SOME FIXED POINT THEOREMS IN ANALYSIS

KANHAYA LAL SINGH

SOME FIXED POINT THEOREMS IN ANALYSIS

by

KANHAYA LAL SINGH

A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF ARTS.

DEPARTMENT OF MATHEMATICS

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

AUGUST, 1968.

This thesis has been examined and approved by:

....
Supervisor

....
Internal Examiner

....
External Examiner

Date ....

TABLE OF CONTENTS

ABSTRACT

The main object of this thesis is to study the Contraction Mapping Principle given by Banach. The principle states:

Theorem. Let f be a self mapping of a complete metric space X. If there exists a real number $\lambda \in (0, 1)$ such that the condition

$$d(f(x), f(y)) < \lambda d(x, y)$$

holds for every pair of points $x, y \in X$, then f has a unique fixed point.

This theorem has been used extensively in proving existence and uniqueness theorems of differential and integral equations. Some examples have been given to illustrate its applications.

Several generalizations of Banach's contraction principle have been given in recent years. We have tried to give some further generalizations in Chapter II.

We have also studied Contractive mappings and Eventually contractive mappings. A few new results have been investigated

related to these mappings.

The converse statements of Banach's contraction principle have been given by a few mathematicians. We have also obtained a few new results on the converse of the Banach contraction principle.

A few simple but interesting results related to commuting functions and common fixed points have been given. Some new results on commuting polynomials and common fixed points have been obtained.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my sincere thanks to my supervisor, Dr. S.P. Singh, for his able guidance. I would also like to thank Professors W.J. Blundon and A.E. Fekete for the inspiration and training I received from them. Finally, my thanks go to Mrs. Janet Buchanan for typing the entire manuscript.

INTRODUCTION

In Chapter I of this thesis we present a survey of known results concerning fixed-point theorems. In addition we include a few new results.

In Chapter II we are concerned with the classical fixed point theorem of Banach, commonly known as the contraction mapping principle, which states:

Theorem (A) Let T be a mapping of a complete metric space X into itself. If for every pair of points $x, y \in X$ and some fixed α, $0 \leq \alpha < 1$,

$$(1) \quad d(T(x), T(y)) \leq \alpha d(x, y).$$

Then T has a unique fixed point, and the sequence of iterates $\{T^n(x)\}$ for each $x \in X$ converges to this unique fixed point.

A mapping satisfying (1) is called a contraction mapping and α is called the contractive constant for T with respect to d. Theorem (A) has been used extensively in proving the existence and uniqueness of solutions to various functional equations, particularly integral and differential equations (Kolmogorov and Fomin [47], Nemizki [57] and Zarantonello [76]). It has been applied to prove the convergence of successive approximations of solutions to ordinary differential equations (Luxemburg [50]) and integral equations to L_p-spaces (Willet [74]), to prove the Frobenius-Perron theorem on positive matrices (Birkhoff [8], and Samuelson [61]), and to develop many otherwise difficult existence and uniqueness theorems in various function spaces (Mathews [52], and Thompson [72]).

Because of its widespread applicability there has been a search for generalizations of the Banach's contraction principle. Here we have the work of Edelstein ([31], [32], [33], [34], [35]) Rakotch ([58], [59]) Chu and Diaz ([25], [26]) Janos [42], Naimpally [56] and Browder [20]. Generalizations due to Edelstein's have been applied by Edwards [36].

The major contribution to the subject in Banach and Hilbert spaces is due to Browder, Petryshyn and Kirk. Further, the notion of contraction has been extended to more general spaces (mostly in uniform spaces) and the corresponding fixed point theorems have been given by Knill [46], Davis [28], Mathews and Curtis [52], Edelstein [34], and Kammerer and Kasriel [45]. The first attempt to generalize the contraction principle in uniform spaces was due to Brown and Comfort [21].

Luxemburg [50], Diaz and Margolis [30], Margolis [51], Monna [55] and also Edelstein [35] have given the contraction principle in generalized metric spaces, in which the concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance.

The contraction mapping principle has also been widely used by numerical analysts in the study of convergence and error estimates (Schroder [62]). In each section of this chapter we have tried to give some new results, Section 2.6 contains the results of one paper which has been accepted for publication.

In Chapter III we discuss the results related to commuting functions and common fixed points. We also present some new results on commuting polynomials and common fixed points. In the end of this chapter we prove some theorems related to the conjecture which generalizes the result of DeMarr [29]. Two papers have been accepted for publication from this Chapter.

CHAPTER I

FIXED POINT THEOREMS

1.1 FIXED POINTS.

1.1.1 Definition. A point x is said to be a fixed point for the transformation T if $T(x) = x$. In other words, a point which remains invariant under a transformation is known as a fixed point.

Examples.

1. The mapping of the interval $[0, 1]$ into itself defined by $f(x) = x^m$, where m is a positive integer different from one, has two fixed points, namely 0 and 1.

2. The mapping of the open interval $(0, 1)$ onto itself defined by $f(x) = x^m$, where m is a positive integer different from one, has no fixed point.

3. The unit transformation $f(x) = x$ fixes every point.

4. The transformation $w = \frac{1+i}{z-i}$ has two fixed points, namely -1 and $1+i$.

1.1.2 Fixed Points of Linear Functions.

Theorem. The linear functions of the form $f(x) = ax+b$,

$a \neq 1$, have unique fixed points.

Proof. Consider $f(x) = ax + b$, $a \neq 1$, and suppose that $f(x_0) = x_0$ for some real number x_0. Hence $x_0 = f(x_0) = ax_0 + b$, which implies $x_0(1-a) = b_0$, since $a \neq 1$ implies $1-a \neq 0$. Therefore we have shown that there is a unique fixed point $\frac{b}{1-a}$. Substituting for x_0, $\frac{b}{1-a}$ in relation $f(x_0) = ax_0 + b$ we have $f(\frac{b}{1-a}) = \frac{b}{1-a}$. Hence $\frac{b}{1-a}$ is a fixed point and therefore is unique.

We know that the linear function $f(x) = x$ fixes all points.

From the above facts it follows that these are only linear functions with fixed points.

Remark. The only linear functions which have no fixed points are of the form $f(x) = x + b$, $b \neq 0$.

1.1.3 Fixed Points of a Linear Fraction.

Theorem. Every linear fraction has two fixed points, which in certain cases coalesce into a single fixed point.

Proof. (1) Let $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$, be a linear fraction. Then we have the following cases:

Case I. Suppose that $c = 0$. Then (1) reduces to

$$f(z) = \frac{az + b}{d} = \frac{a}{d} z + \frac{b}{d} = \alpha z + \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where $\alpha = \frac{a}{d}$, $\beta = \frac{b}{d}$.

It is clear from the equation (2) that $f(\infty) = \infty$, and there is a fixed point at infinity.

If $\alpha \neq 1$, then there exists another fixed point determined by $z = \alpha z + \beta$, which implies $z = \frac{\beta}{1-\alpha}$; hence the point $\frac{\beta}{1-\alpha}$ is a fixed point. But if $\alpha = 1$, $\beta \neq 0$, there is no finite fixed point.

If $\alpha \neq 1$, $\beta \neq 0$, the finite fixed point $\frac{\beta}{1-\alpha}$ approaches ∞ as α tends to 1. Therefore, in the case of the transformation

$$f(z) = z + \beta \qquad (\beta \neq 0)$$

the point at infinity can be regarded as two fixed points which coincide.

Case II. Let $c \neq 0$. Then

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad \text{gives} \quad f(\infty) = \frac{a}{c} \neq \infty.$$

Therefore the point at infinity is not a fixed point. Similarly, the pole $\delta = -\frac{d}{c}$ of the transformation is not a fixed point, since

$$f(\delta) = \infty \neq \delta.$$

Assuming that $z \neq \infty$ and $z \neq \delta$, we solve the equation

$$z = \frac{az + b}{cz + d}$$

or $$cz^2 - (a-d)z - b = 0,$$

obtaining

$$z = \frac{a-d \pm\sqrt{(a-d)^2 + 4bc}}{2c}.$$

If $(a-d)^2 + 4bc \neq 0$, we obtain two different finite fixed points; if $(a-d)^2 + 4bc = 0$, these two fixed points coalesce to form a single finite fixed point $\frac{a-d}{2c}$.

1.1.4 <u>On the Fixed Points of</u> $f(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$, $ad - bc \neq 0$.

We know that the number of fixed points of

(1) $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ is 1 or 2, except in the case of the identity transformation, which fixes all points. These are no longer true in the case of

(2) $f(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$, $ad - bc \neq 0$.

In transformation (2) the following cases will arise:-

(i) no fixed point (example:- $f(z) = -1/\bar{z}$);

(ii) one fixed point (example:- $f(z) = \bar{z} + 1$, the point at infinity);

(iii) two fixed points (example:- $f(z) = 2\bar{z}$; $z = 0$ and $z = \infty$);

(iv) an infinite number of fixed points (example:- $f(z) = \bar{z}$, all the points on the real axis).

The transformation (2) may be factored into $f(z) = \bar{z}$, and $f(z) = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$. Hence generalized circles (i.e. Euclidean circles and straight lines) will be transformed into generalized circles, and the angles will be preserved in magnitude but reversed in sense.

In this section we prove the following result, which is more general in notion and form.

1.1.5 Theorem. Let T^n (n is a positive integer) be a function defined on a non-empty set X into itself, and let K be another function, also defined on X into itself, such that K possesses a right inverse K^{-1} (that is, a function K^{-1} such that $KK^{-1} = I$, where I is the identity mapping of X). Then the function T^n has a fixed point if and only if the composite function $K^{-1}T^nK$ has a fixed point.

<u>Proof</u>. Suppose that ξ is a fixed point of T^n. Then

$$T^n(\xi) = \xi.$$

Now $K^{-1}\xi = K^{-1}T^n(\xi) = K^{-1}T^n(KK^{-1})(\xi) = K^{-1}T^nK(K^{-1}\xi)$.

Therefore $K^{-1}\xi$ is a fixed point for $K^{-1}T^nK$.

Conversely, suppose that η is a fixed point of $K^{-1}T^nK$. Then

$$K^{-1}T^nK\eta = \eta$$

or $$KK^{-1}T^nK\eta = K\eta$$

or $$T^nK\eta = K\eta.$$

Therefore $K\eta$ is a fixed point of T^n.

<u>Corollary</u>. In particular case, when $n = 1$, we get a well-known result due to Chu and Diaz [25].

In this section we prove a similar result to that of Chu and Diaz [25] by considering left inverse.

1.1.6 <u>Theorem</u>. Let T and K be two functions defined on a non-empty set X into itself, such that K possesses a left inverse (i.e., a function K^{-1} such that $K^{-1}K = I$, where I is the identity mapping of X). Then the function T has a fixed point if and only if KTK^{-1} has a fixed point.

Proof. Let x be a fixed point of T. Then $Tx = x$, implies that $T(K^{-1}K)x = x$;

or $\quad KT(K^{-1}K)x = Kx$

or $\quad (KTK^{-1})(Kx) = Kx.$

i.e., Kx is a fixed point of KTK^{-1}.

Conversely, let us assume that y is a fixed point of KTK^{-1}.

Then

$$KTK^{-1}y = y$$

$$\text{or} \quad K^{-1}KTK^{-1}y = K^{-1}y$$

$$\text{or} \quad TK^{-1}y = K^{-1}y$$

i.e., $K^{-1}y$ is a fixed point of T.

Thus the proof.

1.2 CONTINUOUS FUNCTIONS AND FIXED POINTS.

1.2.1 Theorem. Let f be a continuous function from the closed interval $[-1, 1]$ into itself. Then there must exist a point x_0 in $[-1, 1]$ such that $f(x_0) = x_0$.

<u>Proof</u>. We prove this fact by taking a function $F(x) = f(x) - x$. We note that $F(-1) \geq 0$ and $F(1) \leq 0$. Therefore, by Weierstrass intermediate value theorem we find that there exists a point x_0 in $[-1, 1]$ such that $F(x_0) = 0$. This implies $f(x_0) = x_0$.

We prove the following result by using Weierstrass intermediate value theorem.

1.2.2 <u>Theorem</u>. Let I be the closed unit interval on the real line. Let f and g be two continuous functions from I into itself, where f is onto. Then there always exists a point p in I for which $f(p) = g(p)$.

<u>Proof</u>. Let $h(x) = f(x) - g(x)$ for all x in I. Then $h(x)$ is continuous in I. The following cases will arise.

(1) Let $h(x) = 0$ for all x in I.

In this case, obviously $f(p) = g(p)$ for all p in I.

(2) Let $h(x) > 0$ for all x in I.

i.e. $f(x) > g(x)$ for all x in I.

The function $f(x)$ is onto, and therefore it takes value 0 and consequently $g(x) < 0$, contradicting the fact that $g(x)$

lies in I. Therefore $h(x)$ cannot be positive for all x in I. Let there be a point x_1 for which $h(x) < 0$.

(3) Let $h(x) < 0$ for all x in I.

i.e. $f(x) < g(x)$ for all x in I.

Taking $f(x) = 1$ we again get a contradiction, and thus $h(x)$ cannot be negative for all x in I. Let there be a point x_2 for which $h(x) > 0$.

Thus the continuous function $h(x)$ takes negative and positive values in I; and therefore by Weierstrass intermediate value theorem $h(p) = 0$ where $x_2 < p < x_1$, and hence $f(p) = g(p)$.

Remark. Let I be the closed interval of real numbers. Let f and g be two continuous functions from $I^n = I \times I \times I \times \ldots \times I$ into itself, where f is onto. Then for $n > 1$ there need not exist a point p such that $f(p) = g(p)$, as will be seen from the following example [1].

It suffices to show this for $n = 2$; the same situation holds for $n > 2$. Let

$$f(x,y) = \begin{cases} (2x,y), & 0 \le x \le \frac{1}{6} \text{ or } \frac{1}{3} \le x \le \frac{1}{2} \\ (2x, y[\frac{11}{12} + |x - \frac{1}{4}|]), & \frac{1}{6} \le x \le \frac{1}{3} \\ (2-2x,y), & \frac{1}{2} \le x \le \frac{2}{3} \text{ or } \frac{5}{6} \le x < 1 \\ (2-2x, y[\frac{11}{12} + |x - \frac{1}{4}|] + \frac{1}{12} - |x - \frac{1}{4}|), & 2/3 \le x \le 5/6 \end{cases}$$

$$= (f^1(x,y),\ f^2(x,y));$$

$$g(x,y) = \begin{cases} f(x + \frac{1}{2}, y), & 0 \le x \le \frac{1}{2} \\ f(x - \frac{1}{2}, y), & \frac{1}{2} \le x \le 1 \end{cases}$$

$$= (g^1(x,y),\ g^2(x,y)).$$

f and g both are continuous.

If for some (x,y), $f(x,y) = g(x,y)$ then either $f(x,y) = f(x + \frac{1}{2}, y)$ or $f(x,y) = f(x - \frac{1}{2}, y)$. We need to consider only one of these equations. Suppose $f(x,y) = f(x + \frac{1}{2}, y)$; then $2 - 2(x + \frac{1}{2}) = 2x$ implies $x = \frac{1}{4}$, and $f(\frac{1}{4}, y) = f(3/4, y)$ implies $\frac{11}{12}y = \frac{11}{12}y + \frac{1}{12}$ which is impossible. Thus f and g never take the same value simultaneously. Moreover, both are onto and at most 2 to 1.

1.3 FIXED POINT SPACE.

1.3.1 Definition. A topological space X is said to have a fixed point property if and only if each continuous function f of X into itself has at least one fixed point. Or, we say that a topological space X is a fixed point space if every continuous mapping f of X into itself has a fixed point.

Examples.

1. The Theorem 1.2.1 shows that $[-1, 1]$ is a fixed point space.

2. The closed disc $\{(x, y): x^2 + y^2 \leq 1\}$ in the Euclidean plane R^2 is also a fixed point space.

1.3.2 Theorem. The fixed point property is topological invariant.

Proof. Let h be a homeomorphism from a space X onto a space Y; let X have a fixed point property and f be a continuous function from Y into Y. Since X has a fixed point property, therefore there exists a point x in X such that $h^{-1}(f(h(x))) = x$. Hence $h(h^{-1}(f(h(x)))) = h(x)$, or $f(h(x)) = h(x)$. Let $h(x) = y$. Then $f(y) = y$; hence f has a fixed point.

1.3.3 Both of the examples of section 1.3.1 are special cases of

Brouwer's Fixed Point Theorem. The closed unit sphere $S = \{x : ||x|| \leq 1\}$ in R^n is a fixed point space.

Brouwer's theorem itself is a special case of

Schauder's Fixed Point Theorem. Every convex compact subspace of a Banach space is a fixed point space.

The proofs of these theorems, together with a discussion of other related results, may be found in Bers [5 pp. 86, and pp. 93-97]. Schauder's theorem was foreshadowed by the work of Birkhoff and Kellogg[9] on existence theorem in analysis. Shortly afterwards Tychonoff * extended Schauder's result from Banach spaces to arbitrary locally convex topological spaces. In both cases Brouwer's theorem was used as a starting point.

Recently Browder [14] gave generalizations of Schauder and Tychonoff fixed point theorems. He also gave several generalizations to Schauder fixed point theorem ([15], [16], [17], [18], [19]) which center around the concept of asymptotic fixed point theorems and of deformation of non-compact mappings.

* Tychonoff, A: Ein Fixpunktzatz, Math. Ann. 111 (1935) 767-776.

CHAPTER II

THE CONTRACTION MAPPING THEOREM

2.1.1 Definition. A metric space is a pair consisting of a set X and a mapping $(x, y) \longrightarrow d(x, y)$ of $X \times X$ into R having the following properties:

(1) $d(x, y) \geq 0$ if $x \neq y$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$ (symmetry);

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The function d is called a metric and $d(x, y)$ is called the distance between the points x, y.

2.1.2 Definition. A sequence $\{x_n\}$ in a metric space X is said to converge to an element x of X if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

2.1.3 Definition. A sequence $\{x_n\}$ of elements of a metric space X is called a Cauchy sequence if given $\varepsilon > 0$ there exists N such that for $p, q \geq N$, $d(x_p, x_q) < \varepsilon$.

2.1.4 Definition. A metric space X is said to be complete if every

Cauchy sequence of points of X is convergent in X.

2.1.5 Definition: Given a vectorspace E, a norm on E is a map $x \longrightarrow ||x||$ from E into the set R of positive real numbers which satisfies the following axioms:

(1) $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda| \, ||x||$ for all $\lambda \in K$ and $x \in E$, where K is either the field of real numbers or the field of complex numbers.

(3) $||x+y|| \leq ||x|| + ||y||$ (the triangle inequality).

A vector space on which a norm is defined is called a normed vectorspace, or simply a normed space.

2.1.6 Definition. A normed vectorspace E is called a Banach space if it is complete as a metric space.

2.1.7 Definition. A vectorspace E over K is called an inner product space if there is defined a map $(x,y) \longrightarrow (x|y)$ from $E \times E$ into K which has the following properties:

(i) $(x|x) \geq 0$ for every $x \in E$.

(ii) $(x|x) = 0$ if and only if $x = 0$.

(iii) $(x|y) = \overline{(y|x)}$ for every $x \in E$, $y \in E$.

(iv) $(\lambda x + \mu y|z) = \lambda(x|z) + \mu(y|z)$ for every $\lambda, \mu \in K$, x, y and $z \in E$.

The value $(x|y)$ is called the inner product or scalar product of the vectors x and y.

2.1.8 Definition. Let E be an inner product space and $||x||$ be the norm defined by $||x|| = \sqrt{(x|x)}$. If E is complete for this norm (i.e. E is a Banach space), then E is said to be a Hilbert space.

2.1.9 Definition. Let X and X' be two metric spaces with the metrics d and d'; let $T:x \longrightarrow x'$ be a bijection of X to X'.

Then T is called an isometry if for all $x, y \in X$,

$$d(x,y) = d'(x',y').$$

2.1.10 Definition. A mapping T of a metric space X into itself is said to satisfy a Lipschitz condition, with Lipschitz constant α, if

(1) $d(T(x), T(y)) \leq \alpha d(x,y)$; $(x, y \in X)$.

In case $0 \leq \alpha < 1$, then T is called a contraction mapping. Thus in the contraction mapping, the distance between the images of any two points is less than the distance between the points.

Example 1. If $x = \{x_n\}_{n=1}^{\infty}$ in l^2, let $T = \{\frac{x_n}{2}\}_{n=1}^{\infty}$.

Then T is contraction on l^2. For if $y = \{y_n\}_{n=1}^{\infty}$ is any other point in l^2, then

$$d(T(x), T(y)) = ||T(x) - T(y)||_2 = \left[\sum_{n=1}^{\infty}\left(\frac{x_n}{2} - \frac{y_n}{2}\right)^2\right]^{1/2}$$

$$= \frac{1}{2}||x-y||_2$$

$$= \frac{1}{2}\, d(x,y).$$

Thus in this example, α may be taken to be $\frac{1}{2}$. For this T it is obvious that there is one and only one sequence $s \in l^2$ such that $T(s) = s$, namely the sequence $0, 0, 0 \ldots$.

Example 2. Let T be a function on Euclidean 2-space R^2, i.e. $T: R^2 \longrightarrow R^2$ defined by $T(x) = x/2$. Then T is a contraction for ,

$$d(T(x), T(y)) = ||T(x) - T(y)||$$

$$= ||\frac{x}{2} - \frac{y}{2}||$$

$$= \frac{1}{2}||x-y|| = \frac{1}{2}d(x,y).$$

Example 3. Let $T = T(x)$ be a real valued function of a real variable and suppose that for all x_1 and x_2 in the domain of T

$$|T(x_2) - T(x_1)| \leq \alpha |x_2 - x_1|$$

with $0 \leq \alpha < 1$. Then T is a contraction mapping in R^1.

2.1.11 Definition. A function $T{:}R \longrightarrow R$ is continuous at a point x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \text{ implies } |T(x) - T(x_0)| < \varepsilon.$$

The function T is continuous on R if it is continuous at every point of R.

2.1.12 Theorem. If T is a contraction mapping on a metric space X, then T is continuous on X.

Proof. Let $\varepsilon > 0$ be given and let x_1 be any point in X. Then if $\alpha = 0$ in (1) we have

$$d(T(x_1), T(y)) = 0 < \varepsilon,$$

for all y in X; and T is continuous at x_1. Otherwise let $\delta = \varepsilon/\alpha$, and let y be any point in X such that $d(x_1, y) < \delta$. Then

$$d(T(x_1), T(y)) \leq \alpha d(x_1, y) \leq \alpha.\varepsilon/\alpha = \varepsilon;$$

and T is again continuous at x_1.

Remark. The converse of the above statement is not true, i.e. a continuous function need not be a contraction. For example, $f(x) = 2x$ is continuous, but it is not a contraction.

2.2 THE CONTRACTION MAPPING PRINCIPLE (CACCIOPOLI BANACH)

The most elementary and by far the most fruitful method for proving theorems on the existence and uniqueness of solutions is the principle formulated by Banach [4 in 1922] and first applied to the proof of an existence theorem by Cacciopoli [22 in 1930].

2.2.1 Theorem. Every contraction mapping of a complete metric space X into itself has a unique fixed point (i.e. the equation $Tx = x$ has a unique solution).

Proof. Let x_0 be an arbitrary point. Set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, and in general let $x_n = T^n x_0$. We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. In fact,

$$d(x_n, x_m) = d(T^n x_0, T^m x_0) \leq \alpha^n d(x_0, x_{m-n})$$

$$\leq \alpha^n\{d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{m-n-1}, x_{m-n})$$

$$\leq \alpha^n d(x_0, x_1) \{1 + \alpha + \alpha^2 + \dots + \alpha^{m-n-1}\}$$

$$\leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1).$$

Since $\alpha < 1$ this quantity is arbitrarily small for sufficiently large n, thus the sequence is Cauchy. Since X is complete, $\lim\limits_{n\to\infty} x_n$ exists. We set $x = \lim\limits_{n\to\infty} x_n$. Then by virtue of the continuity of the mapping T

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Thus the existence of a fixed point is proved.

We shall now prove the uniqueness. Let $Tx = x$, and $Ty = y$, $x \neq y$. Then $d(x, y) = d(T(x), T(y)) \leq \alpha d(x, y)$. But $\alpha < 1$, therefore $d(x, y) = 0$ i.e. $x = y$.

Remarks

(1) The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximations.

(2) The error of approximation can be estimated as

$$d(x, x_n) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0).$$

(3) The proof of this theorem does not involve any topological machinery.

(4) It guarantees the existence and uniqueness of a fixed point. However the requirement that T be contraction is a severe restriction.

(5) If X is not a complete metric space, a contraction mapping of X into itself may have no fixed point; for example, the mapping $x \longrightarrow x/2$ of $(0,1]$ into itself has no fixed point.

2.2.2 The principle of contraction mapping can be applied to the proof of the existence and uniqueness of solution obtained by the method of successive approximations. We shall consider the following simple examples.

2.2.3 Picard's Theorem. Consider the differential equation $\frac{dy}{dx} = T(x, y)$, where $T(x, y)$ satisfies a Lipschitz condition $|T(x, y_1) - T(x, y_2)| \leq M|y_1 - y_2|$. Then on the interval $|x-x_0| < d$, there exists a unique solution $y = \Phi(x)$ of the equation which satisfies the condition $\Phi(x_0) = y_0$.

Proof. The given differential equation can be written in the form of an integral equation as,

$$y = y_0 + \int_{x_0}^{x} T(t, y)dt.$$

Consider the set of all continuous functions $\{\phi(x)\}$ on the interval $a \le x_0 \le x \le b$. This set is a complete metric space if the distance between two functions ϕ_1 and ϕ_2 is defined by

$$d(\phi_1, \phi_2) = \max_{a \le x \le b} |\phi_1 - \phi_2|.$$

We shall consider the right-hand side of the above integral equation as an operator

$$A(\phi) = y_0 + \int_{x_0}^{x} T(t, \phi)dt,$$

defined on $\{\phi(x)\}$. Since the operation of integration is a continuous function of the upper limit, this operator transforms points of $\{\phi(x)\}$ into the same space. Estimating $d(A(\phi_1), A(\phi_2))$, we have

$$\begin{aligned} d(A(\phi_1), A(\phi_2)) &= \max|A(\phi_1) - A(\phi_2)| \\ &= \max \left| \int_{x_0}^{x} [T(t, \phi_1) - T(t, \phi_2)dt \right| \\ &\le M \max|\phi_1 - \phi_2| \; |x-x_0|. \end{aligned}$$

If we take $|x-x_0| < \alpha|M$, where $\alpha < 1$, then

$$d(A(\phi_1), A(\phi_2)) < \alpha d(\phi_1, \phi_2),$$

and hence a unique solution $\overline{\phi}(x)$, of the equation $A(\phi) = \phi$, exists. This solves the given differential equation. It follows from the same theorem, that this solution can be obtained by iterating the operator

$A(\phi)$, starting with any continuous function.

2.2.4 Cauchy's Theorem. Consider the differential equation $\frac{dy}{dx} = T(x,y)$, where $T(x,y)$ is an analytic function of x and y, that is $T(x,y) = \Sigma a_{\alpha\beta} x^{\alpha} y^{\beta}$ in the domain $|x-x_0| < \varepsilon$, $|y-y_0| < \varepsilon$. Then there exists a unique solution $y = \phi(x)$ which can be expanded in powers of $x-x_0$ in some neighbourhood of the point x_0, and which satisfies the condition $\phi(x_0) = y_0$. Here x can be either a real or a complex variable.

To prove the theorem with analytic functions, we must recall the following propositions from the theory of power series.

Proposition 1. Let the series $\Sigma\, a_{\alpha\beta\ldots\gamma} x^{\alpha} y^{\alpha} \ldots u^{\gamma}$ converge inside some sphere $(x-x_0)^2 + (y-y_0)^2 + \ldots + (u-u_0)^2 < d$. If the variables $x, y \ldots u$ are replaced by powerseries which converge in a sphere of radius d', the resulting power series converges in a sphere whose radius is the smaller of the numbers d and d'.

Proposition 2. The limit of a uniformly convergent sequence of analytic functions in any number of variables is analytic in any domain enterior to the domain of convergence of the members of this sequence (Weierstrass' theorem).

<u>Proof of 2.2.4.</u> Let $M = \max\left|\frac{\partial T}{\partial x}\right|$ for $|x-x_0| \leq \varepsilon' < \varepsilon$, $|y-y_0| \leq \varepsilon' < \varepsilon$. Consider the set of analytic function $\{\phi(x)\}$ which are holomorphic in the circle $(x-x_0)^2 + (y-y_0)^2 < d^2$ of radius $d = \min\{\alpha|M, \varepsilon'\}$, where α is some fixed number less than one.

The differential equation may be transformed into an integral equation by

$$y = y_0 + \int_{x_0}^{x} T(t, y)dt = y_0 + \int_{x_0}^{x} \Sigma a_{\alpha\beta} t^{\alpha} y^{\beta} dt$$

and consider the right-hand side of the equation as an operator A defined on the set $\{\phi(x)\}$. By proposition 1 and well-known theorem on the integration of power series, we conclude that the result of applying the operator A is a new function in the set $\{\phi(x)\}$.

In addition, if we take $\max|\phi_1 - \phi_2|$ for $d(\phi_1, \phi_2)$, then by the Weierstrass theorem [proposition 2], the set $\{\phi(x)\}$ forms a complete metric space.

Estimating $d(A(\phi_1), A(\phi_2))$, we have

$$d(A(\phi_1), A(\phi_2)) = \max\left| \int_{x_0}^{x} [T(x, \phi_1) - T(x, \phi_2)]\, dx \right|$$

$$\leq \left| \int_{x_0}^{x} \max\left|\frac{\partial T}{\partial \phi}\right| . |\phi_1 - \phi_2| dx \right|$$

$$\leq M \max|\phi_1 - \phi_2| . |x-x_0| .$$

Taking $|x-x_0|$ less than $\alpha|M$, we have

$$d(A(\phi_1), A(\phi_2)) < \alpha d(\phi_1, \phi_2).$$

Consequently Banach's theorem is applicable, and this proves Cauchy's theorem.

In an analogous manner we can prove the following theorem of Poincaré.

2.2.5 Poincaré's Theorem. Suppose that in the equation $\frac{dy}{dx} = T(x, y; \lambda)$, the function $T(x, y; \lambda)$ can be expanded in a power series $\Sigma a_{\alpha\beta\gamma} x^\alpha y^\beta \lambda^\gamma$ in x, y and λ which converges in the region $|x| < \varepsilon$, $|y| < \varepsilon$, $|\lambda| < \varepsilon$. Then there exists a solution of the form

$$y = \lambda u_1(x) + \lambda^2 u_2(x) + \ldots + \lambda^n u_n(x) + \ldots .$$

Proof. Let $M = \max \left|\frac{\partial T}{\partial y}\right|$, where M does not depend on x,

y and λ. Now consider the set of functions $\phi(x, \lambda) = \Sigma c_{\alpha\beta} x^{\alpha} \lambda^{\beta}$, which are analytic in the domain $|x| \leq \min\{\alpha|M, \varepsilon\}$, $|\lambda| \leq \{\alpha|M, \varepsilon\}$, where $\alpha < 1$. This set is a complete metric space if the distance is taken as

$$\max|\phi_1(x, \lambda) - \phi_2(x, \lambda)|.$$

Consider the operator

$$A(\phi) = \int_0^x \Sigma a_{\alpha\beta\gamma} x^{\alpha} \phi^{\beta} \lambda^{\gamma} dx.$$

Because of propositions 1 and 2 we conclude that $A(\phi)$ is also a function of the set.

Estimating $d(A(\phi_1), A(\phi_2))$ as in Cauchy's theorem, we obtain

$$d(A(\phi_1), A(\phi_2)) < \alpha d(\phi_1, \phi_2),$$

which proves Poincaré's theorem.

We prove as a typical and last application illustrating the use of Banach's contraction principle a version of the classical implicit function theorem.

2.2.6 <u>Theorem</u>. Let $T(x, y)$ be a continuous real valued function defined on the rectangle $I_a \times I_b \subset R^2$ where $I_a = \{x \mid |x-x_0| \leq a\}$ and $I_b = \{y \mid |y-y_0| \leq b\}$. Assume that $T(x_0, y_0) = 0$ and that there is a $\alpha < 1$ such that $|T(x, y) - T(x, y')| \leq \alpha|y-y'|$ for all $x \in I_a$, $y, y' \in I_b$. Then there exists a positive $s \leq a$ and a unique continuous function $h: I_s \longrightarrow I_b$ such that $h(x_0) = y_0$ and $h(x) = y_0 + T(x, h(x))$ on I_s.

Before giving the proof of Theorem 2.2.6 we need the following definition and theorem.

2.2.7 Definition. For any two spaces Y, Z the map $w: Z^Y \times Y \longrightarrow Z$ defined by $(f, y) \longrightarrow f(y)$ is called the evaluation map of Z^Y.

2.2.8 Theorem. Let X be an arbitrary space, and Y be d-complete. Then $C(X, Y; d)$ is d^+-complete. Topology by Dugundji

Proof. Let $\{T_n\}$ be any d^+-Cauchy sequence, so that

$$\forall\ \varepsilon > 0 \quad \exists\ N(\varepsilon) \quad \forall\ n, m \geq N(\varepsilon): d^+(T_n, T_m) < \varepsilon.$$

Since $d(T_n(x), T_m(x)) \leq d^+(T_n, T_m)$, it follows that $\{T_n(x)\}$ is a d-Cauchy sequence in Y for each x, and therefore converges to some element, which we denote by $F(x)$. Furthermore, we have $T_n(x)$ in $B(T_m(x), \varepsilon)$ for all x and $n, m \geq N(\varepsilon)$, consequently $F(x)$ in $\overline{B(T_m(x), \varepsilon)}$ for each x and all $m,n \geq N(\varepsilon)$, which shows that the sequence $\{T_n\}$ converges to the function F uniformly on Y. Therefore F is continuous and $F \in C(X, Y; d)$; since $T_n \longrightarrow F$. This concludes the proof.

Proof of 2.2.6. For any fixed positive $\gamma \leq a$, consider the space $C(I_\gamma, I_b; d_e)$, which is d_e^+-complete (By 2.2.8), and let C_γ be the subspace $\{\phi | \phi(x_0) = y_0\}$; C_γ is closed, since it is the inverse image of y_0 under the evaluation map wx_0, and so C_γ is d_e^+-complete. For $\phi \in C_\gamma$ define $F(\phi)$ to be the function $F(\phi(x)) = y_0 + T(x, \phi(x))$ on I_γ; then always $F(\phi)(x_0) = y_0 + T(x_0, \phi(x_0)) = y_0$, and the problem reduces to showing that in a suitable C_γ, there is an h such that $F(h) = h$. To apply Banach's contraction principle we must first determine a C_γ that is mapped by F into itself; that is, for each ϕ satisfying $|y_0 - \phi(x)| \leq b$ on I_γ, $F(\phi)$ satisfies the same conditions. Now,

$$\begin{aligned} |y_0 - F(\phi(x))| &= |T(x, \phi(x))| \\ &\leq |T(x, \phi(x)) - T(x, y_0)| \\ &\quad + |T(x, y_0)| \\ &\leq \alpha|\phi(x) - y_0| + |T(x, y_0)| \\ &\leq \alpha b + |T(x, y_0)|. \end{aligned}$$

Since $T(x_0, y_0) = 0$ and T is continuous, by choosing $\gamma = s$ so small that $|T(x, y_0)| \leq b(1 - \alpha)$ for all x in I_s, we shall indeed have that F maps C_s into itself. Next, for ϕ, $\psi \in C_s$, we have

$$|F(\phi(x)) - F(\psi(x))| = |T(x, \phi(x)) - T(x, \psi(x))|$$
$$\leq \alpha|\phi(x) - \psi(x)|,$$

so that $d_e^+(F\phi, F\psi) \leq \alpha d_e^+(\phi, \psi)$; since $\alpha < 1$, $F{:}C_s \longrightarrow C_s$ is contraction and has a unique fixed point h.

2.3 EXTENSIONS OF BANACH'S CONTRACTION PRINCIPLE.

The contraction theorem of Banach remains the most fruitful means for proving and analysing the convergence of iterative processes. For this reason extensions of the theorem are of continuing interest.

The following extension of Banach's contraction principle is given in [12], [37] and [47].

2.3.1 Theorem. If T is a continuous mapping of a complete metric space X into itself and if, for some positive integer n, the iterate T^n is a contraction, then T has a unique fixed point.

Proof. If we take an arbitrary point $x \in X$ and consider the sequence $T^{kn}x$ $(k = 0, 1, 2, \ldots)$, a repetition of the arguments introduced in the proof of Banach's contraction principle yields the convergence of this sequence. Let $x_0 = \lim_{k\to\infty} T^{kn}x$. Then $Tx_0 = x_0$. In fact $Tx_0 = \lim\limits_{k\to\infty} T^{kn}Tx$. Since the mapping T^n is contraction we have

$$d(T^{kn}(Tx), T^{kn}(x)) \leq \alpha d(T^{(k-1)n}(Tx), T^{(k-1)n}(x))$$

$$\leq \ldots \qquad \leq \alpha^k d(T(x), (x)).$$

Consequently

$$\lim_{k\to\infty} d(T^{kn}(Tx), T^{kn}(x)) = 0$$

i.e. $Tx_0 = x_0$.

Remark The proof of theorem 2.3.1 may be simplified somewhat, as follows: Since T^n is a contraction, it possesses by Banach's contraction principle, a unique fixed point, say x_0, such that $T^n x_0 = x_0$. It will now be shown that $Tx_0 = x_0$. Since

$$d(Tx_0, x_0) = d(TT^n x_0, T^n x_0) = d(T^n Tx_0, T^n x_0)$$
$$\leq \alpha d(Tx_0, x_0),$$

and $\alpha < 1$, one has $d(Tx_0, x_0) = 0$ i.e. $Tx_0 = x_0$.

Thus the argument just given shows that the assumption that T itself is continuous made in the hypothesis of theorem 2.3.1 is superfluous. However, it should be noticed that the proof of theorem 2.3.1, nevertheless, does make use of the continuity of T. Specifically, when it is asserted that since $x_0 = \lim_{k\to\infty} T^{kn}x$, one has

$$Tx_0 = T(\lim_{k\to\infty} T^{kn}x) = \lim_{k\to\infty} T^{kn}Tx.$$

Therefore the following theorem is an extension of theorem 2.3.1.

2.3.2 Theorem. If T is a (single valued) function defined on a complete metric space X into itself, such that the function T^n is a contraction for some (positive integer) n, then T has a unique fixed point.

Remark. The conclusion that T has a fixed point can be reached in an even more direct manner, still without assuming that T itself is continuous. Since T^n is a contraction, it follows from Banach's contraction principle that T^n has a unique fixed point x_0, such that $T^n x_0 = x_0$. Hence,

$$Tx_0 = TT^n x_0 = T^n T(x_0),$$

which means that Tx_0 is also a fixed point of T^n. But T^n has a unique fixed point, and therefore $Tx_0 = x_0$. Thus, x_0 is a fixed point of T. The uniqueness of the fixed point of T is obvious, since each fixed point of T is also a fixed point of T^n.

<u>Remark</u>. An examination of the preceeding argument shows that there is no need to assume that T^n is a contraction and defined on a complete metric space. All that is used in obtaining the conclusion of theorem 2.3.2 is that T^n has exactly one fixed point. Hence one has

2.3.3 Theorem. Let S be any non empty set of elements (called "points") and T be a single valued function defined on S. Suppose that for some positive integer n, the function T^n has a unique fixed point x_0. Then T also has a unique fixed point, namely x_0.

<u>Proof</u>. T^n has a unique fixed point; therefore

$$T^n(x_0) = x_0.$$

But $$T^{n+1} = T.T^n = T^n.T;$$

therefore $$T^nT(x_0) = T(T^n(x_0)) = T(x_0).$$

Hence $T(x_0)$ is a fixed point of T^n. The uniqueness of this point shows that $T(x_0) = x_0$; in otherwords, x_0 is also a fixed point of T. The uniqueness of the fixed point of T is obvious, since each fixed point of T is also a fixed point of T^n.

(When S is a complete metric space X, and T a single valued function on X to X, such that T^n, for some positive integer n, is a contraction then 2.3.3 reduces to 2.3.2).

In order to illustrate Theorem 2.3.2 we take the following examples:

<u>Example 1</u>. Let $T:[0, 2] \longrightarrow [0, 2]$ be defined by

$$T(x) = \begin{cases} 0 & x \in [0, 1] \\ 1 & x \in (1, 2] . \end{cases}$$

Then $T^2(x) = 0$ for all $x \in [0, 2]$; hence T^2 is a contraction on $[0, 2]$, although T is not continuous.

<u>Example 2</u>. Let T map $[0, 1]$ into itself according to the formula

$$T(x) = \begin{cases} 1/2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Then T^2 is a contraction, although T is not continuous.

Example 3. The metric space R is taken to be the Banach space of all real valued continuous functions $C([0, 1])$, on the closed interval $0 \leq x \leq 1$, with the norm of the function $f(t)$ being the maximum of $|f(x)|$ for x in this interval. Consider the linearly independent elements (i.e. such that any finite subset is linearly independent) of $C([0, 1])$:

$$e^x,\ 1,\ x,\ x^2,\ \ldots\ldots\ldots\ldots,$$

and extend this linearly independent set to a Hamel basis H (i.e. maximal linearly independent set;). The transformation T is defined, for elements of H, as follows:

$$T(e^x) = \frac{1}{2}.1, \quad \text{and} \quad T(1) = \frac{1}{2}.e^x.$$

While $T(h) = \frac{1}{2}h$ for any element of H which is different from 1 or e^x (notice that, therefore, $T(x^n) = \frac{1}{2}x^n$ for $n = 1,2,\ldots$). Since H is a basis for $C([0, 1])$, the definition of T may be extended, from H to all of $C([0, 1])$, merely by defining $T(y) = \sum_{i=1}^{n} \alpha_i T(h_i)$ whenever $y = \sum_{i=1}^{n} \alpha_i h_i$ (with n positive integer, real numbers $\alpha_i \neq 0$ for $i = 1,2,\ldots, n$, and h_i in H for $i = 1,2,\ldots,n$); further, let $T(0) = 0$. Then $T^2 = \frac{1}{4}I$, where I is the identity mapping. Thus, T^2 is a contraction. But T is not continuous at e^x; that is,

$$\lim_{n\to\infty} T\left(\sum_{k=0}^{n} \frac{1}{k!}\, x^k\right) \neq T(e^x) = \frac{1}{2},$$

because

$$\lim_{n\to\infty} [T(1 + \sum_{k=1}^{n} \frac{1}{k!} x^k)] = \lim_{n\to\infty} [\frac{1}{2}e^x + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k!} x^k]$$

$$= \frac{1}{2} e^x + \frac{1}{2} (e^x - 1)$$

$$= e^x - \frac{1}{2}.$$

Example 4. It is of interest to notice that an example of a discontinuous transformation T, with T^2 a contraction, can be given even when the metric space R is the set of real numbers. Let the numbers 1 and Π be contained in a Hamel basis H for the real numbers (ie, a set H of rationally independent real numbers such that every non zero real number may be uniquely written as a finite sum, $y = \sum_{i=1}^{n} \alpha_i h_i$, where n is a positive integer, the α_i are non zero rational numbers, and the h_i are numbers of H; [G-Hamel [39]]. The transformation T will be defined, for elements of H, as follows:

$$T(1) = \pi/2, \quad \text{and} \quad T(\pi) = \frac{1}{2}\cdot 1,$$

while $T(h) = \frac{1}{2} h$ for any number of H which is different from 1 or π. The definition of T may be extended, from H to all the real numbers, by defining $T(y) = \sum_{i=1}^{n} \alpha_i T(h_i)$ for any non zero real number $y = \sum_{i=1}^{n} \alpha_i h_i$; and by putting $T(0) = 0$. The transformation T satisfies $T(T(y)) = \frac{1}{4} y$ for every real y; hence T^2 is a contraction. But T cannot be continuous, For, from the way it was defined, T satisfies the Cauchy functional equation

$$T(x) + T(y) = T(x + y) .$$

If the function T were continuous, then it would have to be linear; that is,

$$T(y) = Cy,$$

for some real number C, and any real y. Since $C = T(1) = \pi/2$, one would then have that $T(\pi) = C.\pi = \pi^2/2$, contradicting the original definition of T which states that $T(\pi) = \frac{1}{2}$.

2.3.4 Considering the simplicity and usefulness (43-52 for example [47]) of Banach's contraction principle, it is surprising that only recently have there been attempts to generalize it. Probably the most natural generalization that one can make is to localize condition (1), given by Edelstein [31].

In the local version of Banach's contraction principle the following definitions were introduced by Edelstein [31].

2.3.5 Definition. A mapping T of X into itself is said to be locally contractive if for every x in X, there exist ε and λ $(\varepsilon > 0, 0 \leq \lambda < 1)$; which may depend on x, such that

(2) $\quad p, q$ in $S(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ implies

$$d(T(p), T(q)) < \lambda d(p, q), \; p \neq q.$$

2.3.6 Definition. A mapping T of X into itself is said to be (ε, λ)-uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Remark 1. A globally contractive (contraction) mapping can be regarded as a (∞, λ) - uniformly locally contractive mapping.

Remark 2. For some special spaces every locally contractive mapping is globally contractive. For example:

2.3.7 If X is convex, complete metric space, then every mapping T of X into itself which is (ε, λ)-uniformly contractive is also globally contractive with the same λ.

To show that the condition (1) of 2.1.10 and 2.3.6 are equivalent we need the following definition, which is due to Bing [7].

Definition. A metric space X is said to be convex provided x and y in X imply there exists z in X such that $d(x, z) = d(z, y) = (\frac{1}{2})d(x, y)$.

Proof of 2.3.7. A theorem by Menger [11 p.41] states that a convex and complete metric space contains, together with x and y, a metric segment whose extremities are x and y; that is, a subset isometric to an interval of length $d(x, y)$.

Using this fact we see that if $x, y \in X$, then there are points $x = x_0, x_1, \ldots, x_n = y$ such that

$$d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i) \quad \text{and} \quad d(x_{i-1}, x_i) < \varepsilon.$$

Hence
$$d(T(x), T(y)) \leq \sum_{i=1}^{n} d(T(x_{i-1}), T(x_i))$$
$$< \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) = \lambda d(x, y).$$

Hence the theorem.

Remark. It is quite easy to exhibit spaces which admit locally contractive, or even uniformly locally contractive mappings which are not globally contractive. The following is a simple example:

Consider the circular arc described in the complex z-plane by

$$X = \{\exp(it) : 0 \leq t \leq 3\pi/2\},$$

and let (X, d) consist of X with the metric induced by that of the Euclidean plane. The map $T:X \longrightarrow X$ given by

$$T(\exp(it)) = \exp(it/2),$$

is not globally contractive, since

$$|\exp(i3\pi/2) - \exp(0)| < |\exp(i3\pi/4) - \exp(0/2)|;$$

but it is easily shown to be uniformly locally contractive.

2.3.8 Definition. A metric space X will be said to be η-chainable if for every $a, b \in X$ there exists an η-chain; that is, a finite set of points $a = x_0, x_1, \ldots, x_n = b$ (n may depend on both a and b) such that $d(x_{i-1}, x_i) < \eta$ $(i = 1, 2, \ldots, n)$.

The concept of η-chainability is apparently due to Fréchet [38]. Fréchet and his contemporaries, and later Whyburn [73], made use of this concept in its role as a generalization of connectedness in the context of a metric space. It has also been found useful in several papers (see [31] - [32]) on various extensions of Banach's contraction mapping theorem.

Before stating and proving the main result connected with the extension of Banach's contraction principle we would like to give here some results related to well-chained metric spaces given in Choquet [24].

2.3.9 Theorem. Every connected metric space X is well-linked.

Proof. Let $a \in X$ and let $X(a, \varepsilon)$ be the set of points x of X which can be joined to a by a chain of steps at most equal to ε. This set is not empty, as it contains a; it is open, since if $x \in X(a, \varepsilon)$, the same is true for every y such that $d(x,y)<\varepsilon$; it is closed, since if x is an accumulation point of $X(a, \varepsilon)$, there exist points y of $X(a, \varepsilon)$ such that $d(x,y)<\varepsilon$.

Since X is connected we have $X(a, \varepsilon) = X$; in other words, every point b of X can be joined to a by chain of steps at most equal to ε. Thus X is well-linked.

Remark. A well-linked metric space need not be connected. For example, the set Q of rationals is well-linked but not connected.

However, this converse holds if X is compact.

2.3.10 Theorem. For a compact metric space, the properties of being connected and being well-linked are equivalent.

Proof. We have to show only one half of this equivalence. Thus let X be a compact metric space. If it is not connected, there exists a partition of X into two non empty closed sets X_1 and X_2. Since X_1 and X_2 are compact the distance between them is non zero. A point of X_1 cannot be joined to a point of X_2 by a chain of steps less than $\delta/2$, for if $(a_1, a_2, \ldots, a_n)$ is such a chain, let i be the smallest index such that $a_i \in X_2$; then $a_{i-1} \in X_1$ and $d(a_{i-1}, a_i) < \delta/2$, in contradiction with $d(X_1, X_2) = \delta$.

In other words, if X is well-linked, it is also connected.

Edelstein [31] has given the following theorem:

Let X be a complete ε-chainable metric space and T be a mapping of X into itself which is (ε, λ)-uniformly locally contractive. Then there exists a unique point ξ in X such that $T(\xi) = \xi$.

In case T does not satisfy the above condition but a suitable power T^p of T does, then we have the following:

2.3.11 <u>Theorem</u>. Let T be a mapping of a complete ε-chainable metric space X into itself and suppose that T^p is (ε, λ)-uniformly locally contractive; then there exists a unique point a_0 in X such that $T(a_0) = a_0$.

<u>Proof</u>. We set $T^p = g$.

Let x be an arbitrary point of X. Consider the ε-chain

$$x = x_0, x_1, \ldots, x_k = g(x).$$

By the triangle inequality $d(x, g(x)) \leq \sum_{i=1}^{k} d(x_{i-1}, x_i)$

$$< k\varepsilon \quad \ldots\ldots\ldots\ldots(1).$$

For a pair of consecutive points of the ε-chain the condition p, q in $s(x, \varepsilon) = \{y | d(x, \varepsilon) < \varepsilon\}$ implies that $d(g(p), g(q)) < \lambda d(p, q)$, $p \neq q$, is satisfied.

Let a be an arbitrary point in X. Set $a_1 = ga$, $a_2 = ga_1 = g^2a$ and in general let $a_n = ga_{n-1} = g^n a$. We shall show that the sequence $\{a_n\}$ is a Cauchy. In fact,

$$
\begin{aligned}
d(a_n, a_m) = d(g^n(a), g^m(a)) &< \lambda^n d(a, a_{m-n}) \\
&< \lambda^n\{d(a, a_1) + d(a_1, a_2) + \ldots + d(a_{m-n-1}, a_{m-n}). \\
&< \lambda^n d(a, a_1)\{1 + \lambda + \lambda^2 + \ldots + \lambda^{m-n-1}\} \\
&< \frac{\lambda^n}{1-\lambda} d(a, a_1) \\
&< \frac{\lambda^n}{1-\lambda} \, d(a, g(a)) \\
&< \frac{\lambda^n}{1-\lambda} \, k\varepsilon \qquad \text{from (1).}
\end{aligned}
$$

Since $\lambda < 1$, this quantity is arbitrarily small for the sufficiently large n. Thus $\{a_n\}$ is a Cauchy sequence. Since X is complete, $\lim\limits_{n\to\infty} a_n$ exists. We set $a_0 = \lim\limits_{n\to\infty} a_n$. Then, by virtue of continuity of g, $g(a_0) = g \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} ga_n =$ $\lim\limits_{n\to\infty} a_{n+1} = a_0$. Thus g has a fixed point a_0 i.e. $g(a_0) = a_0$.

In order to complete the proof we have to show that $a_0 = \lim\limits_{n\to\infty} a_n$ is a unique fixed point satisfying $g(a_0) = a_0$. Let a_0 and b_0 be two different fixed points i.e., $g(a_0) = a_0$ and $g(b_0) = b_0$. Then $a_0 \neq b_0$ implies $d(a_0, b_0) > 0$.

Now $d(g(a_0), g(b_0)) = \lim d(g(a_n), g(b_n))$

$$= d(g(a_n), g(b_n)) \quad \text{when} \quad n \to \infty$$

$$< \frac{\lambda^n}{1-\lambda} d(a_0, b_0)$$

$$= \frac{\lambda^n}{1-\lambda} d(a_0, g(b_0)) \quad \ldots\ldots\ldots\ldots(2)$$

Let a be an arbitrary point of X. Consider the chain

$$a_0 = x_1, x_2, \ldots, x_{k+1} = b_0 = g(b_0).$$

Then, by the triangle inequality, $d(a_0, g(b_0))$

$$\leq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) < k\varepsilon \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Therefore by (3),(2) reduces to

$$d(g(a_0), g(b_0)) < \frac{\lambda^n}{1-\lambda} k\varepsilon \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Thus $d(a_0, b_0) = 0$, which is impossible unless $a_0 = b_0$. Therefore a_0 is a unique fixed point for g.

Now a_0 is the unique fixed point of g. The relation $g(a_0) = a_0$ gives

$$T(g(a_0)) = T(a_0);$$

but $$T^{p+1} = T(g) = g(T).$$

Therefore $$g(T(a_0)) = T(a_0).$$

Hence $T(a_0)$ is a fixed point for g. The uniqueness of this point shows that $T(a_0) = a_0$; in other words, a_0 is also a fixed point for T. The uniqueness of the fixed point of T follows from by the fact that every point of T is a fixed point for T^p.

Corollary. Let X be a convex complete metric space and T an (ε, λ)-uniformly locally contractive mapping of X into itself. Then T has a unique fixed point.

Proof. By 2.3.7. T becomes a globally contractive mapping of a complete metric space X into itself. Hence, by Banach's contraction principle T has a unique fixed point.

From Theorem 2.3.11, there follows a corollary regarding expansive mappings. These mappings can be defined in a natural way.

2.3.12 Definition. A mapping T of X into itself is said to be locally expansive if for every $x \in X$ there exist ε and λ $(\varepsilon > 0, \lambda > 1)$, which may depend on x, such that

$$p, q \text{ in } S(x, \varepsilon) = \{y | d(x, y) < \varepsilon\} \text{ implies}$$
$$d(T(p), T(q)) > \lambda d(p, q), \quad p \neq q.$$

2.3.13 Definition. A mapping T of X into itself is said to be (ε, λ)-uniformly locally expansive if both ε and λ do not depend on x.

2.3.14 Corollary. If T is a one to one (ε, λ)-uniformly locally expansive mapping of a metric space Y onto an ε-chainable complete metric space $X \supset Y$, then there exists a unique ξ such that $T(\xi) = \xi$.

Proof. This assertion is an immediate consequence of the fact that for the inverse mapping $T^{-1}(x)$ all the assumptions of the Theorem are satisfied.

Remark. It is obvious that a connected metric space is ε-chainable for every $\varepsilon > 0$ (as proved in Theorem 2.3.9). Now suppose that X is a connected and complete metric space. It is natural to ask whether there exists a fixed point if the condition of (ε, λ)-uniformly contractivity is replaced by the following one.

for every x_0 in X there exists a sphere

$S_0 = S(x_0, \varepsilon(x_0))$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y),$$

for every x, y in S_0, where $\alpha < 1$. Such a mapping will be said to be α-locally contractive.

The counter example given in [58] shows that this is not the case. In the same paper it is also proven that some special assumption is necessary in order to guarantee the existence of a fixed point. The required condition is the following:

2.3.15 Theorem. Let T be an α-locally contractive mapping of a complete metric space into itself such that

(1) For some x_0 in X the point x_0 and Tx_0 are connected by an arc $C \subset X$ of finite length. Then there exists a fixed point.

Remark. If assumption (1) is dropped, or if C is not of finite length, then a fixed point may fail to exist. The counter example may be found in [58].

2.4 CONTRACTIVE MAPPING AND FIXED POINTS.

2.4.1 Definition. A mapping T of a metric space X into itself is said to be contractive if for every two distinct points x, y in X,

$$(1) \qquad d(T(x), T(y)) < d(x, y).$$

A contractive mapping is clearly continuous; and if such a mapping has a fixed point, then this fixed point is unique. Contraction (i.e., $|T(x) - T(y)| \leq \alpha |x-y|$ for all x, y in X and some fixed α, $0 \leq \alpha < 1$) is an example of contractive mapping.

Remark. It is interesting to observe that the condition (1) is not sufficient for the existence of a fixed point, as will be seen in the following examples.

Example 1. Let X be the set of real numbers with the usual definition of distance. Let

$$T(x) = x + \frac{\pi}{2} - \arctan x.$$

Since $\arctan x < \frac{\pi}{2}$ for every x, the operator T has no fixed point. At the same time, if $x < y$, then

$$T(y) - T(x) = y - x\ (\arctan y - \arctan x),$$

and by Lagrange's formula,

$$T(y) - T(x) = y - x - \frac{y - x}{1 + z^2} \qquad (x < z < y).$$

If we had

$$|T(y) - T(x)| \geq |y - x|,$$

then this would mean that

$$\left|1 - \frac{1}{1 + z^2}\right| \geq 1,$$

but this inequality is not satisfied for any z. Therefore we always have

$$|T(y) - T(x)| < |y - x|.$$

<u>Example 2.</u> Let X be the space of all real numbers, and define the function

$$T(x) = \log\ (1 + e^x).$$

Differentiating we obtain

$$T'(x) = \frac{e^x}{1 + e^x} < 1,$$

i.e. T is a contractive mapping, and it is easy to see that T has no fixed point.

Example 3. Let $X = \{x | x \geq 1\}$ with usual distance $d(x, y) = |x - y|$. Let $T: X \longrightarrow X$ be given by $T(x) = x + \frac{1}{x}$. Then T is contractive, but it has no fixed point.

Whether directly influenced by Banach's result or not several authors have examined mappings which satisfy condition (1). Some of these are listed in the bibiliography ([32], [33], [34], and [23]). The most recent result concerning mappings satisfying condition (1) is that of Edelstein [33], given below.

2.4.2 Theorem. If T takes R^n into itself, if T satisfies (1), and if there is an $x \in R^n$ such that the subsequence of $\{T^n(x)\}$ converges, then there exists a fixed point under T.

We note that in both conditions (1,2.1.10),(2,2.3.5) λ is independent of x and y. This suggests that one might generalize these conditions by letting λ vary with x and y. Cheney and Goldstein [23] as well as Edelstein [32] did this in the following manner. They required that the mapping T satisfy

(2) $0 < d(x, y)$ implies $d(T(x), T(y)) < d(x, y)$.

Note that this is equivalent to requiring that for $d(x, y)$ greater than 0, $d(T(x), T(y)) \leq \lambda(x, y)\, d(x, y)$ where $\lambda(x, y) < 1$ for all x and y in X. In both [23] and [32] we find the following theorem.

2.4.3 Theorem. Suppose T satisfies (2) and there exists x in X such that some subsequence of $\{T^n(x)\}$ converges. Then there exists a unique z in X such that $T(z) = z$.

In [32] Edelstein also considered mappings satisfying the following localized version of (2).

(3) There exists $\varepsilon > 0$ such that $0 < d(x, y) < \varepsilon$ implies $d(T(x), T(y)) < d(x, y)$.

He obtained the following two results, among others.

2.4.4. Theorem. If T satisfies (3) and there exists x in X such that $\{T^{n_i}(x)\} \longrightarrow z$ in X, then there exists at least one periodic point under T.

2.4.5. <u>Theorem.</u> If X is compact and ε-chainable and if T satisfies (3), then there exists a unique fixed point under T.

Here we [69] would like to give direct, rather simple proofs of the above theorems. For that purpose we need the following definitions introduced by Edelstein [33].

2.4.6. <u>Definition.</u> A mapping $T:X \longrightarrow X$ of a metric space X into itself is said to be nonexpansive (ε-non expansive) if the condition

$$(4) \qquad d(T(p), T(q)) \leq d(p, q)$$

holds for all $p, q \in X$ (for all p, q with $d(p, q) < \varepsilon$). Isometry (i.e. $|T(x) - T(y)| = |x - y|$ for all $x, y \in X$) is a simple example of non expansive mapping.

Mappings as above satisfying (4) with the strict inequality sign for all p, q in X, $p \neq q$ (for all p, q with $0 < d(p, q) < \varepsilon$) are called contractive (ε-contractive).

2.4.7 <u>Definition.</u> A point $y \in Y \subset X$ is said to belong to T-closure of Y, $y \in Y^T$, if $T(Y) \subset Y$ and there is a point $\eta \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < \ldots < n_i < \ldots)$, so that $\lim T^{n_i}\{n\} = y$.

2.4.8. <u>Definition</u>. A sequence $\{x_i\} \subset X$ is said to be an isometric (ε-isometric) sequence if the condition

$$d(x_m, x_n) = d(x_{m+k}, x_{n+k})$$

holds for all $k, m, n = 1, 2, \ldots$; (for all $k, m, n = 1, 2, \ldots$; with $d(x_m, x_n) < \varepsilon$). A point x in X is said to generate an isometric (ε-isometric) sequence under T, if $\{T^n(x)\}$ is such a sequence.

<u>Example</u>. In R^2 the sequence $\{\cos n\phi, \sin n\phi | n = 0, 1, 2, \ldots\}$ is a simple example of an isometric sequence. When $\pi^{-1}\phi$ is rational, the range of $\{x_i\}$ is the set of vertices of a regular polygon, otherwise it is a dense subset of the unit circle. We may, then, think of an isometric sequence as a generalization of a regular polygon.

The following results are due to Edelstein [33].

2.4.9. <u>Theorem</u>. If $T:X \longrightarrow X$ is ε-nonexpansive and x in X^T, then a sequence $\{m_J\}$, $(m_1 < m_2 < \ldots)$, of positive integers exists so that $\lim\limits_{J\to\infty} T^{m_J}(x) = x$. [Hence, in particular, $(X^T)^T = X^T$.]

2.4.10 <u>Theorem</u>. If $T:X \longrightarrow X$ is an ε-nonexpansive mapping of X into itself, then each x in X^T generates an ε-isometric sequence.

2.4.11 <u>Theorem</u>. If $T:X \longrightarrow X$ is a nonexpansive and $x \in X^T$, then x generates an isometric sequence.

Here we would like to remark that one could prove more than that given by Edelstein [ ], i.e. it can be proved that even in this case the following holds:

2.4.12 <u>Theorem</u>. If $T:X \longrightarrow X$ is nonexpansive and x in X^T, then x generates an isometric sequence, and T has a unique fixed point, equality holds when $x = y$, $x, y \in X$.

<u>Proof</u>. By Theorem 2.4.11 T generates an isometric sequence; therefore $d(x, T(x)) = d(T(x), T^2(x))$, but T is nonexpansive, so that $d(T(x), T^2(x)) \leq d(x, T(x))$. This shows that $d(x, T(x)) = 0$, so that $x = T(x)$. Also, if y in X and $y = T(y)$ then $d(T(x), T(y)) = d(x, y)$, contradicting the fact that T is non-expansive unless $x = y$. Thus x is unique fixed point for T.

<u>Corollary</u>. If $T:X \longrightarrow X$ is an ε-nonexpansive mapping of

X into itself, then each x in X^T generates an ε-isometric sequence, and T has a periodic fixed point i.e., there exists a positive integer k such that $T^k(x) = x$.

Remark. It is natural to ask whether Theorem 2.4.4 would remain true if (2) is substituted by a localized version such as $p \neq q$; p, q in $S(x, \varepsilon(x))$ implies $d(T(p), T(q)) < d(p, q)$ where

$$S(x, \varepsilon(x)) = \{z | d(z, x) < \varepsilon(x)\}.$$

The following example serves to show that this is not the case.

Example.

$$X = \{(\tfrac{1}{n}, i) | n = 2i, 2i+1, \ldots,\} \cup X_0 \cup Y_0$$

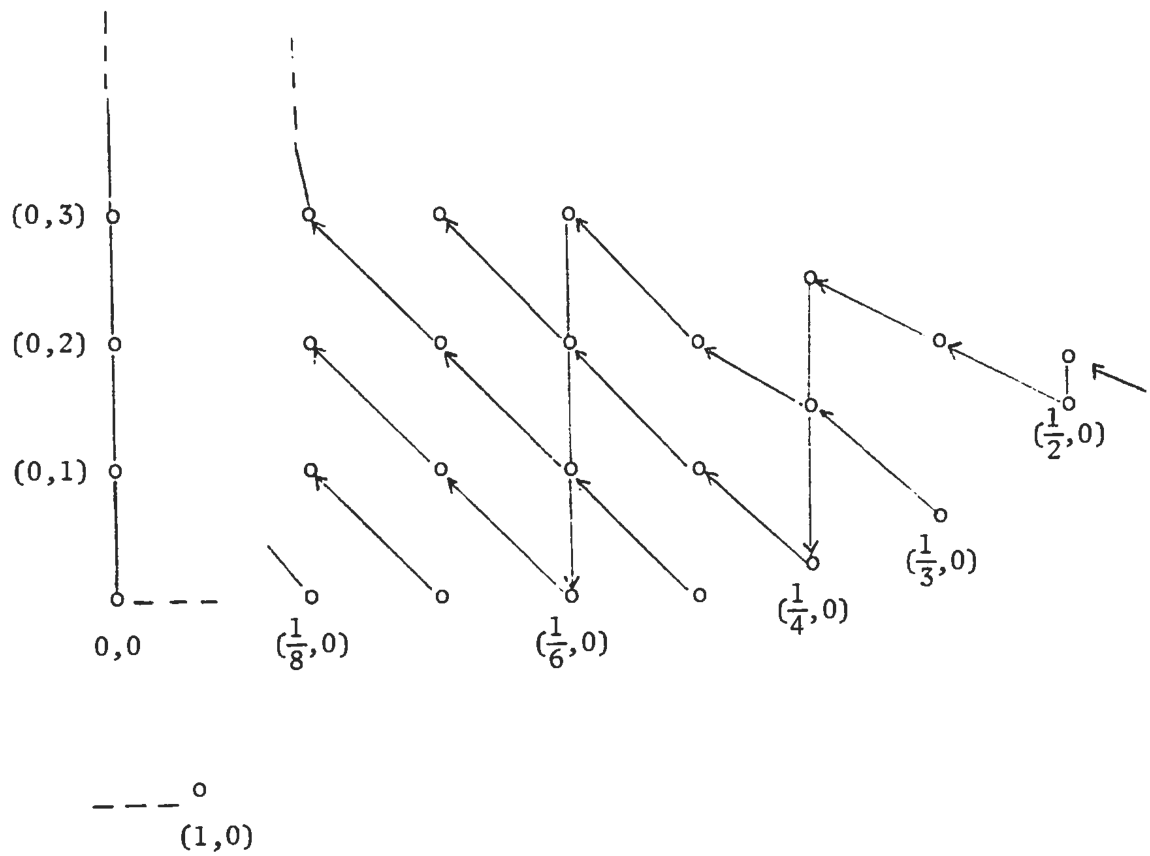
$$X_0 = \{(\tfrac{1}{n}, 0) | n = 1, 2, \ldots\}, \; Y_0 = \{(0, i) | i = 0, 1, 2, \ldots\}.$$

$$T(\tfrac{1}{n}, i) = \begin{cases} (\tfrac{1}{n+1}, i + 1) & \text{if } n \neq 2i \\ (\tfrac{1}{n}, 0) & \text{if } n = 2i \end{cases}$$

$$T(0, i) = (0, i + 1); \quad i = 0, 1, 2, \ldots .$$

X is taken in the metric of the euclidean plane. Here condition

(2) and (5) are satisfied, and although $T^n(1, 0)$ contains a subsequence which converges to $(0, 0)$ this last point is not periodic.



In [23] Cheney and Goldstein have proved the following theorem.

Let T be a map of a metric space X into itself such that

(i) $d(T(x), T(y)) \leq d(x, y)$;

(ii) if $x \neq T(x)$, then $d(T(x), T^2(x)) < d(x, T(x))$;

(iii) for each x, the sequence $T^n(x)$ has a cluster point.

Then for each x the sequence $T^n(x)$ converges to a fixed point of T.

Here we would like to remark that by relaxing conditions (ii) and (iii) we get a unique fixed point. Although the theorem has already been given by Edelstein [32], we prefer the direct rather simple proof here.

2.4.13 Theorem. Let T be a map of compact metric space X into itself such that

(i) $d(T(x), T(y)) \leq d(x, y)$, equality holds when $x = y$.

Then T has a unique fixed point.

Proof. The compactness of X and the condition (i) imply that each x in X^T generates an isometric sequence, [33 Theorem 1']. Therefore, by the definition of isometric sequence,

$d(x, T(x)) = d(T(x), T^2(x))$; but from condition (i) we have $d(T(x), T^2(x)) \leq d(x, T(x))$. This shows that $d(x, T(x)) = 0$, which implies $x = T(x)$ i.e., x is a fixed point for T. To prove the uniqueness, let us assume that y is another point such that $y \neq x$ and $T(y) = y$. Then $d(T(x), T(y)) = d(x, y)$ contradicting the condition (i) unless $x = y$. Thus x is a unique fixed point for T.

In the same vein Rakotch [59] allowed λ to vary in the restricted way and was able to obtain a fixed point theorem on complete metric spaces. The exact conditions Rakotch imposed on λ are following:

2.4.14 <u>Definition</u>. Denote by F the family of functions $\lambda(x, y)$ satisfying the following conditions:

(i) $\lambda(x, y) = \lambda(d(x, y))$, i.e., λ is dependent on the distance between x and y only.

(ii) $0 \leq \lambda(d) < 1$ for every $d > 0$.

(iii) $\lambda(d)$ is monotonically decreasing function of d.

We give here the localized form of Rakotch's theorem.

2.4.15 <u>Theorem</u>. If T is a contractive mapping of a complete

ε-chainable metric space X into itself satisfying

$$0<d(x,y)<\varepsilon \Longrightarrow d(T(x), T(y)) \leq \lambda(x, y)d(x, y)$$

for every x, y in X and $\lambda(x, y) \in F$, then T has a unique fixed point.

Proof. Since (X, d) is ε-chainable we define, for every x, y in X

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i),$$

where the infimum is taken over all ε-chains $x_0, x_1, x_2, \ldots, x_n$ joining $x = x_0$ and $y = x_n$. Then d is a distance function on X satisfying

(i) $d(x, y) \leq d_\varepsilon(x, y)$

(ii) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$.

From (ii) it follows that a sequence $\{x_n\}$ in X is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect to d. Hence since (X, d) is complete, (X, d_ε) is also a complete metric space. Moreover T is a contractive mapping with respect to d_ε. Given x, y in X, and any

ε-chain $x_0, x_1, \ldots, x_n$ with $x_0 = x$, and $x_n = y$, we have

$$d(x_{i-1}, x_i) < \varepsilon \qquad (i = 1,2,\ldots,n),$$

so that

$$\begin{aligned} d(T(x_{i-1}),T(x_i)) &\leq \lambda(x_{i-1}, x_i)\, d(x_{i-1}, x_i) \\ &= \lambda(d(x_{i-1}, x_i))d(x_{i-1}, x_i) \\ &< \lambda(\varepsilon).\varepsilon \qquad i = 1,2,\ldots,n. \end{aligned}$$

Hence $Tx_0, Tx_1, \ldots, Tx_n$ is an ε-chain joining Tx and Ty, and

$$\begin{aligned} d_\varepsilon(T(x), T(y)) &\leq \sum_{i=1}^{n} d(T(x_{i-1}), T(x_i)) \\ &\leq \sum_{i=1}^{n} \lambda(d(x_{i-1}, x_i))\, d(x_{i-1}, x_i), \end{aligned}$$

since $x_0, x_1, \ldots, x_n$ is an arbitrary ε-chain, we have

$$d_\varepsilon(T(x), T(y)) \leq \lambda(d_\varepsilon(x, y))\, d_\varepsilon(x, y).$$

Therefore, by corollary to Theorem 2 [59], T has a unique fixed point.

The following definitions and theorems related to Banach's contraction principle are due to Janos [42].

2.4.16 Definition. Let X be a completely regular space. By $D = \{f_i | i \in F\}$ we always understand a family of pseudometrics on X inducing the given topology on X. Let $T:X \longrightarrow X$. It is natural to say that T is contractive under D if and only if $\forall\, i \in F\ \exists\, \alpha_i$ in $(0, 1)$, $\forall\, x, y \in X$, $d_i(T(x), T(y)) \leq \alpha_i d_i(x, y)$.

2.4.17 Definition. Let X be a metrizable topological space, m the set of all metrics on X inducing the given topology, and $T:X \longrightarrow X$ a continuous mapping. If $a \in X$ is a fixed point of T, we will say a is of contractive character, if, for some $T \in (0, 1)$, there exists a neighbourhood $N(a)$ of an invariant under T and a metric d in m such that $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in N(a)$.

2.4.18 Theorem. Let X be a complete regular and $T;X \longrightarrow X$. Let D be a family of pseudometrics on X with respect to which X is complete and under which T is contractive. Then T has a unique fixed point on X.

2.4.19 Theorem. Let X be a compact Hausdorff, $T:X \longrightarrow X$, and $D = \{f_i | i \in F\}$ such that $\forall\, i \in F$ and $\forall\, x, y \in X$, $d_i(T(x), T(y)) \leq d_i(x, y)$. Then T has a unique fixed point on X. Moreover, if X is metrizable, then for each $\alpha \in (0, 1)$ there exists a metric d on

X, inducing the given topology such that $\forall$ x, y in X:
$d(T(x), T(y)) \leq \alpha d(x, y)$.

2.4.20 Theorem. Let X be compact and connected and let $T: X \longrightarrow X$ be such that for some d in m and some $\varepsilon > 0$ the following condition holds:

$$0 < d(x, y) < \varepsilon \Longrightarrow d(T(x), T(y)) < d(x, y).$$

Then T has a unique fixed point a, a is of contractive character and $T^n(x) \longrightarrow a$ for all $x \in X$.

Recently Sims [66] has given the following generalization of Banach's contraction principle:

If T is a contraction on a bounded and complete pseudometric space (X, d), then T has a unique fixed point.

In the end of this section we give a generalization of the following theorem of Bers [5]. For that we need the following:

2.4.21 Definition. Let K be a subset of a normed vectorspace. A mapping T of K into itself is called a contracting mapping if, for all $x, y \in K$,

$$||T(x) - T(y)|| \leq \alpha||x - y|| \quad \text{where} \quad 0 < \alpha < 1.$$

2.4.22 Definition. A mapping T of a subset K of a normed vectorspace into itself is called a nonexpanding mapping if, for all $x, y \in K$,

$$||T(x) - T(y)|| \leq ||x - y||.$$

Bers [5 p.81] has given the following theorem:

Theorem. Let $K = \{x \mid ||x|| \leq 1\}$ be a subset of a Banach space, and let T be a continuous/contractive mapping of K into itself. Then T has one and only one fixed point.

Here we would like to remark that the above theorem may be put in even a general set as follows. At the same time the condition of continuity is superfluous.

Theorem. Let K be a closed subset of a Banach space and let T be a contracting mapping of K into itself. Then T has one and only one fixed point.

Proof. Let x_0 be an arbitrary point in K and consider the sequence $\{x_n\}$. Set $x_1 = T(x_0)$, $x_2 = T(x_1)$, ...

$$\text{Now}\quad ||x_{n+1} - x_n|| = ||T(x_n) - T(x_{n-1})||$$

$$\leq \alpha||x_n - x_{n-1}||,$$

$$\text{and}\quad ||x_n - x_{n-1}|| = ||T(x_{n-1}) - T(x_{n-2})||$$

$$\leq \alpha||x_{n-1} - x_{n-2}||.$$

$$\text{Hence}\quad ||x_{n+1} - x_n|| \leq \alpha\cdot\alpha||x_{n-1} - x_{n-2}||$$

$$= \alpha^2||x_{n-1} - x_{n-2}||.$$

Therefore by continuing this process we have $||x_{n+1} - x_n|| \leq \alpha^n||x_1 - x_0|| = \alpha^n M$, where $M = ||x_1 - x_0||$.

Using this inequality we will show that the sequence $\{x_n\}$ is a Cauchy sequence.

$$||x_{n+p} - x_n|| \leq ||x_{n+p} - x_{n+p-1}|| + ||x_{n+p-1} - x_{n+p-2}||$$

$$+ \ldots\ldots + ||x_{n+1} - x_n||$$

$$\leq M\alpha^{n+p-1} + M\alpha^{n+p-2} + \ldots\ldots + M\alpha^n$$

$$\leq M\alpha^n \{1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{p-1}\}$$

$$\leq M\,\frac{\alpha^n}{1-\alpha}\,.$$

Since $\alpha < 1$, $\frac{\alpha^n}{1-\alpha} M$ tends to zero as n tends to ∞, and hence $\{x_n\}$ is a Cauchy sequence.

Since K is closed subset of a Banach space, K is complete. Hence x_n converges to some point $x.$ in K. Set $x. = T \lim_{n\to\infty} x_n$.

Then by the virtue of continuity of T, $Tx. = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$. Thus the existence of a fixed point is proved.

Uniqueness. Assume that x and y are two fixed points of T, i.e., $T(x) = x$ and $T(y) = y$. Then since T is a contracting mapping we have

$$||x - y|| = ||T(x) - T(y)|| \leq \alpha||x - y||$$

i.e. $(1 - \alpha)||x - y|| = 0$, whence $||x - y|| = 0$, so that x coincides with y. The theorem is proved.

2.5 EVENTUALLY CONTRACTIVE MAPPINGS

In [2] Bailey considered continuous mappings obeying the following generalization of conditions (2) and (3) of 2.4. Throughout this section X will denote a compact metric space and T will denote the mapping of X into X.

2.5.1 Definition. A continuous mapping T is eventually contractive if for every distinct pair $x, y \in X$ there exists $n(x, y)$, a member of I^+ (the positive integers), such that

(1) $$d(T^n(x), T^n(y)) < d(x, y).$$

2.5.2 Definition. A continuous mapping T is ε-eventually contractive (locally contractive) if there exists $\varepsilon > 0$ such that if x and y are distinct and $d(x, y) < \varepsilon$ then there is $n(x, y)$, a member of I^+, such that

(2) $$d(T^n(x), T^n(y)) < d(x, y), \quad \text{whenever} \quad d(x, y) < \varepsilon.$$

2.5.3 Definition. x is proximal to y under T if for each $\alpha > 0$ there exists n, a member of I^+, such that $d(T^n(x), T^n(y)) < \alpha$. If x and y are not proximal under T they are said to be distal under T. If for each $\alpha > 0$, there exists

n in I^+ such that $d(T^m(x), T^m(y)) < \alpha$ for all $m \geq n$, then x and y are said to be asymptotic under T. Note that we need not require $x \neq y$.

Bailey [3] obtained the following results among others:

2.5.4 Theorem. Suppose T satisfies condition (1). Then every pair of points in X is proximal under T.

2.5.5 Theorem. If T satisfies (2) then $d(x, y) < \varepsilon$ implies x and y are proximal under T.

2.5.6 Theorem. Suppose T satisfies (2) and $0 < \theta < \delta < \varepsilon$. Then there exists $N(\theta, \delta)$ in I^+ such that $\theta < d(x, y) < \delta$ and $d(T^k(x), T^k(y)) < d(x, y)$ imply $d(T^{k+J}(x), T^{k+J}(y)) < d(x, y)$ for some J such that $0 < J \leq N(\theta, \delta)$.

2.5.8 Theorem. If T satisfies (2) and X is convex then T satisfies (1).

2.5.9 Theorem. If T satisfies [3,2.4] and X is ε-chainable, then T satisfies(1).

We prove the following theorems:

2.5.10 Thecrem. Let X be a convex, ε-chainable/complete metric space, and T be a mapping of X into itself which is (ε, λ)-uniformly locally contractive. Then T is also eventually contractive.

Proof. A theorem by Menger [11 p. 41] states that a convex and complete metric space contains, together with a and b, a metric segment whose extremities are a and b, that is, a subset isometric to an interval of length $d(a, b)$.

Using this fact we see that if $p, q \in X$ then there are points $p = x_1, x_2, \ldots, x_n = q$ such that $d(p, q) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$ and $d(x_{i-1}, x_i) < \varepsilon$. Hence $d(T(p), T(q)) \leq \sum_{i=1}^{n} d(T(x_{i-1}, T(x_i))$ $< \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) = \lambda d(p, q)$.

By definition it is clear that every globally contractive mapping is a contractive mapping i.e., the mapping $d(T(p), T(q)) < \lambda d(p, q)$, $(0 \leq \lambda < 1)$ may be regarded as $0 < d(p, q) \Longrightarrow d(T(p), T(q)) < d(p, q)$. Also by definition every contractive mapping in the convex compact metric space may be regarded as an ε-contractive mapping.

As X is ε-chainable, therefore for distinct points p and q there exists $p = p_1, p_2, \ldots, p_n = q$ such that $d(p_i, p_{i+1}) < \varepsilon$ for $i = 0, 1, 2, \ldots, n-1$. By Corollary 1 to Theorem 2 [2] p_i is asymptotic to p_{i+1} under T for $i = 0, 1, 2, \ldots, n-1$. Hence there exists $m \in I^+$ such that

$d(T^m(p_i), T^m(p_{i+1})) < \dfrac{d(p, q)}{n}$ for $i = 0, 1, \ldots, n-1$. Therefore

$$d(T^m(p), T^m(q)) \leq \sum_{i=0}^{n-1} d(T^m(p_i), T^m(p_{i+1})) < \frac{nd}{n}(p, q) = d(p, q).$$

2.5.11 <u>Theorem.</u> Let X be an ε-chainable metric space and T be an ε-contractive mapping

i.e., $0 < d(x, y) < \varepsilon \Longrightarrow d(T(x), T(y)) < d(x,y)$.

Then every pair of points is asymptotic under T.

<u>Proof.</u> Since X is ε-chainable, we define for $p, q \in X$

$$d(p, q) = \inf_{C(p,\, q)} \sum_{i=1}^{n} d(x_{i-1}, x_i),$$

where $C(p, q)$ denotes the collection of all ε-chains $p = x_0, x_1, \ldots, x_n = q$, ($n$ arbitrary), $d(x_{i-1}, x_i) < \varepsilon$, holds. Indeed, T is ε-contractive. We have

$$d(T(x_{i-1}), T(x_i)) < d(x_{i-1}, x_i) \text{ provided}$$

$d(x_{i-1}, x_i) < \varepsilon$.

Hence,

$$d(T(p),\ T(q)) < \inf_{C(p,q)} \sum_{i=1}^{n} d(T(x_{i-1}),\ T(x_i))$$

$$< \inf_{C(p,q)} \sum_{i=1}^{n} d(x_{i-1},\ x_i) = d(p,\ q)$$

for all p, q. Thus the mapping is contractive.

Now since X is compact and T is a contractive mapping of X into itself, therefore by Theorem 2.4.13 T ~~contains~~ has a unique fixed point x. Also the property of compactness implies that each sequence $\{T^n(x)\}$ converges to x. Therefore it follows that every pair of points is asymptotic under T.

2.5.12 Theorem. If T satisfies (1), and K is a homeomorphism of X onto X, then KTK^{-1} satisfies (1). In addition T has a unique fixed point.

Proof. By Theorem 1.3 [3] $K^{-1}(x)$ and $K^{-1}(y)$ are proximal. Also since K is a homeomorphism, therefore K and K^{-1} both are continuous. Since K^{-1} is continuous and X is compact, there exists $\delta > 0$ such that $d(w, z) < \delta$ implies $d(K^{-1}(w), K^{-1}(z)) <$ $d(x, y)$. Now $K^{-1}(x)$ and $K^{-1}(y)$ are proximal under T; therefore for each $\delta > 0$ there exists n a member of I^+ such that $d(T^n(K^{-1}(x)), T^n(K^{-1}(y)) < \delta$. Hence $d(KTK^{-1})^n(x), (KTK^{-1})^n(y))$ $= d(KT^nK^{-1}(x), KT^nK^{-1}(y)) < d(x, y)$.

Again, by Theorem 1.3 [3] x and $KTK^{-1}(x)$ are proximal under T. Now choose $\{n_i\} \subseteq I^+$ such that $n_i < n_{i+1}$ and $d((KTK^{-1})^{n_i}(x), (KTK^{-1})^{n_{i+1}}(y)) < \frac{1}{i}$.

By the compactness of X, we may assume that $\{(KTK^{-1})^{n_i}(x)\} \longrightarrow \xi$ and $(KTK^{-1})^{n_{i+1}}(y)\} \longrightarrow \eta$, for some ξ and η in X. Clearly $\xi = \eta$. Also the continuity of KTK^{-1} implies $KTK^{-1}\eta = \eta$, so that η is a fixed point of KTK^{-1}. That this point is unique is immediate.

Since KTK^{-1} has a unique fixed point η, $KTK^{-1}\eta = \eta$, or $K^{-1}KTK^{-1}(\eta) = K^{-1}\eta$, or $TK^{-1}\eta = K^{-1}\eta$. Thus $K^{-1}\eta$ is a unique fixed point of T.

Remark. A similar result for $K^{-1}TK$ has been given by Bailey in the following form:

2.5.13 Theorem. If T satisfies (1), and K is a homeomorphism of X onto X, then $K^{-1}TK$ satisfies (1).

Sehgal [63] has given the following definition and generalization of Banach's contraction principle.

2.5.14 Definition. A continuous self mapping is called an eventual global contraction if for each $x \in X = (X, d)$ and for some integer $n = n(x)$

$$d(T^n(y), T^n(x) \leq \lambda d(x, y)$$

for every $y \in X$ and some $\lambda \in [0, 1)$.

2.5.15 Theorem. An eventual global contraction of a complete metric space has a unique fixed point θ, and for every $x \in X$, $T^n(x) \longrightarrow \theta$.

As an application of the foregoing material we state and prove the following theorem concerning holomorphic mappings.

Theorem. Let T be a holomorphic mapping of a compact, convex subset M of the plane into itself such that given z in M there exists $n(z)$ in I^+ such that $|DT^n(z)| < 1$, then there exists a unique fixed point in M.

Proof. Note that since $DT(z)$ exists for all z in M and DT is continuous in M, DT^n is continuous on M for all n in I^+. Given z in M, let $n(z)$ be the smallest member of I^+

such that $|DT^{n(z)}(z)| < 1$. Also for z in M let $0(z)$ be an open sphere about z such that w in $\overline{0(z)}$ implies $|DT^{n(z)}(w)| < 1$. Now since M is compact a finite number of these $0(z)$ cover M, say $0(z_1)$, $0(z_2)$, ..., $0(z_n)$. Now let ε be the Lebesgue covering number of the above covering. $|s - t| < \varepsilon$ implies s and t are in $0(z_i)$ for some $1 \leq 1 < n$ which implies the line segment L joining s and t is in $0(z_i)$. Thus if z is on L, $|DT^{n(z_i)}(z)| < 1$. Therefore

$$|T^{n(z_i)}(s) - T^{n(z_i)}(t)| = \left| \int_s^t DT^{n(z_i)}(z)\, dz \right| \leq \int_s^t |DT^n(z)|\,|dz|$$

$< |s - t|$. Hence T satisfies (2), and since M is convex Theorem 1.18 [3] implies that T satisfies (1). Therefore by Theorem 2.5.12 there exists a unique fixed point in M.

2.6 SOME FURTHER EXTENSIONS OF BANACH'S CONTRACTION PRINCIPLE

In [67] we proved the following theorems by taking T as a mapping of a metric space X into itself such that there exists a mapping K of X into itself which has a right inverse and which makes $K^{-1}TK$ a contraction. Corollary to Theorem 1.1.5 has been used in the proof of these theorems. One can also prove these theorems easily for KTK^{-1} (where K^{-1} is left inverse of K such that $K^{-1}K = I$) by applying Theorem 1.1.6.

2.6.1 <u>Theorem</u>. Let X be a complete ε- chainable metric space; let T be a self-mapping of X into itself such that there exists a mapping K of X into itself, which has the right inverse K^{-1} and which makes the mapping $K^{-1}TK$ an (ε, λ)-uniformly contractive [i.e., there exists a real number λ with $0 \leq \lambda < 1$ such that

$$0 < d(x, y) < \varepsilon \Longrightarrow d(K^{-1}TK(x), K^{-1}TK(y))$$

$$< \lambda d(x, y), \ (x, y \in X, x \neq y)].$$

Then T has a unique fixed point.

<u>Proof</u>. Since (X, d) is ε-chainable, we define for $x, y \in X$,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

where the infimum is taken over all ε-chains $x_0, x_1, x_2, \ldots, x_n$ joining $x_0 = x$ and $x_n = y$. Then d is a distance function on X satisfying

$$\text{(1)} \quad d(x, y) \leq d_\varepsilon(x,y)$$

$$\text{(2)} \quad d(x, y) = d_\varepsilon(x, y) \quad \text{for } d(x, y) < \varepsilon.$$

From (2) it follows that a sequence $\{x_n\}$, $x_n \in X$ is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it is convergent with respect to d. Since (X, d) is complete, therefore (X, d_ε) is also complete metric space. Given $x, y \in X$ and any ε-chain $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = x$ and $x_n = y$ we have

$$d(x_{i-1}, x_i) < \varepsilon \quad (i = 1,2,\ldots,n);$$

so that

$$\begin{aligned} d(K^{-1}TK(x_{i-1}), K^{-1}TK(x_i)) &< \lambda d(x_{i-1}, x_i) \\ &< \lambda\varepsilon \quad (i = 1,2,\ldots,n). \end{aligned}$$

Since $\lambda < 1$ therefore

$$d(K^{-1}TK(x_{i-1}), \ K^{-1}TK(x_i)) < \varepsilon.$$

Hence $K^{-1}TK(x_0)$, $K^{-1}TK(x_1)$, ..., $K^{-1}TK(x_n)$ is an ε-chain joining $K^{-1}TK(x)$ and $K^{-1}TK(y)$ and,

$$d_\varepsilon(K^{-1}TK(x), K^{-1}TK(y)) < \sum_{i=1}^{n} d(K^{-1}TK(x_{i-1}), K^{-1}TK(x_i))$$

$$< \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

$x_0, x_1, x_2, \ldots, x_n$ being an arbitrary ε-chain, we have

$$d_\varepsilon(K^{-1}TK(x), K^{-1}TK(y)) < \lambda d_\varepsilon(x, y).$$

Thus $K^{-1}TK$ is a contraction with respect to (X, d_ε). Therefore $K^{-1}TK$ has a unique fixed point $\xi \in X$. Hence by Corollary to Theorem [1.1.5] T has a unique fixed point.

2.6.2 Theorem. Let X be a complete ε-chainable metric space. Let T be a mapping of X into itself such that there exists a mapping K of X into itself which has a right inverse K^{-1} and which for some positive integer n makes the mapping $K^{-1}T^nK(\varepsilon,\lambda)$-uniformly locally contractive (where T^n is taken as the n^{th} iterate of T). Then T has a unique fixed point.

Proof. By Theorem 2.6.1 T^n has a unique fixed point ξ

in X such that $T^n(\xi) = \xi$.

Now the relation $T^n(\xi) = \xi$ gives

$$T(T^n(\xi)) = T(\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

but $T^{n+1} = T(T^n) = T^n(T)$

therefore $T^{n+1}(\xi) = T(T^n(\xi)) = T^n(T(\xi))$

which by (1) reduces to

$$T^n(T(\xi)) = T(\xi).$$

Thus $T(\xi)$ is a fixed point for T^n. But T^n has a unique fixed point ξ. Therefore $T(\xi) = \xi$ i.e., ξ is a fixed point of T. The uniqueness of the fixed point of T is obvious, because each fixed point of T is also a fixed point of T^n.

Corollary. If $K^{-1}TK$ is a one to one (ε, λ)-uniformly locally expansive mapping of a metric space Y onto an ε-chainable complete metric space $X \supset Y$. Then there exists a unique fixed point $\xi \in X$ such that $T(\xi) = \xi$.

Proof. This assertion is a direct consequence of the fact that for the inverse mapping $K^{-1}T^{-1}K(x)$ all the assumptions of the above theorem are satisfied. Therefore $K^{-1}T^{-1}K$ has a unique fixed point

ξ, such that $K^{-1}T^{-1}K(\xi) = \xi$, and by Corollary to Theorem 1.1.5 it follows that ξ is a unique fixed point for T.

2.6.3 <u>Theorem</u>. Let X be a metric space. Let T be a self mapping of X into itself such that there exists a mapping K of X into itself

which has the right inverse K^{-1} and which makes the mapping $K^{-1}TK$ contractive, that is

$$d(K^{-1}TK(x),\ K^{-1}TK(y)) < d(x,\ y), \quad (x,\ y \in X,\ x \neq y),$$

such that there exists a point $x \in X$ whose sequence of iterates $\{(K^{-1}TK)_n(x)\}$ contains a convergent subsequence $\{(K^{-1}TK)_{n_i}(x)\}$; then $\xi = \lim_{i\to\infty} (K^{-1}TK)_{n_i}(x)$ is a unique fixed point for T.

<u>Proof</u>. Suppose $(K^{-1}TK)(\xi) \neq \xi$, and consider the sequence $(K^{-1}TK)_{n_i}(x)$ which easily can be verified to converge to $(K^{-1}TK)(\xi)$.

The mapping $r(p,\ q)$ of $Y = X \times X - \Delta$ (Δ denotes the diagonal $\{(x,\ y) | x = y\}$ into the real line defined by

$$r(p,\ q) = \frac{d((K^{-1}TK)(p),\ (K^{-1}TK)(q))}{d(p,\ q)} \quad \ldots\ldots\ldots \quad (2)$$

is clearly continuous. Hence there exists a neighbourhood U of

$\{\xi, (K^{-1}TK)(\xi)\} \in Y$, such that $p, q \in U$ implies

$$0 \leq r(p, q) < R < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Let $S_1 = S_1(\xi, \rho)$ and $S_2 = S_2((K^{-1}TK)(\xi), \rho)$ be open discs with centres ξ and $(K^{-1}TK)(\xi)$ respectively and radius $\rho > 0$ such that

$$\rho < \frac{1}{3}(\xi, (K^{-1}TK)(\xi)) \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Since $K^{-1}TK$ has a convergent subsequence therefore there exists a positive integer N such that $i \geq N$ implies $(K^{-1}TK)_{n_i}(x) \in S_1$ and therefore by definition of contractive mapping $(K^{-1}TK)_{n_i}(x) \in S_2$.

Thus, by (4)

$$d((K^{-1}TK)n_i(x), \ (K^{-1}TK)_{n_{i+1}}(x)) > \rho, \ (i > N) \ \ldots \quad (5)$$

On the otherhand, for each i, it follows from (2) and (3) that

$$d((K^{-1}TK)_{n_{i+1}}(x)), \ ((K^{-1}TK)_{n_{i+2}}(x)))$$

$$< Rd((K^{-1}TK)_{n_i}(x)), \ ((K^{-1}TK)n_{i+1}(x))) \quad ,\ldots\ldots(6)$$

Repeating (6) for $\ell > J > N$ we have

$$d((K^{-1}TK)_{n_\ell}(x)), ((K^{-1}TK)_{n_{\ell+1}}(x)))$$

$$\leq d(((K^{-1}TK)_{n_{\ell-1}}+1(x)), ((K^{-1}TK)_{n_{\ell-1}}+2(x)))$$

$$\leq Rd((K^{-1}TK)_{n_{\ell-1}}(x)), (K^{-1}TK)_{n_{\ell-1}}+1(x)))$$

$$< R^{\ell-J}d(((K^{-1}TK)_{n_J}(x)), (K^{-1}TK)_{n_J}+1(x)))$$

$$\longrightarrow 0 \text{ as } \ell \longrightarrow \infty$$

which is incompatible with (6). Hence

$$(K^{-1}TK)(\xi) = \xi.$$

To prove the uniqueness we assume further, that there exists another point η such that

$$(K^{-1}TK)(\eta) = \eta, \quad \text{whenever} \quad \eta \neq \xi.$$

Therefore $d(K^{-1}TK(\xi),\ K^{-1}TK(\eta)) = d(\xi, \eta)$ which contradicts the definition of contractive mappings. This proves unicity and $K^{-1}TK$ has a unique point ξ, therefore by Corollary to Theorem 1.1.5 T possesses a unique fixed point.

2.6.4 Theorem. Let X be a metric space. Let T be a self mapping of X into itself such that there exists a mapping K of X into itself which has the right inverse K^{-1} and which makes the mapping $K^{-1}TK$ contractive, further assume that there exists a subset $M \subseteq X$ and a point $x_0 \in M$ such that

$$(7) \qquad d(x, x_0) - d((K^{-1}TK)(x),(K^{-1}TK)(x_0)) < 2d(x_0, K^{-1}TK(x_0))$$

and $K^{-1}TK$ maps M into a compact subset of X, then there exists a unique fixed point.

Proof. Assume $K^{-1}TK(x_0) \neq x_0$ and let

$$(8) \qquad x_{n+1} = (K^{-1}TK)(x_n) \qquad n = 0, 1, 2, \ldots$$

$K^{-1}TK$ maps M into a compact subset of X by assumption, therefore to obtain the theorem as a direct consequence of Theorem 2.6.3 it suffices to show that $x_n \in M$ for every n.

Since $K^{-1}TK$ is a contractive mapping, the sequence $d(x_n, x_{n+1})$ is by (8) non increasing and by $K^{-1}TK(x_0) \neq x_0$ it follows that

$$(9) \qquad d(x_n, x_{n+1}) < d(x_0, x_1) \qquad n = 1, 2, \ldots .$$

By the triangle inequality

$$(10) \qquad d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1}).$$

By the help of (9) we can write (10) as

$$d(x_0, x_n) < d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_0, x_1)$$

or $$d(x_0, x_n) < 2d(x_0, x_1) + d(x_1, x_{n+1})$$

or $$d(x_0, x_n) - d(x_1, x_{n+1}) < 2d(x_0, x_1)$$

which by virtue of (8) becomes

$$d(x_0, x_n) - d((K^{-1}TK)(x_0), (K^{-1}TK)(x_n)) < 2d(x_0, K^{-1}TK(x_0))$$

and by (7) it follows that $x_n \in M$ for every n hence the theorem follows.

Corollary. Let X be a metric space. Let T be a self-mapping of X into itself such that there exists a mapping K of X into itself which has the right inverse K^{-1} and which makes the mapping $K^{-1}TK$ contractive such that there exists a point $x_0 \in X$ satisfying

$$(11) \qquad d(K^{-1}TK(x), K^{-1}TK(x_0)) \leq \lambda(x, x_0)d(x, x_0)$$

for every $x \in X$ where $\lambda(x, y) = \lambda(d(x, y)) \in F$, and $K^{-1}TK$ maps $S(x_0, r)$ with

$$r = \frac{2d(x_0, (K^{-1}TK)(x_0))}{1-\lambda[2d(x_0,(K^{-1}TK)(x_0)]}$$

into a compact subset of X, then there exists a unique fixed point of T.

Proof. Taking in Theorem 2.6.4 $M = S(x_0, r)$; then by (11) the monotonicity of $\lambda(d)$ and $r \geq 2d(x_0, K^{-1}TK(x_0))$ it follows that if $d(x, x_0) \geq r$ then

$$\begin{aligned} d(x, x_0) - d((K^{-1}TK)(x),(K^{-1}TK)(x_0)) &\geq d(x, x_0) \\ &\quad - \lambda(d(x, x_0))d(x, x_0) \\ &= [1 - \lambda(d(x, x_0))]d(x, x_0) \\ &\geq [1 - \lambda(r)]r. \\ &\geq [1 - \lambda(2d(x_0, K^{-1}TK(x_0))]r \\ &= 2d(x_0, (K^{-1}TK)(x_0)) \quad \text{for every } x \in X - M, \end{aligned}$$

i.e. (7) holds.

2.6.5 Theorem. Let X be a complete metric space. Let T be a selfmapping of X such that there exists a mapping K of X into itself, which has the right inverse K^{-1} and which makes the mapping $K^{-1}TK$ contractive such that there exists a subset M of X and a point $x_0 \in M$, satisfying the following

$$(12) \qquad d(x, x_0) - d((K^{-1}TK)(x), (K^{-1}TK)(x_0)) < 2d(x_0, K^{-1}TK(x_0))$$

for every $x \in M$,

$$(13) \qquad d((K^{-1}TK)(x), (K^{-1}TK)(y)) \leq \lambda(x, y)d(x, y)$$ for every $x, y \in M$,

where $\lambda(x, y) = \lambda[d(x, y)] \in F$.

Then T has a unique fixed point.

<u>Proof</u>. Assume $(K^{-1}TK)(x_0) \neq x_0$ and let

$$(8') \qquad x_{n+1} = (K^{-1}TK)(x_n) \qquad n = 0, 1, 2, \ldots .$$

Since $K^{-1}TK$ is a contractive mapping, the sequence $d(x_n, x_{n+1})$ is by (8') non increasing and by $K^{-1}TK(x_0) \neq x_0$ it follows that

$$(9') \qquad d(x_n, x_{n+1}) < d(x_0, x_1) \qquad n = 1,2,\ldots .$$

By the triangle inequality

$$(10') \qquad d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1})$$

By the help of (9') we can write (10') as

$$d(x_0, x_n) < d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_{n+1})$$

or $$d(x_0, x_n) < 2d(x_0, x_1) + d(x_1, x_{n+1})$$

or $$d(x_0, x_n) - d(x_1, x_{n+1}) < 2d(x_0, x_1)$$

which by virtue of (8') becomes

$$d(x_0, x_n) - d((K^{-1}TK)(x_0), (K^{-1}TK(x_n)) < 2d(x_0, K^{-1}TK(x_0))$$

and by (12) it follows that $x_n \in M$.

Now we have to prove that the sequence $\{x_n\}$ is bounded.

By equation (13) and definition of $\{x_n\}$,

$$d(x_1, x_{n+1}) = d((K^{-1}TK)(x_0), (K^{-1}TK)(x_n)) \tag{14}$$

$$\leq \lambda(d(x_0, x_n))d(x_0, x_n)$$

and by the triangle inequality

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_{n+1}, x_n).$$

Hence by (9') and (14)

$$[1 - \lambda(d(x_0, x_n))]\, d(x_0, x_n) < 2d(x_0, x_1).$$

Now if $d(x_0, x_n) \geq d_0$ for a given $d_0 > 0$, then by the monotonicity of $\lambda(d)$ it follows that $\lambda(d(x_0, x_n)) \leq \lambda(d_0)$ and therefore

$$d(x_0, x_n) < \frac{2d(x_0, x_1)}{1-\lambda(d(x_0,x_n))} \leq \frac{2d(x_0, x_1)}{1-\lambda(d_0)} = C.$$

Hence

$$(15) \qquad d(x_0, x_n) \leq R \qquad n = 1,2,\ldots,$$

where $R = \max(d_0, C)$ i.e., the sequence $\{x_n\}$ is bounded. Now for $J > 0$ where J is any positive integer by (13) we have,

$$d(x_{\ell+1}, x_{\ell+J+1}) \leq \lambda(x_\ell, x_{\ell+J})\, d(x_\ell, x_{\ell+J}).$$

Therefore taking the product from $\ell = 0$ to $\ell = n-1$ and dividing both sides by the same terms we obtain

$$d(x_n, x_{n+J}) \leq d(x_0, x_J) \prod_{\ell=0}^{n-1} \lambda(x_\ell, x_{\ell+J})$$

which by (15) reduces to

$$(16) \qquad d(x_n, x_{n+J}) \leq R \prod_{\ell=0}^{n-1} \lambda(x_\ell, x_{\ell+J}).$$

Now it remains to prove that $\{x_n\}$ is a Cauchy sequence for that purpose we have to show that for every $\varepsilon > 0$ there exists a number N depending on ε only (not on J) such that for every $J > 0$ there is $d(x_N, x_{N+J}) < \varepsilon$ (since the sequence $d(x_n, x_{n+J})$ is non decreasing.

If $d(x_\ell, x_{\ell+J}) \geq \varepsilon$ for $\ell = 0,1,2,\ldots,n-1$, then by (13) the monocity of $\lambda(d)$ we have $\lambda(x_\ell, x_{\ell+J}) = \lambda(d(x_\ell, x_{\ell+J})) \leq \lambda(\varepsilon)$ and by (16) it follows that

$$d(x_n, x_{n+J}) \leq R[\lambda(\varepsilon)]^n.$$

But $\lambda(\varepsilon) < 1$ and $[\lambda(\varepsilon)]^n \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore there exists a positive integer N independent of J such that $d(x_N, x_{N+J}) < \varepsilon$ for every $J > 0$ which proves that $\{x_n\}$ is a Cauchy sequence. By the completion of X it follows that there exists $\xi = \lim_{n\to\infty} x_n$ and by the continuity of $K^{-1}TK$, ξ is a fixed point for $K^{-1}TK$, and by the Corollary to Theorem 1.15 ξ is a fixed point for T. The uniqueness is obvious.

Corollary. Putting $M = X$ we have, if $d(K^{-1}TK(x), K^{-1}TK(y)) < \lambda(x, y)d(x, y)$ for every $x, y \in X$ (X complete) where $\lambda(x, y) \in F$, then there exists a unique fixed point.

2.7 CONVERSE OF BANACH'S CONTRACTION PRINCIPLE

The natural converse statement of the Banach's contraction principle is the following. "Let X be a complete metric space, and let T be a mapping of X into itself such that for $x \in X$, the sequence of iterates $\{T^n(x)\}$ converges to a unique fixed point $w \in X$. Then there exists a complete metric on X in which T is a contraction". This is in fact, true even in stronger sense. The following converse of Theorem 2.2.1 was due to Bessaga [6].

2.7.1 <u>Theorem</u>. Let X be an abstract set and T be a mapping of X into itself such that for each positive integer $k > 0$ the equation $T^k(x) = x$ holds for some x in X implies $x = Tx$, the unique fixed point of T. Then for each λ, $0 \leq \lambda < 1$, there exists a complete metric on X such that $d(T(x), T(y)) \leq \lambda d(x, y)$ for all x, y in X.

A weaker form of Theorem 2.7.1 in case X is compact metrizable space was also given by Janos [43].

2.7.2 <u>Theorem</u>. Let X be a metrizable compact space and T be a continuous mapping of X into itself such that $\bigcap_{n=0}^{\infty} T^n(x)$ is a one-point set. Then for every $\lambda \in (0, 1)$ there exists a metric $d(x, y)$

such that

$$d(T(x), T(y)) \leq \lambda d(x, y).$$

The following generalization of Theorem 2.7.1 is due to Wong [75].

2.7.3 Theorem. Let X be an abstract set with n mutually commuting mappings $T_1, T_2, \ldots, T_n$ defined on X into itself such that each iterate $T_1^{k_1}, T_2^{k_2}, \ldots, T_n^{k_n}$ (where $k_1, k_2, \ldots, k_n$ are non-negative integers not all equal to zero) possesses a unique fixed point which is common to every choice of $k_1, k_2, \ldots, k_n$. Then for each $\lambda \in (0, 1)$ there exists a complete metric d on X such that $d(T_i(x), T_i(y)) \leq \lambda d(x, y)$ for $i \leq i \leq n$, and for all $x, y \in X$.

Probably the most natural generalizatinn to the converse of Banach's contraction principle that one can make is due to Meyers [54] given below:

2.7.4 Theorem. If T is an $(\varepsilon\text{-}\lambda)$-uniform local contraction on a complete ε-chainable metric space (X, d) then there exists a metric d^* topologically equivalent to d such that T is a contraction on (X, d^*) and the Banach contraction theorem can be applied.

Corollary. If T is a local contraction and X is compact, then the conclusion of Theorem 2.7.4 holds.

2.7.5 Theorem. If T is an ε-uniform local contraction, and if there exists $x \in X$ such that $T(x) = x$, then the conclusion of Theorem 2.7.4 holds.

2.7.6 Theorem. Let X be a topological space admitting a metric (complete metric), and $T:X \longrightarrow X$ a continuous map with a fixed point x_0 obeying the purely topological conditions:

(i) $T^n(x) \longrightarrow x_0$ for all x in X.

(ii) $T(U) \subset U$, $T^n(U) \longrightarrow \{x_0\}$ for some neighbourhood of x_0.

Then X admits a metric (complete metric) for which T is contraction.

2.7.7 Theorem. If T is a continuous self mapping of (X, d) and if T^n is a contraction on (X, d), then there is a metric d^* under which T and T^n are contractions.

2.7.8 Theorem. If T_t, $t \geq 0$ is a family of continuous maps of a metric space satisfying $t_{t_1} . T_{t_2} = T_{t_1+t_2}$ and if $\lim_{t \to 0} T_t = T_{t_0}$

i.e., $\lim_{t\to t_0} \sup_{x\in X} d(T_t(x), T_{t_0}(x)) = 0$, then a necessary and sufficient condition for each T_t to be a contraction (with respect to some metric d_t equivalent to d) is that some one T_t be a contraction.

We also present here a generalization of the converse of Banach's contraction principle.

Let (X, d) be a complete metric space and T a contractive mapping of X into itself, i.e., $d(T(x),T(y)) \leq \lambda d(x,y)d(x,y)$, where $x, y \in X$, $\lambda d(x, y) \in F$.

Then it follows from a theorem due to Rakotch [59] that the iterated images $T^n(x)$ of X shrink to the point ξ of X. This can be written in the form

$$\bigcap_{n=1}^{\infty} T^n(X) = \{\xi\}.$$

Since this formula does not involve the metric and has a topological character, it is natural to ask the following question:

Let X be a compact metrizable topological space and T a continuous mapping of X into itself which has the property that $\cap T^n(X) = \{\xi\}$. Is it possible to find a metric $d(x, y)$ generating the given Topology on X such that the mapping T is contractive with respect to d?

The answer is yes. We now construct such a metric and denote it by d^*.

2.7.9 <u>Definition</u>. Let $X = A_0$, $T(X) = A_1, \ldots, T^n(X) = A_n$, and introduce the functions $n(x)$ and $n(x, y)$ as follows:

$$n(x) = \max \{n; \; x \in A_n\}$$

$$n(x, y) = \min \{n(x), n(y)\}.$$

2.7.10 <u>Theorem</u>. For $\lambda(d(x, y)) \in F$ there exists a distance function d^* such that

$$d^*(T(x), T(y)) \leq \lambda(d(x, y)) \, d^*(x, y).$$

<u>Proof</u>. By Theorem 1 of Janos [43] there exists a metric $d(x, y)$ with respect to which the mapping T is non expansive. Let

$$\alpha(x, y) = \{\lambda(d(x, y))\}^{n(x,y)} d(x, y).$$

$$A^n = T^n(X)$$

$$T^{n+1}(X) = T(T^n(X)).$$

Hence $x \in A_n$ implies $T(x) \in A_{n+1}$.

Let i = max subscript for x. Then $i + 1$ = max subscript for $T(x)$.

Let J = max subscript for y. Then $J + 1$ = max subscript of $T(y)$.

Thus $n(T(x), T(y)) = \min\{i + 1, J + 1\}$

$$= \min\{i, J\} + 1$$

$$= n(x, y) + 1.$$

$$\alpha(T(x), T(y)) \leq [\lambda(d(x, y))]^{n(T(x),T(y))} d(T(x), T(y)).$$

Now T is contractive; thus

$$d(T(x), T(y)) \leq d(x, y).$$

Hence $\alpha(T(x), T(y)) \leq [\lambda(d(x, y))]^{n(x,y)+1} d(x, y)$

$$= [\lambda(d(x, y))]^{n(x,y)} [\lambda(d(x, y))]\ d(x, y)$$

$$\leq \lambda(d(x, y))\ \alpha(x, y).$$

The function $\alpha(x, y)$ is not in general a metric. However, a derived metric $d^*(x, y)$ can be defined as

$$d^*(x, y) = \inf \sum_{i=1}^{n} \alpha(x_i, x_{i-1}),$$

where the infimum is taken over all possible finite system of elements $x_1, x_2, x_3, \ldots, x_n \in X$ such that $x_1 = x$, and $x_{n+1} = y$.

From the definition of $d^*(x, y)$ it is clear that $d^*(x, y) \leq \alpha(x, y) \leq d(x, y)$. The same method as used by Janos in [43] shows that $d^*(x, y)$ is a metric.

Now we have only to prove that

$$d^*(T(x), T(y)) \leq \lambda(d(x, y))d^*(x, y).$$

Let $\varepsilon > 0$ be given. From the definition of $d^*(x, y)$ there exists a representative of $d^*(x, y)$ in the form
$d^*(x, y) = \inf \sum_{i=1}^{n} \alpha(x_i, x_{i-1})$.

$$\text{Thus } d^*(T(x), T(y)) \leq \inf \sum_{i=1}^{n} \alpha(T(x_i), T(x_{i-1}))$$

$$\leq \inf \sum_{i=1}^{n} \lambda(d(x_i, x_{i-1})\alpha(x_i, x_{i+1})$$

$$= \lambda \inf \sum_{i=1}^{n} d(x_i, x_{i-1}) \inf \sum_{i=1}^{n} \alpha(x_i, x_{i-1})$$

$$= \lambda(d(x, y))\ d^*(x, y).$$

Corollary. When λ is constant with $0 \leq \lambda < 1$, we get the result of Janos [43].

CHAPTER III

COMMUTING FUNCTIONS AND FIXED POINTS

3.1 COMMUTING POLYNOMIALS AND COMMON FIXED POINTS

Let $f = f(x)$ and $g = g(x)$ be two continuous and commuting functions (under substitution), each mapping the closed interval $[a, b]$ into itself. Isbell [41] has conjectured that f and g must have a common fixed point, or equivalently, that f and the composite function $h = fg = f(g(x))$ must have a common fixed point. Except in the special cases the conjecture has not been verified.

One interesting special case of the conjecture was investigated a number of years ago by Ritt [60]. He proved that if f and g are polynomials which do not belong to a certain class (f and g do not come from the multiplication theorem of e^z and $\cos z$, c.f. [60] for definition), then neglecting a linear transformation they are both iterates of a third polynomial p. Thus they would have as common fixed points the fixed points of p. Also in this case, f and g would be a member of a semi-group of commuting functions formed from the iterates of p.

Among the commuting polynomials excluded from the Ritt theorem are the Tchebysheff's polynomials defined by $T_n(x) = \cos(n \arccos x)$ for $-1 \leq x \leq 1$. It appears that even in this case the polynomials could be embeded in a one parameter semi-group of commuting functions defined by

(1) $$f_t(x) = T_\alpha(x) = \cos(\alpha \text{ arc } \cos x), \quad \alpha = \exp(t).$$

This suggests that one method for attacking the Isbell conjecture is to try to embed the commuting functions in a semi-group and thereby hope to prove the existence of the common fixed point.

Block and Thielman [10] have given the following theorem-

If $h_2(x) = \alpha x^2 + \beta x + \gamma$ and $h_3(x)$ commute, then $\Delta(h_2) = 2\beta$, or $2\beta + 8$, where $\Delta(h_2) = \beta^2 - 4\alpha\gamma$.

We prove the following theorem on commuting polynomials.

3.1.1 Theorem. If $h_2(x) = \alpha x^2 + \beta x + \gamma$, and $h_3(x) = C_0x^3 + C_1x^2 + C_2x + C_3$ are two polynomials, then they commute if and only if the following conditions hold:

(i) $C_0 = \alpha^2$

(ii) $C_1 = \frac{3}{2}\alpha\beta$

(iii) $C_2 = \frac{3}{8}(2\beta^2 + 2\beta - \Delta)$

(iv) $C_3 = \dfrac{\beta}{16\alpha}\ (2\beta^2 + 6\beta - 8 - 3\Delta)$

and (v) $\Delta = 2\beta$, or $2\beta + 8$, where $\Delta = \Delta(h_2)$

$= \beta^2 - 4\alpha\gamma.$

<u>Proof</u>. Let $h_2(x) = \alpha x^2 + \beta x + \gamma$, and $h_3(x) = C_0x^3 + C_1x^2 + C_2x + C_3$ commute, i.e., $h_2h_3(x) = h_3h_2(x)$.

Then $\alpha(C_0x^3 + C_1x^2 + C_2x + C_3)^2 + \beta(C_0x^3 + C_1x^2 + C_2x + C_3) + \gamma = C_0(\alpha x^2 + \beta x + \gamma)^3 + C_1(\alpha x^2 + \beta x + \gamma)^2 + C_2(\alpha x^2 + \beta x + \gamma) + C_3.$

Comparing coefficients of like powers we get

$$
\text{(A)}\quad \begin{cases}
\alpha C_0{}^2 &= \alpha^3 C_0. \\
2\alpha C_0C_1 &= 3\alpha^2\beta C_0. \\
\alpha C_1^2 + 2\alpha C_0C_2 &= (3\alpha^2\gamma + 3\alpha\beta^2)C_0 + \alpha^2 C_1. \\
2\alpha C_0C_3 + 2\alpha C_1C_2 + \beta C_0 &= (\beta^3 + 6\alpha\beta\gamma)C_0 + 2\alpha\beta C_1. \\
\alpha C_2^2 + 2\alpha C_1C_3 + \beta C_1 &= (3\beta^2\gamma + 3\alpha\gamma^2)C_0 + (\beta^2 + 2\alpha\gamma)C_1 + \alpha C_2. \\
2\alpha C_2C_3 + \beta C_2 &= 3\beta\gamma^2 C_0 + 2\beta\gamma C_1 + \beta C_2. \\
\alpha C_3^2 + \beta C_3 + \gamma &= \gamma^3 C_0 + \gamma^2 C_1 + \gamma C_2 + C_3.
\end{cases}
$$

The first four equations give

$$C_0 = \alpha^2$$

$$C_1 = \frac{3}{2}\alpha\beta$$

$$C_2 = \frac{3}{8}(2\beta^2 + 2\beta - \Delta)$$

$$C_3 = \frac{\beta}{16\alpha}(2\beta^2 + 6\beta - 8 - 3\Delta)$$

Each of the three remaining equations gives

$$\Delta^2 - 4\beta\Delta - 8\Delta + 4\beta^2 + 16\beta = 0$$

or $\quad (\Delta - 2\beta)(\Delta - 2\beta - 8) = 0$

or $\quad \Delta = 2\beta$ or $2\beta + 8$.

Assuming that the given conditions hold, we have to prove that $h_2(x)$ and $h_3(x)$ commute,

i.e., $h_2h_3(x) = h_3h_2(x)$.

Here $h_2h_3(x) = h_2[h_3(x)] = h_2[C_0x^3 + C_1x^2 + C_2x + C_3]$.

$$\begin{aligned} &= \alpha[C_0x^3 + C_1x^2 + C_2x + C_3]^2 \\ &+ \beta[C_0x^3 + C_1x^2 + C_2x + C_3] + \gamma. \\ &= \alpha C_0^2x^6 + 2\alpha C_0C_1x^5 + (\alpha C_1^2 + \alpha C_0C_2)x^4 \\ &\quad + (2\alpha C_0C_3 + 2\alpha C_1C_2 + \beta C_0)x^3 + \\ &\quad (\alpha C_2^2 + 2\alpha C_1C^3 + \beta C_1)x^2 \\ &\quad + (2\alpha C_2C_3 + \beta C_2)x + \alpha C_3^2 + \beta C_3 + \gamma. \end{aligned}$$

which by (A) reduces to

$$\alpha^3C_0x^6 + 3\alpha^2\beta C_0x^5 + \{(3\alpha^2\gamma + 3\alpha\beta^2)C_0 + \alpha^2C_1\}x^4$$

$$+ \{(\beta^3 + 6\alpha\beta\gamma)C_0 + 2\alpha\beta C_1\}x^3 + (3\beta^2\gamma + 3\alpha\gamma^2)C_0$$

$$+ (\beta^2 + 2\alpha\gamma)C_1 + \alpha C_2\}x^2 + (3\beta\gamma^2C_0 + 2\beta\gamma C_1 + \beta C_2)x$$

$$+ \gamma^3C_0 + \gamma^2C_1 + \gamma C_2 + C_3 = h_3h_2(x).$$

Corresponding to two different values of Δ, we take two examples to illustrate the theorem.

Example 1. Let $h_2(x) = x^2 + 6x + 4$, and $h_3(x) = x^3 + 9x^2 + 24x + 15$. Then $h_2h_3(x)$ $= h_2[x^3 + 9x^2 + 24x + 15] = [x^3 + 9x^2 + 24x + 15]^2$ $+ 6[x^3 + 9x^2 + 24x + 15] + 4 = x^6 + 18x^5 + 129x^4$ $+ 468x^3 + 800x^2 + 864x + 319$, and $h_3h_2(x)$ $= h_3[x^2 + 6x + 4] = [x^2 + 6x + 4]^3 + 9[x^2 + 6x + 4]^2$ $+ 24[x^2 + 6x + 4] + 15 = x^6 + 18x^5 + 129x^4$ $+ 468x^3 + 800x^2 + 864x + 319$. Thus $h_2h_3(x)$ $= h_3h_2(x)$ i.e., h_2 and h_3 commute.

Example 2. Let $h_2(x) = x^2 + 6x + 6$, and $h_3(x) = x^3 + 9x^2 + 27x + 24$. Then $h_2h_3(x) =$ $h_3h_2(x) = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 726$. Therefore $h_2(x)$ and $h_3(x)$ commute.

3.1.2 Theorem. Let $h_2(x) = \alpha x^2 + \beta x + \gamma$, and $h_3(x) = C_0x^3 + C_1x^2 + C_2x + C_3$ be two polynomials. If they commute, then they have a common fixed point.

Proof. We know that if $h_2(x)$, and $h_3(x)$ commute then $\Delta = 2\beta$ or $2\beta + 8$.

Case I. $\Delta = 2\beta$.

$$h_2(x) = \alpha(x + \frac{\beta}{2\alpha})^2 - \beta/2\alpha.$$

$$h_3(x) = \alpha^2(x + \frac{\beta}{2\alpha})^3 - \frac{\beta}{2\alpha}.$$

$h_2(x) = x$ implies $\alpha x + \frac{1}{2}\beta = 0$ or -1.

$h_3(x) = x$ implies $\alpha x + \frac{1}{2}\beta = 0$ or 1, or -1.

$h_2(x) = x = h_3(x)$ implies $\alpha x + \frac{1}{2}\beta = 0$ or -1,

so that the common fixed points are

$$-\frac{\beta}{2\alpha}, \quad \frac{2-\beta}{2\alpha}.$$

Case II. $\Delta = 2\beta + 8$.

$$h_2(x) = \alpha(x + \frac{\beta}{2\alpha})^2 - \frac{\beta+4}{2\alpha}.$$

$$h_3(x) = \alpha^2(x + \frac{\beta}{2\alpha})^3 - 3x - \frac{2\beta}{\alpha}.$$

$h_2(x) = x$ implies $\alpha x + \frac{1}{2}\beta = 2$ or -1.

$h_3(x) = x$ implies $\alpha x + \frac{1}{2}\beta = 2$ or -2, or 0.

Hence, $h_2(x) = x = h_3(x)$ implies $\alpha x + \frac{1}{2}\beta = 2$, so that the common fixed point is $\frac{4-\beta}{2\alpha}$.

Example 1. (Case I). Let $h_2(x) = x^2 + 6x + 6$, and $h_3(x) = x^3 + 9x^2 + 27x + 24$ be two polynomials. Then $h_2(x)$ and $h_3(x)$ commute and have -2 and -3 as common fixed points.

Example 2. (Case II). Let $h_2(x) = x^2 + 6x + 4$, and $h_3(x) = x^3 + 9x^2 + 24x + 15$ be two polynomials. Then $h_2(x)$ and $h_3(x)$ commute and have -1 as a common fixed point.

An alternate proof of Theorem 3.1.2 is the following.

Proof. Case 1. $\Delta = 2\beta$.

Now $h_2(x) = \alpha x^2 + \beta x + \gamma$.

Hence $h_2(x) = x$ implies $x = \alpha x^2 + \beta x + \gamma$

or $\alpha x^2 + (\beta - 1)x + \gamma = 0$;

hence $x = \dfrac{-(\beta - 1) \pm \sqrt{(\beta - 1)^2 - 4\alpha\gamma}}{2\alpha}$

or $x = \dfrac{-(\beta - 1) \pm \sqrt{\beta^2 - 2\beta + 1 - 4\alpha\gamma}}{2\alpha}$.

Here we have $\beta^2 - 4\alpha\gamma = 2\beta$.

Hence $x = \dfrac{-(\beta - 1) \pm 1}{2\alpha}$

or $x = \dfrac{2-\beta}{2\alpha}$, or $\dfrac{-\beta}{2\alpha}$.

We would like to show that $x = \frac{-\beta}{2\alpha}$ is also a fixed point of $h_3(x)$.

i.e., $h_3(x) = x$ implies $x = C_0x^3 + C_1x^2 + C_2x + C_3$.

After substituting $x = \frac{-\beta}{2\alpha}$ we have

$$\frac{-\beta}{2\alpha} = C_0\left(\frac{-\beta}{2\alpha}\right)^3 + C_1\left(\frac{-\beta}{2\alpha}\right)^2 + C_2\left(\frac{-\beta}{2\alpha}\right) + C_3.$$

Substituting the values of C_0, C_1, C_2 and C_3 from Theorem 3.1.1, and 2β for Δ we get

$$\frac{-\beta}{2\alpha} = \alpha^2\left(\frac{-\beta}{2\alpha}\right)^3 + \frac{3}{2}\alpha\beta\left(\frac{-\beta}{2\alpha}\right)^2 + \frac{3}{8}(2\beta^2)\left(\frac{-\beta}{2\alpha}\right) + \frac{\beta}{16\alpha}(2\beta^2 - 8).$$

or $$-\frac{\beta}{2\alpha} = \frac{-\beta^3}{8\alpha} + \frac{3\beta^3}{8\alpha} - \frac{3\beta^3}{8\alpha} - \frac{\beta^3}{8\alpha} - \frac{\beta}{2\alpha} = \frac{-\beta}{2\alpha}.$$

Thus $\frac{-\beta}{2\alpha}$ is a fixed point for $h_3(x)$. Similarly we can easily show that $\frac{-\beta+2}{2\alpha}$ is also a fixed point for $h_3(x)$. Therefore $h_2(x)$ and $h_3(x)$ have $\frac{-\beta}{2\alpha}$ and $\frac{-\beta+2}{2\alpha}$ as common fixed points.

<u>Case II.</u> $\Delta = 2\beta + 8$.

Now $h_2(x) = \alpha x^2 + \beta x + \gamma$.

Hence $h_2(x) = x$ implies $x = \alpha x^2 + \beta x + \gamma$,

or $\alpha x^2 + (\beta - 1)x + \gamma = 0.$

Hence $x = \dfrac{-(\beta - 1) \pm \sqrt{(\beta - 1)^2 - 4\alpha\gamma}}{2\alpha}$

or $x = \dfrac{-(\beta - 1) \pm \sqrt{\beta^2 - 2\beta + 1 - 4\alpha\gamma}}{2\alpha}$

Here we have $\beta^2 - 4\alpha\gamma = 2\beta + 8.$

Hence $x = \dfrac{-(\beta - 1) \pm 3}{2\alpha}$

or $x = \dfrac{4-\beta}{2\alpha}$, or $-\dfrac{\beta-2}{2\alpha}$.

We would like to show that $x = \dfrac{4-\beta}{2\alpha}$ is also a fixed point of $h_3(x)$.

i.e., $h_3(x) = x$ implies $x = C_0x^3 + C_1x^2 + C_2x + C_3$.

After substituting $x = \dfrac{4-\beta}{2\alpha}$ we have

$$\frac{4-\beta}{2\alpha} = C_0\left(\frac{4-\beta}{2\alpha}\right)^3 + C_1\left(\frac{4-\beta}{2\alpha}\right)^2 + C_2\left(\frac{4-\beta}{2\alpha}\right) + C_3.$$

Having in mind that $\Delta = 2\beta + 8$ and substituting the values of C_0, C_1, C_2 and C_3 from Theorem 3.1.1 we have,

$$\frac{4-\beta}{2\alpha} = \alpha^2 \left(\frac{4-\beta}{2\alpha}\right)^3 + \frac{3}{2}\alpha\beta\left(\frac{4-\beta}{2\alpha}\right)^2 + \frac{3}{8}\,(2\beta^2-8)$$

$$\left(\frac{4-\beta}{2\alpha}\right) + \frac{\beta}{16\alpha}\,(2\beta^2 - 32)$$

$$= \frac{8}{\alpha} - \frac{6\beta}{\alpha} + \frac{3\beta^2}{2\alpha} - \frac{\beta^3}{8\alpha} + \frac{6\beta}{\alpha} - \frac{3\beta^2}{\alpha} + \frac{3\beta^3}{8\alpha}$$

$$+ \frac{3\beta^2}{2\alpha} - \frac{3\beta^3}{8\alpha} - \frac{6}{\alpha} + \frac{3\beta}{2\alpha} + \frac{\beta^3}{8\alpha} - \frac{2\beta}{\alpha}$$

$$= \frac{2}{\alpha} - \frac{\beta}{2\alpha} = \frac{4-\beta}{2\alpha}.$$

Thus $\frac{4-\beta}{2\alpha}$ is a fixed point for $h_3(x)$. Similarly we can show that $-\frac{\beta-2}{2\alpha}$ is not a fixed point for $h_3(x)$. Thus $h_2(x)$ and $h_3(x)$ have only $\frac{4-\beta}{2\alpha}$ as a common fixed point.

3.2 COMMUTING FUNCTIONS AND COMMON FIXED POINTS

3.2.1 Definition. Let $f:R \longrightarrow R$ and $g:R \longrightarrow R$. Then we define the composite of f and g (denoted by gof) by,

$$gof(x) = g[f(x)], \quad x \in R.$$

For example, if

$$f(x) = 1 + \sin x \qquad (-\infty < x < \infty),$$
$$g(x) = x^2 \qquad (0 \leq x < \infty),$$

then $gof(x) = 1 + 2\sin x + \sin^2 x \qquad (-\infty < x < \infty)$.

3.2.2 Definition. The functions f and g are said to be commutative if $fog(x) = gof(x)$ for all $x \in R$.

Remark. In general the binary operation is not commutative. For example, if $f(x) = x^2 + 1$ and $g(x) = 2 - x$. Then $gof(x) = g[f(x)] = 2 - (x^2 + 1) = 1 - x^2$ and $fog(x) = f[g(x)] = (2 - x)^2 + 1 = 5 - 4x + x^2$. Therefore $gof \neq fog$.

3.2.3 Definition. A point x_0 is said to be a common fixed point for the functions f and g if $f(x_0) = x_0 = g(x_0)$ for all $x_0 \in R$.

Under what conditions then, if any, is it true that $f \circ g = g \circ f$ - that is, $f[g(x)] = g[f(x)]$ for all $x \in R$? This is, in general, a very difficult question. Therefore we specialize and consider only linear functions on R - that is a function of the form $f(x) = ax + b$ where a and b are real constants. The following easy theorem is often given as an exercise at this point. (For example, Levi [49] Chapter I, Exercise 13).

3.2.4 Theorem. Let f and g be linear functions on R. Then $f \circ g = g \circ f$ if and only if $f \circ g(0) = g \circ f(0)$.

Proof. Let $f(x) = ax + b$ and $g(x) = cx + d$. By definition $f \circ g = g \circ f$ if and only if $f(g(x)) = g(f(x))$. But $f(g(x)) = f(cx + d) = d(cx + d) + b = acx + ad + b$ and $gf(x) = g(ax + b) = c(ax + b) + d = acx + bc + d$. Therefore $f \circ g(x) = g \circ f(x)$ if and only if $acx + ad + b = acx + bc + d$, or $ad + b = bc + d$. Now $(f \circ g)(0) = f(g(0)) = f(d) = ad + b$ and $(g \circ f)(0) = g(f(0)) = g(b) = cb + d$. Hence $f \circ g = g \circ f$ if and only if $f \circ g(0) = g \circ f(0)$, which proves the theorem.

This theorem states that, for linear functions f and g, the function $f \circ g$ and $g \circ f$ take on the same value at each x if and only if they take on the same value at 0. The following Corollary, while not as formally elegant as the theorem, states the condition $f \circ g(0) = g \circ f(0)$ in an equivalent form which will be more useful in applications.

Corollary. If $f(x) = ax + b$ and $g(x) = cx + d$, then $f \circ g = g \circ f$ if and only if $f(d) = g(b)$.

Proof. This follows immediately from the Theorem 3.2.4, since $(f \circ g)(0) = f(d)$ and $g \circ f(0) = g(b)$; actually these equations have already appeared in the second last sentence in the proof above.

The following results for analytic functins have been given by Shields [65], Edelstein [3], and Singh [71], respectively.

Before stating the above results we would like to give the following lemma related to analytic functions which is due to Shields [65]. We also add a theorem related to linear fractions by Singh [71].

Lemma. Let f be a linear map of D onto itself. Then three cases are possible

(i) $f = z$;

(ii) f has exactly one fixed point in the closed disc;

(iii) f has two distinct points on the boundary of D and the iterates of f converge to one of these points.

Theorem. Let $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ and $g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$ be two linear fractions. Then f and g have a common fixed point, provided they commute and $(a - d)^2 + 4bc = 0$.

3.2.5 Theorem. If f and g map the closed unit disc $|z| \leq 1$ in the complex plane into itself in a continuous manner, if they are analytic in the open disc and if they commute, then they have a common fixed point $f(z_0) = z_0 = g(z_0)$. In general any commuting family of such functions has a common fixed point.

3.2.6 Theorem. Let $f(z)$ be an analytic function in a domain D of the complex z-plane; let $f(z)$ map a compact and connected subset C of D into itself. If in addition $|f'(z)| < 1$ for all $z \in C$, then the equation $f(z) = z$ has a unique solution.

3.2.7 Theorem. If f and g are two analytic functions in a domain D of the complex z-plane that map a compact and connected subset C of D into itself and if f and g commute then they have a common fixed point provided that $|f'(z)| < 1$ for all z in C.

We [70] proved the following theorems related to complex valued functions.

3.2.8 Theorem. If f and g are two mappings of the closed unit disc of the complex z-plane into itself, such that $fg(z) = gf(z)$ for all z in the closed unit disc, $|f(z_1) - f(z_2)| \leq \alpha|z_1 - z_2|$ and $|g(z_1) - g(z_2)| \leq \beta|z_1 - z_2|$ where β is any positive real number and $0 \leq \alpha < 1$; then f and g have a common fixed point.

Proof. Since a closed subset of a complete metric space is complete, a closed unit disc is a complete metric space in the z-plane. The condition $|f(z_1) - f(z_2)| \leq \alpha|z_1 - z_2|$ for all z_1, z_2 in the closed unit disc, where $0 \leq \alpha < 1$, implies that f is a contraction mapping. Thus f is a contraction mapping of a complete metric space into itself. Therefore by Banach's contraction principle f has a unique fixed point in the closed unit disc, i.e., there exists a unique point z_0 in the closed unit disc such that $f(z_0) = z_0$.

It is given that

$$fg(z) = gf(z) \quad \text{for all } z.$$

Therefore $fg(z_0) = gf(z_0) = g[f(z_0)] = g(z_0)$.

i.e., $f[g(z_0)] = g(z_0)$.

Thus $g(z_0)$ is a fixed point for f. But f has a unique fixed point, say z_0. Therefore $g(z_0) = z_0$, and thus z_0 is a fixed point for g.

Example. Let $f(z) = \frac{1}{2} - \frac{z}{2}$, and $g(z) = z$ be two functions. Then $f(z)$ and $g(z)$ commute, because $f[g(z)] = f[z] = \frac{1}{2} - \frac{z}{2}$, $g[f(z)] = g[\frac{1}{2} - \frac{z}{2}] = \frac{1}{2} - \frac{z}{2}$. They have $z = \frac{1}{3}$ as a common fixed point.

3.2.9 Theorem. If f and g are two continuous functions from a closed unit disc into itself such that $fg(z) = gf(z)$ for all z in the closed unit disc, $|f(z_1) - f(z_2)| > \alpha|z_1 - z_2|$ is a one to one mapping of a subset of a closed unit disc onto the closed unit disc, and $|g(z_1) - g(z_2)| \leq \beta|z_1 - z_2|$, where β is any positive real number with $\alpha > 1$. Then f and g have a common fixed point.

Proof. Since a closed subset of a complete metric space is complete, a closed unit disc is complete metric space in the z-plane. The mapping $|f(z_1) - f(z_2)| > \alpha|z_1 - z_2|$, where $\alpha > 1$ is an expansive mapping of a subset of the closed unit disc onto the closed unit disc. Since f is one to one and onto, therefore the inverse function exists. Thus all the assumptions of Banach's contraction principle for $f^{-1}(x)$ are satisfied. Therefore there exists a unique fixed point ξ in the closed unit disc such that $f^{-1}(\xi) = \xi$, or $\xi = f(\xi)$. Thus f has a unique fixed point ξ. It is given that

$$fg(z) = gf(z) \quad \text{for all } z.$$

Therefore $fg(\xi) = gf(\xi) = g[f(\xi)] = g(\xi)$.

Thus $g(\xi)$ is also a fixed point for f. But f has a unique fixed point, say ξ.

Therefore $g(\xi) = \xi$; and thus ξ is a fixed point for g. Thus the theorem.

3.2.10 __Theorem.__ Let $f(z) = az + b$, $a \neq 1$, and $g(z)$ be any continuous function. If f and g commute, then they have a common fixed point.

__Proof.__ If f and g commute then

$$fg(z) = gf(z) \quad \text{for all } z.$$

Consider $f(z) = az + b$, $a \neq 1$, and suppose that $f(z_0) = z_0$

for some z_0. Hence $z_0 = f(z_0) = az_0 + b$ which implies $z_0(1-a) = b$, since $a \neq 1$, implies $1 - a \neq 0$. Therefore w e have shown there is a unique fixed point $\frac{b}{1-a}$. Substituting for z_0, $\frac{b}{1-a}$ in relation $f(z_0) = az_0 + b$ we have

$$f\left(\frac{b}{1-a}\right) = \frac{b}{1-a} .$$

Hence $\frac{b}{1-a}$ is a fixed point and therefore is unique.

Thus the function $f(z) = az + b$, $a \neq 1$, has a unique fixed point, say $z_0 = \frac{b}{1-a}$.

i.e., $f(z_0) = z_0$ and z_0 is unique.

Using the method of the previous theorem we can easily see that z is also a fixed point for g. Thus the theorem.

We take the following examples to illustrate the Theorem 3.2.10.

Example 1. If $f(z) = 3z + 6$ and $g(z) = 2z + 3$. Then $fog(z) = f[g(z)] = f[g(z)] = f[2z + 3] = 3[2z + 3] + 6 = 6z + 15$, and $gof(z) = g[f(z)] = g[3z + 6] = 2[3z + 6] + 3 = 6z + 15$. Thus $fg(z) = gf(z)$. The functions f and g commute and have -3 as a common fixed point.

Example 2. Let $f(z) = 2z + 5$, and $g(z) = z$. Since the identity function commutes with every function, f and g commute and have -5 as a common fixed point.

3.2.11 Theorem Let $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$, be a linear fraction and $g(z)$ be any analytic function. Then f and g have a common fixed point, provided they commute and $(a - d)^2 + 4bc = 0$.

Proof. The linear fraction $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ has a unique fixed point say $z_0 = \frac{a - d}{2c}$, under the condition $(a - d)^2 + 4bc = 0$ (Theorem 1.1.3). The remaining part of the proof follows on the same line as given in the above theorem.

In order to illustrate the Theorem 3.2.11 we take the following example.

Example. Let $f(z) = \frac{6z + 4}{-z + 2}$ be a linear fraction and $g(z) = z$ be an analytic function. Then $f(z)$ and $g(z)$ commute and have a common fixed point, say $z_0 = -2$.

Corollary. Let the Mobius trandformation $f(z) = e^{i\lambda} \frac{z - \alpha}{1 - \bar{\alpha}z}$ ($|z| < 1$) map the closed unit disc $|z| \leq 1$ onto itself, and let $g(z)$ be any analytic function which commutes with $f(z)$. Then $f(z)$ and $g(z)$ have a common fixed point, provided $|\alpha| = (\text{Sin } \lambda/2)$.

On examining the basis of the proof of Theorem 2.3.3 reveals that the essential property (besides uniqueness of the fixed point for T^n) employed is that T^n and T commute with each other. This suggests immediately the following:

3.2.12 Theorem. Let S be any non empty set of elements and K be a single valued function defined on S and with values in S. Suppose further that K possesses a unique fixed point x_0. Then, if T is a single valued function on S to S which commutes with K, that is such that $KT = TK$, then T also has x_0 as a fixed point (not necessarily unique; however, if K happens to be an iterate of T, that is $K = T^n$, with n positive integer, then it is unique.

Proof. The proof is immediate, starting from the equation $Kx_0 = x_0$, upon noticing that

$$Tx_0 = TKx_0 = KTx_0.$$

This means that Tx_0 is a fixed point of K, but K has only x_0 as a fixed point by hypothesis.

Remark. Theorems 2.3.1, 2.3.2 and 2.3.3 are particular cases of Theorem 3.2.12.

Before closing this section we would like to give a simple and interesting result of Seguin [64], related to commuting linear functions and common fixed points.

3.2.14 Theorem. Let $f(x) = ax + b$, $a \neq 1$. Then $g(x) = cx + d$ commute with f if and only if f and g have a common fixed point.

Proof. Suppose first of all that $f \circ g = g \circ f$, and let k be the unique fixed point of f that is, $f(k) = k$. Now $f \circ g = g \circ f$ implies that $(f \circ g)(k) = (g \circ f)(k)$. But $f \circ g(k) = f[g(k)]$ and $(g \circ f)(k) = g[f(k)] = g(k)$. Therefore, $f[g(k)] = g(k)$. Hence $g(k)$ is a fixed point of f. But since k is also a fixed point and the fixed point is unique, we must have $g(k) = k$, which means that k is also a fixed point of g. Therefore f and g have k as a common fixed point.

Conversely, suppose that $f(k) = k$ and $g(k) = k$; that is, f and g have a common fixed point. From Theorem 3.2.10 we know that the unique fixed point of f is $\frac{b}{1-a}$. Therefore $k = \frac{b}{1-a}$. Since $k = g(k) = ck + d$, we have $d = \frac{b}{1-a}(1 - c)$, which implies that $ad + b = cb + d$. But $ad + b = f(d)$ and $cb + d = g(b)$ so that $f(d) = g(b)$. Therefore, by the Corollary to Theorem 3.2.4 $f \circ g = g \circ f$.

3.3 SOME FIXED POINTS RELATED TO A CONJECTURE

3.3.1 The well-known conjecture that if f and g are two continuous functions which map a closed interval of real line into itself and if they commute then they have a common fixed point, has been given by Eldon Dyer in 1954, by Allen Shields in 1955 and Lester Dubins in 1956 independently. The partial proofs of the conjecture have been given by Cohen [27], Jungck [44] DeMarr [29] and others. The conjecture has been disproved very recently by Boyce [13] and Huneke [40] independently. In the present section the following theorems related to this conjecture have been given.

3.3.2 Theorem. Let f and g be two continuous functions which map the closed unit interval into itself such that $fg(x) = gf(x)$ for all x in I. Then they have a common fixed point provided $f(x)$ is differentiable in the open interval $(0, 1)$ and $|f'(x)| < 1$.

Proof. Since f is continuous in the closed interval $I = [0, 1]$ and

f has derivative in the open interval $(0, 1)$,

therefore by mean value theorem there exists a point $\xi \in (0, 1)$ such that

$$f(x) - f(y) = f'(\xi)(x - y) \quad \text{for} \quad x, y \in I.$$

i.e. $|f(x) - f(y)| = |f'(\xi)(x - y)|$

or $|f(x) - f(y)| = |f'(\xi)||x - y|$

or $|f(x) - f(y)| \leq \alpha|x - y|$.

Because $|f'(\xi)| \leq \alpha < 1$.

Therefore f is a contraction operator.

Since I being a closed subset of a complete metric space R is itself complete and f is a contraction mapping of I into itself. Therefore, by Banach contraction principle f has a unique fixed point in I. i.e. there exists a point $x_0 \in I$ such that $f(x_0) = x_0$.

Given that f and g commute, therefore

$$fg(x) = gf(x) \quad \text{for all} \quad x \quad \text{in} \quad I.$$

Now $fg(x_0) = gf(x_0) = gf(x_0) = g(x_0)$.

Thus $g(x_0)$ is a fixed point for f. But f has a unique fixed point say x_0. Therefore $g(x_0) = x_0$ and thus x_0 is a fixed point for g.

3.3.3 Theorem. If f and g are mappings of $I = [0, 1]$ into itself such that $f(g(x)) = g(f(x))$ for all $x \in I$, and $|f(x) - f(y)| \leq \alpha|x - y|$ and $|g(x) - g(y)| \leq \beta|x - y|$ for all $x, y \in I$, where β is any positive real number and $0 \leq \alpha < 1$, then there exists a common fixed point for both f and g.

Proof. Since I is complete with respect to usual metric, the condition $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in I$, where $0 \leq \alpha < 1$ implies that f is a contraction mapping. Thus f is a contraction mapping of a complete metric space I into itself. Therefore by Banach's contraction principle f has a unique fixed point in I. i.e. there exists a unique point $x_0 \in I$ such that $f(x_0) = x_0$.

Using the method of previous theorem we can easily see that x_0 is also a fixed point for g. Thus the theorem.

Example. Let $f(x) = \frac{1}{2} - \frac{x}{2}$, and $g(x) = x$ be two functions then $f(x)$ and $g(x)$ commute; because $f[g(x)] = f(x) = \frac{1}{2} - \frac{x}{2}$, and $g[f(x)] = g[\frac{1}{2} - \frac{x}{2}] = \frac{1}{2} - \frac{x}{2}$, and they have $\frac{1}{3}$ as a common fixed point.

3.3.4 <u>Theorem</u>. If f and g are two continuous functions from $I = [0, 1]$ into itself such that $fg(x) = gf(x)$ for all $x \in I$ and $|f(x) - f(y)| > \alpha|x - y|$ is a one to one mapping of a subset of I onto I and $|g(x) - g(y)| \leq \beta|x - y|$, where β is any positive real number, and $\alpha > 1$. Then f and g have a common fixed point.

<u>Proof</u>. We know that $[0, 1]$ is well-linked or ε-chainable complete metric space. The mapping $|f(x) - f(y)| > \alpha|x - y|$ where $\alpha > 1$ is an expansive mapping of a subset I onto I. Since the mapping f is one to one and onto, the inverse $f^{-1}(x)$ exists. Thus the mapping $f^{-1}(x)$ satisfies all the conditions of Banach's contraction principle and therefore by Banach's contraction principle there exists a unique fixed point $x_0 \in I$ such that $f(x_0) = x_0$.

The remaining part of the proof follows on the same line as given in the above theorem.

<u>Remark</u>. In Theorem 3.3.3 and Theorem 3.3.4 the Lipschitz condition on g can be dropped altogether. Moreover, it suffices, apart from the commuting property, merely to assume that f has a unique fixed point.

BIBLIOGRAPHY

1. Amer. Math. Monthly. 75 (1968), pp. 417. [Advanced Problem 5482]

2. Bailey, D.F. "Some Theorems on Contractive Mappings", Jour. London Math. Soc. 41 (1966), pp. 101-106.

3. Bailey, D.F. "On Contractive and Expansive Mappings", Doctoral dissertation, Vanderbilt University (1965).

4. Banach, S. "Surles Opérations dans les Ensembles Abstraits et leur application aux Équations Integrales", Fund. Math. 3(1922), pp. 133-181.

5. Bers, L. "Topology", Lecture Notes, New York University (1956-57).

6. Bessaga, C. "On the Converse of the Banach Fixed Point Principle", Coll. Math. VII (1959), pp. 41-43.

7. Bing, R.H. "A Convex Metric for Locally Connected Continuum", Bull. Amer. Math. Soc. 55 (1949), pp. 812-819.

8. Birkhoff, G. "Extensions to Jentch's Theorem", Trans. Amer. Math. Soc. 85 (1957), pp. 219-227.

9. Birkhoff, G. and Kellogg, O.D. "Invariant Points in Function Spaces", Trans. Amer. Math. Soc. 23 (1922), pp. 96-115.

10. Block, H.D. and Thielman, H.P. "Commutative Polynomials", Quart. Jour. Math. Oxford. Ser. (2), 2(1951), pp. 241-243.

11. Blumenthal, L.M. "Theory and Applications of Distance Geometry", Oxford, Clarendon Press (1953).

12. Bonsall, F.F. "Lectures on Some Fixed Point Theorems of Functional Analysis", Tata Institute of Fundamental Research, Bombay (India). (1962).

13. Boyce, W.M. "Commuting Functions with no Common Fixed Points", Notices Amer. Math. Soc. 14 (1967), pp. 280.

14. Browder, F.E. "A new generalization of the Schauder Fixed Point Theorem", Math. Annalen 174 (1967), pp. 285-290.

15. Browder, F.E. "On a Generalization of the Schauder Fixed Point Theorem", Duke Math. Jour. 26 (1959), pp. 291-304.

16. Browder, F.E. "Another Generalization of the Schauder Fixed Point Theorem", Duke Math. Jour. 32 (1965), pp. 399-406.

17. Browder, F.E. "A Further Generalization of the Schauder Fixed Point Theorems", Duke Math. Jour. 32 (1965), pp. 575-580.

18. Browder, F.E. "Fixed Point Theorems on Infinite Dimensional Manifolds", Trans. Amer. Math. Soc. 119 (1965), pp. 179-194.

19. Browder, F.E. "Nonlinear Operators in Banach Spaces", Math. Ann. 162 (1966), pp. 280-283.

20. Browder, F.E. "On the Convergence of Successive Approximations for Nonlinear Functional Equations", Indag. Math., 30, No. 1, (1968), pp. 27-35.

21. Brown, T.A. and Comfort, W.W. "New Method for Expansion and Contraction Maps in Uniform Spaces", Proc. Amer. Math. Soc. 11 (1960), pp. 483-486.

22. Cacciopoli, R. "Un teorema generale sull' esistenza di elementi in unatransformazione functionale", Rend. Accad. Lincei 11 (1930), pp. 794-799.

23. Cheney, W. and Goldstein, A.A. "Proximity Maps for Convex Sets", Proc. Amer. Math. Soc. 10 (1959), pp. 448-450.

24. Choquet, G. "Topology" Academic Press. New York (1966).

25. Chu, S.C. and Diaz, J.B. "A Fixed Point Theorem for << In the Large >> Application of the Contraction Principle", Atti. della Accademia delle Scienze di Torino 99 (1964-65), pp. 351-363.

26. Chu, S.C. and Diaz, J.B. "Remarks on a Generalization of Banach's Principle of Contraction Mappings", Jour. Math. Anal. Appl. 11 (1965), pp. 440-446.

27. Cohen, H. "On Fixed Point Theorems for Commuting Mappings", Proc. Amer. Math. Soc. 15 (1964), pp. 293-296.

28. Davis, A.S. "Fixed Point Theorem for Contraction of Well-chained Topological Spaces", Proc. Amer. Math. Soc. 14 (1963), pp. 981-985.

29. DeMarr, R. "Common Fixed Point Theorems for Commuting Mappings", Amer. Math. Monthly. LXX (1963), pp. 535-537.

30. Diaz, J.B. and Margolis, B. "A Fixed Point Theorem of the Alternative for Contractions on a Generalized Complete Metric Space", Bull. Amer. Math. Soc. 74 (1968), pp. 305-309.

31. Edelstein, M. "An Extension of Banach's Contraction Principle", Proc. Amer. Math. Soc. 12 (1961), pp. 7-10.

32. Edelstein, M. "On Fixed and Periodic Points Under Contractive Mappings", Jour. London Math. Soc. 73 (1962), pp. 74-79.

33. Edelstein, M. "On Non Expansive Mappings", Proc. Amer. Math. Soc. 15 (1964), pp. 689-695.

34. Edelstein, M. "On Non Expansive Mappings of Uniform Spaces", Indag. Math. 27, No. 1 (1965), pp. 47-51.

35. Edelstein, M. "A Remark on a Theorem of A.F. Monna", Indag. Math., 26, No. 1 (1964), pp. 88-89.

36. Edwards, R.E. "A Fixed Point Theorem with Application to Convolution Equations", Jour. Australian Math. Soc. III (1963), pp. 385-395.

37. Edwards, R.E. "Functional Analysis Theory and Applications", Holt, Rinehart and Winston, New York (1965).

38. Frétchet, M. "Surles Ensembles Abstraits", Ann. Ée. Norm. Sup; 38 (1921), pp. 341-387.

39. Hamel, G. "Eene Basis Aller Zahlen und die Unstetige Lösungen der Funktionalcichung $f(x+y) = f(x) + f(y)$. Math. Ann. 60 (1905), pp. 459-462.

40. Huneke, J.P. "Two counter Examples to a Conjecture on Commuting Continuous Functions of the Closed Unit Interval", Notices Amer. Math. Soc. 14 (1967), pp. 284.

41. Isbell , J.R. "Commuting Mappings of Trees", Bull, Amer. Math. Soc. 63 (1957), pp. 919.

42. Janos, L. "Two Operations on Distance Functions", Notices Amer. Math. Soc. 11 (1964), pp. 588.

43. Janos, L. "A Converse of Banach's Contraction Theorem", Proc. Amer. Math. Soc. 18 (1967), pp. 287-289.

44. Jungck, G. "Commuting Mappings and Common Fixed Points", Amer. Math. Monthly 73 (1966), pp. 735-737.

45. Kammerer, W.J. and Kasriel, R.H. "On Contractive Mappings in Uniform Spaces", Proc. Amer. Math. Soc. 15 (1964), pp. 288-290.

46. Knill, R.J. "Fixed Point of Uniform Contractions", Jour. Math. Anal. Appl. 12 (1965), pp. 449-455.

47. Kolmogorov, A.N. and Fomin, S.V. "Elements of the Theory of Functions and Functional Analysis", Volume 1 (Metric and Normed Spaces), Graylock Press (1957). New York.

48. Kransnoleski, M.A. "Two Remarks Concerning the Method of Successive Approximation", Uspekhi Mathematicheskie Nauk 10 (1955), pp. 123-127.

49. Levi, H. "Foundation of Geometry and Trigonometry", Prentice-Hall (1960), New Jersey.

50. Luxemburg, W.A.J. "On the Convergence of Successive Approximations in the Theory of Ordinary Differential Equations II", Indag. Math. 20 (1958), pp. 540-546.

51. Margolis, B. "On Some Fixed Point Theorems in Generalized Complete Metric Spaces", Bull. Amer. Math. Soc. 74 (1968), pp. 275-282.

52. Mathews, J.C. "Fixed Point Theorems for Contraction Mapping in a Regular Developable Spaces", Doctoral Dissertation, Iowa State University, Ames. Iowa (1959).

53. Mathews, J.C. and Curtis, D.W. "Rationally Contractive Relations in Pair of Generalized Uniform Spaces", Jour. London. Math. Soc. (To Appear).

54. Meyers, P.R. "On the Converse to the Contraction Mapping Principle", Doctoral Dissertation, University of Maryland, (1966).

55. Monna, A.F. "Sur un Theoreme dem Luxemburg concernant les points fixes dume classed applications dune spaces mitrique dans lui meme", Indag. Math. 23 (1961), pp. 89-96.

56. Naimpally, S.A. "A Note on Contraction Mappings", Akad. Van. Wetenschappen Proc. Series A (1964), pp. 275-279.

57. Nemizki, V.V. "Method nepodviznikh totchekv analiza", Nauk. Matem. 1 (1936), pp. 141-174.

58. Rakotch, E. "A Note on α-locally Contractive Mappings", Bull. Res. Counc. of Israel. $10F_4$ (1962), pp. 188-191.

59. Rakotch, E. "A Note on Contractive Mappings", Proc. Amer. Math. Soc. 13 (1962), pp. 459-465.

60. Ritt, J.F. "Permutable Rational Functions", Trans. Amer. Math. Soc. 29 (1923), pp. 399-448.

61. Samuelson, H. "On the Perron-Frobenius Theorem", Michigan. Math. Jour. 6 (1957), pp. 57-59.

62. Schröder, J. "A Unified Theory for Estimation and Iteration in Metric Spaces and Partially Ordered Spaces", MRC No. 237. University of Wisconsin, (1961).

63. Sehgal, V.M. "A Fixed Point Theorem for Local Contraction Mappings", Notices Amer. Math. Soc. 12 (1965), pp. 481.

64. Seguin, C.P. "Commuting Linear Functions and Fixed Points", Math. Teacher December (1965), pp. 702-704.

65. Shields, A.L. "On Fixed Points of Commuting Analytic Functions", Proc. Amer. Math. Soc. 15 (1964), pp. 703-706.

66. Sims, B.T. "Some Generalizations of the Contraction Mapping Theorem", Notices Amer. Math. Soc. 15 (1968), pp. 620.

67. Singh, K.L. "Contraction Mappings and Fixed Point Theorems", Annales de la Societe Scienti fique de Bruxells. (Accepted for Publication).

68. Singh, K.L. "On Commuting Functions and Common Fixed Points", Bull. Mathematique Rumania (Accepted for Publication).

69. Singh, K.L. "On Some Fixed Point Theorems", Rivista di Matematica (Submitted).

70. Singh, K.L. and Singh, S.P. "On Fixed Point Theorems", Rivista di Matematica (Accepted for Publication).

71. Singh, S.P. "On Commuting Analytic Functions with Fixed Points", Notices Amer. Math. Soc. 14 (1967), pp. 673.

72. Thompson, A.C. "On certain Contraction Mappings in Partially Ordered Vectorspaces", Proc. Amer. Math. Soc. 14 (1963), pp. 438-443.

73. Whyburn, G.T. "Topological Analysis", Princeton University Press, Princeton, New Jersey (1958).

74. Willet, D.W. "Nonlinear Vector Integral Equations as a Contraction Mapping", Arch. for Rat. Mech. and Anal. 15 (1964), pp. 79-86.

75. Wong, J.S.W. "Generalizations to the Converse of Contraction Mapping Principle", Doctoral Dissertation, California Institute of Technology (1965).

76. Zarantonello, E.H. "Solving Functional Equations by Contractive Averaging", MRC No. 160 University of Wisconsin, June (1960).