BUCKLING ANALYSIS OF SEMI-INFINITE ICE SHEET

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BUCKLING ANALYSIS OF SEMI-INFINITE ICE SHEET

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ABSTRACT

A finite element analysis has been performed to study the buckling of a semi-infinite ice sheet due to an in-plane distributed load. This work has relevance in the determination of the forces generated on marine structures due to buckling failure of ice sheets. In this analysis, an ice sheet is treated as an isotropic plate resting on an elastic foundation.

A computer program is developed and tested to analyse this problem using a 16 degree-of-freedom rectangular plate element.

Since the major portion of the deformation takes place within a few characteristic lengths of the plate on elastic foundation, a large size plate is used as a model of the semi-infinite ice sheet. In order to check this simulation, it is found that there is very little effect on the buckling loads and modes of buckling due to imposition of different types of boundary conditions on the edges which are supposed to be at infinity. The results of this analysis are presented in a non-dimensional form.

To demostrate the usefulness of the computer program for finite size plates, the results of buckling analysis of a clamped rectangular plate resting on an elastic foundation are also presented and compared with those obtained by Rayleigh-Ritz procedure.

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A	Length of the rectangular plate (m)
b	Width of the partially distributed in-plane load on one-half of the semi-infinite plate (m)
B	Width of the rectangular plate(m)
C	$\frac{Eh}{1-\nu^2}$, stretching rigidity (N/m)
d	Width of the beam (m)
D	$\frac{Eh^3}{12(1-\nu^2)}$, flexural rigidity (N.m)
E	Modulus of elasticity (N/m^2)
h	Thickness of the plate (m)
I	Beam moment of inertia (m^4)
[I]	Unit matrix
k	Stiffness of the elastic foundation (N/m^3)
[K]	Linear flexural stiffness matrix
$[K_G]$	Geometric stiffness matrix
$[K_u]$	In-plane stiffness matrix
$K_{\theta w}, K_{ww}, K_{\theta\theta}$	Partitioned matrices of matric [K]
ℓ	The characteristic length (m)
L	Length of the beam (m)
m	Number of half waves in the buckling mode in x-direction
M	Bending moment (N)
$\bar{M}$	Prescribed bending moment on the boundary S_F (N)
n	Number of half waves in the buckling mode in y-direction
N_{xx}, N_{yy}, N_{xy}	In-plane stress resultants (N/m)

P	Buckling load (N)
p_e	$\frac{P}{2bh}$, effective pressure (N/m^2)
$\{q_u\}$	In-plane displacement vector
$\{q\}$	Out-of-plane displacement vector
$\{\bar{Q}_u\}$	In-plane prescribed load vector
$\{\bar{Q}\}$	Out-of-plane prescribed load vector
u,v,w	Displacements in x,y and z directions respectively
U	Strain energy
U_e	In-plane strain energy
U_f	Flexural strain energy
V	Normal force (N)
$\bar{V}$	Prescribed normal force on the boundary S_F (N)
W_e	The in-plane work done (N.m)
W_f	The flexural work done (N.m)
x,y,z	Rectngular Cartesian Co-ordinates
$\bar{X},\bar{Y}$	Prescribed in-plane loads in x and y directions respectively
$\varepsilon_{xx},\varepsilon_{yy}$	In-plane strain in x and y directions respectively
γ_{xy}	In-plane shear strain
ν	Poisson's ratio
π_e	In-plane potential energy
π_f	Flexural potential energy
λ	Non-dimensional buckling load
ξ,η	Non-dimensional co-ordinates
$,ij$	$\equiv \frac{\partial^2}{\partial x_i \partial x_j}$

CHAPTER 1

INTRODUCTION

One of the most important forces in the design of structures in ice infested water is the horizontal thrust applied to the structures by ice. Neill [5], in his review paper, has classified the ice action on structures into four principal modes as described in Table 1.1.

The problem of ice forces on structures has received considerable attention in the recent past, but the amount of experimental data is still very small, and the available data has various deficiencies which make them difficult to use directly for design or for verification of analytical methods. Designers have generally relied on empirical formulae, design codes and their own individual judgement.

The mechanical properties of ice are so complex and variable that the analytical approach of estimating ice forces on a structure will only be acceptable when it is supported by the full-scale measurements and evaluation of structure performance. The number of variables influencing ice forces is so great that field data alone cannot be expected to give enough information for design unless they can be fitted into a logical analytical framework. The gap between theoretical analysis and full-scale data may be bridged by small-scale experiments.

We shall restrict our discussion to consideration of horizontal ice forces on single vertical structures such as piers, piles and towers. The mode of failure of ice sheet is due to crushing (see Figure 1.1) when the aspect ratio (structure width/ice thickness) is small. It has been observed by many investigators [1; 2, 3 and 22] during small-scale tests that the ice sheet fails in the buckling mode when the aspect ratio is greater than 6. The determination of buckling load by analytical approach does not exist for a partially distributed load acting on a semi-infinite plate on elastic foundation. Hence, a numerical approach is followed to determine the buckling loads and modes of buckling by finite element method. The initial motivation of this study is to determine the horizontal ice forces generated on structures due to buckling only, but this approach is quite general to perform buckling analysis of plates on elastic foundation.

1.1 Failure of ice sheet in crushing mode

The failure of ice sheet in crushing mode is a complex phenomenon because of the fact that many parameters are affecting the ice forces generated on the marine structures. Much research work has been done in estimating the ice forces due to crushing, but there still remains much work to be done in this field to treat the various deficiencies in the available experimental data and to get enough information, especially from full-scale measurements, to be used by the designers and for verification of analytical results. Small-scale tests in laboratory are providing some insight into this complex problem.

Korzhavin [21] presented the following empirical formula to estimate the effective ice pressure ($p_e = P/2bh$, see Figure 1.1) in a steady crushing mode.

$$\text{Effective pressure } p_e = \frac{C_i m K \sigma_c}{(V/V_0)^{1/3}} \qquad (1.1\text{-}1)$$

where C_i is the indentation coefficient, m the coefficient for plan shape of the nose, K the contact coefficient, V the velocity of ice sheet in meters per second, V_0 the reference velocity equals to 1 meter per second and σ_c the compressive (cube) strength of ice. A discussion on the values of various parameters in Korzhavin's formula is given in Reference 5.

Schwarz et al. [1, 2 and 10] and Nevel et al. [4] have conducted small-scale experiments to determine the factors which affect the ice forces exerted by ice sheets on vertical marine structures. There seems to be general consensus that ice forces depend upon the aspect ratio, strain rate and the compressive strength of ice. Neill [5] has given the following simplified relation between four dimensionless groups for isotropic infinite uniform ice sheet against rigid vertical pile or pier.

$$\frac{P}{2bh\sigma_c} = f\left(\frac{b}{h}, \frac{V}{2b\dot{\varepsilon}}, \frac{E}{\sigma_c}\right) \qquad (1.1\text{-}2)$$

where P is the force developed, h the ice sheet thickness, 2b the pile diameter or pier width, V the velocity of ice sheet, σ_c the compressive strength, $\dot{\varepsilon}$ the corresponding test strain rate, and E the modulus of elasticity. Schwarz et al. [1] have suggested some more parameters in their dimensional analysis, e.g. temperature, ratio of structure width to grain size, etc.

Neill [5] has discussed briefly some of the full scale tests done in North America and Europe, some correlation with empirical formulae has been indicated by the full-scale measurements, but the complete verification of analytical results is not possible due to complexities of full-scale tests and variability of ice properties.

1.2 Failure of ice sheet in buckling mode

As mentioned earlier, the ice sheet fails in buckling mode when the aspect ratio (2b/h) is larger than six.

The theoretical formulation of buckling of plate on elastic foundation is the same as that of a plate except for the extra term related to the elastic foundation. Vlasov and Leuten [18] have discussed the theoretical formulation of the buckling problems of beams, plates and shells resting on elastic foundation. They have presented solutions of beams, plates and shells considering only the simply supported boundary conditions. Hetenyi [25] presented the deflection and buckling analysis of beams on elastic foundation.

Recently, some attention is being paid to the buckling problem of plate on elastic foundation in order to determine the ice forces generated on marine structures. Kivisild [11] has presented a few formulae to calculate the buckling load without giving any derivation. Kerr [16] has solved the buckling problem of tapered beam floating on water. The justification of considering a tapered beam is due to the fact that radial vertical cracks have been observed to originate from the loading region and radiate outward in the ice sheet. Takagi (17] developed a theoretical solution to the buckling problem of a floating plate stressed uniformly along the periphery of an internal hole.

There is no extensive reference in literature to the buckling analysis of finite or semi-infinite plate on elastic foundation for arbitrary boundary conditions, plate geometry or loading. The analytical solutions of these problems are complex, and numerical solutions of a few problems are available in literature [12, 16, 17, 18, 21].

1.3 Statement of the problem

The general differential equation governing the buckling problem of thin plate on elastic foundation due to in-plane loading only, can be written in the following manner [18]:

$$D\nabla^4 w + kw = N_{xx}\, w_{,xx} + 2N_{xy}\, w_{,xy} + N_{yy}\, w_{,yy} \qquad (1.3\text{-}1)$$

where w - out-of-plane deflection

D - flexural rigidity of the plate

k - foundation stiffness

N_{xx}, N_{xy}, N_{yy} - in-plane stress resultants (N/m)

∇^4 - biharmonic operator in x and y co-ordinates

$_{,xy} \equiv \dfrac{\partial^2}{\partial x \partial y}$

The semi-infinite plate on elastic foundation, as shown in Figure 1.2, is used as a mathematical model to study the buckling of ice sheet under a compressive in-plane load of intensity P which is distributed over a length 2b. The boundary conditions for this problem are as follows:

$$\left.\begin{array}{llll} \text{at } x = 0 & M_{xx} = V_x = 0 & \text{(free edge)} \\ \text{and at } x = \infty & w = w_{,r} = 0 & \text{(clamped edge)} \end{array}\right\} \qquad (1.3\text{-}2)$$

where $r = \sqrt{x^2 + y^2}$

M_{xx} and V_x are the bending moment and the effective shear force.

The expressions for in-plane stress resultants N_{ij} due to the in-plane compressive loading can be written in the following manner [19]:

$$N_{xx} = \frac{P}{2b\pi}\left[\, \theta_2 - \theta_1 + \frac{1}{2}\left(\sin 2\theta_2 - \sin 2\theta_1 \right)\right]$$
$$N_{yy} = \frac{P}{2b\pi}\left[\, \theta_2 - \theta_1 - \frac{1}{2}\left(\sin 2\theta_2 - \sin 2\theta_1 \right)\right] \qquad (1.3\text{-}3)$$
$$N_{xy} = \frac{P}{4b\pi}\left[\cos 2\theta_2 - \cos 2\theta_1\right]$$

Where θ_1 and θ_2 have been defined in Figure 1.2. The expressions for $\frac{2bN_{ij}}{P}$ are non-dimensional but these depend upon the width of the in-plane loading (2b).

The differential equation (1.3-1) is transformed into non-dimensional co-ordinates $\xi = x/\ell$ and $\eta = y/\ell$ where $\ell = \sqrt[4]{D/k}$, the characteristic lenght of the plate, thus we have

$$\bar{\nabla}^4 w + w = \lambda \left[\frac{\ell}{P} \left(N_{\xi\xi}\, w_{,\xi\xi} + 2N_{\xi\eta}\, w_{,\xi\eta} + N_{\eta\eta}\, w_{,\eta\eta} \right)\right] \qquad (1.3\text{-}4)$$

Where $\bar{\nabla}^4$ - biharmonic operator in ξ and η co-ordinates.

$\lambda = \frac{P}{k\ell^3}$,non-dimensional buckling load

The non-dimensional buckling load λ will depend upon the mode of buckling and the ratio b/ℓ. Hence,for any given mode of buckling , the non-dimensional buckling load λ is a function of the ratio b/ℓ only.

The finite element method is used to perform the buckling analysis of the above-mentioned problem. As a numerical method, it can be used to solve the differetial equation for any type of plate geometry; boundary conditions and loading. To demonstrate this capability, the buckling analysis of a clamped rectangular plate on elastic foundation is performed and its results are then compared with those obtained by Rayleigh-Ritz procedure.

CHAPTER 2

BUCKLING OF PLATE ON ELASTIC FOUNDATION

2.1 Introduction

The classical differential equation governing the buckling problem of thin plate on elastic foundation has been formulated for a long time [18]. The exact theoretical solutions [18] and approximate numerical solutions [12] using different approximate numerical approaches, are available for only few special cases of loading, plate geometry and boundary conditions.

A finite element formulation for buckling of a plate not resting on elastic foundation is presented using different kinds of finite elements by Holand and Moan [8], Carson and Newton [9], and Gallagher [13]. The finite element formulation of buckling of plate on elastic foundation is almost the same as that for a plate without elastic foundation except for an additional linear stiffness which is due to the elastic foundation.

In this chapter, we discuss briefly the general formulation of buckling of plate on elastic foundation and the finite element formulation of the same. The results of the finite element analysis are presented in the next chapter.

2.2 Formulation of buckling of plate on elastic foundation

Based on the well-known theory by Von Kármán [7], the energy expression for large deflection of a plate is written in terms of the displacements u, v and w as follows:

$$\underset{\sim}{V} = \frac{1}{2}\iint C[u_{,x}^2 + v_{,y}^2 + 2\nu u_{,x}v_{,y} + \frac{1-\nu}{2}(u_{,y} + v_{,x})^2]\,dxdy$$

$$+ \frac{1}{2}\iint D[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)\,w_{,xy}^2]\,dxdy$$

$$+ \frac{1}{2}\iint C[(u_{,x} + \nu v_{,y})w_{,x}^2 + (v_{,y} + \nu u_{,x})\,w_{,y}^2$$

$$+ (1-\nu)(u_{,y} + v_{,x})\,w_{,x}w_{,y}]\,dxdy \qquad (2.2\text{-}1)$$

Where E - modulus of elasticity

h - thickness of the plate

ν - Poisson's ratio

$C = \dfrac{Eh}{1-\nu^2}$

$D = \dfrac{Eh^3}{12(1-\nu^2)}$

$u_{,x}^2 \equiv (u_{,x})^2$, etc

Under the assumption of independent prebuckling analysis for in-plane loading, the in-plane stress resultants are related to the in-plane strain component by the following relations :

$$\left.\begin{aligned} N_{xx} &= C\,(u_{,x} + \nu\, v_{,y}) \\ N_{yy} &= C\,(v_{,y} + \nu\, u_{,x}) \\ N_{xy} &= C\,(\tfrac{1-\nu}{2})\,(u_{,y} + \nu\, v_{,x}) \end{aligned}\right\} \qquad (2.2\text{-}2)$$

The strain energy expression (2.2-1) can now be written as a sum of extensional and flexural energy expression in the following manner:

$$U = U_e + U_f \qquad (2.2\text{-}3)$$

where $$U_e = \frac{1}{2}\iint C\,[u^2_{,x} + v^2_{,y} + 2\nu\, u_{,x}\, v_{,y} + \frac{1-\nu}{2}(u_{,y} + v_{,x})^2]\, dxdy$$

$$U_f = \frac{1}{2}\iint D\,[w^2_{,xx} + w^2_{,yy} + 2\nu\, w_{,xx}\, w_{,yy} + 2(1-\nu)\, w^2_{,xy}]\, dxdy$$

$$+ \frac{1}{2}\iint [N_{xx}\, w^2_{,x} + 2N_{xy}\, w_{,x}\, w_{,y} + N_{yy}\, w^2_{,y}]\, dxdy\ .$$

In order to determine the critical intensity of the in-plane load system to cause elastic instability, the in-plane stress resultants N_{xx}, N_{yy} and N_{xy}) are determined for an arbitrary chosen load intensity, and these are then used to determine the critical load intensity (taken to be ω times the arbitrary chosen intensity) for neutral stability.

I In-plane Analysis:

The principle of minimum potential energy is used to determine the stress resultants due to arbitrary chosen load system. The pertinent functional for the principle of minimum potential energy takes the following form:

$$\pi_e = \frac{1}{2}\iint C\,[u^2_{,x} + v^2_{,y} + 2\nu\, u_{,x}\, v_{,y} + \frac{1-\nu}{2}(u_{,y} + v_{,x})^2]\, dxdy$$

$$- \int_{S_F} (u_n\, \bar{N}_{nn} + u_t\, \bar{N}_{nt})\, ds \qquad (2.2\text{-}4)$$

where u, v - the admissible displacement function which satisfy the kinematic boundary conditions.

u_n, u_t - displacement in the normal and tangential direction at the boundary.

$\overline{N}_{nn}$, $\overline{N}_{nt}$ - prescribed in-plane stress resultants with respect to normal and tangential coordinate system at the boundary.

S_F - the part of the boundary on which loads are prescribed.

It has been shown [15] that, when $\delta \pi_e = 0$, one obtains the following equilibrium equations in the domain after taking the constitutive relations (2.2-2) into account.

$$\left.\begin{aligned} N_{xx,x} + N_{xy,y} &= 0 \\ N_{xy,x} + N_{yy,y} &= 0 \end{aligned}\right\} \qquad (2.2\text{-}5)$$

along with the associated boundary conditions on S_F

$$\left.\begin{aligned} N_{nn} &= \overline{N}_{nn} \\ N_{nt} &= \overline{N}_{nt} \end{aligned}\right\} \qquad (2.2\text{-}6)$$

For the finite element formulation, the domain is discretized into rectangular elements of sides a and b in the x and y directions, respectively. A non-dimensional local co-ordinate system is chosen for each element, as shown in Figure 2.1, in the following manner:

$$\xi = \frac{x}{a}$$

$$\eta = \frac{y}{b}$$

The displacement functions u and v are assumed within each element in the following bilinear form:

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$$
$$v = \alpha_5 + \alpha_6 \xi + \alpha_7 \eta + \alpha_8 \xi \eta \qquad (2.2\text{-}7)$$

The above displacement functions can be expressed in terms of nodal displacements instead of α_i (i = 1 to 8). The generalized nodal displacement vector, for every element, contains the in-plane nodal displacements in the following manner:

$$\{q_u\}^t = [u_1 \; u_2 \; u_3 \; u_4 \; \vdots \; v_1 \; v_2 \; v_3 \; v_4] \qquad (2.2\text{-}8)$$

The displacement functions (2.2-7) can now be substituted into the functional π_e (2.2-4) to derive the in-plane stiffness matrices and the consistent load vectors for each element. This derivation is given in many textbooks [6, 13 and 14] and hence it is not repeated here. The system equations for the in-plane stress analysis of the composite system are obtained by assembling the elements stiffness matrices and the consistent load vectors and these are written as follows:

$$[K_u^*] \left\{ q_u^* \right\} = \left\{ Q_u^* \right\} \qquad (2.2\text{-}9)$$

where $\left\{ q_u^* \right\}$ - represents the generalized displacement vector containing all the nodal degrees of freedom of the system.

$[K_u^*]$ - the stiffness matrix of the system.

$[Q_u^*]$ - the consistent load vector of the system.

The above linear algebraic equations are solved and the average in-plane strain components are then obtained within each element in terms of nodal displacements of the element [7], as follows:

$$\varepsilon_{xx} = u_{,x} = \frac{1}{2a} \left[u_2 - u_1 + u_3 - u_4 \right]$$

$$\varepsilon_{yy} = v_{,y} = \frac{1}{2b} \left[v_4 - v_1 + v_3 - v_2 \right] \qquad (2.2\text{-}10)$$

$$\gamma_{xy} = u_{,y} + v_{,x} = \frac{1}{2b} \left[u_4 - u_1 + u_3 - u_2 \right] + \frac{1}{2a} \left[v_2 - v_1 + v_3 - v_4 \right] .$$

The average in-plane stress resultants within each element can now be determined using equations (2.2-2) and (2.2-10).

II- Buckling Analysis:

The principle of minimum potential energy is again used to perform the buckling analysis. The pertinent functional takes the following form:

$$\pi_f = \frac{1}{2} \iint D \left[w^2_{,xx} + w^2_{,yy} + 2\nu\, w_{,xx}\, w_{,yy} + 2(1-\nu)\, w^2_{,xy} \right] dx\, dy$$
$$+ \frac{1}{2} \iint \left[N_{xx}\, w^2_{,x} + 2N_{xy}\, w_{,x}\, w_{,y} + N_{yy}\, w^2_{,y} \right] dx\, dy$$
$$- \left[\int_{SF} (w\, \bar{V}_n - w_{,n}\, \bar{M}_{nn})\, ds - \frac{1}{2} \iint k\, w^2\, dx\, dy \right] \qquad (2.2\text{-}11)$$

where w - admissible transverse displacement function which satisfies the kinematic boundary conditions.

$\bar{V}_n$ - prescribed effective shear force at the boundary.

$\bar{M}_{nn}$ - prescribed normal bending moment at the boundary.

S_F - the part of the boundary where $\bar{V}_n$ and $\bar{M}_{nn}$ are prescribed.

It can be shown [15] that, when $\delta\, \pi_f = 0$, one obtains the following equilibrium equations in the domain

$$D\, \nabla^4 w + kw = (N_{xx}\, w_{,xx} + 2N_{xy}\, w_{,xy} + N_{yy}\, w_{,yy}) \qquad (2.2\text{-}12)$$

along with the associated boundary conditions on S_F

$$V_n = \bar{V}_n \qquad M_{nn} = \bar{M}_{nn} \tag{2.2-13}$$

The critical load is obtained when the equilibrium equation (2.2-12) is satisfied.

For the finite element formulation, the domain is discretized into the same rectangular element mesh as in the case of in-plane stress analysis. Using the same local non-dimensional co-ordinate system, the displacement function w is assumed, within each element, in the following bicubic polynomial which yields a conforming plate element:

$$w = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi^2 + \alpha_5 \xi \eta + \alpha_6 \eta^2 + \alpha_7 \xi^3 + \alpha_8 \xi^2 \eta + \alpha_9 \xi \eta^2 + \alpha_{10} \eta^3 + \alpha_{11} \xi^3 \eta + \alpha_{12} \xi^2 \eta^2 + \alpha_{13} \xi \eta^3 + \alpha_{14} \xi^3 \eta^2 + \alpha_{15} \xi^2 \eta^3 + \alpha_{16} \xi^3 \eta^3 \tag{2.2-14}$$

The generalized nodal displacement vector for an element is taken in the following form:

$$\{ q \}^t = [w_1\ w_2\ w_3\ w_4 \mid w_{,x1}\ w_{,x2}\ w_{,x3}\ w_{,x4} \mid w_{,y1}\ w_{,y2}\ w_{,y3}\ w_{,y4} \mid w_{,xy1}\ w_{,xy2}\ w_{,xy3}\ w_{,xy4}] \tag{2.2-15}$$

The displacement function w (2.2-14) can be expressed in terms of nodal displacements, i.e. in terms of $\{ q \}$ instead of α_i (i = 1 to 16). The derivation of the stiffness matrices and the consistent load vectors, is given in many textbooks [6, 7, 8, 13 and 14]. The functional π_f (2.2-11) can be expressed in the following form:

$$\pi_f = \frac{1}{2} \{ q \}^t [K] \{ q \} + \frac{\omega}{2} \{ q \}^t [K_G] \{ q \} - \{ q \} \{ \bar{Q} \} \qquad (2.2\text{-}16)$$

where $\{ q \}$ - the generalized displacement vector containing all the nodal degrees of freedom of the system.

$[K]$ - the generalized stiffness matrix of the system which includes the stiffness of the plate and the foundation.

$[K_G]$ - the geometric stiffness matrix derived from the in-plane stress resultants due to arbitrary in-plane compressive load density.

$\{ \bar{Q} \}$ - the generalized consistent load vector of the system.

ω - proportionality factor associated with the critical buckling loads.

For the equilibrium of the system, the first variation of the functional must be equal to zero which results in the following system of equations:

$$[K] \{ q \} + \omega [K_G] \{ q \} = \{ \bar{Q} \} \qquad (2.2\text{-}17)$$

For the condition of neutral stability of the system, we must have the second variation of the functional equals to zero

$$\delta^2 \pi_f = \delta \{ q \}^t \{ [K] + \omega [K_G] \} \delta \{ q \} = 0 \qquad (2.2\text{-}18)$$

or

$$| [K] + \omega [K_G] | = 0 \qquad (2.2\text{-}19)$$

where $| \quad |$ - symbolizes the determinant.

We can interpret equation (2.2-19) from the point of view of bifurcation in the stability analysis. Bifurcation will only occur if $\{ \bar{Q} \}$ is identically equal to zero. Then, we look for non-trivial solution of equation (2.2-17) if there is a vector $\{ q \}$ and scalar quantity ω such that

$$[K] \{ q \} + \omega [K_G] \{ q \} = \{ 0 \} \qquad (2.2\text{-}20)$$

The eigenvalues ω and the eigenvectors $\{ q \}$ of the above equation can be obtained using the standard subroutines in the computer.

The displacement vector $\{ q \}$, as it exists in equation (2.2-20), contains the nodal values of transverse deflection, rotation in x and y directions and the twist at all nodes. From the point of view of both computer time and core requirements, it is very expensive to solve the eigenvalue problem with all degrees of freedom.

With a suitable condensation procedure [13], the eigenvalue problem can be solved with less degrees of freedom. The full details of the condensation procedure are given in Reference 13.

CHAPTER 3

RESULTS

3.1 Introduction

A finite element computer program has been developed (see Appendix B) to solve the buckling problem of plate on elastic foundation using rectangular finite element described ealier. The computer program is capable of incorporting any kind of boundary conditions provided the domain is rectangular. However, this method can be used for any type of domain geometry, boundary conditions and in-plane loading using appropriately shaped finite element.

The results of the finite element method are first checked by comparing them with the available exact theoretical solutions in literature for simply supported beam and rectangular plate on elastic foundation. After the finite element approach had shown very good agreement with the exact solutions the program was then used to perform a numerical buckling analysis of semi-infinite plate on elastic foundation due to partially distributed in-plane load. The program was also used to perform the buckling analysis of a finite rectangular clamped plate on elastic foundation. The agreement between the results by the finite element method and the Rayleigh-Ritz procedure are found to be good

3.2 Checking of the Computer Program

I. Buckling of Simply Supported Beam on Elastic Foundation:

The differential equation which governs this problem is written as follows :

$$EI \frac{d^4w}{dx^4} + kd\,w = P \frac{d^2w}{dx^2} \qquad (3.2\text{-}1)$$

Where d - the width of the beam.

The exact solution of the above equation for the case of simply supported beam is given in Reference 15, and the buckling load can be expressed in the following non-dimensional form :

$$\lambda = \left(m \pi \frac{\ell}{L}\right)^2 + \frac{1}{\left(m \pi \frac{\ell}{L}\right)^2} \qquad (3.2\text{-}2)$$

Where $\lambda = \frac{P}{kd\ell^2}$, the non-dimensional buckling load.

$\ell = \sqrt[4]{\frac{EI}{kd}}$, the characteristic length of the beam.

m - number of half-waves in the buckling mode.

L - length of the beam.

The finite element analysis is performed by using a string of long rectangular elements. The results of the finite element analysis is plotted in non-dimensional form in Figure 3.1 for different values of L/ℓ along with the plot of equation (3.2-2).

The agreement between the exact results and the finite element results is excellent not only in the values of the first few buckling loads, but also in the modes of buckling.

The buckling load for infinitely long simply supported beam on elastic foundation was first given by Hetenyi [25] to be equal to $2kd\ell^2$ and this load can be derived from equation (3.2-2) by deriving the condition for which λ is minimum. Differentiating the equation (3.2-2) with respect to $(m \pi \frac{\ell}{L})$ and equating it to zero, one obtain the following condition for λ_{min}

$$\left(\frac{m\pi\ell}{L}\right) = 1$$

Therefore, $\lambda_{min} = 2$ for $\frac{L}{\ell} = m\pi$, $m = 1, 2, \ldots, \infty$. This means that all the curves relating λ and $\left(\frac{L}{\ell}\right)$ for different values of m must touch the line $\lambda = 2$. One can notice this in Figure 3.1. Hence, one can conclude that the buckling load for an infinite long simply supported beam is $2k\ell^2$ (i.e. $\lambda_{min} = 2$).

II- Buckling of simply supported rectangular plate on elastic foundation with all four edges simply supported :

The differential equation which governs this problem for uni-directional in-plane loading, is written as follows:

$$D\left\{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} + kw = (P/B)\frac{\partial^2 w}{\partial x^2} \qquad (3.2\text{-}3)$$

The exact solution of the above differential equation for the case of simply supported rectangular plate is given in Reference 18., The buckling load can be expressed in the following non-dimensional form:

$$\lambda_{mn} = \frac{B}{\ell}\left[\frac{1}{\left(m\frac{B}{A}\frac{\ell}{B}\right)^2} + \left(m\frac{B}{A}\frac{\ell}{B}\right)^2\left(1 + \left(\frac{nA}{mB}\right)^2\right)^2\right] \qquad (3.2\text{-}4)$$

where $\lambda_{mn} = \frac{P}{k\ell^2}$, non-dimensional buckling load

$\ell = \sqrt[4]{\frac{D}{k}}$, characteristic length of the plate

m,n - the number of half-waves of the mode of buckling in x and y directions

A,B - length and width of rectangular plate respectively

One quarter of a square plate is discretized as shown in Figure 3.2 and proper boundary conditions are imposed along x and y axes for the analysis of symmetric and anti-symmetric modes of buckling.

The exact solution given by equation (3.2-4) and the results of finite element analysis for different values of B/ℓ when A=B, are presented in non-dimensional form in Figure 3.2. The agreement between the exact and approximate non-dimensional loads is good for the first few modes of buckling.

3.3 Buckling of semi-infinite plate on elastic foundation due to partially distributed in-plane load

The buckling of semi-infinite plate on elastic foundation due to partially distributed in-plane load is used to simulate the failure of ice sheet in buckling mode. The fluid foundation to the ice sheet can be considered as an elastic foundation of Winkler type. This type of foundation produces a vertical resisting force proportional to the vertical displacement of the point at the plate-foundation interface. The contact between the ice sheet (the plate) and the fluid foundation (the elastic foundation) is assumed to be frictionless.

Because of the limitations of the computer capacity, it is quite difficult to perform the buckling analysis for a semi-infinite domain using the finite element approach. So, a finite large size plate resting on elastic foundation has been used in order to simulate the semi-infinite domain of a floating ice sheet. The finite element computer program is used, first, to check whether the finite size of the plate has any effect on the simulation to the semi-infinite domain. Two different kinds of boundary conditions, simply supported and clamped, are imposed on the edges CE, EF and CD (Figure 3.3) which are simulating the edges which are supposed at infinity. The resulting effect of imposing the two different kinds of boundary conditions on the buckling loads and modes is found to be very small (see Table 3.1).

Due to the symmetry of the problem about the centre line, on half of the plate is discretized into different meshes. Small size finite elements have been used near the in-plane load due to the rapid changes in the stresses and deflection, and large size finite elements have been used in the regions which are further away from the load. The symmetric and anti-symmetric boundary conditions, for the deflection w, are imposed along the line of symmetry to determine the buckling loads for the symmetric and anti-symmetric modes of buckling for the whole plate.

Since we are using an approximate numerical approach, we must make sure that the numerical results of the analysis converge as the plate is discretized into more number of finite elements. Meshes of 3x3, 4x4, 5x5, 6x6 and 7x7 elements are used to discretize one-half of the plate as shown in Figure 3.4. The convergence of the first two buckling loads is presented in Table 3.1 and in Figure 3.5. It can be noticed that the convergence is good and the mesh of 6x6 elements gives as good results as that of 7x7 mesh. The mesh of 6x6 elements has been used to perform the buckling analysis of the semi-infinite plate resting on elastic foundation. The numerical results for the buckling loads, using the finite element computer program, have been calculated for different values of b/ℓ ratio which is varied by changing the value of the modulus of the elastic foundation for low values of b/ℓ ratio and the width of the applied in-plane load for high values of b/ℓ ratio.

The results of the lowest buckling load is presented in Table 3.2 and plotted in non-dimensional form (λ_1) with respect to b/ℓ ratio in Figure3.6. One can notice that the curve tends to approach the value of the non-dimensional concentrated buckling load ($\lambda = \frac{P}{k\ \ell^3}$ = 3.32, where P is the concentrated buckling load) as the value of b/ℓ ratio approaches zero. For high values of b/ℓ ratio, the curve tends to approach asymtotically the Hetenyi's buckling load of semi-infinite beam on elastic foundation [25] .

The first three typical buckling modes have been sketched in Figure3.7 which depict two symmetric and one anti-symmetric mode about the centre line.

The buckling effective pressure p_e ($=\frac{P}{2bh}$) corresponding to the lowest buckling load has been calculated and plotted in non-dimensional form ($\frac{2hp_e}{k\ell^2}$) with respect to b/ℓ ratio (see Figure 3.8). One can notice that, for a particular thickness and crushing strength of ice sheet, it will fail in the buckling mode when the aspect ratio exceeds a certain value. Also, for a given kind of ice, thin sheets will fail in the buckling mode at lower values of the aspect ratio than thick ice sheets, i.e thin ice sheet needs smaller width of the in-plane load to fail in the buckling mode than thick ice sheet: This agrees with the exprimental observations made by many investigators while performing small scale expriments that thin ice sheet tends to fail in the buckling mode rather than in the crushing mode.

3.4 Buckling of clamped rectangular plate on elastic foundation

To the knowledge of the author, the exact theoretical solution to this problem is not available and no mention of approximate solution has been found in literature.

Rayleigh-Ritz procedure is used to solve the buckling problem of clamped rectangular plate resting on elastic foundation. The deflection at any point of the plate is assumed to be a function of a product of two characteristic functions for a clamped beam. The deflection function can be expressed in terms of non-dimensional co-ordinates ξ and η as follows:

$$w = \sum_{m=1}^{p} \sum_{n=1}^{q} A_{mn} \; X_m(\xi) \; Y_n(\eta) \qquad (3.4\text{-}1)$$

where $\xi = x/A$, $\eta = y/B$.

A, B - length and width of the rectangular plate

$$\left.\begin{aligned} X_m(\xi) &= \operatorname{Cosh}\varepsilon_m\xi - \operatorname{Cos}\varepsilon_m\xi - \alpha_m(\operatorname{Sinh}\varepsilon_m\xi - \operatorname{Sin}\varepsilon_m\xi) \\ Y_n(\eta) &= \operatorname{Cosh}\varepsilon_n\eta - \operatorname{Cos}\varepsilon_n\eta - \alpha_n(\operatorname{Sinh}\varepsilon_n\eta - \operatorname{Sin}\varepsilon_n\eta) \end{aligned}\right\} \qquad (3.4\text{-}2)$$

The values of ε_m, α_m and the numerical values of the necessary integrals are available in Reference [20]. The above deflection function satisfies the clamped boundary conditions along the edges of the plate.

We introduce the following notation:

$$E_{im} = \int_0^1 X_i \frac{d^2 X_m}{d\xi^2} \, d\xi$$

$$E_{mi} = \int_0^1 X_m \frac{d^2 X_i}{d\,\xi^2}\, d\,\xi$$

$$F_{kn} = \int_0^1 Y_k \frac{d^2 Y_n}{d\,\eta^2}\, d\,\eta$$

$$F_{nk} = \int_0^1 Y_n \frac{d^2 Y_k}{d\,\eta^2}\, d\,\eta$$

$$H_{im} = \int_0^1 \frac{dX_i}{d\xi} \frac{dX_m}{d\xi}\, d\,\xi$$

$$K_{kn} = \int_0^1 \frac{dY_k}{d\eta} \frac{dY_n}{d\eta}\, d\,\eta$$

We also note the following relations [20]:

$$\int_0^1 X_r\ X_s = 1 \qquad \text{for } r = s$$
$$= 0 \qquad \text{for } r \neq s$$

$$\int_0^1 Y_r\ Y_s = 1 \qquad \text{for } r = s$$
$$= 0 \qquad \text{for } r \neq s$$

$$\int_0^1 \frac{d^2 X_r}{d\,\xi^2} \frac{d^2 X_s}{d\,\xi^2}\, d\,\xi = \varepsilon_r^4 \qquad \text{for } r = s$$
$$= 0 \qquad \text{for } r \neq s$$

$$\int_0^1 \frac{d^2Y_r}{d\eta^2}\frac{d^2Y_s}{d\eta^2} = \varepsilon_r^4 \qquad \text{for } r = s$$
$$= 0 \qquad \text{for } r \neq s$$

Substituting equation (3.4-1) in the following energy expression

$$\pi = \frac{1}{2} D \iint [\, w_{,xx}^2 + w_{,yy}^2 + 2\nu\, w_{,xx} w_{,yy} + 2(1-\nu)\, w_{,xy} \,]\, dx\, dy$$

$$+ \frac{1}{2} k \iint w^2\, dx\, dy + \frac{1}{2} \iint (\, w_{,x} N_{xx} w_{,x} \,)\, dx\, dy \qquad (3.4\text{-}3)$$

we get

$$= \frac{1}{2} \{A\}^t [C] \{A\} + \frac{1}{2} \{A\}^t [S] \{A\} + \frac{1}{2} \{A\}^t [T] \{A\} \qquad (3.4\text{-}4)$$

where

$$\{A\}^t = [A_{11}\ A_{12}\ A_{13} \ldots A_{21}\ A_{22}\ A_{23} \ldots A_{31}\ A_{32} \ldots A_{66}]$$

$$C_{mn}^{(ij)} = \frac{D}{A^4} [\, \frac{B}{A} \varepsilon_m^4 + \frac{A^3}{B^3} \varepsilon_n^4 + 2\,(1-\nu)\, H_{mm} H_{nn}$$
$$+ 2\nu \frac{A}{B} E_{mm} F_{nn} \,] \qquad \text{for } mn = ij$$

$$= \frac{D}{A^2} [\, 2(1-\nu) \frac{A}{B} H_{mi} H_{nj} + \nu \frac{A}{B} E_{mi} F_{nj}$$
$$+ \nu \frac{A}{B} E_{im} F_{jn} \,] \qquad \text{for } mn \neq ij$$

$$S_{mn}^{(ij)} = k\, AB\, \delta_{mn}^{(ij)}$$

where $\delta_{mn}^{(ij)} = 1$ for $mn = ij$

$= 0$ for $mn \neq ij$

$$\bar{T}_{mn}^{(ij)} = \frac{B}{A} N_{xx} H_{mi} \qquad \text{for } mn = ij$$

$$= 0 \qquad \text{for } mn \neq ij$$

The first variation of functional π given in equation (3.4-4) is written as follows:

$$\delta \pi = \delta \{A\}^t \{ \frac{D}{A^2} [P] + A B k [I] - \frac{B}{A} N_{xx} [Q] \} \{A\} = 0$$

in which the elements of matrices [P] and [Q] are nondimensional and [I] is the identity matrix.

For arbitrary variation of $\{A\}$, we get the following eigenvalue problem:

$$[N] \{A\} = \lambda [B] \{A\} \qquad (3.3\text{-}5)$$

where

$$[N] = (\frac{\ell}{A})^4 (\frac{A}{B}) [P] + [I]$$

$$= \frac{P}{k \ell^3}$$

$$[B] = (\frac{\ell}{A})^3 (\frac{A}{B}) [Q]$$

As mentioned earlier in the case of simply supported, the nondimensional buckling load depends upon the buckling mode m,n, B/ℓ and A/B ratios.

The finite element computer program is also used to determine the buckling loads and the mode of buckling by discretising one quarter of a square clamped plate into a mesh of 4 x 4 = 16 elements and imposing proper boundary conditions for different combinations of symmetric and anti-symmetric buckling modes about x and y axes.

The results of the finite element analysis and Rayleigh-Ritz procedure for different values of foundation stiffness, are presented in non-dimensional form in Figure 3.9. The results obtained by the two approximate numerical analysis are close to each other.

It can be noticed that the non-dimensional load , for a particular mode of buckling, decreases to a certain minimum value as the width ratio (the width of the applied load to the characteristic length of the plate) increases, and then starts to increase. For high values of width ratio, which correspond to high values of foundation stiffness, the buckling load is a function of the number of half-waves in the load direction only.

CHAPTER 4

CONCLUDING REMARKS

The ice forces on structures are caused due to failure of ice sheet either in crushing mode or in buckling mode. The purpose of this study is to determine the buckling load of an ice sheet which is modelled as a semi-infinite plate on elastic foundation.

Only a few references exist in literature regarding the exact and approximate buckling analysis of plates on elastic foundation. In this study, the finite element method is chosen to perform the buckling analysis of such plates.

A finite element computer program is written and tested to determine the loads and modes of buckling using conforming rectangular plate element. Although this computer program can only analyse the stability of rectangular plate, the method is quite general to handle any geometry and boundary conditions of the plate.

The buckling loads of a semi-infinite plate on elastic foundation are determined and the lowest buckling load is plotted against the width of the partially distributed load in non-dimensional form. From this graph, the effective pressure is determined which will cause the plate to buckle. For a particular crushing strength of ice sheet, one can determine the value of aspect ratio above which the ice sheet fails in the buckling mode.

The buckling loads of a clamped rectangular plate on elastic foundation have been determined by two methods--finite element method and Rayleigh-Ritz procedure. The comparison is good between the results obtained by the two methods.

Further research in this area is suggested to obtain the exact solution of the buckling problem of a semi-infinite plate on elastic foundation. A systematic experimental work is also suggested to verify the theoretical results with the experimental data.

REFERENCES

1. Schwarz, J., Hirayama, K. and Wu, H. "Effect of Ice Thickness on Ice Forces", proceedings of The Offshore Technology Conference, Houston, Texas, paper No. OTC 2048, 1974.

2. Hirayama, K., Schwarz, J. and Wu, H. "Model Technique For The Investigation of Ice Force on Structures", proceedings of The Second International Conference on Port and Ocean Engineering Under Arctic Conditions, University of Iceland, 1972.

3. Zabilansky, L. J., Nevel, D. E. and Haynes, F. D.", Ice Forces on Model Structures", Second Canadian Hydrotechnical Conference, Burlington, Ontario, Canada, May 1975.

4. Nevel, D. E., Perham, R. E. and Hogue, G. B.", Ice Forces on Vertical Piles", Proceeding of U.S. Army Science Conference, 1972.

5. Neil, C. R. "Dynamic Ice Forces on Piers and Piles. An Assessment of Design Guidelines in The Light of Recent Research", Canadian Journal of Civil Engineering, Vol. 3, 1976.

6. Zienkiewicz, O. C. "The Finite Element Method in Engineering Science", McGraw Hill, 1971.

7. Yang, H. T. Y. "Flexible Plate Finite Element on Elastic Foundation", ASCE, Struc. Div., Vol. 96, No. ST. 10, pp. 2033-2101, Oct. 1970.

8. Holand, I., Moan, T. "The Finite Element Method in Plate Buckling", Edited by I. Holand and K. Bell, Trondheim, Norway, Tapir, 1969.

9. Carson, W. G. and Newton, R. E. "Plate Buckling Analysis Using a Fully Compatible Finite Element", AIAA, Vol. 7, No. 3, pp. 527-529, March 1969.

10. Hirayama, K., Schwarz, J. and Wu, H. C. "An Investigation of Ice Forces on Vertical Structures", IAHR Report No. 158, June 1974.

11. Kivisild, M. R. "Ice Impact on Marine Structures", Ice Seminar Sponsored by The Petroleum Society of Canadian Institute of Mining and Metallurgy and The American Petroleum Institute, Calgary, Alberta, 6 pp., 1969.

12. Hand Book of Structure Stability, Edited by Column Research Committee of Japan, Corona Publishing Company, Tokyo, Japan, 1971.

13. Gallagher, R. H. "Finite Element Analysis, Fundamentals", Prentice Hall, 1975.

14. Desai, C. S. and Abel, I. F. "Introduction to The Finite Element Method, a Numerical Method For Engineering Analysis", Van Nostrand Reinhold Company, 1972.

15. Timoshenko, S. P., Gere, J. M. "Theory of Elastic Stability" McGraw Hill, Second Edition, 1961.

16. Kerr, A. D. " The Determination of Horizontal Forces on a Floating Ice Plate Exerts on a Structure", Personal Communications.

17. Takagi, S., "The Buckling Pressure of an Elastic Plate Floating on Water and Stressed Uniformly Along The Periphery of an Internal Hole", The Trans. of The 22nd. Conf. of Army Mathematicians, 1976.

18. Vlasov, V. Z., Leont'Ev, V. N. "Beams, Plates and Shells on Elastic Foundation", Isreal Program For Scientific Translation, IPST Cat. No. 1453, 1966.

19. Flugge, W. (Ed), Handbook of Engineering Mechanics, Ch. 44

20. Young, D. "Vibration of Rectangular Plate By The Ritz Method", ASME, Applied Mechanics Division, March 1950.

21. Korzhavin, K. N. "Action of Ice on Engineering Structures" U.S.S.R. Acad. Science Siberian Branch Droft; Translated by CRREL, 1971.

22. Afanas'ev, V. P.; Dolgopolov, Yu. V. and Shreishtein, Z. J. "Ice Pressure on Isolated Structures in The Sea, in Russian", Trudy Leningrad, Arkt. i Antarkt. Inst., Vol. 300, pp. 61-80, English Translation by U.S.A. CRREL, DT 346, 1972.

23. Kheishin, D. E. "Determination of External Loads Which Act on The Hull of a Ship Caused by Compression of Ice Cover, in Russian" Problemy Arktiki i Antarktiki, No. 7, pp. 25-31, 1971.

24. Rose, E. "Thrust Exerted by Expanding Ice Sheets" Transactions ASCE, Vol. 112, pp. 871-885, 1947.

25. Hetenyi, M. "Beams on Elastic Foundation", The University of Michigan Press, Chapter VII, 1946.

APPENDIX (A)

Table 1.1 Principal modes of ice action

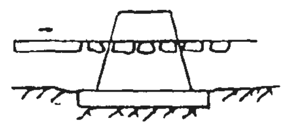
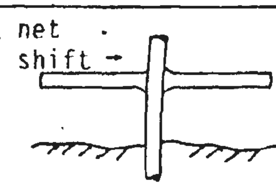
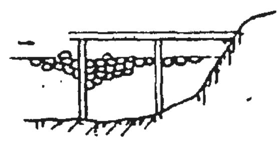
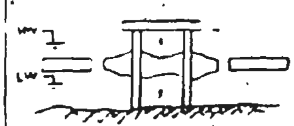
No.	DESCRIPTION	TYPICAL ENVIRONMENT	ILLUSTRATION
1	Impact of moving sheets and floes	Rivers at break-up coastal water with appreciable currents	
2	Static pressure from expanding or contracting sheets	Lakes, sheltered coastal water temp. changes and jacking by refreezing of cracks.	
3	Slow pressure from ice pack or jam	Exposed coastal waters, rivers .	
4	Vertical movement	Tidal locations with heavy ice build-up .	

Table 3.1

Non-dimensional partially distributed buckling load of semi-infinite plate for different discretizations of the plate*

No.	3x3 Mesh	Mode	4x4 Mesh	Mode	5x5 Mesh	Mode	6x6 Mesh	Mode	7x7 Mesh	Mode
1	5.0625 *[illegible]*	S	3.7210 *[illegible]*	S	3.5836 *[illegible]*	S	3.4646 *[illegible]*	S	*3.4548*	S
2	14.908 *14.932*	S	13.875 *13.926*	S	12.501 *12.424*	S	11.990 *11.951*	S	*11.616*	S
3	15.535 *15.52*	A	17.651 *17.555*	A	16.898 *16.702*	A	17.006 *17.00*	A	*17.016*	A
4	42.208 *42.170*	S	28.167 *28.147*	A	19.868 *[illegible]*	S	18.295 *18.357*	S	*17.978*	S
5	43.986 *43.916*	A	29.098 *29.072*	S	27.093 *27.084*	A	25.529 *25.510*	A	*24.937*	A
6	51.958 *51.552*	S	44.161 *44.551*	S	31.773 *31.959*	S	25.676 *25.555*	S	*25.076*	S
7	59.399 *59.051*	A	54.530 *54.574*	A	32.171 *32.455*	A	31.315 *31.153*	A	*31.407*	A
8	76.609 *77.247*	A	61.129 *61.021*	S	43.847 *44.222*	S	39.232 *39.524*	A	*32.784*	S
9	82.026 *85.282*	S	61.856 *63.220*	A	49.423 *49.541*	A	39.444 *39.697*	S	*36.555*	A
10	86.678 *87.190*	A	94.288 *95.507*	A	70.361 *70.874*	A	63.931 *64.555*	A	*49.021*	A

*Non-dimensional buckling load (·) = P/k·

S symmetric mode about the centre line.

A anti-symmetric mode about the centre line.

5.0625 simply supported edges.

5.0615 clamped edges.

h = 0.0254 m, b = 0.1524 m, b/ℓ = 0.355, ν = 0.3, k = 9.126 kN/m^3

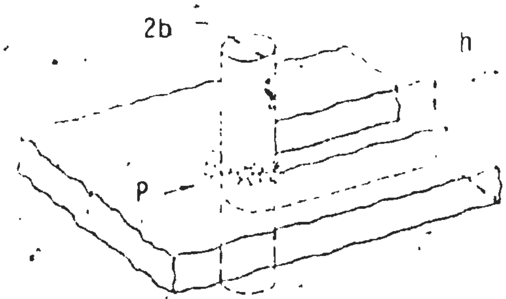
E = 206.700 kN/m^2

Table 3.2

Non-dimensional partially distributed buckling load of a semi-infinite plate for different [illegible] ratio

[illegible]	0.[illegible]		0.3[illegible]		0.[illegible]		1.7[illegible]		3.9031		7.0[illegible]		12.[illegible]		[illegible]	
	λ*	Mode	λ	Mode	λ	Mode	λ	Mode	λ	Mode	λ*	Mode	λ	Mode	λ	Mode
1	3.3[illegible]	S	3.5[illegible]	S	4[illegible]	S	5.4745	S	9.1602	S	15.3642	S	26.32[illegible]	S	35.2842	S
2	11.403	S	13.277	S	14.974	S										
3	17.801	S	19.[illegible]43	A	17.[illegible]	A										
4	20.339	A	1[illegible].935	S	23.[illegible]	A										
5	2[illegible]	S	2[illegible].728	A	31[illegible]	S										
6	27.[illegible]	A	26.321	S	39.[illegible]	S										
7	30.761	A	32.668	A	42.9[illegible]	S										
8	31.40[illegible]	S	40.604	A	51.564	A										
9	34.394	A	42.904	S	56.[illegible]42	A										
10	46.135	A	56.790	A	69.985	A										

* $\lambda = p/kt^3$



Effective pressure, $p_e = P/2bh$

Figure 1.1 Definition of effective pressure

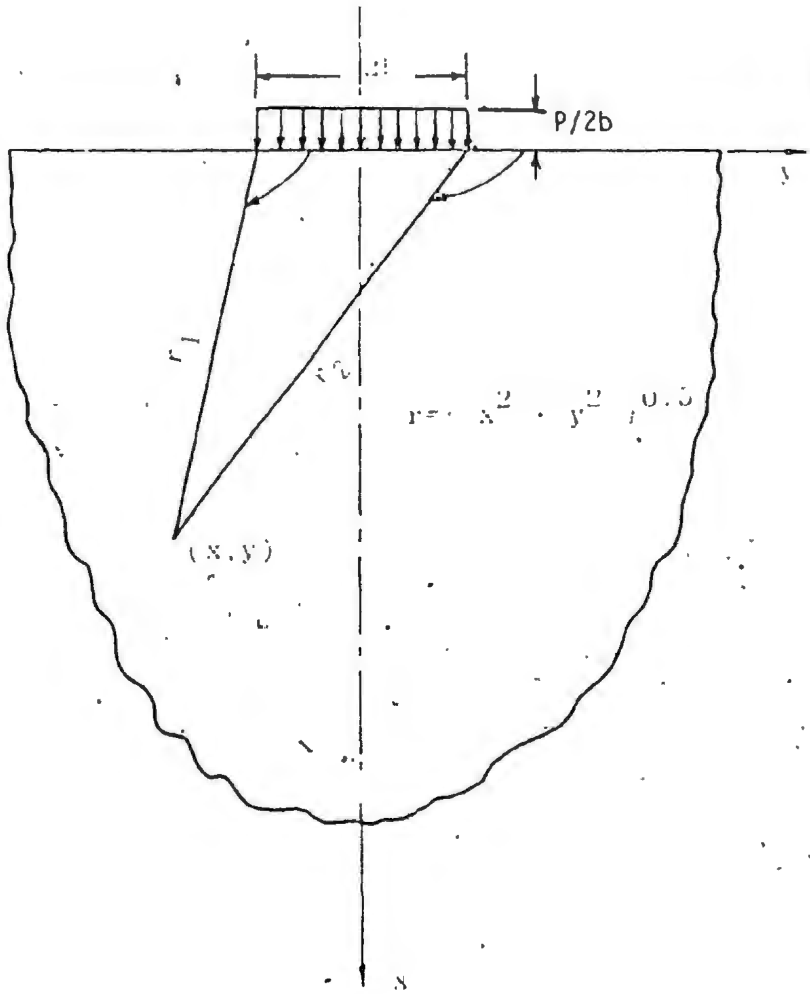


Figure 1.2 Semi-infinite plate under partially distributed in-plane load

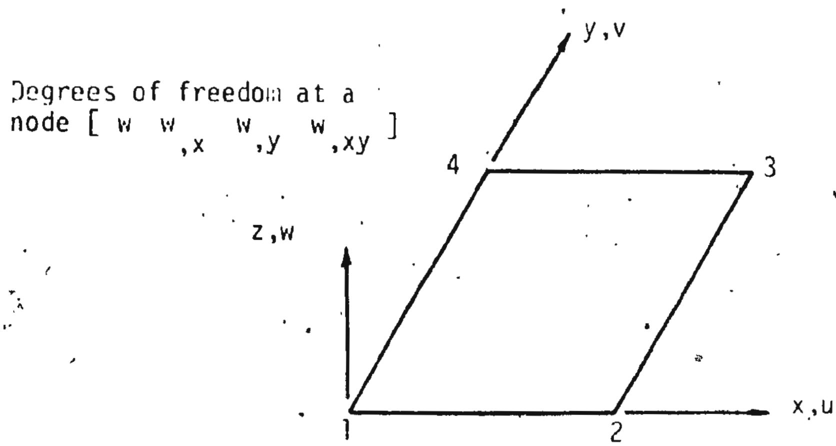


Figure 2.1 Rectangular finite element with degrees of freedom

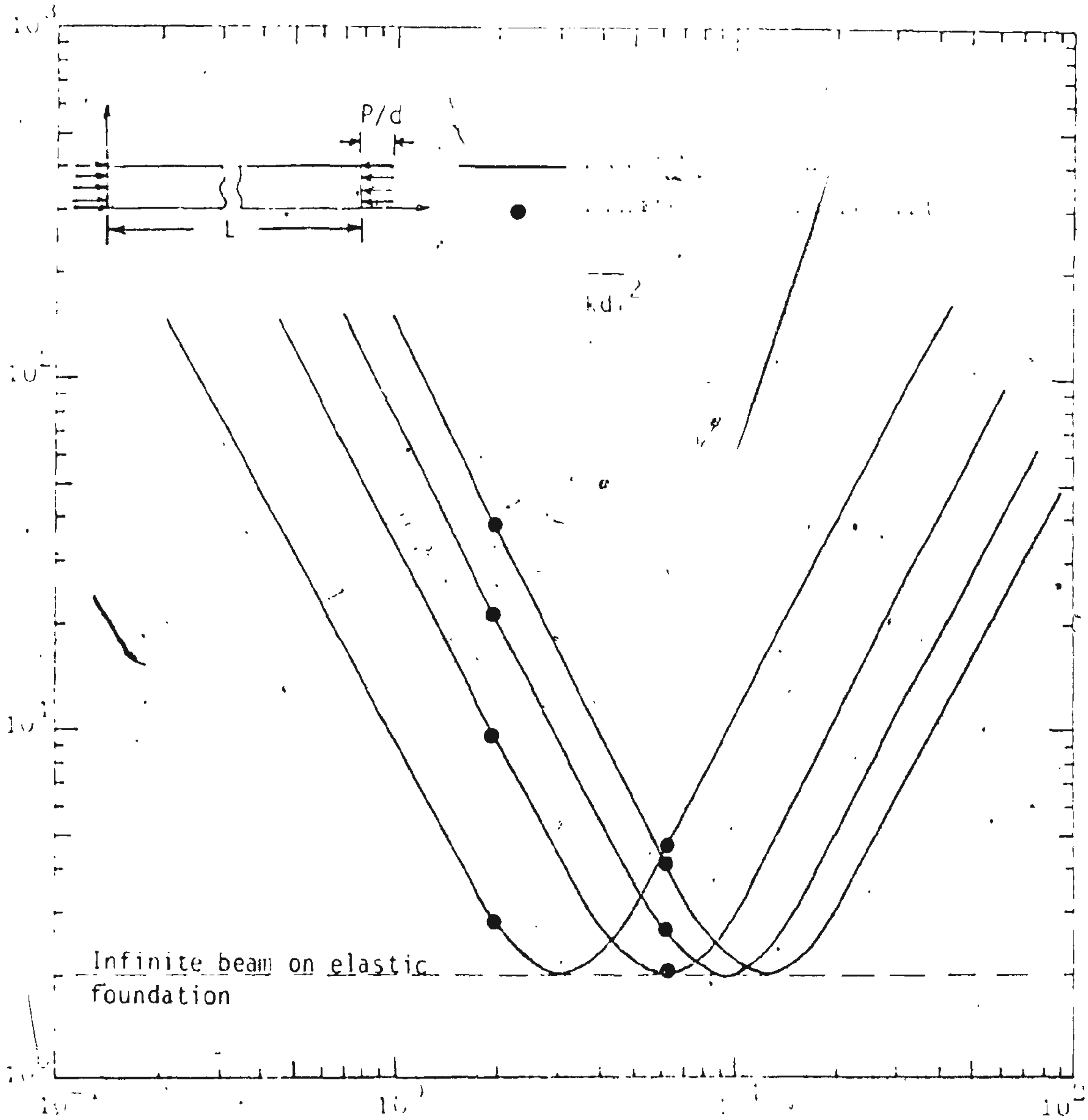


Figure 3.1 Non-dimensional buckling loads for a simply supported beam on an elastic foundation

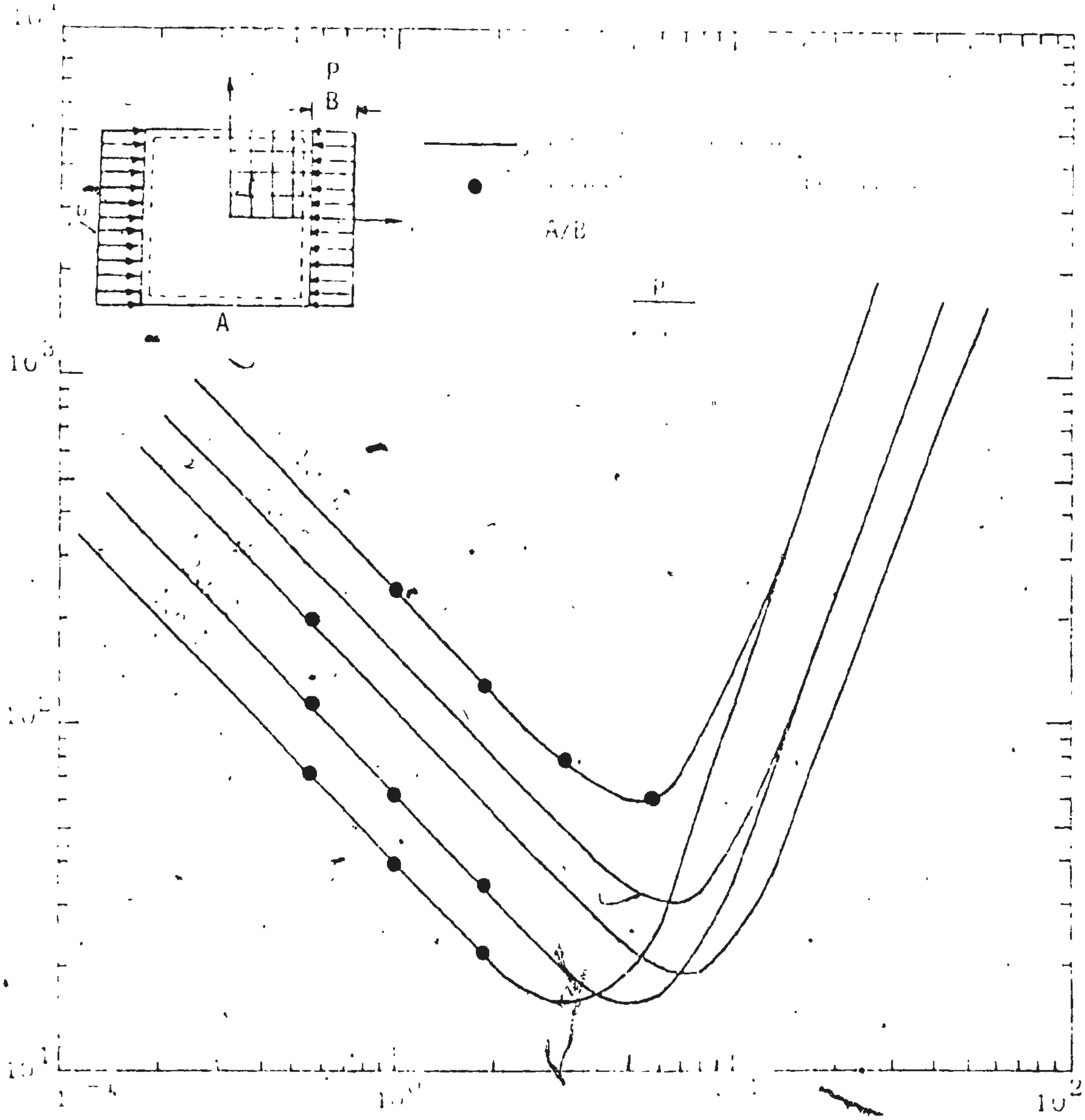


Figure 3.6 Non-dimensional buckling loads for a simply supported square plate on an elastic foundation

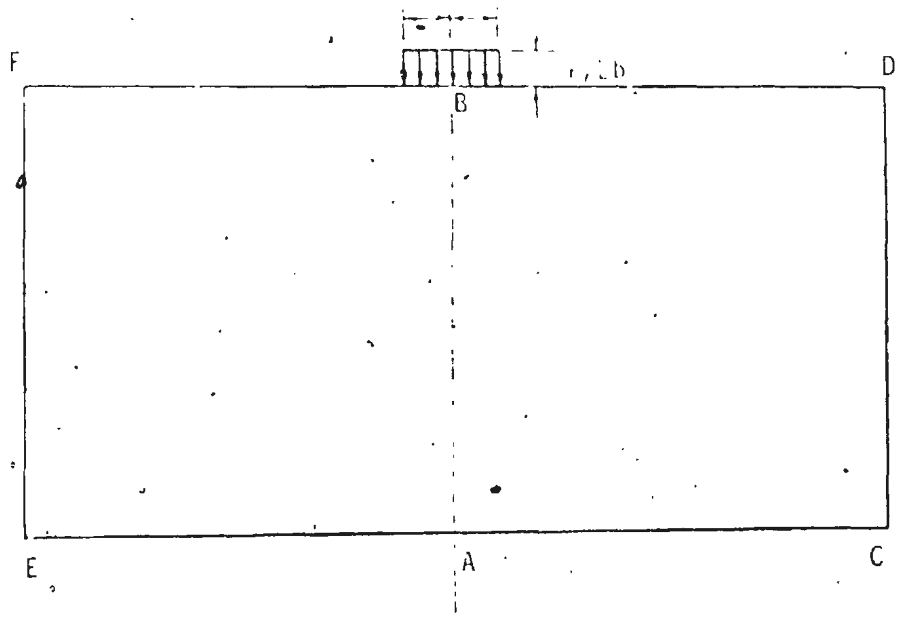


Figure 3.3 Finite element model for an ice sheet under partially distributed load.

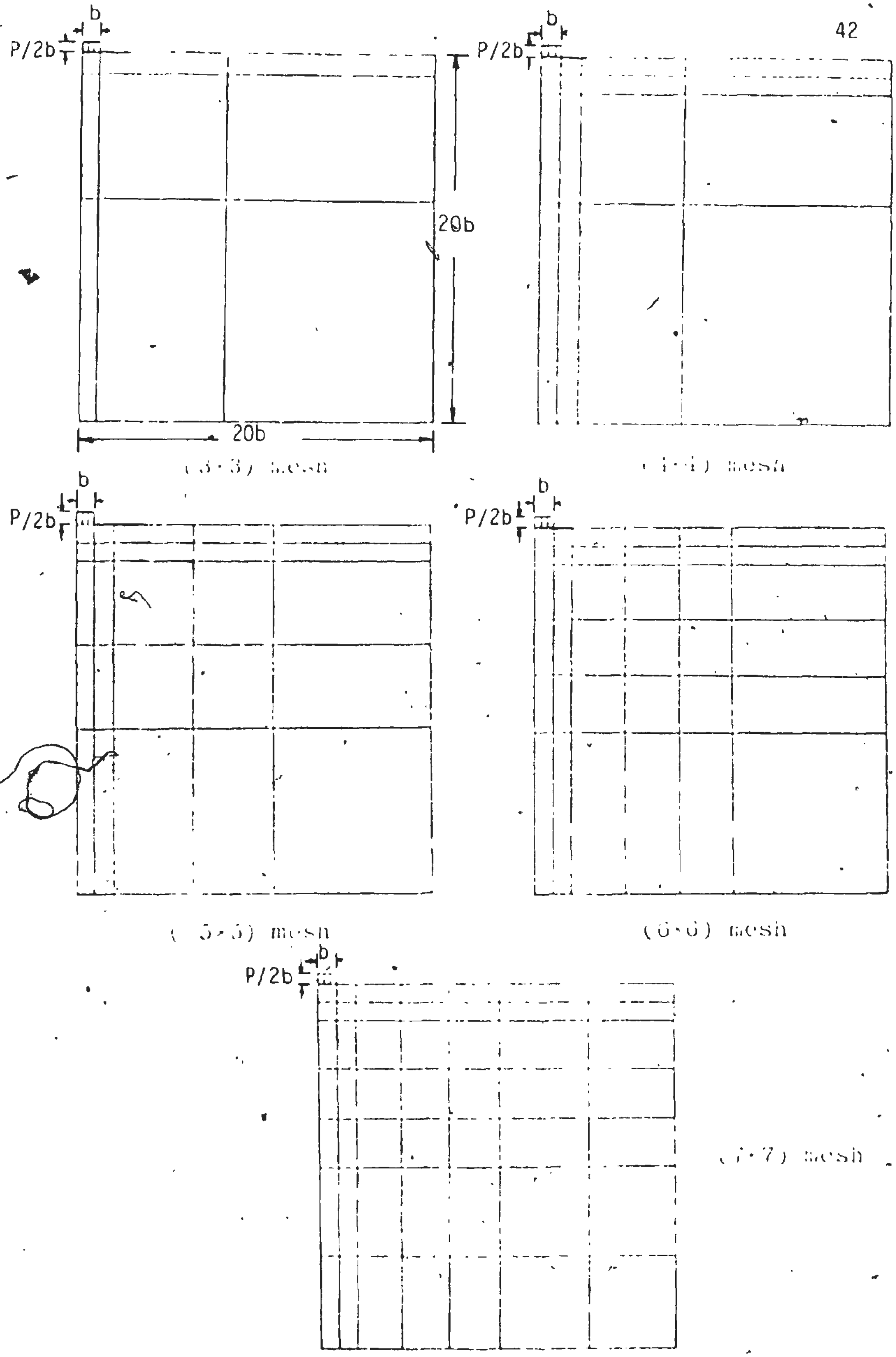


Figure [illegible].1 Discretization of one-half of a semi-infinite plate

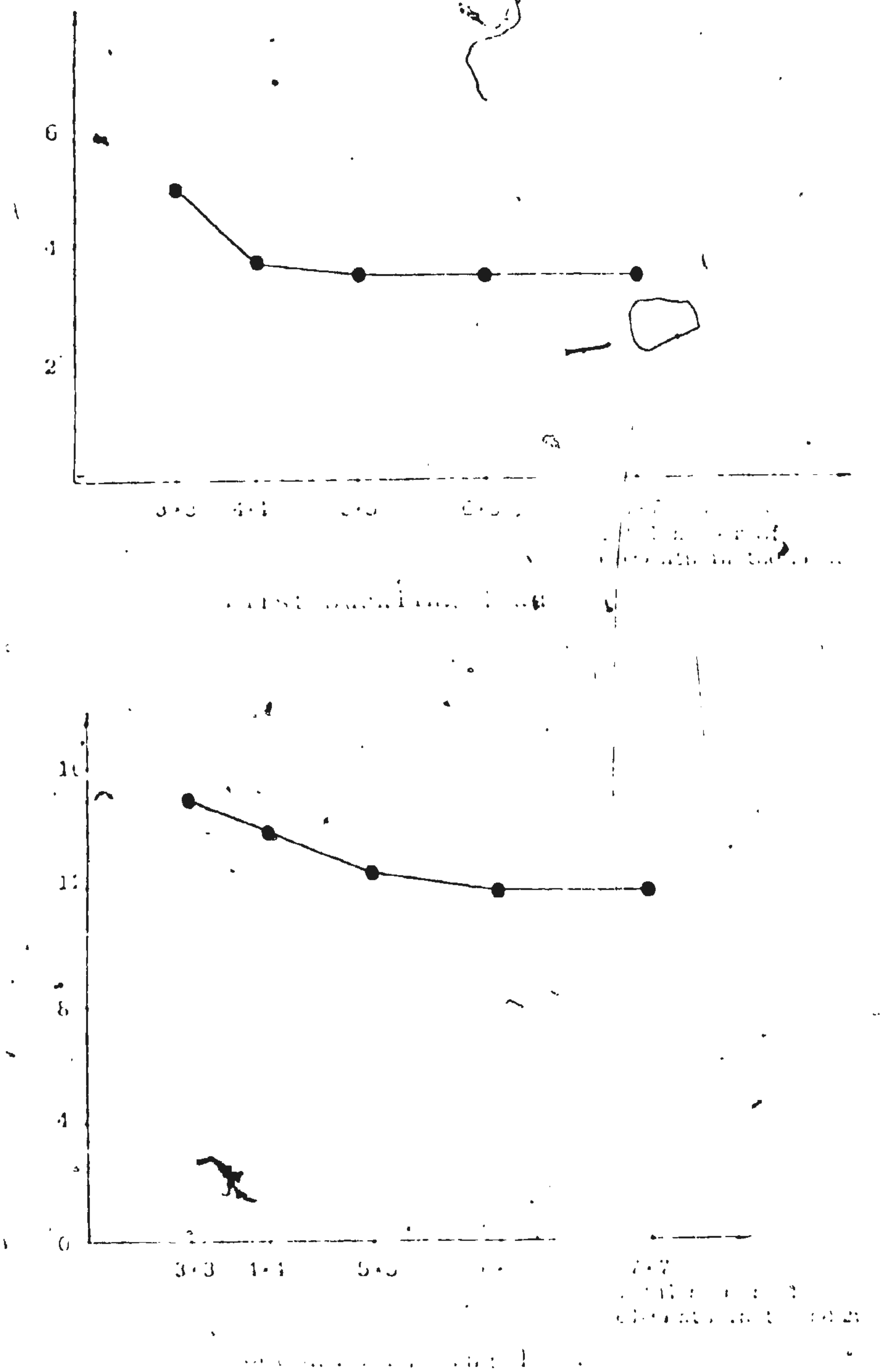
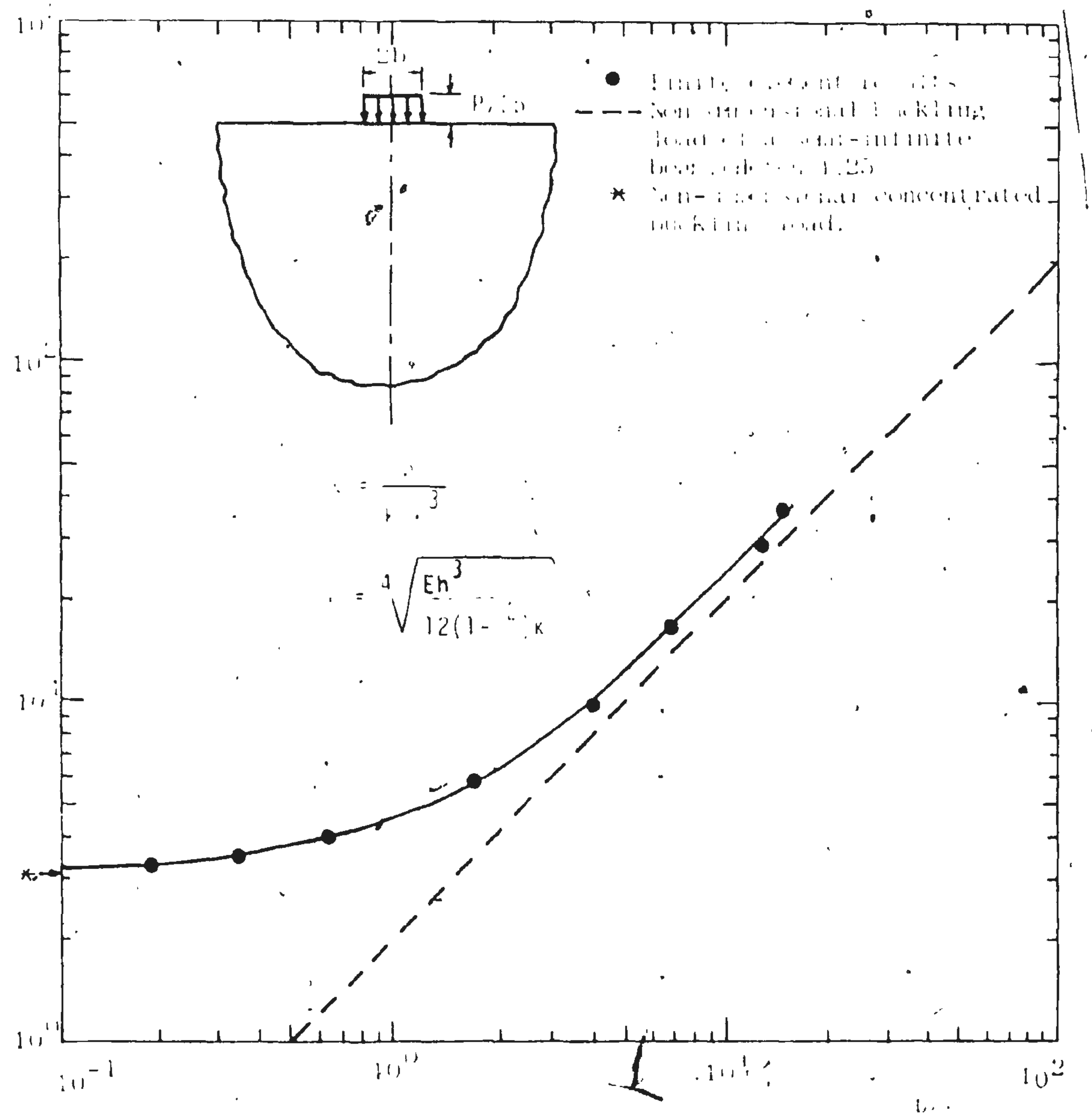


Figure 3.5 Convergence of non-dimensional buckling loads of a semi-infinite plate on an elastic foundation



[illegible] Plot [illegible]

buckling [illegible] with resp[illegible] for a

semi-infinite plate on an elastic foundation

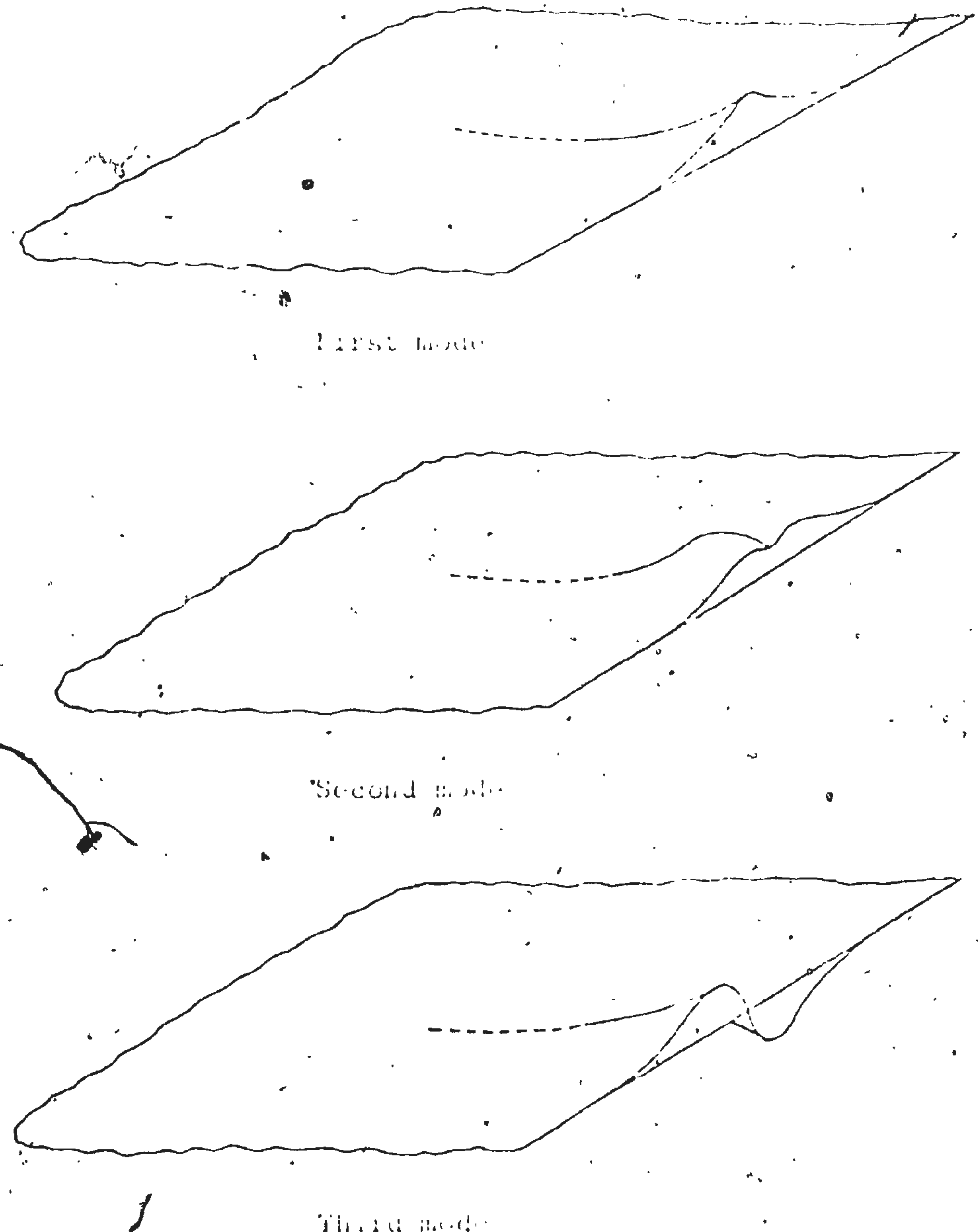


Figure 3.7 First three buckling modes for a semi-infinite plate on an elastic foundation

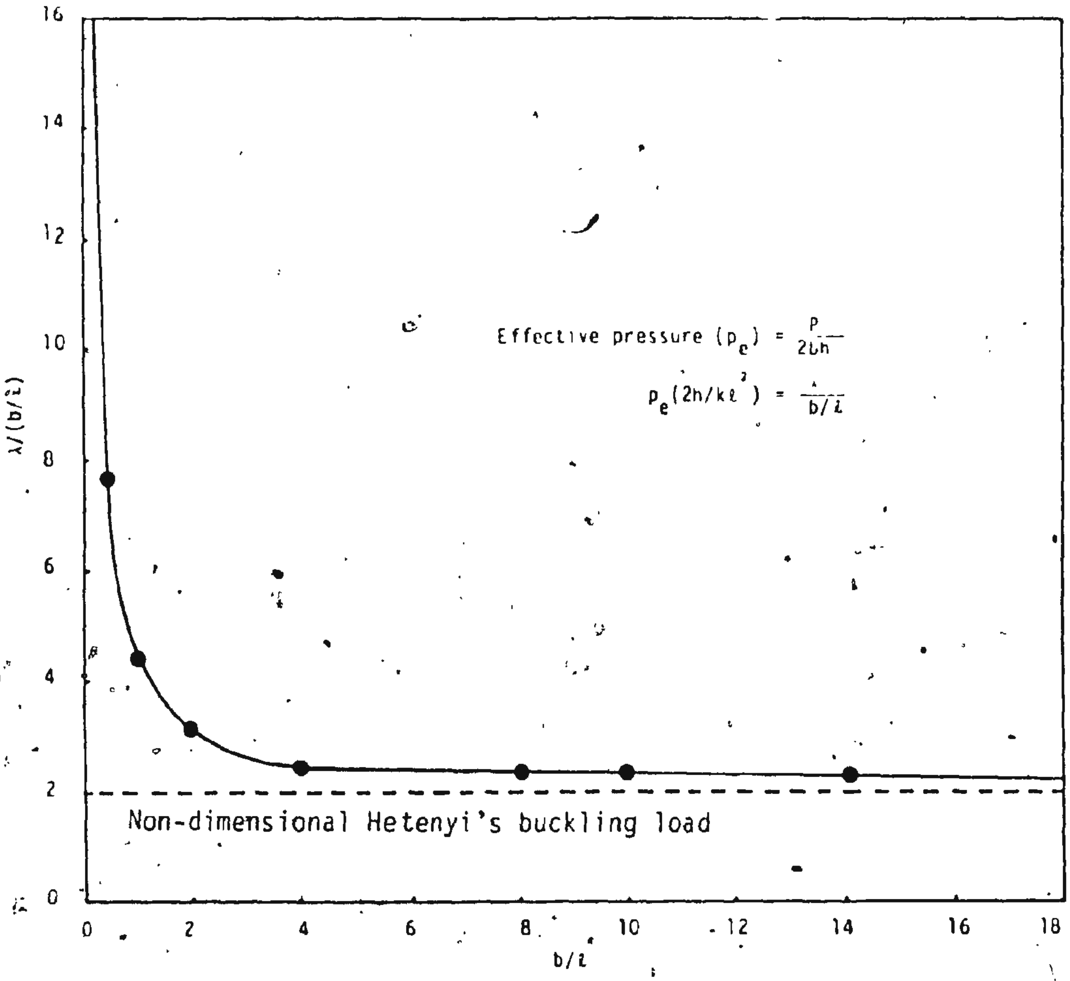


Figure 3.8 The lowest non-dimensional buckling effective pressure $(\frac{2hp_e}{k\ell^3})$ of a floating ice sheet with respect to b/ℓ ratio

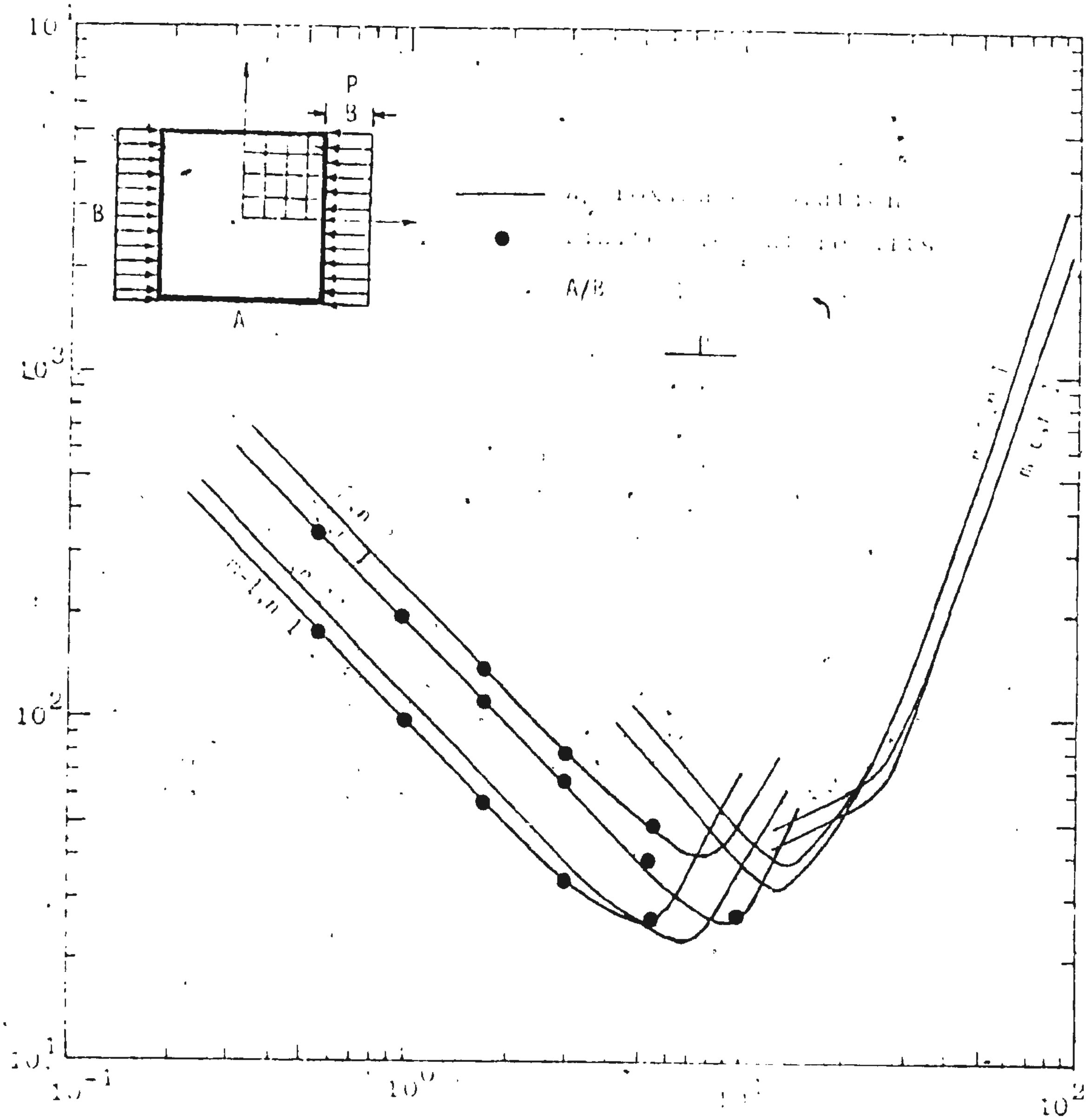
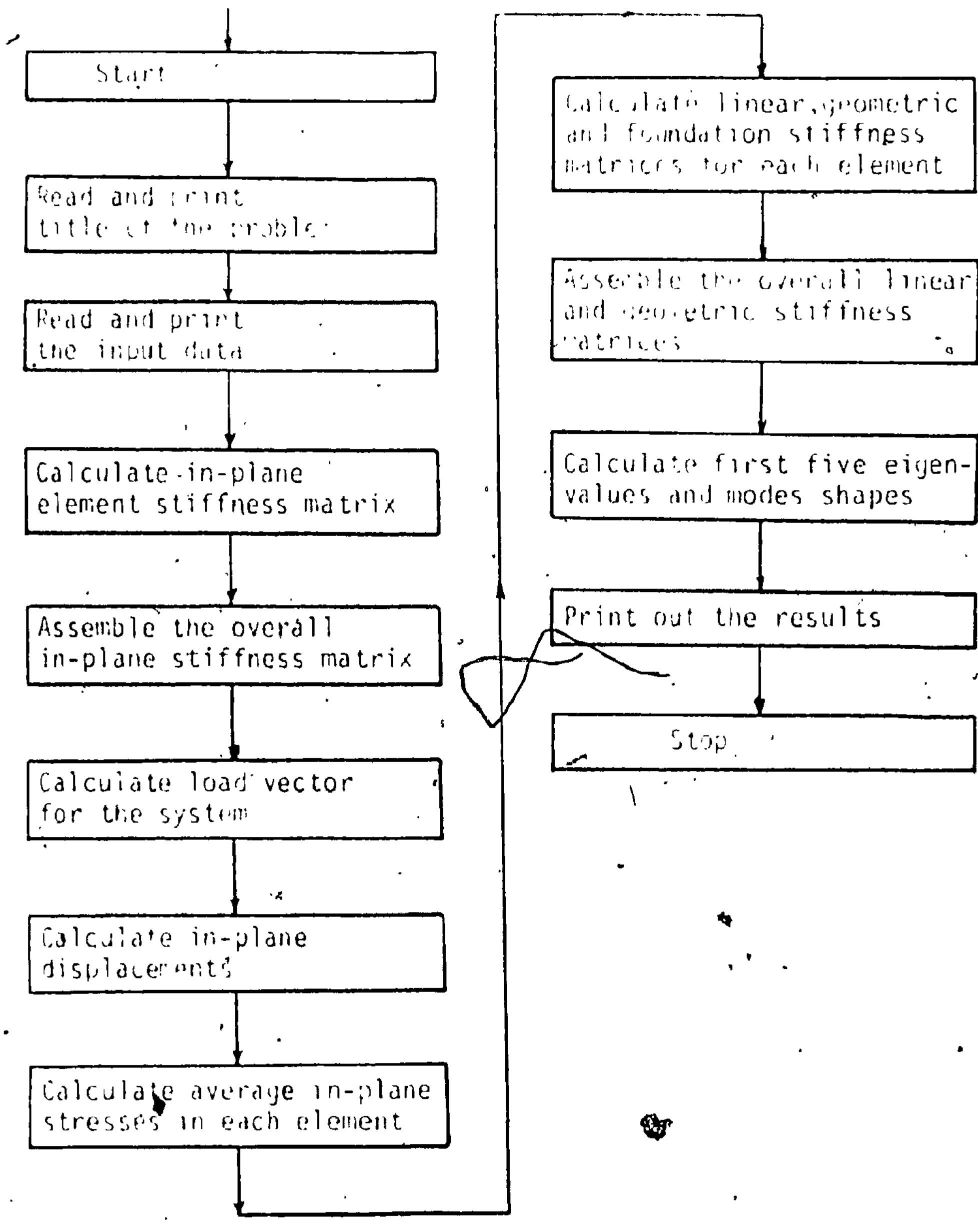


Figure 3.9 Non-dimensional buckling loads for a clamped square plate on an elastic foundation

APPENDIX (B)

The flow chart of the computer program :



```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AK(160,160),AKNG(160,160),Z(160,160),AK1(80,80)
      CALL YARABA (AK,AKNG,Z,160,AK1,80,KW,KWW,KTT)
  999 STOP
      END
      SUBROUTINE YARABA(AK,AKNG,Z,ND1,AK1,ND2,KW,KWW,KTT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION TN1(8,8),TZ(4,4),T(16),U(16),TN(16,16),E(20),PR(20),
     1TH(20),ROA(20),AA(20),BB(20),KTYPE(90),IE(90,4),R1(300),NU(90,8),
     2S1(8,8),P(16,16),Q(16,16),PQ(16,16),TR(16,16),PP(16,16),QQ(16,16),
     3PPQQ(16,16),FXX(90),FYY(90),FXY(90),SX(4),SY(4),SM(8,8),NR(8),
     4S2(16,16),S(16,16),KW(80,16),AN(16,16),NODE(90,4),NDU(300),
     5NOW(500),EIGEN(500),DD(500),E1(500),E2(500),DD(500),EIGE(10)
      DIMENSION W(16,16),W1(8,8)
      DIMENSION AK(ND1,ND1),AKNG(ND1,ND1),Z(ND1,ND1),AK1(ND2,ND2)

      DIMENSION FOUND(10)
      DIMENSION TITLE (18)
C
C
C THIS SUBROUTINE TO SOLVE THE BUCKLING PROBLEM OF A PLATE ON ELASTIC
C FOUNDATION USING EIGEN VALUE APPROACH
C  THE FOMULATION USED IN THIS ANALYSIS IS LARGE DEFLECTION APPROACH
C AND LINEAR (ELASTIC) MATERIAL BEHAVIOUR
C
C
C
      FOUND(1)=0.0336
      FOUND(2)=0.00336
      FOUND(3)=0.336
      FOUND(4)=3.36
      FOUND(5)=33.600
      FOUND(6)=336.0
      FOUND(7)=3360.0
      FOUND(8)=33600.0
      FOUND(9)=336000.0
      FOUND(10)=0.336E 07
      NKK=ND1
      NKI=ND2
      NKJ=ND2
      READ1,TZ
      PRINT1,TZ
      CALL INVERS(4,4,TZ,1)
      DO 457 I=1,4
      DO 457 J=1,4
      TN1(I+0,J+0)=TZ(I,J)
      TN1(I+4,J+4)=TZ(I,J)
      TN1(I+4,J+0)=0.0
      TN1(I+0,J+4)=0.0
  457 CONTINUE
      READ 1,(T(I),I=1,16)
      READ 1,(U(I),I=1,16)
    1 FORMAT (16F5.1)
      READ 1,((TN (I,J),J=1,16),I=1,16)
 9999 READ 100,NPROB,TITLE
      IF(NPROB.EQ.0) GO TO 999
  100 FORMAT (I5,3X,18A4)
      PRINT 101,NPROB,TITLE
  101 FORMAT (1H1,I5,5X,18A4)
      PRINT 717
C
C READING THE INPUT DATA
C
      READ 111,NNP,NEL,NTYPE,FOUN,NCONG,DEM
```

```
111 FORMAT (3I5,E15.8,I5,E15.8)

      PRINT 2,NNP,NEL,NTYPE,FOUN
    2 FORMAT (10X,4HNNP=,I5/10X,4HNEL=,I5/10X,6HNTYPE=,I5/10X,5HFOUN=,
     1E15.7//////)
      READ 3,(E(I),PR(I),TH(I),ROA(I),AA(I),BB(I),I=1,NTYPE)
    3 FORMAT (6F10.3)
      PRINT 4,(E(I),PR(I),TH(I),ROA(I),AA(I),BB(I),I=1,NTYPE)
    4 FORMAT (10X,17HMODULUS OF ELAC.=,E15.7/10X,14HPOISSON RATIO=,
     1E15.7/10X,10HTHICKNESS=,E15.7/10X,10HB/A RATIO=,E15.7/10X,
     27HLENGTH=,E15.7/10X,6HWIDTH=,E15.7//////)
      READ 5,(NTYPE(K),(IE(K,I),I=1,4),K=1,NEL)
    5 FORMAT (5I5)
      PRINT 6,(NTYPE(K),(IE(K,I),I=1,4),K=1,NEL)
      PRINT 717
    6 FORMAT (5I5/)
      DO 57 I=1,NNP
      READ 58,(NODE(I,J),J=1,6)
      PRINT 58,(NODE(I,J),J=1,6)
   57 CONTINUE
      PRINT 717
   58 FORMAT (6I5)
C
C
C CALCULATION OF IN-PLANE PROBLEM
C
C
      NUV=2*NNP
      DO 99 I=1,NKI
      R1(I)=0.
      DO 99 J=1,NKJ
      AK1(I,J)=0.
   99 CONTINUE
      K1=1
      KK=0
      DO 81 JJ=1,2
      DO 81 II=1,NNP
      IF(NODE(II,JJ).EQ.0) GO TO 87
      KK=KK+1
      NDU(K1)=KK
      GO TO 81
   87 NDU(K1)=0
   81 K1=K1+1
      PRINT 59,(NDU(I),I=1,NUV)
   59 FORMAT (20I5/)
      PRINT 717
      KUV=KK
      PRINT 59,KUV
      PRINT 717
C
C
      DO 19 MTYPE=1,NTYPE
      A=AA(MTYPE)
      B=BB(MTYPE)
      PO=PR(MTYPE)
      BE=ROA(MTYPE)
      D1=E(MTYPE)*TH(MTYPE)/(1.0-PO**2)
      ZB=(1.0-PO)/2.0
      ZA=(1.0+PO)/2.0
      DO 458 I=1,8
      DO 458 J=1,8
  458 S1(I,J)=0.0
      S1(2,2)=BE
      S1(2,4)=BE/2.0
      S1(2,7)=PO
```

```
      S1(2,8)=P0/2.0
      S1(3,3)=ZB/BE
      S1(3,4)=S1(3,3)/2.0

      S1(3,6)=ZB
      S1(3,8)=ZB/2.0
      S1(4,4)=(BE+ZB/BE)/3.0
      S1(4,6)=ZB/2.0
      S1(4,7)=P0/2.0
      S1(4,8)=ZA/4.0
      S1(6,6)=ZB*BE
      S1(6,8)=ZB*BE/2.0
      S1(7,7)=1.0/BE
      S1(7,8)=0.5/BE
      S1(8,8)=(1./BE+ZB*BE)/3.0
      DO 459 I=1,8
      DO 459 J=1,8
      S1(I,J)=S1(I,J)*D1
  459 S1(J,I)=S1(I,J)
      CALL REDUCE(*1,S1,TN1,8,8)
C
      DO 22 K=1,NEL
      IF(KTYPE(K).NE.MTYPE) GO TO 22
      DO 15 I=1,4
      NX=  IE(K,I)
      NXX=  IE(K,I)         +NNP
      NU(K,  I  )=NDU(NX)
      NU(K,4+I)=NDU(NXX)
   15 CONTINUE
      DO 72 I=1,8
      LX=NU(K,I)
      DO 73 J=1,8
      LY=NU(K,J)
      IF (LX.EQ.0.OR.LY.EQ.0) GO TO 73
      LQ=LY-LX+1
      IF(LQ.LE.0) GO TO 73
      AK1(LX,LQ)=AK1(LX,LQ)+S1(I,J)
   73 CONTINUE
   72 CONTINUE
   22 CONTINUE
   19 CONTINUE
C
C
      DO 26 MN=1,NCOND
   25 READ 44,I,R1(I)
   26 CONTINUE
      PRINT 333,(R1(I),I=1,NUV)
      PRINT 717
   44 FORMAT (I5,F10.4)
C
C
      CALL BANSOL (1,AK1,R1,KUV,KUV,NK1,NKJ)
      CALL BANSOL (2,AK1,R1,KUV,KUV,NK1,NKJ)
C
  333 FORMAT (10X,E15.7,10X,E15.7,10X,E15.7,10X,E15.7/)
      PRINT 717
C
C
C CALCULATION OF THE AVERAGE STRAIN STRESSES IN EACH ELEMENT
C
C
      DO 991 MTYPE=1,NTYPE
      A=AA(MTYPE)
      B=BB(MTYPE)
      EF=E(MTYPE)
```

```
      PO=PR(MTYPE)
      DO 79 K=1,NEL
      IF(KTYPE(K).NE.MTYPE) GO TO 79
      DO 777 I=1,4
      LX=NU(K,I)


      IF(LX.EQ.0) GO TO 771
      SX(I)=R1(LX)
      GO TO 772
  771 SX(I)=0.
  772 LY=NU(K,I+4)
      IF (LY.EQ.0) GO TO 773
      SY(I)=R1(LY)
      GO TO 777
  773 SY(I)=0.
  777 CONTINUE
C
      EXX=(1./(2.*A))*(-SX(1)+SX(2)+SX(3)-SX(4))
      EYY=(1./(2.*B))*(-SY(1)-SY(2)+SY(3)+SY(4))
      EXY=(-1./(2.*B))*(SX(1)+SX(2)-SX(3)-SX(4))+(-1./(2.*A))*
     1(SY(1)-SY(2)-SY(3)+SY(4))
C
      FXX(K)=(EE/(1.-PO**2))*(EXX+PO*EYY)*TH(MTYPE)
      FYY(K)=(EE/(1.-PO**2))*(EYY+PO*EXX)*TH(MTYPE)
      FXY(K)=(EE/(2.*(1.+PO)))*EXY*TH(MTYPE)
  335 FORMAT (10X,I5,3E15.7///)
   79 CONTINUE
  991 CONTINUE
C
C
C CALCULATION OF OUT OF PLAN PROBLEM
C
C
      DO 17 I=1,16
      DO 17 J=1,16
      P(I,J)=       FF(T(I)+T(J)-2., U(I)+U(J))*T(I)*T(J)
      Q(I,J)=      FF(T(I)+T(J),U(I)+U(J)-2.)*U(I)*U(J)
      PQ(I,J)=      (FF(T(I)+T(J)-1.,U(I)+U(J)-1.)*T(I)*U(J)+
     1        FF(T(I)+T(J)-1.,U(I)+U(J)-1.)*T(J)*U(I))
   17 CONTINUE
      DO 103 NFOUN=6,10
      FOUN=FOUND(NFOUN)
      K1=1
      KK=0
      DO 887 J=1,4
      DO 86 I=1,NNP
      IF ( NODE(I,J+2).EQ.0) GO TO 88
      KK=KK+1
      ND*(K1)=KK
      GO TO 86
   88 ND*(K1)=0
   86 K1=K1+1
      IF (J.EQ.1) K**=KK
  887 CONTINUE
      KW=KK
      DO 98 I=1,KW
      DO 98 J=1,KW
      AKNL(I,J)=0.
      AK(I,J)=0.
   98 CONTINUE
C
C
      DO 31 MTYPE=1,NTYPE
      A=AA(MTYPE)
      B=BB(MTYPE)
```

```
      EE=E(MTYPE)
      PO=PR(MTYPE)
      D=(EE*TH(MTYPE)**3)/(12.*(1.-PO**2))
      DO 29 I=1,4
      DO 29 J=1,16
      TR(I,J)=TN(I,J)
      TR(I+4,J)=TN(I+4,J)/A
      TR(I+8,J)=TN(I+8,J)/B
      TR(I+12,J)=TN(I+12,J)/(A*B)
   29 CONTINUE
      CALL INVERS (16,16,TR,1)
C
C
C CALCULATION OF OUT OF PLAN STIFFNESS MATRIX AND ELASTIC FOUND. MATRIX
C
C
      V=PR(MTYPE)
      VW=2.*(1.-V)
      DO 43 I=1,16
       DO 43 J=1,16
      F4=(1.0 /(A*B))*((B/A)**2*T(I)*(T(I)-1.)*T(J)*(T(J)-1.)*FF(T(I)+T(
     1J)-4.,U(I)+U(J))+(A/B)**2*U(I)*(U(I)-1.)*U(J)*(U(J)-1.)*FF(T(I)+T(
     2J),U(I)+U(J)-4.)+FF(T(I)+T(J)-2.,U(I)+U(J)-2.)*(VW*T(I)*U(I)*T(J)*
     3U(J)+V*T(I)*(T(I)-1.)*U(J)*(U(J)-1.)+V*U(I)*(U(I)-1.)*T(J)*(T(J)-1
     4.)))
      S(I,J)=F4*D
      S(J,I)=S(I,J)
C
C
C CALCULATE ELASTIC FOUNDATION MATRIX
C
C
      F2=FF(T(I)+T(J),U(I)+U(J))*A*B
      S2(I,J)=F2*FOUN
      S2(J,I)=S2(I,J)
   43 CONTINUE
      CALL REDUCE (M,S,TR,16,16)
      CALL REDUCE (M,S2,TR,16,16)
C
C
C CALCULATION OF NON-LINEAR OUT OF PLAN STIFFNESS MATRIX
C
C
      DO 32 K=1,NEL
      IF (MTYPE(K).NE.MTYPE) GO TO 32
      DO 779 I=1,4
      N1=IE(K,I)
      N2=IE(K,I)+NNP
      N3=IE(K,I)+2*NNP
      N4=IE(K,I)+3*NNP
      NW(K,I)=NDW(N1)
      NW(K,I+4)=NDW(N2)
      NW(K,I+8)=NDW(N3)
      NW(K,I+12)=NDW(N4)
  779 CONTINUE
C
      DO 55 I=1,16
      DO 55 J=1,16
      AN(I,J)= P(I,J)*(B/A)*FXX(K)+ O(I,J)*(A/B)*FYY(K)+  PQ(I,J)*FXY(K)
   55 CONTINUE
      CALL REDUCE(M,AN,TR,16,16)
C
C ASSEMBLY OF STIFFNESS MATRIX
C
```

```
      DO 76 I=1,16
      LX=NW(K,I)
      DO 76 J=1,16
      LY=NW(K,J)
      IF (LX.EQ.0.OR.LY.EQ.0) GO TO 76
      AK(LX,LY)=AK(LX,LY)+S(I,J)+S2(I,J)
      AKNL(LX,LY)=AKNL(LX,LY)+AN(I,J)
   76 CONTINUE
   32 CONTINUE

   31 CONTINUE
C
      KTT=K*-KWW
C
      CALL MATRDI (AK,AKNL,KWW,KTT,NKK)
C
C
C CALLING THE EIGEN VALUE SUBROUTINE
C
C
      PRINT 717
      PRINT 333,FOUN
      PRINT 717
      CALL DEVIN (AKNL,AK,EIGEN,Z,DL,DD,E1,E2,5,5,KWW,NKK)
C
      PRINT 717
      PRINT 59,KTT,K*,KWW
      PRINT 717
C
C
      DO 102 I=1,5
      EIGE(I)=-1.0/EIGEN(I)
      PRINT 335,I,EIGE(I)
  102 CONTINUE
  103 CONTINUE
      GO TO 9999
  717 FORMAT (/////)
  999 RETURN
      END
      SUBROUTINE BANSOL(KKK,AK,R,NEQ,IBAND,NDIM,MDIM)
      IMPLICIT REAL*8(A-H,O-Z)
C SYMMETRIC BAND MATRIX EQUATION SOLVER.   (REF. 2)
C
C KKK = 1 TRIANGULARIZES THE BAND MATRIX AK, EQ. (2-2)
C KKK = 2 SOLVES FOR RIGHT HAND SIDE R, SOLUTION RETURNS IN R, EQ.(2-3)      2
C
      DIMENSION AK(NDIM,MDIM), R(NDIM)
            NRS = NEQ - 1
            NR = NEQ
         IF(KKK.EQ.2) GO TO 200
         DO 130  N=1,NRS
            M= N-1
            MR = MIN0(IBAND,NR-M)
           PIVOT = AK(N,1)
         DO 120 L=2,MR
            CP= AK(N,L)/PIVOT
          IF (CP) 100,120,100
  100       J = 0
            I = M+L
          DO 110 K=L,MR
            J=J+1
  110       AK(I,J)= AK(I,J) -CP*AK(N,K)
  120       AK(N,L) = CP
  130     CONTINUE
      GO TO 400
```

```
  200     DO 220 N=1,NRS
               M= N-1
               MR = MIN0(IBAND,NR-M)
               CP= R(N)
               R(N)=CP/AK(N,1)
          DO 220  L=2,MR
               I  =  M + L
  220          R(I)= R(I) - AK(N,L)*CP
               R(NR) = R(NR)/AK(NR,1)
          DO 320 I =1,NRS
               N= NR- I


               M= N-1
               MR = MINO(IBAND,NR-M)
          DO 320  K = 2,MR
               L = M+K
C STORE COMPUTED DISPLACEMENTS IN LOAD VECTOR R
  320          R(N)= R(N)- AK(N,K)*R(L)
  400 RETURN
      END
      SUBROUTINE REDUCE (A,B,C,NB,NA)
      IMPLICIT REAL*8(A-H,O-Z)
C     SUBROUTINE OVER WRITES  B=C'*B*C  WHERE C'=C TRANSPOSE      NA<NB
C     A=B*C
      DIMENSION A(NB,NA),B(NB,NB),C(NB,NA)
      DO 1 I=1,NB
      DO 1 J=1,NA
      A(I,J)=0.0
      DO 1 K=1,NB
    1 A(I,J)=A(I,J)+B(I,K)*C(K,J)
      DO 2 I=1,NA
      DO 2 J=1,NA
      B(I,J)=0.
      DO 2 K=1,NB
    2 B(I,J)=B(I,J)+C(K,I)*A(K,J)
      RETURN
      END
      FUNCTION FF(X,Y)
      IMPLICIT REAL*8(A-H,O-Z)
      IF(X.EQ.-1..OR.Y.EQ.-1.) GO TO 2
      FF=1./((X+1.)*(Y+1.))
      GO TO 1
    2 FF=1.0
    1 RETURN
      END
      SUBROUTINE MATRU1(A,B,NA,NB,IDIM)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(IDIM,IDIM),B(IDIM,IDIM)
      IF(NB.EQ.0) GO TO 999
      NC=NA+NB
      NAP1=NA+1
       CALL INVERS(NC,IDIM,A,NAP1)
      DO 92 I=1,NB
      DO 92 J=1,NA
      A(I+NA,J)=0.0
      DO 92 K=1,NB
   92 A(I+NA,J)=A(I+NA,J)+A(I+NA,K+NA)*A(J,K+NA)
      DO 93 I=1,NA
      DO 93 J=1,NA
      DO 93 K=1,NB
   93 A(I,J)=A(I,J)-A(I,K+NA)*A(K+NA,J)
      DO 94 I=1,NB
      DO 94 J=1,NA
      B(I+NA,J)=0.0
      DO 94 K=1,NB
```

```
 94 B(I+NA,J)=B(I+NA,J)+B(I+NA,K+NA)*A(K+NA,J)
    DO 95 I=1,NA
    DO 95 J=1,NA
    DO 95 K=1,NB
 95 B(I,J)=B(I,J)+A(K+NA,I)*(B(K+NA,J)-B(J,K+NA))-B(I,K+NA)*A(K+NA,J)
999 RETURN
    END
```

SUBROUTINE DEVIN

is a eigenvalue subroutine to calculate the first five eigenvalues and the corresponding modes shapes .