FIBRATIONS, COFIBRATIONS AND HOMOTOPY EQUIVALENCES

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BY

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A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

AUGUST 1973

This thesis has been examined and approved by

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ACKNOWLEDGEMENTS

I would like to thank Dr. S. Thomeier for his helpful advice and encouragement during the preparation of this thesis and Mrs. Rosalind Wells for her careful typing of the manuscript.

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NOTATION : *The symbol* $\square$ *is placed at the end of each proof and at the end of a statement if no proof is forthcoming.*

INTRODUCTION

The first chapter is an attempt to unify some standard results concerning mapping cylinders and mapping cones by means of a construction which includes them both as special cases; namely, the double mapping cylinder for two maps with a common domain. If one of the maps is a cofibration, then this space is homotopically equivalent to the push-out space of the two maps. Consequently, under this hypothesis, we can obtain information about the push-out space from properties of the double mapping cylinder.

Given two maps with a common codomain we can form the pull-back space. There is a construction dual to the double mapping cylinder which provides a space which is homotopically equivalent to the pull-back space when one of the maps is a fibration. This construction has properties analagous to those of the double mapping cylinder.

In the second chapter one of the main results of chapter one is used to show that every finite dimensional connected CW complex is homotopically equivalent to a CW complex constructed "around a point".

The third chapter is devoted to the construction of some of the basic exact sequences of homotopy theory. The constructions of the Puppe sequence and its dual are included here for the sake of completeness. The exact sequences for a triple are obtained by geometrical arguments as opposed to the construction by Eckmann in [1]. It is possible to associate two similar looking exact sequences with a fibration. These two exact sequences are shown to be essentially the same by means of a "transgression square" of spaces and maps.

CHAPTER ONE: THE DOUBLE MAPPING CYLINDER AND ITS DUAL

We shall work in the category Top_* of topological spaces with base points and base point preserving maps. We always suppose that the base point x_0 of a space X is closed as a subspace of X. All homotopies keep the base point fixed.

If X and Y are two based spaces with base points x_0 and y_0 respectively, their product $X \times Y$ has the base point (x_0,y_0). The wedge of X and Y is the subspace $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ of $X \times Y$. $X \vee Y$ may also be described as the quotient space of the disjoint union of X and Y obtained by identifying the base points. There are injections $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ which map X and Y homeomorphically onto closed subspaces of $X \vee Y$. We identify X and Y with these subspaces.

The (reduced) cylinder, cone and suspension of a space X are the quotient spaces

$$ZX = X \times I/\{x_0\} \times I$$

$$CX = X \times I/\{x_0\} \times I \cup X \times \{1\}$$

$$SX = X \times I/\{x_0\} \times I \cup X \times \{0,1\}$$

where I is the unit interval $[0,1]$ with the usual topology. In each case we denote the class of (x,t), where $x \in X$ and $t \in I$, by $[x,t]$. With the obvious assignments for maps we obtain from these constructions functors Z, C and S from Top_* to Top_*.

The path space of a space X, denoted by PX, is the space of all maps $I \to X$ with the compact open topology. The subspace of PX consisting of those paths ω for which $\omega(1) = x_0$ is denoted by EX. The loop space of X, denoted by ΩX, is the subspace of PX consisting of those paths ω for which $\omega(0) = \omega(1) = x_0$. We now have functors P, E and Ω which are adjoint to Z, C and S respectively.

If F is any of the functors Z, C, S, P, E, Ω we define F^n for each integer $n \geq 0$ inductively by

$$F^0 X = X$$

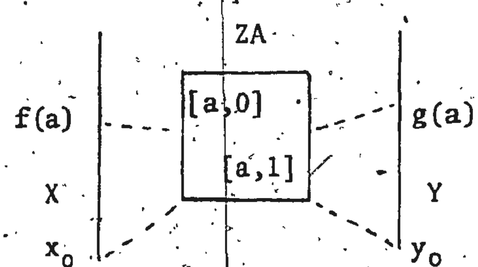
$$\text{and} \quad F^n X = F(F^{n-1}X).$$

If F is Z, C or S, $F^n X$ is a quotient space of $X \times I^n$ and we have $F^n(X \vee Y) = F^n X \vee F^n Y$. If F is P, E or Ω, $F^n X$ may be interpreted as a subspace of the space of maps $I^n \to X$ and we have $F^n(X \times Y) = F^n X \times F^n Y$.

Let

$$X \xleftarrow{f} A \xrightarrow{g} Y, \qquad (1.1)$$

be a diagram of based spaces and maps. The <u>double mapping cylinder</u> of this diagram, denoted by $Z(f,g)$, is the quotient space of $X \vee ZA \vee Y$ obtained by identifying $[a,0] \in ZA$ with $f(a) \in X$ and $[a,1] \in ZA$ with $g(a) \in Y$ for all $a \in A$.

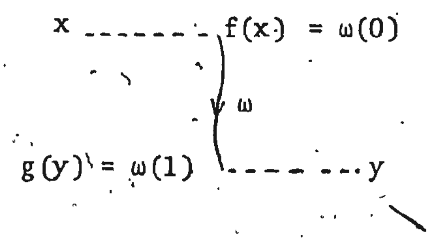


If $Y = A$ and g is the identity map on A, then $Z(f,g) = Z_f$, the mapping cylinder of f. If $Y = *$, a point, then $Z(f,g) = C_f$, the mapping cone of f. Z_f and C_f may also be described as the quotient spaces of $ZA \vee X$ and $CA \vee X$ respectively obtained by identifying $[a,0]$ with $f(a)$ for all $a \in A$. The wedge, cylinder, cone and suspension may also be regarded as special cases of this construction.

We have a dual or adjoint construction for a diagram

$$X \xrightarrow{f} A \xleftarrow{g} Y \qquad (1.1')$$

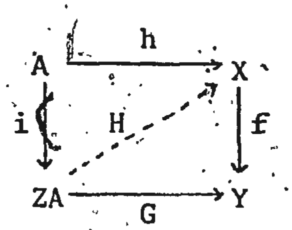
of based spaces and maps. The <u>double mapping track</u> of this diagram, denoted by $P(f,g)$, is the subspace of $X \times PA \times Y$ consisting of triples (x,ω,y) such that $f(x) = \omega(0)$ and $g(y) = \omega(1)$.



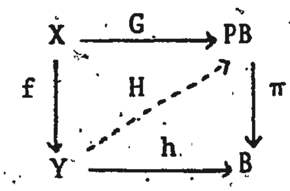
If $Y = A$ and g is the identity map on A, then $P(f,g) = P_f$, the mapping track of f. P_f may also be described as the subspace of $X \times PA$ consisting of pairs (x,ω) such that $f(x) = \omega(0)$. We denote by E_f the subspace of $X \times EA$ consisting of pairs (x,ω) such that $f(x) = \omega(0)$. If $Y = *$, then $P(f,g) = E_f$. The spaces $X \times Y$, PA, EA and ΩA can all be obtained as special cases of this construction.

The double mapping cylinder and double mapping track constructions are each functorial on a category of diagrams of the appropriate type.

A map $f:X \to Y$ is called a <u>fibration</u> if for each space A, given a map $h:A \to X$ and a homotopy $G:A \times I \to Y$ such that $G(-,0) = Fh$, there is a homotopy $H:A \times I \to X$ such that $H(-,0) = h$ and $fH = G$. Equivalently, $f:X \to Y$ is a fibration if each commutative square



where i is the inclusion $a \mapsto [a,0]$, can be completed as shown with the map H so that the whole diagram is commutative. A map $f:X \to Y$ is called a <u>cofibration</u> if for each space B, given a map $h:Y \to B$ and a homotopy $G: X \times I \to B$ such that $G(-,0) = hf$, there is a homotopy $H: Y \times I \to B$ such that $H(-,0) = h$ and $H(f(x),t) = G(x,t)$ for all $x \in X$, $t \in I$. Equivalently, $f: X \to Y$ is a cofibration if each commutative square



where π is the projection $\omega \mapsto \omega(0)$, can be completed as shown with the map H so that the whole diagram is commutative. A pair of spaces (X,A) with $A \subset X$ is said to have the homotopy extension property H E P if the inclusion $A \to X$ is a cofibration.

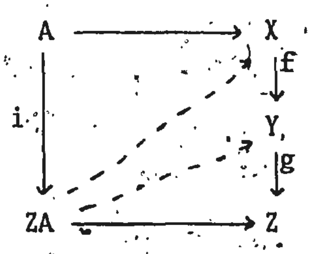
The kernel of a fibration $f: X \to Y$, $\ker f = \{x \in X \mid f(x) = y_0\}$, is called the <u>fibre</u> of f. If $X = Y \times F$ for some space F, the projection $Y \times F \to Y$ onto the first factor of the product is a fibration with fibre F since we can define $H: A \times I \to Y \times F$ by $H(a,t) = (G(a,t), ph(a))$ where p is the projection $Y \times F \to F$ onto the second factor.

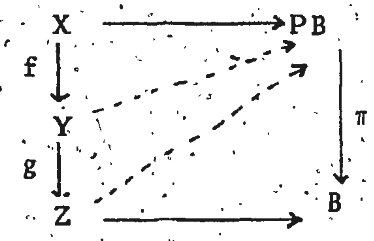
The cokernel of a cofibration $f: X \to Y$, $\text{cok} f = Y/f(X)$ is called the <u>cofibre</u> of f. If $Y = X \vee C$ for some space C, the inclusion $X \to X \vee C$ is a cofibration with cofibre C since we can define $H: (X \vee C) \times I \to B$ by $H(x,t) = G(x,t)$, $x \in X$, $t \in I$ and $H(c,t) = h(c)$, $c \in C$, $t \in I$.

<u>Lemma</u> (1.2') If $f: X \to Y$ and $g: Y \to Z$ are fibrations, then so is $gf: X \to Z$.

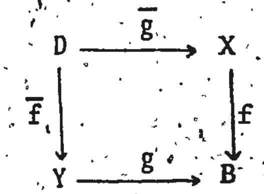
<u>Lemma</u> (1.2) If $f: X \to Y$ and $g: Y \to Z$ are cofibrations, then so is $gf: X \to Z$

The proofs of these lemmas are indicated by the diagrams



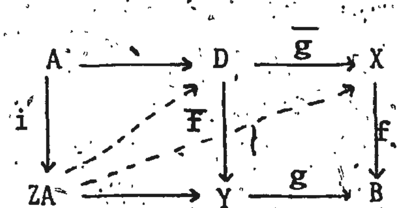


<u>Lemma</u> (1.3) If the square

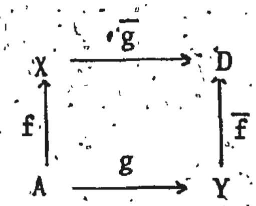


is a pull-back square in which f is a fibration, then $\bar{f}$ is also a fibration.

<u>Proof</u> We use the property of the pull-back square as indicated by the diagram



<u>Lemma</u> (1.3) If the square



is a push-out square in which f is a cofibration, then $\bar{f}$ is also a cofibration.

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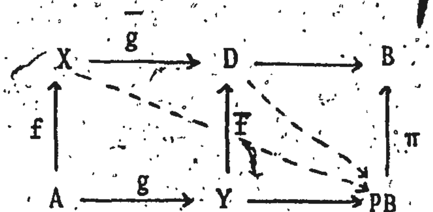


It is appropriate to remark at this stage that the evident duality involved in this work [2] is related to the formal duality of category theory but is itself not formal. A theorem should therefore be stated and proved independently of its dual. In practice we shall often merely sketch the proof of a dual result when the procedure for dualization is clear.

Proposition (1.4) Given a diagram (1.1), the inclusion $X \vee Y \to Z(f,g)$ is a cofibration.

Proof. Let B be any space and suppose that we have a map $h: Z(f,g) \to B$ and a homotopy $G: (X \vee Y) \times I \to B$ such that $G(-,0) = h|X \vee Y$. Let $U = I \times \{0\} \cup \{0,1\} \times I \subset I^2$ and define $\phi: A \times U \to B$ by

$$\phi(a,s,t) = \begin{cases} G(f(a),t), & \text{if } s = 0 \\ h[a,s], & \text{if } t = 0 \\ G(g(a),t), & \text{if } s = 1 \end{cases}$$

Since U is a retract of I^2, $A \times U$ is a retract of $A \times I^2$ and ϕ has an extension $\Phi: A \times I^2 \to B$. Define $H: Z(f,g) \times I \to B$ by

$$H(x,t) = G(x,t), \quad x \in X,\ t \in I$$
$$H([a,s],t) = (a,s,t), \quad a \in A,\ s,t \in I$$
$$H(y,t) = G(y,t), \quad y \in Y,\ t \in I$$

Since I is locally compact we can use the adjoint map $Z(f,g) \to PB$ to see that H is well defined and continuous. We have $H(-,0) = h$ and $H|X \vee Y = G$ as required.

Proposition (1.4') Given a diagram (1.1'), the projection $p: P(f,g) \to X \times Y$ given by $p(x,\omega,y) = (x,y)$ is a fibration.

Proof Let B be any space and suppose that we have a map $h:B \to P(f,g)$ and a homotopy $G:B \times I \to X \times Y$ such that $G(-,0) = ph$. Let q be the projection $P(f,g) \to PA$ given by $q(x,\omega,y) = \omega$ and let p_1, p_2 be the projections of $P(f,g)$ onto X and Y respectively. Then $h(a) = (p_1 h(a), qh(a), p_2 h(a))$ for all $a \in A$. Define $\phi:B \times U \to A$, where $U = I \times \{0\} \cup \{0,1\} \times I$, by

$$\phi(b,s,t) = \begin{cases} fp_1 G(b,t) & , \text{ if } s = 0 \\ qh(b)(s) & , \text{ if } t = 0 \\ gp_2 G(b,t) & , \text{ if } s = 1 \end{cases}$$

Then ϕ has an extension $\Phi:B \times I^2 \to A$. Define $\hat{\Phi} = B \times I \to PA$ by $\hat{\Phi}(b,t)(s) = \Phi(b,s,t)$ and $H:B \times I \to P(f,g)$ by $H(b,t) = (p_1 G(b,t), \hat{\Phi}(b,t), p_2 G(b,t))$. Then $H(-,0) = h$ and $pH = G$ as required. ▯

Corollary (1.5) The inclusions $X \to Z(f,g)$, $Y \to Z(f,g)$ are cofibrations.

Proof Each inclusion is the composite of two inclusions each of which is a cofibration. Hence the result follows from lemma (1.2). ▯

Corollary (1.5') The projections $P(f,g) \to X$, $P(f,g) \to Y$ are fibrations. ▯

Given a diagram (1.1) we can form the push-out space $U(f,g)$. We may regard $U(f,g)$ to be the quotient space of $X \vee A \vee Y$ obtained by identifying $a \in A$ with $f(a) \in X$ and $g(a) \in Y$ for all $a \in A$. Since each point in $U(f,g)$ is represented by a point in X or Y we may also regard $U(f,g)$ to be a quotient space of $X \vee Y$.

There is a projection $q:Z(f,g) \to U(f,g)$ given by

$$\begin{aligned} q(x) &= [x] \quad , \quad x \in X \\ q[a,s] &= [a] \quad , \quad a \in A,\ s \in I \\ q(y) &= [y] \quad , \quad y \in X \end{aligned}$$

q is induced by the map $X \vee ZA \vee Y \to X \vee A \vee Y$ which collapses ZA onto A.

<u>Proposition</u> (1.6) If f or g is a cofibration, then q is a homotopy equivalance.

<u>Proof</u> By symmetry it is sufficient to consider the case where f is a cofibration.

Define $G:A \times I \to Z(f,g)$ by $G(a,t) = [a,t]$. Then $G(-,0) = i_1 f$ where i_1 is the inclusion $X \to Z(f,g)$. Since f is a cofibration there is a homotopy $H:X \times I \to Z(f,g)$ such that $H(-,0) = i_1$ and $H(f(a),t) = G(a,t) = [a,t]$ for all $a \in A$. Define $q':u(f,g) \to Z(f,g)$ by

$$q'[x] = H(x,1)$$
$$q'[y] = y$$

Then $q'[f(a)] = H(f(a),1) = [a,1] = g(a)$ so q' is well defined. We show that q' is a homotopy inverse for q.

Define $F:U(f,g) \times I \to U(f,g)$ by

$$F([x],t) = qH(x,t) \quad , \quad x \in X,\ t \in I$$
$$F([y],t) = [y] \quad , \quad y \in Y,\ t \in I$$

Then $F([f(a)],t) = qH(f(a),t) = q[a,t] = [a] = [g(a)]$ and so via the corresponding map $U(f,g) \to PU(f,g)$ we see that F is well defined and continuous. F is a homotopy from the identity map on $U(f,g)$ to qq'.

Define $F':Z(f,g) \times I \to Z(f,g)$ by

$$F'(x,t) = H(x,t)$$
$$F'([a,s],t) = [a,(1-t)s + t]$$
$$F'(y,t) = y$$

Then $F'([a,1],t) = [a,1] = g(a)$ and $F'([a,0],t) = [a,t] = H(f(a),t)$ so F' is well defined and continuous. F' is a homotopy from the identity map on $Z(f,g)$ to $q'q$. ▯

Remark The fact that q' is well defined implies that f is injective.

Corollary (1.7) If $f:X \to Y$ is a cofibration then the map $q:C_f \to$ cofibre f given by

$$q[x,t] = *$$
$$q(y) = [y]$$

is a homotopy equivalence.

Proof $\operatorname{cok} f = Y/f(X)$ and C_f are respectively the push-out space and double mapping cylinder of the diagram

$$* \longleftarrow X \xrightarrow{f} Y .$$

Given a diagram (1.1') we can form the pull-back space $I(f,g)$. We may regard $I(f,g)$ to be the subspace of $X \times Y$ consisting of pairs (x,y) such that $f(x) = g(y)$. There is an inclusion $j:I(f,g) \to P(f,g)$ given by $j(x,y) = (x,\omega(x,y),y)$ where $\omega(x,y) \in PA$ is the constant path onto the point $f(x) = g(y) \in A$.

Proposition (1.6') If f or g is a fibration, then j is a homotopy equivalence.

Proof We consider the case where f is a fibration.

Define $G:P(f,g) \times I \to A$ by $G(x,\omega,y,t) = \omega(t)$. Then $G(-,0) = fp_1$ where p_1 is the projection $P(f,g) \to X$. Since f is a fibration, there is a homotopy $H:P(f,g) \times I \to X$ such that $H(-,0) = p_1$ and $fH = G$.

Define $j':P(f,g) \to I(f,g)$ by $j'(x,\omega,y) = (H(x,\omega,y,1), y)$. Then $fH(x,\omega,y,1) = \omega(1) = g(y)$ so j' is well defined.

Define $F:I(f,g) \times I \to I(f,g)$ by $F(x,y,t) = (H(j(x,y),t),y)$. Then $fH(j(x,y),t) = \omega(x,y)(t) = g(y)$ so F is well defined. F is a homotopy from the identity on $I(f,g)$ to $j'j$.

Define $F':P(f,g) \times I \to P(f,g)$ by $F'(x,\omega,y,t) = (H(x,\omega,y,t), \phi(\omega,t),y)$

where $\phi: PA \times I \to PA$ is given by $\phi(\omega,t)(s) = \omega((1-t)s+t)$. Then $\phi(\omega,t)(0) = \omega(t) = fH(x,\omega,y,t)$ and $\phi(\omega,t)(1) = \omega(1) = g(y)$ so F' is well defined. F' is a homotopy from the identity map on $P(f,g)$ to jj'. Hence j' is a homotopy inverse for j.

<u>Corollary</u> (1.7') If $f: X \to Y$ is a fibration with fibre F, then the map $j: F \to E_f$ given by $j(x) = (x,*)$ is a homotopy equivalence. ▯

<u>Proposition</u> (1.8) If $f \simeq f': A \to X$ and $g \simeq g': A \to Y$, then there is a homotopy equivalence $Z(f,g) \to Z(f',g')$ which extends the identity on $X \vee Y$.

<u>Proof</u> Let $F: A \times I \to X$ be a homotopy from f to f'. Define $\phi: Z(f,g) \to Z(f',g)$ by

$$\phi(x) = x$$

$$\phi[a,s] = \begin{cases} F(a,2s), & 0 \le s \le 1/2 \\ [a,2s-1], & 1/2 \le s \le 1 \end{cases}$$

$$\phi(y) = y$$

and $\phi': Z(f',g) \to Z(f,g)$ by

$$\phi'(x) = x$$

$$\phi'[a,s] = \begin{cases} F(a,1-2s), & 0 \le s \le 1/2 \\ [a,2s-1], & 1/2 \le s \le 1 \end{cases}$$

$$\phi'(y) = y$$

Then $\phi'\phi: Z(f,g) \to Z(f,g)$ is given by

$$\phi'\phi(x) = x$$

$$\phi'\phi[a,s] = \begin{cases} F(a,2s), & 0 \le s \le 1/2 \\ F(a,3-4s), & 1/2 \le s \le 3/4 \\ [a,4s-3], & 3/4 \le s \le 1 \end{cases}$$

$$\phi'\phi(y) = y$$

Define $H: Z(f,g) \times I \to Z(f,g)$ by

$$H(x,t) = x$$

$$H([a,s],t) = \begin{cases} F(a,2s), & 0 \le s \le 1/2 - 1/2\,t \\ F(a,3 - 4s - 3t), & 1/2 - 1/2\,t \le s \le 3/4 - 3/4\,t \\ [a, \frac{4s + 3t - 3}{1 + 3t}], & 3/4 - 3/4\,t \le s \le 1 \end{cases}$$

$$H(y,t) = y$$

Then H is a homotopy from $\phi'\phi$ to the identity on $Z(f,g)$. There is a similar homotopy from $\phi\phi'$ to the identity on $Z(f',g)$, hence ϕ is a homotopy equivalence with homotopy inverse ϕ'.

Let G be a homotopy from g to g' and define $\psi:Z(f',g) - Z(f',g')$ by

$$\psi(x) = x$$

$$\psi[a,s] = \begin{cases} [a,2s], & 0 \le s \le 1/2 \\ G(a,2 - 2s), & 1/2 \le s \le 1 \end{cases}$$

$$\psi(y) = y$$

Then ψ is a homotopy equivalence by a similar argument to the above. Thus $\psi\phi:Z(f,g) \to Z(f',g')$ given by

$$\psi\phi(x) = x$$

$$\psi\phi[a,s] = \begin{cases} F(a,2s), & 0 \le s \le 1/2 \\ [a, 4s - 2], & 1/2 \le s \le 3/4 \\ G(a, 4 - 4s), & 3/4 \le s \le 1 \end{cases}$$

$$\psi\phi(y) = y$$

is a homotopy equivalence.

<u>Proposition</u> (1.9) Given a diagram (1.1) and homotopy equivalences $h:X \to X'$ and $k:Y \to Y'$, the map $Z(f,g) \to Z(hf,kg)$ induced by h and k is a homotopy equivalence.

<u>Proof</u>. Let $h':X' \to X$ be a homotopy inverse for h and let $H:X \times I \to X$ be a homotopy from $h'h$ to 1_X. From the proof of proposition (1.8) we see that the map $\phi:Z(h'hf,g) \to Z(f,g)$ given by

$$\phi(x) = x$$

$$\phi[a,s] \begin{cases} H(f(a), 2s), & 0 \le s \le 1/2 \\ [a, 2s - 1], & 1/2 \le s \le 1 \end{cases}$$

$$\phi(y) = y$$

is a homotopy equivalence.

Let $\eta:Z(f,g) \to Z(hf,g)$ and $\eta':Z(hf,g) \to Z(h'hf,g)$ be the maps induced by h and h' respectively. We show that η is a homotopy equivalence with homotopy inverse $\xi = \phi\eta'$.

$\xi\eta:Z(f,g) \to Z(f,g)$ is given by

$$\xi\eta(x) = h'h(x)$$

$$\xi\eta[a,s] = \begin{cases} H(f(a), 2s), & 0 \leq s \leq 1/2 \\ [a, 2s - 1], & 1/2 \leq s \leq 1 \end{cases}$$

$$\xi\eta(y) = y$$

Define $F:Z(f,g) \times I \to Z(f,g)$ by

$$F(x,t) = H(x,t)$$

$$F([a,s],t) = \begin{cases} H(f(a), 2s + t), & 0 \leq x \leq 1/2 - 1/2\, t \\ \left[a, \dfrac{2s - 1 + t}{1 + t}\right], & 1/2 - 1/2\, t \leq s \leq 1 \end{cases}$$

$$F(y,t) = y$$

Then F is a homotopy from $\xi\eta$ to $1_{Z(f,g)}$.

There is a homotopy from hh' to 1_X, which we can use to obtain a homotopy equivalence $\psi:Z(hh'hf,g) \to Z(hf,g)$.

Let $\eta'':Z(h'hf,g) \to Z(hh'hf,g)$ be the map induced by h and let $\xi' = \psi\eta''$. Then $\xi'\eta' \simeq 1$ by a similar homotopy to F. We have $\phi\eta'\eta \simeq 1$ and since ϕ is a homotopy equivalence $\eta'\eta\phi \simeq 1$. Thus $\xi' \simeq \xi'\eta'\eta\phi \simeq \eta\phi$ and therefore $\eta\xi = \eta\phi\eta' \simeq \xi'\eta' \simeq 1$.

By a similar argument it follows that the map $Z(hf,g) \to Z(hf,kg)$ induced by k is a homotopy equivalence. □

Proposition (1.10) Given a diagram (1.1) and a homotopy equivalence $e:B \to A$, the map $Z(fe,ge) \to Z(f,g)$ induced by e is a homotopy equivalence.

Proof. Let $e':A \to B$ be a homotopy inverse for e, and let $E:A \times I \to A$ be a homotopy from ee' to 1_A. From the proof of proposition (1.8) we see that the map $\theta:Z(fee',gee') \to Z(f,g)$ given by

$$\theta(x) = x \qquad \theta[a,s] = \begin{cases} fE(a, 1 - 2s), & 0 \le s \le 1/2 \\ [a, 4s - 2], & 1/2 \le s \le 3/4 \\ gE(a, 4s - 3), & 3/4 \le s \le 1 \end{cases} \qquad \theta(y) = y$$

is a homotopy equivalence.

Let $\varepsilon : Z(fe,ge) \to Z(f,g)$ and $\varepsilon' : Z(fee',gee') \to Z(fe,ge)$ be the maps induced by e and e' respectively. We show that ε is a homotopy equivalence with homotopy inverse $\lambda = \varepsilon'\theta$.

$\varepsilon\lambda : Z(f,g) \to Z(f,g)$ is given by

$$\varepsilon\lambda(x) = x \qquad \varepsilon\lambda[a,s] = \begin{cases} fE(a, 1 - 2s), & 0 \le s \le 1/2 \\ [ee'(a), 4s - 2], & 1/2 \le s \le 3/4 \\ gE(a, 4s - 3), & 3/4 \le s \le 1 \end{cases} \qquad \varepsilon\lambda(y) = y$$

Define $F : Z(f,g) \times I \to Z(f,g)$ by

$$F(x,t) = x \qquad F([a,s], t) = \begin{cases} fE(a, 1 - 2s), & 0 \le s \le 1/2 - 1/2\,t \\ \left[E(a,t), \dfrac{4s - 2 + 2t}{1 + 3t}\right], & 1/2 - 1/2\,t \le s \le 3/4 + 1/4\,t \\ gE(a, 4s - 3), & 3/4 + 1/4\,t \le s \le 1 \end{cases} \qquad F(y,t) = y$$

Then F is a homotopy from $\varepsilon\lambda$ to $1_{Z(f,g)}$. By a similar argument to that used in the previous proposition we also have $\lambda\varepsilon \simeq 1$. ▯

<u>Proposition</u> (1.8'). If $f \simeq f' : X \to A$ and $g \simeq g' : Y \to A$ then there is a homotopy equivalence $P(f',g') \to P(f,g)$ which covers the identity on $X \times Y$.

<u>Proof</u> Let $F : X \times I \to A$ and $G : Y \times I \to A$ be homotopies from f to f' and g to g' respectively. Let $\hat{F} : X \to PA$ and $\hat{G} : Y \to PA$ be the maps corresponding to F and G. Define $\phi : P(f',g) \to P(f,g)$ by

$$\phi(x,\omega,y) = (x, \hat{F}(x) + \omega, y)$$

where $+$ denotes the usual addition of paths, and $\phi' : P(f,g) \to P(f',g)$

by $\phi'(x,\omega,y) = (s, -\hat{F}(x) + \omega, y)$. Then it is straightforward to show that ϕ' is a homotopy inverse for ϕ. Similarly, $\psi: P(f',g') \to P(f',g)$ given by

$$\psi(x,\omega,y) = (x, \omega + (-\hat{G}(y)), y)$$

is a homotopy equivalence. Thus $\phi\psi: P(f',g') \to P(f,g)$ given by

$$\phi\psi(x,\omega,y) = (x, \hat{F}(x) + (\omega + (-\hat{G}(y))), y)$$

is a homotopy equivalence. $\square$

<u>Proposition</u> (1.9') Given a diagram (1.1') and homotopy equivalence $h: X' \to X$ and $k: Y' \to Y$, the map $P(fh,gk) \to P(f,g)$ induced by h and k in a homotopy equivalence. $\square$

<u>Proposition</u> (1.10') Given a diagram (1.1') and a homotopy equivalence $e: A \to B$, the map $P(f,g) \to P(ef,eg)$ induced by e is a homotopy equivalence. $\square$

From propositions 1.6, 1.8, 1.9 and 1.10 we obtain the following result.

<u>Theorem</u> (1.11) Given a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\ h\downarrow & & e\downarrow & & \downarrow k \\ X' & \xleftarrow{f'} & A' & \xrightarrow{g'} & Y' \end{array} \qquad (1.12)$$

there is a commutative square

$$\begin{array}{ccc} Z(f,g) & \xrightarrow{\phi} & Z(f',g') \\ q\downarrow & & \downarrow q' \\ U(f,g) & \xrightarrow{\psi} & U(f',g') \end{array}$$

in which ϕ and ψ are each induced by the triple of maps (h,e,k) and q,q' are the projections.

(a) If h, e and k are homotopy equivalences, then ϕ is a homotopy equivalence.

(b) If one map in each of the pairs (f,g), (f',g') is a cofibration, then q, q' are homotopy equivalences. Consequently, ψ is a homotopy equivalence.

(c) If the diagram (1.12) is only homotopy commutative, then we still have $Z(f,g) \simeq Z(f',g')$ by a map which extends $h \vee k$ on $X \vee Y$. If the hypothesis of (b) holds, then $U(f,g) \simeq U(f',g')$.

Corollary (1.12) If (X,A) has the HEP and $f \simeq g : A \to X$, then the adjunction spaces $Y \cup_f X$ and $Y \cup_g X$ are homotopically equivalent.

Corollary (1.13) If (X,A) has the HEP and A is contractible, then the projection $X \to X/_A$ is a homotopy equivalence.

Proof Consider the commutative diagram

$$\begin{array}{ccccc} X & \supset & A & = & A \\ \| & & \| & & \downarrow \\ X & \supset & A & \longrightarrow & * \end{array}$$

The push-out space of the top line is X while that of the bottom line is $X/_A$.

From propositions 1.6', 1.8', 1.9' and 1.10' we obtain the dual

Theorem (1.11') Given a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & A & \xleftarrow{g} & Y \\ h\downarrow & & e\downarrow & & \downarrow k \\ X' & \xrightarrow{f'} & A' & \xleftarrow{g'} & Y' \end{array} \qquad (1.12')$$

there is a commutative square

$$\begin{array}{ccc} I(f,g) & \xrightarrow{\psi} & I(f',g') \\ j \downarrow & & \downarrow j' \\ P(f,g) & \xrightarrow{\phi} & P(f',g') \end{array}$$

in which ϕ and ψ are induced by the triple of maps (h,e,k) and j,j' are the inclusions.

(a) If h,e and k are homotopy equivalences, then ϕ is a homotopy equivalence

(b) If one map in each of the pairs (f,g), (f',g') is a fibration, then j,j' are homotopy equivalences. Consequently, ψ is a homotopy equivalence.

(c) If the diagram (1.12') is only homotopy commutative, then we still have $P(f,g) \simeq P(f',g')$ by a map which covers $h \times k$ on $X \times Y$. If the hypothesis of (b) holds, then $I(f,g) \simeq I(f',g')$.

CHAPTER II: APPLICATIONS TO CW COMPLEXES

We now summarize some basic ideas concerning CW complexes. More details can be obtained from [4], [5], [7] and [9].

A cell complex X is a Hausdorff space which is a (set theoretic) disjoint union of subsets e_α called cells, indexed by some set J, for each of which there is a map $\phi_\alpha : E^n \to X$ for some $n \geq 0$, where n is the dimension of e_α, such that:

(1) $\phi_\alpha | E^n \setminus S^{n-1}$ is bijective onto e_α,

(2) $\phi_\alpha(S^{n-1})$ meets only cells of dimension less than n.

Here E^n is the subspace of Euclidean n-space R^n consisting of all vectors x such that $|x| \leq 1$ and S^{n-1} is its boundary $\dot{E}^n$.

The map ϕ_α is called the characteristic map of e_α. The subspace of X consisting of all the cells of X whose dimension does not exceed n is called the n-skeleton of X and is denoted by X^n. If X has no m-cells for $m > n$, then $X = X^n$ and we say that X has dimension n. A cell complex which consists of a finite number of cells is called a finite cell complex. A finite cell complex is clearly finite dimensional.

Lemma (2.1) For each $\alpha \in J$

(1) $\phi_\alpha(E^n) = \bar{e}_\alpha$,

(2) $\bar{e}_\alpha$ has the quotient topology determined by ϕ_α,

(3) $\phi_\alpha | E^n \setminus S^{n-1}$ is a homeomorphism onto e_α. ▯

The closures $\bar{e}_\alpha$ of the cells e_α of X are called the closed cells of X. A union S of cells of a cell complex X is called a subcomplex of X iff for each cell e of X contained in S, S contains the closed cell $\bar{e}$.

A map $f:X \to Y$ between cell complexes is called cellular iff $f(X^n) \subset Y^n$ for all $n \geq 0$. For example, the inclusion of a subcomplex is a cellular map.

A cell complex is called closure finite iff the closure of each cell of X meets only finitely many cells of X, or equivalently, iff the boundary of each n-cell of X meets only finitely many cells of X of dimension less than n. A cell complex is said to have the weak topology iff it has the weak topology with respect to its closed cells. A closure finite cell complex with the weak topology is called a CW complex.

Lemma (2.2) A subcomplex of a CW complex is a CW complex in the induced topology. $\square$

Lemma (2.3) A subcomplex of a CW complex X is closed as a subspace of X. $\square$

Lemma (2.4) Every compact subset of a CW complex is contained in a finite subcomplex. $\square$

Let X be a topological space and J be any set. The disjoint union of copies of X indexed by J, denoted by $\coprod_{\alpha \in J} X_\alpha$, is the product space $X \times J$, where J has the discrete topology. Each subspace $X_\alpha = X \times \{\alpha\}$ of $X \times J$ is a homeomorphic copy of X. $\coprod_{\alpha \in J} X_\alpha$ has the weak topology with respect to the subspaces X_α.

If X has a base point x_0 the wedge $\bigvee_{\alpha \in J} X_\alpha$ is the quotient space $X \times J/_{\{x_0\} \times J}$.

Theorem (2.5) A space X is a CW complex iff there is a nested sequence of subspaces

$$X^0 \subset X^1 \subset \ldots \subset X^n \subset \ldots$$

of X such that:

(1) X^0 is discrete,

(2) X^n is obtained from X^{n-1} by attaching n-cells; that is, for each $n \geq 1$ there is a push-out diagram

$$\begin{array}{ccc} \coprod_{\alpha \in J^n} E^n_\alpha & \longrightarrow & X^n \\ \uparrow & & \uparrow \\ \coprod_{\alpha \in J^n} S^{n-1}_\alpha & \longrightarrow & X^{n-1} \end{array}$$

where $J^n \subset J$ is the indexing set for the n-cells of X,

(3) $X = \bigcup_{n \geq 0} X^n$ and X has the weak topology with respect to the subspaces X^n.

The elements of X^0 are the 0-cells of X and are called <u>vertices</u>.

<u>Proposition</u> (2.6) Let A be a subcomplex of a CW complex X and let $f:A \to Y$ be a cellular map into a CW complex Y. Then the adjunction space $Y \cup_f X$ is a CW complex with n-skeleton $Y^n \cup_{f^n} X^n$ where f^n is the restruction of f to A^n.

<u>Proposition</u> (2.7) Let X be a CW complex with a vertex as base point and let A be a subcomplex of X which contains the base point. Then (X,A) has the HEP.

<u>Theorem</u> (2.8) Let $f:X \to Y$ be a map between CW complexes such that $f|A$ is cellular, where A is a subcomplex of X. Then f is homotopic relative to A to a cellular map.

<u>Corollary</u> (2.9) Let $f:X \to Y$ be a pointed map between CW complexes with vertex base points. Then f is homotopic (relative to the base point of X) to a pointed cellular map.

Corollary (2.10) Let A be a subcomplex of a based CW complex X and let $f:A \to Y$ be an arbitrary pointed map into a based CW complex Y. Then the adjunction space $Y \cup_f X$ has the homotopy type of a CW complex.

Proof The result follows by the theorem from proposition (2.7) and corollary (1.12).

Corollary (2.11) A CW complex X is path connected iff X^1 is path connected.

Proof Suppose X is path connected and let $f:I \to X$ be a path in X from a point x to a point x' where $x,x' \in X^1$. Give I the obvious cellular structure of two 0-cells and one 1-cell. Then f is homotopic rel. $\{0,1\}$ to a path in X^1 from x to x'.

Conversely, suppose that X^1 is path connected. Any map $* \to X$ is homotopic to a map $* \to X^o \subset X^1$. Hence X is path connected.

Lemma (2.12) If X is a CW complex, then the path components of X are subcomplexes. If X is connected, then it is path connected.

Proof Let P be a path component of X and let e be a cell of X which is contained in P. Since each point of $\bar{e}$ is connected by a path to a point in e, we have $\bar{e} \subset P$. Hence, P is a subcomplex of X. The path components of X form a family of disjoint subcomplexes whose union is X. If X is not path connected, then there is more than one path component. By selecting one and taking the union of the remainder, X can be expressed as the union of two disjoint subcomplexes each of which is a closed subspace of X.

From corollary (2.11) and lemma (2.12) we obtain the following

Proposition (2.13) For a CW complex X the following statements are equivalent

(1) X is connected ;

(2) X is path-connected ;

(3) X^1 is path-connected.

A topological space X is called n-connected iff every map $S^i \to X$, $i \leq n$ has a continuous extension over E^{i+1}. Thus X is 0-connected iff X is path connected. A 1-connected space is also called simply connected.

A one dimensional CW complex is called a graph and a simply connected graph is called a tree.

Lemma (2.14) A graph is a tree iff it is contractible.

Proof Since a contractible space is obviously simply connected, a contractible graph is a tree. Conversely, let X be a tree and let x_0 be a point of X. We define a homotopy F from the identity map of X to the constant map of X onto x_0. Since S is path connected there is a path $\hat{v}$ from each vertex v of X to x_0. We define F on $v \times I$ by $F(v,t) = \hat{v}(t)$. For each 1-cell e^1 of X, F is now defined on the subset $\overline{e^1} \times \{0,1\}$ $\dot{e}^1 \times I$ of $e^1 \times I$. Since X is simply connected we can extend F over $e^1 \times I$. Thus we obtain a function $F: X \times I \to X$ whose restriction to $e^1 \times I$ is continuous for each 1-cell e^1 of X. To see that F is continuous, consider the corresponding function $X \to PX$ which is continuous by virtue of the fact that X has the weak topology with respect to its closed cells.

The set of trees contained in a graph X is partially ordered by inclusion.

Lemma (2.15) Let X be a connected graph. Then X contains a maximal tree and any maximal tree contains all the vertices of X.

Proof Let L be a linearly ordered set of trees in X and let $\overline{T} = \bigcup_{T \in L} T$. $\overline{T}$ is clearly a subcomplex of X. We show that $\overline{T}$ is a tree. Let f be a map $S^i \to \overline{T}$ where $i = 0$ or 1. Since $f(S^i)$ is compact, it is contained in a finite subcomplex F of $\overline{T}$. Then F can intersect only finitely many distinct elements $T_1, T_2, \ldots T_n$ of L since L is linearly ordered. Then $T = \bigcup_{j=1}^{n} T_i$ is a tree in L and we have $f(S^i) \subset F \subset T$. Since T is a tree, the map $f: S^i \to T \subset \overline{T}$ has an extension over E^{i+1}. This shows that $\overline{T}$ is simply connected and is therefore a tree. It now follows by Zorn's Lemma that X contains a maximal tree.

Let T be a maximal tree in X and suppose that there is a vertex v of X which is not in T. Then there is a path in X from v to some vertex of T. Let v_1 and v_2 be the last vertex outside T and the first vertex inside T intersected by this path. Then there is a 1-cell e^1 of X such that $\dot{e}^1 = \{v_1, v_2\}$. Let $T_1 = T \cup \overline{e}^1$. Then T is a deformation retract in T_1 and so T_1 is contractible. Thus T_1 is a tree strictly larger than T, contradicting the maximality of T.

Proposition (2.16) Every connected graph is homotopically equivalent to a wedge of 1-spheres.

Proof Let X be a connected graph and let T be a maximal tree in X. Take a vertex in T as base point. Since T contains every vertex of X it follows that $X/_T$ is homeomorphic to a wedge of 1-spheres. Then, since T is a subcomplex of X, (X,T) has the HEP, and so by corollary (1.13) the projection $X \to X/_T$ is a homotopy equivalence.

In the following we identify S^n with the quotient space $E^n/\dot{E}^n$. We denote the class in S^n containing $x \in E^n$ by $[x]$ and if $|x| = 1$ we write $[x] = 0 \in S^n$. We form the space $S^n \vee I$ with the base point of the unit interval $I = [0,1]$ taken as 0.

It is clear that the inclusion $S^n \to S^n \vee I$ is a homotopy equivalence, but we wish to make use of a different homotopy equivalence. Change the base point of $S^n \vee I$ to $1 \in I$ and define $h:S^n \to S^n \vee I$ by

$$h[x] = \begin{cases} [2x] , & 0 \le |x| \le 1/2 \\ 2|x| - 1, & 1/2 \le |x| \le 1 \end{cases}$$

so that h pulls a portion of the "balloon" S^n down the "string" I. Let $k:S^n \vee I \to S^n$ be the projection given by

$$k[x] = [x] , \quad x \in E^n$$
$$k(u) = 0 , \quad u \in I$$

h and k are both pointed maps with respect to the new base point for $S^n \vee I$.

<u>Lemma</u> (2.17) h is a homotopy equivalence with homotopy inverse k.

<u>Proof</u> $hk:S^n \vee I \to S^n \vee I$ is given by

$$hk[x] = h[x]$$
$$hk(u) = 1$$

Define $F:(S^n \vee I) \times I \to S^n \vee I$ by

$$F([x],t) = \begin{cases} \left[\dfrac{2x}{2-t}\right], & 0 \leq |x| \leq 1 - 1/2\ t \\ 2|x| - 2 + t, & 1 - 1/2\ t \leq |x| \leq 1 \end{cases}$$

$$F(u,t) = u + t - ut$$

Then F is a homotopy from $1_{S^n \vee I}$ to hk.

$kh: S^n \to S^n$ is given by

$$kh[x] = \begin{cases} [2x], & 0 \leq |x| \leq 1/2 \\ 0, & 1/2 \leq |x| \leq 1 \end{cases}$$

Define $G: S^n \times I \to S^n$ by

$$G([x],t) = \begin{cases} \left[\dfrac{2x}{2-t}\right], & 0 \leq |x| \leq 1 - 1/2\ t \\ 0, & 1 - 1/2\ t \leq |x| \leq 1 \end{cases}$$

Then G is a homotopy from 1_{S^n} to kh.

Identifying E^n with CS^{n-1}, the cone of S^{n-1}, we define $\bar{h}: E^n \to E^n \vee I$ by

$$\bar{h}[[x], u] = \begin{cases} [[2x], u], & 0 \leq |x| \leq 1/2 \\ 2|x| - 1, & 1/2 \leq |x| \leq 1 \end{cases}$$

and $\bar{k}: E^n \vee I \to E^n$ by

$$\bar{k}[[x], v] = [h[x], v]$$

$$\bar{k}(u) = 0$$

Here the base point of $E^n \vee I$ is the point $1 \in I$. $\bar{h}$ and $\bar{k}$ are extensions of the maps h and k defined previously.

<u>Lemma</u> (2.18) $(\bar{h},h)$ is a homotopy equivalence of pairs

$$(E^n, S^{n-1}) \to (E^n \vee I, S^{n-1} \vee I)$$

with homotopy inverse $(\bar{k},k)$.

<u>Proof</u> It is obvious how to define extensions of the homotopies

F and G of the previous lemma so as to provide homotopies $\bar{F}$ from $1_{E^n \vee I}$ to $\bar{h}\bar{k}$ and $\bar{G}$ from 1_{E^n} to $\bar{k}\bar{h}$.

Theorem (2.19). If X is a connected CW complex with a vertex as base point, then for each integer $n \geq 2$ there is a pointed map $\phi: \bigvee_{\alpha \in J^n} S_\alpha^{n-1} \to X^{n-1}$, where J^n is the indexing set for the n-cells of X, and a homotopy equivalence $\lambda: C_\phi \to X^n$ such that the diagram

$$\begin{array}{ccccc} \bigvee_{\alpha \in J^n} S^{n-1} & \xrightarrow{\phi} & X^{n-1} & \hookrightarrow & C_\phi \\ & & & \searrow & \downarrow \\ & & & & X^n \end{array}$$

is commutative.

Proof We have a push-out diagram

$$\begin{array}{ccc} \coprod_{\alpha \in J^n} E^n & \longrightarrow & X^n \\ \uparrow & & \uparrow \\ \coprod_{\alpha \in J^n} S^{n-1} & \longrightarrow & X^{n-1} \end{array}$$

in which the vertical maps are inclusions and f is the attaching map of the n-cells of X.

For each $\alpha \in J^n$ let

$$(\bar{h}_\alpha, h_\alpha): (E_\alpha^n, S_\alpha^{n-1}) \to ((E^n \vee I)_\alpha, (S^{n-1} \vee I)_\alpha)$$

be the relative homotopy equivalence of lemma (2.18).

Let $\eta = \bigvee h_\alpha : \bigvee S_\alpha^{n-1} \longrightarrow \bigvee (S^{n-1} \vee I)_\alpha$

and $\bar{\eta} = \bigvee \bar{h}_\alpha : \bigvee E_\alpha^n \longrightarrow \bigvee (E^n \vee I)_\alpha$.

Then $(\bar{\eta}, \eta) : (\bigvee E_\alpha^n, \bigvee S_\alpha^{n-1}) \longrightarrow (\bigvee (E^n \vee I)_\alpha, \bigvee (S^{n-1} \vee I)_\alpha)$

is a homotopy equivalence of pairs.

Since X is connected, X^{n-1} is path connected and for each $\alpha \in J_\circ^n$ there is a path p_α in X^{n-1} from the image of $0 \in S_\alpha^{n-1}$

under f to the base point of X. These paths together with the map f define in an obvious way a map

$$g:\bigvee(S^{n-1} \vee I)_\alpha \longrightarrow X^{n-1}$$

and it is clear that the push-out space of the system

$$\bigvee(E^n \vee I)_\alpha \supset \bigvee(S^{n-1} \vee I)_\alpha \longrightarrow X^{n-1}$$

is homeomorphic to X^n.

Let $\phi = g\eta:\bigvee S^{n-1}_\alpha \to X^{n-1}_\alpha$. Then ϕ is base point preserving and we have a commutative diagram

$$\begin{array}{ccccc} \bigvee E^n_\alpha & \longleftarrow & \bigvee S^{n-1}_\alpha & \xrightarrow{\phi} & X^{n-1} \\ \bar{\eta}\downarrow & & \eta\downarrow & & \| \\ \bigvee(E^n \vee I)_\alpha & \longleftarrow & \bigvee(S^{n-1} \vee I)_\alpha & \xrightarrow{g} & X^{n-1} \end{array}$$

in which the push-out space of the top row is homeomorphic to C_ϕ. It is clear that the two inclusions are cofibrations since, for example, they are inclusions of subcomplexes of CW complexes. Since the three vertical maps are homotopy equivalences, it follows by theorem (1.11) that they induce a homotopy equivalence $\lambda:C_\phi \to X^n$ with the desired properties. ▯

We shall say that a CW complex X <u>has pointed attaching maps</u> if it has a single vertex which is taken as the base point and for each $n \geq 1$ there is a push-out diagram

$$\begin{array}{ccc} \bigvee_{\alpha \in J^n} E^n_\alpha & \longrightarrow & X^n \\ \uparrow & & \uparrow \\ \bigvee_{\alpha \in J^n} S^{n-1} & \longrightarrow & X^{n-1} \end{array}$$

of pointed maps.

<u>Theorem</u> (2.20) Every finite dimensional connected CW complex X is homotopically equivalent to a CW complex $\bar{X}$ with pointed attaching maps.

<u>Proof</u> The 1-skeleton of X is a connected graph and as such is homotopically equivalent to a wedge of spheres which we take as the 1-skeleton $\bar{X}^1$ of $\bar{X}$. Let $\lambda_1 : \bar{X}^1 \to X^1$ be a homotopy equivalence.

Now suppose that we have constructed the $(n-1)$-skeleton $\bar{X}^{n-1}$ of $\bar{X}$ and have a homotopy equivalence $\lambda_{n-1} : \bar{X}^{n-1} \to X^{n-1}$ with a homotopy inverse $\mu_{n-1} : X^{n-1} \to \bar{X}^{n-1}$. From the proof of theorem (2.19) we see that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\bigvee E^n & \hookleftarrow & \bigvee S^{n-1} & \xrightarrow{\mu_{n-1}\phi} & \bar{X}^{n-1} \\
\bar{\eta}\downarrow & & \eta\downarrow & \overset{\phi}{\searrow} & \downarrow\lambda_{n-1} \\
\bigvee(E^n \vee I) & \hookleftarrow & \bigvee(S^{n-1} \vee I) & \longrightarrow & X^{n-1}
\end{array}$$

in which the three vertical maps are homotopy equivalences. The push-out space of the bottom row is homeomorphic to X^n and we let $\bar{X}^n$ be the push-out space of the top row. Since the two inclusions are cofibrations, it follows by theorem (1.11) that there is a homotopy equivalence $\lambda_n : \bar{X}^n \to X^n$. It is clear, therefore, that the theorem follows by induction. $\square$

CHAPTER III: SOME EXACT SEQUENCES

Let X and Y be two based spaces. The set $[X,Y]$ of homotopy classes of maps from X to Y has a distinguished element given by the class of the constant map; in fact $[\,,\,]$ is a bifunctor with values in the category of pointed sets.

If X is a suspension, say $X = SX'$, then $[SX',Y]$ has a natural group structure given by the "pinch map" $SX' \to SX' \vee SX'$ with identity element given by the class of the constant map. If Y is a loop space, say $Y = \Omega Y'$, then $[X,\Omega Y']$ has a natural group structure given by the map $\Omega Y' \times \Omega Y' \to \Omega Y'$ corresponding to addition of loops. If $X = SX'$ and $Y = \Omega Y'$, then the two group structures on $[SX',\Omega Y']$ coincide and the group structure is necessarily abelian.

S and Ω, considered as functors from the homotopy category of based spaces to itself, are adjoint. The adjunction isomorphism $[SX,Y] \approx [X,\Omega Y]$ is actually a group isomorphism. We write $\pi_n[X,Y] = [S^nX,Y] = [S^{n-1}X,\Omega Y] = \ldots = [X,\Omega^n Y]$. In particular $\pi_n(X) = [S^n,X] = \pi_n[S^o,X]$ is the nth homotopy group of X if $n \geq 1$, and the set of path components of X if $n = 0$.

Let (X,A) be a pair of based spaces with $A \subset X$ and let $i:A \to X$ be the inclusion. We may identify E_i with the space of paths in X which start in A and end at the base point, which we denote by $E(X,A)$. The nth relative homotopy group $\pi_n(X,A)$ of the pair (X,A), $n \geq 2$, is the group $\pi_{n-1}(E(X,A))$. $\pi_n(X,A)$ has an obvious interpretation by means of maps $(E^n,S^{n-1}) \to (X,A)$. More generally, the nth homotopy group $\pi_n(f)$ of a map $f:X \to Y$, $n \geq 2$, is the group $\pi_{n-1}(E_f)$.

Proposition (3.1) A sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f$$

where i is the inclusion, induces an exact sequence

$$[C_f,B] \xrightarrow{i^*} [Y,B] \xrightarrow{f^*} [X,B]$$

for each space B.

Proof The map $F:X \times I \to C_f$ given by $F(x,t) = [x,t]$ is a homotopy from if to the constant map $X \to C_f$. Hence $f^*i^* = 0$. Let g be a map $Y \to B$ such that $[g] \in \ker f^*$. Then there is a homotopy $G:X \times I \to B$ from gf to the constant map $X \to B$. Define $h:C_f \to B$ by $h[x,t] = G(x,t)$ and $h(y) = g(y)$. Then $i^*[h] = [hi] = [g]$. []

Proposition (3.1') A sequence

$$E_f \xrightarrow{p} X \xrightarrow{f} Y$$

where p is the projection $(x,\omega) \mapsto x$, induces an exact sequence

$$[A,E_f] \xrightarrow{p_*} [A,X] \xrightarrow{f_*} [A,Y]$$

for each space A.

Proof The map $f:E_f \times I \to Y$ given by $F(x,\omega,t) = \omega(t)$ is a homotopy from fp to the constant map $E_f \to Y$. Hence $f_*p_* = 0$. Let g be a map $A \to X$ such that $[g] \in \ker f_*$. Then there is a homotopy $G:A \times I \to Y$ from fg to the constant map $A \to Y$. Define $\phi:A \to EY$ by $\phi(a)(t) = G(a,t)$ and $h:A \to E_f$ by $h(a) = (g(a), \phi(a))$. Then $p_*[h] = [ph] = [g]$. []

Proposition (3.2) For each map $f:X \to Y$ there is a homotopy commutative diagram

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \xrightarrow{j} & C_i & \xrightarrow{j_1} & C_j \\ & & & & & \searrow^{p} & \downarrow & & \downarrow \\ & & & & & & SX & \xrightarrow{Sf} & SY \end{array} \qquad (3.3)$$

in which i, j and j_1 are the inclusions, p is the projection and the vertical maps are homotopy equivalences.

<u>Proof</u> We may regard C_i to be the quotient space of $CX \vee CY$ obtained by identifying $[x,0] \in CX$ with $[f(x),0] \in CY$ for all $x \in X$. Since i is a cofibration, the projection $q:C_i \to SX$ given by $q[x,u] = [x,u]$, $q[y,v] = *$ is a homotopy equivalence. Clearly $qj = p$.

Let p_1 be the projection $C_i \to SY$ given by $p_1[x,u] = *$, $p_1[y,v] = [y,v]$. Replacing f by i in the previous argument we see that there is a homotopy equivalence $q_1:C_j \to SY$ such that $q_1 j_1 = p_1$.

For each space A there is a natural homeomorphism $\sigma_A:SA \to SA$ given by $\sigma_A[a,u] = [a, 1-u]$. If g is a map into SA we write $-g = \sigma_A g$ and if h is a map from SA, we write $-h = h\sigma_A$. To complete the proof of the proposition, we show that $-Sf \circ q \simeq p_1$. Then we can take $q:C_i \to SX$ and $-q_1:C_j \to SY$ to be the vertical maps in the diagram (3.3).

We have $(-Sf \circ q)[x,u] = [f(x), 1-u]$

and $(-Sf \circ q)[y,v] = *$

Define $F:C_i \times I \to SY$ by

$$F([x,u],t) = [f(x), (1-u)(1-t)]$$
$$F([y,v],t) = [y, vt + 1 - t]$$

Then F is a homotopy from $-Sf \circ q$ to p_1.

<u>Proposition</u> (3.2') For each map $f:X \to Y$ there is a homotopy commutative diagram

$$\begin{array}{ccccccccc} E_q & \xrightarrow{q_1} & E_p & \xrightarrow{q} & E_f & \xrightarrow{p} & X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow & \nearrow_{i} & & & & & \\ \Omega X & \xrightarrow{\Omega F} & \Omega Y & & & & & & \end{array} \qquad (3.3')$$

in which p, q and q_1 are the projections, i is the inclusion and the vertical maps are homotopy equivalences.

<u>Proof</u> We may regard E_p to be the subspace of $EY \times EX$ consisting of pairs (ω,σ) such that $f\sigma(0) = \omega(0)$. Since p is a fibration, the inclusion $j:\Omega Y \to E_p$ given by $j(\omega) = (\omega, *)$ is a homotopy equivalence and we have $qj = i$.

Let i_1 be the inclusion $\Omega X \to E_p$ given by $i_1(\sigma) = (*,\sigma)$. Replacing f by p in the previous argument, we see that there is a homotopy equivalence $j_1:\Omega X \to E_q$ such that $q_1 j_1 = i_1$.

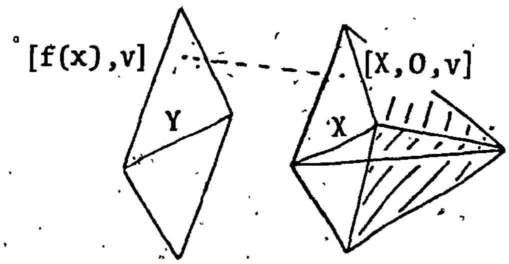
For each space A there is a natural homeomorphism $\rho_A:\Omega A \to \Omega A$ given by $\rho_A(\omega) = -\omega$. If g is a map into ΩA, we write $-g = \rho_A g$ and if h is a map from ΩA we write $-h = h\rho_A$. To complete the proof of this proposition we show that $j(-\Omega f) \simeq i_1$. Then we can take $j:\Omega Y \to E_p$ and $-j_1:\Omega X \to E_q$ to be the vertical maps in diagram (3.3').

We have $j(-\Omega f)(\sigma) = (-f\sigma,*)$. Define $\mu:\Omega X \times I \to EY$ by $\mu(\sigma,t)(u) = f\sigma((1-u)(1-t))$ and $\nu:\Omega X \times I \to EX$ by $\nu(\sigma,t)(u) = \sigma(1-t+ut)$. Then $F:\Omega X \times I \to E_p$ given by $F(\sigma,t) = (\mu(\sigma,t), \nu(\sigma,t))$ is a homotopy from $j(-\Omega f)$ to i_1.

<u>Lemma</u> (3.4) $SC_f = C_{Sf}$

<u>Proof</u> C_{Sf} is the quotient space of $CSX \vee SY$ obtained by identifying $[x,u,0] \in CSX$ with $[f(x),u] \in SY$ for all $x \in X$, $u \in I$, while SC_f is the quotient space of $S(CX \vee Y) = SCX \vee SY$ obtained by identifying $[x,0,v] \in SCX$ with $[f(x),v] \in SY$ for all $x \in X$, $v \in I$. We may

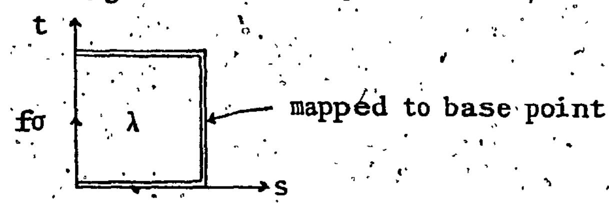
identify CSX with SCX by interchanging coordinates. Hence it is clear that we may identify C_{Sf} with SC_f.

SC_f 

(3.5)

<u>Lemma</u> (3.4') $\Omega E_f = E_{\Omega f}$

<u>Proof</u> $E_{\Omega f}$ is the subspace of $\Omega X \times E\Omega Y$ consisting of pairs (σ,μ) where μ is a map $I^2 \to Y$ such that $\lambda(0,-) = f\sigma$, $\lambda(1,t) = y_0$ for all $t \in I$ and $\lambda(s,0) = \lambda(s,1) = y_0$ for all $s \in I$.



ΩE_f is the subspace of $\Omega(X \times EY) = \Omega X \times \Omega EY$ consisting of pairs (σ,μ) where μ is a map $I^2 \to Y$ such that $\mu(-,0) = f\sigma$, $\mu(s,1) = y_0$ for all $s \in I$ and $\mu(0,t) = \mu(1,t) = y_0$ for all $t \in I$. It is clear that we can identify the two spaces.

<u>Theorem</u> (3.6') For each map $f:X \to Y$ and each space A there is an infinite exact sequence

$$\longrightarrow \pi_n[A,X] \xrightarrow{f_*} \pi_n[A,Y] \xrightarrow{i_*} \pi_{n-1}[A,E_f] \xrightarrow{p_*} \pi_{n-1}[A,X] \xrightarrow{f_*} \pi_{n-1}[A,Y] \ldots$$

$$\longrightarrow \pi_0[A,X] \xrightarrow{f_*} \pi_0[A,Y] \qquad (3.7')$$

<u>Proof</u> It is clear from propositions (3.1') and (3.2') that there is an exact sequence

$$[A,\Omega X] \xrightarrow{f_*} [A,\Omega Y] \xrightarrow{i_*} [A,E_f] \xrightarrow{p_*} [A,X] \xrightarrow{f_*} [A,Y]$$

Replacing f by Ωf and identifying ΩE_f with $E_{\Omega f}$ we obtain the exact sequence

$$[A,\Omega^2X] \xrightarrow{f_*} [A,\Omega^2Y] \xrightarrow{i_*} [A,\Omega E_f] \xrightarrow{p_*} [A,\Omega X] \xrightarrow{-f_*} [A,\Omega Y]$$

Iterating this procedure yields the desired long exact sequence. ∎

In particular, for any map $f:X \to Y$ we have an exact sequence

$$\to \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(f) \longrightarrow \pi_{n-1}(X) \quad \cdots \quad \to \pi_o(Y).$$

<u>Theorem</u> (3.6) For each map $f:X \to Y$ and each space B there is an infinite exact sequence

$$\to \pi_n[Y,B] \xrightarrow{f^*} \pi_n[X,B] \xrightarrow{p^*} \pi_{n-1}[C_f,B] \xrightarrow{i^*} \pi_{n-1}[Y,B] \xrightarrow{f^*} \pi_{n-1}[X,B] \cdots$$

$$\longrightarrow \pi_0[Y,B] \xrightarrow{f^*} \pi_0[X,B]. \qquad (3.7)$$

<u>Corollary</u> (3.8') If $f:X \to Y$ is a fibration with fibre F, then for each space A there is an infinite exact sequence

$$\longrightarrow \pi_n[A,X] \xrightarrow{f_*} \pi_n[A,Y] \xrightarrow{\tau} \pi_{n-1}[A,F] \xrightarrow{i_*} \pi_{n-1}[A,X] \xrightarrow{f_*} \pi_{n-1}[A,Y] \cdots$$

$$\longrightarrow [A,X] \xrightarrow{f_*} [A,Y],$$

where i_* is induced by the inclusion $F \to X$ and τ is the composition

$$\pi_{n-1}[A,Y] \longrightarrow \pi_{n-1}[A,E_f] \longrightarrow \pi_{n-1}[A,F]$$

in which the first map is induced by the inclusion $\Omega Y \to E_f$ and the second map is the inverse of the isomorphism induced by the inclusion $F \to E_f$.

<u>Corollary</u> (3.8) If $f:X \to Y$ is a cofibration with cofibre $C = Y/f(X)$, then for each space B there is an infinite exact sequence

$$\longrightarrow \pi_n[Y,B] \longrightarrow \pi_n[X,B] \longrightarrow \pi_{n-1}[C,B] \longrightarrow \pi_{n-1}[Y,B] \longrightarrow \pi_{n-1}[X,B] \cdots$$
$$\longrightarrow \pi_0[Y,B] \to \pi_0[X,B]. \qquad \|$$

Now suppose we have maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and let $h = gf$. The commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{1} & X & \xrightarrow{f} & Y \\ f\downarrow & & h\downarrow & & \downarrow g \\ Y & \xrightarrow[g]{} & Z & \xrightarrow[1]{} & Z \end{array} \tag{3.9}$$

gives rise to a sequence

$$C_f \xrightarrow{\phi} C_h \xrightarrow{\psi} C_g$$

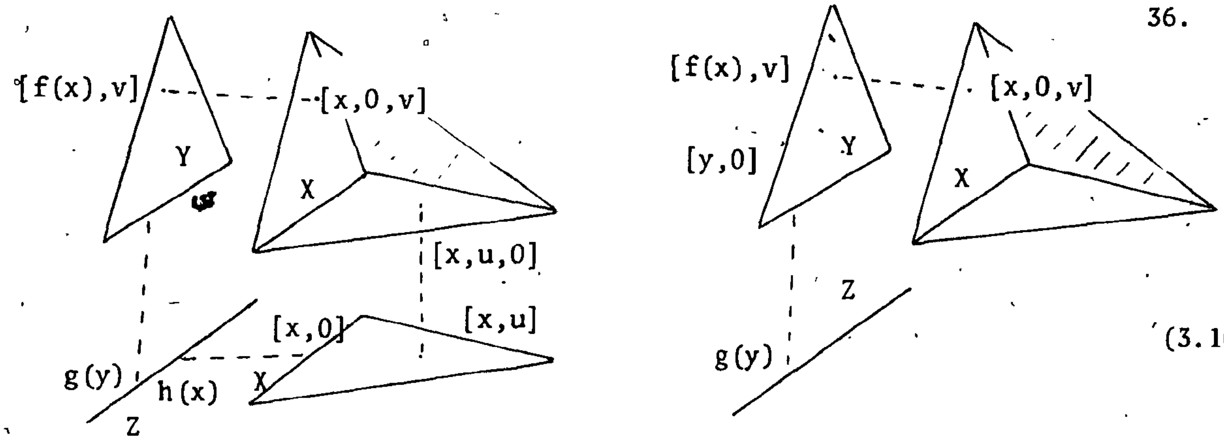
where ϕ is induced by the pair $(1,g)$ and ψ by the pair $(f,1)$.

Consider the mapping cone sequence

$$C_f \xrightarrow{\phi} C_h \xrightarrow{i} C_\phi$$

where i is the inclusion.

C_ϕ is the quotient space of $C^2X \vee CY \vee CX \vee Z$ obtained by identifying $[x,u,0] \in C^2X$ with $[x,u]$ CX, $[y,0] \in CY$ with $g(y) \in Z$, $[x,0,v] \in C^2X$ with $[f(x),v] \in CY$ and $[x,0] \in CX$ with $h(x) \in Z$. We may identify C_ϕ with the quotient space of $C^2X \vee CY \vee Z$ obtained by identifying $[x,0,v] \in C^2X$ with $[f(x),v] \in CY$ and $[y,0] \in CY$ with $g(y) \in Z$.



(3.10)

With this identification the inclusion $i:C_h \to C_\phi$ is given by

$$i[x,u] = [x,u,0]$$
$$i(z) = z$$

Let j be the inclusion $C_g \to C_\phi$ given by

$$j[y,v] = [y,v]$$
$$j(z) = z$$

<u>Proposition</u> (3.11) $j:C_g \to C_\phi$ is a homotopy equivalence with homotopy inverse a retraction $r:C_\phi \to C_g$ such that the diagram

$$\begin{array}{ccccc} C_f & \xrightarrow{\phi} & C_h & \xrightarrow{i} & C_\phi \\ & & & \underset{\psi}{\searrow} & \downarrow r \\ & & & & C_g \end{array}$$

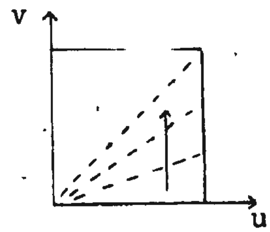
is commutative.

<u>Proof</u> We define a strong deformation retraction $D:C^2X \times I \to C^2X$ of C^2X onto CX by

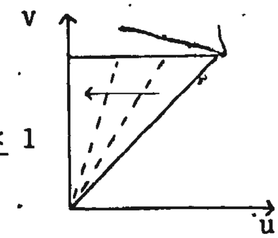
$$D([x,u,v],t) = \begin{cases} [x,u,(1-2t)v + 2tu], & u \geq v, \quad 0 \leq t \leq 1/2 \\ [x,u,v], & u \leq v, \quad 0 \leq t \leq 1/2 \\ [x,(2-2t)u,\ u], & u \geq v, \quad 1/2 \leq t \leq 1 \\ [x,(2-2t)u,\ v], & u \leq v, \quad 1/2 \leq t \leq 1 \end{cases}$$

The action of D is indicated by the diagrams

$0 \leq t \leq 1/2$



$1/2 \leq t \leq 1$



Now define $\Delta : C_\phi \times I \to C_\phi$ by

$$\Delta([x,u,v],t) = D([x,u,v],t)$$
$$\Delta([y,v],t) = [y,v]$$
$$\Delta(z,1) = z$$

Then Δ is a strong deformation retraction of C_ϕ onto C_g.

We have

$$\Delta([x,u,v],1) = \begin{cases} [x,0,u] = [f(x),u], & u \geq v \\ [x,o,v] = [f(x),v], & u \leq v \end{cases}$$
$$\Delta([y,v],1) = [y,v]$$
$$\Delta(z,1) = z$$

Then letting $r = \Delta(-,1)$ we have $rj = 1_C$ and $jr \simeq 1_C$.

$$ri[x,u] = r[x,u,0] = [f(x),u]$$
$$ri(z) = z$$

Hence $ri = \psi$.

In the sequence

$$C_f \xrightarrow{\phi} C_h \xrightarrow{i} C \xrightarrow{p} SC_f \xrightarrow{S\phi} SC_h$$

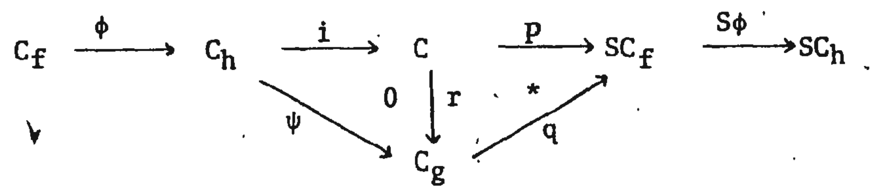
the projection $p : C_\phi \to SC_f$ is given by

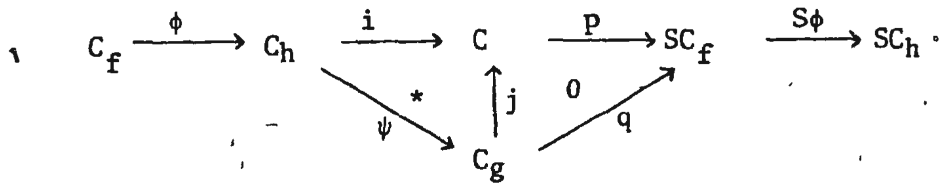
$$p[x,u,v] = [x,u,v],$$
$$p[y,v] = [y,v],$$
$$p(z) = * .$$

(See diagrams (3.5) and (3.10)).

Let $q : C_g \to SC_f$ be the restriction of p to C_g.

Corollary (3.12) In the diagrams





the triangles marked 0 are commutative while those marked * are homotopy commutative.

Proof We have $\psi = ri$ and $q = pj$. Thus $qr = pjr \simeq p$ and $j\psi = jri \simeq i$. ▯

Corollary (3.13) For each space B the sequence

$$\longrightarrow \pi_n[C_g,B] \longrightarrow \pi_n[C_h,B] \longrightarrow \pi_n[C_f,B] \longrightarrow \pi_{n-1}[C_g.B] \quad \dots$$
$$\longrightarrow \pi_o[C_h.B] \longrightarrow \pi_o[C_f,B].$$

is exact. ▯

The commutative diagram (3.9) also gives rise to a sequence

$$E_f \xrightarrow{\psi} E_h \xrightarrow{\phi} E_g$$

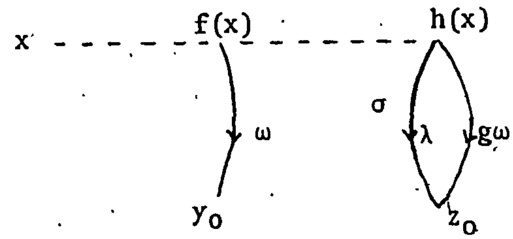
where $\psi:E_f \to E_h$ is given by $\psi(x,\omega) = (x,g\omega)$ and $\phi:E_h \to E_g$ is given by $\phi(x,\sigma) = (f(x),\sigma)$.

Consider the sequence

$$E_\phi \xrightarrow{p} E_h \xrightarrow{\phi} E_g$$

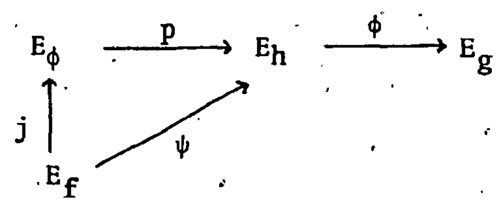
E_ϕ is the subspace of $X \times EZ \times EY \times E Z$ consisting of elements $(x,\sigma,\omega,\lambda)$ such that $\omega(0) = f(x)$, $\sigma(0) = h(x)$, $\lambda(0,-) = \sigma$ and

$\lambda(-,0) = g\omega$. We may identify E_ϕ with the subspace of $X \times EY \times E^2Z$ consisting of elements (x,ω,λ) such that $\omega(0) = f(x)$ and $\lambda(-,0) = g\omega$.



With this identification the projection $p:E_\phi \to E_h$ is given by $p(x,\omega,\lambda) = (x,\lambda(0,-))$. Let r be the projection $E_\phi \to E_f$ given by $r(x,\omega,\lambda) = (x,\omega)$.

<u>Proposition</u> (3.11') $r:E_\phi \to E_f$ is a homotopy equivalence with homotopy inverse an imbedding $j:E_f \to E_\phi$ such that the diagram



is commutative.

<u>Proof</u> Define an injection $e:EZ \to E^2Z$ by

$$e(\sigma)(v,u) = \begin{cases} \sigma(u), & u \geq v \\ \sigma(v), & u \leq v \end{cases}$$

and define $j:E_f \to E_\phi$ by $j(r,\omega) = (x,\omega,e(g\omega))$. Then $rj = 1_{E_\phi}$ so j is an imbedding. Define $D:E^2Z \times I \to E^2Z$ by

$$D(\lambda,t)(v,u) = \begin{cases} \lambda(v - 2tv + 2u,\ u), & u \geq v, \quad 0 \leq t \leq 1/2 \\ \lambda(v,u), & u \leq v, \quad 0 \leq t \leq 1/2 \\ \lambda(u,\ 2u - 2tu), & u \geq v, \quad 1/2 \leq t \leq 1 \\ \lambda(v,\ 2u - 2tu), & u \leq v \quad 1/2 \leq t \leq 1 \end{cases}$$

and $\Delta:E_\phi \times I \to E_\phi$ by $\Delta(x,\omega,\lambda,t) = (x,\omega,D(\lambda,t))$. Then Δ is a homotopy from 1_{E_ϕ} to jr.

We have $pj(x,\omega) = p(x,\omega,e(g\omega)) = (x,g\omega) = \psi(x,\omega)$. Thus $pj = \psi$.

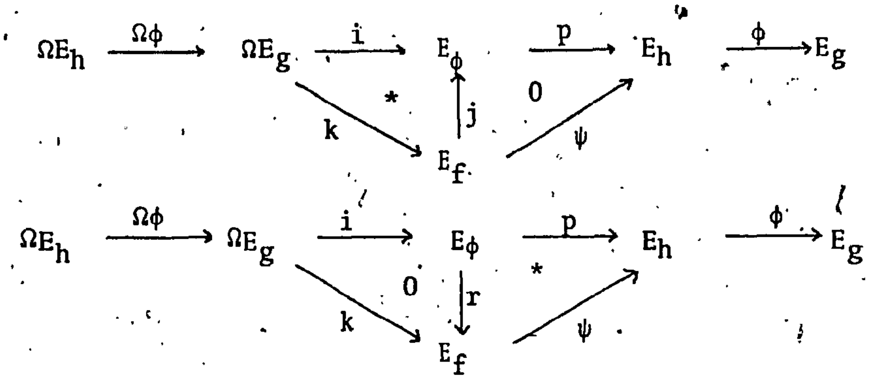
In the sequence

$$\Omega E_h \xrightarrow{\Omega\phi} \Omega E_g \xrightarrow{i} E_\phi \xrightarrow{p} E_h \xrightarrow{\phi} E_g$$

$\ker p = \Omega E_g$ is the subspace of $\Omega Y \times \Omega EZ$ consisting of pairs (ω,λ) such that $\lambda(-,0) = g\omega$ and the inclusion $i:\Omega E_g \to E_\phi$ is given by $i(\omega,\lambda) = (x_o,\omega,\lambda)$.

Define $k:\Omega E_g \to E_f$ by $k(\omega,\lambda) = (x_o,\omega)$.

<u>Corollary</u> (3.12') In the diagrams



the triangles marked 0 are commutative while those marked * are homotopy commutative.

<u>Corollary</u> (3.13') For each space A the sequence

$$\longrightarrow \pi_n[A,E_f] \xrightarrow{\psi_*} \pi_n[A,E_h] \xrightarrow{\phi_*} \pi_n[A,E_g] \xrightarrow{k_*} \pi_{n-1}[A,E_f] \;\ldots.$$

$$\longrightarrow \pi_o[A,E_h] \xrightarrow{\phi_*} \pi_o[A,E_g]$$

is exact.

In particular there is an exact sequence

$$\longrightarrow \pi_n(f) \longrightarrow \pi_n(h) \longrightarrow \pi_n(g) \longrightarrow \pi_{n-1}(f) \;\ldots..$$

For each triple of spaces (X,A,B) with $B \subset A \subset X$ there is an exact sequence

$$\longrightarrow \pi_n(A,B) \longrightarrow \pi_n(X,B) \longrightarrow \pi_n(X,A) \longrightarrow \pi_{n-1}(A,B) \ldots$$

Let f be a map $X \to Y$. A <u>lifting function</u> λ for f is a map $\lambda:P_f \to PX$ from the mapping track of f to the path space of X such that

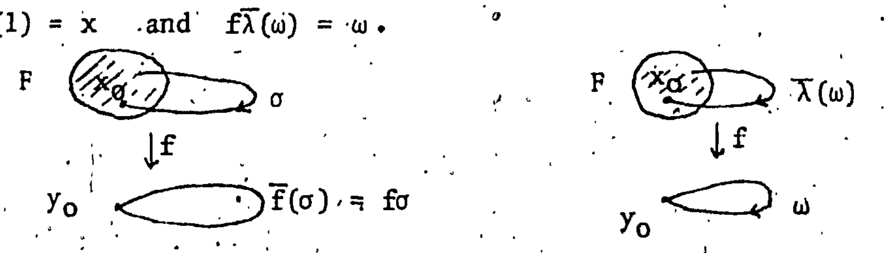
$$\lambda(x,\omega)(0) = x \quad \text{and} \quad f\lambda(x,\omega) = \omega$$

<u>Lemma</u> (3.14) $f:X \to Y$ is a fibration if and only if f has a lifting function.

<u>Proof</u> Suppose that $f:X \to Y$ is a fibration and define $G:P_f \times I \to Y$ by $G(x,\omega,t) = \omega(t)$. Then $G(-,0) = fr$ where r is the projection $P_f \to X$ given by $r(x,\omega) = x$. Since f is a fibration, there is a homotopy $H:P_f \times I \to X$ such that $H(x,\omega,0) = r(x,\omega) = x$ and $fH(x,\omega,t) = G(x,\omega,t) = \omega(t)$. Then $\lambda:P_f \to PX$ given by $\lambda(x,\omega)(t) = H(x,\omega,t)$ is a lifting function for f.

Conversely, suppose that $f:X \to Y$ has a lifting function $\lambda:P_f \to PX$. Let h be a map $A \to X$ and G be a homotopy $A \times I \to Y$ such that $G(-,0) = fh$. Define $\phi:A \to PY$ by $\phi(a)(t) = G(a,t)$ and then define $H:A \times I \to X$ by $H(a,t) = \lambda(h(a),\phi(a))(t)$. We have $H(-,0) = h$ and $fH = G$, as required.

Let $f:X \to Y$ be a fibration with fibre F and let $\lambda:E_f \to PX$ be the restriction to E_f of a lifting function for f. Define $\overline{f}:E(X,F) \to \Omega Y$ by $\overline{f}(\sigma) = f\sigma$ and $\overline{\lambda}:\Omega Y \to E(X,F)$ by $\overline{\lambda}(\omega) = -\lambda(x_o,-\omega)$. We have $\overline{\lambda}(\omega)(1) = x$ and $f\overline{\lambda}(\omega) = \omega$.

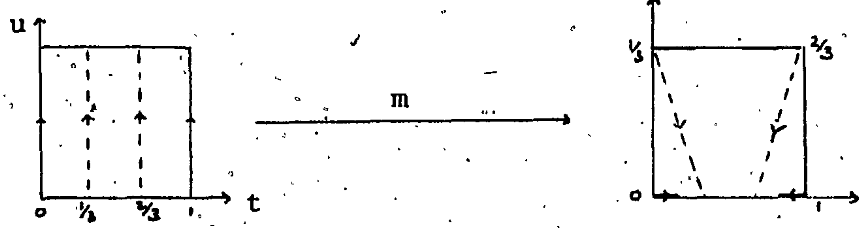


Proposition (3.15) $\bar{f}:E(X,F) \to \Omega Y$ is a homotopy equivalence with homotopy inverse λ.

Proof Clearly $\bar{f}\bar{\lambda} = 1_{\Omega Y}$. We must show that $\bar{\lambda}\bar{f} = 1_{E(X,F)}$.
Define a homeomorphism $m:I^2 \to I^2$ by

$$m(t,u) = \begin{cases} (u(1/6 + t/2),\ 3t(1 - u)), & 0 \le t \le 1/3 \\ ((3t - 1)(1 - u) + ut,\ 1 - u), & 1/3 \le t \le 2/3 \\ (1 - u)(2/3 - t/2),\ 3(1 - t)(1 - u), & 2/3 \le t \le 1 \end{cases}$$

The action of m is indicated by the diagram



Define $\phi:E(X,F) \times I^2 \to Y$ by $\phi(\sigma,m(t,u)) = f\sigma(u)$.
If $g:E(X,F) \times I \to X$ is the map given by

$$g(\sigma,t) = \begin{cases} \sigma(6t), & 0 \le t \le 1/6 \\ x_0, & 1/6 \le t \le 5/6 \\ \bar{\lambda}(f\sigma)(6 - 6t), & 5/6 \le t \le 1 \end{cases}$$

we have

$$fg(\sigma,t) = \begin{cases} f\sigma(6t), & 0 \le t \le 1/6 \\ y_0, & 1/6 \le t \le 5/6 \\ f\sigma(6 - 6t), & 5/6 \le t \le 1 \end{cases}$$

$$= \phi(\sigma,t,0)$$

Then, since f is a fibration, there is a map $\psi:E(X,F) \times I^2 \to X$ such that $\psi(\sigma,t,0) = g(\sigma,t)$ and $f\sigma = \phi$.

Define $\theta:E(X,F) \times I \to E(X,F)$ by $\theta(\sigma,t)(u) = \psi(\sigma,m(t,u))$.
Then $f\theta(\sigma,t)(0) = \phi(\sigma,m(t,0)) = f\sigma(0) = y_0$, so θ is well defined.

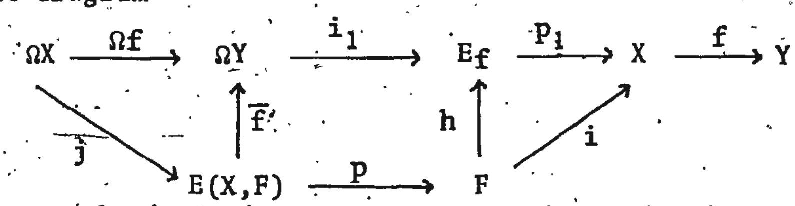
$$\theta(\sigma,0)(u) = \psi(\sigma, 1/6\,u, 0) = g(\sigma, 1/6\,u) = \sigma(u)$$
$$\theta(\sigma,1)(u) = \psi(\sigma, 1 - 1/6\,u, 0) = g(\sigma, 1 - 1/6\,u) = \overline{\lambda}(f\sigma)(u) = \overline{\lambda f}(\sigma)(u)$$

Thus θ is a homotopy from $1_{E(X,F)}$ to $\overline{\lambda f}$.

<u>Corollary</u> (3.16) For each space A, $\overline{f}:E(X,F) \to \Omega Y$ induces a bijection $[A,E(X,F)] \to [A,\Omega Y]$ which is an isomorphism of groups when A is a suspension.

In particular, we have an isomorphism $f_*:\pi_n(X,F) \to \pi_n(Y)$ for each $n \geq 2$.

Consider the diagram



(3.17)

where i, i_1, j are the inclusions, p, p_1 are the projections and h is given by $h(x) = (x,*)$.

<u>Lemma</u> (3.18) The two triangles in the diagram (3.17) are commutative and the square becomes homotopy commutative on replacing i_1 by the map $-i_1$ given by $-i_1(\omega) = i_1(-\omega)$.

<u>Proof</u> The first part is clear. By corollary (1.7'), h is a homotopy equivalence and from the <u>proof</u> of proposition (1.6') we see that $k:E_f \to F$ given by $k(x,\omega) = \lambda(x,\omega)(1)$ is a homotopy inverse for h.

~~Thus we have a commutative diagram~~

$$\begin{array}{ccc} Y & \xrightarrow{-i_1} & E_f \\ {\scriptstyle \bar{\lambda}}\downarrow & & \downarrow{\scriptstyle k} \\ E(X,F) & \xrightarrow{p} & F \end{array}$$

since $p\bar{\lambda}(\omega) = \bar{\lambda}(\omega)(0) = \lambda(x_0,-\omega)(1) = k(x_0,-\omega)$. The result now follows. $\square$

We may call the square

$$\begin{array}{ccc} Y & \xrightarrow{i_1} & E_f \\ {\scriptstyle \bar{f}}\uparrow & & \uparrow{\scriptstyle h} \\ E(X,F) & \xrightarrow[p]{} & F \end{array}$$

the transgression square, [2].

We have seen one way of obtaining an exact sequence for a fibration f. An alternative way of obtaining an exact sequence is to make use of the sequence

$$\Omega F \longrightarrow \Omega X \longrightarrow E(X,F) \longrightarrow F \longrightarrow X$$

and the homotopy equivalence $\bar{f}: E(X,F) \to \Omega Y$. By means of the transgression square we see that the resulting exact sequences are essentially the same.

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