TOPOLOGICAL METHODS IN NUMBER THEORY: A DISCUSSION OF GAUSSIAN INTEGERS AND PRIMES

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TOPOLOGICAL METHODS IN NUMBER THEORY: A DISCUSSION OF GAUSSIAN INTEGERS AND PRIMES

by

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ABSTRACT

A problem which has enthralled mathematicians through the ages is that of deciding the cardinality of the set of primes of the form n^2+1. This thesis deals with this problem from a topological standpoint.

Chapter I discusses the hereditary properties of topological spaces which are most applicable to the spaces used and the problem at hand. Its objective is to make available, information which is useful for producing counterexamples.

In Chapter II several topological structures on the rational integers are discussed and extended to the Gaussian integers. Also, mention is made of how these topological structures can be extended to general algebraic number fields. The properties of these topologies, along with those of another one, are given, and also various subspaces are discussed.

A discussion of the applications of the topological structures described in the previous chapter make up the contents of Chapter III. Also, some properties of the topologies on the rational integers are discussed and their generalizations are given.

Chapter IV changes our problem from one of finding infinitely many prime numbers to that of discussing which properties a topological structure should have in order to be useful for solving the problem.

The final chapter, Chapter V, gives some different approaches for tackling the problem. One of these approaches involves some algebraic topology while others remain entirely within the field of number theory.

Although no major conclusions are drawn from the paper it attempts to use topological methods and ideas in dealing with this and related problems in number theory.

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TABLE OF CONTENTS

Introduction 1

Chapter I
Hereditary Properties of Topological Spaces 3

Chapter II
Three Useful Topological Structures on the Gaussian
Integers 10

Chapter III
Applications of the Three Topologies 31

Chapter IV
Requirements for a Useful Topology 35

Chapter V
Some New Approaches to the Problem 38

Bibliography 45

LIST OF TABLES

Table 1.1
Table of Hereditary Properties of Topological Spaces 6

Table 2.1
Properties of Topological Spaces on $\not{C}$ and Induced
Topologies on its Subspaces 29

LIST OF FIGURES

Figure 2.1
The Gaussian Plane 11

LIST OF ABBREVIATIONS

$\mathbb{Z}$ = the set of rational integers

$\mathbb{N}^+$ = the set of natural numbers

$\mathbb{N} = \mathbb{N}^+ \cup \{0\}$

$\mathbb{N}_k = \{a+ki \mid a \in \mathbb{N}^+\}$

$\mathbb{G}$ = the set of Gaussian integers

$\mathbb{G}^+ = \{a+bi \mid a \in \mathbb{N}^+, b \in \mathbb{N}\}$

$\mathbb{P}$ = the set of prime integers

$\mathbb{S}$ = an arbitrary finite subset of $\mathbb{P}$

$\overline{U} = \mathrm{Cl}(U)$ = the closure of the set U

$\mathrm{Int}(U)$ = the interior of the set U

δU = the boundary of the set U

Introduction

This paper was written as an attempt to answer two questions. The first is one of antiquity: "Are there infinitely many primes of the form n^2+1?". This question is referred to throughout the paper as the "problem in question" or the "problem at hand". The other question arose from a paper by S. W. Golomb which appeared in Volume 66 of the American Mathematical Monthly in 1959. In his paper entitled "A Connected Topology for the Integers", Golomb used topological methods to prove the infinitude of the primes and the fact that the existance of a prime in all arithmetic progressions of the form a+bn where a and b are relatively prime, is equivalent to the statement of Direchlet's Theorem. From this work came the question: "Can topological methods be used to answer similar questions about the integers?".

In this thesis we work towards answers to these two questions by considering topological structures on the Gaussian integers and those induced on such subspaces as the rational integers.

Definitions of required topological properties and types of heredity are stated in chapter I. We then proceed with these definitions to prove which properties are hereditary and which ones are not. Our conclusions are tabulated within the chapter to make easy reference possible.

In Chapter II we discuss several topological structures on the Gaussian integers and suggest ways of extending them to general algebraic number fields. The first topology is due to H. Furstenberg and appeared in 1955 in Volume 62 of the American Mathematical Monthly. The second is the one by S. W. Golomb which is mentioned above.

Both these topologies are defined on the rational integers but we extend them to the Gaussian integers and also consider the topologies of several subsets of this space. The third topology is similar to the others but it is defined using closed sets rather than open sets. The properties of all spaces and subspaces are examined and tabulated at the end of the chapter.

These three topologies are investigated further in Chapter III. Here we look at their usefulness in solving particular problems by discussing the density of the primes, the number of primes in each open set and other similar topics. We again refer to Golomb's paper and extend his theorems to the Gaussian integers.

Chapter IV gives a different outlook on the problem. In this chapter, rather than discuss a particular topology, we decide which properties we require on a topology in order that it be useful for proving the problem at hand. Our problem then becomes one of proving the existance of a single topology instead of proving the existance of an infinity of primes.

The final chapter deals with different approaches to the problem. Here we mention such things as the use of cohomology theory, the application of Pythagorean triples and the properties of Egyptian fractions, all of which suggest ways of solving the problem.

The paper draws no concrete conclusions with respect to the first question but does illustrate several different ways of viewing it. A partial answer to the second question is given by actually extending the topology to the Gaussian integers and answering questions which are analogue to those posed by Golomb in his paper.

Chapter I

Hereditary Properties of Topological Spaces

Topology can be defined as the study of the properties of spaces. So, because of their importance to this branch of mathematics, this chapter deals exclusively with topological properties.

We define an hereditary property as a topological property which is preserved by every subspace of a space originally having the property.

Using hereditary properties we can discover much information about subspaces and also we can produce counterexamples.

To illustrate the effectiveness of hereditary properties we have the following:

Example: We let X represent a space and A represent one of its subspaces. We wish to prove that A has the property p. By assuming the negative of p we find that A will satisfy certain topological properties and will not satisfy others. Now if q is an hereditary property and X has q, but under our negative hypothesis A does not have q, we have a proof of our hypothesis.

Hence, a table of hereditary properties would be quite helpful for using the idea mentioned above.

The properties which are listed below are those which are most applicable to the problem at hand and because of the inconsistency in the literature of the definitions of these properties, they are also stated. Thus we have the following:

Definition 1: A space, (X, τ), is T_0 if for every pair of points $a, b \in X$ there exists an open set $O \in \tau$ such that either $a \in O$ and $b \notin O$ or $a \notin O$ and $b \in O$.

Definition 2: A space, (X, τ), is T_1 if for every pair of points $a, b \in X$ there exist open sets O_a and O_b in τ such that $a \in O_a$ and $b \in O_b$ but $a \notin O_b$ and $b \notin O_a$.

Definition 3: A space, X, is T_2 if for every pair of points $a, b \in X$ there exist disjoint open sets O_a and O_b which contain a and b respectively.

Definition 4: A space, X, is T_3 if for every closed set A and point b not in A, there exist disjoint open sets O_A and O_b which contain A and b respectively.

Definition 5: A space, X, is T_4 if for every pair of disjoint closed sets A and B in X, there exists a pair of disjoint open sets O_A and O_B which contain A and B respectively.

Definition 6: A space, X, is T_5 if for every pair of seperated sets A and B in X, there exist disjoint open sets O_A and O_B containing A and B respectively.

Definition 7: A space is regular if and only if it is both T_0 and T_3.

Definition 8: A space is normal if and only if it is both T_1 and T_4.

Definition 9: A space is compact if every open cover contains a finite subcover.

Definition 10: A space is locally compact if each point is contained in a compact neighbourhood.

Definition 11: A space is paracompact if every open cover has an open locally finite refinement.

Definition 12: A space is connected if and only if it is not the union of two seporated sets.

Definition 13: A space is path connected if for every pair of points a and b there exists a continuous function, f, from the unit interval such that f(0)=a and f(1)=b.

Definition 14: A space is locally connected if it has a basis consisting of connected sets.

We also state the following:

Definition A: A topological property is hereditary if every subspace has the property whenever the space has it.

Definition B: A topological property is F-hereditary if every closed subspace has the property whenever the space has it.

Definition C: A topological property is G-hereditary if every open subspace has the property whenever the space has it.

Definition D: A topological property is C-hereditary if every continuous function preserves it.

Definition E: A topological property is q-hereditary if every quotient space has the property whenever the space has it.

Definition F: A topological property is finite P-hereditary if every finite product of spaces with the property also has the property.

Table 1.1

Table of Hereditary Properties of Topological Spaces

Property	Hereditary	F Hereditary	G Hereditary	C Hereditary	q Hereditary	finite P Hereditary
T_0	+	+	+	−	−	+
T_1	+	+	+	−	−	+
T_2	+	+	+	−	−	+
T_3	+	+	+	−	−	+
T_4	−	+		−	−	−
T_5	+	+	+	−		−
Regular	+	+	+	−		+
Normal	−	+	−	−	−	−
Compact	−	+	−	+	+	+
Locally Compact	−	+	−	−	−	+
Paracompact	−	+	−	−		−
Connected	−	−	−	+	+	+
Path Connected	−	−	−	+	+	+
Locally Connected	−		+	−	+	+

On the previous page in Table 1.1 we have listed the topological properties and indicated whether or not they are hereditary. In the table a "+" indicates that the property is hereditary, a "-" that it is not hereditary and a "blank" that it has not been determined.

If we now number the rows and columns of the table in the same way as we did the definitions we see that the entries:

A1, A2, A3, A7, A8, A11, B8, B9, B10, B11, C8, C11, D9, D10, D11, D12, D14, F1, F2, F3, F7, F8, F9, F10, F11, and F12 can all be found in <14>.

A4, A6, C10, C14, E10, F4, F14, can all be found in <6>.

B5, E1, E8, and E14 can be found in <20>.

D13 and F13 can be found in <9>.

E3, E4 and E5 can be found in <10>.

B1, B2, B3, B4, B6, B7, C1, C2, C3, C4, C6 and C7 all follow from the fact that every hereditary property is also F-hereditary and G-hereditary.

B12 and B13 are proven by giving X the particular point topology (the only open sets are those containing a particular point), which is both connected and path connected. We take Y such that it does not contain the particular point. Now Y is closed and has the discrete topology which is neither connected nor path connected.

E2 is proven by giving X the finite complement topology (the only open sets are those having finite complements) which is T_1. We take a finite set U and partition X by taking each point of U as a component and its complement as a component. Now any set V with $(X\setminus U) \subset V$ is open and any set W with $(X\setminus U) \cap W = \emptyset$ is closed. This is just the particular point topology on a finite set and it is not T_1.

F5 and F6 are proven by giving X the right half-open interval topology (basis given by sets of the form [a,b)) which is both T_4 and T_5. However, the product topology on X x X is neither T_4 nor T_5.

C9 is proven by giving X the excluded point topology (the only open sets other than X are those which do not include a given point) which is compact. Now any subset which does not contain the excluded point is open and has the discrete topology which is not compact.

A5 is proven by considering the divisor topology on N^+ (basis given by sets of the form U_n = the set of all divisors of n) which is T_4. If we now consider the prime multiples of 2 as a subspace, it has the particular point topology which is not T_4.

D1, D2, D3, D7 and D8 all follow from the fact that every function into the indiscrete topology is continuous and this topology is neither T_0, T_1, T_2, regular nor normal.

D4, D5, and D6 follow from the fact that every function from the discrete topology is continuous and this topology is T_3, T_4 and T_5.

C12 and C13 are proven by giving X the excluded point topology which is connected and path connected, whereas any open set is neither connected nor path connected.

E9, E12 and E13 follow from the fact that all C-hereditary properties are q-hereditary.

A9, A10, A12 and A13 follow from the fact that if a property is not G-hereditary then it is not hereditary.

A14 is proven by giving [-1,1] the overlapping interval topology.(generated by sets of the form [-1,b) for b>0 and (a,1] for a<0) which is locally connected. Then we have that the subspace topology on [0,1] is the right half-open interval topology which is not locally connected.

Thus, we have the table completed and we illustrate its usefulness by the following:

Example: Suppose we have a space X which is both T_2 and connected. However, we require a topology on X which is connected but not T_2. Using the table we see that X x X will be connected and T_2, but now if we project this topology onto X we have that the resulting topology is connected since the property is C-hereditary but the space is not necessarily T_2. This gives a possible, easy construction of our required space. Another construction which gives the same results is that of taking the quotient topology on X.

Chapter II

Three Useful Topological Structures on the Gaussian Integers

In this chapter we will consider extensions of topologies on the rational integers to topological structures on the Gaussian integers. Because the Gaussian integers are more complex than the rational integers, we have to be careful of our definitions since a seemingly non-trivial basis may lead to a trivial topology.

To illustrate how such a trivial topology can be defined we take a proof which was given by Solomon W. Golomb in the American Mathematical Monthly in 1959, change it slightly but still leave it valid, and then extend it to the Gaussian integers and note the consequences. Hence, we state the following:

Theorem: There are infinitely many primes.
Proof: Since the set of integers is symmetric about the origin we need only consider the set $\mathbb{N}^+$ of positive integers. We define a basis for a topology on $\mathbb{N}^+$ by declaring open all sets $B(a,b)$ = the set of integers of the form $a+bn$ where $a,b \in \mathbb{N}^+$, $n \in \mathbb{N}$, and $(a,b)=1$. We see that $\mathcal{B} = \cup B(a,b)$ covers $\mathbb{N}^+$ and $B(a,b) \cap B(a',b') = \emptyset$ or $B(a,b) \cap B(a',b') = B(q,[b,b'])$ where q is the smallest element in the intersection and $[b,b']$ is the least common multiple of b and b'. Hence, we see that $\mathcal{B}$ does indeed form a basis for a topological structure on $\mathbb{N}^+$. Now since $\{1\} = \mathbb{N}^+ \setminus B(2,1)$ it is a closed subset of $\mathbb{N}^+$. We consider $A_p = \{p, 2p, 3p, \ldots\}$ where p is prime, and we note that $\mathbb{N}^+ \setminus A_p = B(1,p) \cup B(2,p) \cup \ldots \cup B(p-1,p)$ and hence A_p is closed. If we assume there are only finitely many primes, $\{p_1, p_2, \ldots, p_k\}$ we have that $Ap_1 \cup Ap_2 \cup \ldots \cup Ap_k$ is closed, but $\mathbb{N}^+ \setminus Ap_1 \cup \ldots \cup Ap_k = \{1\}$ which is also closed. Now this impossible since $\mathbb{N}^+$ is

connected. Therefore, we have a contradiction and hence there are infinitely many primes.

An extension of this proof to the Gaussian integers results in the following:

Theorem: There are infinitely many Gaussian primes.
Proof: If we consider the Gaussian integers in the same way we considered the rational integers we see that the Gaussian plane can be divided into four symmetric parts, namely, the four quadrants. We take $\mathcal{G}^+$ to be the quadrant in which the integers a+bi are such that $a \in \mathcal{N}^+$ and $b \in \mathcal{N}$.

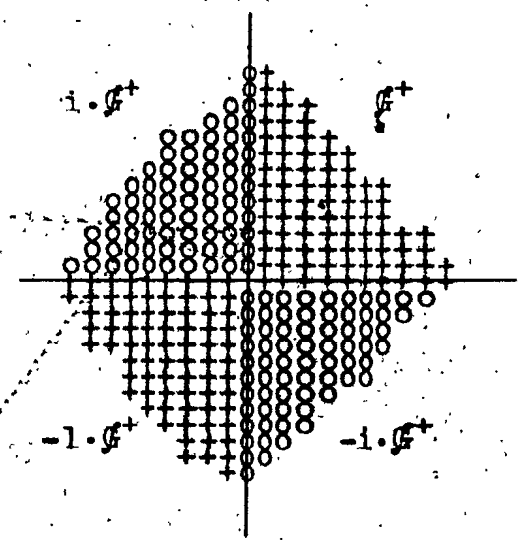

Figure 2.1
The Gaussian Plane

As we can see, if we take the union $\mathcal{G}^+ \cup i \cdot \mathcal{G}^+ \cup -1 \cdot \mathcal{G}^+ \cup -i \cdot \mathcal{G}^+$ we see that we have all the Gaussian integers except 0. (This is analogue to $\mathcal{N}^+ \cup -1 \cdot \mathcal{N}^+$.) Hence, we need only extend our topology on $\mathcal{N}^+$ to a topology on $\mathcal{G}^+$. We define a topology on $\mathcal{G}^+$ by declaring open all sets $B(\alpha,\beta)$ = the set of all Gaussian integers of the form $\alpha+\delta\beta$ where $\alpha, \beta \in \mathcal{G}^+$, $(\alpha,\beta) = 1$, $\delta \in \mathcal{G}^+ \cup \langle 0 \rangle$ and $\alpha+\delta\beta \in \mathcal{G}^+$. The restriction that $\alpha+\delta\beta \in \mathcal{G}^+$ is necessary since $\mathcal{G}^+$ is not closed under multiplication whereas $\mathcal{N}^+$ is. Now $\mathcal{B} = \cup B(\alpha,\beta)$ covers $\mathcal{G}^+$ so to show that $\mathcal{B}$ is a basis we need to show that $B(\alpha,\beta) \cap B(\alpha',\beta') = B(\bar{\alpha},\bar{\beta})$ or $\emptyset$.

If at this point we assumed that $\mathcal{B}$ was indeed a basis for the topology our proof would continue as it did in the rational case. However, we look more closely at $\mathcal{B}$. We find that $\mathcal{N}_k = \{a+ki \mid a \in \mathbb{N}^+\} = B((k+1)i,1) \cap B(ki+1,-i)$, and $\mathcal{M}_k = \{k+ai \mid a \in \mathbb{N}\} = B(k-1,i) \cap B(k+i,1)$. Hence, we have that for all k, $\mathcal{N}_k$ is open and $\mathcal{M}_k$ is open. Now if we take a point $a+bi \in \mathbb{G}^+$ then we have that $\{a+bi\} = \mathcal{N}_b \cap \mathcal{M}_a$ and is therefore open. So we see that $\mathbb{G}^+$ has the discrete topology, one quite different from the one we wished to extend.

Since the topology arose from one on the space of rational integers and our problems resulted from the fact that the positive rational integers are closed with respect to multiplication, whereas $\mathbb{G}^+$ is not. Therefore, it seems reasonable that, if we wish to define a topology on $\mathbb{G}^+$, we first define one for the whole of the Gaussian integers and then consider the subspace topology.

This is what we will do with the topology Harry Furstenberg used in his paper On the"Infinitude of Primes".[1] Furstenberg introduces a topology on the set of rational integers by using arithmetic progressions as a basis. With this topology, which makes $\mathbb{Z}$ normal and metrizable, he proves that there are infinitely many primes.

We will now look closely at this space and discuss the topological properties it has. In our discussion B(a,b) will represent the arithmetic progression a+nb. Hence, we have the following:

Theorem 1: The space is T_0, T_1, and T_2.
Proof: Pick c such that $(a-b,c) = 1$. Then $a \in B(a,c)$, $b \in B(b,c)$ and $B(a,c) \cap B(b,c) = \emptyset$. Hence, the space is T_2 and therefore T_1 and T_0.

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1. American Mathematical Monthly, 62 (1955), 353.

Theorem 2: The space is T_3, T_4, T_5, regular and normal.
Proof: The space is metrizable and hence it is T_5. A T_5 space is T_4. Since the space is T_1 and T_4, it is T_3 and normal. A T_0, T_3 space is regular.

Theorem 3: The space is not compact.
Proof: We cover $\mathbb{Z}$ by open sets consisting of the multiples of primes. This is an infinite open cover having no finite subcover and hence the space is not compact.

Theorem 4: The space is not locally compact.
Proof: We take an arbitrary neighbourhood $B(a,b)$ of c and show that it is not compact. First, we cover part of $B(a,b)$ with $B(a,2b)$. Then we take the number closest to a which is not covered. If there are several we take either, say d. We then use the open set $B(d,3b)$ as part of our cover. Continuing in this manner we have an infinite open cover which has no finite subcover and hence we find that the space is not locally compact.

Theorem 5: The space is neither connected nor path connected.
Proof: The sets $B(1,2)$, $B(2,2)$ form a non-trivial separation of the space and hence it is not connected. A space which is not connected is not path connected.

Theorem 6: The space is not locally connected.
Proof: $B(a,b) = B(a,2b) \cup B(a+b, 2b)$, i. e. we have a non-trivial seperation of an arbitrary neighbourhood.

We now consider an extension of this topology to the Gaussian integers and we will call this new topology the Open Set Topology (O. S. T.). We define this topology by taking as a basis the open sets $B(\alpha,\beta)$ = Gaussian integers of the form $\alpha+\delta\beta$ where α, β, $\delta \in \mathbb{G}$. We see that this is indeed a basis when we note that if $B(\alpha,\beta) \cap B(\alpha',\beta')$

$\neq \emptyset$ then it is $B(\bar{\alpha},\bar{\beta})$ where $\bar{\alpha}$ is some element in the intersection and $\bar{\beta}$ is the least common multiple of β and β'. Looking at the properties of this topology we have the following:

Theorem 7: The space is T_0, T_1 and T_2.
Proof: Pick $\gamma \in \mathcal{G}$ such that $(\alpha-\beta,\gamma) = 1$. Then $\alpha \in B(\alpha,\gamma)$, $\beta \in B(\beta,\gamma)$ and $B(\alpha,\gamma) \cap B(\beta,\gamma) = \emptyset$. Hence, the space is T_2 and therefore T_1 and T_0.

Theorem 8: The space is T_3 and regular.
Proof: Let A be closed in $\mathcal{G}$ and let a be a point which is in the complement of A. If A is finite we take the difference between all pairs of elements in $A \cup \langle a \rangle$. We then consider the set of norms of all these differences and pick the largest element in this set, say 1. Now $a \in B(a,1)$ and $A \subset \mathcal{G}\backslash B(a,1)$, both of which are disjoint open sets. If A is infinite then $A = \mathcal{G}\backslash(\cup B(\alpha,\beta)) = \cap(\mathcal{G}\backslash B(\alpha,\beta))$. If the intersection is finite then A is also open and $\mathcal{G}\backslash A$ is open giving us the required seperation. If the intersection is infinite we take an intersection of only finitely many of the sets, each not containing the point a. This gives us an open set containing A and having an open complement containing a. Hence, the space is T_3 and since it is also T_0 it is regular.

Theorem 9: The space is second countable.
Proof: Every basic open set is specified by a pair of Gaussian integers and this is a countable set. Hence, the space has a countable basis.

Theorem 10: The space is T_5, T_4, T_3, regular and normal.
Proof: According to <14> a second countable T_3 space is T_5 and hence T_4. A T_4, T_1 space is T_3 and normal.

Theorem 11: The space is not compact.

Proof: We cover $\mathbb{Z}$ by the open sets consisting of multiples of primes. Since there are infinitely many primes this is an infinite cover having no finite subcover, and hence the space is not compact.

Theorem 12: The space is not locally compact.
Proof: We take an arbitrary neighbourhood $B(\alpha,\beta)$ of γ and show that it is not compact. First, we cover part of the neighbourhood with $B(\alpha,2\beta)$. Then, we take the point closest to α which has not been covered. If there are several we use either, say δ. We then use $B(\delta,3\beta)$ in our cover. Continuing the process we arrive at an infinite cover for $B(\alpha,\beta)$ which has no finite subcover and hence the space is not locally compact.

Theorem 13: The space is neither connected nor path connected.
Proof: The sets $B(1,1+i)$, $B(2,1+i)$ form a non-trivial seperation for the space and hence it is not connected. Since the space is not connected it is not path connected.

Theorem 14: The space is not locally connected.
Proof: An arbitrary neighbourhood $B(\alpha,\beta)$ can be covered by the disjoint neighbourhoods $B(\alpha,(i+1)\beta)$ and $B(\alpha+\beta,(i+1)\beta)$.

We now consider the subspace topology on $\mathbb{Z}$ which is introduced by the Open Set Topology and decide how it compares with the topology used by Furstenberg. We find that all properties are the same and hence formulate the following:

Theorem: The "Furstenberg" Topology on $\mathbb{Z}$ and the subspace topology introduced by the Open Set Topology are the same.
Proof: First we show that every open set in the Furstenberg Topology is open in the O. S. T. on $\mathbb{Z}$. Take

$O = \cup B(a,b)$. We want to show that there is a $U \in$ O. S. T. on $\mathcal{G}$ such that $U \cap \mathbb{Z} = O$. For each $B(a,b) \subset O$ we pick $B(a,b) \subset U$ and then $B(a,b) \cap \mathbb{Z} = B(a,b)$ and $[\cup B(a,b)] \cap \mathbb{Z} = \cup B(a,b) = O$.
Now we show that every open set in the O. S. T. on $\mathbb{Z}$ is open in the "Furstenberg" Topology. Take $O = U \cap \mathbb{Z}$ where $U = \cup B(\alpha,\beta)$ and $U \in$ O. S. T. on $\mathcal{G}$. Now $B(\alpha,\beta) \cap \mathbb{Z} = B(a,|\beta|)$ where $a = \alpha+\delta\beta$ for some δ (if a does not exist then $B(\alpha,\beta) \cap \mathbb{Z} = \emptyset$). Hence, O has the form $\cup B(a,b)$ which is open in the "Furstenberg" Topology.

Therefore, since the two topologies are the same, the O. S. T. on $\mathcal{G}$ is indeed a generalization of the "Furstenberg" Topology.

Other useful subspaces of $\mathcal{G}$ are $\mathcal{G}^+$, $\mathbb{N}$, and $\mathbb{N}_1$, and these all have the same properties as the total space $\mathcal{G}$. We will return to this topological structure later and discuss its value in being applicable to particular problems in number theory.

The next topology we consider is the one used by Solomon W. Golomb in a paper entitled "A Connected Topology for the Integers".[2] He defined a topology on $\mathbb{Z}$ by taking as a basis arithmetic progressions of the form $an+b$ where a and b are relatively prime. This topological structure on $\mathbb{Z}$ was then used to prove the infinitude of the primes and the space was found to be T_2 and connected but neither T_3 nor compact.

We add to Golomb's analysis of this topology the following:

Theorem 14: The space is T_1 and T_0, but it is neither

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2. American Mathematical Monthly, 66 (1959), 663.

T_4, T_5, regular, normal, locally compact, nor paracompact.
Proof: A T_2 space is T_1 and T_0. The space is T_1 but not T_3 and hence not T_4. Since the space is not T_4 it is neither T_5 nor normal and since it is not T_3 it's not regular. The space is T_2 but not T_3 and hence neither locally compact nor paracompact.

Theorem 15: The space is not locally connected.
Proof: Suppose N is an open neighbourhood of 1 contained in $B(1,2)$; let $1+2n \in N$ for some $n > 0$. Then $U = B(1,2^{n+1})$ is an open subset of $B(1,2)$ whose relative complement V is open and contains $1+2n$ (since $V = B(1,2)\setminus B(1,2^{n+1}) = \cup_{i=1}^{2^n-1} B(1+2i,2^{n+1})$). Thus $U \cap N$ and $V \cap N$ seperate N, so $B(1,2)$ cannot contain any open connected neighbourhood of 1. Thus the space is not locally connected.[3]

Theorem 16: The space is not path connected.
Proof: Assume the space is path connected and $p:[0,1] \to N$ is a path connecting a and b. Now since N is T_1, p passes through a countable number of closed sets and their preimages under p must be disjoint closed subsets of [0,1]. Hence we have that [0,1] can be written as a countable union of disjoint closed sets which is a contradiction and therefore the space is not path connected.

We again extend this topology to one on the Gaussian integers. This new topology will be called the Relatively Prime Open Set Topology (R. P. O. S. T.) and is defined by open sets which are arbitrary unions and finite intersections of the sets $B(\alpha,\beta)$ = Gaussian integers of the form $\alpha+\delta\beta$ where $\alpha,\beta,\delta \in \mathcal{G}$ and $(\alpha,\beta) = 1$. Now we

3. Counterexamples in Topology, Lynn A. Steen and J. Arthur Seebach, Holt, Rinehart and Winston, Inc. (1970), 83.

prove the following:

Theorem: Sets of the form $B(\alpha,\beta)$ form a basis for the topology on $\mathscr{G}$.
Proof: Obviously $\mathscr{B} = \cup B(\alpha,\beta)$ covers $\mathscr{G}$. We need only prove that $B(\alpha,\beta) \cap B(\alpha',\beta') = B(\overline{\alpha},\overline{\beta})$ or $\emptyset$. Now either $B(\alpha,\beta) \cap B(\alpha',\beta') = \emptyset$ or $B(\alpha,\beta) \cap B(\alpha',\beta') \neq \emptyset$. If the intersection is not empty we have $\varphi \in B(\alpha,\beta) \cap B(\alpha',\beta')$ and we claim that $B(\alpha,\beta) \cap B(\alpha',\beta') = B(\varphi,[\beta,\beta'])$ where $[\beta,\beta']$ is the least common multiple of β and β' and it is defined because of unique factorization in $\mathscr{G}$. We now give a proof of the above claim. First, we show that $B(\alpha,\beta) \cap B(\alpha',\beta') \subset B(\varphi,[\beta,\beta'])$. We assume $x \in B(\alpha,\beta) \cap B(\alpha',\beta')$ and hence $x = \alpha+\delta\beta = \alpha'+\delta'\beta'$. But $\varphi \in B(\alpha,\beta) \cap B(\alpha',\beta')$ which gives $\varphi = \alpha+\delta_0\beta = \alpha'+\delta_0'\beta'$. Now we have $x-\varphi = (\delta-\delta_0)\beta = (\delta'-\delta_0')\beta'$. Therefore, $[\beta,\beta'] \mid (x-\varphi)$ and hence $x = \varphi+\overline{\delta}[\beta,\beta']$ for some $\overline{\delta} \in \mathscr{G}$. Hence $x \in B(\varphi,[\beta,\beta'])$. Next, we show that $B(\varphi,[\beta,\beta']) \subset B(\alpha,\beta) \cap B(\alpha',\beta')$. We assume $x \in B(\varphi,[\beta,\beta'])$ and $[\beta,\beta'] = \gamma\beta = \gamma'\beta'$. Now $x = \varphi+\theta[\beta,\beta'] = \alpha+\delta_0\beta+\theta\gamma\beta = \alpha'+\delta_0'\beta'+\theta\gamma'\beta'$ and therefore $x \in B(\alpha,\beta) \cap B(\alpha',\beta')$. Hence $\mathscr{B}$ is a basis for our topology.

As before, we now consider the following:

Theorem 17: The space is T_0 and T_4, but is neither T_1, T_2 nor T_3.
Proof: If $\alpha \neq 0 \neq \beta$ we pick γ such that $(\alpha-\beta,\gamma) = (\alpha,\gamma) = 1$. Then $\alpha \in B(\alpha,\gamma)$ and $\beta \notin B(\alpha,\gamma)$. If $\alpha = 0$ we pick γ such that $(\beta,\gamma) = 1$ and then $\beta \in B(\beta,\gamma)$ and $\alpha \notin B(\beta,\gamma)$. Similarly if $\beta = 0$. Hence the space is T_0. Since the only open set containing zero is the total space, the space is neither T_1 nor T_2. The space is not T_3 since every closed set contains zero and the only open set containing zero is the total space. The space is T_4 vacuously since there are no disjoint closed sets.

Theorem 18: The space is not T_5.
Proof: We consider the sets $\langle 1\rangle$ and $\langle 2\alpha\backslash 0\rangle$ which are seperated in $\not\mathcal{G}$. We have $\langle\overline{1}\rangle = \langle 1,0\rangle$ and $\langle\overline{2\alpha\backslash 0}\rangle = \langle 2\alpha\rangle$. If the set $\langle 1\rangle$ is covered by $\beta+\delta\gamma$ then we require that $\langle\beta+\delta\gamma\rangle \cap \langle 2\alpha\rangle = \emptyset$ and therefore we find that this open set must contain the progression $\langle 1+\delta\gamma\rangle$ where γ is even and hence $\gamma \in \langle 2\alpha\rangle$. Now if $\varphi+\delta\theta$ is in the open cover of $\langle 2\alpha\rangle$ and it contains γ we have that for some δ_0, $\gamma = \varphi+\delta_0\theta$. Since $(\theta,\varphi) = 1$ we have $(\theta,\gamma) = 1$ and hence $\langle\varphi+\delta\theta\rangle \cap \langle\beta+\delta\gamma\rangle$ is infinite. Hence, no disjoint neighbourhoods exist.

Theorem 19: The space is compact, locally compact and paracompact.
Proof: The total space must be contained in every open cover and hence every cover has a finite subcover. Thus, the space is compact and therefore locally compact and paracompact.

Theorem 20: The space is connected.
Proof: The only open - closed set in this topology is the total space.

Theorem 21: The space is not path connected.
Proof: Assume $f:[0,1] \longrightarrow \not\mathcal{G}$ is a path connecting $a,b \in \not\mathcal{G}$. If f does not pass through zero, since it passes through a countable number of points which are closed, we have that $[0,1]$ can be expressed as a countable union of disjoint closed sets, which is a contradiction. If f passes through zero, say $f(n) = 0$ then we consider $f((0,n))$ and if this is never zero we arrive at the same contradiction as above. In this way we can eliminate any zeros and thus always arrive at a contradiction which proves our theorem.

Theorem 22: The space is not locally connected.
Proof: If $(i+1) \not| \alpha$ and $(i+1) \not| \alpha+\beta$ then $B(\alpha,\beta) = B(\alpha,(i+1)\beta)$

$\cup\ B(\alpha+\beta,(i+1)\beta)$. Hence, we have a seperated neighbourhood and therefore the space is not locally connected.

We now consider the subspace topology inherited by $\mathcal{N}$ from the R. P. O. S. T. on $\mathcal{G}$. We find that this space has the same properties as those of the "Golomb" Topology on $\mathcal{N}$ and therefore we state and prove the following:

Theorem: The "Golomb" Topology on $\mathcal{N}$ is the same as the topology on $\mathcal{N}$ inherited from the R. P. O. S. T. on $\mathcal{G}$.
Proof: First we show that an open set in the "Golomb" Topology is open in the subspace topology. An open set in the "Golomb" Topology is a union of sets of the form $\langle a+nb\rangle$ and these sets are simply $\langle a+\delta b\rangle \cap \mathcal{N}$ which are open in the subspace topology. Now we show that an open set in the subspace topology is open in the "Golomb" Topology. We take $B(\alpha,\beta) = \langle\alpha+\delta\beta\rangle$ which is open in $\mathcal{G}$. Now we consider $B(\alpha,\beta) \cap \mathcal{N}$. If this empty then it is open in the "Golomb" Topology. If it is not empty then it contains an element a which is relatively prime to β. Our intersection is simply $\langle a+n|\beta|\rangle$ which is also open in the "Golomb" Topology. Hence, the two topologies are the same and the R. P. O. S. T. on $\mathcal{G}$ is indeed an extension of the "Golomb" Topology.

Now we look at the subspace topology on $\mathcal{G}^+$ and consider the following:

Theorem 23: The space is T_0.
Proof: $\mathcal{G}$ itself is T_0 and this property is hereditary.

Theorem 24: The space is T_2 and T_1.
Proof: We can seperate α,β by $B(\alpha,\gamma)$ and $B(\beta,\gamma)$ where $(\alpha-\beta,\gamma) = (\alpha\beta,\gamma) = 1$. Thus the space is T_2 and T_1.

Theorem 25: The space is not T_3.

Proof: We take the point 1 and the closed set $\{2\delta \mid \delta \in \mathscr{I}^+\}$. Now an open covering of 1 which does not meet $\{2\delta\}$ must contain a progression $\{1+\gamma\delta\}$, where γ is even. That is, $\gamma \in \{2\delta\}$. Let $\{\alpha+\delta\beta\}$ be the member of the open covering of $\{2\delta\}$ which contains γ. Thus we have $\gamma = \alpha+\delta_0\beta$ and since $(\alpha,\beta) = 1$ we have $(\gamma,\beta) = 1$, and hence $\{\alpha+\delta\beta\} \cap \{1+\delta\gamma\}$ is infinite. Therefore, we cannot find disjoint neighbourhoods.

Theorem 26: The space is neither T_4, T_5, regular, normal, compact, locally compact nor paracompact.
Proof: The space is T_1 and not T_3 so therefore it is not T_4. Since it is not T_4 it is neither T_5 nor normal. Since it is not T_3 it is not regular and since it is T_2 but not T_3 it is neither compact, locally compact nor paracompact.

Theorem 27: The space is connected.
Proof: We take $A = \{\alpha+\delta\beta\}$ and $B = \{\gamma+\delta\varphi\}$ to be disjoint open sets in $\mathscr{I}$. Their closures contain multiples of β and γ respectively and hence are not disjoint. Now $A' = A \cap \mathscr{I}^+$ and $B' = B \cap \mathscr{I}^+$ are disjoint open sets with intersecting closures and since every basic open set in $\mathscr{I}^+$ can be expressed in the same way as A' above, we have that the space is connected (see <1>).

Theorem 28: The space is not path connected.
Proof: The proof of this theorem is analogue to the proof of Theorem 16.

Theorem 29: The space is not locally connected.
Proof: If $(i+1) \nmid \alpha$ and $(i+1) \nmid (\alpha+\beta)$ then $B(\alpha,\beta) \cap \mathscr{I}^+ = (B(\alpha,(i+1)\beta) \cap \mathscr{I}^+) \cup (B(\alpha+\beta,(i+1)\beta) \cap \mathscr{I}^+)$. Hence, we have a seperated neighbourhood and therefore the space is not locally connected.

Finally we look at the subspace topology on the subset $\mathbb{N}_1$ and we have the following:

Theorem 30: The space is T_0.
Proof: $\mathcal{G}$ is T_0 and this property is hereditary.

Theorem 31: The space is T_2 and T_1.
Proof: The open sets have rational common difference and starting point in $\mathbb{N}_1$. We seperate a+i and b+i by the sets B(a+i,c) and B(b+i,c) where $(c,(a+i)(b+i)) = 1 = (c,a-b)$. Hence the space is T_2 and therefore T_1.

Theorem 32: The space is not T_3.
Proof: We consider the point i and the set $\{(i+1)\delta\} \cap \mathbb{N}_1$. Now $\{(i+1)\delta\} \cap \mathbb{N}_1 = \{(i+1) + 2k\}$ and $i \notin \{(i+1) + 2k\}$. We assume O covers i and $O \cap \{(i+1) + 2k\} = \emptyset$. Now O contains $\{\gamma\delta+1\}$ where γ has the form $(i+1)+2k$. Let $\{\alpha+\delta\beta\}$ be in the open covering of $\{(i+1)\delta\} \cap \mathbb{N}_1$ such that for some δ_0, $\gamma = \alpha+\delta_0\beta$. Now $(\alpha,\beta) = 1 \longrightarrow (\gamma,\beta) = 1$ and hence $\{\alpha+\delta\beta\} \cap \{1+\delta\gamma\}$ is infinite and thus the space is not T_3.

Theorem 33: The space is neither T_4, T_5, regular, normal, compact, locally compact nor paracompact.
Proof: The space is T_1 but not T_3 and hence not T_4. A space which is not T_4 is neither T_5 nor normal. A space which is not T_3 is not regular. The space is T_2 but not T_3 and hence neither compact, locally compact nor paracompact.

Theorem 34: The space is connected.
Proof: The closure of any basic open set contains the multiples of the common difference defining the set, and therefore two disjoint open sets do not have disjoint closures which makes the space connected.

Theorem 35: The space is not path connected.

Proof: Again the proof of this theorem is analogue to the proof of Theorem 16.

Theorem 36: The space is not locally connected.
Proof: If $(i+1) \nmid \alpha$ and $(i+1) \nmid (\alpha+\beta)$ then $B(\alpha,\beta) \cap \mathbb{N}_1 = (B(\alpha,(i+1)\beta) \cap \mathbb{N}_1) \cup (B(\alpha+\beta,(i+1)\beta) \cap \mathbb{N}_1)$. Hence, we have a seperated neighbourhood and therefore the space is not locally connected.

The final topology we wish to discuss in this chapter is not one which has been motivated by some paper as were the previous two, but it appears to be of some value in this present discussion. This topology will be called the Straight Line Closed Set Topology (S. L. C. S. T.) and is defined by the closed sets $B(\alpha,\beta)$ = Gaussian integers of the form $\alpha+n\beta$ where $(\alpha,\beta) = 1$, $\alpha,\beta \in \mathbb{G}$ and $n \in \mathbb{Z}$, and arbitrary intersections and finite unions of such sets.

In this topology we see that all points are closed and hence all finite subsets are closed. The topological properties of this space are given in the following:

Theorem 37: The space is T_1 and T_0.
Proof: All points in the space are closed.

Theorem 38: The space is neither T_2 nor T_3.
Proof: There are no disjoint open sets.

Theorem 39: The space is neither T_4, T_5, regular nor normal.
Proof: The space is T_1 and not T_3 and therefore not T_4. A space which is not T_4 is neither T_5 nor normal. The space is not regular because it is not T_3.

Theorem 40: The space is compact, locally compact and paracompact.

Proof: An open set is the complement of a finite union of closed sets and since $\not{\mathscr{G}}$ cannot be covered by a finite union of straight lines (basic closed sets), every open cover is finite. Thus, the space is compact and therefore locally compact and paracompact.

Theorem 41: The space is connected.
Proof: There are no disjoint open sets and hence there is no non-trivial seperation.

Theorem 42: The space is not path connected.
Proof: Assume $f:[0,1] \longrightarrow \not{\mathscr{G}}$ is a map with $f(0) = \alpha$ and $f(1) = \beta$. Now f must pass through a countable number of points which are closed sets in $\not{\mathscr{G}}$ and therefore the preimages of these points must be disjoint closed sets. Hence, we can write $[0,1]$ as a countable union of disjoint closed sets which gives us a contradiction.

Theorem 43: The space is locally connected.
Proof: Assume U is open in $\not{\mathscr{G}}$. Then $\not{\mathscr{G}}\backslash U$ is closed and is a finite union of basic closed sets. If U has a non-trivial seperation, say V, W, then $\not{\mathscr{G}}\backslash V$ and $\not{\mathscr{G}}\backslash W$ are closed in $\not{\mathscr{G}}$ and hence are the finite unions of basic closed sets. This gives us the fact that $\not{\mathscr{G}}$ can be expressed as a finite union of closed sets which leads to a contradiction.

As before we will now look at the properties of the three subspaces $\not{\mathscr{N}}$, $\not{\mathscr{G}}^+$, and $\not{\mathscr{N}}_1$. We have:

Theorem 44: The subspace $\not{\mathscr{N}}$ is T_0 and T_1.
Proof: Both these properties are hereditary.

Theorem 45: The subspace $\not{\mathscr{N}}$ is not T_2.
Proof: We prove this by showing that $\not{\mathscr{N}}$ cannot be covered by a finite number of closed sets. Hence, we must show

that all but a finite number of points cannot be covered by a finite number of arithmetic progressions in $\mathcal{N}$. Assume $\mathcal{N}$ = a finite union of sets of the form $B(a,b)$. If we call these sets $B_1, B_2, \ldots, B_n$ and have them generated respectively by $(a_1,b_1), (a_2,b_2), \ldots, (a_n,b_n)$ then neither set contains $b_1b_2\ldots b_n$ or any of its multiples. Thus we have an infinite set which is not contained in the union of these sets. This shows us that $\mathcal{N}$ cannot be expressed as the finite union of closed sets and therefore we cannot find disjoint open sets to seperate any two points.

Theorem 46: The subspace $\mathcal{N}$ is neither T_3, T_4, T_5, regular nor normal.
Proof: The space is T_0 but not T_2 and hence not T_3. The space is T_1 and not T_3 and therefore not T_4. A space which is not T_4 is neither T_5 nor normal. The space is not T_3 and therefore not regular.

Theorem 47: The subspace $\mathcal{N}$ is compact, locally compact and paracompact.
Proof: $\mathcal{Z}$ is a closed subset of the S. L. C. S. T. on $\mathcal{G}$ and compactness is weakly hereditary. Therefore, $\mathcal{Z}$ is compact. Now, since every open cover of $\mathcal{N}$ can be defined by taking the intersection of the elements of the open cover of $\mathcal{Z}$ with $\mathcal{N}$, we have that $\mathcal{N}$ is compact. A compact space is locally compact and paracompact.

Theorem 48: The subspace $\mathcal{N}$ is connected.
Proof: Assume A,B seperates $\mathcal{N}$. Then $A = \mathcal{N}\backslash B$ = a finite union of closed sets. Similarly B is a finite union of closed sets. Hence, $\mathcal{N}$ can be expressed as a finite union of closed sets which is a contradiction. Therefore the subspace is connected.

Theorem 49: The subspace $\mathcal{N}$ is not path connected.

Proof: If we assume it is path connected then this implies that $[0,1]$ can be expressed as a countable union of disjoint closed sets which is impossible. This contradiction tells us that the subspace is not path connected.

Theorem 50: The subspace $\mathcal{S}$ is locally connected.
Proof: If $U = \mathcal{S} \backslash$(finite union of closed sets) is disconnected then $U =$ a finite union of closed sets. Hence, $\mathcal{S}$ can be expressed as a finite union of closed sets which is a contradiction.

Theorem 51: The subspace $\mathcal{S}^+$ is both T_0 and T_1.
Proof: $\mathcal{S}$ is both T_0 and T_1, and both these properties are hereditary.

Theorem 52: The subspace $\mathcal{S}^+$ is neither T_2, T_3, T_4, T_5, regular nor normal.
Proof: There are no disjoint open sets and hence the subspace is not T_2. The subspace is T_0 but it is not T_2 and therefore not T_3. The subspace is not T_4 since it is T_1 but not T_3. A space which is not T_4 is neither T_5 nor normal. The space is not T_3 and hence not regular.

Theorem 53: The subspace $\mathcal{S}^+$ is compact, locally compact and paracompact.
Proof: Any open set is the complement of a finite union of closed sets and since $\mathcal{S}^+$ cannot be covered by a finite union of closed sets, every open cover must be finite. Hence, $\mathcal{S}^+$ is compact and therefore locally compact and paracompact.

Theorem 54: The subspace $\mathcal{S}^+$ is connected.
Proof: There are no disjoint open sets and hence no non-trivial seperation.

Theorem 55: The subspace $\mathcal{S}^+$ is not path connected.

Proof: If we assume the subspace is path connected then this implies that $[0,1]$ can be written as a countable union of disjoint closed sets which is a contradiction.

Theorem 56: The subspace $\mathcal{G}^+$ is locally connected.
Proof: Assuming that for an arbitrary open neighbourhood we can find a non-trivial seperation we have that $\mathcal{G}^+$ can be written as a finite union of closed sets which is a contradiction. Hence, the subspace $\mathcal{G}^+$ is locally connected.

Theorem 57: The subspace $\mathcal{N}_1$ is both T_0 and T_1.
Proof: $\mathcal{G}$ is both T_0 and T_1 and both these properties are hereditary.

Theorem 58: The subspace $\mathcal{N}_1$ is T_2.
Proof: If $a+i$ and $b+i$ are two distinct points then we pick $p = 4k+3$, which is both a rational prime and a Gaussian prime, such that $(a-b,p) = 1$. Then $B(a+i,p)$ and $B(b+i,p)$ are open sets seperating $a+i$ and $b+i$.

Theorem 59: The subspace $\mathcal{N}_1$ is compact, locally compact and paracompact.
Proof: $\mathcal{N}_1$ is a closed subset of $\mathcal{G}^+$ and these three properties are weakly hereditary.

Theorem 60: The subspace $\mathcal{N}_1$ is T_3, T_4, regular and normal. The subspace is both T_2 and paracompact and therefore T_3 and T_4. A T_0, T_4 space is regular and a T_1, T_4 space is normal.

Theorem 61: The subspace $\mathcal{N}_1$ is T_5.
Proof: We note that a basic closed set gives rise to a subbasic open set. The set consisting of basic closed sets is countable since the basic closed sets are defined by elements from the countable set $\mathbb{Z} \times \mathbb{Z}$. Hence, we have

a countable open subbasis which makes the space second countable. Now, since the space is regular it is metrizable and therefore T_5.

Theorem 62: The subspace $\mathcal{N}_1$ is neither connected nor path connected.
Proof: $B(i,3) \cup B(i+1,3) \cup B(i+2,3)$ is a non-trivial seperation of $\mathcal{N}_1$ and hence the subspace is not connected and therefore not path connected.

Theorem 63: The subspace $\mathcal{N}_1$ is not locally connected.
Proof: The open set $\mathcal{N}_1 \setminus B(i,3)$ has the non-trivial seperation given by $B(i+1,3) \cup B(i+2,3)$.

Thus, we have surveyed three topologies on the set of Gaussian integers and the structures they induce on several subspaces. Several properties have been given and proven for each of the spaces and they are tabulated below for easy reference. It should be noted that although the topologies have been defined in similar ways their properties are quite different. Our table follows:

Table 2.1

Properties of Topologies on $\mathcal{G}$ and Induced Topologies on its Subspaces

	Furstenberg Top	Open Set Topology on the Spaces:					Golomb Topology	Relatively Prime Open Set Topology on the Spaces:				Straight Line Closed Set Topology on the Spaces:			
		$\mathcal{G}$	$\mathcal{Z}$	$\mathcal{G}^+$	$\mathcal{N}$	$\mathcal{N}_1$		$\mathcal{G}$	$\mathcal{N}$	$\mathcal{G}^+$	$\mathcal{N}_1$	$\mathcal{G}$	$\mathcal{N}$	$\mathcal{G}^+$	$\mathcal{N}_1$
T_0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
T_1	+	+	+	+	+	+	+	–	+	+	+	+	+	+	+
T_2	+	+	+	+	+	+	+	–	+	+	+	–	–	–	+
T_3	+	+	+	+	+	+	–	–	–	–	–	–	–	–	+
T_4	+	+	+	+	+	+	–	+	–	–	–	–	–	–	+
T_5	+	+	+	+	+	+	–	–	–	–	–	–	–	–	+
Regular	+	+	+	+	+	+	–	–	–	–	–	–	–	–	+
Normal	+	+	+	+	+	+	–	–	–	–	–	–	–	–	+
Compact	–	–	–	–	–	–	–	+	–	–	–	+	+	+	+
Loc. Compact	–	–	–	–	–	–	–	+	–	–	–	+	+	+	+
Paracompact							–	+	–	–	–	+	+	+	+
Connected	–	–	–	–	–	–	+	+	+	+	+	+	+	+	–
Path Conn.	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
Loc. Conn.	–	–	–	–	–	–	–	–	–	–	–	+	+	+	–

As a final note in this chapter we look at the problem of extending these three topological structures to a general algebraic number field. Since the O. S. T. makes no restrictions on any of the integers defining the basic open sets we extend it to a general field by the following equivalence relation:

$$\alpha \ \overline{\theta} \ \beta \text{ iff } \alpha = \delta\theta + \beta \text{ for fixed } \theta \text{ and arbitrary } \delta$$

However, this approach cannot be used when extending the R. P. O. S. T. and the S. L. C. S. T. because in order to define these two structures we require that the defining integers be relatively prime. Before we can discuss the greatest common divisor of two elements we need unique factorization and this property is not always present in arbitrary algebraic number fields. If however, we restrict our discussion to Euclidean number fields, our extension is analogue to those given.

The details of each extension varies with the field under consideration and will not be given here.

Chapter III
Applications of the Three Topologies

In this chapter we again look at the three topologies which we defined and discussed in the previous chapter, but this time we view their usefulness with respect to particular problems in number theory.

The first topology, the Open Set Topology, was used by Furstenberg to prove the infinitude of the rational primes. We use the extended topological structure on $\mathcal{G}$ to prove the:

Theorem: There are infinitely many Gaussian primes.
Proof: We consider the set $A = \cup A_\pi$ where A_π consists of all multiples of π and π runs through the set of Gaussian primes. The only numbers not belonging to A are 1, -1, i, and -i and since $\{1, -1, i, -i\}$ is clearly not open, A cannot be closed. Since A_π is closed for all π then A must be an infinite union of such sets and hence there is an infinity of primes.

We realize, of course, that the proof above is analogue to that of Furstenberg.

Looking more closely at the O. S. T. we see that because the multiples of a compound number form an open set, the primes are not dense in this topology. Hence, this topological structure on $\mathcal{G}$ has little value when discussing the local properties of prime numbers.

By making the restriction that the pair α, β, defining an open set in the O. S. T., be relatively prime, we have the R. P. O. S. T. which can also be used to prove the:

Theorem: There are infinitely many Gaussian primes.

Proof: If π is prime the progression $\langle\delta\pi\rangle$ is closed, since its complement is a union of open sets. Consider the union $X = \cup_\pi\langle\delta\pi\rangle$ extended over all primes. If this is a finite union of closed sets, then X is closed. But the complement of X is $\{1, -1, i, -i\}$, which is neither empty nor infinite. Since the complement of X is not open, X cannot be closed, the union is not a finite one, and the number of primes is infinite.

This proof is analogue to that used by Golomb in his proof of infinitely many rational primes. Also, we have the following:

Theorem: If $\mathcal{S}$ is a finite subset of $\mathcal{P}$ then the assertion that $\mathcal{P}\backslash\mathcal{S}$ is dense in $\mathcal{G}$ is equivalent to the assertion that $\mathcal{P}$ is dense in $\mathcal{G}$.

Proof: If $\mathcal{P}\backslash\mathcal{S}$ is dense in $\mathcal{G}$ for all $\mathcal{S}$ then $\mathcal{P}$ is dense. If $\mathcal{P}$ is dense in $\mathcal{G}$ then every open set contains a prime. i.e. for all pairs α, β with $(\alpha,\beta) = 1$ in $\mathcal{G}$ we have that there exists $\delta_0 \in \mathcal{G}$ such that $\alpha+\delta_0\beta = \pi_0$ = a prime. Now $(\beta,\alpha+\delta_0\beta) = 1$ so there exists $\delta_1 \in \mathcal{G}$ such that $\alpha+\delta_0\beta+\delta_1\beta = \pi_1$ = a prime. i.e. $\alpha+(\delta_0+\delta_1)\beta = \pi_1$. Continuing this process we have that there are infinitely many primes in $B(\alpha,\beta)$. As mentioned in the proof of the rational case this theorem is the same as saying that if the closure of the primes is the Gaussian integers then the derived set of the primes is the Gaussian integers and vice versa.

Theorem: In this topology the interior of the set of primes is empty.

Proof: Assume $B(\alpha,\beta)$ is an open set consisting entirely of primes. If we let $\delta = \alpha+\beta+1$ then $\alpha+\delta\beta = \alpha+\alpha\beta+\beta^2+\beta = (\alpha+\beta)(\beta+1)$ which is composite and produces a contradiction.

where the proofs are again analogue to Golomb's.

In this topology we find that $\mathcal{P}$ and $\mathcal{P}\setminus\mathcal{S}$ are both dense and this follows from a theorem by Direchlet in which he proves that:

> there are infinitely many primes of the form $\alpha+\delta\beta$ where δ is an arbitrary Gaussian integer and α and β are given Gaussian integers with $(\alpha,\beta) = 1$.[1]

Since our basic open sets in this topology are such arithmetic progressions in Gaussian integers, the density of $\mathcal{P}$ and $\mathcal{P}\setminus\mathcal{S}$ follows.

The third topology, the S. L. C. S. T., is different from the other two topological structures on $\mathcal{G}$ in that here we have that $\mathcal{N}_1$ is a closed subspace. In the other two topologies this subspace was neither open nor closed, and because of the importance of this subspace in the solution of the problem at hand all the information we find on it could prove valuable.

In this space we also find that $\mathcal{P}$ and $\mathcal{P}\setminus\mathcal{S}$ are dense. These facts follow since the complement of every basic closed set must contain an arithmetic progression of the type described by Direchlet above, and hence we have that every open set contains an infinitude of primes.

Although this chapter draws on concrete conclusions about the three topologies, the discussion given illustrates more properties of the structures. These properties are discussed in the following chapter when we try to discover which topologies are useful for solving our problem. We find that these three topologies quite closely fit the requirements for a useful topology and

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1. <12>, pg. 511.

this seems to indicate that some minor changes in the structure may be the key to solving the problem.

Chapter IV
Requirements for a Useful Topology

In the previous two chapters we were mainly interested in three topological structures on the Gaussian integers, and we considered their value for solving the problem at hand and related problems.

Now, rather than discuss a particular topology we want to discuss which properties a topology must have in order to be useful for solving the problem and then try to decide its existence. Thus we have the following:

Theorem 1: If there exists a topology on $\mathscr{G}$ with the following two properties:

1. $\overline{\mathscr{P}\setminus\mathscr{S}} = \mathscr{G}$ i.e. $\mathscr{P}\setminus\mathscr{S}$ is dense in $\mathscr{G}$
2. $\mathscr{N}_1$ is open in $\mathscr{G}$

then there are infinitely many primes of the form n^2+1.
Proof: Assume there are only finitely many primes of the form n^2+1. Then there are only finitely many Gaussian primes of the form $n+i$, i.e. $\mathscr{N}_1$ contains only finitely many primes. If we denote this finite set by $\mathscr{S}$ then $\mathscr{P}\setminus\mathscr{S} = \mathscr{P}\setminus\mathscr{N}_1 \subset \mathscr{G}\setminus\mathscr{N}_1$ which is closed since $\mathscr{N}_1$ is open by property 2 of the topology. Now we have $\overline{\mathscr{P}\setminus\mathscr{S}} \subset \mathscr{G}\setminus\mathscr{N}_1$ which contradicts property 1 of the topology and hence the theorem is proven.

A statement quite similar to that of the above theorem is that if there exists a topological structure on $\mathscr{G}$ in which $\mathscr{P}\setminus\mathscr{S}$ is dense and the interior of $\mathscr{N}_1$ is non-empty, then there are infinitely many primes of the form n^2+1. This follows since the interior of $\mathscr{N}_1$ is an open subset of $\mathscr{N}_1$ containing infinitely many Gaussian primes.

We have the existence of topologies satisfying one of the properties of the topology, but not satisfying the other. For example if we take the topology generated by finite complements we have that $\mathcal{P}\setminus\mathcal{S}$ is dense but $\mathcal{N}_1$ is not open. The excluded point topology (the open sets are those which do not include a given point) on $\mathcal{G}$ with $2i+1$ being the excluded point, has $\mathcal{N}_1$ open but $\mathcal{P}\setminus\mathcal{S}$ not dense. Now the minimal topology generated by these two topologies is called Fort Space and is defined by declaring open any set whose complement is finite or includes a given point. In this space $\mathcal{N}_1$ is again open but $\mathcal{P}\setminus\mathcal{S}$ still fails to be dense. Although this example did not prove the existence of the required topology it illustrates a method which probably could be used to achieve this end.

Another topology worth mentioning in this connection is the particular point topology (the only open sets are those which contain a given point) on $\mathcal{G}$ with $1+i$ being the particular point. Now $\mathcal{N}_1$ is open since it contains $1+i$, and $\mathcal{P}$ is dense because $1+i$ is a prime contained in every open set. However, $\mathcal{P}\setminus\mathcal{S}$ still fails to be dense.

If we now consider closed sets instead of open ones, we have the following:

Theorem 2: If there exists a topology on $\mathcal{G}$ with the following properties:

1. For all closed sets C in $\mathcal{G}$, $C\cap\mathcal{P}$ is infinite
2. $\mathcal{N}_1$ is closed in $\mathcal{G}$

then there are infinitely many primes of the form n^2+1.

We note that $\mathcal{N}_1$ is closed in the S. L. C. S. T. of Chapter 2 but since every point is closed in this topology property 1 of the theorem fails.

Thus far in this chapter we have tried to illustrate the relationship between the proof of a single existence (one topology) and the proof of an infinite existence (infinity of primes of the form n^2+1). We now change this theme slightly by stating and proving the following:

Theorem 3: If for infinitely many pairs a,b of rational integers we can define a particular point topology, τ_a^b, on $\mathcal{G}$ in which $\mathcal{P}$ is dense and $\mathcal{N}_1$ is open then we have proven the existance of an infinitude of primes of the form n^2+1.

Proof: We define τ_a^b using a+bi as the particular point. Now if $\mathcal{P}$ is dense then a+bi must be prime, and if $\mathcal{N}_1$ is open then it must contain a+bi. These two facts imply that the particular point must be a prime of the form n+i. Now, an infinitude of these topologies gives an infinitude of such primes and hence an infinitude of primes of the form n^2+1.

Another useful topological structure on $\mathcal{G}$ is given by the following:

Theorem 4: If there exists a connected topology on $\mathcal{G}$ having $\mathcal{P}\setminus\mathcal{Z}$ dense, $\mathcal{N}_1$ closed, and $\mathcal{G}\setminus\mathcal{N}_1$ seperated, we have the existance of infinitely many primes of the form n^2+1.

The proof of this theorem involves the Excision Theorem in Cohomology Theory andis discussed in detail in the next chapter.

This chapter is not intended to give a complete list of all useful topologies but only to illustrate some of those which may be effective.

Chapter V

Some New Approaches to the Problem

In this final chapter, we look at a series of discussions on problems arising as a result of the problem in question. Although no conclusions have been drawn from these discussions, they are included because they provide ideas which may be helpful in solving the problem.

1. Our first discussion involves some algebraic topology. We define a topology, t, on $\mathcal{G}$ by declaring open any set, U, whose complement contains finitely many primes, or is empty. Hence, a set, C, is closed if and only if it contains finitely many primes. This topology has the property that $\mathcal{P}\backslash\mathcal{S}$ (where $\mathcal{S}$ is any finite subset of $\mathcal{P}$) is dense. If we assume, now, that $\mathcal{M}_1$ has only finitely many primes then it is closed. We take U to be any open superset of $\mathcal{M}_1$ and by the Excision Theorem of Cohomology Theory we have that:

$$H^q(\mathcal{G},U) = H^q(\mathcal{G}\backslash\mathcal{M}_1,U\backslash\mathcal{M}_1)$$

If we could show that this result leads to a contradiction then we would have that there are infinitely many primes of the form n^2+1.

Now, because of the complicated structure of the topology on $\mathcal{G}$, especially with regard to discussing cohomology, we turn to the Alexander Cohomology and find that according to <8> we have:

$H^0(X,A)$ is isomorphic to the group of all locally constant functions from X into $\mathcal{G}$ and hence $H^0(X,A) = 0$ if X is connected and $A \neq \emptyset$.

We see that for the topology defined above the theorem does not give the required contradiction since both $\mathcal{N}$ and $\mathcal{N}\backslash\mathcal{N}_1$ are connected. However, by referring to Table 1.1 in Chapter I we find that connectedness is not hereditary and hence it may be possible to find a topology on $\mathcal{N}$ which is connected but having $\mathcal{N}\backslash\mathcal{N}_1$ disconnected.

Hence, we see that if we can prove the existance of a connected topology on $\mathcal{N}$ having $\mathcal{P}\backslash\mathcal{S}$ dense, $\mathcal{N}_1$ closed, and $\mathcal{N}\backslash\mathcal{N}_1$ seperated we have the solution to our problem. This result was already mentioned in the previous chapter.

2. We now consider two problems in number theory, both of which are unproven, and try to establish a relationship between them. Before we state these problems we consider the following problem by Robert Spira which appeared in the American Mathematical Monthly in 1956. the problem has the following statement:

> Consider the two propositions:
> I. If $(a,b) = 1$ then $ax+b$ assumes infinitely many prime values.
> II. If $(a,b) = 1$ then $ax+b$ assumes at least one prime value.
> I is Direchlet's Theorem. Clearly I implies II. Show that II implies I.[1]

The proof of this theorem using number theory is quite easy, but below we restate the theorem in topological terms and present a topological discussion of the proof. Thus, we state the following:

Theorem: The set of rational primes, $\mathcal{P}$, is dense in $\mathcal{N}$ with the topology given by arithmetic progressions (Golomb Topology) if and only if Direchlet's Theorem holds.

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1. American Mathematical Monthly, 63 (1956), 342.

Discussion: Assume Direchlet's Theorem holds, i.e. for all $a,b \in \mathcal{N}$ with $(a,b) = 1$ the sequence $\langle a+bn \rangle$ contains infinitely many primes. Since $\langle a+bn \rangle = B(a,b) =$ a basic open set in our topology we have that $\mathcal{P}$ meets every basic open set and is therefore dense.
Assume $\mathcal{P}$ is dense in $\mathcal{N}$ and assume $\mathcal{S}$ is a finite subset of $\mathcal{P}$. Now since $\mathcal{S}$ is finite and all nonempyt open sets in $\mathcal{N}$ are infinite we have that $\mathrm{Int}(\mathcal{S}) = \emptyset$. Now, if it were true that:

$$\mathrm{Cl}(\mathcal{P}\backslash\mathcal{S}) = \mathrm{Cl}\mathcal{P}\backslash\mathrm{Int}\mathcal{S} = \mathcal{N}\backslash\emptyset = \mathcal{N}$$

we would have that $\mathcal{P}\backslash\mathcal{S}$ is dense in $\mathcal{N}$. If we now assume that $B(a,b)$ is a basic open set containing only finitely many primes, say $\mathcal{S}$, then $\mathcal{N}\backslash B(a,b)$ is a closed set containing $\mathcal{P}\backslash\mathcal{S}$ and hence $\mathrm{Cl}(\mathcal{P}\backslash\mathcal{S}) \subset \mathcal{N}\backslash B(a,b)$ and therefore $\mathcal{P}\backslash\mathcal{S}$ is not dense. This is a contradiction implying that $B(a,b)$ contains infinitely many primes for all relatively prime pairs a,b and hence Direchlet's Theorem holds.

The mistake in this discussion is the fact that $\mathrm{Cl}(\mathcal{P}\backslash\mathcal{S}) \neq \mathrm{Cl}\mathcal{P}\backslash\mathrm{Int}\mathcal{S}$ in general. If however, we have that $\mathcal{P}$ is dense and $\mathcal{S}$ is open or $\delta\mathcal{P} \cap \delta\mathcal{S} = \emptyset$ (where $\delta\mathcal{P}$ is the boundary of $\mathcal{P} = \mathrm{Cl}\mathcal{P}\backslash\mathrm{Int}\mathcal{P}$) then the theorem holds.

We are now ready to discuss the connection between the two problems in number theory. The first is the infinitude of primes of the form n^2+1, and the second is the existance of an integer b for every given integer a such that a^2+b^2 is a prime of the form $4k+1$. Hence we state the:

Conjecture: If for every $a \in \mathcal{Z}$ there exists a $b \in \mathcal{Z}$ such that a^2+b^2 is a rational prime of the form $4k+1$ then there are infinitely many primes of the form n^2+1.
Discussion: We topologize the Gaussian integers with the

product of the indiscrete topology on the rational integers and the particular point topology on the rational integers (in this topology a set is open if and only if it contains the point 1). Now we let $\mathcal{N}_b = \{n+bi \mid n \in \mathbb{Z}$ and b is fixed in $\mathbb{Z}\}$, and $\mathcal{M}$ be an arbitrary subset of $\mathbb{Z}$, then sets of the form $\mathcal{N}_1 \cup \{\cup_{b\in\mathcal{M}}\mathcal{N}_b\}$ are open in our topology. We see that these open sets form a basis for the topology and since $1+i$ is a Gaussian prime common to every basic open set, the primes are dense in this topological space. Now if $\mathcal{S}$ is a finite subset of $\mathcal{P}$, then, since all non-empty open sets are infinite, $\mathrm{Int}\,\mathcal{S} = \emptyset$ and if we again assume that:

$$\mathrm{Cl}(\mathcal{P}\backslash\mathcal{S}) = \mathrm{Cl}\mathcal{P}\backslash\mathrm{Int}\mathcal{S} = \mathcal{G}\backslash\emptyset = \mathcal{G}$$

we have that $\mathcal{P}\backslash\mathcal{S}$ is dense in $\mathcal{G}$. Hence, we have that every basic open set contains infinitely many primes, and therefore there are infinitely many primes of the form n^2+1.

In this discussion the same fallacy occurred as previously and although this does not supply a proof of the conjecture it indicates that one may not be impossible. It is worth noting that although neither of the two problems have been solved we have that primes of the form a^2+b^2 have been tabulated in <11> for values of a up to 3136, and there are no counterexamples up to this point.

3. Our next discussion will concern the decomposition of numbers of the form n^2+1 into prime factors. We have some results in this direction already. For example, it has been proved that there are infinitely many integers of the form n^2+1 having at most three prime factors.[2]

2. <15>, pg. 103.

We state a conjecture which, if proven, gives us the above result for two factors.

Conjecture: There are infinitely many gaussian primes of the form $x+(x+1)i$.

Theorem: If there are infinitely many Gaussian primes of the form $x+(x+1)i$ then there are infinitely many rational integers of the form n^2+1 with only two factors.
Proof: An infinitude of Gaussian primes of the form $x+(x+1)i$ implies an infinitude of rational primes of the form $2x^2+2x+1$. Now, multiplication by 2 gives us numbers of the form $4x^2+4x+2 = 4x^2+4x+1+1 = (2x+1)^2+1 = n^2+1$. Hence, we have infinitely many rational integers of the form n^2+1 with only two factors.

As a matter of fact, for an arbitrary natural number k, we have that if $(1+2i)+n(1+(2k+1)i)$ is prime infinitely often then there are infinitely many integers of the form n^2+1 with exactly $k+2$ factors.

Another result is one of Bredihin, that there are infinitely many primes of the form n^2+m^2+1.[3] Since we wish to find an infinitude of primes of the form x^2+1, the solution follows if the two sets of integers mentioned, ($\{$primes of the form $n^2+m^2+1\}$, $\{$integers of the form $x^2+1\}$), have infinite intersection. We note that a discussion of the intersection of these two sets leads to a discussion of Pythagorean triples.

Also worth noting is the fact that every proper fraction can be expressed as a sum of Egyptian fractions (fractions with numerator 1). If we now consider a/b and rewrite it as $b+ai$ we see that the elements of $\mathcal{N}_1$

3. <15>, pg. 103.

can be used to "generate" the elements of $\mathcal{G}^+$. This suggests a mapping which in turn suggests a possible topology for $\mathcal{G}^+$. Hence, by giving $\mathcal{G}_1$ a particular topology, some useful results may arise from the topology introduced on $\mathcal{G}^+$.

4. In this discussion we consider the following proven:

Theorem: If $f(x)$ is a polynomial of degree n with integer coefficients and k is an arbitrary integer, then the greatest common divisor of the numbers $f(x)$, x running over all integers, is equal to the greatest common divisor of the following $n+1$ integers: $f(k)$, $f(k+1)$, ..., $f(k+n)$.[4]

and the:

Conjecture: If $f(x)$ is an irreducible polynomial with integral coefficients and if N denotes the greatest common divisor of the numbers $f(x)$, x running over all integers, then the polynomial $f(x)/N$ takes prime number values for infinitely many x's.[5]

The above two conditions together, imply the infinitude of primes of the form n^2+1, and also many similar questions are answered as consequences of these two conditions.

In 1958 A. Schinzel formulated the following:

Conjecture H: If s is a natural number and if $f_1(x)$, $f_2(x)$, ..., $f_s(x)$ are polynomials with integral coefficients satisfying the condition that each of the polynomials $f_i(x)$ is irreducible, its leading coefficient

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4. <16>, pg. 11.
5. <16>, pg. 127.

is positive and there is no natural number $d>1$ that is a divisor of each of the numbers $P(x) = f_1(x)f_2(x)\ldots f_s(x)$, x being an integer, then there exist infinitely many natural values of x for which each of the numbers $f_1(x)$, $f_2(x)$, ..., $f_s(x)$ is prime.[6]

Some consequences of this conjecture follow:

1. For $f_1(x) = x^{2^n}+1$, $f_2(x) = x^{2^n}+3$, $f_3(x) = x^{2^n}+7$, $f_4(x) = x^{2^n}+9$, the conjecture holds and this gives us infinitely many quadruplets of primes, and hence infinitely many prime twins.

2. There are infinitely many primes of the form x^2+1, and x^4+1.

3. Applying the conjecture to $f_1(x) = x$ and $f_2(x) = x+2k$ we have that every even number has infinitely many representations as the difference of two primes.

4. If $(a,b) = (a,b(b+2)) = 1$, then there exist infinitely many prime twins in the arithmetic progression $\langle an+b \rangle$.

5. Every odd integer has infinitely many representations as the difference of a prime and the double of a prime.

6. There exist arbitrarily long arithmetic progressions whose terms are consecutive prime numbers.

Thus, we see that if we assume the validity of this conjecture, the solution to our problem follows as a simple consequence.

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6. <16>, pg. 128.

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