x$, called opposite of x, such that $x + (-x) = 0$.

(v) $\alpha(x + y) = \alpha x + \alpha y$

(vi) $(\alpha + \beta)x = \alpha x + \beta x$

(vii) $\alpha(\beta x) = (\alpha\beta)x$

(viii) $1x = x$.

Remark 1.2.2. The elements of X are called 'points' or 'vectors' while the numbers α, β, ... are often called 'scalars'.

Definition 1.2.3. A set X is said to be a topological vector space if

(i) X is a vector space over field K

(ii) X is a topological space

(iii) the map $(x,y) \mapsto x + y$ from $X \times X$ into X is continuous

(iv) the map $(\alpha,x) \mapsto \alpha x$ from $K \times X$ into X is continuous.

Definition 1.2.4. Given a vector space X, a seminorm on X is a map $p : x \longmapsto p(x)$ from X into $\mathbb{R}$ which satisfies the following axioms

(i) $p(x) \geq 0$ for every $x \in X$.

(ii) $p(x + y) \leq p(x) + p(y)$ for every $x,y \in X$ (subadditivity).

(iii) $p(\alpha x) = |\alpha| p(x)$ for every $\alpha \in \mathbb{R}$ and for every $x \in X$.

Definition 1.2.5. A set K in a vector space X is convex if for every $x,y \in K$ and $0 \leq \alpha < 1$ we have $\alpha x + (1 - \alpha)y \in K$. In other words, K is convex if for $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ we have $\alpha x + \beta y \in K$ for every $x,y \in K$.

It can be easily seen that if K is convex in a vector space X then $K + x (x \in X)$ and αK are also convex.

Definition 1.2.6. A topological vector space X is said to be locally convex if every neighbourhood of 0 includes a convex neighbourhood of 0.

By this we mean a topological vector space in which every open set containing 0 contains a convex open set containing 0.

Remark 1.2.7. The notion of seminorm is of fundamental importance in discussing linear topological spaces. In fact, the seminorm of a vector in a linear space gives a kind of length for the vector. To introduce a topology in a linear space of infinite dimension suitable for application to classical and modern analysis, it is sometimes necessary to make use of a system of an infinite number of seminorms. It is one of the merits of the Bourbaki group that they stressed the importance, in functional analysis of locally convex topological vector spaces which are defined through a system of seminorms satisfying the axiom of separation. If the system reduces to a single seminorm, the corresponding linear space is called a normed linear space.

Remark 1.2.8. It can be seen that the topology of a locally convex topological vector space is given by a set of seminorms as follows:

Let U be a convex open set containing 0. Then $V = U \cap (-U)$ is also a convex open set containing 0. It is easy to see that for every $x \in X$ there exists an $a \in \mathbb{R}$ such that $x \in aV$. Moreover, $x \in aV \iff -x \in aV$.

Let $p(x) = \sup\{a \mid x \notin aV,\ a \geq 0\}$, if $x \neq 0$

$p(0) = 0$.

It is now a routine matter to verify that p is a seminorm and the sets

$U_{p,r} = \{x | p(x) < r\}$ for every p and for every $r > 0$

obtained in this way, form a base for the topology in X at 0.

Thus, in a locally convex topological vector space, the topology is given by a system p_i of seminorms. The requirement that for every $x \neq 0$ there is an open set $K \subset X$ such that $0 \in K$ and $x \notin K$ is translated into the requirement that for every $x \neq 0$ we have $p_i(x) \neq 0$.

Definition 1.2.9. A topological space X is said to be Hausdorff if for every two points $x,y (x \neq y)$ of X there exists neighbourhoods U and V respectively such that $U \cap V = \phi$.

Remark 1.2.10. A locally convex topological vector space with the topology described in Remark 1.2.8 is not in general Hausdorff.

Definition 1.2.11. In the Definition 1.2.4 if the condition (i) is replaced by

(i*) $p(x) \geq 0$ for every $x \in X$ where $p(x) = 0 \iff x = 0$ then p is called a norm on X.

Definition 1.2.12. A linear space X, equipped with the norm $p(x) = ||x||$, is called a normed linear space. In this case we have

(i^o) $|| x || \geq 0$ for every $x \in X$ where $||x|| = 0 \iff x = 0$

(ii^o) $||x + y|| \leq ||x|| + ||y||$ for every $x,y \in X$ (triangle inequality)

(iii^o) $||\alpha x|| = |\alpha| \, ||x||$ for every $x \in X$ and for every α .

It can be easily seen that every normed linear space X becomes a metric space if we set $d(x,y) = ||x - y||$.

Definition 1.2.13. A complete normed linear space is called a Banach space.

Definition 1.2.14. A mapping f of a vector space X into $\mathbb{R}$ is called a linear functional on X if

(i) $f(x + y) = f(x) + f(y) \qquad (x,y \in X)$

(ii) $f(\alpha x) = \alpha f(x) \qquad (x \in X, \ \alpha \in \mathbb{R})$

Definition 1.2.15. A functional f is said to be continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $||x_1 - x_2|| < \delta$.

Continuity and boundedness are equivalent.

Definition 1.2.16. A subset K of a normed space X is said to be bounded if there exists a constant M such that $||x|| \leq M (x \in K)$.

Definition 1.2.17. A linear operator f mapping a Banach space X into itself is said to be completely continuous if

(i) f is continuous, and

(ii) it maps every bounded set into a relatively compact set.

Remark 1.2.18. If X is finite-dimensional then every linear operator is completely continuous, while in an infinite-dimensional space, complete continuity of an operator is a stronger requirement than merely being continuous (i.e., bounded).

We now state the celebrated fixed point theorem of Brouwer the proof of which may be found in Dunford & Schwartz [15].

Theorem 1.2.19. (Brouwer's Fixed Point Theorem). A continuous map of a ball in E^n into itself has at least one fixed point.

Remark 1.2.20. The Brouwer fixed point theorem in the form stated above does not hold in infinite dimensional spaces as the following example shows:

Consider the space ℓ^2 of sequences $x = (x_1, x_2, \ldots)$ with $\Sigma|x_i|^2 < \infty$. Define T as a map of the closed solid sphere into itself as follows: For $x = (x_1, x_2, \ldots .)$ let $Tx = (\sqrt{1 - |x|^2}, x_1, x_2, \ldots)$.

$$|Tx|^2 = 1.$$

Suppose x is a fixed point. Then $|x| = |Tx| = 1$. But then $x_1 = 0$ and one sees in turn that $x_2 = 0, x_3 = 0, \ldots,$ and hence $x = 0$. Therefore, T has no fixed point. This is due to S. Kakutani [35].

Schauder [46] extended Brouwer's theorem to infinite-dimensional spaces in the following way:

Theorem 1.2.21. (Schauder's Fixed Point Theorem - 1st. form). A continuous map of a compact convex set K in a normed linear space X into itself has at least one fixed point.

Theorem 1.2.22. (Schauder's Fixed Point Theorem - 2nd. form). A completely continuous map of closed convex set K in a complete normed linear space X into itself has at least one fixed point.

The proofs of the above two theorems may be found in Nirenberg [35].

It has been shown by Tychonoff that the 1st. form of Schauder's fixed point theorem holds if X is a locally convex topological vector space.

Theorem 1.2.23. (Schauder-Tychonoff Fixed Point Theorem). Let K be a non-empty compact convex subset of a Hausdorff locally convex topological vector space X, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.

The proof of the above theorem may be found in Bonsall [3].

Definition 1.2.24. A Banach space X is called uniformly convex if for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$ then $\left|\left|\frac{x + y}{2}\right|\right| \leq 1 - \delta$.

Definition 1.2.25. A Banach space X is called strictly convex if for any $x,y \in X$, $||x + y|| = ||x|| + ||y|| \Rightarrow x = \lambda y$, $\lambda > 0$.

Remark 1.2.26. Every uniformly convex Banach space is strictly convex. But the converse is not true.

Definition 1.2.27. Let X be a Banach space and X^* denote its first dual space. For any fixed vector $x \in X$, the mapping of X^* into $\mathbb{R}$ which assigns to every $u \in X^*$ the value (u,x) of u at x is a linear continuous functional in the space X^*, i.e., an element of X^{**}. Moreover the norm of this functional is equal to $||x||$. Also the canonical mapping of X into X^* defined by this correspondence between elements of X and linear continuous functional on X^* is linear and one to one. Therefore, it is an isometrical imbedding of X into X^{**}.

Now, a Banach space is called reflexive if $X = X^{**}$, i.e., the canonical mapping of X into X^{**} is onto.

Remark 1.2.28. Every uniformly convex Banach space is reflexive.

Definition 1.2.29. A point $x \in K \subset X$ is a diametral point of K if

$$\delta(K) = \sup\{||x - y|| \; | y \in K\}$$

where $\delta(K)$ denotes the diameter of K.

Definition 1.2.30. A convex set $K \subset X$ is said to have a normal structure if for each bounded convex subset H of K which contains more than one point, there is some point $x \in H$ which is not a diametral point of H.

Remark 1.2.31. Every uniformly convex space X has a normal structure.

Definition 1.2.32. Let X be a vector space over K (real or complex). A mapping of $X \times X$ into K which takes ordered pair $\{x,y\} \in X \times X$ into the number $(x,y) \in K$ is called an inner product in X if

(i) $(x,y) = \overline{(y,x)}$

(ii) $(x + y, z) = (x,z) + (y,z)$

(iii) $(\alpha x,y) = \alpha(x,y)$

(iv) $(x,x) > 0$ if $x \neq 0$.

A vector space X, together with an inner product in X, is called an inner product space or pre-Hilbert space.

Definition 1.2.33. A Hilbert space is a pre-Hilbert space which is complete w.r.t. the norm derived from the inner product. In this case the norm and the inner product are related by $||x|| = (x,x)^{\frac{1}{2}}$.

Remark 1.2.34. Every Hilbert space is relfexive.

We now state without proof the following fixed point theorem due to Browder [5]:

Theorem 1.2.35. Let X be a uniformly convex Banach space, T a non-expansive mapping of the bounded closed convex subset K of X to itself. Then T has a fixed point in K.

Remark 1.2.36. (i) If K is compact or T is completely continuous, it becomes a particular case of the Schauder fixed point theorem.

(ii) If T is contraction, then the result follows from the Banach contraction principle.

(iii) The following example shows that the result cannot be extended to the general Banach spaces.

Let $X = C_o$, the space of sequences converging to 0, C the unit ball in the maximum norm, e_1 the unit vector given by $e_1 = (1,0,0,0, \ldots)$, $s(x) = (0,x_1,x_2, \ldots)$.

Then the mapping $Tx = e_1 + s(x)$ maps C into itself, is non-expansive, and has no fixed point in C.

Kirk [27] gave the following generalization of the above theorem:

Theorem 1.2.37. Let X be a reflexive Banach space and K a nonempty bounded closed convex subset of X. Furthermore, suppose that K has normal structure. Then a non-expansive mapping T of K into itself has a fixed point.

In the following examples it has been shown that the restrictions on K are necessary.

Example 1. (Boundedness of K). A translation in a Banach space is an isometry and obviously has no fixed points.

Example 2. (Closedness of K). Let $X = \mathcal{R}$ be a Hilbert space. Let C be the interior of the unit ball, i.e., $C = \{x \mid ||x|| < 1\}$. Consider T the mapping of C into itself defined by

$$Tx = \frac{x}{2} + \frac{a}{2}$$

where $a \in \mathcal{R}$ is a vector of unit norm. In this case T has no fixed point in C.

Example 3. (Convexity of K). Let $X = \mathcal{R}$ be a Hilbert space. Let C be a set containing just two distinct points a and b. Define $T : C \to C$ as $Ta = b$ and $Tb = a$. Clearly T is an isometry and has no fixed point.

The following example indicates that one cannot expect existence of fixed points for non-expansive mappings in the most general class of Banach spaces.

Example 4. Let C[0,1] be a Banach space with

$$||f|| = \max_{x \in [0,1]} |f(x)|$$

It is known that C[0,1] is not a reflexive Banach space.

Let $C = \{f \in C[0,1] \mid f(0) = 0,\ f(1) = 1,\ 0 \leq f(x) \leq 1\}$. Then C is bounded, closed and convex.

Let T be a mapping defined as follows:

$$T : C \to C$$

$$f(x) \to xf(x), \text{ i.e. } Tf(x) = xf(x)$$

and T is non-expansive.

It is easy to verify that $T(C) \subset C$ and T has no fixed point.

Example 5. (Normal structure of K). The mapping $T : C_o \to C_o$ defined by

$$T : (c_1, c_2, \ldots.) \to (1, c_1, c_2, \ldots.)$$

maps the unit ball c_o into itself but does not have any fixed point, since $(c_1, c_2, \ldots.) = (1, c_1, \ldots)$ would simply mean that $c_1 = c_2, \ldots, = 1$ and this is impossible.

Browder & Petryshyn [7] and Kachurovskii [25] independently proved the following fixed point theorem in Hilbert spaces:

Theorem 1.2.38. Let K be a closed bounded convex subset of a Hilbert space X and $T : K \to K$ a non-expansive mapping. Then T has at least one fixed point in K.

Remark 1.2.39. It may be noted that the proofs of Theorem 1.2.35 and Theorem 1.2.37 are based on a transfinite argument due to Brodsky & Milman [4] while in the case of Hilbert spaces, i.e., in Theorem 1.2.38 the proof is given by using a connection with monotone operators.

Definition 1.2.40. Let X be a metric space and A be a bounded subset of X. Then we define measure of non-compactness of A, denoted by $\alpha(A)$, as

$$\alpha(A) = \{\varepsilon > 0 \mid A \text{ can be covered by a finite number of subsets of diameter} < \varepsilon\}.$$

The above concept was introduced by Kuratowski [32]. This measure of non-compactness α satisfies the following properties:

(i) $0 \leq \alpha(A) \leq \delta(A)$, where δ is the diameter of A.

(ii) $\alpha(A) = 0 \iff A$ is totally bounded.

(iii) $\alpha(\lambda A) = |\lambda|\alpha(A)$, where λ is a real number.

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(v) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(vi) If $A \subset B$, then $\alpha(A) \leq \alpha(B)$.

(vii) If $\bar{A}$ is the closure of A, then $\alpha(\bar{A}) = \alpha(A)$.

(viii) $\alpha(N_r(A)) \leq \alpha(A) + 2r$, where $N_r(A) = \{x \in X | d(x,A) < r\}$ is a neighbourhood of A.

(ix) $\alpha(\overline{co}(A)) = \alpha(A)$, where $\overline{co}(A)$ denotes the convex closure of A.

(x) $\alpha(B) = \alpha(S) = 2$, where $B = \{x \in X| \ ||x|| \leq 1\}$ and $S = \{x \in X| \ ||x|| = 1\}$ in an infinite-dimensional Banach space X.

These properties are discussed in details by Darbo [13], Nussbaum [36] and Sadovskii [45]. Closely associated with the measure of non-compactness is k-set-contraction mapping introduced by Darbo [13].

<u>Definition 1.2.41</u>. Let A be a subset of a metric space X and $T : A \to X$ be continuous. Then T is said to be a k-set-contraction mapping if given any bounded subset D of A we have

$$\alpha(T(D)) \leq k\alpha(D)$$

for some $k \geq 0$.

It may be noted that every contraction mapping is k-set-contraction with $k < 1$.

The following result is due to Darbo [13].

Theorem 1.2.42. Let K be a non-empty closed convex bounded subset of a Banach space X and $T : K \to K$ be a k-set-contraction mapping with $k < 1$. Then T has a fixed point in K.

A slightly extended result of the above theorem was given by Nussbaum [36] as follows:

Theorem 1.2.43. Let X and K be as in Theorem 1.2.42 and $T : K \to K$ be continuous. Further let (i) $K_1 = \overline{co}T(K)$, (ii) $K_n = \overline{co}T(K_{n-1})$, $n > 1$ and (iii) $\alpha(K_n) \to 0$ as $n \to \infty$. Then T has a fixed point.

Definition 1.2.44. In the definition of k-set-contraction if $k = 1$, i.e., if $\alpha(T(D)) \leq \alpha(D)$ for every bounded subset D of A, then we call this mapping to be 1-set-contraction.

Remark 1.2.45. Clearly, a nonexpansive mapping is 1-set-contraction, but the converse is not true. The difference between these two types of mappings may well be illustrated by comparing Theorem 1.2.38 with the following conjecture.

Conjecture: Let K be a non-empty closed convex bounded subset of a Hilbert space X and $T : K \to K$ be a 1-set-contraction mapping. Then T has no fixed point.

To see this let us consider the following example:

Consider the ℓ^2-space and a mapping $f : c \to c$, where c is the unit ball in ℓ^2-space, be defined by

$$f(x) = (\sqrt{1 - ||x||^2}, x_1, x_2, \dots x_n, \dots.)$$

where $x = (x_1, x_2, \dots.)$ and $||x|| = (\Sigma|x_i|^2)^{\frac{1}{2}}$.

Clearly, f is continuous. Suppose now that f has a fixed point. Then,

$$f(x) = x = (x_1, x_2, \ldots) = (\sqrt{1 - ||x||^2}, x_1, x_2, \ldots).$$

Let us take the elementary case,

i.e., $f(0) = 0 = (0,0, \ldots) = (\sqrt{1 - ||0||^2}, 0, 0, \ldots)$

i.e., $(0,0, \ldots) = (1,0,0, \ldots)$, which is impossible.

Hence f has no fixed point.

The following useful theorem was given by Nussbaum [36]:

Theorem 1.2.46. Let $B = \{x \in X | \; ||x|| \leq 1\}$ be a ball in a Banach space X and let $R : X \to B$ be a radial projection (also called, radial retraction), i.e.,

$$Rx = \begin{cases} \dfrac{x}{||x||} & \text{for } ||x|| \geq 1 \\ x & \text{for } ||x|| \leq 1. \end{cases}$$

then R is a 1-set-contraction.

Definition 1.2.47. Let $T : X \to X$ be a continuous mapping of a metric space X into itself. If for any bounded set $A \subset X$ with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$ then the mapping T is said to be densifying.

This definition was introduced by Furi & Vignoli [21]. Sadovskii [45] called this as condensing map.

It may be noted that contraction mappings and completely continuous mappings are densifying. Also sums of contraction and completely continuous mappings defined on Banach spaces are densifying.

The following theorem was given by Furi & Vignoli [22].

Theorem 1.2.48. Let $T : K \to K$ be a densifying mapping from a non-empty closed convex bounded subset of a Banach space X into itself. Then T has at least one fixed point.

The above important fixed point principle for densifying operators was generalized to topological vector spaces by Swaminathan & Thompson [51] as follows:

Theorem 1.2.49. Let K be a complete, convex bounded subset of a locally convex topological vector space X and $T : K \to K$ be densifying. Then T has a fixed point in K.

The following theorem was given by Petryshyn [37].

Theorem 1.2.50. Let B be an open ball about the origin in a Banach space X. If $T : \bar{B} \to X$ is a densifying mapping which satisfies the boundary condition

(*) If $Tx = \alpha x$ for some x in ∂B, then $\alpha \leq 1$,

then T has a fixed point.

Proof. We define first a radial retraction mapping $R : X \to \bar{B}$, by

$$Rx = \begin{cases} x & \text{if } ||x|| \leq r \\ \dfrac{rx}{||x||} & \text{if } ||x|| \geq r. \end{cases}$$

Then by Theorem 1.2.46, R is a 1-set-contraction.

We now define a mapping T_1 on $\bar{B}$ by $T_1(x) = RT(x)$, for every $x \in \bar{B}$. Then T_1 maps $\bar{B}$ into itself which is also densifying, for $\alpha(T_1\bar{B}) = \alpha(RT\bar{B}) \leq \alpha(T\bar{B}) < \alpha(\bar{B})$. Therefore by Theorem 1.2.48, T_1 has a fixed point in $\bar{B}$, say x_o, i.e., $T_1 x_o = x_o$. We claim that $Tx_o = x_o$. Indeed, if $x_o \in B$, then $Rx_o = x_o$ and $T_1 x_o = RTx_o = x_o$, therefore $Tx_o = x_o$. And, if $x_o \in \partial B$, then

$Rx_o = \dfrac{rx_o}{||x_o||}$ and $T_1 x_o = RTx_o = x_o$; therefore, $r\dfrac{Tx_o}{||Tx_o||} = x_o$, i.e.,

$Tx_o = x_o \dfrac{||Tx_o||}{r}$, i.e., $\alpha x_o = x_o \dfrac{||Tx_o||}{r}$, i.e., $\alpha = \dfrac{||Tx_o||}{r} > 1$,

which is a contradiction to $(*)$. Hence the proof.

The following three corollaries were given by Petryshyn [37].

Corollary 1.2.51. Suppose $T : \bar{B} \to X$ is densifying such that

(i) $T(\bar{B}) \subset \bar{B}$, or

(ii) $T(\partial B) \subset \bar{B}$, or

(iii) $||Tx - x||^2 \geq ||Tx||^2 - ||x||^2$ for all $x \in \partial B$, or

(iv) $(Tx, Jx) \leq (x, Jx)$ for all $x \in \partial B$, where J is a duality mapping of X into its dual X^* (or rather into the set 2^{X^*} of all subsets of X^*) such that

$$(Jx, x) = ||x||^2 \text{ and } ||Jx|| = ||x|| \text{ for all } x \in X.$$

Then T has a fixed point.

Corollary 1.2.52. Let $T : \bar{B} \to H$ be any mapping and $T_o : \bar{B} \to H$ be densifying (H is the Hilbert space), such that

(i) $(Tx, x) \leq ||x||^2$

(ii) $||Tx - T_o x|| \leq ||x - Tx||$ for all $x \in \partial B$.

Then T has a fixed point.

Corollary 1.2.53. Let $T = S + C$ be a map from $\bar{B}$ to X such that S is contraction on $\bar{B}$ and C is compact on $\bar{B}$. Suppose also that T satisfies condition (*) of Theorem 1.2.50 on ∂B. Then T has a fixed point.

The following theorem was also given by Petryshyn [37].

Theorem 1.2.54. Let C be a bounded open subset of a Banach space X and $T : \bar{C} \to X$ be a 1-set-contraction mapping satisfying either of the following two conditions:

(a) there exists an $x_o \in C$ such that if $Tx - x_o = \alpha(x - x_o)$ holds for some $x \in \partial C$, then $\alpha \leq 1$.

(b) C is convex and $T(\partial C) \subseteq \bar{C}$.

Then T has a fixed point if $(I - T)\bar{C}$ is closed.

Proof. Define $Q = C - x_o = \{x - x_o | x \in C\}$.

Then it follows that Q is bounded, open, $0 \in Q$, $\partial Q = \partial C - x_o$ and $\bar{Q} = \bar{C} - x_o$. Furthermore, Q is convex if C is convex.

Now define the map $T'(y)$ for y in $\bar{Q}$ and $y = \{x - x_o | x \in \bar{C}\}$ by $T'(y) = Tx - x_o$. Then T' maps $\bar{Q}$ into X and T' is 1-set-contraction and T' satisfies condition (*) of Theorem 1.2.50 on ∂Q. Furthermore, $(I - T')\bar{Q}$ is closed since $(I - T')\bar{Q} = (I - T)\bar{C}$. Thus T' and Q satisfy all the conditions of Theorem 7 of Petryshyn [37]. Hence there exists a y in $\bar{Q}$ such that $T'(y) = y$, i.e., $Tx - x_o = x - x_o$ with $x \in \bar{C}$ or $Tx = x$.

Next we show that (b) implies (a). Suppose (b) is given and let

$x_o \in C$ be any fixed element. Then $Q = C - x_o$ is convex, $0 \in Q$, and $T'(\partial Q) \subseteq \bar{Q}$ since $T'(\partial Q) = T(\partial C) - x_o \subseteq \bar{C} - x_o = \bar{Q}$ and C is convex. Hence T' satisfies boundary condition on ∂Q, i.e., condition (a) of this theorem is satisfied.

Hence the proof.

CHAPTER II

Some Fixed Point Theorems for Quasibounded Mappings

2.1. Quasibounded Mappings & Fixed Point Theorems

Definition 2.1.1. Let X be a Banach space and $T : X \to X$ be continuous. If $|T| = \lim_{\rho\to\infty} \{\sup_{||x||\geq\rho} \frac{||T(x)||}{||x||}\}$ is finite then T is called quasibounded and $|T|$ is called quasinorm of T.

Example 2.1.2. Any bounded linear mapping is quasibounded and its norm coincides with its quasinorm.

Remark 2.1.3. The notion of quasibounded mappings was first introduced by Granas [24]. The same mapping was termed as linearly upper bounded by Kolomy [28] & Srinivasacharyulu [50]. It is easy to see that T is quasibounded if and only if there exist $\alpha, \beta > 0$ such that $||Tx|| \leq \beta||x||$ for $||x|| \geq \alpha$.

The following known result is due to Granas [24].

Theorem 2.1.4. Let $T : X \to X$ be a quasibounded completely continuous mapping of a Banach space X into itself. If $|T| < 1$ then the equation $y = x - Tx$ has a solution for every $y \in X$.

Proof. Let $y^* \in X$ be arbitrary. We define a mapping $\bar{T}$ by

$$\bar{T}x = y^* + Tx \quad \text{for every} \quad x \in X.$$

Clearly $\bar{T}$ is completely continuous.

Since $|T| < 1$, therefore it follows that $\frac{||Tx||}{||x||} < \delta < 1$ holds for every x with $||x|| \geq r_1$, where δ and r_1 are some constants.

Let $\varepsilon > 0$ be such that $\varepsilon + \delta < 1$ and let $r_2 = \frac{||y^*||}{\varepsilon}$.

Now, for every x with $||x|| \geq r_2$ we have $\frac{||y^*||}{||x||} \leq \varepsilon$.

Let $r = r_1 + r_2$, $K_r = \{x \in X| \; ||x|| \leq r\}$ and $S_r = \{x \in X| \; ||x|| = r\}$. Also, let $x \in S_r$. Then $||x|| \geq r_1$, $||x|| \geq r_2$ and hence

$$\frac{||\bar{T}x||}{||x||} \leq \frac{||y^*||}{||x||} + \frac{||Tx||}{||x||} \leq \varepsilon + \delta < 1.$$

It then follows that $\bar{T}(S_r) \subset K_r$.

Now by a fixed point theorem of Rothe [44], $\bar{T}$ has a fixed point in K_r, say x^*. Therefore, $\bar{T}x^* = x^* = y^* + Tx^*$, i.e., $y^* = x^* - Tx^*$. The theorem is now proved.

The following corollary is due to Granas [24].

Corollary 2.1.5. Let $T : X \to X$ be a quasibounded completely continuous mapping. If $||Tx|| = 0(||x||)$ (as $||x|| \to +\infty$) then the equation $y = x - \lambda Tx$ with the real parameter λ has a solution for every $y \in X$ and for every λ .

Proof. Clearly, for every λ, the mapping λT is completely continuous and quasibounded; also the quasinorm $|\lambda T|$ is equal to 0. Hence the corollary follows from the last theorem.

The following theorem due to Nashed and Wong [34], may be treated as a perturbation theorem where completely continuous quasibounded mappings are perturbed by contraction mappings.

Theorem 2.1.6. Let $S : X \to X$ be a contraction mapping and $T : X \to X$ be completely continuous and quasibounded. If $|T| < 1 - \nu$, where ν is the contraction constant, then the equation $y = x - Sx - Tx$ has a solution for every $y \in X$.

<u>Proof</u>. For any fixed element $z \in X$, we define an operator $\bar{S}$ by

$$\bar{S}x = Sx + Tz + y^* \quad \ldots\ldots\ldots\ldots (1)$$

where $y^* \in X$ is arbitrary. A simple computation shows that $\bar{S}$ is again a contraction mapping. So we may define a mapping G which associates to each $z \in X$ the unique fixed point of $\bar{S}$.

In other words, from (1), we have,

$$Gz = \bar{S}Gz = SGz + Tz + y^* \quad \ldots\ldots\ldots\ldots (2)$$

Now for any $u,v \in X$, we obtain from (2), the following estimate:

$$||Gu - Gv|| \leq \frac{1}{1 - \nu} ||Tu - Tv|| \quad \ldots\ldots\ldots\ldots (3)$$

It clearly follows from (3) that G is completely continuous.

In order to establish that G has a fixed point, we need to show that G maps a certain closed ball into itself. Denote by $S_n(y) = \{x \in X | \; ||x - y|| \leq n\}$ where n is a positive integer. We claim that there exists a positive integer $N > 0$ such that $G(S_N(y)) \subseteq S_N(y)$. Assume the contrary, then there must exist for each $n > 0$, $u_n \in S_n(y)$ such that $||Gu_n - y|| > n$. Since G is completely continuous, we must then have $||u_n|| \to \infty$ as $n \to \infty$. From (2), we may estimate $||Gu_n - y||$ as follows:

$$||Gu_n - y|| \leq ||SGu_n - Sy|| + ||Sy|| + ||Tu_n|| \quad \ldots\ldots (4)$$

For each $\varepsilon > 0$, choose n_o such that for $n \geq n_o$, we have

$$||Tu_n|| \leq (|T| + \tfrac{\varepsilon}{3})||u_n|| , \quad ||Sy|| < \varepsilon||u_n||/3$$

$$\text{and} \quad ||Gu_n - y|| > (1 - \frac{\varepsilon}{3(1 - \nu)})||u_n|| ,$$

which is possible by the choice of $\{u_n\}$. Dividing (4) by $||u_n||$, and substitute these estimates for $n \geq n_o$ in (4), we obtain

$$(1 - \nu) \left[1 - \frac{\varepsilon}{3(1 - \nu)}\right] < \frac{2\varepsilon}{3} + |T|$$

or, $(1 - \nu) < \varepsilon + |T|$.

Since ε is arbitrary, we conclude that $|T| \geq (1 - \nu)$, which is a contradiction to our hypothesis that $|T| < 1 - \nu$.

Thus $G : S_n(y) \rightarrow S_n(y)$ is a completely continuous operator. Hence by Schauder's [46] fixed point theorem, G has a fixed point in $S_n(y)$, say x^*.

Therefore, $Gx^* = x^* = Sx^* + Tx^* + y^*$

i.e., $y^* = x^* - Sx^* - Tx^*$.

Hence the proof is complete.

Corollary 2.1.7. We obtain Theorem 2.1.4 of Granas [24] when $S = 0$ in Theorem 2.1.6.

The following result was given by Nashed & Wong [34]:

Theorem 2.1.8. Let S be a bounded linear operator on X such that S^q is a contraction mapping (with contraction constant ν, $0 \leq \nu < 1$) for some $q > 1$, and T be quasibounded and completely continuous on X. If $|T| < 1 - \nu$ then the equation $y = x - Sx - Tx$ has a solution for every $y \in X$.

Proof. Proceeding in the same manner as that of Theorem 2.1.6, we define for each $z \in X$ the operator $\bar{S}$ by (1). Again we may show by the linearity of S that $\bar{S}^q$ is a contraction, hence we may define a mapping G which maps z to the unique fixed point of $\bar{S}$ in such a way that (2) of Theorem 2.1.6 holds. Note that instead of (3) we have from

$$Gu - Gv = S^q(Gu - Gv) + \sum_{j=0}^{q-1} S^j(Tu - Tv) ,$$

the following estimate

$$||Gu - Gv|| \leq \frac{||Tu - Tv||}{1 - \nu} \sum_{j=0}^{q-1} ||S||^j ,$$

which establishes the complete continuity of G. A similar argument as that of Theorem 2.1.6 applied to the balls $S_n(u)$ where $u = \sum_{j=0}^{q-1} S^j y$ completes the proof.

Remarks 2.1.9. (i) Theorems 2.1.6 and 2.1.8 may be considered as variants of a fixed point theorem of Krasnoselskii [29].

(ii) The utility of Theorems 2.1.6 and 2.1.8 result from the fact that, unlike the standard form of Schauder Theorem, they do not require a priori that a certain closed bounded convex set is mapped into itself by the completely continuous operator.

(iii) The hypotheses of Theorems 2.1.6 and 2.1.8 guarantee the existence of some closed ball which is mapped into itself by a certain completely continuous operator G whose fixed point coincide with the fixed point of the operator $\bar{S}x = Sx + Tx + y^*$.

Theorem 2.1.4 was extended for densifying mappings by Vignoli [52] as follows:

Theorem 2.1.10. Let $T : X \to X$ be a quasibounded densifying mapping of a Banach space X into itself. If $|T| < 1$ then the equation $y = x - Tx$ has a solution for every $y \in X$.

Proof. Let $y^* \in X$ be arbitrary. We define $Gx = y^* + Tx$ for every $x \in X$. Clearly G is densifying. We consider now the following family of balls with center y^*:

$$Q(k) = \{x \in X \mid ||x - y^*|| \leq k\} \quad , \quad k = 1,2, \ldots .$$

We want to show that for some integer $q > 0$, the mapping G maps $Q(q)$ into itself. Assume the contrary. Then for any positive integer k there exists an element x_k such that

$$||Gx_k - y^*|| > k.$$

But

$$||Gx_k - y^*|| = ||Tx_k|| \quad .$$

Hence, $\dfrac{||Tx_k||}{||x_k||} > \dfrac{k}{||x_k||}$.

On the other hand,

$$||x_k|| \leq ||y^*|| + k \quad .$$

Then it follows that

$$1 > |T| = \lim_{||x|| \to \infty} \sup \frac{||Tx_k||}{||x_k||} \geq \lim_{k \to \infty} \frac{k}{||x_k||} \geq \lim_{k \to \infty} \frac{k}{||y^*|| + k} = 1,$$

which is a contradiction.

This contradiction shows that for some $q > 0$, $G : Q(q) \to Q(q)$ is a densifying mapping. Then, by Theorem 1.2.48 of Furi & Vignoli [22], G has a fixed point in $Q(q)$, say x^*.

Therefore $Gx^* = x^* = y^* + Tx^*$

i.e., $y^* = x^* - Tx^*$.

Hence the proof is complete.

The following four corollaries are due to Vignoli [52]:

Corollary 2.1.11. Let $T : X \to X$ be a quasibounded densifying mapping from a Banach space X into itself. Let $|\lambda|\ |T| < 1$ where λ is a real number such that $|\lambda| \leq 1$. Then the equation $y = x - \lambda Tx$ has a solution for every $y \in X$.

Corollary 2.1.12. Let $T : X \to X$ be a densifying mapping from a Banach space X into itself. Let $||Tx|| = 0(||x||)$ as $||x|| \to \infty$. Let λ be a real number such that $|\lambda| \leq 1$. Then the equation $y = x - \lambda Tx$ has at least one solution for each $y \in X$.

Remark 2.1.13. (i) In the above two corollaries, the condition $|\lambda| \leq 1$ is required in order that λT is densifying.

(ii) If the mapping T is assumed to be completely continuous then both the above corollaries can be proved without the assumption $|\lambda| \leq 1$ (see Granas [24] for Corollary 2.1.11 and Dubrovskii [14] for Corollary 2.1.12).

Corollary 2.1.14. Let $S : X \to X$ be a quasibounded densifying mapping from a Banach space X into itself with quasinorm $|S| \leq \alpha$, $0 \leq \alpha < 1$, and let $T : X \to X$ be completely continuous with quasinorm $|T| < 1 - \alpha$. Then the equation $y = x - Sx - Tx$ has a solution for every $y \in X$.

Remark 2.1.15. In Corollary 2.1.14, if in particular S is assumed to be a contraction mapping with constant $\alpha < 1$ then S is densifying and satisfies the condition $|S| \leq \alpha$. Indeed,

$$\frac{||Sx||}{||x||} \leq \frac{||Sx - S(0)||}{||x||} + \frac{||S(0)||}{||x||} \leq \alpha + \frac{||S(0)||}{||x||} \quad \text{for every } x \in X$$

and hence $|S| \leq \alpha$.

Corollary 2.1.16. Let $S : X \to X$ be a quasibounded densifying mapping from a Banach space X into itself with quasinorm $|S| \leq \alpha$, $0 \leq \alpha < 1$, and let $T : X \to X$ be quasibounded and completely continuous. Let λ be a real number such that $|\lambda||T| < 1 - \alpha$. Then the equation $y = x - Sx - \lambda Tx$ has a solution for every $y \in X$.

The following theorem due to Petryshyn [38] is the generalization of the results of Granas [24] for quasibounded compact maps and of Vignoli [52] for quasibounded densifying mappings.

Theorem 2.1.17. Suppose $T : X \to X$ is quasibounded 1-set-contraction such that $(I - T)(\bar{B}(0,r))$ is closed for each $r > 0$ and $|T| < 1$. Then $(I - T)$ is surjective.

Definition 2.1.18. Let X,Y be two Banach spaces, f be a mapping of an open subset V of X into Y and let $x_o \in V$; if there exists a bounded linear operator $S : X \to X$ such that

$$\lim_{t \to 0} \frac{f(x_o + tx) - f(x_o)}{t} = S(x)$$

for every $x \in X$, we say that f has the Gateaux derivative S at x_o.

The following result is due to Srinivasacharyulu [50]:

Theorem 2.1.19. Let $f : X \to X$ be a mapping of a uniformly convex Banach space X into itself such that it has the Gateaux derivative $f'(x)$ for every $x \in X$. Let $T : X \to X$ be a linear mapping of X onto X having an inverse and let $F = I - Tf$, where I is the identity mapping of X. Assume further that $\sup_{x \in X} ||F'(x)|| < 1$. If $|F| < 1$, then the equation $f(x) = y$ has at least one solution for each y in X.

Proof. By definition, $F(x) = x - Tf(x)$ and $F(x)$ has the Gateaux derivative $F'(x)$ given by $F'(x) = I - Tf'(x)$; since $||Fx - Fy|| \leq ||F'(z)||\ ||x - y||$ for some z on the segment $[x,y]$ and $\sup_{x \in X} ||F'(x)|| < 1$, we see that F is a non-expansive mapping. Let y^* be an arbitrary point of X and let $T(y^*) = z^*$; the equation $f(x) = y^*$ is equivalent to $x - F(x) = z^*$.

We prove that $f(x) = y^*$ has a solution x^* in X; to prove this, we define a mapping $\bar{F} : X \to X$ by $\bar{F}(x) = F(x) + z^*$, $x \in X$.

Since $|F| < 1$, we have $\frac{||Fx||}{||x||} < \varepsilon < 1$ for all x with $||x|| \geq \alpha_1$. Let $\delta > 0$ be such that $\varepsilon + \delta < 1$ and let $\alpha_2 = \frac{||z^*||}{\delta}$; put $\alpha = \alpha_1 + \alpha_2$, $B = \{x \in X \mid ||x|| \leq \alpha\}$. Clearly, B is bounded, closed, and convex and

$$\begin{aligned} ||\bar{F}x|| &\leq ||Fx|| + ||z^*|| \\ &\leq (\varepsilon + \delta)\ ||x|| \\ &< ||x|| , \end{aligned}$$

for $x \in B$.

Moreover, $||\bar{F}x_1 - \bar{F}x_2|| \leq ||x_1 - x_2||$ for every $x_1, x_2 \in B$; hence $\bar{F}$ has at least one fixed point $x^* \in B$ by Theorem 1.2.35 of Browder [5].

Therefore $\bar{F}x^* = x^*$ or equivalently $f(x^*) = T^{-1}(z^*) = y^*$. Thus the theorem.

We next give an existence theorem for nonlinear problem due to Kolomy [28].

We shall say that a linear continuous mapping $A : X \to X$ of Hilbert space X is normal if $AA^* = A^*A$, where A^* denotes the mapping adjoint to A.

<u>Theorem 2.1.20</u>. Let $F : X \to X$ be a mapping of a Hilbert space X into itself such that, for every $x \in X$ it has the Gateaux derivative $F'(x)$. Let $PF'(x)$ be a normal mapping for every $x \in X$ such that $(PF'(x)h,h) \geq 0$ for every $x \in X$, $h \in X$, where P is a linear mapping of X onto X having an inverse P^{-1}, $||P|| \leq \left(\sup_{x \in X} ||F'(x)||\right)^{-1}$. If $|I - PF| < 1$, where I is the identity mapping of X, then the equation $F(x) = y$ has at least one solution for every $y \in X$.

<u>Proof</u>. For every $x \in X$ the mapping $G(x) = x - PF(x)$ has the Gateaux derivative $G'(x)$ and $G'(x) = I - PF'(x)$. Because $G'(x)$ is a normal mapping for every $x \in X$, then

$$\begin{aligned} ||G'(x)|| = \sup_{||h||=1} |(G'(x)h,h)| &= \sup_{||h||=1} |(h - PF'(x)h,h)| \\ &= \sup_{||h||=1} [1 - (PF'(x)h,h)] \\ &\leq 1, \end{aligned}$$

since $0 \leq (PF'(x)h,h) \leq ||P||(\sup_{x \in X} ||F'(x)||) < 1$ for every $x \in X$ and $h \in X$ with $||h|| = 1$. Because $||Gx - Gy|| \leq ||G'(\bar{x})||\ ||x - y||$, where $\bar{x}$ is an element which lies on the line-segment connecting the points $x,y \in X$ and $\sup_{x \in X} ||G'(x)|| \leq 1$, we conclude that $G : X \to X$ is Lipschitzian mapping with constant one.

Now let y^* be an arbitrary point in X and set $z^* = P(y^*)$. The equation $F(x) = y^*$ is equivalent to $x - G(x) = z^*$. We shall show that there exists an element $x^* \in X$ such that $F(x^*) = y^*$.

Define a mapping $\bar{G} : X \to X$ by $\bar{G}(x) = G(x) + z^*$ for every $x \in X$. Since $|G| < 1$, it follows that the inequality $||G(x)|| \, ||x||^{-1} < \varepsilon < 1$ holds for all x with norm $||x|| \geq \rho_1$, where ε, ρ_1 are some constants. Now choose a positive number ν such that $\varepsilon + \nu < 1$ and let $\rho_2 = ||z^*||\nu^{-1}$. Put $r = \rho_1 + \rho_2$, $D = \{x \in X| \; ||x|| \leq r\}$, $S = \{x \in X| \; ||x|| = r\}$. Let $x \in S$, then

$$||\bar{G}(x)|| \leq ||z^*|| + ||G(x)|| \leq (\varepsilon + \nu)||x|| < ||x||.$$

Thus $||\bar{G}(x)|| < ||x||$ for every $x \in S$. Also, $||\bar{G}x_1 - \bar{G}x_2|| \leq ||x_1 - x_2||$ for every $x_1, x_2 \in D$. Hence $\bar{G}$ is Lipschitzian with constant one on D, $\bar{G} : D \to X$ and $\bar{G}(S) \subset D$. Since all the assumptions of Browder's theorem [6] are fulfilled, there exists at least one $x^* \in D$ such that $\bar{G}(x^*) = x^*$. Hence $x^* = G(x^*) + z^*$ and therefore $F(x^*) = P^{-1}(z^*)$. Because $P^{-1}(z^*) = y^*$, there is $F(x^*) = y^*$, which completes the proof.

Remark 2.1.21. The condition $|I - PF| < 1$ is equivalent to the following assumption: there exist numbers $\alpha, \nu > 0$, $\nu < 1$ such that $||x - PF(x)|| \leq \nu||x||$ whenever $||x|| \geq \alpha$.

The following corollary is also due to Kolomy [28]:

Corollary 2.1.22. Let $\phi : X \to X$ be a mapping of a Hilbert space X into X such that, for every $x \in X$ it has the Gateaux derivative $\phi'(x)$. Let $\phi'(x)$ be a normal mapping for every $x \in X$ such that

$|(\lambda\phi'(x)h,h)| \leq ||h||^2$ for every $x \in X$, $h \in X$. If the mappings $\lambda\phi$ is linearly upper bounded (i.e. quasibounded) with a constant $\nu < 1$ (λ is a real parameter), then the equation $x - \lambda\phi(x) = y$ has at least one solution for every $y \in X$.

2.2. p-Quasibounded Mappings & Fixed Point Theorems

Throughout this section, X will denote a Hausdorff locally convex topological vector space and P the family of seminorms that generates the topology of X.

Definition 2.2.1. Let $D \subset X$ and $p \in P$. A mapping $T : D \to D$ is said to be a p-contraction if there is a ν_p, $0 \leq \nu_p < 1$, such that for all $x,y \in D$, $p(Tx - Ty) \leq \nu_p\, p(x - y)$.

The above definition is due to Cain & Nashed [8]. They also mentioned the following:

Let $\mathcal{U}$ be the neighbourhood system of the origin obtained from P, the system of seminorms. Then for any given $U \in \mathcal{U}$ there exist a finite number of seminorms in P, say $p_1, p_2, \ldots, p_n$ and $r_i > 0$, $i = 1,2, \ldots, n$ such that

$$U = \bigcap_1^n r_i V(p_i), \quad \text{where} \quad V(p) = \{x | p(x) < 1\} .$$

The following theorem due to Cain & Nashed [8] generalizes Banach's fixed point theorem to Hausdorff locally convex topological vector spaces:

Theorem 2.2.2. Suppose D is a sequentially complete subset of X and the mapping $T : D \to D$ is a p-contraction for every $p \in P$. Then T has a unique fixed point $\bar{x}$ in D, and $T^n x \to \bar{x}$ for every $x \in D$.

Proof. Let $x \in D$ and $U = \bigcap_1^n r_i V(p_i)$ be given. For $y \in D$ and $k \geq 1$, we have

$$p_i(T^k y - y) \leq (1 - \nu_i)^{-1} p_i(Ty - y).$$

Choose N so that for $m \geq N$,

$$\nu_i{}^m (1 - \nu_i)^{-1} p_i(Tx - x) \leq r_i, \quad i = 1, \ldots, n.$$

Thus
$$\begin{aligned} p_i(T^{m+k}x - T^m x) &\leq (1 - \nu_i)^{-1} p_i(T^{m+1}x - T^m x) \\ &\leq \nu_i{}^m (1 - \nu_i)^{-1} p_i(Tx - x) \\ &\leq r_i . \end{aligned}$$

Hence $\{T^k x\}$ is a Cauchy sequence in D and therefore converges to a point $\bar{x}$ in D. Clearly, $T\bar{x} = \bar{x}$, and the uniqueness of the fixed point follows as usual since X is Hausdorff. Therefore, the theorem is proved.

The following definitions are also due to Cain & Nashed [8].

Definition 2.2.3. For $p \in P$ and $r > 0$, the set $\{x \in X | p(x - x_o) \leq r\}$ is denoted by $S_p(x_o, r)$. The closure of this set is denoted by $\bar{S}_p(x_o, r)$, and its boundary by $\partial S_p(x_o, r)$.

A continuous mapping $T : X \to X$ is said to be p-completely continuous for $p \in P$ if the closure of $T[\bar{S}_p(\theta, n)]$ is compact for each positive integer n, where θ is the zero element of X.

Definition 2.2.4. For an operator T, a point $x_o \in X$, and a real number $r > 0$ we define for each $p \in P$,

$$R_p(x_o,T,r) = r^{-1} \sup\{p(Tx - Tx_o) | p(x - x_o) \le r\}$$

and $$Q_p(x_o,T,a) = \{r | R_p(x_o,T,r) < a\} .$$

Now consider $Q_p(x_o,T,a)$ as a subset (possibly empty) of $[0,\infty]$, the one-point compactification of $[0,\infty)$, and let $\overline{Q_p(x_o,T,a)}$ denote the closure of $Q_p(x_o,T,a)$ relative to $[0,\infty]$.
We define

$$\beta_p(x_o,T) = \inf\{a | \infty \in \overline{Q_p(x_o,T,a)}\}$$

We shall say that T is p-quasibounded at x_o if $\beta_p(x_o,T)$ exists. T is called quasibounded at x_o if it is p-quasibounded at x_o for each $p \in P$.

It may be noted that this notion of quasiboundedness generalizes that of Granas [24].

In order to prove the following theorem of Loc [33] we require the following two lemmas, also due to Loc [33]:

<u>Lemma 2.2.5</u>. Let $S : X \to X$ be p-completely continuous and T be a self-map of X. Then the composite mapping $T \circ S : X \to X$ is p-completely continuous.

<u>Lemma 2.2.6</u>. Let X_1 and X_2 be closed subsets of a topological space Z and let T_1 and T_2 be mappings of X_1 and X_2 into a topological space Y such that $T_1x = T_2x$ on $X_1 \cap X_2$. Then

$$Tx = \begin{cases} T_1x & \text{for } x \in X_1 \\ T_2x & \text{for } x \in X_2 \end{cases}$$

is a mapping of $X_1 \cup X_2$ into Y.

<u>Theorem 2.2.7</u>. Let $T : X \to X$ be a p-completely continuous mapping. If T maps $\partial S_p(x_o,r)$ into $\bar{S}_p(x_o,r)$, then T has a fixed point in $\bar{S}_p(x_o,r)$.

<u>Proof</u>. We define a mapping $S : X \to X$ by

$$Sx = \begin{cases} Tx & , \text{ if } p(x - x_o) \geq r \\ T_1x = Tx & , \text{ if } x \in X_1 \\ T_2x = x_o + r \dfrac{Tx - x_o}{p(Tx - x_o)} & , \text{ if } x \in X_2 \end{cases}$$

where $X_1 = \{x | p(Tx - x_o) \leq r\} \cap \bar{S}_p(x_o,r)$

$X_2 = \{x | p(Tx - x_o) \geq r\} \cap \bar{S}_p(x_o,r)$.

Then by the continuity of T, the sets X_1 and X_2 are closed subsets of $\bar{S}_p(x_o,r)$. Furthermore, for $x \in X_1 \cap X_2$, $T_1x = T_2x$. Hence by Lemma 2.2.6, Sx is continuous on $\bar{S}_p(x_o,r)$. Moreover on $\partial S_p(x_o,r)$, we have $T_1x = T_2x = Tx$ and then another application of Lemma 2.2.6 shows that S is continuous on X.

Since $X \equiv \{x | p(x - x_o) \geq r\} \cup \bar{S}_p(x_o,r)$, and T is p-completely continuous, it follows from Lemma 2.2.5 that S is also p-completely continuous.

But $p(x - x_o) \leq r$ implies that $p(x) \leq p(x_o) + r$. Hence there exists a positive integer n such that $p(x) \leq p(x_o) + r \leq n$. This means $\bar{S}_p(x_o,r) \subset nU$, where U is a closed convex balanced neighbourhood of 0. Therefore $\overline{S[\bar{S}_p(x_o,r)]} \subset \overline{S(nU)}$ is compact. Hence S maps the closed convex set $\bar{S}_p(x_o,r)$ into a compact subset of itself.

Then by Schauder-Tychonoff fixed point theorem, S has a fixed point in $\bar{S}_p(x_o,r)$, say x^*, i.e. $Sx^* = x^*$. Supposing

$Sx^* = T_2x^*$, we get $r\frac{Tx^* - x_o}{p(Tx^* - x_o)} = x^* - x_o$

Therefore, $p(x^* - x_o) = r$, i.e. $x^* \in \bar{S}_p(x_o,r)$.

Since $T(\partial S_p(x_o,r)) \subset \bar{S}_p(x_o,r)$, it follows that $p(Tx^* - x_o) \leq r$, and then by the definition of Sx we have $x^* = Sx^* = Tx^*$. Hence the proof.

We now prove a theorem that generalizes Theorems 2.1.4 and 2.1.10.

<u>Theorem 2.2.8</u>. Let $T : X \to X$ be densifying. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $\beta_p(T) < 1$ then the equation $y = x - Tx$ has a solution for every $y \in X$.

<u>Proof</u>. Let y^* be an arbitrary element of X and let us define an operator G on X by

$$Gx = y^* + Tx$$

for every $x \in X$.

It is then clear that G is also densifying.

Since T is p-quasibounded at x_o and $\beta_p(T) < 1$, therefore by the definition of p-quasiboundedness, we choose $a \in \beta_p$ such that $a < 1$ and $\infty \in \overline{Q_p(x_o,T,a)}$.

Let $u_o = (I - T)x_o$ and choose c such that

$$c \in Q_p(x_o,T,a) \quad \text{and} \quad c > \frac{p(y^* - u_o)}{1 - a}$$

Then, $\quad R_p(x_o,T,c) < a$

or $\quad c^{-1}\sup\{p(Tx - Tx_o)\} < a$

or $\sup\{p(Tx - Tx_o)\} < ca$

or $p(Tx - Tx_o) < ca.$

We define a ball D with center x_o and radius c as follows:

$$D = \{x \in X | p(x - x_o) \leq c\}$$

Clearly then D is closed and convex; also D being a closed subset of the complete topological vector space X is complete. Furthermore, D is bounded.

We now show that G maps D into itself.

Let $x \in D$. Then,

$$\begin{aligned} p(Gx - x_o) &= p(y^* + Tx - u_o - Tx_o) \\ &\leq p(y^* - u_o) + p(Tx - Tx_o) \\ &< c(1 - a) + ca \end{aligned}$$

i.e., $p(Gx - x_o) < c.$

Thus, $G : D \to D$ is a densifying mapping which maps the complete, convex, bounded subset D of a locally convex topological vector space X into itself.

Now, by Theorem 1.2.49 of Swaminathan & Thompson [51], G has a fixed point in D, say x^*.

Therefore $Gx^* = x^* = y^* + Tx^*$

i.e. $y^* = x^* - Tx^*.$

Hence the proof is complete.

Corollary 2.2.9. If X is a Banach space, $T : X \to X$ is densifying and quasibounded and $|T| < 1$ then we obtain Theorem 2.1.10 due to Vignoli [52].

Corollary 2.2.10. If X is a Banach space, $T : X \to X$ is completely continuous and quasibounded and $|T| < 1$ then we obtain Theorem 2.1.4 due to Granas [24].

Corollary 2.2.11. Let (X,P) be a Hausdorff locally convex topological vector space and $T : X \to X$ be densifying. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $|\lambda|\beta_p(T) < 1$, where λ is a real number with $|\lambda| \leq 1$ then $y = x - \lambda Tx$ has a solution for every $y \in X$.

Proof. Since $\lambda \leq 1$, therefore λT is densifying; also λT is p-quasibounded at x_o and it is given that $|\lambda|\beta_p(T) < 1$.

Therefore the corollary follows from the Theorem 2.2.8.

Corollary 2.2.12. Let (X,P) be a Hausdorff locally convex topological vector space, $S : X \to X$ be densifying and $T : X \to X$ be completely continuous. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that both S and T are p-quasibounded at x_o, $\beta_p(S) < \alpha$, $0 \leq \alpha < 1$ and $\beta_p(T) < 1 - \alpha$ then the equation $y = x - Sx - Tx$ has a solution for every $y \in X$.

Proof. Since every completely continuous mapping is also densifying, therefore $S + T$ is densifying.

Also $\beta_p(S) + \beta_p(T) < 1$.

Therefore the corollary now follows from Theorem 2.2.8.

Exactly in the same lines as in the proof of Theorem 2.2.8, one can prove the following theorem using Theorem 2.2.7 of Loc [33]:

Theorem 2.2.13. Let $T : X \to X$ be p-completely continuous. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $\beta_p(T) < 1$ then the equation $y = x - Tx$ has a solution for every $y \in X$.

The following result due to Cain & Nashed [8] generalizes a fixed point theorem of Krasnoselskii [29] to locally convex spaces.

Theorem 2.2.14. Let D be a convex and complete subset of X, and S,T be operators on D into X such that $Sx + Ty \in D$ for all $x,y \in D$. Suppose S is a p-contraction for every $p \in P$, and T is continuous and $T(D)$ is contained in a compact set. Then there is a point x^* in D such that $Sx^* + Tx^* = x^*$.

Proof. For every $y \in D$, the mapping $\hat{S}$ defined by $\hat{S}x = Sx + Ty$ is a p-contraction for every $p \in P$ and maps D into D. So by Theorem 2.2.2, it has a fixed point, say Ly. In other words,

$$Ly = \hat{S}Ly = SLy + Ty.$$

Thus for every $u,v \in D$,

$$Lu - Lv = SLu - SLv + Tu - Tv.$$

So for every $p \in P$, we have

$$p(Lu - Lv) \leq \nu_p\, p(Lu - Lv) + p(Tu - Tv) \quad (\nu_p \text{ is p-contraction constant}).$$

i.e., $$p(Lu - Lv) \leq (1 - \nu_p)^{-1}\, p(Tu - Tv) \qquad \cdots\cdots\cdots\cdots \qquad (1)$$

It is clear from (1) that the operator L is continuous.

To see that $L(D)$ is contained in a compact set, let Lx_a be a net in $L(D)$. Then $\{Tx_a\}$ has a convergent subset $\{Tx'_a\}$, since $T(D)$ is contained in a compact set. Thus $\{Tx'_a\}$ is a Cauchy net, and by

(1), so also is $\{Lx'_a\}$. Hence $L(D)$ is contained in a compact set.

Then, by Schauder-Tychonoff fixed point theorem, L has a fixed point in D, say x^*.

Therefore, $Lx^* = x^* = SLx^* + Tx^* = Sx^* + Tx^*$.

Hence the proof.

Corollary 2.2.15. If X is a Banach space, S is contraction, T is completely continuous and $Sx + Ty \in D$, then we obtain the result of Krasnoselskii [29].

The following theorem given by Cain and Nashed [8] generalizes Theorem 2.1.6.

Theorem 2.2.16. Suppose $S : X \to X$ be a p-contraction mapping for every $p \in P$, with p-contraction constant ν_p, and suppose the mapping $T : X \to X$ is continuous and $\overline{T(X)}$ is compact. If X is complete and if there is a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $\nu_p + \beta_p(T) < 1$ then $y = x - Sx - Tx$ always has a solution for every $y \in X$.

Proof. Choose a so that $\nu_p + a < 1$ and $\infty \in \overline{Q_p(x_o,T,a)}$.

Let $y^* \in X$ be arbitrary, $u_o = (I - S - T)x_o$ and choose c such that $c \in Q_p(x_o,T,a)$ and $c > \dfrac{p(y^* - u_o)}{1 - (\nu_p + a)}$.

Then $R_p(x_o,T,a) < a$. Now we define the set

$$D = \{x \in X \mid p(x - x_o) \leq c\} .$$

If then follows that for $x, y \in D$, $Sx + Ty + y^* \in D$:

$$p(Sx + Ty + y^* - x_o) = p(Sx + Ty + y^* - u_o - Sx_o - Tx_o)$$

$$\leq p(Sx - Sx_o) + p(Ty - Tx_o) + p(y^* - u_o)$$

$$\leq \nu_p\, p(x - x_o) + p(Ty - Tx_o) + p(y^* - u_o)$$

$$< \nu_p\, c + ca + c[1 - (\nu_p + a)]$$

i.e., $p(Sx + Ty + y^* - x_o) < c$.

It now follows from Theorem 2.2.14 of Cain & Nashed [8] that there is an x^* in D such that

$$Sx^* + Tx^* + y^* = x^*$$

$$\text{i.e.,}\quad y^* = x^* - Sx^* - Tx^*.$$

Hence the proof.

Theorem 2.2.17. Let $S : X \to X$ be p-contraction and $T : X \to X$ be p-completely continuous. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $\beta_p(T) < 1 - \nu_\rho$, where ν_p is the p-contraction constant, $0 \leq \nu_p < 1$ then the equation $y = x - Sx - Tx$ has a solution for every $y \in X$.

Proof. Let $y^* \in X$ be arbitrary. We define an operator

$$\bar{S}x = Sx + Tz + y^*$$

for every $x \in X$ and for any fixed $z \in X$.

Clearly, $\bar{S}$ is a p-contraction mapping and hence by the Theorem 2.2.2 of Cain & Nashed [8], $\bar{S}$ has a unique fixed point, say Gz. In other words, we define a mapping G which associates to each $z \in X$, the unique fixed point of $\bar{S}$.

Thus, $Gz = \bar{S}Gz = SGz + Tz + y^*$.

Now, for every $u,v \in X$,

$$p(Gu - Gv) = p(SGu + Tu + y^* - SGv - Tv - y^*)$$
$$\leq p(SGu - SGv) + p(Tu - Tv)$$
$$\leq \nu_p \, p(Gu - Gv) + p(Tu - Tv)$$

or, $p(Gu - Gv) \leq (1 - \nu_p)^{-1} p(Tu - Tv)$,

which shows that G is p-completely continuous.

Since T is p-quasibounded at x_o and $\beta_p(T) < 1 - \nu_p$, therefore we choose $a \in \beta_p(T)$ such that

$a < 1 - \nu_p$ and $\infty \in \overline{Q_p(x_o,T,a)}$.

Let $u_o = (I - S - T)x_o$ and choose c such that

$$c \in Q_p(x_o,T,a) \text{ and } c > \frac{p(y^* - u_o)}{1 - (\nu_p + a)} .$$

Then, as in Theorem 2.2.8, we have $p(Tx - Tx_o) < ca$.

We define now a ball D and its boundary ∂D with center x_o and radius c as follows:

$$D = \{x \in X | p(x - x_o) \leq c\}$$

and $\partial D = \{x \in X | p(x - x_o) = c\}$.

We then show that $G(\partial D) \subset D$. Let $x \in \partial D$. Then,

$$p(Gx - x_o) = p(SGx + Tx + y^* - u_o - Sx_o - Tx_o)$$
$$\leq p(SGx - Sx_o) + p(Tx - Tx_o) + p(y^* - u_o)$$
$$\leq \nu_p \, p(Gx - x_o) + p(Tx - Tx_o) + p(y^* - u_o)$$

or, $$p(Gx - x_o) \leq (1 - \nu_p)^{-1}[p(Tx - Tx_o) + p(y^* - u_o)]$$
$$< (1 - \nu_p)^{-1}[ca + c(1 - (\nu_p + a))]$$

i.e., $p(Gx - x_o) < c$.

Thus, G is a p-completely continuous mapping which maps ∂D into D. Hence, by Theorem 2.2.7 of Loc [33], G has a fixed point in D, say x^*.

Therefore, $Gx^* = x^* = SGx^* + Tx^* + y^*$

$$= Sx^* + Tx^* + y^*$$

i.e., $y^* = x^* - Sx^* - Tx^*$.

The proof is complete.

Corollary 2.2.18. Theorem 2.2.13 follows from Theorem 2.2.17 when $S = 0$.

Corollary 2.2.19. If X is a Banach space, $S : X \to X$ is contraction, $T : X \to X$ is completely continuous and quasibounded and $|T| < 1 - \nu$, where ν is the contraction constant, $0 \leq \nu < 1$, then we obtain Theorem 2.1.6 of Nashed and Wong [34].

Corollary 2.2.20. Let (X,P) be a Hausdorff locally convex topological vector space, $S : X \to X$ p-contraction with p-contraction constant ν_p, $0 \leq \nu_p < 1$ and $T : X \to X$ be p-completely continuous. If X is complete and if there exist a $x_o \in X$ and a $p \in P$ such that T is p-quasibounded at x_o and $|\lambda|\beta_p(T) < 1 - \nu_p$, where λ is a real number with $|\lambda| \leq 1$ then $y = x - Sx - \lambda Tx$ has a solution for every $y \in X$.

Proof. Since $\lambda \leq 1$ therefore λT is p-completely continuous and also λT is p-quasibounded at x_o. It is also given that $|\lambda|\beta_p(T) < 1 - \nu_p$.

Thus the corollary follows immediately from the Theorem 2.2.17.

Theorem 2.2.21. Let a convex function $f : X \to X$ be such that f has Gateaux derivative $f'(x)$ for every $x \in X$, and let $F = I - f$, where I is the identity mapping of X. If X is complete and if there exist $x_o \in X$, $p \in P$ such that F is p-quasibounded at x_o, $\beta_p(F) < 1$ and $p(F'(x)) \leq \frac{1}{2}$ for every $x \in X$ then the equation $y = f(x)$ has a unique solution for every $y \in X$.

Proof. By definition, for every $x \in X$, the mapping

$$F(x) = x - f(x)$$

has the Gateaux derivative $F'(x)$ which is given by

$$F'(x) = I - f'(x).$$

Since f is differentiable and f is also convex, therefore by a well-known characterization of convexity,

$$\begin{aligned} f'(x)(y - x) &\leq f(y) - f(x) \\ &= y - F(y) - x + F(x) \\ &= (y - x) + (F(x) - F(y)) \end{aligned}$$

for every $x,y \in X$.

Thus $f'(x) \leq \dfrac{(y - x) + (F(x) - F(y))}{y - x}$,

and therefore $F'(x) \geq I - \dfrac{(y - x) + (F(x) - F(y))}{y - x}$

i.e., $F'(x)(y - x) \geq F(y) - F(x)$.

By hypothesis, since $p(F'(x)) \leq \frac{1}{2}$, for every $x \in X$, therefore, $p(F(y) - F(x)) \leq \frac{1}{2}\, p(y - x)$ which shows that F is p-contraction.

Let $y^* \in X$ be arbitrary and define a mapping $G : X \to X$ by $G(x) = y^* + F(x)$ for every $x \in X$. Clearly, G is also p-contraction.

Since $\beta_p(F) < 1$, choose $a \in \beta_p(F)$ such that $a < 1$ and $\infty \in \overline{Q_p(x_o,F,a)}$. Let $u_o = (I - F)x_o$ and choose c such that $c \in Q_p(x_o,F,a)$ and $c \ngtr \frac{p(y^* - u_o)}{1 - a}$.

Therefore, $R_p(x_o,F,c) < a$. Then $p(Fx - Fx_o) < ca$. Now, define a ball D with center x_o and radius c as follows:

$$D = \{x \in X | p(x - x_o) \leq c\} \quad , \quad p \in P.$$

Clearly, D is closed. Further, D being a closed subset of a complete topological space X is also complete.

We claim that $G(D) \subset D$. Let $x \in D$. Then

$$\begin{aligned} p(Gx - x_o) &= p(y^* + F(x) - u_o - F(x_o)) \\ &\leq p(y^* - u_o) + p(F(x) - F(x_o)) \\ &< c(1 - a) + ca \end{aligned}$$

i.e., $p(Gx - x_o) < c$.

Thus, G is a p-contraction mapping that maps a complete subset D of a Hausdorff locally convex topological vector space X into itself. Hence by Theorem 2.2.2 of Cain & Nashed [8], G has a unique fixed point in D, say x^*. Therefore,

$$G(x^*) = x^* = y^* + F(x^*) = y^* + x^* - f(x^*)$$

i.e., $\qquad y^* = f(x^*)$.

2.3. Intersection Theorems for Quasibounded Mappings

Let a Banach space $X = A \oplus B$ be a direct sum of two subspaces A and B of X, i.e. each element x of X can be uniquely represented in the form

$$x = a + b \quad , \quad \text{where} \quad a \in A \quad \text{and} \quad b \in B.$$

Let P_A denote projection of X onto A and P_B projection of X onto B, i.e., for each $x = a + b \in X$, $P_A(x) = a$ and $P_B(x) = b$. Clearly, the mappings $P_A : X \to A$ and $P_B : X \to B$ are linear and hence we have

$$||P_A(x)|| \leq ||P_A|| \; ||x|| \quad \text{and} \quad ||P_B(x)|| \leq ||P_B|| \; ||x|| \;, \quad \text{for each} \quad x \in X \quad \text{.......(1)}$$

where $||P_A||$ and $||P_B||$ are norms of P_A and P_B respectively.

The following result is due to Granas [24].

Theorem 2.3.1. Let $X = A \oplus B$ and let the mappings $f : A \to X$, $g : B \to X$ be completely continuous such that

$$f(a) = a - F(a) \quad \text{and} \quad g(b) = b - G(b) \quad \text{..............(2)}$$

If the mappings $F : A \to X$ and $G : B \to X$ are quasibounded and

$$|F| \; ||P_A|| + |G| \; ||P_B|| < 1 \quad \text{.................(3)}$$

then $f(A) \cap g(B) \neq \phi$.

Proof. Each element $x \in X$ can be uniquely represented in the form

$$x = a - b \quad , \quad a \in A \quad \text{and} \quad b \in B.$$

Let us put $H_1(x) = F(a)$, $H_2(x) = -G(b)$, and $H(x) = H_1(x) + H_2(x)$ for $x \in X$.

Clearly, $H : X \to X$ is completely continuous. Let us define a completely continuous mapping $h : X \to X$ by

$$h(x) = x - H(x) = a - b - F(a) + G(b) \quad \ldots\ldots\ldots\ldots\ldots (4)$$

We shall prove that H is quasibounded mapping and $|H| < 1$.

By (3), there is a positive ε such that

$$|F| \, ||P_A|| + |G| \, ||P_B|| + \varepsilon(||P_A|| + ||P_B||) < 1 \quad \ldots\ldots\ldots (5)$$

There are such constants r_1 and r_2 that the inequalities

$$\frac{||F(a)||}{||a||} \leq |F| + \varepsilon \quad \text{and} \quad \frac{||G(b)||}{||b||} \leq |G| + \varepsilon \quad \ldots\ldots\ldots (6)$$

hold whenever $||a|| \geq r_1$ and $||b|| \geq r_2$.

For every $x \in X$ we have

$$\frac{||H(x)||}{||x||} \leq \frac{||H_1(x)||}{||x||} + \frac{||H_2(x)||}{||x||} = \frac{||F(a)||}{||x||} + \frac{||G(b)||}{||x||}$$

From this taking into account (1) we obtain that the inequality

$$\frac{||H(x)||}{||x||} \leq \frac{||F(a)||}{||a||} \cdot \frac{||a||}{||x||} + \frac{||G(b)||}{||b||} \cdot \frac{||b||}{||x||}$$

$$\leq \frac{||F(a)||}{||a||} \cdot ||P_A|| + \frac{||G(b)||}{||b||} \cdot ||P_B||$$

holds for $x \in X$ such that $a \neq 0$ and $b \neq 0$.

From this, taking into account (5) and (6) we conclude that the inequality

$$\frac{||H(x)||}{||x||} < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

holds for all $x \in M_o$, where $M_o = \{x \in X | \; ||a|| \geq r_1 \; , \; ||b|| \leq r_2\}$.

Let us put

$$M_1 = \{x \in X | \; ||a|| < r_1 \; , \; ||b|| \geq 0\}$$

and $$M_2 = \{x \in X | \; ||a|| \geq 0 \; , \; ||b|| < r_2\}.$$

Since F is completely continuous on A we conclude that the values of $H_1(x)$ are bounded for all $x \in M_1$. Similarly, since G is completely continuous on B, the values of $H_2(x)$ are bounded for all $x \in M_2$.

From this we infer that for sufficiently large positive numbers r_1^* and r_2^* the inequality (7) holds

for all $x \in M_1^* = \{x \in M_1 \mid ||b|| \geq r_2^*\}$

and also for all $x \in M_2^* = \{x \in M_2 \mid ||a|| > r_1^*\}$.

Putting $M = (M_1 - M_1^*) \cup (M_2 - M_2^*)$ one can observe that the set M is bounded and $M = X - (M_o \cup M_1^* \cup M_2^*)$.

Thus the inequality (7) holds for all points $x \in X$ which do not belong to the bounded set M; hence H is quasibounded mapping and $|H| < 1$.

By Theorem 2.1.4, $h(x) = x - H(x)$ has a solution. Hence there exists an element $x = a - b$ such that $h(x) = 0$. By (4), we have

$$0 = a - b - F(a) + G(b)$$

i.e., $$f(a) = g(b)$$

which completes the proof.

Granas [24] proved the following:

Theorem 2.3.2. Let $X = A \oplus B$ and let $f : A \to X$, $g : B \to X$ be compact mappings. Then the image $f(A)$ of A under f intersects the image $g(B)$ of B under g, i.e., $f(A) \cap g(B) \neq \phi$.

Proof. This follows from the preceding Theorem 2.3.1 because the quasinorm of a compact mapping is equal to zero.

Definition 2.3.3. A continuous mapping $T : X \to X$ from a metric space X into itself is said to be α-Lipschitz with constant L, if for any bounded set $A \subset X$ we have

$$\alpha(T(A)) \leq L\alpha(A) \quad , \quad 0 \leq L < +\infty .$$

Clearly any completely continuous mapping is α-Lipschitz with constant $L = 0$.

The above was given in Kuratowskii (see [32]). The following result due to Vignoli [53] is a generalization of the Theorem 2.3.1.

Theorem 2.3.4. Let $X = A \oplus B$ and let $f : A \to X$ and $g : B \to X$ be such that

$$f(a) = a + F(a) \quad , \quad \text{for every } a \in A,$$
$$g(b) = b + G(b) \quad , \quad \text{for every } b \in B,$$

where the mappings F and G are α-Lipschitz with constants L and L' respectively which satisfy

$$L||P_A|| + L'||P_B|| < 1.$$

If the mappings F and G are also quasibounded with quasinorms satisfying

$$|F| \, ||P_A|| + |G| \, ||P_B|| < 1,$$

then $f(A) \cap g(B) \neq \phi$.

Proof: Since $X = A \oplus B$, any element $x \in X$ can be represented as

$$x = a - b \quad , \quad a \in A \text{ and } b \in B.$$

Let $T = T_1 + T_2$, where $T_1 = F \circ P_A$ and $T_2 = -G \circ P_B$.

Clearly the mapping $T : X \to X$ is densifying. Indeed, for any bounded subset D of X with $\alpha(D) > 0$, we have

$$\begin{aligned} \alpha(T(D)) &\leq \alpha(FP_A(D)) + \alpha(GP_B(D)) \\ &\leq L\alpha(P_A(D)) + L'\alpha(P_B(D)) \\ &\leq L||P_A||\alpha(D) + L'||P_B||\alpha(D) \\ &< \alpha(D). \end{aligned}$$

Now using the same argument as that in Theorem 2.3.1, it can be proved that T is also quasibounded with $|T| < 1$.

Hence by Theorem 2.1.10, there exist at least one element $x^* \in X$ such that

$$x^* + Tx^* = 0$$

$$\text{i.e.,} \quad 0 = a^* - b^* + F(a^*) - G(b^*)$$

$$\text{i.e.,} \quad f(a^*) = g(b^*).$$

This proves the theorem.

Remark 2.3.5. The above theorem contains as a particular case the Theorem 2.3.1 of Granas [24] for F and G completely continuous. In this case the condition $L||P_A|| + L'||P_B|| < 1$ is trivially satisfied since $L = L' = 0$.

The following corollaries are due to Vignoli [53].

<u>Corollary 2.3.6</u>. Let $X = A \oplus B$ and let $f : A \to X$, $g : B \to X$ be such that

$$\left.\begin{aligned} f(a) &= a + F(a) + M(a), \quad \text{for each } a \in A \\ g(b) &= b + G(b) + N(b), \quad \text{for each } b \in B \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

where the mappings M and N are completely continuous and the mappings F and G are α-Lipschitz with constants L and L' respectively such that

$$L||P_A|| + L'||P_B|| < 1.$$

Let the mappings F and G be also quasibounded with quasinorms satisfying

$$|F|\ ||P_A|| + |G|\ ||P_B|| \leq \beta \quad , \quad 0 \leq \beta < 1,$$

and let the mappings M and N be quasibounded with quasinorms such that

$$|M|\ ||P_A|| + |N|\ ||P_B|| < 1 - \beta .$$

Then $f(A) \cap g(B) \neq \phi$.

<u>Proof</u>. Let $T = T_1 + T_2$, where $T_1 = F \circ P_A + M \circ P_A$, $T_2 = -G \circ P_B - N \circ P_B$. Clearly then T is densifying. Indeed

$$\begin{aligned} \alpha(T(D)) &\leq \alpha(F \circ P_A(D)) + \alpha(M \circ P_A(D)) + \alpha(G \circ P_B(D)) + \alpha(N \circ P_B(D)) \\ &\leq L\alpha(P_A(D)) + L'\alpha(P_B(D)) \\ &< \alpha(D). \end{aligned}$$

Also the mapping T is quasibounded and $|T| < 1$.

Hence by Theorem 2.3.4, we get the result.

<u>Remark 2.3.7</u>. In Corollary 2.3.6, instead of (1) we could consider the following mappings:

$$\left.\begin{aligned} f(\mu;a) &= a + F(a) + \mu M(a) \\ g(\mu;b) &= b + G(b) + \mu N(b) \end{aligned}\right\} \quad \ldots\ldots (1^*)$$

where the mappings F,G satisfy the same hypothesis of the Corollary 2.3.6 and the quasinorms of M, N and the real number μ are such that

$$|\mu|(|M|\ ||P_A|| + |N|\ ||P_B||) < 1 - \beta \ , \quad 0 \leq \beta < 1.$$

Then $f(\mu;A) \cap g(\mu;B) \neq \phi$. (Evidently for $\mu = 1$ we obtain Corollary 2.3.6.).

Corollary 2.3.8. Let $X = A \oplus B$ and let $f : A \to X$, $g : B \to X$ be such that

$$f(\lambda;a) = a + \lambda F(a) \ , \quad \text{for every } a \in A,$$
$$g(\lambda;b) = b + \lambda G(b) \ , \quad \text{for every } b \in B,$$

where λ is a real number such that $|\lambda| \leq 1$ and the mappings F,G are α-Lipschitz with constants L, L' respectively, satisfying

$$L||P_A|| + L'||P_B|| < 1.$$

If the mappings F,G are also quasibounded with quasinorms satisfying

$$|\lambda|(|F|\ ||P_A|| + |G|\ ||P_B||) < 1, \quad \ldots\ldots (I)$$

then $f(\lambda;A) \cap g(\lambda;B) \neq \phi$.

Corollary 2.3.9. Let X,f,g and λ be as in Corollary 2.3.8. If instead of the condition (I) in Corollary 2.3.8 the mappings F,G satisfy

$$\left.\begin{aligned} ||F(x)|| &= 0\ (||x||) \quad \text{as } ||x|| \to \infty \\ ||G(x)|| &= 0\ (||x||) \quad \text{as } ||x|| \to \infty \end{aligned}\right\} \quad \ldots\ldots (I^*)$$

then $f(\lambda;A) \cap g(\lambda;B) \neq \phi$.

Remarks 2.3.10. (i) The condition $|\lambda| \leq 1$ in Corollaries 2.3.8 and 2.3.9 is required in order that λT is densifying.

(ii) In Corollary 2.3.9, a particular case for $\lambda = 1$ and F, G completely continuous was proved by Granas [24].

The following result is due to Nashed and Wong [34].

Theorem 2.3.11. If we have

$$\zeta = \frac{1}{|\mu|} (\nu||P_A|| + \nu'||P_B||) < 1 \quad \ldots\ldots\ldots\ldots (1)$$

and

$$|F_2|\ ||P_A|| + |G_2|\ ||P_B|| < 1 - \rho \quad \ldots\ldots\ldots\ldots (2)$$

then $f_\mu(A) \cap g_\mu(B) \neq \phi$, where for each $\mu \neq 0$, $f_\mu : A \to X$, $g_\mu : B \to X$ be such that

$$f_\mu(a) = \mu a + F_1(a) + F_2(a), \quad \text{for every } a \in A,$$

$$g_\mu(b) = \mu b + G_1(b) + G_2(b), \quad \text{for every } b \in B,$$

with F_1, G_1 Lipschitzian operator, ν, ν' Lipschitz norms of F_1, G_1 respectively and F_2, G_2 quasibounded completely continuous operators from A, B into X respectively.

Proof. Define an operator T on X by

$$Tx = G_1(P_B(-x)) + G_2(P_B(-x)) - F_1(P_A(x)) - F_2(P_A(x)).$$

It is easy to show that $f_\mu(A) \cap g_\mu(B) \neq \phi$ if and only if the equation $Tx = \mu x$ has a solution in X. Let

$$M = \frac{1}{\mu}\{G_1(-P_B) - F_1(P_A)\}$$

and

$$N = \frac{1}{\mu}\{G_2(-P_B) - F_2(P_A)\}.$$

It is then readily verified that (1) and (2) imply M is contraction and $|N| < 1 - \rho$. Hence the existence of solution of $Tx = \mu x$ follows as an immediate consequence of Theorem 2.1.6 of Nashed & Wong [34].

Remark 2.3.12. Theorem 2.3.11 reduces to Theorem 2.3.1 of Granas [24] by taking $\nu = \nu' = 0$ and $\mu = 1$.

Another interesting intersection theorem has been obtained by Petryshyn [40] where he extended Theorem 2.3.1 of Granas [24] to the following:

(a) either F or G is P-compact

and (b) condition (3) of Theorem 2.3.1 is replaced by a much weaker condition.

We first give the following preliminaries and definition:

Let X be a real Banach space with the property that there exists a sequence $\{X_n\}$ of finite dimensional subspaces X_n of X, a sequence of linear projections $\{P_n\}$ defined on X, and a constant $K > 0$ such that

$$P_nX = X_n \quad , \quad X_n \subset X_{n+1}, \quad n = 1,2,3, \ldots, \quad \overline{\bigcup_{n=1}^{\infty} X_n} = X \quad . \; . \; . \; . \; . \quad \text{(I)}$$

$$||P_n|| \leq K \; , \quad n = 1,2,3, \ldots \qquad . \; . \; . \; . \; . \quad \text{(II)}$$

Let B_r denote the closed ball in X of radius $r > 0$ about the origin and let S_r denote the boundary of B_r. Let the symbol "$\to$" denote the strong convergence in X.

Definition 2.3.13. A nonlinear operator T mapping X into itself is called projectionally-compact (P-compact) if P_nT is continuous in X_n for all sufficiently large n and if for any constant $p > 0$ and any bounded sequence $\{x_n\}$ with $x_n \in X_n$ the strong convergence of the sequence $\{g_n\} = \{P_nTx_n - px_n\}$ imply the existence of a strongly convergent subsequence

$\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \to x$ and $P_{n_i} Tx_{n_i} \to Tx$.

This definition is due to Petryshyn [40]. We now state some results without proof on which our next intersection theorem is essentially based.

Theorem 2.3.14. Suppose that T is P-compact. Suppose further that for given $r > 0$ and $\mu > 0$ the operator T satisfies both the following conditions:

(Λ): There exists a number $c(r) > 0$ such that if, for any n, $P_n Tx = \lambda x$ holds for x in S_r with $\lambda > 0$, then $\lambda \leq c(r)$.

(Π_μ): If for some x in S_r the equation $Tx = \alpha x$ holds, then $\alpha < \mu$.

Then there exists at least one element u in $(B_r - S_r)$ such that

$$Tu - \mu u = 0.$$

The above result is due to Petryshyn [40].

Remarks 2.3.15. The assertion of Theorem 2.3.14 remains valid if condition (Λ) is replaced, for example, by any one of the following stronger conditions whose degree of generality increases in the given order:

(i) T is bounded, i.e., T maps bounded sets in X into bounded sets.

(ii) For any given $r > 0$ the set $T(S_r)$ is bounded.

(iii) X is a Hilbert space and, for any given $r > 0$, $(Tx, x) \leq c\,||x||^2$ for every $x \in S_r$ and some $c > 0$.

The following result is due to Petryshyn [39].

Theorem 2.3.16. The class of P-compact operators with $p < 0$ contains, among others, the following operators:

(a) closed precompact operators (and, in particular, completely continuous and strongly continuous operators) in X.

(b) Quasicompact operators in X.

(c) Continuous, demicontinuous, and weakly continuous monotone increasing operators in X, when X is a Hilbert space.

Now we give the intersection theorem due to Petryshyn [40].

Theorem 2.3.17. Let $G : B \to X$ be a nonlinear mapping such that $G(-P_B)$ is P-compact and such that to a given $r > 0$ there corresponds a number $c(r) > 0$ with the property that for all x in S_r

$$||G(-P_B x)|| \leq c(r). \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (1)$$

Let $F : A \to X$ be a completely continuous mapping and $f_\mu : A \to X$, $g_\mu : B \to X$ be defined by $f_\mu(a) = \mu a + F(a)$, $g_\mu(b) = \mu b + G(b)$ respectively for every $a \in A$ and $b \in B$. If for given $r > 0$ and $\mu > 0$ the operators F and G satisfy the condition:

(Π) : If $Fa + \alpha a = Gb + \alpha b$ for some a in A and b in B with $||a - b|| = r$, then $\alpha < \mu$,

then $f_\mu(A) \cap g_\mu(B) \neq \phi$.

Proof. Let us define a nonlinear mapping $T : X \to X$ by

$Tx = G(b) - F(a)$ with $b = -P_B x$ and $a = P_A x$, for every $x \in X$ . . . . (2)

and observe that $f_\mu(A) \cap g_\mu(B) \neq \phi$ if and only if the equation

$$Tx = \mu x \quad \ldots \ldots \ldots \ldots \ldots \quad (3)$$

has a solution in X. Indeed, if x is a solution of (3), then x has a unique representation $x = P_A x + P_B x = a - b$ and, in view of (2), (3) implies that $G(b) - F(a) = \mu(a - b)$ or that $\mu b + G(b) = \mu a + F(a)$, i.e., $f_\mu(A) \cap g_\mu(B) \neq \phi$. On the other hand, if $a \in A$ and $b \in B$ are two elements such that $f_\mu(a) = g_\mu(b)$, then $\mu(a - b) = G(b) - F(a)$. Hence if we put $x = a - b$, then (2) implies that x is a solution of (3).

Thus, to prove Theorem 2.3.17 it is sufficient, in view of the above observation and Theorem 2.3.14, to show that the operator T defined by (2) is P-compact and satisfies conditions (Λ) and (Π_μ) of Theorem 2.3.14.

Let us first show that T is P-compact. Now, by our conditions on G and F, P_nT is certainly continuous in X_n for all sufficiently large n. Further, let $\{x_n\}$ be a bounded sequence so that for any $p > 0$

$$g_n = P_nTx_n - px_n = P_nG(-P_Bx_n) - px_n - P_nF(P_Ax_n) \to g, \quad \text{for every } x_n \in X_n \ldots (4)$$

Since $\{v_n\} = \{P_Ax_n\}$ is bounded and F is completely continuous, there exists a subsequence, which we again denote by $\{x_n\}$ such that $F(v_n) = F(P_Ax_n) \to v$ and $P_nF(v_n) \to v$ where $v \in X$. This and (4) imply that

$$\tilde{g}_n \equiv P_nG(-P_Bx_n) - px_n = g_n + P_nF(P_Ax_n) \to g + v, \quad (n \to \infty).$$

Since $G(-P_B)$ is P-compact, therefore there exists a subsequence again denoted by $\{x_n\}$, such that $x_n \to x$ and $P_nG(-P_Bx_n) \to G(-P_Ax)$. This and the continuity of F imply that

$$P_nTx_n = P_nG(-P_Bx_n) - P_nF(P_Ax_n) \to G(P_Bx) - F(P_Ax) = Tx.$$

i.e., T is P-compact.

Suppose now that $Tx = \alpha x$ for some x in S_r. This then means that $G(b) + \alpha b = F(a) + \alpha b$ with $||a - b|| = ||P_Ax + P_Bx|| = r$. Hence our condition ($\Pi$) implies that $\alpha < \mu$; i.e., T satisfies condition (Π_μ). Finally we see that for any x in S_r condition (1) and the complete continuity of F imply the inequality

$$||Tx|| \leq ||G(-P_B x)|| + ||F(P_A x)|| \leq c(r) + c, \quad x \in S_r,$$

where $c > 0$ is such that $||F(P_A x)|| \leq c$ for all x in S_r. Thus the set $T(S_r)$ is bounded and therefore, by Remark 2.3.15 (ii), T satisfies condition (Λ). Hence, by Theorem 2.3.14, equation (3) has at least one solution in B_r or, equivalently, the intersection $f_\mu(A) \cap g_\mu(B) \neq \phi$. The proof is complete now.

Petryshyn [40] gave the following two corollaries.

Corollary 2.3.18. Suppose that F and G satisfy all conditions of Theorem 2.3.17 except that condition (Π) is replaced by the condition

$$||\mu b + G(b) - (\mu a + F(a))||^2 \geq ||Fa - Gb||^2 - \mu^2||a - b||^2 \quad \ldots \quad (5)$$

for $a \in A$, $b \in B$ with $||a - b|| = r$. Then $f_\mu(A) \cap g_\mu(B) \neq \phi$.

Proof. We may assume, without loss of generality, that there is no elements a in A and $b \in B$ with $||a - b|| = r$ such that $f_\mu(a) = g_\mu(b)$. Suppose now that for some x in S_r or equivalently for some a in A and b in B with $||a - b|| = r$ we have $Fa + \alpha a = Gb + \alpha b$. Then

$$||Gb - Fa - \mu(a - b)||^2 = ||\alpha(a - b) - \mu(a - b)||^2 = (\alpha - \mu)^2||a - b||^2 ,$$

and $||Gb - Fa||^2 - \mu^2||a - b||^2 = (\alpha^2 - \mu^2)||a - b||^2$.

Hence, by (5), $(\alpha - \mu)^2 \geq \alpha^2 - \mu^2$ or $2\mu^2 \geq 2\mu\alpha$. Since $\mu > 0$, our assumption then implies that $\alpha < \mu$ and, consequently, (4) implies the condition (Π). Corollary 2.3.18 then follows from Theorem 2.3.17.

Remark 2.3.19. In case X is a Hilbert space, condition (5) is equivalent to the requirement

$$(G(-P_B x) - F(P_A x), x) \leq \mu||x||^2 , \quad x \in S_r. \quad \ldots\ldots\ldots\ldots (6)$$

Corollary 2.3.20. Suppose that F and G are completely continuous and quasibounded, i.e., there exists four constants $M_1 > 0$, $M_2 > 0$, $r_1 > 0$ and $r_2 > 0$ such that

$$||Fa|| \leq M_1||a|| \text{ for every } a \text{ in } A \text{ with norm } ||a|| \geq r_1 \quad . . . . (7)$$

$$||Gb|| \leq M_2||b|| \text{ for every } b \text{ in } B \text{ with norm } ||b|| \geq r_2 \quad . . . . (8)$$

Suppose further that M_1 and M_2 satisfy the inequality

$$M_1||P_A|| + M_2||P_B|| \leq 1 \quad . . . . (9)$$

Then $f_1(A) \cap g_1(B) \neq \phi$.

Proof. Let us first remark that, as was shown by Granas [24], the conditions of Corollary 2.3.20 imply the existence of a constant $r > 0$ such that

$$||G(-P_Bx) - F(P_Ax)|| \leq ||x|| \text{ for every } x \text{ in } X \text{ with } ||x|| \geq r. \quad . . . (10)$$

i.e., the operator $T(x) = G(-P_Bx) - F(P_Ax)$ is quasibounded. Assuming without loss of generality, that there are no elements a in A and b in B with $||a - b|| = r$ such that $f_1(a) = g_1(b)$, it is easy to see that whenever $Fa + \alpha a = Gb + \alpha b$ for some $a \in A$ and $b \in B$ with $||a - b|| = r$, then (10) implies that $\alpha < 1$. Hence condition (Π) of Theorem 2.3.17 holds for $\mu = 1$. Furthermore, since G is completely continuous, (1) of Theorem 2.3.17 is clearly satisfied and, by Theorem 2.3.16, $G(-P_W)$ is P-compact. Consequently, Corollary 2.3.20 follows from Theorem 2.3.17.

Remark 2.3.21. For the sake of completeness let us show that the conditions of Corollary 2.3.20 imply the validity of (10) for some $r > 0$. First let $r_o = \max\{r_1, r_2\}$ and let $c > 0$ be a constant such that

$||Fa|| \leq c$ for all $a \in A$ with $||a|| \leq r_o$

and $||Gb|| \leq c$ for all $b \in B$ with $||b|| \leq r_o$

Taking $r = \max\left\{2r_o \ , \ \frac{c}{1 - M_o}\right\}$ where $M_o = \max\{M_1||P_A||, M_2||P_B||\} < 1$, we obtain (10). Indeed, (10) follows trivially from (7), (8), (9) if for $x = a - b$ with $||x|| \geq r$ we have $||a|| \geq r_o$ and $||b|| \geq r_o$. On the other hand, if for $||x|| \geq r$ one of the conditions $||a|| \geq r_o$ or $||b|| \geq r_o$ is not satisfied (e.g. $||b|| \not\geq r_o$), then by our definition of c and M_o we get the desired inequality.

$$\begin{aligned} ||G(-P_Bx) - F(P_Ax))|| &\leq ||G(b)|| + ||F(b)|| \\ &\leq c + M_o||x|| \\ &\leq (1 - M_o)||x|| + M_o||x|| \\ &= ||x|| . \end{aligned}$$

We now give an intersection theorem in Hilbert spaces by Kolomy [28].

Let X be a Hilbert space, Y,Z non-trivial subspaces of X such that X is their direct sum, i.e., $X = Y \oplus Z$. Denote by P_Y, P_Z the linear projection of X onto Y,Z respectively. Set $f(x) = x + AF(x)$, $g(x) = x + AG(x)$, where $A : X \to X$ is a linear continuous mapping of X into X and F,G are non-linear mappings of Y,Z into X respectively.

Theorem 2.3.22. Let $X = Y \oplus Z$ and let $f : Y \to X$, $g : Z \to X$ be defined as above, where $F : Y \to X$, $G : Z \to X$ are Lipschitzian mappings with constants α_1, α_2 respectively. Furthermore, let F,G linearly upper bounded (i.e., quasibounded) with bounds β_1, β_2 respectively such that $\varepsilon = ||A||(\beta_1||P_Y|| + \beta_2||P_Z||) \leq 1$. If $(\alpha_1||P_Y|| + \alpha_2||P_Z||)||A|| \leq 1$, then the intersection $f(Y) \cap f(Z)$ is non-void.

Proof. Put $\phi(x) = A(G(-P_Zx) - F(P_Yx))$ for every $x \in X$. Then for all $x_1, x_2 \in X$ we have

$$\begin{aligned}||\phi(x_1) - \phi(x_2)|| &\leq ||A||(||G(-P_Zx_1) - G(-P_Zx_2)|| + ||F(P_Yx_2) - F(P_Yx_1)||)\\ &\leq ||A||(\alpha_2||P_Zx_1 - P_Zx_2|| + \alpha_1||P_Yx_2 - P_Yx_1||)\\ &\leq ||A||(\alpha_1||P_Y|| + \alpha_2||P_Z||)||x_1 - x_2||\\ &\leq ||x_1 - x_2||.\end{aligned}$$

Thus the mapping $\phi : X \to X$ is Lipschitzian with constant one. Under our assumptions, F, G are linearly upper bounded with β_1, β_2 respectively. Therefore,

$$||Fy|| \leq \beta_1||y|| \quad \text{for every } y \in Y \text{ with } ||y|| \geq \rho_1 \quad \ldots\ldots(1)$$

$$||Gz|| \leq \beta_2||z|| \quad \text{for every } z \in Z \text{ with } ||z|| \geq \rho_2 \quad \ldots\ldots(2)$$

for some $\rho_1, \rho_2 > 0$. Put $\rho = \max(\rho_1, \rho_2)$; then (1), (2) are fulfilled for every $y \in Y$ with $||y|| \geq \rho$ and every $z \in Z$ with $||z|| \geq \rho$. Put $K_\rho = \{y \in Y|\ ||y|| \leq \rho\}$ and $\Omega_\rho = \{z \in Z|\ ||z|| \leq \rho\}$. Since F, G are Lipschitzians on Y, Z respectively, then there exist positive numbers K_1, K_2 such that $||Fy|| \leq K_1$, $||Gz|| \leq K_2$ for all $y \in K_\rho$, $z \in \Omega_\rho$ respectively (cf. [28]). Set $K = \max(K_1, K_2)$,

$$N = \max(\beta_1||A||\ ||P_Y||, \beta_2||A||\ ||P_Z||), \quad \rho_o = \max(2\rho, ||A||K/(1 - N)),$$

$$K_{\rho_o} = \{x \in X|\ ||x|| \leq \rho_o\}, \quad S_{\rho_o} = \{x \in X|\ ||x|| = \rho_o\}.$$

If for $x = P_Yx + P_Zx \in X$ with $||x|| \geq \rho_o$ there is also $||P_Yx|| \geq \rho$, $||P_Zx|| \geq \rho$, then

$$||\phi(x)|| \leq ||A||(||F(P_Y x)|| + ||G(-P_Z x)||)$$
$$\leq \varepsilon \, ||x||$$
$$\leq ||x||.$$

If for $||x|| \geq \rho_o$ one of the inequalities $||P_Y x|| \geq \rho$, $||P_Z x|| \geq \rho$ is not fulfilled (for instance the first), then

$$||\phi(x)|| \leq ||A||K + N \, ||x||$$
$$\leq ||x||.$$

Hence for every $x \in S_{\rho_o}$ there is $||\phi(x)|| \leq \rho_o$. Therefore, $\phi(S_{\rho_o}) \subset K_{\rho_o}$. According to Browder's theorem, the mapping $\phi : K_{\rho_o} \to X$ has at least one point $x^* \in K_o$ such that $\phi(x^*) = x^*$.

Hence, $P_Y x^* + P_Z x^* = A(G(-P_Z x^*) - F(P_Y x^*))$

and $f(P_Y x^*) = g(-P_Z x^*)$.

This concludes the proof.

2.4. Some Further Results for Quasibounded Mappings

In this section we give some mapping theorems for quasibounded mappings by means of topological argument.

Recall that a nonlinear mapping T from a Banach space X into itself is quasibounded if there exist two constants $M > 0$ and $q_o > 0$ such that $||Tx|| \leq M||x||$ for all x in X with $||x|| \geq q_o$. (Petryshyn [40]).

We state without proof the following result, due to Petryshyn [40].

Theorem 2.4.1. Suppose that T is P-compact. Suppose further that there exist a sequence of spheres $\{S_{r_p}\}$ with $r_p \to \infty$, as $p \to \infty$, and two sequences of positive numbers $c_p = c(r_p)$ and $k_p = k(r_p)$ with $k_p \to \infty$, as $r_p \to \infty$, such that the following conditions hold:

(Λ_f) : Whenever for any given f in B_k and any n the equation $P_n Tx - \lambda x = P_n f$ holds for x in S_{r_p} with $\lambda > 0$ then $\lambda \leq c_p$.

(Π_p) : $||Tx - \eta x|| \geq k_p$ for any $\eta \geq \mu > 0$ and any x in S_{r_p}.

Then for every f in X there exists an element u in X such that

$$Tu - \mu u = f.$$

The following result is due to Petryshyn [40].

Theorem 2.4.2. Suppose that T is P-compact and quasibounded mapping of X into itself. If $\mu > M$, then $(T - \mu I)$ is onto.

Proof. Let $\{r_p\}$ be a sequence of real numbers such that $r_p \geq q_o$ for all p and such that $r_p \to \infty$, as $p \to \infty$. Then, in view of our conditions, for all x in S_{r_p} and $\eta \geq \mu$,

$$\begin{aligned} ||Tx - \eta x|| &\geq \eta||x|| - ||Tx|| \\ &\geq \mu||x|| - M||x|| \\ &= (\mu - M)||x|| \\ &= (\mu - M)r_p . \end{aligned}$$

Thus condition (Π_p) of Theorem 2.4.1 is satisfied with $k_p = (\mu - M)r_p$. Now suppose that for any f in B_{k_p} and any n the equation $P_n Tx - \lambda x = P_n f$ holds for x in S_{r_p} with $\lambda > 0$. Then by (II) before the Definition 2.3.13 and the quasiboundness of T, the latter equation implies that

$$\begin{aligned}\lambda r_p = \lambda||x|| &= ||P_n(Tx - f)|| \\ &\leq K||Tx - f|| \\ &\leq K\{||Tx|| + ||f||\} \\ &\leq K\{Mr_p + k_p\}\end{aligned}$$

Hence, $\lambda \leq \mu K$, i.e., condition (Λ_f) of Theorem 2.4.1 is satisfied with $c_p = \mu K$ for each p. Consequently, Theorem 2.4.2 follows from Theorem 2.4.1.

Remark 2.4.3. It is not hard to see that Theorem 2.4.2 remains valid if instead of assuming that $\mu > M$ we assume that $\mu > |T|$.

Petryshyn [40] also gave the following corollary:

Corollary 2.4.4. Suppose that T is quasibounded and P-compact with $p < 0$. If $\mu > M$, then $(\mu I + T)$ maps X onto itself.

Proof. The conditions of Corollary 2.4.4 imply that $\bar{T} = -T$ is quasibounded and P-compact with $\bar{p} = -p > 0$. Hence, by Theorem 2.4.2, $(\bar{T} - \mu I)$ or equivalently the operator $(\mu I + T)$ is onto.

Remark 2.4.5. When T is completely continuous and $\mu = 1$, then the Corollary 2.4.4 was proved by Granas [24].

The following surjectivity result due to Vignoli [54] will be useful in our next theorem:

Theorem 2.4.6. Let X be a Banach space, $T : X \to X$ be α-contractive with constant k $(0 \leq k < 1)$ and let there exist a sequence $\{\partial B(0,\beta_n)\}$ $(\partial B(0,\beta_n) = \{x \in X|\ ||x|| = \beta_n\})$ of spheres and a sequence $\{\nu_n\}$ of positive real numbers $\nu_n \to \infty$ as $n \to \infty$ such that for any $\lambda > \mu$ (where μ satisfies $0 \leq k < 1 - |1 - \mu|$) and any $x \in \partial B(0,\beta_n)$

$$||Tx - \lambda x|| \geq \nu_n \quad \ldots\ldots\ldots\ldots \quad (1)$$

Then the mapping $(T - \mu I)$ is surjective.

The following theorem is due to Vignoli [54]:

Theorem 2.4.7. Let $T : X \to X$ be quasibounded and α-contractive with constant k, $(0 \leq k < 1)$. If μ satisfies

(i) $0 \leq k < 1 - |1 - \mu|$,

(ii) $\mu > M$,

then the mapping $T - \mu I$ is surjective.

Proof. Let $\{\beta_n\}$ be a sequence of real numbers such that $\beta_n \geq c$ for all n, and $\beta_n \to \infty$ as $n \to \infty$. Then for all $x \in \partial B(0,\beta_n)$ and any $\lambda > \mu$

$$\begin{aligned} ||Tx - \lambda x|| &\geq \lambda||x|| - ||Tx|| \\ &> \mu||x|| - M||x|| \\ &= (\mu - M)||x|| \\ &= (\mu - M)\beta_n , \end{aligned}$$

hence by putting $\nu_n = (\mu - M)\beta_n$ condition (i) of Theorem 2.4.6 is satisfied. Therefore Theorem 2.4.7 now follows from Theorem 2.4.6.

Remark 2.4.8. In this case when T is P-compact mapping and X satisfies some special conditions, then Theorem 2.4.7 was proved by Petryshyn [40].

The following result was given by Fucik [20]:

Theorem 2.4.9. Let X be a real Banach space, $h : X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$ and $0 \leq k < 1$. The following hypotheses are fulfilled:

(I) For every $y_o \in X$ and $R > 0$ with the property $(y_o - H)(S_R) \subset K_R$ (S_R or K_R denote the sets of all x such that $||x|| = R$ or $||x|| \leq R$ respectively) there exists $x_{y_o,R} \in X$ such that $Hx_{y_o,R} = y_o - x_{y_o,R}$.

(II) H is the quasibounded operator with the constant K.

Then h is a surjective operator.

<u>Proof</u>. Let $y_o \in X$, $\varepsilon > 0$ be such that $k + \varepsilon < 1$, $\rho_1 = \frac{||y_o||}{\varepsilon}$ and $\rho_2 = \rho_o + \rho_1$ (ρ_o is from the definition of quasiboundedness of H given by Granas [ ], i.e. $|H| = \inf_{0 \leq \rho_o < \infty} \{ \sup_{||x|| \geq \rho_o} \frac{||Hx||}{||x||} \}$).

For every $x \in X$, $||x|| \geq \rho_2$ we have $\frac{||y_o||}{||x||} \leq \frac{||y_o||}{\rho_1} \leq \varepsilon$.

For $x \in S_{\rho_2}$, we obtain from the triangle inequality and the hypothesis (II),

$$\frac{||y_o - Hx||}{||x||} \leq \frac{||y_o||}{||x||} + \frac{||Hx||}{||x||} \leq \varepsilon + k < 1$$

i.e. $||y_o - Hx|| \leq ||x||$

and by hypothesis (I), there exists $x_{y_o,\rho_2} \in X$ such that

$$Hx_{y_o,\rho_2} = y_o - x_{y_o,\rho_2} \text{ , i.e., } hx_{y_o,\rho_2} = y_o \text{ .}$$

<u>Remark 2.4.10</u>. If $H = -I$ (1 denotes the identity mapping) we see that if $k = 1$, the Theorem 2.4.9 is not valid.

The following two theorems due to Fucik [20] are the consequences of Theorem 2.4.9:

<u>Theorem 2.4.11</u>. Let X be a real Banach space and $h : X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$, where H is completely continuous. Let the hypothesis (II) of Theorem 2.4.9 with $0 \leq k < 1$ be fulfilled. Then H is a surjective operator.

<u>Theorem 2.4.12</u> . Let X be a uniformly convex Banach space and H be a nonexpansive mapping on X satisfying the hypothesis (II) of Theorem 2.4.9 with $0 \leq k < 1$. Then $hx = x + Hx$ is surjective.

<u>Definition 2.4.13</u>. Let H be a real Hilbert space. We call a mapping $T : H \to H$ coercive if for all $x \in H$, $(Tx,x) \geq c(||x||)||x||$, where c is a real-valued continuous function defined on $\mathbb{R}_+$ and such that $c(t) \to \infty$ as $t \to \infty$.

We now state without proof the following result and its corollary due to Edmunds & Webb [18] which will be used in the proof of the next mapping theorem.

<u>Theorem 2.4.14</u>. Let $T : H \to H$ be densifying and suppose $I - T$ is coercive, where I is the identity map. Then $I - T$ is surjective, i.e., given any $y \in H$ there exist x in H such that $x - Tx = y$.

<u>Corollary 2.4.15</u>. Let T be a 1-set-contraction such that $(I - T)$ is coercive and maps closed balls into closed sets. Then $I - T$ is surjective.

The following result is due to Edmunds & Webb [18].

<u>Theorem 2.4.16</u>. Let H be a real Hilbert space and let $T : H \to H$ be a densifying quasibounded operator with $|T| < 1$. Then $I - T$ is surjective.

Proof. Let $\delta > 0$ be such that $|T| + \delta < 1$. Then since

$$\frac{((I - T)x, x)}{||x||} \geq (1 - |T| - \delta)\, ||x|| \;.$$

for large enough $||x||$, $I - T$ is coercive. The result now follows immediately from Theorem 2.4.14.

Edmunds & Webb [18] gave the following corollary:

Corollary 2.4.17. If T is a 1-set-contraction such that $I - T$ maps closed balls into closed sets and $|T| < 1$, then $I - T$ is surjective.

Proof. This follows easily from the Corollary 2.4.15.

Remarks 2.4.18. (i) Various special cases of Theorem 2.4.16 are known apart from the classical result with T linear and $||T|| < 1$; for example, the situation for T completely continuous was dealt by Granas [24] while Nashed & Wong [34] and Fucik [20] dealt with sums of completely continuous and contraction maps.

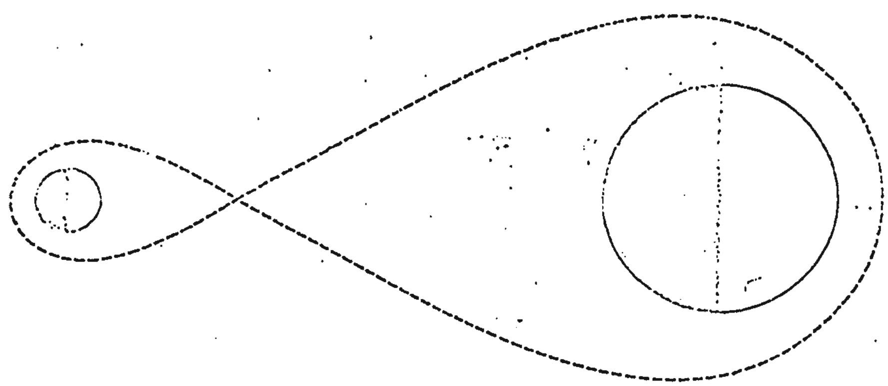
(ii) Theorem 2.4.16 is also true for k-set contractions ($k < \frac{1}{2}$) in Banach spaces. Obviously, when X is a uniformly convex space, it is enough to assume in these results that T is a k-set-contraction with $k < \frac{1}{2 - 2\delta}$.

CHAPTER III

Some Applications of Fixed Point Theorems

In this chapter we shall be discussing a few applications of some of the fixed point theorems established in the last two chapters.

The following illustration due to Shrinbot [48] will give us some intuitive idea of how fixed point theorems can be applied to various everyday situations:



Feasibility of an orbit by which a satellite would revolve around earth and moon is the type of question to which mathematicians apply fixed point theorems for infinite-dimensional surfaces. The element of time in any equation for the orbit makes the problem infinite-dimensional, reducing such simple theorems as Brouwer's theorem inapplicable.

3.1. Application of Contraction Mapping Theorem

Here we apply the principle of contraction mapping to prove the existence and uniqueness of solutions obtained by the method of successive approximations.

Example 3.1.1. Given $\frac{dy}{dx} = f(x,y)$, $y(x_o) = y_o$ . . . . (*)
where $f(x,y)$ is continuous in a plane region G containing (x_o,y_o) and satisfies a Lipschitz condition with respect to y as follows:

$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2| .$$

We shall show, by means of the principle of contraction maps, that on some closed interval $|x - x_o| < d$ there exists a unique solution $y = \phi(x)$ of equation (*) satisfying the initial condition.
This important statement is in fact known as Picard's theorem.

Equation (*) is equivalent to the integral equation

$$\phi(x) = y_o + \int_{x_o}^{x} f(t,\phi(t))dt.$$

Since $f(x,y)$ is continuous, we have $|f(x,y)| \leq k$ in some region $G' \subseteq G$ and which contains the point (x_o,y_o). Select a $d > 0$ such that

(i) $(x,y) \in G'$ if $|x - x_o| \leq d$, $|y - y_o| \leq kd$

(ii) $Md < 1$.

Let C^* be the space of continuous functions $\phi^*(x)$ which are defined on $|x - x_o| \leq d$ and are such that
$|\phi^*(x) - y_o| \leq kd$. Use the metric $d(\phi_1,\phi_2) = \sup_x|\phi_1(x) - \phi_2(x)|$ on C^*.
C^* is a complete space since it is a closed subspace of the complete space of continuous functions on a closed interval with the sup. metric.

Now consider the map $\psi = F\phi$ defined by

$$\psi(x) = y_o + \int_{x_o}^{x} f(t,\phi(t))dt \quad \text{where } |x - x_o| \leq d.$$

We claim: F is a contraction map of the complete metric space C^* into itself.

Let $\phi \in C^*$ and $|x - x_o| \leq d$. Then

$$|\psi(x) - y_o| = \left| \int_{x_o}^{x} f(t,\phi(t))dt \right| \leq kd.$$

Thus, $\psi(x) \in C^*$ or $F(C^*) \subset C^*$.

Moreover, $|\psi_1(x) - \psi_2(x)| \leq \int_{x_o}^{x} |f(t,\phi_1(t)) - f(t,\phi_2(t))| dt$

$$\leq Md \sup_x |\phi_1(x) - \phi_2(x)|$$

and so $\sup_x |\psi_1(x) - \psi_2(x)| \leq Md \sup_x |\phi_1(x) - \phi_2(x)|$

i.e., $d(F\phi_1, F\phi_2) \leq \alpha \cdot d(\phi_1,\phi_2)$,

where $\alpha = Md < 1$.

Hence, F is a contraction mapping and thus there exists a unique $\phi \in C^*$ such that $F\phi = \phi$, or the integral equation has a unique solution, or the differential equation (*) has a unique solution satisfying the given initial condition.

Example 3.1.2. By a Fredholm equation (of the second kind) is meant an integral equation of the form

$$f(x) = \lambda \int_a^b K(x,y)f(y)dy + \phi(x) \quad \dots\dots\dots\dots\dots \quad (1)$$

involving two given functions K and ϕ , an unknown function f and an arbitrary parameter λ . The function K is called the kernel of the equation, and the equation is said to be homogeneous if $\phi = 0$ (but otherwise non-homogeneous).

Suppose $K(x,y)$ and $\phi(x)$ are continuous on the square $a \leq x \leq b$, $a \leq y \leq b$, so that in particular

$$|K(x,y)| \leq M(a \leq x \leq b, \quad a \leq y \leq b).$$

Consider the mapping $g = Tf$ of the complete metric space $C_{[a,b]}$ into itself given by

$$g(x) = \lambda \int_a^b K(x,y)f(y)dy + \phi(x).$$

Clearly, if $g_1 = Tf_1$, $g_2 = Tf_2$, then

$$\begin{aligned} d(g_1,g_2) &= \max_x |g_1(x) - g_2(x)| \\ &\leq |\lambda| M(b - a) \max_x |f_1(x) - f_2(x)| \\ &= |\lambda| M(b - a) d(f_1,f_2), \end{aligned}$$

so that T is a contraction mapping if

$$|\lambda| < \frac{1}{M(b - a)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It follows from Theorem 1.1.12 that the integral equation (1) has a unique solution for any value of λ satisfying (2). The successive approximations $f_o, f_1, \ldots, f_n, \ldots$ to this solution are given by

$$f_n(x) = \lambda \int_a^b K(x,y)f_{n-1}(y)dy + \phi(x) \qquad (n = 1,2, \ldots),$$

where any function continuous on $[a,b]$ can be chosen as f_o. Note that the method of successive approximations can be applied to the equation (1) only for sufficiently small $|\lambda|$.

<u>Example 3.1.3</u>. Next consider the Volterra equation

$$f(x) = \lambda \int_a^x K(x,y)f(y)dy + \phi(x), \qquad \ldots\ldots\ldots\ldots (3)$$

which differs from the Fredholm equation (1), by having the variable x rather than the fixed number b as the upper limit of integration. It is easy to see that the method of successive approximations can be applied to the Volterra equation (3) for arbitrary λ , not just for sufficiently small $|\lambda|$ as in the case of the Fredholm equation (1).

In fact, let T be the mapping of $C_{[a,b]}$ into itself defined by

$$Tf(x) = \lambda \int_a^x K(x,y)f(y)dy + \phi(x),$$

and let $f_1, f_2 \in C_{[a,b]}$. Then

$$|Tf_1(x) - Tf_2(x)| = \lambda \int_a^x K(x,y)[f_1(y) - f_2(y)]dy$$
$$\leq \lambda M(x - a) \max_x |f_1(x) - f_2(x)| ,$$

where $M = \max_{x,y} |K(x,y)|$.

It follows that

$$|T^2f_1(x) - T^2f_2(x)| \leq \lambda^2M^2 \max_x |f_1(x) - f_2(x)| \int_a^x (x - a)dx$$
$$= \lambda^2M^2 \frac{(x - a)^2}{2} \max_x |f_1(x) - f_2(x)| ,$$

and in general

$$|T^nf_1(x) - T^nf_2(x)| \leq \lambda^nM^n \frac{(x - a)^n}{n!} \max_x |f_1(x) - f_2(x)|$$
$$\leq \lambda^nM^n \frac{(b - a)^n}{n!} \max_x |f_1(x) - f_2(x)| ,$$

which implies

$$d(T^nf_1, T^nf_2) \leq \lambda^nM^n \frac{(b - a)^n}{n!} d(f_1,f_2).$$

But given any λ , we can always choose n large enough to make

$$\lambda^nM^n \frac{(b - a)^n}{n!} < 1,$$

i.e., T^n is a contraction mapping for sufficiently large n. It follows from Theorem 1.1.15 that the integral equation (3) has a unique solution for arbitrary λ .

Remark 3.1.4. Equation (3) can be regarded formally as a special case of (1) by extending the definition of the kernel, i.e., by setting $K(x,y) = 0$ if $y > x$.

Example 3.1.5. In this example, we apply the Corollary 1.1.18 to show the existence of a unique solution to the scalar differential equation

$$\frac{dx(t)}{dt} = f(x(t),t) \quad , \quad x(a) = c \; , \quad t \in [a,b] \quad \ldots\ldots\ldots \quad (4)$$

where $f(u,t)$ is continuous in t for any $u \in \mathbb{R}$ and it satisfies the Lipschitz condition

$$|f(u_1,t) - f(u_2,t)| \leq k|u_1 - u_2| \; , \quad u_1,u_2 \in \mathbb{R} \; , \quad t \in [a,b] \quad \ldots \quad (5)$$

and b is any finite number bigger than a. The mapping $z = Fx$ defined by

$$z(t) = c + \int_a^t f(x(s),s)ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

maps $C(a,b)$ into itself. The mapping $z = Kx$ defined by

$$z(t) = e^{\lambda k(t-a)}x(t) \quad , \quad \lambda > 1, \quad t \in [a,b] \; , \quad \ldots\ldots\ldots\ldots \quad (7)$$

also maps $C(a,b)$ into itself and has an inverse defined by the function $e^{-\lambda k(t-a)}$. To show that $K^{-1}FK$ is a contraction, let $y_1,y_2 \in C(a,b)$ and then

$$||K^{-1}FKy_1 - K^{-1}FKy_2|| = \max_{t\in[a,b]} |e^{-\lambda k(t-a)} \int_a^t [f(e^{\lambda k(s-a)}y_1(s), s$$

$$- f(e^{\lambda k(s-a)}y_2(s),s)]ds|$$

$$\leq \max_{t\in[a,b]} e^{-\lambda k(t-a)} \int_a^t ke^{\lambda k(s-a)}|y_1(s) - y_2(s)|ds$$

$$\leq \max_{t\in[a,b]} \lambda^{-1}(1 - e^{-\lambda k(t-a)})||y_1 - y_2||$$

$$\leq \lambda^{-1}||y_1 - y_2|| \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Since $\lambda > 1$, $K^{-1}FK$ is a contraction.

Example 3.1.6. Let us consider the integro-differential equations, with the following form in the simplest case:

$$\dot{x}(t) = \int_0^\infty K(t,s,x(t - s))ds \,, \quad \ldots\ldots\ldots\ldots\ldots \quad (**)$$

or in the integral form:

$$x(t) = \phi(t_o) + \int_{t_o}^{t} dt \int_0^\infty K(t,s,x(t - s))ds$$

with the initial conditions $x(t - s) = \phi(t - s)$ for $t - s \leq t_o$, where the continuous function ϕ can differ from zero only on a finite segment.

If in equation (**) the functions f and K are continuous, $\int_0^\infty K(t,s,0)ds$ converges uniformly for $t_o \leq t \leq t_o + h_1$ where $h_1 > 0$, and K satisfies a Lipschitz condition in the third argument, then there exists a unique continuous solution of (**) for $t_o \leq t \leq t_o + h$, where h is sufficiently small, and this solution may be found by the method of successive approximations.

Contraction mapping principle is used in the proof of the above which can be found in Èlśgolě [19].

Next, we shall apply the contraction mapping theorem to the calculation of the inverse of a bounded linear operator. In this connection it may be noted that if X is a Banach space then the space of bounded linear operators $B(X,X)$ is also a Banach space.

Theorem 3.1.7. Let L be a map in $B(X,X)$ that has a bounded linear inverse in $B(X,X)$. Then for every $M \in B(X,X)$ such that $||L - M|| \leq 1/||L^{-1}||$, M has a bounded linear inverse M^{-1} in $B(X,X)$, and $||L^{-1} - M^{-1}|| \leq (||L - M||\ ||L^{-1}||^2)/(1 - ||L - M||\ ||L^{-1}||)$.

Proof. Put $A = M - L$. Then $N = L^{-1} + B$ is an inverse for $M \iff MN = NM = I$. We can write the equation $MN = I$ as $(L + A)(L^{-1} + B) = I$, or $LL^{-1} + LB + AL^{-1} + AB = I$, or $LB = -AB - AL^{-1}$, or $B = -L^{-1}AB - L^{-1}AL^{-1}$. If we put $f(B) = -L^{-1}AB - L^{-1}AL^{-1}$, this equation has the form $B = f(B)$. Since $||f(B) - f(B')|| = ||-L^{-1}A(B - B')|| \leq ||L^{-1}||\ ||A||\ ||B - B'||$, and $||L^{-1}||\ ||A|| = ||L^{-1}||\ ||L - M|| < 1$, f is a contraction on $B(X,X)$. It has a unique fixed point B. Then $(L + A)(L^{-1} + B) = MN = I$. Similarly, the equation $N'M = I$ can be expressed as $B' = -B'AL^{-1} - L^{-1}AL^{-1}$ if we put $N' = L^{-1} + B'$. By the same argument, it has a solution as well. Now $N' = N'I = N'(MN) = (N'M)N = IN = N$. Therefore $N = N'$ is the unique inverse of M. Moreover, the contraction mapping theorem tells us that

$$||M^{-1} - L^{-1}|| = ||N - L^{-1}|| = ||B|| = ||B - 0|| \leq \frac{1}{1 - ||A||\ ||L^{-1}||}\ ||f(0) - 0||$$

$$= \frac{||L^{-1}AL^{-1}||}{1 - ||A||\ ||L^{-1}||} \leq \frac{||L^{-1}||^2\ ||A||}{1 - ||A||\ ||L^{-1}||}.$$

3.2 Application of Schauder's Theorem and Monotonically Decomposible Operators

It may be noted that for applying Brouwer's Theorem (Theorem 1.2.19) and Schauder's Theorem (Theorem 1.2.21, and 1.2.22), no information is needed concerning the norms, distances, Lipschitz constants, etc. These theorems have been used very often, perhaps the Schauder theorem is one of the most important theorems for the numerical treatment of equations occuring in analysis.

Examples are given for finite systems of linear algebraic equations (Schröder [47]), for nonlinear vibrations (Reissig [43]), etc. Let us give a simple example.

Example 3.2.1. Consider the vector-differential equation

$$x'(t) = A(t)x(t) + g(t,x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

with a given matrix $A(t)$ and a function g satisfying

$$|g(t,x) - g(t,0)| \leq s(t)q(|x|) \ ,$$

where $s(t)$ is a continuous function of period p and $q(x)$ is a continuous, monotone, non-decreasing, bounded function with $q(0) = 0$.

Then the Brouwer's fixed point theorem for a sufficiently large sphere gives the existence of a periodic solution of (9) (Reissig [43]).

3.2.2. Monotonically decomposible operators: The operator T mapping the domain D of a partially ordered space R_1 into a partially ordered space R_2 is called syntone, if $v \leq w$ for all $v,w \in D$ implies $Tv \leq Tw$ and is called antitone, if from $v \leq w$, it follows that $Tv \geq Tw$ for all $v,w \in D$.

The following theorem is due to Collatz [12]:

Theorem 3.2.3. In the equation

$$u = Tu + r = \bar{T}u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

in a partially ordered Banach space R we suppose, that T has the form

$T_1 + T_2$ where T_1 is syntone and T_2 is antitone, and that T_1, T_2 are continuous and defined in a convex domain D of R. Let the iteration procedure

$$\left.\begin{aligned} v_{n+1} &= T_1 v_n + T_2 w_n + r \\ w_{n+1} &= T_1 w_n + T_2 v_n + r \\ (n &= 0,1,2, \ldots.) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11)$$

start with elements $v_o, w_o \in D$ with

$$v_o \leq v_1 \leq w_1 \leq w_o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

If $\bar{T}$ maps the interval $M_n = [v_n, w_n]$ for some $n \geq 0$ into a relatively compact set, then there exists at least one element $u \in M_n$ with $u = \bar{T}u$.

It is quite elementary that $\bar{T}$ maps M_n into itself; (M_n is the set of all elements z with $v_n \leq z \leq w_n$) one has

$$M_n \supset M_{n+1} \supset \bar{T}M_n \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$
$$(n = 0,1,2, \ldots.)$$

The existence of a fixpoint is given by Schauder's theorem. The condition of compactness is often satisfied for integral operators.

<u>Example 3.2.4. (Basic Example)</u>: Let $A = (a_{jk})$ $(j,k = 1, \ldots, n)$ be a matrix with real elements a_{jk}. In the case $a_{jk} \geq 0$ is A syntone, for $a_{jk} \leq 0$ is A antitone; but every real matrix A is monotonically decomposible: $A = A_1 + A_2$ with A_1 syntone, A_2 antitone. Let $\bar{A} = (|a_{jk}|)$ be the matrix of the absolute modules of elements a_{jk}, then one can choose $2A_1 = A + \bar{A}$, $2A_2 = A - \bar{A}$.

In the same way, every real kernel $K(x,t)$ can be written as $K(x,t) = K_1(x,t) + K_2(x,t)$ with $K_1 \geq 0$, $K_2 \leq 0$. Then the operator

$$T_j u = \int_B K_j(x,t)u(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

is syntone for $j = 1$ and antitone for $j = 2$.

Every Hammerstein-operator of the form

$$f(x) + \int_B K(x,t)\phi(u(t))dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

with real K and a function ϕ of bounded variation is monotonically decomposible. $\phi(z)$ can be written as $\phi(z) = \phi_1(z) + \phi_2(z)$ with monotone, non-decreasing ϕ_1 and monotone, non-increasing ϕ_2.

Then the operator

$$\bar{T}u = \int_B K(x,t)\phi(u(t))dt = T_1 u + T_2 u \quad \ldots\ldots\ldots\ldots \quad (16)$$

with $\quad T_1 u = \int_B [K_1\phi_1(u) + K_2\phi_2(u)]dt$

and $\quad T_2 u = \int_B [K_1\phi_2(u) + K_2\phi_1(u)]dt$

is monotonically decomposible.

Example 3.2.5. (Numerical Example): Consider the problem of solving the Hammerstein-Equation

$$u(x) = 1 + \int_0^1 |x - t|[u(t) - \frac{1}{2}u^2(t)]dt \quad \ldots\ldots\ldots\ldots \quad (17)$$

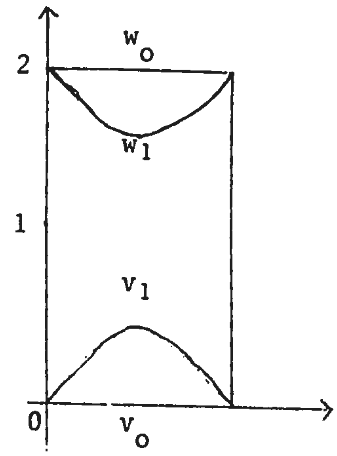


Fig. 1

In the range $u(x) \geq 0$,

$T_1 v = \int_0^1 |x - t| v(t)dt$ is syntone, and

$T_2 v = -\frac{1}{2} \int_0^1 |x - t| v^2(t)dt$ is antitone.

Here $v_o = 0$, $w_o = 2$ give

with (11), $v_1 = 1 - \int_0^1 |x - t| \frac{1}{2} \cdot 4dt = 2(x - x^2)$, $w_1 = 2(1 - x + x^2)$

and (12) is satisfied, see Fig. 1, and a solution $u(x)$ exists with $v_1 \leq u \leq w_1$.

Example 3.2.6. The following example shows the direct applicability of Schauder's theorem. In the determination of the stationary temperature distribution in the presence of chemical reactions, we are faced with the boundary value problem $(r^2 = x^2 + y^2)$:

$$- \nabla^2 u = r^2 + ae^u \quad \text{on} \quad B(r < 1) \quad \ldots\ldots\ldots\ldots \quad (18)$$
$$u = 1 \quad \text{on the boundary } \partial B \text{ of } B(r = 1)$$

We consider first functions u_o and u_1 which satisfy

$$-\nabla^2 u_1 = r^2 + a \exp(x_o) \text{ on } B \quad , \quad u_o = u_1 = 1 \text{ on } \partial B.$$

The trial solution

$$u_1 = 1 + b(1 - r^2) + c(1 - r^4)$$

yields

$$a \exp(u_o) = 4b + (16c - 1)r^2 .$$

The condition $u_o = 1$ for $r = 1$ requires that $c = \frac{1}{16}(1 + ae) - \frac{1}{4} b$.

It turns out that (this was done by computer)

$u_o(r;\ 0.54) = v_o$, $u_1(r;\ 0.54) = v_1$, $u_o(r;\ 0.64) = w_o$, $u_1(r;\ 0.64) = w_1$

satisfy the condition (12), see Fig. 2. Thus the inclusion $v_1 \leq u \leq w_1$ holds; the greatest deviation between v_1 and w_1 occurs at the origin where we find

$$v_1 = 1.552,446 \leq u \leq w_1 = 1.627,446.$$

Cases where the contraction theorem fails. The application of Banach contraction principle is sometimes met with difficulties, since a suitable Lipschitz constant has to be found; there are also cases where suitable starting elements of Schauder's theorem are difficult to find. Examples are easily given where the contraction principle theorem does not apply at all, but Schauder's theorem does; for example, the boundary value problem

$$y'' = f(x,y) \quad , \quad y(0) = y_o \quad , \quad y(1) = y_1 \quad . . . . \quad (19)$$

can be written in the form $y = Ty$ by introducing a Green's function $G(x,s)$

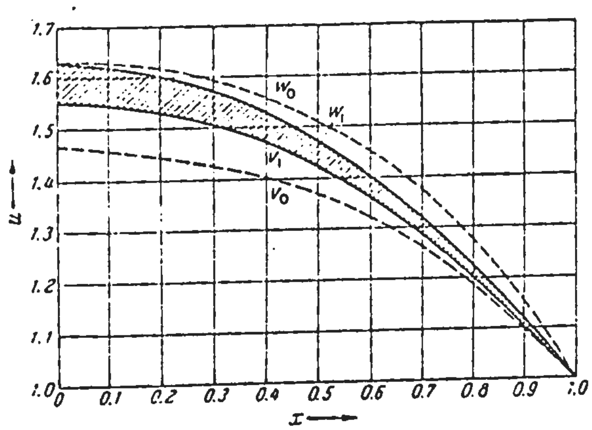


Fig. 2.

$$Tz(x) = g(x) + \int_0^1 G(x,s)f(s,z(s))ds \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

where $g(x)$ is a given function. If $f(x,z)$ for fixed x, is monotone in z, but $\partial f/\partial z$ is unbounded, then there is no finite Lipschitz constant and the contraction theorem is not applicable whereas Schauder's theorem may be. A simple example for this case is the following problem:

$$-y'' = x + \sqrt{y} \quad , \quad y(0) = 0 \quad , \quad y(1) = 1 \quad \ldots\ldots\ldots\ldots \quad (21)$$

It is easy to find functions v_o, v_1, w_o, w_1 which satisfy condition (12) and all of which satisfy the boundary conditions and

$$-v_1'' = x + \sqrt{v_o} \quad , \quad -w_1'' = x + \sqrt{w_o}$$

(the positive square root not to be taken); for example,

$$v_o = x^2 \ , \ v_1 = \frac{4}{3}x - \frac{x^3}{3} \quad , \quad w_1 = \frac{23}{15}x - \frac{8}{15}x^{5/2} \quad , \quad w_o = (2\sqrt{x} - x)^2 \ .$$

Since all the assumptions of Schauder's theorem are satisfied by these functions, the following inequality holds for some solution of (21):

$$v_1 \le y(x) \le w_1 \ .$$

3.3. Application of Densifying Mappings

In this section we study the question of the existence of the solutions of the following equation of neutral type:

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))]. \quad \ldots\ldots \quad (22)$$

If the function $f(t,x,y,z)$ satisfies a Lipschitz condition in the variables x,y and z, with constants k_x, k_y and k_z, respectively, with $k_z < 1$, then, under minor additional assumptions, the question of the existence of the solution is easily reduced to the contraction mapping principle.

Here we shall dispense with the Lipschitz condition in the variable x and y. To prove the existence theorem in this case we shall apply the fixed point principle for densifying operators due to Furi & Vignoli [22] (see Theorem 1.2.48).

We shall consider (22) in conjunction with the initial condition

$$x(t) = x_o(t) \quad (-h < t \leq 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

where $x_o(t)$ is a fixed function defined on the (finite or infinite) semi-interval $(-h,0]$. By a solution of the problem (22)-(23) we shall mean a function $x(t)$ $(-h < t \leq H]$ that satisfies the initial condition (23) and the following three requirements:

(a) $x(t)$ is continuous on $(-h,H]$;

(b) $x'(t)$ exists almost everywhere on $(-h,H]$ and is p^{th} power integrable, $p \geq 1$;

(c) almost everywhere on $[0,H]$

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))].$$

We shall denote by $E(0,H)$ the set of continuous functions on $[0,H]$ having a derivative that is p^{th} power integrable; this set becomes a Banach space with the natural linear operations if we put $||x||_E = ||x||_C + ||x'||_{\alpha_p}$, (see [1]). For any function $x(t) \in X(0,H)$, where X is a Banach space, we put

$$\tilde{x}(t) = \begin{cases} x_o(t) , & -h < t < 0 \\ x(t) , & 0 \leq t \leq H. \end{cases}$$

Together with the problem (22)-(23) we consider the following operator equation in the space $E(0,H)$:

$$y = Iy \;, \qquad \ldots\ldots\ldots\ldots\ldots \qquad (24)$$

where the operator I is defined by the formula

$$Iy(t) = x_o + \int_0^t f[s, y(s) \;,\; \tilde{y}(s - h_1(s)) \;,\; \tilde{y}'(s - h_2(s))]ds$$

$$(x_o = x_o(0)) \;.$$

It is not difficult to verify that if the function $x_o(t)$ is continuous and its derivative is p^{th} power integrable, then the equation (24) is equivalent to the problem (22)-(23) in the following sense: if $x(t)$ is a solution of the problem (22)-(23) then its restriction $y(t)$ to the segment $[0,H]$ is a solution of the equation (24) and, conversely if $y(t)$ is a solution of the equation (24) then the function $x(t) = \tilde{y}(t)$ is a solution of the problem (22)-(23).

The following lemma due to Badoev & Sadovskii [1] gives some properties of the operator I.

Lemma 3.3.1. Let E_o be the set of functions in $E(0,H)$ that satisfy the condition $x(0) = x_o$. Suppose that the functions $x_o(t)$, $h_1(t)$ and $f(t,x,y,z)$ satisfy the following requirements:

(I) $x_o(t)$ is continuous and bounded and $x_o'(t)$ is p^{th} power integrable on $(-H,0]$;

(II) $-H + t \leq h_i(t) < h + t$ $(i = 1,2 \;;\; 0 \leq t \leq H)$;

(III) $h_1(t)$ and $h_2(t)$ are measurable on $[0,H]$:

(IV) the function $q(t) = t - h_2(t)$ is such that

(a) the inverse image of every set of measure zero is measurable,

(b) for any measurable subset $M \subseteq [0,H]$ satisfy the condition $q(M) \subseteq [0,H]$, we have the inequality $\mu M \leq r\mu q(M)$ (where the number r does not depend on M);

(V) $f(t,x,y,z)$ is defined for $0 \leq t \leq H$ and all real x,y and z;

(VI) $f(t,x,y,z)$ is measurable in t for any fixed x,y and z;

(VII) $f(t,x,y,z)$ is measurable in the pair x,y for fixed t and z;

(VIII) $f(t,x,y,z)$ satisfies a Lipschitz condition in z :

$$|f(t,x,y,z_1) - f(t,x,y,z_2)| \leq k|z_1 - z_2| ;$$

(IX) for any $R > 0$ we can find a function $m_R(T) \in L_p(0,H)$ such that

$$|f(t,x,y,z)| \leq m_R(t)$$

$$(0 \leq t \leq H ; |x - x_o| , |y - x_o| \leq R ; -\infty < z < \infty).$$

Then the operator I is continuous from E_o into E_o .

Our next task is the construction of the set K that appears in Theorem 1.2.48. We fix $R > 0$ (so that, when $-h < t \leq 0$, we have the inequality $|x_o(t) - x_o| \leq R$) and let K_1 denote the set of functions in E_o that satisfy the inequality $||x - x_o||_E \leq R$. We choose $H > 0$ to be so small that the following inequality holds:

$$\int_0^H m_R(t)dt + ||m_R(t)||_{L_p} \leq R.$$

Then $IK_1 \subseteq K_1$. Indeed,

$$||Ix - x_o||_E = ||Ix - x_o||_C + ||(Ix)'||_{L_p}$$

$$\leq \int_0^H |f[s,x(s), \tilde{x}(s - h_1(s)), \tilde{x}'(s - h_2(s))] ds + ||m_R(t)||_{L_p}$$

$$\leq R.$$

The set K_1 is convex closed and bounded, but the operator I is not densifying on this set, in general. We put $K = \overline{co}\, IK_1$, where $\overline{co}$ denotes the convex closure. It is not difficult to see that the set K is also convex, closed and bounded, with $IK \subseteq K$.

The following lemma is due to Badoev & Sadovskii [1]:

Lemma 3.3.2. Suppose that the conditions of Lemma 3.3.1 are satisfied. Suppose in addition that the following condition is satisfied:

$$(*) \; . \; . \; . \; . \quad kr^{\frac{1}{p}} < \begin{cases} 1, & \text{if } p > 1, \\ \frac{1}{2}, & \text{if } p = 1. \end{cases}$$

Then the operator I is densifying on K, if H is sufficiently small.

From Lemma 3.3.1 and 3.3.2 and Theorem 1.2.48 we obtain the following result due to Badoev & Sadovskii [1] on the solvability of the problem (22)-(23).

Theorem 3.3.3. Let the functions $x_o(t)$, $h_1(t)$, $h_2(t)$ and $f(t,x,y,z)$ satisfy the conditions (I)-(IX) and (*). Then the problem (22)-(23) has a solution $x(t)$ that is defined on some semi-interval $(-h,H]$ $(H > 0)$.

3.4. Application of Quasibounded Mappings

We consider nonlinear integral equation of the form

$$x(t) = w(t) + \mu \int_a^t F(t,s,x(s))ds + \lambda \int_a^b G(t,s,x(s))ds \quad . \; . \; . \; . \; . \; . \; . \quad (25)$$

where λ and μ are real parameters, a and b are finite numbers, and the functions F, G and w satisfy certain conditions to be specified later. Clearly, the usual nonlinear Volterra equation and the Urysohn equation are special cases of (25). Integral equations of the form (25) arise in a number of problems in ordinary differential equations, in particular, certain classes of singular perturbation problems. One can consider equation (25) in the operator form $x = Tx$. However, it turns out that the applicability of the method of successive approximations directly to (25) as well as general existence theorems for fixed points of T is rather limited. See Willet [55] for an example and further discussions of this point.

The purpose of this section is to obtain some existence theorems for integral equations of the Volterra-Hammerstein and the Volterra-Urysohn type in the function spaces $L_2[a,b]$ and $c[a,b]$ using the fixed point theorems established in Section 2.1. Similar results can be obtained in other function spaces, e.g. L_p, imposing appropriate conditions which ensure the complete continuity, quasiboundedness and other properties of the operators under consideration. For the sake of simplicity, we confine ourselves to conditions which seem useful in practice and do not attempt to impose the weakest possible ones. The extension of our results to system of equations, to integral operators where the integration is over some measurable subset in $\mathbb{R}^n$, as well as to the case when x is a function of t with values in some Banach space requires minor modification.

3.4.1. An Existence Theorem for the Volterra-Hammerstein Equation in $L_2[a,b]$.

We consider here a special case of (25) in the following form:

$$x(t) = w(t) + \mu \int_a^t V(t,s)x(s)ds + \lambda \int_a^b k(t,s)g(s,x(s))ds \quad \ldots\ldots \quad (26)$$

where we assume the following conditions on V, k and g:

(A1) $k(t,s)$ is measurable in both t and s and satisfies

$$||k||^2 = \int_a^b \int_a^b k^2(t,s)dtds < \infty \quad ,$$

(A2) $g(s,u)$ is continuous in u, measurable in s, and satisfies

$$|g(s,u) - u| \leq \sum_{i=1}^{n} g_i(s)|u|^{1-\beta_i} + g_o(s)$$

for $a \leq s \leq b$, and $-\infty < u < \infty$, where $g_o \in L_2$ and $g_i \in L_{2/\beta_i}$, $0 \leq \beta_i < 1$, for $i = 1,2, \ldots, n$.

(A3) $V(t,s)$ is measurable in both t and s, and satisfies

$$||V||^2 = \int_a^b \int_a^t V^2(t,s)dsdt < \infty \quad . \ldots\ldots\ldots\ldots \quad (27)$$

The following result is due to Nashed & Wong [34]:

<u>Theorem 3.4.2.</u> Under the assumption (A1)-(A3), the integral equation (26) has a solution $x \in L_2[a,b]$ for each $w \in L_2$ and each pair of real numbers μ and λ with $|\lambda| < \frac{1}{||k||}$.

<u>Proof.</u> Define the operators V,H,T in the space $w_2[a,b]$ as follows:

$$Vx = \int_a^t V(t,s)x(s)ds \ ,$$

$$Hx = \int_a^b k(t,s)g(s,x(s))ds \ ,$$

$$Tx = w + \mu Vx.$$

It is well known under assumption (A3) that for each μ, the Volterra equation $x = \mu Vx$ has only the trivial solution. Using this fact and the Fredholm theory, it may then be shown that the Neumann series fir the inhomogeneous Volterra equation $x = w + \mu Vx$ is convergent in the mean for any μ and for each $w \in L_2$ if and only if the kernel $V(t,x)$ satisfies assumption (A3). Since V is bounded, it follows from the theory of resolvents that the radius of convergence of the Neumann series is equal to $\left(\lim_{n\to\infty} ||V^n||^{\frac{1}{n}}\right)^{-1}$. Thus, in this case we have

$$\lim_{n\to\infty} ||V^n||^{\frac{1}{n}} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28).$$

Denote $Nu(t) = g(t,u(t))$ and $Ku(t) = \int_a^b k(t,s)u(s)ds$;

thus $H = KN$. Under assumptions (A1) and (A2), it is known that N is continuous and bounded and K is completely continuous on L_2, from which it follows that H is completely continuous.

From (A2), we obtain

$$||Hx - Kx|| = \left\{\int_a^b \left(\int_a^b k(t,s)(g(s,x(s)) - x(s))ds\right)^2 dt\right\}^{\frac{1}{2}}$$

$$\leq ||k|| \left\{\int_a^b \left(\sum_{i=1}^{n} g_i(s)|x(s)|^{1-\beta_i} + g_o(s)\right)^2 ds\right\}^{\frac{1}{2}}$$

$$\leq ||k|| \left\{\sum_{i=1}^{n} \left(\int_a^b g_i^{2/\beta_i}(s)ds\right)^{\beta_i} ||x||^{1-\beta_i} + ||g_o||\right\},$$

here $|| \; ||$ denotes the L_2 norm unless otherwise specified. Thus the operator H is quasibounded with quasinorm $|H| = ||K||$. For given λ with $|\lambda| < \frac{1}{|H|}$, we may choose n so large in (28) such that

$$||V^n|| < \frac{1}{2}(1 - |\lambda| \; |H|).$$

Now that the assumptions of Theorem 2.1.8 are all realized, the conclusion follows immediately.

Remark 3.4.3. Consider (25) with $G(t,s,x(s)) = 0$ and $F(t,s,x(s))$ satisfies the following condition:

$$|F(t,s,x(s)) - F(t,s,y(s))| \leq V(t,x)|x(s) - y(s)|$$

for all t,s and x in their respective domains of definition. If we assume that the Lipschitz constant $V(t,s)$ satisfies instead of (27) the following stronger condition:

$$\sup_{a\leq t\leq b} \int_a^t |V(t,s)|^2 ds = M^2 < \infty ,$$

then a simple induction yields

$$||V^n u - V^n v|| \leq |\mu|^n M^n \left(\frac{(b - a)^n}{n!}\right)^{\frac{1}{2}} ||u - v|| ,$$

where V is the operator defined by the right-hand side of (25). Consequently, the operator V^n for sufficiently large n becomes a contraction and the existence of a unique solution of (25) in this case follows from the classical contraction mapping principle.

3.4.4. An Existence Theorem for the Volterra-Urysohn Equation in the Space c[a,b].

Here we consider the existence of solution of the nonlinear Volterra-Urysohn equation (25) under the following assumptions:

(B1) $F(t,s,x(s))$ satisfies the Lipschitz condition:

$$|F(t,s,x(s)) - F(t,s,y(s))| \leq \alpha(s)|x(s) - y(s)| \quad \ldots\ldots \quad (29)$$

for some integrable function α on $[a,b]$.

(B2) $G(t,s,x(s))$ is such that the Urysohn operator U defined by

$$Ux(t) = \int_a^b G(t,s,x(s))ds,$$

maps the space $C[a,b]$ into itself, is completely continuous and quasi-bounded (see Remark 3.4.6).

The following theorem is due to Nashed & Wong [34]:

Theorem 3.4.5. Under the assumptions (B1) and (B2), if for some positive number $\eta > |\mu|$,

$$|\lambda| \; |U| \exp\left[\eta \int_a^b \alpha(s)ds\right] < 1 - \frac{|\mu|}{\eta} \quad , \quad \ldots\ldots\ldots \quad (30)$$

then equation (25) has a solution in $C[a,b]$.

Proof. Define a new norm on $C[a,b]$ by

$$N(x) = \sup_{a\leq t\leq b} \left\{ \exp\left[-\eta \int_a^t \alpha(s)ds\right] |x(t)| \right\} , \quad \ldots\ldots \quad (31)$$

where η is a positive number, $\eta > |\mu|$ and satisfies (30). Clearly the norm $N(x)$ is equivalent to the sup norm $||x|| = \sup_{a\leq t\leq b} |x(t)|$.

Let $Tx = w + \mu \int_a^t F(t,s,x(s))ds$. Using the definition of T, (29) and (31) we obtain the following estimate:

$$|Tx(t) - Ty(t)| \leq \frac{|\mu|}{\eta} \int_a^t \eta\alpha(s) \left(|x(s) - y(s)| \exp\left[-\eta\int_o^s \alpha(\tau)\right] d\tau\right) \exp\left[\eta \int_o^s \alpha(\tau)d\tau\right] ds$$

$$\leq \frac{|\mu|}{\eta} N(x-y) \left\{ \exp\left[\eta \int_a^t \alpha(\tau)d\tau\right] - 1 \right\} ,$$

from which we have

$$N(Tx - Ty) \leq \frac{|\mu|}{\eta} N(x - y),$$

i.e., T is a contraction with respect to the new norm $N(x)$. We complete the proof by applying Theorem 2.1.6 to the space $C[a,b]$ with the norm $N(x)$. From (31) and Definition 2.1.1, we have

$$|U|_N = \inf_{0<\rho<\infty} \sup_{||x||\geq\rho} \frac{N(Ux)}{N(x)}$$

$$\leq |U| \exp\left[\eta \int_a^b \alpha(s)ds\right] .$$

Since λ,μ,η satisfy (30), so $|\lambda U|_N < 1 - \frac{|\mu|}{\eta}$; and the theorem follows immediately.

Remark 3.4.6. Conditions guaranteeing complete continuity of the Urysohn and Hammerstein operators in L_2 and $C[a,b]$ may be found in Krasnoselskii [30], [31].

Remark 3.4.7. Theorems 2.2.13 and 2.2.17 can be applied to obtain existence theorems for mixed nonlinear integral equations of Urysohn-Volterra and Hammerstein-Volterra types in locally convex topological vector spaces in the similar way as done in this section.

3.5. Application to Stability of Fixed Points and Solutions of Nonlinear Operator Equations

In this section we study the stability of fixed points and solutions where we apply some fixed point theorems already proved in Section 2.2.

All throughout this section X denotes a Hausdorff locally convex topological vector space and P be a family of seminorms that defines the topology of X.

Let K be a collection of continuous maps on X whose domains are such that if $A_o \in K$, $x_o \in$ domain of A_o, then $S_p(x_o,r) \subset$ domain of A_o for r sufficiently small. Let T be a topology on K. Suppose $A_o \in K$, $y_o \in X$ and $A_o x_o = y_o$.

Definition 3.5.1. The solution x_o of $A_o u = y_o$ is called p-stable with respect to (K,T) if for each $r > 0$ there exist $d > 0$ and a neighbourhood Ω of A_o such that for all $y \in S_p(y_o,d)$ and $A \in \Omega$, there exists an $x \in S_p(x_o,r)$ such that $Ax = y$. The solution x_o is said to be a stable solution with respect to (K,T) if it is p-stable solution for every $p \in P$.

For $A \in K$, (x_o,A,r) will be called a p-admissible triple if $\bar{S}_p(x_o,r)$ is contained in the domain of A.

Let K_p be the class of all continuous maps B from open subsets of X into X which are such that $I - B$ is p-completely continuous. If (x_o,B_o,r) is a p-admissible triple and $b > 0$, then $\Omega_U(x_o,B_o,r,p,b)$ will denote the collection of all $b \in K_p$ such that (x_o,B,r) is a p-admissible triple and $p(Bx - B_o x) \leq b$ for all $x \in \bar{S}_p(x_o,r)$. Let T_p be the topology on K_p generated by taking the collection of all such Ω_U as a subbase.

Now define

$$\tilde{R}_p(x_o,T,r) = r^{-1}\sup\{p(Tx - Tx_o) \mid p(x - x_o) = r\}$$

and $\eta_p(x_o,T) = \inf\{r \mid \tilde{R}_p(x_o,T,r) < 1\}$.

Note that stability for the class K can be reduced to consideration of equations of the form $A_o x = \theta$.

The above definitions are due to Cain & Nashed [8]. The following result is also due to Cain & Nashed [8].

<u>Theorem 3.5.2</u>. Let $B_o \in K_p$ and suppose $B_o x_o = \theta$. If $\eta_p(x_o, I - B_o) = 0$, then x_o is a p-stable solution of $B_o x = \theta$ with respect to (K_p, T_p).

<u>Proof</u>. Let $e > 0$ be given. There is an r, $0 < r < e$, such that $R = \tilde{R}_p(x_o, I - B_o, r) < 1$. Let a and d be positive numbers such that $a + d < (1 - R)r$. Let $B \in \Omega_U(x_o, B_o, r, p, a)$ and $y \in S_p(\theta, d)$. Consider the mapping F on $\bar{S}_p(x_o, r)$ defined by $Fx = x - Bx + y$.

Clearly F is p-completely continuous since $B \in K_p$. If F maps $\partial S_p(x_o, r)$ into $\bar{S}_p(x_o, r)$, then by Theorem 2.2.7 it has a fixed point $\bar{x} \in \bar{S}_p(x_o, r)$. Then $B\bar{x} = y$, with $\bar{x} \in \bar{S}_p(x_o, r) \subset S_p(x_o, e)$, which proves the theorem. Now we show that F indeed maps $\partial S_p(x_o, r)$ into $\bar{S}_p(x_o, r)$:

$$p(Fx - Fx_o) \leq p(x - B_o x - x_o) + p(Bx - B_o x) + p(y)$$

and $p(x - B_o x - x_o) \leq \tilde{R}_p(x_o, I - B_o, r)r = Rr$.

Hence $p(Fx - x_o) \leq Rr + a + d \leq Rr + r - Rr = r$.

If K_C is the class of all continuous operators B from open subsets of X into X which are such that $(I - B)$ is completely continuous, and if T_C is the topology of K_C generated by taking as a subbase the sets $\Omega_U(x_o, B_o, r, p, b)$ for all $p \in P$, then we have the following result due to Cain & Nashed [8]:

<u>Corollary 3.5.3</u>. If $B_o \in K_C$ and $B_o x_o = \theta$, and if $\eta_p(x_o, I - B_o) = 0$ for every $p \in P$, then x_o is a stable solution of $B_o x = \theta$ with respect to (K_C, T_C).

We next turn our attention to the question of stability of sums of operators.

If $x_o \in X$, A_o is a continuous operator, and $U \in \mathcal{U}$, then we shall say (x_o, A_o, U) is an admissible triple if $x_o + \bar{U} \subset$ domain A_o. (Recall th-: $\mathcal{U}$ is the neighbourhood system of the origin obtained from P). Let C_1 be the collection of all continuous operators A which are such that $(I - A)$ is a p-contraction for every $p \in P$. (Hereafter called simply a contraction). For A_o in C_1, $p \in P$, a and b real numbers, and (x_o, A_o, U) an admissible triple, we define $\Omega_1(x_o, A_o, U, p, a, b)$ to be the collection of all A in C_1 such that

(i) (x_o, A, U) is an admissible triple,

(ii) $p((A - A_o)x - (A - A_o)x_o) \leq bp(x - x_o)$ for all $x \in x_o + \bar{U}$,

(iii) $p(Ax_o - A_o x_o) \leq a$.

We define T_1 to be the topology on C_1 obtained by taking all such Ω_1 as a subbase.

Let C_2 be the collection of all continuous operators B which are such that $(I - B)$ has its range contained in a compact set. For $B_o \in C_2$, $p \in P$, r a real number, (x_o, B_o, U) an admissible triple, we define $\Omega_2(x_o, B_o, U, p, r)$ to be the collection of all $B \in C_2$ such that

(i) (x_o, B, U) is an admissible triple, and

(ii) $p(Bx - Bx_o) \leq r$ for all $x \in x_o + \bar{U}$.

We define T_2 to be the topology on C_2 with all such Ω_2 as a subbase.

Next let $C = C_1 \times C_2$ be the cartesian product of C_1 and C_2 endowed with the product topology $T = T_1 \times T_2$. Suppose K_o is an operator such that $I - K_o = S_o + T_o$ for $(I - S_o, I - T_o)$ in C.

Our next definition is also due to Cain & Nashed [8].

Definition 3.5.4. The solution x_o of $K_o u = y_o$ is called stable with respect to (C,T) if for each $U \in \mathcal{U}$, there is a neighbourhood Ω of $(I - S_o, I - T_o)$ and a $W \in \mathcal{U}$ such that for all $y \in y_o + W$ and $(I - S, I - T) \in \Omega$, there exists an $x \in x_o + U$ so that $Kx = y$, where $I - K = S + T$.

Recall the definition of $R_p(x_o,T_o,r)$ and $Q_p(x_o,T_o,a)$ from Definition 2.2.4. For $p \in P$ define

$$\alpha_p(x_o,T_o) = \inf \{a | 0 \in \overline{Q_p(x_o,T_o,a)}\} .$$

The following result is also due to Cain & Nashed[8]:

Theorem 3.5.5. Let X be complete. Suppose $K_o x_o = y_o$, where $I - K_o = S_o + T_o$ for $(I - S_o, I - T_o)$ in C. If $\nu_p + \alpha_p < 1$ for every $p \in P$, then x_o is a stable solution with respect to (C,T). (ν_p is p-contraction constant of S_o and $\alpha_p \equiv \alpha_p(x_o,T_o)$).

Proof. Once again we shall, without loss of generality, take $y_o = \theta$. Let $U = \bigcap_1^n r_i V(p_i) \in \mathcal{U}$ be given. (ref. the observation made after the Definition 2.2.1). For each $i = 1,2, \ldots, n$, there is a $\zeta_i > 0$ such that $\zeta_i + \nu_i < 1$ and $0 \in \overline{Q_i(x_o,T_o,\zeta_i)}$, where ν_i denotes ν_{p_i} etc. Choose $s_i \leq r_i$ so that $R_i(x_o,T_o,s_i) < \zeta_i$. Now choose positive constants a_i, b_i, c_i, d_i, for each $i = 1,2, \ldots, n$, so that

$$b_i s_i + a_i + 2c_i + d_i < (1 - \zeta_i - \nu_i)s_i .$$

Let $\quad B = I - T \in \bigcap_1^n \Omega_2(x_o, I - T_o, U, p_i, c_i),$

and $\quad A = I - S \in \bigcap_1^n \Omega_1(x_o, I - S_o, U, p_i, a_i, b_i).$

Also let $\quad W = \bigcap_1^n d_i V(p_i).$

Suppose $y \in W$ and consider $Sx + Tz + y$ for all x and z in $x_o + U^*$, where $U^* = \bigcap_1^n s_i V(p_i)$. We shall show that $Sx + Tz + y \in x_o + U^*$:

$$\begin{aligned} Sx + Tz + y - x_o &= Sx + Tz + y - S_o x_o - T_o x_o \\ &= (Sx - S_o x_o) + (Tz - T_o x_o) + y \\ &= (A - A_o)x - (A - A_o)x_o + S_o x - S_o x_o + (A - A_o)x_o \\ &\quad + (Tz - T_o z) + (T_o x_o - Tx_o) + (T_o z - T_o x_o) + y \end{aligned}$$

where $A_o = I - S_o$. Now for each $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} p_i(Sx + Tz + y - x_o) &\leq p_i((A - A_o)x - (A - A_o)x_o) + p_i(S_o x - S_o x_o) \\ &\quad + p_i(A - A_o)x_o) + p_i(Tz - T_o z) + p_i(T_o x_o - Tx_o) \\ &\quad + p_i(T_o z - T_o x_o) + p_i(y) \\ &\leq b_i p_i(x - x_o) + \nu_i p_i(x - x_o) + a_i + c_i + c_i \\ &\qquad + R_i(x_o, T_o, s_i)s_i + p_i(y) \\ &\leq (1 - \zeta_i - \nu_i)s_i + (\nu_i + \zeta_i)s_i = s_i. \end{aligned}$$

So, for every x, $z \in x_o + U^*$, we have $Sx + Tz + y \in x_o + U^*$; thus by Theorem 2.2.14, there is a point $\bar{x} \in x_o + U^*$ so that $S\bar{x} + T\bar{x} + y = \bar{x}$ or $K\bar{x} = y$, where $I - K = S + T$.

Remark 3.5.6. If we take $T_o = 0$ in Theorem 3.5.5, we get a stability theorem for the fixed point of a contraction mapping on a complete locally convex Hausdorff topological vector space X. We note, however, that it is possible to formulate other notions of "contraction" for which the fixed point is not necessarily stable. Let W_o be an open neighbourhood of $\theta \in X$, $x_o \in X$ and $W = x_o + W_o$. Let $F : W \to X$. We say that F is a weak contraction if there exists a convex, closed and bounded $V \subset W_o$ such that $x,y \in W$ and $y - x \in \lambda V$ imply $F(y) - F(x) \in \lambda \cdot \alpha V$ for some $0 < \alpha < 1$. Let F be a weak contraction on W into X, and $F(x_o) - x_o \in (1 - \alpha)V$. Then there exists a unique fixed point $\bar{x}$ of F, $\bar{x} \in x_o + V$. However, this fixed point is obviously not necessarily stable.

We now obtain as an application of Theorem 2.2.14, a sufficient condition for a mapping to be open, which generalizes conditions given in Reichback [41], [42], Kasriel & Nashed [26].

Recall that a mapping $F : X \to Y$ is open at a point $y_o \in F(x)$ if y_o is an interior point of $F(X)$; that is, if there is a neighbourhood N of y_o such that $N \subset F(X)$. It follows easily from Definition 3.5.4 that if $Ku = y_o$ has a stable solution with respect to (C,T), then K is open at y_o. We can, however, find much weaker conditions which insure that K is open at y_o. To this end, define

$$\phi_p(x_o,T) = \inf \{a \mid Q_p(x_o,T,a) \neq \phi\}$$

and suppose K is as defined earlier in this section, i.e. $I - K = S + T$ for $(I - S, I - T)$ in C.

The following result is given by Cain & Nashed [8]:

Theorem 3.5.7. Assume X is complete. If $Kx_o = y_o$ and for some $p \in P$ it is true that $\nu_p + \phi_p < 1$, then K is open at y_o.

Proof. We may without loss of generality, take $y_o = \theta$. Choose ζ so that $Q_p(x_o,T,\zeta) \neq \phi$ and $\nu_p + \zeta < 1$. Let $s \in Q_p(x_o,T,\zeta)$ and choose $d < (1 - \zeta - \nu_p)s$. We shall now show that $S_p(\theta,d)$ is contained in the range of K.

Let $u \in S_p(\theta,d)$ and consider $p(Sx + Ty + u - x_o)$ for x and y in $\bar{S}_p(x_o,s)$;

$$\begin{aligned} p(Sx + Ty + u - x_o) &= p(Sx + Ty + u - Sx_o - Tx_o) \\ &\leq p(Sx - Sx_o) + p(Ty - Tx_o) + p(u) \\ &\leq \nu_p s + \zeta s + d < s. \end{aligned}$$

Thus, by Theorem 2.2.14, there is an $\bar{x} \in \bar{S}_p(x_o,s)$ such that $S\bar{x} + T\bar{x} + u = \bar{x}$, which proves the theorem.

An immediate application of this result is the following theorem due to Cain & Nashed [8] giving sufficient conditions for certain operators to be onto maps.

Theorem 3.5.8. Let $B : X \to X$ be a continuous operator such that $T(X)$ is contained in a compact set, where $T = I - B$. Suppose for each $x \in X$, there is a $p \in P$ such that $\phi_p(x,T) < 1$. Then the range of B is X.

Proof. B is open at each point of $B(X)$ from the previous theorem, so $B(X)$ is an open subset of X. We shall show that $B(X)$ is also a closed subset of X, and hence $B(X)$ must be all of the connected space X.

To show $B(X)$ is closed, let $\bar{x}$ be an accumulation point of $B(X)$ and let $\{y_a\}$ be a net in $B(X)$ such that $y_a \to \bar{x}$. Let x_a be such that $Bx_a = y_a$. Then $\{Tx_a\}$ has a convergent subset, say $\{Tx'_a\}$. Since $Bx'_a = x'_a - Tx'_a$, and $\{Bx'_a\}$ and $\{Tx'_a\}$ converge, we then know that $\{x'_a\}$ converges. But $Bx'_a \to \bar{x}$, so $\bar{x} \in B(X)$. Thus $B(X)$ is closed, and the theorem is proved.

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