MEASURE OF NONCOMPACTNESS, DENSIFYING MAPPINGS AND FIXED POINTS

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MEASURE OF NONCOMPACTNESS, DENSIFYING MAPPINGS

and FIXED POINTS

by

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ABSTRACT

The purpose of this thesis is to set forth some fixed point theorems in metric and Banach spaces for densifying mappings and k-set-contractions.

In Chapter I, we discuss the Banach Contraction Principle and give an extensive coverage of its generalizations. Results are also given for contractive and nonexpansive mappings, and for mappings of an iterative type.

In Chapter II, we introduce the concepts of α-measure (and χ-measure) of noncompactness; some geometric properties are given.

In Chapter III and IV we make a detailed study of some of the recent fixed point results in metric and Banach spaces respectively.

Particular attention is given to the works of Furi, Vignoli, Nussbaum, and Petryshyn.

In Chapter V, we give applications of α-measure to problems in optimization theory.

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INTRODUCTION

The Banach Contraction Principle was formulated in 1922. Since then there have been diverse generalizations of this basic principle. Edelstein [26] and Rakotch [62] have studied contractive mappings. F.E. Browder [10, 11], W.A. Kirk [42], L.P. Belluce and W.A. Kirk [4, 56], D.W. Boyd and J.S.W. Wong [7], R. deMarr [23], and others have studied more general mappings. All this in the search for fixed points; and thus to find solutions of integral and differential equations.

In recent years efforts have been made in extending the classical theory of fixed points. Using measure of noncompactness several mappings have been investigated; among them k-set-contractions, 1-set-contractions, and densifying mappings are worth mentioning. The notion of α-measure of noncompactness was introduced by C. Kuratowskii [47] in 1955 and a generalization of the Cantor Intersection Theorem was given. The proof relies on the notion of α-measure.

In 1955 G. Darbo [19] gave the first fixed point theorem in this light.

> If $T : C \to C$ is a k-set-contraction defined on a closed bounded convex subset C of a Banach space X, then T has at least one fixed point in C.

M. Furi and A. Vignoli [31] were the first to introduce the concept of a densifying mapping in 1969, although in 1967 B.N. Sadovskii [65] had published a paper using condensing mappings and the closely related notion of χ-measure.

The major and significant contributors to date in the area of α-measure and noncompact mappings are R.D. Nussbaum [54, 55], W.V. Petryshyn [56, 60], and J.R.L. Webb [77].

CHAPTER I

General Fixed Points in Metric Spaces

1.1. Metric Spaces.

Definition 1.1.1. Let X be any nonempty set, and let $\mathbb{R}_+$ denote the nonnegative real numbers. A distance function $d : X \times X \to \mathbb{R}_+$ is said to be a metric if the following conditions are satisfied for all $x, y, z \in X$:

(i) $d(x,y) \geq 0$; $d(x,y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, (triangle inequality).

If condition (i) is replaced by

(i^*) $d(x,y) \geq 0$; $d(x,y) = 0$ if $x = y$,

then d is called a pseudo-metric.

The set X with metric d is called a metric space and is denoted by the pair (X,d). When the metric d is understood, we will denote the metric space simply by X.

Example 1.1.2. Any nonempty set X can be considered a metric space by imposing the following metric, called the discrete metric,

$$d(x,y) = 1, \quad x \neq y$$
$$= 0, \quad x = y.$$

Example 1.1.3. Let $X = \mathbb{R}$, the real line,and let the metric $d(x,y) = |x - y|$, for all $x,y \in \mathbb{R}$.
$(\mathbb{R},d)$ is a metric space.

Definition 1.1.4. A sequence $\{x_n\}$ of points in a metric space X is called a Cauchy sequence if for given $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$, whenever $m,n \geq N$. In other words, $\lim_{n,m\to\infty} d(x_m, x_n) = 0$.

Definition 1.1.5. A sequence $\{x_n\}$ is said to converge to x if, given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ with the property that

$$n \geq N \Rightarrow d(x_n, x) < \varepsilon ,$$

i.e., $\lim_{n\to\infty} d(x_n, x) = 0.$

It is well known that every convergent sequence is Cauchy, but not conversely.

Definition 1.1.6. A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

1.2. Banach Contraction Principle.

Definition 1.2.1. Let T be a mapping of a set X into itself. A point $x \in X$ is called a fixed point of T if $T(x) = x$.

Definition 1.2.2. A mapping T of a metric space X into itself is said to satisfy a Lipschitz condition if there exists a real number k ($< \infty$) such that

$$d(T(x), T(y)) \leq k d(x,y) , \quad \text{for all } x,y \in X.$$

In case, when $0 \leq k < 1$, then T is called a contraction mapping.

Example 1.2.3. Let $T : \mathbb{R} \to \mathbb{R}$ be a function defined by $Tx = x/2$. Then T is a contraction, since

$$d(Tx,Ty) = |Tx - Ty| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}\, d(x,y).$$

Here $k \in [\frac{1}{2}, 1)$.

Definition 1.2.4. A function $T : X \to X$ is continuous at a point $x_o \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x,x_o) < \delta \Rightarrow d(T(x),T(x_o)) < \varepsilon .$$

A function T is continuous on X if it is continuous at every point of X.

Theorem 1.2.5. If T satisfies a Lipschitz condition on a metric space X, then T is continuous on X.

Proof. Let $\varepsilon > 0$ be given, and let x be any point in X. Since T satisfies Lipschitz condition, we have

$$d(T(x),T(y)) \leq k\,d(x,y), \quad \text{for all } x,y \in X.$$

If $k = 0$, then $d(T(x), T(y)) = 0 < \varepsilon$, for all $y \in X$. And T is therefore continuous at x.

Otherwise, let $\delta = \varepsilon/k$, and let y be any other point in X, such that $d(x,y) < \delta$. Then we have, $d(T(x),T(y)) \leq kd(x,y) \leq k \cdot \varepsilon/k = \varepsilon$. Hence T is continuous at x.

Since x is an arbitrary point of X, T is continuous everywhere.

Corollary 1.2.6. Every contraction mapping $(0 \leq k < 1)$ is continuous.

S. Banach (1892-1945), a famous Polish mathematician, formulated the "Principle of Contraction Mapping" [3] in 1922, which was first applied to the proof of an existence theorem by Cacciopoli [14] in 1930. We are now ready to give this main theorem on contraction mappings.

Theorem 1.2.7. (Banach Contraction Principle).

Every contraction mapping T of a complete metric space X into itself has a unique fixed point.

Proof. Let x_o be an arbitrary point in X.
Let $x_1 = T(x_o)$, $x_2 = T(x_1) = T^2(x_o)$, ...,
and in general, $x_n = T^n(x_o)$.

Then, for $m < n$, we have

$$\begin{aligned} d(x_m,x_n) &= d(T^m(x_o),\ T^n(x_o)) \\ &= d(T^m(x_o),\ T^mT^{n-m}(x_o)) \\ &\leq k^m d(x_o,\ T^{n-m}(x_o)) \\ &= k^m d(x_o,x_{n-m}) \\ &\leq k^m[d(x_o,x_1) + d(x_1,x_2) + \ldots + d(x_{n-m-1},x_{n-m})] \\ &\leq k^m d(x_o,x_1)(1 + k + \ldots + k^{n-m-1}) \\ &< k^m d(x_o,x_1)\left(\frac{1}{1-k}\right) . \end{aligned}$$

Since $k < 1$, this quantity is arbitrarily small for sufficiently large m. Thus the sequence $\{x_n\}$ is Cauchy. Since X is complete, $\lim\limits_{n\to\infty} x_n$ exists in X.

Let $x = \lim_{n\to\infty} x_n$.

Since T is continuous, we have

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Thus the existence of a fixed point is proved.

To prove uniqueness, let $y \neq x$ be another fixed point of T,

i.e., $T(x) = x, \quad T(y) = y \quad , \quad x \neq y.$

Then $d(x,y) = d(T(x),T(y)) \leq kd(x,y)$,

i.e., $k \geq 1$, a contraction.

Hence $x = y$ and uniqueness is established.

Remarks.

1. Both conditions of the theorem are necessary.

(i) The mapping $T : (0,1] \to (0,1]$ defined by $T(x) = x/2$ is a contraction but has no fixed point since $(0,1]$ is not a complete metric space.

(ii) The mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$ is not a contraction and has no fixed point although $\mathbb{R}$ is complete.

2. The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximations.

3. The contraction mapping theorem has the advantage of being constructive.

Its error of approximation can be estimated as
$d(x,x_m) \leq \frac{k^m}{1 - k} d(x_o,x_1)$. It guarantees the existence and uniqueness of a fixed point.

1.3. Generalization of the Banach Contraction Principle.

In this section a few known generalizations have been given.

Theorem 1.3.1. [53] If T is a contraction mapping of a complete pseudo-metric space X into itself, then T has a fixed point, not unique.

Proof. Existence of the fixed point may be justified as in Theorem 1.2.7. If X is a pseudo-metric space, then we have

$$d(x,y) = 0 \quad \text{if} \quad x = y.$$

Hence $d(x,y) = 0 \not\Rightarrow x = y$.

Now, referring to the discussion on uniqueness of the previous theorem,

$$d(x,y) \leq kd(x,y) \Rightarrow d(x,y) - kd(x,y) \leq 0.$$

Suppose $d(x,y) - kd(x,y) = 0$,

then $(1 - k)d(x,y) = 0$.

But $(1 - k) \neq 0$ for $k \in [0,1)$,

therefore $d(x,y) = 0$.

But this does not necessarily imply that $x = y$.

Theorem 1.3.2. [17] If T is a mapping of a complete metric space X into itself such that T^n, for some positive integer n, is a contraction, then T has a unique fixed point.

Proof. Since T^n is a contraction, it follows from Theorem 1.2.7 that T^n has a unique fixed point, say $T^n z = z$.

Then $$T^n(T(z)) = T(T^n(z)) = T(z),$$

i.e., $T(z)$ is a fixed point of T^n.

But T^n has a unique fixed point, hence $T(z) = z$. So T has a unique fixed point, z, and it is unique since any fixed point of T is also fixed for T^n.

Remark. An extension of the preceding argument shows that there is no need to assume that T^n is a contraction and defined on a complete metric space. All that is used in obtaining the conclusion of the theorem is that T^n has exactly one fixed point.

Example 1.3.3. Let $T : [0,2] \to [0,2]$ be defined by

$$\begin{aligned} T(x) &= 0 , \quad x \in [0,1] \\ &= 1 , \quad x \in (1,2]. \end{aligned}$$

Then $T^2x = 0$, for all $x \in [0,2]$. Hence T^2 is a contraction and has a unique fixed point, although T is not continuous.

Theorem 1.3.4. [16] Let T be a mapping defined on a nonempty set S into itself, K another function defined on S to S such that $KK^{-1} = I$, where I is the identity function on S. Let $K^{-1}TK$ be contraction on S. Then T has a unique fixed point.

The following is an immediate corollary to the above theorem.

Corollary 1.3.5. Let X be a complete metric space, $T : X \to X$, and $K : X \to X$ be such that $K^{-1}TK$ is a contraction in X, then T has a unique fixed point.

A further generalization of the Banach Contraction Principle, due to Sehgal [68], has been given below without proof.

Theorem 1.3.6. If X is a complete metric space and $T : X \to X$ a continuous mapping satisfying the condition : there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$.

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leq kd(x,y) ,$$

then T has a unique fixed point z and $T^n(x_o) \to z$ for each $x_o \in X$.

1.4. Contractive Mappings.

Definition 1.4.1. A mapping $T : X \to X$ is called contractive if

$$d(T(x),T(y)) < d(x,y), \quad \text{for all } x,y \in X, \ x \neq y.$$

A contractive mapping on a complete metric space need not have a fixed point, as the following example illustrates.

Example 1.4.2. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x + \frac{\pi}{2} - \text{arc tan } x$. T is contractive but has no fixed point.

The following theorem is due to Edelstein [26].

Theorem 1.4.3. Let T be a contractive mapping on a metric space X, and let $x \in X$ be such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ which converges to a point $z \in X$. Then z is a unique fixed point of T.

A simpler proof of the above theorem than that due to Edelstein can be given in the following way. This proof is similar to that of Cheney and Goldstein [15].

Proof. Since T is contractive, we have,

$$d(T^n(x),T^{n+1}(x)) < d(T^{n-1}(x),T^n(x))$$
$$\cdots\cdots\cdots$$
$$< d(x,T(x)).$$

Thus $\{d(T^n(x),T^{n+1}(x))\}$ is a decreasing sequence of real numbers bounded below by zero, and therefore has a limit.

Since $\{T^{n_i}(x)\} \to z \in X$ and T is continuous, therefore

$\{T^{n_i}(x)\} \to T(z)$

and $\{T^{n_i+2}(x)\} \to T^2(z)$.

Now, if $z \neq T(z)$, we have

$$d(z,T(z)) = \lim_{i\to\infty} d(T^{n_i}(x),T^{n_i+1}(x)).$$

By continuity of T, we have

$$d(z,T(z)) = \lim_{i\to\infty} d(T^{n+1}(x),T^{n_i+2}(x))$$
$$= \lim_{i\to\infty} d(T(\,^{n_i}(x)),T^2(T^{n_i}(x))$$
$$= d(T(z),T^2(z)),$$

contradiction, since T is contractive.

Hence $z = T(z)$.

Uniqueness is obvious. For, let $y \neq z$ be such that $y = T(y)$. Then

$$d(y,z) = d(T(y),T(z)) < d(y,z),$$

a contradiction.

Hence z is the unique fixed point of T.

The following corollary is due to Edelstein [26].

Corollary 1.4.4. If T is a contractive mapping of a metric space X into a compact metric space $Y \subset X$, then T has a unique fixed point.

Bailey [1] has given the following result.

Theorem 1.4.5. Let X be a compact metric space and $T : X \to X$ be a continuous mapping such that there exists $n = n(x,y)$ with

$$d(T^n x, T^n y) < d(x,y), \quad \text{for } x \neq y.$$

Then T has a unique fixed point.

Rakotch defined a family of functions $F = \{k(x,y)\}$ satisfying $0 \leq k(x,y) < 1$, $\sup k(x,y) = 1$ and such that Banach's theorem holds when the constant k is replaced by any $k(x,y) \in F$, and proved the following two theorems to that effect in [62].

Definition 1.4.6. Denote by F_1 the family of functions $k(x,y)$ satisfying the following conditions:

(1) $k(x,y) = k(d(x,y))$, i.e., k is dependent on the distance between x and y,

(2) $0 \leq k(d) < 1$ for every $d > 0$,

(3) $k(d)$ is a monotonically decreasing function of d.

Theorem 1.4.7. If T is a contractive mapping of a metric space X into itself, and there exists a subset $M \subset X$ and a point $x_0 \in M$ such that

$$d(x,x_0) - d(T(x),T(x_0)) \geq 2d(x_0,T(x_0)), \quad \text{for all } x \in X - M,$$

and T maps M into a compact subset of X,

then there exists a unique fixed point of T.

Theorem 1.4.8. Let T be a contraction mapping of a complete metric space X into itself such that there exists a subset $M \subset X$ and a point $x_o \in M$ satisfying the following:

$$d(x,x_o) - d(T(x),T(x_o)) \geq 2d(x_o,T(x_o)), \quad \text{for all } x \in X - M$$

and

$$d(T(x),T(y)) \leq k(x,y)d(x,y), \quad \text{for all } x,y \in M$$

where $$k(x,y) = k(d(x,y)) \in F_1,$$

then there exists a unique fixed point of T.

Boyd and Wong [7] have introduced mappings which satisfy the condition

$$d(T(x),T(y)) \leq \psi(d(x,y)) ,$$

where ψ is some function defined on the closure of the range of d. In this way they extended the results of Rakotch [62]. They proved the following theorem, and have also given an example to justify their claim. They raised a question which was later answered by Meir and Keeler [52].

Theorem 1.4.9. Let X be a complete metric space, and let $T : X \to X$ satisfy

$$d(T(x),T(y)) \leq \psi(d(x,y)) ,$$

where $\psi : \bar{D} \to [0,\infty)$ is upper semicontinuous from the right on $\bar{D}$, and satisfies $\psi(t) < t$ for all $t \in \bar{D} - \{0\}$. Then T has a unique fixed point z, and $T^n(x) \to z$ for each $x \in X$.

Definition 1.4.10 [52]. A mapping T of a metric space X into itself is called uniformly contractive if the following condition holds:

Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(T(x),T(y)) < \varepsilon .$$

The following known result is given by Meir and Keeler [52].

Theorem 1.4.11. Let X be a complete metric space and T a mapping of X into itself. If T is uniformly contractive, then T has a unique fixed point z. Moreover, for all $x \in X$, $\lim_{n\to\infty} T^n(x) = z$.

Observation: We first observe that T is contractive,

i.e., $$x \neq y \Rightarrow d(T(x),T(y)) < d(x,y).$$

Thus T is continuous and has at most one fixed point. Moreover, the following result is known [17] and easily proved.

Lemma 1.4.12. If $T : X \to X$ is contractive and if, for all $x \in X$, the sequence of iterates $\{T^n(x)\}$ is Cauchy, then T has a unique fixed point z and for all $x \in X$, $\lim_{n\to\infty} T^n(x) = z$.

Proof. Since X is complete and $\{T^n(x)\}$ Cauchy, each $\{T^n(x)\}$ has a limit z.

Since T is continuous,

$$T(z) = T(\lim_{n\to\infty} (T^n(x))) = \lim_{n\to\infty} (T^{n+1}(x)) = z.$$

Thus z is a fixed point.

Remark 1. By Lemma 1.4.12,the theorem is established if T uniformly contractive implies that every sequence $\{T^n(x)\} = \{x_n\}$ of iterates is Cauchy. Meir and Keeler go on to show that where T is uniformly contractive $\lim_{n\to\infty} d(x_n,x_{n+1})$ decreases to zero and that every sequence of iterates must be Cauchy.

Remark 2. Rakotch [62] and Boyd and Wong [7] assume the inequalities (nonlinear contractions)

$$d(T(x),T(y)) \leq \psi(d(x,y)) \quad \text{and} \quad \psi(r) < r$$

(as well as other conditions). The following example due to Meir and Keeler [52] shows that these inequalities may be violated while T is uniformly contractive.

Example 1.4.13. Let $X = [0,1] \quad \{3, 4, 6, 7, \ldots, 3n, 3n+1, \ldots\}$ with Euclidean distance, and let T be defined as follows;

$$\begin{aligned} T(x) &= \frac{x}{2} \, , \quad 0 \leq x \leq 1 \\ &= 0 \, , \quad x = 3n \\ &= 1 - \frac{1}{n+2} \, , \quad x = 3n+1 \, . \end{aligned}$$

Example 1.4.14. Let $S_n = \sum_{k=1}^{n} (1 + 1/k)$,

and let $X = \{S_n\}$. Let $T(S_n) = S_{n+1}$, for all n.

Then $d(T(x),T(y)) \leq \psi(d(x,y))$ with

$$\psi(1 + \frac{1}{n}) = 1 + \frac{1}{n+1}$$ and there is no fixed point.

This example shows that T may be a nonlinear contraction in a complete metric space, while T has no fixed point, resolving the question raised by Boyd and Wong [7].

Browder [11] has given the following result.

Theorem 1.4.15. Let X be a complete metric space, M a bounded subset of X, T a mapping of M into M. Suppose that there exists a monotone nondecreasing function $\psi(r)$ for $r \geq 0$ with ψ continuous on the right, such that $\psi(r) < r$ for all $r > 0$, while for all $x,y \in M$,

$$d(T(x),T(y)) \leq \psi(d(x,y)).$$

Then for each $x_o \in M$, $\{T^n(x_o)\}$ converges to an element z of X, independent of x_o, and

$$d(T^n(x_o),z) \leq \psi^n(d_o)$$

where d_o is the diameter of M, ψ^n is the n-th iterate of ψ, and

$$d_n = \psi^n(d_o) \to 0 \quad \text{as} \quad n \to \infty .$$

Corollary 1.4.16. T can be extended in a unique way to a continuous mapping of $\overline{M} \to \overline{M}$ and z is the unique fixed point of this extended mapping.

Krasmoselkii and Stecenko [46] proved the following result.

Theorem 1.4.17. Let X be a complete metric space and $T : X \to X$ be a function satisfying

$$d(T(x),T(y)) \leq d(x,y) - \Delta(d(x,y)), \quad \text{for all } x,y \in X,$$

where Δ is a continuous real valued function on $[0,\infty)$ such that $\Delta(r) > r$ for $r > 0$. Then T has a unique fixed point.

Zitarosa [79] has given the following theorem , which generalizes the Banach Contraction Principle and the theorem due to Rakotch [62].

Let X be a complete metric space and $T : X \to X$ be a continuous mapping. If $\{T^n(x)\}$ converges then $\{T^n(x), T^{n+1}(x)\}$ converges to zero. Let

$$A(T,\delta) = \{x \in X : d(x,T(x)) < \delta\} \quad \text{and}$$

S be the set of all continuous mappings $T : X \to X$ such that for some positive integer n and for each $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$\operatorname{diam}(A(T,\delta) \cap T^n(x)) < \varepsilon \quad .$$

Theorem 1.4.18. If $T \in S$ and $x \in X$ are such that $\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0$, then $\{T^n(x)\}$ converges to a fixed point of T.

1.5. Nonexpansive Mappings.

Definition 1.5.1. A mapping T of a metric space X into itself is said to be nonexpansive (ε-nonexpansive) if

$$d(T(x),T(y)) \leq d(x,y), \quad \text{for all } x,y \in X \quad (\text{for all } x,y \text{ with } d(x,y) < \varepsilon).$$

An isometry, i.e., $|T(x) - T(y)| = |x - y|$ for all $x,y \in X$, is a simple example of a nonexpansive mapping.

Definition 1.5.2. A point $y \in Y \subset X$ is said to belong to the T closure of Y, $y \in Y^T$, if $T(Y) \subset Y$ and there exists a point $x \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < n_3 < \ldots < n_i < \ldots)$, so that the sequence $\{T^{n_i}(x)\}$ converges to y.

Definition 1.5.3. A sequence $\{x_i\} \in X$ is said to be an isometric (ε-isometric) sequence if the condition

$$d(x_m, x_n) = d(x_{m+k}, x_{n+k})$$

holds for all $k,m,n = 1,2, \ldots$

(for all $k,m,n = 1,2, \ldots,$ with $d(x_m, x_n) < \varepsilon$).

A point $x \in X$ is said to generate an isometric (ε-isometric) sequence, under T, if $\{T^n(x)\}$ is such a sequence.

The following theorems are given by M. Edelstein [28] for nonexpansive and ε-nonexpansive mappings on a metric space X.

Theorem 1.5.4. If $T : X \to X$ is ε-nonexpansive and $x \in X^T$ then a sequence $\{m_j\}$, $(m_1 < m_2 < \ldots)$, of positive integers exists so that $\lim_{j\to\infty} T^{m_j}(x) = x$. [Hence, in particular, $(X^T)^T = X^T$].

Theorem 1.5.5. If $T : X \to X$ is nonexpansive (ε-nonexpansive) mapping, then each $x \in X^T$ generates an isometric (ε-isometric) sequence.

Cheney and Goldstein have proved the following theorem [15].

Theorem 1.5.6. Let T be a mapping of a metric space X into itself such that

(i) $d(T(x),T(y)) \leq d(x,y)$,

(ii) if $x \neq T(x)$, then $d(T(x),T^2(x)) < d(x,T(x))$,

(iii) for each $x \in X$, the sequence $\{T^n(x)\}$ has a cluster point.

Then for each x, the sequence $\{T^n(x)\}$ converges to a fixed point of T.

K.L. Singh [70] has proved the above theorem by relaxing conditions (ii) and (iii) in the following way.

Theorem 1.5.7. Let T be a mapping of a compact metric space X into itself such that

$$d(T(x),T(y)) \leq d(x,y), \text{ equality}$$

holding when $x = y$. Then T has a fixed point.

1.6. Mappings of Iterative Type.

The following results have been given by Kannan [40].

Theorem 1.6.1. If T_1 and T_2 are two mappings of a complete metric space X into itself and if

$$d(T_1(x),T_2(y)) \leq \alpha[d(x,T_1(x)) + d(y,T_2(y))],$$

$$\text{for all } x,y \in X, \text{ and } 0 \leq \alpha < \frac{1}{2},$$

then T_1 and T_2 have a common fixed point.

If $T_1 = T_2 = T$, then we obtain the following theorem.

Theorem 1.6.2. If T is a mapping of a complete metric space X into itself and if

$$d(T(x),T(y)) \leq \alpha[d(x,T(x)) + d(y,T(y))],$$

for all $x,y \in X$ and $0 \leq \alpha < \frac{1}{2}$,

then T has a unique fixed point.

The conclusion of Theorem 1.6.1. can also be obtained under different assumptions.

Theorem 1.6.3. Let T_1 and T_2 be mappings of a complete metric space X into itself and let

(i) $d(T_1(x),T_2(y)) \leq \alpha d(x,y)$; for $x,y \in X$, $x \neq y$, and $0 < \alpha < 1$,

(ii) T_2 is a contraction mapping,

(iii) there exists some $x \in X$ such that if

$x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$,

$x_4 = T_2(x_3)$, and so on, then $x_r \neq x_s$ if $r \neq s$.

Then T_1 and T_2 have a common fixed point.

Theorem 1.6.3. has been generalized by Singh [73] in the following way.

Theorem 1.6.4. Let T_1 and T_2 be two mappings of a complete metric space X into itself, and let

(i) $d(T_1(x),T_2(y)) \leq \alpha d(x,y)$, for $x,y \in X$, $x \neq y$, and $0 < \alpha < 1$,

(ii) T_2 has a unique fixed point,

(iii) there exists some $x \in X$ such that if

$x_1 = T_1(x)$, $x_2 = T\ (x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$, ...,

$x_r \neq x_s$ if $r \neq s$.

Then T_1 and T_2 have a common fixed point.

That the above theorem is more general than Theorem 1.6.3. can be demonstrated by the following example of Singh [73].

Example 1.6.5. Let $T_2 : [0,1] \to [0,1]$ be defined by

$$T_2(x) = \frac{x}{4} \quad , \quad x \in [0, \tfrac{1}{2})$$
$$= \frac{x}{5} \quad , \quad x \in [\tfrac{1}{2}, 1] \; .$$

Clearly, T_2 is discontinuous at $x = \frac{1}{2}$ and therefore T_2 is not a contraction mapping, but T_2 has a unique fixed point, $x = 0$.

Singh [73] also gives the following theorem.

Theorem 1.6.6. Let T_1 and T_2 be two mappings of a complete metric space X into itself and let

(i) $d(T_1(x),T_2(y)) \leq \alpha d(x,y)$; $x,y \in X$, $x \neq y$, $0 \leq \alpha < 1$,

(ii) there exists some $x \in X$ such that if $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$, ..., $x_r \neq x_s$ if $r \neq s$.

Then T_1 and T_2 have a common fixed point.

The following theorems are due to Kannan [41] in which the completeness of the space is omitted.

Theorem 1.6.7. Let X be a metric space, and T be a map of X into itself such that

(i) $d(T(x),T(y)) \leq \alpha[d(x,T(x)) + d(y,T(y))]$, for $x,y \in X$, and $0 < \alpha < \frac{1}{2}$,

(ii) T is continuous at a point $z \in X$,

(iii) there exists some point $x \in X$ such that the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ which converges to z.

Then z is a unique fixed point of T.

Theorem 1.6.8. Let T be a mapping of a metric space X into itself such that

(i) $d(T(x),T(y)) \leq \alpha[d(x,T(x)) + d(y,T(y))], \quad 0 < \alpha < \frac{1}{2}$,

$x,y \in M$, where M is an everywhere dense subset of X,

(ii) there exists a point $x \in X$ such that the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ which converges to a point $z \in X$.

Then z is a unique fixed point of T.

Theorem 1.6.9. Let T be a mapping of a metric space X into itself. Suppose T is continuous at a point $x_o \in X$, and there exists some point $x \in X$ whose sequence of iterates $\{T^n x\}$ converges to x_o, then $T(x_o) = x_o$. Moreover, if

$$d(T(x_o),T(y)) \leq \alpha d(x_o,y), \quad y \in X, \quad 0 < \alpha < 1,$$

then x_o is the unique fixed point of T.

Singh [71] has given the following extension of Theorem 1.6.9 by taking $\alpha = \frac{1}{2}$.

Theorem 1.6.10. Let T be a continuous mapping of a metric space X into itself such that

$$d(T(x),T(y)) < \frac{1}{2}[d(x,T(x)) + d(y,T(y))], \quad \text{for } x \neq y.$$

If for some $x \in X$, the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ converging to z, then $\{T^n x\}$ converges to z and z is the unique fixed point of T.

Theorem 1.6.11. Let T be continuous mapping of a metric space X into itself such that

(i) $d(T(x),T(y)) \leq \frac{1}{2}[d(x,T(x)) + d(y,T(y))]$,

(ii) if $x \neq Tx$, then $d(T^2(x),T(x)) < d(T(x),x)$,

(iii) for some $x \in X$, the sequence $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to z.

Then the sequence $\{T^n(x)\}$ converges to z and z is the unique fixed point of T.

Theorem 1.6.12. Let T be a mapping of a metric space X into itself such that

(i) $d(T(x),T(y)) \leq d(x,y)$, for all $x,y \in X$,

(ii) $d(T(x),T(y)) < \frac{1}{2}[d(x,T(x)) + d(y,T(y))]$, for all $x,y \in X$,

(iii) for some $x \in X$, the sequence $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to z.

Then $\{T^n(x)\} \to z$ and z is the unique fixed point of T.

Simeon Reich [63] has generalized Kannan's result [41] in the following way.

Theorem 1.6.13. Let T be a mapping of a complete metric space X into itself such that

$$d(T(x),T(y)) \leq ad(x,T(x)) + bd(y,T(y)) + cd(x,y), \quad x,y \in X,$$

where a, b, c are nonnegative and $a + b + c < 1$.

Then T has a unique fixed point.

That Reich's theorem is stronger than Banach's (Theorem 1.2.7) and Kannan's (Theorem 1.6.2) can be seen in the following example.

Example 1.6.14. Let $X = [0,1]$ and let T be defined by

$$T(x) = \frac{x}{3}\ , \quad 0 \leq x < 1,$$

$$T(1) = \frac{1}{6}\ .$$

Clearly, T does not satisfy Banach's condition since it is not continuous at 1. Kannan's condition also cannot be satisfied because

$$d(T(0),T(\tfrac{1}{3})) = \tfrac{1}{2}d(0,T(0)) + d(\tfrac{1}{3},\ T(\tfrac{1}{3})).$$

But T satisfies Reich's condition if we put $a = \frac{1}{6}$, $b = \frac{1}{9}$, $c = \frac{1}{3}$; and these are not the smallest possible values.

In [62], Rakotch proved the following result.

Theorem 1.6.15. Let T be a mapping of a complete metric space X into itself such that

$$d(T(x),T(y)) \leq k(d(x,y))d(x,y), \quad x,y \in X, \quad x \neq y,$$

where $k : [0,\infty) \to [0,1)$ as defined in 1.4.5.

Then T has a unique fixed point.

Reich [64] further generalized Rakotch's result and Theorem 1.6.13.

Theorem 1.6.16. Let X be a complete metric space and let a, b, c be monotonically decreasing functions from $(0,\infty)$ into $[0,1)$ such that $a(t) + b(t) + c(t) < 1$. Suppose that $T : X \to X$ satisfies

$$d(T(x),T(y)) \leq a(d(x,y))d(x,T(x))+b(d(x,y))d(x,T(y))+c(d(x,y))d(x,y),$$

where $x,y \in X$, $x \neq y$.

Then T has a unique fixed point.

In [72] Singh has shown the relationship between the Banach Contraction Principle (Theorem 1.2.7) and Kannan's fixed point theorem (Theorem 1.6.2) in the following way:

For $k < \frac{1}{3}$, $d(T(x),T(y)) \leq kd(x,y)$, $x,y \in X$,

implies that

$$d(T(x),T(y)) \leq \alpha[d(x,T(x)) + d(y,T(y))], \quad x,y \in X,$$

$$\text{and} \quad 0 < \alpha < \frac{1}{2}.$$

Proof. $d(T(x),T(y)) \leq kd(x,y)$

$$\leq k[d(x,T(x)) + d(T(x),T(y)) + d(T(y),y)].$$

This implies that

$$d(T(x),T(y)) \leq \frac{k}{1-k}[d(x,T(x)) + d(y,T(y))].$$

Now, $k < \frac{1}{3}$ implies $\alpha = \frac{k}{1-k} < \frac{1}{2}$.

CHAPTER II

The Measure of Non Compactness and Related Results

2.1. Preliminaries of Banach Spaces.

Definition 2.1.1. A linear (vector) space X is said to be normed if to each element $x \in X$ there is made to correspond a nonnegative real number $||x||$, called the norm of x, such that

(i) $||x|| = 0$ if and only if $x = 0$,

(ii) $||\lambda x|| = |\lambda| \cdot ||x||$, for all $\lambda \in K$, $x \in X$,
where K is either the field of real numbers or the field of complex numbers,

(iii) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality) .

Remark. It is easy to see that every normed space is also a metric space, with metric $d(x,y) = ||x - y||$.

Definition 2.1.2. A normed space X is called a Banach space if it is complete as a metric space.

Example 2.1.3. The space $C[a,b]$, of all functions continuous on the interval $[a,b]$ with norm

$$||f|| = \max_{a \leq t \leq b} |f(t)| \quad ,$$

is a normed linear space.

Remark. The above space is complete in the given norm, and hence is a Banach space. However, the space $C^2[a,b]$ with norm

$$||f||_2 = \left(\int_a^b (f(t))^2 dt \right)^{\frac{1}{2}}$$

is incomplete. A proof may be found in [44 , p. 59].

Definition 2.1.4. A set C in a normed space X is said to be convex if $\alpha x + (1 - \alpha)y \in C$ whenever $x,y \in C$ and $0 \leq \alpha \leq 1$.

The following definitions are due to Clarkson [18].

Definition 2.1.5. A Banach space X is called uniformly convex if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$,

then $$\left|\left|\frac{x + y}{2}\right|\right| \leq 1 - \delta .$$

In other words, X is uniformly convex if for any two points x,y on the unit sphere, $S(0,1)$, the midpoint of the segment joining x and y can be close to but not on that sphere only if x and y are sufficiently close to each other.

Every Hilbert space is uniformly convex.

Definition 2.1.6. A Banach space X is called strictly convex if, for any pair of vectors $x,y \in X$, and $||x + y|| = ||x|| + ||y||$, it follows that $x = \lambda y$, $\lambda > 0$.

It is worth noting that every uniformly convex Banach space is strictly convex; however, the converse is not true.

Definition 2.1.7. Let A be a bounded subset of a metric space X. Then the diameter of A, denoted by $\delta(A)$, is $\sup\{d(x,y) : x,y \in A\}$.

Definition 2.1.8. Let C be a convex bounded set in a Banach space X of diameter d. A point $x \in X$ is said to be diametral for C if

$$\sup\{||x - y|| : y \in C\} = d.$$

In the Banach space $C[0,1]$ every point of the convex and bounded set

$$\{x(t) : 0 \leq t \leq 1, x(0) = 0, x(1) = 1 \text{ and } 0 \leq x(t) \leq 1 \text{ for } 0 \leq t \leq 1\}$$

is a diametral point.

Definition 2.1.9. [8] A convex set K in a Banach space X is said to have normal structure if for each bounded convex subset C of K which contains more than one point there exists a point $x \in C$ which is not diametral for C.

Geometrically, K has normal structure if for every bounded and convex subset C of K there exists a ball of radius less than the diameter of C centred at a point of C and containing C.

We would like to give two well-known results related to normal structure.

Theorem 2.1.10.[8] Every convex and compact subset of a Banach space has normal structure.

Theorem 2.1.11.[27,10] Every uniformly convex Banach space X has normal structure.

Definition 2.1.12. For a given Banach space X, X* will denote its first conjugate (dual) space, i.e., the linear space of all linear continuous functionals $T : X \to \mathbb{R}$ or $\mathbb{C}$, with the usual norm (denote by (T,x) the value T(x) of T at x) :

$$||T|| = \sup\{|(T,x)| : ||x|| \leq 1\} .$$

Remark. For any fixed vector x in a Banach space X, the mapping of X* into $\mathbb{R}$ (or $\mathbb{C}$, if X is a complex Banach space) which to every T in X* assigns the value (T,x) of T at x is a linear continuous functional in the space X*, i.e. an element of the space (X*)* noted also as X^{**}. Moreover, the norm of this functional is equal to the norm $||x||$. It may be easily verified that the canonical mapping of X into X^{**} defined by this correspondence between elements of X and linear continuous functionals on X^* is linear and one-to-one. Therefore, it is an isometrical imbedding of X into X^{**}.

Definition 2.1.13. A Banach space X is called reflexive if the canonical imbedding of X into X^{**} is onto.

It is clear that every Hilbert space is reflexive. Also, a uniformly convex Banach space is reflexive.

2.2. Densifying Mappings and k-set Contractions.

Definition 2.2.1. [47,p. 318] Let A be a bounded subset of a metric space X. By the real number $\alpha(A)$ we denote the infimum of all numbers $\varepsilon > 0$ such that A admits a finite covering consisting of subsets of A with diameter less than ε.

We state the following properties of α, and give appropriate proofs. (For detailed discussion see Nussbaum [54]).

Theorem 2.2.2. [19, 47] Let A be a bounded subset of a metric space X, and let $N_r(A) = \{x \in X : d(x,A) < r\}$.

Then (i) $\alpha(A) = \varepsilon \leq \delta(A)$, where $\delta(A)$ = diameter of A,

(ii) if $A \subset B$, then $\alpha(A) \leq \alpha(B)$,

(iii) $\alpha(N_r(A)) \leq \alpha(A) + 2r$,

(iv) let $\bar{A}$ denote the closure of A, then $\alpha(\bar{A}) = \alpha(A)$,

(v) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

Proof. (1) It clearly follows from the definition of $\alpha(A)$.

(2) Suppose $\alpha(B) = d$.

Then, given any $\varepsilon > 0$,

we can write $B = \bigcup_{j=1}^{n} T_j$, where $\delta(T_j) \leq d + \varepsilon$.

Since $A \subset B$, we have $A = \bigcup_{j=1}^{n} (T_j \cap A)$,

and $\delta(T_j \cap A) \leq d + \varepsilon$.

Since ε is arbitrary, we have $\alpha(A) \leq d$.

Hence $\alpha(A) \leq \alpha(B)$.

(3) Let $\alpha(A) = d$.

Then given any $\varepsilon > 0$, we have $A = \bigcup_{j=1}^{n} S_j$,

where $\delta(S_j) \leq d + \varepsilon$.

It follows that $N_r(A) = \bigcup_{j=1}^{n} N_r(S_j)$,

and since $\delta(N_r(S_j)) \leq d + 2r + \varepsilon$,

we have $\alpha(N_r(A)) \leq \alpha(A) + 2r$.

(4) Since $A \subset \bar{A}$, by part (2) above, we

have $\alpha(A) \leq \alpha(\bar{A})$.

Now $\bar{A} \subset N_r(A)$, for all $r > 0$,

therefore $\alpha(\bar{A}) \leq \alpha(N_r(A)) \leq \alpha(A) + 2r$, for all $r > 0$.

Hence $\alpha(\bar{A}) \leq \alpha(A)$ (Since r is arbitrarily small).

And hence, $\alpha(\bar{A}) = \alpha(A)$.

(5) We have $A \subset (A \cup B)$ and $B \subset (A \cup B)$,

therefore, by (2), we have

$$\alpha(A) \leq \alpha(A \cup B), \text{ and } \alpha(B) \leq \alpha(A \cup B);$$

i.e. $\max\{\alpha(A), \alpha(B)\} \leq \alpha(A \cup B)$.

It remains to show the inequality in the opposite sense.

Let $d = \max\{\alpha(A), \alpha(B)\}$,

then given any $\varepsilon > 0$, we can find sets

S_j , for $j = 1,\ldots,m$, and T_i , for $i = 1,\ldots,n$,

such that $A = \bigcup_{j=1}^{m} S_j$ and $B = \bigcup_{i=1}^{n} T_i$,

with $\delta(S_j) \leq d + \varepsilon$ and $\delta(T_i) \leq d + \varepsilon$.

Since $(A \cup B) \subset (\bigcup_{j=1}^{m} S_j) \cup (\bigcup_{i=1}^{n} T_i)$,

and ε is arbitrarily small,

we have $\alpha(A \cup B) \leq d = \max\{\alpha(A), \alpha(B)\}$.

Hence $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

Corollary 2.2.3. If A, a subset of X, is totally bounded (or precompact), then $\alpha(A) = 0$.

Proof. A is totally bounded. Therefore, for any $\varepsilon > 0$, A can be covered by a finite union of subsets with diameter smaller than or equal to ε. And, since ε is arbitrarily small, $\alpha(A) = 0$.

Definition 2.2.4. [44] Given any subset A of a normed linear space X, there is a smallest convex set containing A, i.e., the intersection of all convex sets containing A (there is at least one convex set containing A, namely X itself). This minimal set containing A is called the convex hull of A, denoted by $\bar{\omega}A$. For example, the convex hull of three noncollinear points is the triangle with these three points as vertices.

Lemma 2.2.5. [19] If A is a bounded subset of X, and $\bar{\omega}A$ its convex hull, then $\delta(\bar{\omega}A) = \delta(A)$.

Proof. For convenience, let us denote the $\bar{\omega}A$ by A^*.

Since $A \subset A^*$, we have $\delta(A) \leq \delta(A^*)$.

To prove equality, let us assume $\delta(A) < \delta(A^*)$.

Then $\exists\ x,y \in A^*$, $x,y \notin A$

such that $\delta(A) < ||x - y||$.

Consider $S_x = S(x,\delta(A))$, the sphere about x with radius $\delta(A)$.

Now, either $A \subset S_x$ or $A \not\subset S_x$.

Suppose $A \subset S_x$, then $S_x \cap A^*$ is convex

$(S_x \cap A^*) \supset A$

$(S_x \cap A^*)$ is a proper subset of A^*

$y \notin (S_x \cap A^*)$, which is absurd,

since A^* is the smallest closed convex set containing A.

Suppose, on the other hand, $A \not\subset S_x$

then $\exists\ z \in A$ such that $z \notin S_x$.

Let $S_z = S(z,\delta(A))$,

then $S_z \supset A$

and $x \in A^*$, $x \notin S_z$;

i.e. $A^* \not\subset S_z$,

which is absurd, since A^* is the smallest closed convex set containing A.

Hence, we conclude $\delta(A) < \delta(\bar{\omega}A)$ is false;

i.e. $\delta(A) = \delta(\bar{\omega}A)$.

We state the next theorem without proof. A proof can be found in R.D. Nussbaum [54].

Theorem 2.2.6. If A is a bounded subset of X and $\bar{\omega}A$ is its convex hull, then $\alpha(\bar{\omega}A) = \alpha(A)$.

Lemma 2.2.7. [34] Let S^{n-1} be the unit sphere of the real Euclidean space E^n.

Let $\{A_1, A_2, \ldots, A_q\}$ be a finite family of closed sets with the following two properties:

(a) $\bigcup_{i=1}^{q} A_i = S^{n-1}$,

(b) none of the sets A_i, $(i = 1,2,\ldots,n)$ contains two antipodal (symmetric with respect to the origin) points.

Then $q > n$.

Lemma 2.2.8. [34] Let S be the unit sphere in a finite dimensional real normed linear space X.

If $\{A_1, A_2, \ldots, A_q\}$ is a finite closed covering of S, and none of the sets A_i has a pair of antipodal points, then $q > n$.

Proof. It is easily seen that there exists a homeomorphism $\tau : X \to S^{n-1}$, such that $\tau(-x) = -\tau(x)$.

From this and Lemma 2.2.7, the result is immediate.

Theorem 2.2.9. [34,54] Let X be an infinite dimensional linear normed space.

Let $B = \{x \in X : ||x|| \leq 1\}$, be its unit ball

and $S = \{x \in X : ||x|| = 1\}$, its unit sphere.

Then $\alpha(B) = \alpha(S) = 2$.

Proof. Since $B = \bar{\omega}S$, by Theorem 2.2.6, we have $\alpha(B) = \alpha(S)$; and since $\delta(S) = 2$, it is certain that $\alpha(S) \leq 2$.

Let us assume $\alpha(S) < 2$, and proceed by contradiction.

If $\alpha(S) < 2$, we can write $S = \bigcup_{i=1}^{n} T_i$,

with $\delta(T_i) < 2$.

We assume T_i closed, for if not, we can take its closure.

Let F be an n-dimensional subspace of X, and consider $S \cap F = \bigcup_{i=1}^{n} (T_i \cap F)$.

By the Lusternik-Schnirelman-Borsuk Theorem [24,p. 349], if the unit sphere in an n-dimensional vector space is covered by n closed sets, then at least one of them contains a pair of antipodal points; i.e. points x and $-x$ for some x on the unit sphere.

This means that some $(T_i \cap F)$ contains a pair of antipodal points, so that for this $(T_i \cap F)$ we have

$$2 \leq \delta(T_j \cap F) \leq \delta(T_j). \quad \text{Contradiction.}$$

Hence $\alpha(S) = 2$.

i.e. $\alpha(B) = \alpha(S) = 2$.

Definition 2.2.10. [2*] Let T be a continuous mapping of a metric space X into itself. Then T is called a k-set contraction if

(i) for all $A \subset X$, with A bounded, we have $T(A)$ bounded, and

(ii) $\alpha(T(A)) \leq k\alpha(A)$, where $0 \leq k < 1$.

Definition 2.2.11. [31] If condition (ii) in the above definition is replaced by

$$\alpha(T(A)) < \alpha(A), \quad \text{for } \alpha(A) > 0,$$

then we call T densifying.

*(see added reference)

Definition 2.2.12. [19] The modulo of a k-set-contraction T, denoted by k_T, is the smallest k in order that T is a k-set-contraction.

Remark. A k-set-contraction with $k \in [0,1)$ is densifying, but the converse is not true.

Consider the following example given by Nussbaum in [54].

Let $\ell : [0,1] \to \mathbb{R}$ be a strictly decreasing nonnegative function with $\ell(0) = 1$.

Let B be the unit ball about the origin 0 in an infinite dimensional Banach space X,

$$B = \{x \in X : ||x|| \leq 1\} .$$

And, consider the map $T : B \to B$,

as defined by $$T(x) = x\ell(x).$$

Then, T is denisfying, but T is not a k-set-contraction for any $k \in [0,1)$.

To see this, consider $T(V_r(0))$, for $0 < r \leq 1$, where $V_r(0)$ is the closed ball of radius r.

Clearly $V_{r\ell(r)}(0) \subset T(V_r(0))$.

By Theorem 2.2.9, $\alpha(V_r(0)) = 2r$,

and $\alpha(V_{r\ell(r)}(0)) = 2r\ell(r)$.

Hence T is at best a $\ell(r)$-set-contraction.

Since $\ell(r) \to 1$ as $r \to o$, T cannot be a k-set contraction for any $k \in [0,1)$.

T, however, is a 1-set-contraction.

Let A be a bounded subset of B.

Then $T(A) \subset \bar{\omega}(A \cup \{0\})$,

therefore $\alpha(T(A)) \leq \alpha(\bar{\omega}(A \cup \{0\})) = \alpha(A)$,

i.e. $\alpha(T(A)) \leq \alpha(A)$.

Hence T is a 1-set contraction.

However, more can be said about T.

Let $A \subset B$ and $\alpha(A) = d > 0$.

Choose $r < \frac{d}{2}$,

and define $A_1 = A \cap V_r(0)$

and $A_2 = A \cap V_r^1(0)$,

where $V_r^1(0)$ is the complement of $V_r(0)$.

Then $A = A_1 \cup A_2$,

and $T(A) = T(A_1) \cup T(A_2)$.

Since T is a 1-set contraction, we have

$$\alpha(T(A_1)) \leq 2r < d = \alpha(A).$$

Since ℓ is strictly decreasing,

and $||x|| \geq r$, for $x \in A_2$, we have

$$\alpha(T(A_2)) \leq \ell(r)\alpha(A) < \alpha(A).$$

Hence $\alpha(TA) = \max\{\alpha(T(A_1)), \alpha(T(A_2))\} < \alpha(A)$.

and hence T is densifying.

Example 2.2.13. [31] Let T be a completely continuous mapping of a metric space X into itself, then T is a k-set-contraction.

Proof. T maps bounded sets to precompact sets.

For A a bounded subset of X, we have $\alpha(T)A)) = 0 \leq k\alpha(A)$, for all $k \in [0,1)$.

In this case, $k_T = 0$, and by the previous remark, T is obviously densifying.

Example 2.2.14. [31] Any contraction mapping T of a metric space X into itself is a k-set contraction, with k_T = the Lipschitz constant k of T.

Proof. Let A be a bounded subset of X such that $0 < \alpha(A)$. Choose $\varepsilon > \alpha(A)$.

A can be covered by a finite number of subsets $\{A_1, A_2, \ldots, A_n\}$ of A, such that $\delta(A_i) \leq \alpha(A) + \varepsilon$, $i = 1,2, \ldots, n$;

i.e. $A \subset \bigcup_{i=1}^{n} A_i$, and $\delta(A_i) \leq \alpha(A) < \varepsilon$, for all i.

Clearly, $T(A) \subset \bigcup_{i=1}^{n} TA_i$ (by continuity of T).

Let $1 \leq i \leq n$ be fixed, and take $x,y \in A_i$,

then $d(Tx,Ty) \leq kd(x,y) \leq k\delta(A_i) \leq k\alpha(A) < \varepsilon$.

Therefore $\delta(TA_i) \leq k\delta(A_i)$, $i = 1, 2, \ldots, n$;

i.e. $\alpha(TA) \leq k\alpha(A)$.

Hence T is a k-set-contraction.

Example 2.2.15. [32] Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a right continuous non-decreasing function defined on $[0,\infty)$ such that $\phi(r) < r$, for $r > 0$.

Let $T : X \to X$ be a mapping of a complete metric space X into itself such that $d(T(x),T(y)) \leq \phi(d(x,y))$, for all $x,y \in X$.

Then T is densifying.

Proof of this example follows on similar lines as the previous example, and was given by Furi and Vignoli in [32].

Theorem 2.2.16. [19] $F(T) = \{x \in X : x = T(x),\ T$ is a k-set contraction$\}$, the space of solutions of the equation $x = T(x)$ in a metric space X, is locally compact.

Proof. $F(T)$ is, in fact, closed.

Let H be a bounded subset of $F(T)$.

Then $H = T(H)$, and $\alpha(H) = \alpha(T(H)) \leq k_T\ \alpha(H)$.

Hence $\alpha(H) = 0$;

i.e. H is precompact.

Theorem 2.2.17. [19] Let T be a k-set-contraction of a metric space X into itself and let $U : X \to X$ be such that

$$d(U(x),U(y)) \leq d(T(x),t(y)), \quad \text{for all } x,y \in X. \tag{*}$$

Then U is a k-set-contraction with $k_U \leq k_T$.

Proof. $d(U(x),U(y)) \leq d(T(x),T(y))$, for all $x,y \in X$,

hence U is continuous and maps bounded sets to bounded sets.

Now, let A be a bounded subset of X, and choose $\varepsilon > 0$.

$A \subset \bigcup_{i=1}^{n} A_i$, with $\delta(A_i) \leq \alpha(A) + \varepsilon$.

Since T is continuous, we have

$$T(A) \subset \bigcup_{i=1}^{n} T(A_i), \quad \text{and} \quad \delta(T(A_i)) \leq \alpha(T(A)) + \varepsilon$$
$$\leq k_T\alpha(A) + \varepsilon .$$

Let $A_j = \{x \in A : T(x) \in T(A_j)\}$,

then $A = \bigcup_j A_j$.

by (*), $\delta(U(A_j)) \leq \delta(T(A_j)) \leq k_T\alpha(A) + \varepsilon$.

U is continuous, and therefore

$$U(A) \subset \bigcup_{j=1}^{n} U(A_j), \quad \text{with} \quad \delta(U(A_j)) \leq k_T\alpha(A) + \varepsilon \quad ;$$

i.e. $\alpha(U(A)) \leq k_T\alpha(A) + \varepsilon$.

Since ε is arbitrarily small, $\alpha(U(A)) \leq k_T\alpha(A)$.

Hence U is a k-set-contraction with $k_U \leq k_T$.

Theorem 2.2.18 [54] If T and U are k-set-contractions of a metric space X into itself, then $V = T \circ U$ is a k-set-contraction and $k_V = k_T k_U$.

Proof. Let A be a bounded subset of X,

then
$$\begin{aligned}\alpha(V(A)) &= \alpha((T \circ U)(A)) \\ &= \alpha(T(U(A))) \\ &= k_T\alpha(U(A)) \\ &= k_T k_U \alpha(A), \quad 0 \leq k_T k_U < 1.\end{aligned}$$

That is, V is a k-set-contraction with $k_V = k_T k_U$.

Lemma 2.2.19 [19]. Let A,B be a bounded subset of a Banach space X and denote $\{a + b : a \in A, b \in B\}$ by $A + B$.

Then $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

Proof. Suppose $\alpha(A) = d_1$ and $\alpha(B) = d_2$.

Then given $\varepsilon > 0$, we can write

$$A = \bigcup_{i=1}^{n} S_i \quad , \quad \text{with} \quad \delta(S_i) \leq d_1 + \varepsilon/2 \ ,$$

and
$$B = \bigcup_{i=1}^{n} T_j \quad , \quad \text{with} \quad \delta(T_j) \leq d_2 + \varepsilon/2.$$

Clearly $A + B = \bigcup_{i,j} (S_i + T_j)$.

Therefore, to show $\alpha(A + B) \leq d_1 + d_2 + \varepsilon$, it is sufficient to show $\delta(S_i + T_j) \leq d_1 + d_2 + \varepsilon$.

To this purpose, take $x,y \in S_i + T_j$.

We can write $x = a + b$ and $y = c + d$,

where $a,c \in S_i$ and $b,d \in T_j$.

It follows that

$$||x - y|| \leq ||a - c|| + ||b - d||$$
$$\leq (d_1 + \varepsilon/2) + (d_2 + \varepsilon/2) = d_1 + d_2 + \varepsilon \quad .$$

Hence $\delta(S_i + T_j) \leq d_1 + d_2 + \varepsilon$.

Since ε can be taken arbitrarily small,

we have $\delta(S_i + T_j) \leq d_1 + d_2 = \alpha(A) + \alpha(B)$.

Hence $\delta(A + B) \leq \alpha(A) + \alpha(B)$.

Theorem 2.2.20 [19]. Let X be a metric space and Y a Banach space. Let $T : X \to Y$ be a k_1-set-contraction, and $U : X \to Y$ a k_2-set-contraction. Then $(T + U) : X \to Y$ is a $(k_1 + k_2)$-set-contraction.

Proof. Let A be a bounded subset of X

Then $(T + U)(A) \subseteq T(A) + U(A)$.

By the above lemma,

$$\begin{aligned} \alpha((T + U)(A)) &= \alpha(T(A) + U(A)) \\ &\leq \alpha(T(A)) + \alpha(U(A)) \\ &\leq k_1\alpha(A) + k_2\alpha(A) \\ &= (k_1 + k_2)\alpha(A). \end{aligned}$$

Hence $(T + U)$ is a $(k_1 + k_2)$-set-contraction.

We state without proof the following generalization of the Cantor

Intersection Theorem [69,p. 73].

Lemma 2.2.21 [2*]p. 518] Let X be a complete metric space and $A_1, A_2, \ldots$ be a decreasing sequence of non-empty closed subsets of X.

Assume that $\alpha(A_n) \to 0$, as $n \to \infty$.

Then $A_\infty = \bigcap_{n \geq 1} A_n$ is a non-empty compact set, and A_n approaches A_∞ in the Hausdorff** metric.

Theorem 2.2.22. [54] Let C be a closed bounded convex subset of a Banach space X.

Let $T : C \to C$ be a continuous map.

Let $C_1 = \bar{\omega}(T(C))$, and $C_n = \bar{\omega}(T(C_{n-1}))$, for $n > 1$.

Further, assume $\alpha(C_n) \to 0$, as $n \to \infty$.

Then $F(T) \neq \phi$; i.e. T has at least one fixed point.

Proof. Clearly C_n is closed, bounded, convex, and nonempty, and $C_n > C_{n+1}$, for $n \geq 1$.

Then, by the above lemma,

$C_\infty = \bigcap_{n \geq 1} C_n$ is nonempty and compact, and C_∞ is convex.

By our construction, $T : C_n \to C_{n+1}$,

so that, $T : C_\infty \to C_\infty$.

Hence it follows by Schauder-Tychonoff Fixed Point Theorem [19,p. 456] that T has a fixed point;

i.e. $F(T) \neq \phi$.

**$D(A,B)$, where A and B are closed subsets of a metric space, is $\max\{d(A,B), d(B,A)\}$, with $d(A,B) = \sup\{d(x,B) : x \in A\}$.

*(see added reference)

Corollary 2.2.23 [19]. Let C be a closed, bounded, convex subset of a Banach space X, and $T : C \to C$ a k-set-contraction. Then T has a fixed point.

Proof. Define C_n as above.
To show T has a fixed point, it is sufficient to show that $\alpha(C_n) \to 0$ as $n \to \infty$.
Consider $\alpha(C_1) = \alpha(\bar{\omega}(T(C))) = \alpha(T(C))$

$$\leq k\alpha(C).$$

This implies that $\alpha(C_n) \leq k^n\alpha(C) \to 0$

as $n \to \infty$.

2.3. Condensing Mappings.

In [65] Sadovskii introduced the concept of a condensing operator for mappings defined on subsets of a Banach space and thereby obtained a generalization of the Schauder fixed point theorem [67].

Definition 2.3.1. Let A be a bounded subset of a Banach space X. Denote by $Q(A)$ the set of all $\varepsilon > 0$ such that A has a finite ε-net. Then the measure of noncompactness of the set A is the number $\chi(A) = \inf Q(A)$*. It is clear that $\chi(A)$, the measure of noncompactness, is zero if and only if the set A is compact.

*Note: $\chi(A)$ will, where ambiguity may arise, be denoted by $\chi_X(A)$ if A is considered as a subset of X, and by $\chi_A(A)$ if A is considered as a subset of itself.

Remark . Nussbaum [60] called attention to the fact that, strictly speaking, the measure of noncompactness used by Sadovskii (i.e., $\chi(A)$ in [65]) is slightly different from the definition of $\alpha(A)$ used by Darbo [19], and others [54], [47], [31], since Sadovskii defined $\chi(A)$ to be the infimum of all real $\varepsilon > 0$ such that A admits a finite ε-net. It should be added that $\chi(A)$ does not have all the properties of $\alpha(A)$ since $\chi(A)$ does not depend intrinsically on the bounded set A. (see added reference [10] for details).

Nevertheless, α and χ do have a good deal in common. Thus, for example,

$$\chi_X(A) = \chi_X(\bar{A}),$$

$$\chi_X(A) \leq \chi_X(B) \quad \text{if} \quad A \subset B,$$

$$\chi_X(A \quad B) = \max\{\chi_X(A), \chi_X(B)\},$$

$$\chi_X(A) = 0 \iff \bar{A} \text{ is compact}$$

$$\chi_X(\bar{\omega}A) = \chi_X(A),$$

$$\chi_X(A + B) \leq \chi_X(A) + \chi_X(B).$$

However, if C is a subset of X and A is a bounded subset of C, then $\alpha(A)$ is a number which is independent of whether A is regarded as a subset of C or not; but, in general, $\chi_X(A)$ need not be equal $\chi_C(A)$.

For example [33], $A = \{x \in \ell_2 : ||x|| = 1\}$, then $\chi_A(A) = \sqrt{2}$ while $\chi_X(A) = 1$, where $X = \ell_2$.

Thus, Sadovskii established his theorem for a related class of continuous mappings T of a set C into C such that $\chi(T(C)) < \chi(C)$

and not for densifying maps as we have defined in Section 2.2. Apparently, Furi and Vignoli were first to introduce formally the notion of densifying mappings in [31]. It seems that the fixed point theorem for denisfying mappings T of a closed bounded convex subset C of X into C has been established independently by Nussbaum [54] and Furi and Vignoli [33].

Definition 2.3.2. An operator T from a Banach space X into a Banach space Y is called condensing if it is continuous and if for every bounded noncompact set $A \subset X$ the inequality $\chi(T(A)) < \chi(A)$ holds.

Sadovskii [65] has established the following theorem.

Theorem 2.3.3. If a condensing operator T maps a convex closed bounded set C of a Banach space X into itself, with $T(C) \subseteq C$, then T has at least one fixed point in C.

If in the definition of a condensing operator the strict inequality is replaced by "less than or equal to", the conclusion of the theorem may not hold evenin a Hilbert space, as the following example illustrates.

Example 2.3.4. In the unit ball of the space ℓ_2 let the operator T be defined by the formula

$$T(x) = (\sqrt{1 - ||x||^2}\ , x_1, x_2, \ldots, x_n, \ldots).$$

The continuous operator T transforms the unit ball into the unit sphere. Moreover, it leaves the measure of noncompactness of each set unchanged. Indeed, if the elements

(*) $\qquad x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \ldots), \quad i = 1, 2, \ldots, r,$

form an ε-net of the set A, the compact set S consisting of the elements

$$y = (x_o, x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \ldots),$$
$$x_o \in [0,1], \quad i = 1,2, \ldots, r,$$

forms an ε-net of the set $T(A)$. On the other hand, if the elements (*) form an ε-net of the set $T(A)$ then a finite ε-net of A can be formed from the vectors

$$y_i = (x_{i_2}, x_{i_3}, \ldots) \quad , \quad i = 1,2, \ldots, r.$$

Thus $\chi(T(A)) = \chi(A)$ for any set A. Nevertheless, the operator T does not have any fixed points. In fact any solution $x = T(x)$ must satisfy the equalities $x_1 = x_2 = \ldots = x_n = \ldots$. For an element in ℓ_2 this is equivalent to an equation $x = \theta = (0,0, \ldots, 0, \ldots)$. But $T(\theta) = (1,0,0, \ldots) \neq \theta$, [see 39].

Now, we would like to give some results related to commuting mappings and their common fixed points.

Definition 2.3.5. A family F of mappings of a set A into itself is called commutative if $T_\lambda T_\mu = T_\mu T_\lambda$, for all $\lambda,\mu \in \Delta$.

We state without proof the following theorem due to Kakutani [39] and Markov [49].

Theorem 2.3.6. Let X be a Hausdorff topological space over the reals, and C a convex compact subset of X. Let F be a commutative family of linear continuous mappings in X. Suppose that $T(C) \subset C$ for all $T \in F$. Then there exists a point $x_o \in C$ which is fixed for all $T \in F$.

A similar result, due to Browder [10], is valid for nonexpansive mappings.

Theorem 2.3.7. Let C be a bounded closed subset with normal structure of a reflexive and strictly convex Banach space X. If $\{T_\mu\}$, $\mu \in \Delta$ is a commutative family of nonexpansive mappings of C into itself, then $\{T_\mu\}$ has a common fixed point in C. In particular, every commutative family of nonexpansive mappings of a uniformly convex Banach space into itself has a common fixed point.

We state the following theorem without proof. A proof can be found in [20].

Theorem 2.3.8. Let X be a strictly convex Banach space and T a nonexpansive mapping on X. Then the set $F(T)$ of fixed points of T is convex.

The following result of deMarr[23] states the Markov-Kakutani theorem in another direction.

Theorem 2.3.9. Every commutative family F of nonexpansive mappings of a compact convex subset C of a Banach space X into itself has a common fixed point in C.

The next theorem is an extension of the result of deMarr[23]. It is due to Belluce and Kirk [4].

Theorem 2.3.10. Let C be a bounded closed convex set in a Banach space X, and F a family of commuting nonexpansive mappings of C into itself. Let M be a compact subset of C with the property that

$$M \cap \{T_\mu^n(x) : n = 1, 2, \ldots\} = \phi \quad , \text{ for some } T_\mu \in F$$
$$\text{and all } x \in X.$$

Then there exists $x_o \in C$ such that $T(x_o) = x_o$ for all $T \in F$.

Using the notions of commutative nonexpansive operators and condensing operators, Bakhtin [2] introduced the following two theorems.

Theorem 2.3.11. Let F be a commutative family of nonexpansive operators defined on a closed convex bounded subset C of a Banach space X. Further, let $T(C) \subset C$ for all $T \subset F$, and let there be in this set at least one condensing operator. Then the operators $T \in F$ have in C a common fixed point.

Theorem 2.3.12. If the norm in the Banach space is strictly convex and if in the commutative family F of nonexpansive operators T on X there exists at least one condensing operator T_o with the properties

(i) T_o has at least one fixed point,

(ii) outside a ball $||x|| \leq r$ there are no fixed points of the operator T_o.

Then the operators $T \in F$ have a common fixed point.

Remark. A nonexpansive condensing operator T_o on a Banach space X has at least one fixed point if for some $x_o \in X$ the sequence

$$x_n = T_o x_{n-1} \quad , \quad n = 1, 2, \ldots ,$$

is bounded. It can be shown, as in [65], that the set $L = L(x_o)$ of all limit points of the sequence $\{x_n\}$ is compact and that $T(L) = L$.

Obviously, if L consists of one point z then $T_o(z) = z$. Bakhtin [2] has shown that L must consist of at least one point.

Lifsic and Sadovskii further generalized the concept of condensing operators in [48].

Definition 2.3.13. Let X be a locally convex linear topological space. A continuous operator $T : X \to X$ is called a generalized condensing operator on a set $C \subset X$ if for an arbitrary set A in C the condition $T(A) \subset A$ and the compactness of the set $A - \bar{\omega}(T(A))$ together imply compactness of $\bar{A}$.

Theorem 2.3.14. Let $C(\subset X)$ be a closed convex set. Suppose that a generalized condensing operator T maps C into itself, with $T(C) \subset C$, then T has at least one fixed point in C.

Remark . We also have a third measure of noncompactness, namely, the ball (β) measure. $\beta(A) = \inf\{\varepsilon > 0$ such that A can be covered by a finite number of balls of diameter $\varepsilon\}$. For example, if B denotes the closed ball in an infinite dimensional space X, Nussbaum [55] has shown that $\alpha(B) = \beta(B) = 2$.

For example [77], Webb illustrates the difference between (α) measure and (β) measure.

Let $X = \ell^2$, the space of square summable sequence, and let $\{e_n\}$ be the orthonormal basis : e_n has 1 in the nth place and zeros elsewhere. Take A to be $\{e_n : n \geq 1\}$. It is easily seen that $\alpha(A) = \sqrt{2}$ while $\beta(A) = 2$.

CHAPTER III

Densifying Mappings and Fixed Point Theorems

3.1. The Furi-Vignoli Fixed Point Theorem and Related Results.

Definition 3.1.1.[31] Let F be a real-valued lower semicontinuous function defined on $X \times X$. The mapping T on X is said to be F-contractive if

$$F(T(x), T(y)) < F(x,y), \quad \text{for all } x,y \text{ in } X, \quad x \neq y.$$

If $F(x,y) = d(x,y)$, where d is the distance function on X, then T is simply called contractive.

Definition 3.1.2. [74] Let $F : X \times X \to [0,\infty)$ be a continuous mapping such that

$$F(T(x), T(y)) \leq F(x,y), \quad \text{for all } x,y \text{ in } X.$$

Then T is called F-nonexpansive.

In case, when $x \neq y$, there is some integer $n = n(x,y)$ such that

$$F(T^n(x), T^n(y)) < F(x,y),$$

then T is called iteratively F-contractive at (x,y).

The following result is due to Singh and Zorzitto [74].

Lemma 3.1.3. Let X be a Hausdorff space and T a continuous and F-nonexpansive mapping of X into itself, and let T be also iteratively F-contractive at (x,y), $x \neq y$. If for some $x \in X$, the sequence of iterates $\{x_n\} = \{T^n(x)\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. We have the monotone nonincreasing sequence of nonnegative real numbers,

$$F(x, T(x)) \geq F(T(x), T^2(x)) \geq \ldots \geq F(T^n(x), T^{n+1}(x)) \geq \ldots$$

which must converge along with all its subsequences to some $\alpha \in \mathbb{R}$. We also have a convergent subsequence $\{T^{n_k}(x)\}$ in X which converges to some point, say, z in X. For some $n = n(z, T(z))$,

$$F(z, T(z)) > F(T^n(z), T^{n+1}(z)), \quad \text{if} \quad z \neq T(z).$$

But we also have

$$\begin{aligned} F(z, T(z)) &= F(\lim_k T^{n_k}(x), \lim_k T^{n_{k+1}}(x)) \\ &= \lim_k F(T^{n_k}(x), T^{n_{k+1}}(x)) \\ &= \alpha \\ &= \lim_k F(T^{n_{k+n}}(x), T^{n_{k+n+1}}(x)) \\ &= F(T^n(z), T^{n+1}(z)), \end{aligned}$$

giving a contradiction, unless $z = T(z)$.

To prove uniqueness, let y be another fixed point of T different from z;

i.e. $y, z \in F(T) \subset X$ and $y \neq z$.

Then T is iteratively F-contractive at (y,z). Therefore there exists a positive integer $m = m(y,z)$ such that

$$F(T^m(y), T^m(z)) < F(y,z).$$

This is a contradiction, since $T^m(y) = y$ and $T^m(z) = z$, for all m. Hence $y = z$, and z is the unique fixed point of T.

The following result has been proved by Furi and Vignoli [31].

Theorem 3.1.4. Let T be a densifying and an F-contractive mapping defined on a complete metric space X. If for some $x_o \in X$, the sequence of iterates $\{x_n\}$ is bounded, then T has a unique fixed point in X.

Proof. Uniqueness follows immediately from the F-contractivity of T. We prove the existence.

Let $$A = \bigcup_{n=0}^{\infty} x_n$$

Then $T(A) = \bigcup_{n=1}^{\infty} x_n \subset A = T(A) \cup \{x_o\}$.

Hence A is an invariant set.

Let $\bar{A}$ denote the closure of A. Now, we shall show that $\bar{A}$ is compact, which, by the completeness of X, will be true if $\alpha(\bar{A}) = 0$. Suppose $\alpha(\bar{A}) > 0$, or equivalently, $\alpha(A) > 0$. Then $\alpha(T(A)) < \alpha(A)$, since T is densifying.

But
$$\begin{aligned}\alpha(A) &= \alpha(T(A) \cup \{x_o\}) \\ &= \max\{\alpha(T(A)) , \alpha(\{x_o\})\} \\ &= \max\{\alpha(T(A)) , 0\} \\ &= \alpha(T(A)).\end{aligned}$$

Hence $\alpha(A) = 0$, i.e. $\alpha(\bar{A}) = 0$, and hence $\bar{A}$ is compact.

Now, by the continuity of T, $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$. Hence $\bar{A}$ is invariant under T and is compact. So the space $\bar{A}$ with $T : \bar{A} \to \bar{A}$ satisfying all the assumptions of Lemma 3.1.3, and T has a fixed point in $\bar{A}$.

Corollary 3.1.5. (The Banach Contraction Principle). Let T be a mapping of a complete metric space X into itself such that

$$d(T(x), T(y)) \leq kd(x,y), \quad \text{for all } x,y \in X, \quad 0 \leq k < 1.$$

Then T has a unique fixed point.

Proof. T is a k-set-contraction and hence densifying. (See Example 2.2.14 and remark following Definition 2.2.12).
T is also contractive, since it is a contraction.

Further, if $\{x_n\}$ is a sequence if iterates of T, we have,

$$\begin{aligned} d(x_o, x_n) &\leq \sum_{i=1}^{n} d(x_{i-1}, x_i) \\ &\leq \sum_{i=1}^{n} k^i d(x_o, x_1) \\ &< \frac{d(x_o, x_1)}{1 - k} \end{aligned}$$

i.e. $\{x_n\}$ is bounded, and hence the result follows.

Corollary 3.1.6. [31] A contractive and densifying mapping on a bounded complete metric space has a unique fixed point.

The following result due to Edelstein [26] follows as a corollary.

Corollary 3.1.7. A contractive mapping on a compact metric space X has a unique fixed point.

Remark. Furi and Vignoli [31] have shown that the conditions

(i) there exists a bounded and invariant subset $M \subset X$ (with $T(M) \subset M$),

(ii) T is F-contractive,

(iii) T is densifying

of Theorem 3.1.4 are independent, in the sense that any two of them are not sufficient to ensure the existence of the fixed point.

Example 1. Let $X = \{x \in \mathbb{R} : x \geq 1\}$, and $T(x) = x + \frac{1}{x}$.

We have $|T(x) - T(y)| < |x - y|$. T is also densifying since a bounded subset of $\mathbb{R}$ is relatively compact. But T has no fixed points.

Example 2. Let $S = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the set of complex numbers with norm equal to one. The mapping $T : S \to S$, defined by $T(z) = iz$, does not have any fixed points; but, because of the compactness of S, clearly satisfies both hypothesis (i) and (iii).

Example 3. Let C_o be the Banach space of real sequences, $x = \{x_n\}$, converging to zero, with norm

$$||x|| = \sup_n |x_n| .$$

Let $X = \{x = \{x_n\} \in C_o : 0 \leq x_n \leq 1\}$.

We now define the following mappings of X into itself:

$U : \{x_1, x_2, \ldots\} \to \{1, x_1, x_2, \ldots\}$

$V : \{x_n\} \to \{x'_n\}$, where $1 - x'_n = (1 - x_n) \frac{n}{n+1}$,

i.e. $$x'_n = \frac{1 + nx_n}{n + 1}$$

i.e. $$V : \{x_n\} \to \left\{\frac{1 + nx_n}{n + 1}\right\} .$$

Define $T : X \to X$ by $T = U \circ V$.

Then $T(x) = (U \circ V)(x) = U \circ V(\{x_1, x_2, \ldots, x_n, \ldots\})$

$$= U(\{ \frac{1 + x_1}{2}, \frac{1 + 2x_2}{3}, \ldots, \frac{1 + nx_n}{n + 1}, \ldots \})$$

$$= (\{1, \frac{1 + x_1}{2}, \frac{1 + 2x_2}{3}, \ldots \frac{1 + nx_n}{n + 1}, \ldots\}).$$

U is an isometry, since

$$||U(x) - U(y)|| = \operatorname*{Sup}_n \{1 - 1, x_1 - y_1, \ldots, x_n - y_n, \ldots\}$$
$$= ||x - y||.$$

Therefore $||T(x) - T(y)|| = ||V(x) - V(y)||$

$$= \operatorname*{Sup}_n |x'_n - y'_n|$$

$$= \operatorname*{Sup}_n \left|\frac{1 + nx_n}{n + 1} - \frac{1 + ny_n}{n + 1}\right|$$

$$= \operatorname*{Sup}_n \left|\frac{n}{n + 1} (x_n - y_n)\right|$$

$$< \operatorname*{Sup}_n |x_n - y_n|$$
$$= ||x - y||.$$

i.e. $||T(x) - T(y)|| < ||x - y||$, for all $x,y \in X$, $x \neq y$.

But T has no fixed points. Indeed, if $z \in X$ is such that $z = T(z)$, then $z_n = T(z_n)$ for every positive integer n. Hence $z = \{1,1,1, \ldots\} \notin C_o$. In this example T is not densifying.

The following theorem is given by Furi and Vignoli [32].

Theorem 3.1.8. Let $Q : \mathbb{R}^+ \to \mathbb{R}^+$ be a right continuous nondecreasing function defined on $[0,\infty]$ such that $Q(r) < r$ for $r > 0$. Let $T : X \to X$ be a mapping defined on a complete metric space X such that

$$d(T(x), T(y)) \leq Q(d(x,y)) \quad \text{for all } x,y \in X.$$

Then for any $x_o \in X$ the sequence $\{x_n\}$ of iterates starting from x_o ($x_1 = T(x_o)$, $x_n = T(x_{n-1})$) converges to the unique fixed point of T.

Proof. We shall first show that the mapping T is densifying. Let $A \subset X$ be a bounded set such that $0 < \alpha(A)$, and choose $\varepsilon > \alpha(A)$. Then there exists a finite covering $\{A_1, A_2, \ldots, A_n\}$ of subsets of A, such that $\delta(A_k) < \varepsilon$, $k = 1, 2, \ldots, n$.

Clearly $T(A) \subset \bigcup_{k=1}^{n} T(A_k)$. Let $1 \leq k \leq n$ be fixed, and take $x,y \in A_k$.

Then $d(T(x), T(y)) \leq Q(d(x,y)) \leq Q(\varepsilon)$.

Hence $\delta(T(A_k)) \leq Q(\varepsilon)$.

If $\varepsilon \downarrow \alpha(A)$, then, by the right continuity of Q, we obtain $\alpha(T(A)) \leq Q(\alpha(A)) < \alpha(A)$.

i.e. T is densifying.

Let $x_o \in X$, and let $\{x_n\}$ be the sequence of iterates starting from x_o. Let $\beta_n = d(x_n, T(x_n))$. Let us prove that $\beta_n \to 0$.

Clearly, $\beta_n \leq Q(\beta_{n-1}) < \beta_{n-1}$, if $\beta_{n-1} > 0$.

Hence $\{\beta_n\}$ is a nonnegative nonincreasing sequence. Let $\beta_n \to \beta$. From the right continuity of Q, it follows that $\beta \leq Q(\beta)$. This implies $\beta = 0$, otherwise we would have $Q(\beta) < \beta$.

Since, by our definition, T is contractive and has at most one fixed point, it is sufficient to prove that the sequence $\{x_n\}$ is bounded.

Let r be a positive real number.

Since $\beta_n \to 0$, there exists a positive integer N such that $\beta_N < r - Q(r)$.

Let us prove that the set $B(x_N, r)$ is invariant; i.e. $T(B(x_N, r)) \subset B(x_N, r)$.

Indeed take $y \in X$ such that $d(y, x_N) < r$.

Then
$$\begin{aligned} d(T(y), x_N) &\leq d(T(y), T(x_N)) + d(T(x_N), x_N) \\ &\leq Q(d(y, x_N)) + \beta_N \\ &< Q(r) + (r - Q(r)) \\ &= r. \end{aligned}$$

Clearly, $x_n \in B(x_N, r)$ for all $n \geq N$, i.e. $\{x_n\}$ is bounded.

Thus the theorem follows.

The following result due to Krasnoselskii and Stecenko [46] follows an immediate corollary to Theorem 3.1.8.

Corollary 3.1.9. Let $T : X \to X$ be a mapping defined on a complete metric space X, which satisfies the condition

$$d(T(x), T(y)) \leq d(x,y) - \Delta(d(x,y)), \quad \text{for all } x,y \in X,$$

where Δ is a continuous real function defined on $[0,\infty)$, such that $\Delta(r) > 0$ for $r > 0$.

Then, for any $x_o \in X$, the sequence of iterates $\{x_n\} = \{T^n x_o\}$ converges to the unique fixed point $z \in X$.

Proof. Let $Q(r) = \max \{t - \Delta(t) : t \leq r\}$.

It is easy to see that Q is continuous, nondecreasing and such that

$Q(r) < r$ for $r > 0$. Obviously, $d(T(x), T(y)) \leq Q(d(x,y))$ for all $x,y \in X$. Hence the result follows.

3.2 Generalizations.

In this section, we will give a few generalizations of the results given by Furi and Vignoli [31].

Definition 3.2.1. [75] Let F be a real lower semicontinuous function defined on $X \times X$. The mapping T on X is said to be iteratively F-contractive at the point $x \in X$ if there exists an integer $n(x)$ such that

$$F(T^{n(x)}(x), T^{n(x)}(y)) < F(x,y), \quad \text{for all } y \in X, \quad x \neq y.$$

The following theorem is due to Thomas [75].

Theorem 3.2.2. Let $T : X \to X$ be defined from a complete metric space X into itself such that T^n is densifying for a fixed integer n. Let T be iteratively F-contractive at all points of X, and let $T^{n(x)}$ be continuous at x. If some sequence $\{x_n\}$ of iterates of x_o is bounded, then T has a unique fixed point in X.

Proof. Let $A = \bigcup_{i=1}^{\infty} \{x_i\}$. Then $T^n(A) = \bigcup_{i=n}^{\infty} \{x_i\} \subset A$, and $T^n(\bar{A}) \subset \bar{A}$, where $\bar{A}$ denotes the closure of A.

$\bar{A}$ is compact, since $A = T^n(A) \cup \{x_o, x_1, \ldots, x_{n-1}\}$ implies that $\alpha(T^n(A)) = \alpha(A)$, and if $\bar{A}$ is not compact, $\alpha(T^n(A)) < \alpha(A)$.

Now, define $Q : \bar{A} \to \mathcal{R} : x \mapsto F(x, T^{n(x)}(x))$, defined by $\hat{F} \circ \hat{T}$, where $\hat{F}$ is the restriction of F to $\bar{A} \times \bar{A}$ and $\hat{T} : \bar{A} \to \bar{A} \times \bar{A} : x \mapsto (x, T^{n(x)}(x))$.

Since T is continuous and Q is lower semicontinuous, Q has a minimum point, say z, in A.

Assume $z \neq T(z)$. Since T is iteratively F-contractive for all points of X, we have

$$\begin{aligned} Q(T^{n(z)}(z)) &= F(T^{n(z)}(z), T^{2n(z)}(z)) \\ &< F(z, T^{n(z)}(z)) \\ &= Q(z), \end{aligned}$$

i.e. z is not a minimum point.

Hence $z = T^{n(z)}(z)$.

Since T is iteratively F-contractive, z is the unique fixed point of T^n. It remains to show that z is a fixed point of T. To this purpose, consider

$$T(z) = T(T^{n(z)}(z)) = T^{n(z)}(T(z)),$$

i.e. $T(z)$ is also a fixed point of T^n.
But z is the unique fixed point of T^n,
hence $z = T(z)$ and is unique.

We can give the following result of Chu and Diaz [17] as a corollary to the above theorem.

Corollary 3.2.3. Let T be a mapping of a complete metric space X into itself. If for some positive integer n, T^n is a contraction, then T has a unique fixed point.

Proof. Since T^n is a contraction, we have T^n is densifying, T^n is contractive, T^n is continuous for all x. Any sequence of iterates

$\{x_n\}$ is bounded. The result follows.

The following example may be used to illustrate the generality of Theorem 3.2.2.

Example 3.2.4. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = 0, \quad x \text{ rational}$$
$$= 1, \quad x \text{ irrational.}$$

$T^2(x) = T(T(x))$ is a contraction mapping.

Singh and Zorzitto [74] gave the following generalization of Theorem 3.1.4.

Theorem 3.2.5. Let T be densifying on a complete metric space X, and $F : X \times X \to [0,\infty)$ continuous. For $x \neq y$, there exists some integer n with $F(T^n(x), T^n(y)) < F(x,y)$. If for some point $x_o \in X$, the sequence $\{x_n\}$ of iterates of x_o is bounded, then T has a unique fixed point.

Proof. Let $A = \bigcup_{n=0}^{\infty} x_n$, and let $\bar{A}$ denote its closure. By the completeness of X, we will show that $\bar{A}$ is compact. This will be true if $\alpha(\bar{A}) = 0$, or equivalently $\alpha(A) = 0$. To this purpose, let us assume $\alpha(A) > 0$. Then $\alpha(T(A)) < \alpha(A)$, since T is densifying. But

$$\alpha(A) = \max\{\alpha(T(A)), \ \alpha(\{x_o\})\}$$
$$= \max\{\alpha(T(A)), \ 0\}$$
$$= \alpha(T(A)).$$

Hence $\alpha(A) = \alpha(T(A)) = 0$,

i.e. $\alpha(\bar{A}) = 0$, and hence $\bar{A}$ is compact.

Now, by the continuity of T, we have $T\bar{A} \subset \overline{T\bar{A}} \subset \bar{A}$. So the space $\bar{A}$, with $T : \bar{A} \to \bar{A}$ now satisfies the assumptions of Theorem 3.1.3, and therefore there is a fixed point z in $\bar{A}$. From the condition $F(T^n(x), T^n(y)) < F(x,y)$, for some n, it follows that z is the unique fixed point of T.

3.3 Asymptotically Regular Mappings and Diminishing Orbital Diameters.

Browder and Petryshyn[13] have introduced the following.

Definition 3.3.1. Let T be a mapping of a metric space X into itself. If

$$\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0 , \quad \text{for each } x \in X,$$

then T is asymptotically regular on X.

Singh and Zorzitto [74] have given the following result.

Theorem 3.3.2. Let T be a continuous mapping on a metric space X, such that T is asymptotically regular on X. If for some $x_o \in X$, the sequence $\{x_n\}$ of iterates of x_o contains a subsequence which converges to a point $z \in X$, then z is a fixed point of T.

Proof. Let $\{x_n\} \supset \{x_{n_k}\} \to z \in X$.
Since T is continuous, we have

$$T(x_{n_k}) \to Tz.$$

Now, T is asymptotically regular on X;

i.e. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$$

$$\lim_{k\to\infty} d(x_{n_k}, T(x_{n_k})) = 0$$

$$d(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} T(x_{n_k})) = 0$$

$$d(z, T(z)) = 0.$$

Hence $z = T(z)$,

i.e. z is a fixed point of T.

Furi and Vignoli [32] have given the following result.

Theorem 3.3.3. Let $\{x_n\}$ be a bounded sequence of a complete metric space X, and $T : X \to X$ a densifying mapping such that T is asymptotically regular on $\{x_n\}$. Then $\{x_n\}$ is compact and all its limit points are fixed points of T.

Proof. Let $\{x_n\}$ be a bounded sequence such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $A = \{x_n : n = 1, 2, \ldots\}$, so that $T(A) = \{T(x_n) : n = 1, 2, \ldots\}$. Then, given any $\varepsilon > 0$, it follows that $B(T(A),\varepsilon)$ contains all but a finite number of elements of A, since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Then $\alpha(A) \leq \alpha(B(T(A),\varepsilon)) \leq \alpha(T(A)) + 2\varepsilon$;

hence $\alpha(A) \leq \alpha(T(A))$.

This implies $\alpha(A) = 0$, otherwise we would have $\alpha(T(A)) < \alpha(A)$; and therefore $\{x_n\}$ is compact. And, by the continuity of T, all the limit points of $\{x_n\}$ are fixed for T.

We have found an alternate proof for the above theorem. It runs as follows.

Let $A = \bigcup_{n=0}^{\infty} x_n$, then since T is densifying, X complete, and A bounded, we know that $\bar{A}$ is compact (by Theorem 3.1.4). Hence $T : \bar{A} \to \bar{A}$ satisfies Theorem 3.3.2 and therefore T has a fixed point $z \in \bar{A}$.

The following corollaries are given by Furi and Vignoli [32].

Corollary 3.3.4. Let T be a densifying mapping defined on a bounded complete metric space X. If $\inf\{d(x, T(x)) : x \in X\} = 0$, then T has at least one fixed point z in X.

Corollary 3.3.5. Let T be a completely continuous mapping defined on a bounded complete metric space X. If $\inf\{d(x, T(x)) : x \in X\} = 0$, then T has at least one fixed point z in X.

Definition 3.3.6. [38, p. 57] A topological space X is called a Fréchet space (or a T_1-space) if each set which consists of a single point is closed; or equivalently, X is a Fréchet space is for any pair of distinct points x,y in X, there exists an open-set U of X such that $x \in U$ and $y \notin U$.

Corollary 3.3.7. Let $T : D \to F$ be a mapping defined on a closed subset D of a Fréchet space F such that $T = G + H$, where $G : D \to F$ is a completely continuous mapping and $H : D \to F$ is a contraction (or, more generally, densifying). Then any bounded sequence $\{x_n\}$ such that T is asymptotically regular on $\{x_n\}$` is compact and all the limit points of $\{x_n\}$ are fixed for T. (F is metrizable topological vector space).

Proof. By Theorem 3.3.3, it is sufficient to prove that T is densifying. To this end, let A be a bounded subset of F with $\alpha(A) > 0$. We have

$$\begin{aligned}\alpha(T(A)) &\leq \alpha(G(A) + H(A)) \\ &\leq \alpha(G(A)) + \alpha(H(A)) \\ &= \alpha(H(A)) \\ &< \alpha(A).\end{aligned}$$

The following is due to Belluce and Kirk [4].

Definition 3.3.8. Let T be a mapping of a metric space X into itself. For $x \in X$, let

$O(x) = \{x, T(x), T^2(x), \ldots\}$, and for $A \subset X$,

let the diameter of A be $\delta(A) = \sup\{d(x,y) : x,y \in A\}$.

If, for each $x \in X$, we have

$$\lim_{n\to\infty} \delta(O(T^n(x))) < \delta(O(x)), \quad \text{whenever } \delta(O(x)) > 0,$$

then T is said to have diminishing orbital diameters on X.

Theorem 3.3.9. [43] Let T be a continuous mapping of a compact metric space X into itself. Furthermore, let T have diminishing orbital diameters on X. Then, for each $x \in X$, some subsequence $\{x_{n_k}\}$ of the sequence of iterates of x has a limit which is a fixed point of T.

Proof. Let $x \in X$, and let $L(x)$ denote the set of all points of X which are limits of subsequences of the sequence $\{x_n\}$. Since X is compact, $L(x) \neq \phi$, and it can easily be shown that T maps $L(x)$ into itself.

Since $L(x)$ is closed, we can use Zorn's lemma to obtain a subset K of $L(x)$ which is minimal with respect to being nonempty, closed, and mapped into itself by T.

Suppose $\delta(K) > 0$, and let $x_o \in K$.
$\overline{O(x_o)}$, the closure of $O(x_o)$, is mapped into itself by T and the minimality of T implies that $K = \overline{O(x_o)}$. But if $\delta(K) > 0$, then because $\lim_{n\to\infty} \delta(O(T^n(x))) < \delta(O(x_o))$, for some integer N, the diameter of $O(T^N(x_o))$ is less than $\delta(O(x_o))$. This implies that the closure of $O(T^N(x_o))$ is a proper subset of K, and since it is also mapped into itself by T, the minimality of K is contradicted. Therefore $\delta(K) = 0$ and K consists of a single point which is fixed under T.

Belluce and Kirk [43] gave the following example to show that the compactness of X is essential for the existence of the fixed point.

Example 3.3.10. Let X be the positive integer and assign a metric as follows:

For each pair k,n of positive integers, with $k > n$, let $d(n,k) = 1 + \frac{1}{n} = d(k,n)$, and let $d(n,n) = 0$, $(n = 1,2, \ldots)$. X is complete in the metric, and let $T : X \to X$ be defined by $T(n) = n + 1$, $(n = 1,2, \ldots)$. Then for each integer n, $\lim_{n\to\infty} \delta(O(T^n(n))) = 1$ while $\delta(O(T^{(n)})) = 1 + \frac{1}{n}$. Thus T has dimishing orbital diameters, but no fixed point in X.

We obtain the following generalization.

Theorem 3.3.11. Let T be a densifying mapping of a complete metric space X into itself, and let the sequence $\{x_n\}$ of iterates of x_o

be bounded for some x_0 in X. Further, let T have diminishing orbital diameters on X. Then there exists a point z on $\{x_n\}$ which is a fixed point of T.

Proof. Let $A = \bigcup_{n=0}^{\infty} x_n$, then $T(A) = \bigcup_{n=1}^{\infty} x_n \subset A$.
T is continuous (since densifying) and therefore $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$. $\bar{A}$ is compact, since $A = T(A) \cup \{x_0\}$ implies that $\alpha(T(A)) = \alpha(A)$, and if $\bar{A}$ is not compact, $\alpha(T(A)) < \alpha(A)$.

Now, consider $T : \bar{A} \to \bar{A}$, i.e. the restriction of T to $\bar{A}$. $\bar{A}$ is compact and T is densifying (therefore continuous) with diminishing orbital diameters.
Hence by Theorem 3.3.4, T has a fixed point, say $z = T(z)$, in X.

Remark. If T is densifying, $\{x_n\}$ bounded, and X complete, then $A = \{x_n\}$ is compact and $T : \bar{A} \to \bar{A}$ is densifying. Therefore, the condition that T has diminishing orbital diameters can be replaced by any of the two equivalent conditions given by Belluce and Kirk [6].

(i) T is asymptotically regular

(ii) T is not an isometry on $O(x)$ if $\delta(O(x)) > 0$, $x \in X$.

CHAPTER IV

k-set Contractions in Banach Spaces

4.1 Fixed Points in Banach Spaces.

We state without proof a fundamental result of Kirk [42] for nonexpansive mappings in Banach spaces,

Theorem 4.1.1. If C is a closed bounded convex subset with normal structure of a reflexive Banach space X, then every nonexpansive mapping $T : C \to C$ has a fixed point.

Remark. It is worth noting that all the conditions of Theorem 4.1.1 are necessary. We will give simple examples to support this claim.

1. C closed : Let $X = \mathbb{R}$ be a Hilbert space, and C the interior of the unit ball, i.e. $C = \{x ; \ ||x|| < 1\}$.
   Define $T : C \to C$ by $T(x) = \frac{x + a}{2}$, where $||a|| = 1$, a real.
   Obviously T is nonexpansive, but T has no fixed points.

2. C bounded : A translation in a Banach space is an isometry and obviously has no fixed points.

3. C convex : Let $X = \mathbb{R}$ be a Hilbert space, and C be the set containing just two distinct points a and b. Define $T : C \to C$ as $T(a) = b$ and $T(b) = a$. Clearly T is an isometry and has no fixed points.

4. X reflexive (see Kirk [20]) : Let $X = C[0,1]$ be a Banach space with $||f|| = \max\{|f(t)| : 0 \leq t \leq 1\}$. It is known that $C[0,1]$ is not a reflexive Banach space. Now consider C as the unit ball about the origin in $C[0,1]$,
   i.e. $C = \{f(t) : 0 \leq f(t) \leq 1, \ f(0) = 0, \ f(1) = 1\}$.
   Define $T : C \to C$ by $T(f(t)) = tf(t)$.
   $T(C) \subset C$ and T has no fixed point.

5. C has normal structure : The mapping $T : C_o \to C_o$, where C_o is the unit ball in ℓ^2, defined by $T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ maps the unit ball C_o into itself but does not have any fixed points, since $(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ would simply mean that $x_1 = x_2 = \ldots = 1$ and this is impossible.

An immediate consequence of Theorem 4.1.1 is the following result of Browder [9]. It was independently proved by Kirk, Browder and Gohde.

Theorem 4.1.2. Let $T : C \to C$ be a nonexpansive mapping on a closed bounded convex subset C of a uniformly convex Banach space X. Then T has at least one fixed point in C.

Browder and Petryshyn [13] have given the following result for nonexpansive and asymptotically regular mappings in Banach spaces.

Theorem 4.1.3. Let X be a uniformly convex Banach space and $T : X \to X$ a nonexpansive mapping. If $F(T)$ is nonempty then the mapping $T_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$, is nonexpansive and asymptotically regular. Moreover, $F(T) = F(T_\lambda)$.

Proof. That $F(T) = F(T_\lambda)$ is immediate from the definition of $F(T_\lambda)$. That T_λ is nonexpansive is a consequence of the nonexpansiveness of T. It remains to show that T_λ is asymptotically regular.
Let z be in $F(T)$. Then

$$||T_\lambda^{n+1}(x) - z|| = ||T_\lambda^{n+1}(x) - T_\lambda(z)|| \leq ||T^n_\lambda(x) - z||.$$

So the sequence $\{||T^n_\lambda(x) - z||\}$ is nonincreasing. Thus it converges to some $\alpha \geq 0$. If $\alpha = 0$ then $\{||T^{n+1}_\lambda(x) - T_\lambda^n(x)||\}$ converges to zero trivially.

Therefore, let us assume $\alpha > 0$. We have

$$T_\lambda^{n+1}(x) - z = \lambda T_\lambda^{n}(x) + (1 - \lambda)TT^n_\lambda(x) - z$$

$$= \lambda(T_\lambda^{n}(x) - z) + (1 - \lambda)(TT_\lambda^{n}(x) - z).$$

Since $||T_\lambda^{n+1}(x) - z|| \to \alpha$,

$$||T^n_\lambda(x) - z|| \to \alpha \text{ and } ||TT^n_\lambda(x) - z|| \leq ||T^n_\lambda(x) - z||$$

we have $||(T^n_\lambda(x) - z) - (TT_\lambda^{n}(x) - z)|| \to 0$.

Therefore $||T_\lambda^{n+1}(x) - T_\lambda^{n}(x)|| \to 0$

i.e. T_λ is asymptotically regular.

Remark. The use of T_λ is place of T for the determination of fixed points was considered by Krasnoselskii [45] for $\lambda = \frac{1}{2}$ and completely continuous mappings. The general λ was considered by Schaefer [66] in the case of completely continuous mappings and by Petryshyn [56] in the case of demicompact mappings (see Definition 4.1.9).

The next two results are due to Browder and Petryshyn [13].

Theorem 4.1.4. Let $T : X \to X$ be a nonexpansive asymptotically regular mapping in a Banach space X. If, for some x_o in X, the sequence of iterates $\{x_n\} = \{T^n(x_o)\}$ has a subsequence $\{x_{n_i}\}$ converging to a point z, then z is a fixed point of T and the sequence $\{x_n\}$ converges to z.

Proof. Let us first prove that z is a fixed point of T.

$$T^{n_i}(x_o) \to z \Rightarrow (I - T)T^{n_i}(x_o) \to (I - T)(z).$$

However, $(I - T)T^{n_i}(x_o) = T^{n_i}(x_o) - T^{n_i+1}(x_o) \to 0$ since T is asymptotically regular.

Thus $(I - T)(z) = 0$,

i.e. z is a fixed point of T.

Now, the condition

$$||T^{n+1}(x_o) - z|| \leq ||T^n(x_o) - z||, \text{ for all } n = 1,2, \ldots$$

gives us $\{x_n\} \to z$.

Theorem 4.1.5. Let $T : X \to X$ be a nonexpansive asymptotically regular mapping in a Banach space X. Let $F(T)$, the set of fixed points of T, be nonempty, and let T satisfy the following condition

$(I - T)$ maps bounded closed sets into closed sets. (4.1.5).

Then, for each x_o in X, the sequence $\{T^n(x_o)\}$ converges to some point in $F(T)$.

Proof. If y is a fixed point of T, then

$$||T^{n+1}(x_o) - y|| \leq ||T^n(x_o) - y|| , \quad n = 1,2, \ldots$$

so the sequence $\{T^n(x_o)\}$ is bounded. Let G be the closure of $\{T^n(x_o)\}$. By condition (4.1.5) it follows that $(I - T)(G)$ is closed. This together with the fact that T is asympototically regular gives 0 in $(I - T)(G)$. So there exists a z in G such that $(I - T)(G) = 0$, i.e. $z = T(z)$.

But this implies that either $z = T^n(x_o)$ for some n, or there exists a sequence $\{T^{n_i}(x_o)\}$ converging to z.

Since z is a fixed point of T, we can conclude that, in either case, the sequence $\{T^n(x_o)\}$ converges to z.

Remark. Let λ be such that $0 < \lambda < 1$. Let $T_\lambda = \lambda I + (1 - \lambda)T$. Then T satisfies condition (4.1.5) if and only if T_λ does. To see this, observe that $I - T_\lambda = (1 - \lambda)(I - T)$.

By the above remark and Theorem 4.1.2, we can deduce the following corollaries to Theorem 4.1.5.

Corollary 4.1.6. Let $T : X \to X$ be a nonexpansive mapping of a uniformly convex Banach space X into itself. Suppose $F(T)$ is nonempty and T satisfies condition (4.1.5). Then, for each x_o in X, the sequence $\{x_n\}$ defined by $x_{n+1} = \lambda x_n + (1 - \lambda)T(x_n)$, $0 < \lambda < 1$, converges to some point in $F(T)$.

Corollary 4.1.7. Let $T : X \to X$ be a nonexpansive mapping on a uniformly convex Banach space X. Define $T_\lambda = \lambda I + (I - \lambda)T$, $0 < \lambda < 1$. If $F(T)$ is nonempty and $(I - T)(X)$ is closed, then for every x_o in X the sequence $\{x_n\} = \{T^n(x_o)\}$ converges to some point in $F(T)$.

Corollary 4.1.8. Let $T : C \to C$ be a nonexpansive mapping on a closed bounded convex subset C of a uniformly convex Banach space X. If T is completely continuous, then for every x_o in C, the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1}{2}(x_n + T(x_n))$ converges to a fixed point of T.

Definition 4.1.9. A continuous mapping T of a Banach space X into itself is said to be demicompact if every bounded sequence $\{x_n\}$, such that $\{(I - T)(x_n)\}$ converges, contains a convergent subsequence $\{x_{n_i}\}$.

In [56] Petryshyn proved that the class of demicompact mappings contain, among others, all compact (completely continuous) mappings.

Theorem 4.1.10 [20, p. 47]. A demicompact mapping T of a Banach space X into itself satisfies condition (4.1.5).

Remark. It was claimed in [13] that the converse of Theorem 4.1.10 holds. However, this is not true. Consider, for example, the identity mapping $T = I$. $T = I$ satisfies trivially condition (4.1.5), but it is not demicompact.

In a strictly convex Banach space we have the following result [20].

Theorem 4.1.11. Let X be a strictly convex Banach space and $T : X \to X$ a nonexpansive mapping. Then the set $F(T)$ of fixed points of T is convex.

Proof. Let us assume that $F(T)$ consists of more than one point, otherwise the result is proved. Suppose x and y are in $F(T)$, we will then show that $z = \lambda x + (1 - \lambda)y$, $0 < \lambda < 1$ is also in $F(T)$.
In fact, since T is nonexpansive, we have

$$\begin{aligned} ||x - y|| &= ||T(y) - T(y)|| \\ &\leq ||T(x) - T(z)|| + ||T(z) - T(y)|| \\ &\leq ||x - z|| + ||z - y|| \\ &= ||x - y|| . \end{aligned}$$

Since X is strictly convex, it follows that the vectors $T(x) - T(z)$ and $T(z) - T(y)$ are linearly dependent. But this implies that the vectors $T(z)$ is in a straight line through $T(x) = x$ and $T(y) = y$. On the other hand, $||T(z) - T(x)|| \leq ||z - x||$ and $||T(z) - T(y)|| \leq ||z - y||$. Thus $T(z)$ must coincide with z, and the theorem is proved.

Remark. This theorem is not true in the most general class of Banach spaces, as the following example of deMarr[23] shows.

Let $X = \ell^\infty(2)$, i.e. the space of pairs $x = (a,b)$ with maximum norm $||x|| = \max\{|a|, |b|\}$.
Let T be the mapping defined by

$$T(x) = T(a,b) = (|b|, b).$$

It is easy to see that T is nonexpansive and that $(1,1)$ and $(1, -1)$ are fixed points of T. However, no other point in the segment joining these two points is a fixed point of T.

In [29] Edelstein established the following result, which had been previously proved byKrasnoselskii [45] and Schaefer [66] for uniformly convex Banach spaces. We will omit the proof, which can be found in [20, p. 48].

Theorem 4.1.12. Let X be a strictly convex Banach space and C a closed convex set in X. Let $T : C \to C$ be a nonexpansive mapping such that $T(C)$ is a relatively compact set contained in C. Let $T_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$. The for each point x_0 in C, the sequence $\{T_\lambda^n(x_0)\}$ converges to a fixed point of T.

Remark. We cannot conclude in this case, as we did in Theorem 4.1.3, that T_λ is asymptotically regular. Theorem 4.1.3 does not hold for the wider class of strictly convex spaces. So Theorem 4.1.12 is a special case of Theorem 4.1.5.

Belluce and Kirk [6] have given the following.

Definition 4.1.13. Let C be a convex subset of a Banach space X. A mapping $T : C \to C$ is called convex on C if

$$||T(\frac{x+y}{2})|| \leq \frac{1}{2}(||T(x)|| + ||T(y)||)$$

for all x,y in C.

Theorem 4.1.14. Let C be a nonempty weakly compact convex subset of a Banach space X. Suppose $T : C \to C$ is continuous and $(I - T)$ is convex on C. If $\inf\{||x - T(x)|| : x \in C\} = 0$, then T has a fixed point in C.

Proof. For each $r \geq 0$, let

$$H_r = \{y \in C : ||y - T(y)|| \leq r\} .$$

Because $\inf\{||x - T(x)|| : x \in C\} = 0$, $H_r \neq \phi$, if $r > 0$. Convexity of $(I - T)$ implies H_r is convex, and continuity of T implies H_r is closed. As closed convex subsets of the weakly compact set C, the H_r are also weakly compact and it follows that $\bigcap_{r>0} H_r \neq \phi$, yielding at least one fixed point for T.

Theorem 4.1.15. Let C be a nonempty weakly compact convex subset of a Banach space X. Suppose $T : C \to C$ is nonexpansive and $(I - T)$ is convex on C. Then T has a fixed point in C.

Proof. By a result of Göhde [37] we have $\inf\{||x - T(x)|| : x \in C\} = 0$. Hence the result follows from Theorem 4.1.14.

The following examples are given by Belluce and Kirk in [6].

Example 4.1.16. Let B be the unit ball in any infinite dimensional Banach space X and let $B_{\frac{1}{2}} = \{x \in X : ||x|| \leq \frac{1}{2}\}$.
Let $Q : B \to B_{\frac{1}{2}}$ be any continuous mapping which has no fixed point. (Because $B_{\frac{1}{2}}$ is not compact, such a mapping will exist.) Let $[x, Q(x))$ be the half-line emanating from x which contains $Q(x)$. Then the length of $[x, Q(x)) \cap B$ is at least $\frac{1}{2}$.

Define $T(x)$ to be the point of this ray with distance $\frac{1}{2}$ from x. Then $T : B \to B$ and $||x - T(x)|| \equiv \frac{1}{2}$, for all x in B. Thus $(I - T)$ is convex. T is continuous because Q is. Since X may be chosen so that B satisfies the hypotheses on C in Theorem 4.1.14 (i.e. assume X is reflexive), this example shows that $\inf||x - T(x)|| = 0$ cannot be removed in Theorem 4.1.14, nor can nonexpansiveness of T in Theorem 4.1.15 be replaced by continuity of T.

Example 4.1.17. Consider the space $C[0,1]$ of continuous functions. Let $C = \{f \in C[0,1] : f(0) = 0, \ f(1) = 1, \ 0 \leq f(t) \leq 1\}$.
Define $T : C \to C$ as follows:

$$T(f(t) = tf(t), \quad \text{for all } f \text{ in } C.$$

As seen in [42], T is nonexpansive on C and has no fixed point.
Let $f,g \in C$.

$$||\frac{f + g}{2} - T(\frac{f + g}{2})|| = \text{Sup}\{|\frac{f(t) + g(t)}{2} - t(\frac{f(t) + g(t)}{2})| : 0 \leq t \leq 1\}$$

$$\leq \text{Sup}\ \{\tfrac{1}{2}(1 - t) \ f(t) : 0 \leq t \leq 1\}$$

$$+ \text{Sup}\ \{\tfrac{1}{2}(1 - t)\ g(t) : 0 \leq t \leq 1\}$$

$$= \tfrac{1}{2}\{||f - T(f)|| + ||g - T(g)||\}.$$

Thus $(I - T)$ is convex on C. Also,

$$\begin{aligned} ||T^n(f) - T^{n+1}(f)|| &= \text{Sup}\{|t^n f(t) - t^{n+1} f(t)| : 0 \le t \le 1\} \\ &= \text{Sup}\{(t^n - t^{n+1}) f(t) : 0 \le t \le 1\} \\ &\le \text{Sup}\{(t^n - t^{n+1}) : 0 \le t \le 1\} \\ &\le \frac{1}{n+1} . \end{aligned}$$

So T is asymptotically regular on C.

Therefore the hypothesis of weak compactness is essential in both Theorem 4.1.14 and Theorem 4.1.15.

4.2 Densifying Mappings in Banach Spaces.

In this section we will give some fixed point theorems for densifying mappings in Banach spaces.

The relation between nonexpansive mappings and k-set contractions has been studied by Furi and Vignoli in [33]. We give the following result.

Theorem 4.2.1. Let X be a metric space, and $T : X \to X$ nonexpansive. Then T is a 1-set contraction.

Proof. Let A be a bounded subset of X, and choose $\varepsilon > \alpha(A)$. A can be covered by a finite number of subsets $\{A_1, A_2, \ldots, A_n\}$ of A such that $\delta(A_i) \le \alpha(A) < \varepsilon$, for all $i = 1,2, \ldots, n$.
i.e. $A \; \bigcup_{i=1}^{n} A_i$, $\delta(A_i) \le \alpha(A) + \varepsilon$, for all $i = 1,2, \ldots, n$.
Clearly $T(A) \subset \bigcup_{i=1}^{n} T(A_i)$, by continuity of T.

Let $1 \leq i \leq n$ be fixed and take x,y in A_i. Then $d(T(x), T(y)) \leq d(x,y) \leq \delta(A_i) \leq \alpha(A)$.

Hence $\delta(T(A_i)) \leq \delta(A_i)$, for all i.

i.e. $\alpha(T(A)) \leq \alpha(A)$.

i.e. T is a 1-set-contraction.

The following results have been given by Furi and Vignoli [33].

Theorem 4.2.2. Let $T : C \to C$ be a 1-set-contraction defined on a closed bounded convex subset C of a Banach space X. Then $\inf\{||x - T(x)|| : x \in C\} = 0$.

Proof. Let x_o be a point in C, and define $T_\lambda : C \to C$ by $T_\lambda(x) = (1 - \lambda)x_o + \lambda T(x)$, $0 \leq \lambda < 1$.

The mapping T_λ is a λ-set-contraction, for $0 \leq \lambda < 1$. Indeed, let $A \subset C$. We have

$$T_\lambda(A) = (1 - \lambda)x_o + \lambda T(A).$$

Hence
$$\begin{aligned}\alpha(T_\lambda(A)) &= \alpha((1 - \lambda)x_o + \lambda T(A)) \\ &\leq (1 - \lambda)\alpha(x_o) + \lambda\alpha(T(A)) \\ &= \lambda\alpha(T(A)).\end{aligned}$$

Therefore, it follows, from a result of Darbo (see Corollary 2.2.23) that T_λ has at least one fixed point x_λ in C for any $0 \leq \lambda < 1$.

Furthermore, $T_\lambda(x)$ converges to $T(x)$ uniformly on C as $\lambda \to 1$. But $||x_\lambda - T(x_\lambda)|| = ||T_\lambda(x_\lambda) - T(x_\lambda)||$, therefore $||x_\lambda - T(x_\lambda)|| \to 0$ as $\lambda \to 1$.

Then $\inf\{||x - T(x)|| : x \in C\} = 0$.

We will state the following lemma due to Martelli [50] without proof.

Lemma 4.2.3. Let T be a mapping of a compact topological space X into itself. Then there exists a nonempty subset $K \subset X$ such that $K = \overline{T(K)}$.

Theorem 4.2.4. Let $T : C \to C$ be a densifying mapping defined on a closed bounded convex subset C of a Banach space X.
Then T has at least one fixed point.

Proof. For x_o in C, consider the sequence $\{T^n(x_o) : n = 1,2, \ldots\}$ and let K equal its closure. Then K is invariant and compact. Therefore, by the previous lemma, there exists a nonempty subset $M \subset K$, such that $M = \overline{T(M)}$.

Consider $\mathcal{F} = \{B \subset C : M \subset B$, B closed convex invariant under $T\}$.

Let $A = \cap \{B : B \in \mathcal{F}\}$.
Clearly $A = \overline{\omega}(T(A))$, the convex closure of $T(A)$.
Since $\alpha(\overline{\omega}(T(A))) = \alpha(T(A))$, we get that A is compact.
Then by the Schauder-Tychonoff fixed point theorem (see [25, p. 456]), there exists a point z in A such that $T(z) = z$.

Remark. The proof of Theorem 4.2.4. as given above is due to Martelli [50]. It also follows as a direct consequence of Theorem 4.2.2. and Corollary 3.3.4.

We have the following result.

Theorem 4.2.5. Let $T : C \to C$ be a densifying mapping of a closed bounded subset C of a Banach space X into itself. Further, let T satisfy the following condition

$$x \neq T(x) \Rightarrow ||T(x) - T^2(x)|| < ||x - T(x)||. \qquad (4.2.5)$$

Then T has a fixed point in C.

Proof. Let x_0 be a point of C such that $x_0 \neq T(x_0)$. Consider the sequence $\{T^n(x_0) : n = 0,1,2, \ldots\}$, and let A be its closure. A is bounded since it is a subset of C, and by Theorem 3.1.4. A is compact.

Now, define $Q : A \to \mathbb{R} : x \mapsto ||x - T(x)||$. Q is continuous since the norm in a Banach space is continuous. Hence Q has a minimum, say z, in A.

Assume $z \neq T(z)$. This implies

$$Q(T(z)) = ||T(z) - T^2(z)|| < ||z - T(z)|| = Q(z).$$

i.e. z is not a minimum point.

Hence $z = T(z)$, and the proof is established.

We give the following result, which seems to be new.

Theorem 4.2.6. Let $T : C \to C$ be a densifying and nonexpansive mapping defined on a closed convex subset C of a Banach space X. For $0 \leq \lambda < 1$, define $T_\lambda = \lambda I + (1 - \lambda)T$, and for $x \neq T_\lambda(x)$, let T_λ satisfy condition (4.2.5). If for some x_0 in C, the sequence of iterates $\{x_n\} = \{T_\lambda{}^n(x_0)\}$ is bounded, then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof. By Theorem 4.2.4, we know $F(T) \neq \phi$, and since $F(T) = F(T_\lambda)$, $F(T_\lambda) \neq \phi$, $T_\lambda : C \to C$ is densifying, and $\{x_n\}$ bounded.
Hence $A = \overline{\{x_n\}}$ is compact, and therefore $\{x_n\} \supset \{x_{n_i}\} \to y$ in A.
Since T_λ is continuous, $T_\lambda(x_{n_i}) \to T_\lambda(y)$,

and $T_\lambda^2(x_{n_i}) \to T_\lambda^2(y)$.

Also, T_λ is nonexpansive and thus the sequence $\{||x_n - T_\lambda(x_n)||\}$ is nonincreasing, and therefore converges. So does $\{||x_{n_i} - T_\lambda(x_{n_i})||\}$.

$$\text{Hence} \quad ||y - T_\lambda(y)|| = \lim_i ||x_{n_i} - T_\lambda(x_{n_i})||$$

$$= \lim_i ||T_\lambda(x_{n_i}) - T_\lambda^2(x_{n_i})||$$

$$= ||T_\lambda(y) - T_\lambda^2(y)||.$$

If $y \neq T_\lambda(y)$, then $||T_\lambda(y) - T_\lambda^2(y)|| < ||y - T_\lambda(y)||$;

hence $y = T_\lambda(y)$, i.e. $y \in F(T_\lambda)$.

But $||T_\lambda^{n+1}(x) - y|| < ||T_\lambda^n(x) - y||$, for all n.

Hence $\{x_n\} = \{T_\lambda^n(x_o)\} \to y$.

Remark. For each x_o in C, the sequence $\{x_n\} = \{T_\lambda^n(x_o)\}$ is bounded; and for $x_o \neq T_\lambda(x_o)$, T_λ satisfies Theorem 3.1.4., hence T_λ has a unique fixed point in $A = \overline{\{x_n\}}$. However, $\{T_\lambda^n(y_o)\}$ may converge to a different fixed point for $y_o \neq x_o$. Therefore we cannot conclude that the fixed point of T is unique.

Before we give a few results on star-shaped sets, we want to introduce the following, which can be found in [20, p. 27].

Definition 4.2.7. A set S in a linear space X is said to be star-shaped about a point y in S if for every z in S, there exists a nonnegative number t_z, $0 \leq t_z < +\infty$, such that the set $\{y + tz : 0 \leq t \leq t_z\}$ is in S, and the set $\{y + tz : t_z < t\}$ is outside S. It is clear that every convex set is star-shaped about any of its points.

Theorem 4.2.8. [20]. Let $T : S \to S$ be a nonexpansive mapping of a closed bounded star-shaped subset S of a Banach space X. Further, let the following condition be satisfied:

There exists a compact subset M of X such that for all x in X, the closure of $\{T^n(x)\}$ contains a point of M.

Then T has a fixed point. (4.2.8)

Proof. We may assume, without loss of generality, that the origin 0 is in S, and that S is star-shaped about 0. Then the mapping $T_r = rT$, $0 < r < 1$, is a contraction in S, and consequently it has a unique fixed point x_r in S. We have

$$\begin{aligned}||x_r - T(x_r)|| = || \; rT(x_r) - T(x_r)|| &\leq (1 - r)||T(x_r)|| \\ &\leq d(1 - r),\end{aligned}$$

where d is the diameter of S.

On the other hand, by condition (4.2.8), there exists an integer $n(r)$ and a point y_r in M such that

$$||y_r - T^{n(r)}(x_r)|| \leq 1 - r.$$

Then

$$||y_r - T(y_r)|| \leq ||y_r - T^{n(r)}(x_r)|| + ||T^{n(r)}(x_r) - T^{n(r)+1}(x_r)|| + ||T^{n(r)+1}(x_r) - T(y_r)||$$

$$\leq 2||y_r - T^{n(r)}(x_r)|| + ||x_r - T(x_r)|| .$$

Hence, $||y_r - T(y_r)|| \leq (d + 2)(1 - r)$.

Now, let $\{x_n\}$ be a sequence converging to z. Using the compactness of M, it follows that there exists a subsequence of $\{ y_r\}$ that converges to y in M as $r \to 1$.

From the last inequality, it follows that y is a fixed point of T.

Furi and Vignoli [33] have given the following results.

Theorem 4.2.9. Let $T : S \to S$ be a 1-set-contraction on a closed bounded star-shaped subset S of a Banach space X. Let $p : X \to [0,\infty)$ be a lower semicontinuous function such that

(i) $x \neq 0 \Rightarrow p(x) > 0$,

(ii) $p(tx) \leq t\, p(x)$ for $t \geq 0$.

Let T satisfy the condition

(iii) $p(T(x) - T(y)) \leq p(x - y)$, x,y in S.

Then $\inf\{||x - T(x)|| : x \in S\} = 0$.

Proof. Let S be star-shaped with respect to x_o.
Define $T_\lambda : S \to S$ by $T_\lambda(x) = (1 - \lambda)x_o + \lambda T(x)$, $0 \leq \lambda < 1$. The mapping T_λ is a λ-set-contraction (and therefore densifying) for any $0 \leq \lambda < 1$; and T_λ converges uniformly to T on S as $\lambda \to 1$.
For $0 \leq \lambda < 1$, the mapping T_λ is F-contractive, with $F(x,y) = p(x - y)$.

Indeed, $F(T_\lambda(x), T_\lambda(y)) = p(T_\lambda(x) - T_\lambda(y))$

$$= p(\lambda(T(x) - T(y)))$$

$$\leq \lambda p(x - y)$$

$$< F(x,y),$$

for all $x,y \in S$, $x \neq y$.

Therefore, by Theorem 3.1.4, it follows that T has a unique fixed point x_λ in S for any $0 \leq \lambda < 1$.
Then $\{||x - T(x)|| : x \in S\} = 0$.

Corollary 4.2.10. Let $T : S \to S$ be a densifying mapping defined on a closed bounded star-shaped subset of a Banach space X.
Let $p(T(x) - T(y)) \leq p(x - y)$, where p is as in Theorem 4.2.9. Then T has at least one fixed point.

Proof is immediate from Theorem 3.1.4 and Theorem 4.2.9.

We can deduce the following result.

Theorem 4.2.11. Let $T : S \to S$ be a 1-set-contraction on a closed bounded star-shaped subset S of a Banach space X. Let T satisfy conditions (i) and (ii) of Theorem 4.2.9 and

(iii*) $p(T(x) - T(y)) < p(x - y)$, x,y S, $x \neq y$,

p defined as in Theorem 4.2.9.
Then T has a unique fixed point in S.

Proof. The existence of a fixed point is immediate from Theorem 4.2.9. Uniqueness follows from condition (iii*). Indeed, assume x,y are distinct fixed points of T.

i.e. $T(x) = x$, $T(y) = y$, $x \neq y$.

Then by condition (iii*)

$$p(x - y) = p(T(x) - T(y)) < p(x - y).$$

Hence $p(x - y) = 0$,

i.e. $x = y$, a contraction.

Therefore the fixed point is unique.

Remark. Theorems 4.2.9, 4.2.11, Corollary 4.2.10 hold in particular if $T : S \to S$ is a k-set contraction.

4.3 Mappings With A Boundary Condition.

In this section we will give some fixed point theorems in Banach spaces; $B = \{x \in X : ||x|| < r\}$ will denote the open ball about the origin, $\dot{B} = \{x \in X : ||x|| = r\}$ its boundary, and $\bar{B} = B \cup \dot{B}$ its closure. We will restrict our study, for the most part, to mappings $T : \bar{B} \to X$ which satisfy the following boundary condition:

If $T(x) = ax$, for some x in $\dot{B}$, then $a \leq 1$. $(A \leq)$.

We will make use of the following mapping:

$R : X \to \bar{B}$ defined by the formula

$$R(x) = \frac{r\,x}{||x||}, \quad ||x|| \geq r$$
$$= x, \quad ||x|| \leq r,$$

is called the radial retraction of X onto $\bar{B}$.

The following Lemma is due to Nussbaum [54]. The proof given here is given by Martelli and Vignoli in [51].

Lemma 4.3.1. Let X be a Banach space and B the open unit ball of X about the origin. Then the radial retraction $R : X \to \bar{B}$ is a 1-set-contraction.

Proof. Let $A \subset X$ be a bounded set. Clearly $R(A) \subset \bar{\omega}(\{0\} \cup A)$. Therefore.

$$\alpha(R(A)) \leq \alpha(\bar{\omega}(\{0\} \cup A)) = \alpha(\{0\} \cup A) = \alpha(A).$$

The following results are due to Petryshyn [60].

Theorem 4.3.2. Let B be an open ball about the origin in a Banach space X. If $T : \bar{B} \to X$ is a densifying (in particular, a k-set-contraction, $k < 1$) mapping which satisfies the boundary condition $(A \leq)$, then $F(T)$, the set of fixed points of T in $\bar{B}$, is nonempty and compact.

Proof. Since every k-set-contraction $(k < 1)$ is densifying, it suffices to prove the theorem for the case when T is densifying.
Define $R : X \to \bar{B}$ as in Lemma 4.2.1. Then R is a 1-set-contraction of X onto $\bar{B}$.

Now, if for all x in $\bar{B}$ we define the mapping $T_1(x) = R(T(x))$, then T_1 is a continuous mapping of $\bar{B}$ into $\bar{B}$ which is also densifying. Indeed, $T : \bar{B} \to X$ is densifying, $R : X \to \bar{B}$ is a 1-set-contraction and, therefore $\alpha(T_1(\bar{B})) = \alpha(R(T(\bar{B}))) \leq \alpha(T(\bar{B})) < \alpha(\bar{B})$.
Hence by Furi and Vignoli's fixed point theorem (see Theorem 4.2.4), T_1 has at least one fixed point, z, in $\bar{B}$.

But then z is also a fixed point of T.

Indeed, if $z \in B$, then $T(z) = z$, since the assumption of the equality $T(z) = \frac{||T(z)||}{r} z$ would contradict the fact that $||z|| < r$.

If $z \in \dot{B}$ and z is not a fixed point of T, then

$a = \frac{||T(z)||}{r} > 1$, in contradiction to condition $(A \leq)$.

Thus z is a fixed point of T, and hence $F = F(T)$ is a nonempty set in $\bar{B}$. Since T is continuous, F is obviously a closed subset of $\bar{B}$ such that $T(F) = F$. This also shows that F is compact, for otherwise the assumption $\alpha(F) > 0$ would lead to the contradictory inequality $\alpha(F) = \alpha(T(F)) < \alpha(F)$, which follows from the densifying property of T.

Remark 1. If instead of the boundary condition $(A \leq)$, we assume that T satisfies the following condition on $\dot{B}$.

If $T(x) = ax$ for some x in $\dot{B}$, then $a < 1$, $(A <)$

then the nonempty compact set $F(T)$ is contained in B and hence lies at a positive distance from $\dot{B}$.

2. In Theorem 4.3.2., we have shown that $F(T_1) \subset F(T)$. Now, $F(T)$ is obviously contained in $F(T_1)$, hence $F(T_1) = F(T)$.

Corollary 4.3.3. If $T : \bar{B} \to X$ is densifying (and, in particular) a k-set-contraction) and satisfies any one of the following conditions:

(i) $T(\bar{B}) \subset \bar{B}$,

(ii) $T(\dot{B}) \subset \bar{B}$,

(iii) $||T(x) - x||^2 \geq ||T(x)||^2 - ||x||^2$, for all x in $\dot{B}$, then

the set of fixed points $F(T)$ of T is nonempty and compact.

Proof. By Theorem 4.3.2, it is sufficient to show that each of the given conditions implies condition (A $\leq$).

It is obvious that (i) and (ii) each implies (A $\leq$). Hence we must show (iii) implies (A $\leq$).

Suppose $T(x) = ax$ for some x in $\dot{B}$. Then (iii) implies that $(a - 1)^2 \geq a^2 - 1$ or that $a \leq 1$, i.e. condition (A $\leq$) holds.

Corollary 4.3.4. Let B be the open unit ball in a Hilbert space H, and $T : \bar{B} \to H$ be a mapping of $\bar{B}$ into H. Let $T_o : \bar{B} \to H$ be a densifying mapping such that the following two conditions are satisfied:

(i) $(T(x), x) \leq ||x||^2$, and

(ii) $||T(x) - T_o(x)|| \leq ||x - T(x)||$ for all x on $\dot{B}$.

Then $F(T_o) \subset \bar{B}$ is nonempty and compact.

Proof. By Theorem 4.3.2., it suffices to show that conditions (i) and (ii) imply condition (A $\leq$) for T_o. Thus, suppose $T_o(x) = ax$ for some x in $\dot{B}$ and, without loss of generality, assume $a > 0$.

Then (ii) shows that $||T(x) - ax|| \leq ||x - T(x)||$ or that $2(T(x), x)(1 - a) \leq (1 - a^2)(x, x)$. This implies $a \leq 1$, for the assumption that $a > 1$ would lead to the inequality $2(T(x), x) \geq (1 + a)(x, x) > 2(x,x)$, contradicting (i).

Remark. Because of its usefulness in applications, we shall explicitly state the following special case of Theorem 4.3.2, whose first assertion (i.e., $F(T) \neq \phi$) was obtained independently in [78] and in [59, 61] under some additional conditions (see [12] for a more general result).

Corollary 4.3.5. Let $T = S + C$ be a map from $\bar{B}$ to X such that S is a contraction on $\bar{B}$ and C is completely continuous on $\bar{B}$. If T satisfies condition $(A \leq)$ on $\dot{B}$, then $F(T)$ is nonempty and compact.

We can generalize Theorem 4.3.2. in the following way.

Theorem 4.3.6. Let B be the open ball about the origin in a Banach space X. If $T : \bar{B} \to X$ is a 1-set-contraction which satisfies the boundary condition $(A \leq)$, then $F(T)$, the set of fixed points of T in $\bar{B}$, is nonempty, provided T is demi-compact.

Proof. Let $R : X \to \bar{B}$ be the radial retraction of X onto $\bar{B}$ (see Lemma 4.3.1). Then R is a 1-set-contraction.
Again, for all x in $\bar{B}$ define the mapping $T_1(x) = R(T(x))$. T is a continuous mapping of $\bar{B}$ into $\bar{B}$ and a 1-set-contraction. For, $T : \bar{B} \to X$ is a 1-set-contraction, $R : X \to \bar{B}$ a set contraction, and therefore

$$\alpha(T_1(\bar{B}) = \alpha(R(T(\bar{B}))) \leq \alpha(T(\bar{B})) \leq \alpha(\bar{B}).$$

Now, for x_o in $\bar{B}$ and $0 \leq \lambda < 1$, let $T_\lambda : \bar{B} \to \bar{B}$ be given by $T_\lambda(x) = (1 - \lambda)x_o + \lambda T_1(x)$, for all x in $\bar{B}$. T_λ is a λ-set-contraction $(\lambda < 1)$.
Therefore by Darbo's fixed point theorem (see Corollary 2.2.23), T_λ has at least one fixed point x_λ in $\bar{B}$ for any $0 \leq \lambda < 1$.

Furthermore, $T_\lambda(x)$ converges to $T_1(x)$ uniformly on $\bar{B}$ as $\lambda \to 1$. But $||x_\lambda - T_1(x_\lambda)|| = ||T_\lambda(x_\lambda) - T_1(x_\lambda)||$, therefore $|| x_\lambda - T_1(x_\lambda)|| \to 0$ as $\lambda \to 1$.

Then $\inf \{||x - T_1(x)|| : x \in \bar{B}\} = 0$.
By applying Theorem 4.2.4. to T_1 we may conclude $F(T_1) \neq \phi$ in $\bar{B}$. But $F(T_1) = F(T)$, as was shown in the proof of Theorem 4.3.2, hence $F(T)$ is nonempty in $\bar{B}$.

Remark. We cannot conclude to the compactness of $F(T)$, since T is a 1-set-contraction. It is known, however, that $F(T_\lambda)$ is nonempty and compact for each $\lambda < 1$.

We state the following lemma without proof; the lemma will be used in the proof of Theorem 4.3.8.

Lemma 4.3.7. If D is a bounded open subset of a Banach space X with 0 in D and $T : \bar{D} \to X$ is a densifying mapping which satisfies the boundary condition $(A \leq)$ in $\dot{D}$, then $F(T) \subset X$ is nonempty and compact.

The following results of Petryshyn [60] generalize Theorem 4.3.2.

Theorem 4.3.8. Let D be a bounded open subset of a Banach space X with 0 in D and let $T : \bar{D} \to X$ be a 1-set-contraction satisfying $(A \leq)$ on $\dot{D}$. Then, if $(I - T)(\bar{D})$ is closed, $F(T) \neq \phi$.
In particular, if T is demicompact and a 1-set-contraction, then $F(T)$ is nonempty and compact.
(For definition of demicompact, see Definition 4.1.9).

Proof. For each $0 < t < 1$, consider the t-set-contraction T_t of $\bar{D}$ into X defined by $T_t(x) = tT(x)$. It is easy to see that T_t satisfies condition $(A \leq)$ on $\dot{D}$ for each t in $(0,1)$. Hence, by Lemma 4.3.7, for each $t_n \in (0,1)$, with $t_n \to 1$ as $n \to \infty$, there exists an $x_n \in \bar{D}$

such that $T_{t_n}(x_n) = x_n$. Since $T(x_n) - x_n = (1 - t_n)T(x_n)$ and T is bounded, it follows that

$$T(x_n) - x_n \to 0 \quad \text{as} \quad n \to \infty \ .$$

In view of this and the assumed closedness of $(I - T)(\bar{D})$, we see that $0 \in (I - T)(\bar{D})$. Hence, $F(T) \neq \phi$. If we assume that T is demicompact and 1-set-contraction on $\bar{D}$, then $(I - T)(\bar{D})$ is closed and thus, by the first part of Lemma 4.3.7, $F(T) \neq \phi$. Furthermore, $F(T)$ is also compact since T is demicompact on $\bar{D}$.

Remarks.

1. The set $(I - T)(\bar{D})$ is certainly closed if T is densifying (and, in particular, if T is a k-set-contraction, $k < 1$).

2. If D is also convex, then $(A \leq)$ holds on $\dot{D}$ if $T(\dot{D}) \subset \bar{D}$ and, in particular, if $T(\bar{D}) \subset \bar{D}$.

3. When $0 \notin D$, then the following generalization of Theorem 4.3.8 holds. And in view of Remarks 1 and 2, this generalization includes all the fixed point theorems for densifying and k-set-contraction mappings mentioned to present in this section.

Theorem 4.3.9. Let D be a bounded open subset of Banach space X and $T : \bar{D} \to X$ a 1-set-contraction such that T satisfies any one of the following conditions:

(i) There exists an x_o in D such that if $T(x) - x_o = a(x - x_o)$ holds for some x in $\dot{D}$, then $a \leq 1$.

(ii) D is convex and $T(\dot{D}) \subset \bar{D}$.

Then, if $(I - T)(\bar{D})$ is closed, we have $F(T) \neq \phi$.

In particular, if T is demicompact and 1-set-contraction, then $F(T)$ is nonempty and compact.

Proof. We will first prove the theorem for condition (i).

Consider the set $Q = \ddot{D} - x_o = \{x - x_o : x \in \ddot{D}\}$. It follows that Q is bounded, open, $0 \in Q$, $\dot{Q} = \dot{D} - x_o$, and $\bar{Q} = \bar{D} - x_o$.

Furthermore, Q is convex if D is convex.

Now define the map $T'(y)$ for y in $\bar{Q}$ and

$$y = \{x - x_o : x \in \bar{D}\} \text{ by } T'(y) = T(x) - x_o.$$

Then T' maps $\bar{Q}$ into X, T' is a 1-set-contraction, and T' satisfies condition $(A \leq)$ on $\dot{Q}$.

Furthermore, $(I - T')(\bar{Q})$ is closed since $(I - T')(\bar{Q}) = (I - T)(\bar{D})$.

Thus T' and Q satisfy all the conditions of Theorem 4.3.9. Hence there exists a y in Q such that $T'(y) = y$,

i.e. $\quad T(x) - x_o = x - x_o$ with $x \in \bar{D}$,

i.e. $T(x) = x$.

The second part of Theorem 4.3.1 also follows since the demicompactness of T implies the same for T'.

Now, we will show that condition (ii) implies condition (i).

Suppose condition (ii) holds, and let x_o be any fixed element in $\ddot{D}$.

Then $Q = D - x_o$ is convex, $0 \in Q$, and $T'(\dot{Q}) \subset \bar{Q}$ since

$T'(\dot{Q}) = T(\dot{D}) - x_o \subset \bar{D} - x_o = \bar{Q}$ and D is convex. Hence, by Remark 2 following Theorem 4.3.9, T' satisfies condition $(A \leq)$ on $\dot{Q}$, that is, Theorem 4.3.9 holds.

Based upon results of Petryshyn [60], Vignoli [76] has given the following results.

Theorem 4.3.10. Let $T : \bar{B} \to X$ be a k-set-contraction, and let T satisfy the following condition on $\dot{B}$

(i) If $T(x) = \beta x$ for some x in $\dot{B}$, then $\beta \leq \mu$, where μ is any real number satisfying the inequality;

(ii) $0 \leq k < 1 - |1 - \mu|$, where $0 \leq k < 1$ is the contracting constant of T.

Then there exists x in $\bar{B}$ such that $T(x) = \mu x$.

Proof. Consider the mapping $F : \bar{B} \to X$ defined by $F(x) = T(x) + (1 - \mu)x$, where μ satisfies condition (ii).
The mapping F is a k-set-contraction. For let A be any bounded subset of $\bar{B}$, then we have

$$\begin{aligned}\alpha(F(A)) &= \alpha(T(A) + (1 - \mu)A) \\ &\leq \alpha(T(A)) + |1 - \mu|\alpha(A) \\ &\leq (k + |1 - \mu|)\,\alpha(A).\end{aligned}$$

Condition (ii) implies $0 \leq k + |1 - \mu| < 1$, so F is a $(k + |1 - \mu|)$-set-contraction.
Now, let $R : X \to \bar{B}$ be the radial retraction of X onto $\bar{B}$ (see Lemma 4.3.1). Since R is a 1-set-contraction, the composite mapping $R \circ F : \bar{B} \to \bar{B}$ is a k-set-contraction with contraction constant $(k + |1 - \mu|)$.
Thus by Darbo's fixed point theorem (see Corollary 2.2.23), there exists x in $\bar{B}$ such that $(R \circ F)(x) = x$. But from the definition of a radial

retraction and condition (i) it follows that $\dot{x}$ is also a fixed point of F.

Indeed, if $x \in B$, then $F(x) = x$. Otherwise the equality

$F(x) = \frac{x\|F(x)\|}{r}$ would imply $\|x\| = r$ which contradicts the assumption $\|x\| < r$.

Now suppose $x \in \dot{B}$ and x is not a fixed point of F; this implies

$\frac{\|F(x)\|}{r} > 1$, hence $T(x) = (\frac{\|F(x)\|}{r} - 1 + \mu)x$,

$$\text{thus} \quad \beta = \frac{\|F(x)\|}{r} - 1 + \mu > \mu$$

which contradicts condition (i).

Therefore $F(x) = x$, i.e. $T(x) = \mu x$.

Remarks.

1. Condition (ii) of Theorem 4.3.10 implies $0 < \mu < 2$. This restriction (at least with the present proof) is essentially due to the fact that the class of k-set-contractions is not closed under linear combinations, but under convex ones.

2. For $\mu = 1$ we obtain a fixed point theorem which extends Darbo's theorem (Corollary 2.2.23) in the case of the ball. As shown by Petryshyn [60] in this case F(T) the set of all fixed points of T is nonempty and compact. Actually this is proved in [60] for densifying mappings.

3. If $k = 0$ (i.e. T is completely continuous mapping), Theorem 4.3.10 holds for any $0 < \mu < 2$.

4. Imposing some restrictions on the Banach space X, Petryshyn [57, 58] was able to prove a theorem analogous to Theorem 4.3.10

involving P-compact mappings (see Petryshyn [58]).

Let X be a real Banach space and X^* its dual space. By (x,y), x in X and y in X^*, we will denote the pairing between X and X^*. Let $K : X \to X^*$ be a mapping such that $(x, K(x)) > 0$ if $x \neq 0$. As a consequence of Theorem 4.3.10, Vignoli [76] has obtained the following result.

Theorem 4.3.11. Let $T : \bar{B} \to X$ be a k-set-contraction. Suppose

$(T(x), K(x)) \leq (x, K(x)) - a||K(x)||$, for all x in $\bar{B}$, where $a \geq 0$ is some constant.

Then for any fixed f in X and any real λ such that $\lambda||f|| \leq a$ there exists x in $\bar{B}$ such that $x - T(x) = \lambda f$.

Proof. Let $f \in X$ be arbitrary but fixed. Consider the mapping $F = (\mu - 1)I + T + \lambda f : \bar{B} \to X$, where μ is a real number such that $|\mu - 1| + k < 1$, and λ, $(0 \leq \lambda \leq 1)$, is such that $\lambda||f|| \leq a$. Clearly F is a $(k + |\mu - 1|)$-set-contraction. Furthermore if for some x in $\dot{B}$ we have $F(x) = \beta x$, then

$$
\begin{aligned}
(\beta x, K(x) &= (F(x), K(x)) \\
&= \mu(x, K(x)) + (T(x), K(x)) - (x, K(x)) \\
&\quad + \lambda(f, K(x)) \\
&\leq \mu(x, K(x)) + (T(x), K(x)) - (x, K(x)) \\
&\qquad\qquad + a||K(x)|| \\
&\leq \mu(x, K(x)).
\end{aligned}
$$

i.e. $\beta \leq \mu$.

Hence by Theorem 4.3.10 there exists x in $\bar{B}$ such that

$$F(x) = \mu x$$

$$\text{i.e.} \quad x - T(x) = \lambda f.$$

By choosing $f = 0$ we obtain

Theorem 4.3.12. Let $T : \bar{B} \to X$ be as in Theorem 4.3.11, then there exists at least one element x in $\bar{B}$ such that $T(x) = x$.

It is easy to see that Theorem 4.3.11 holds also for densifying mappings.

Theorem 4.3.13. Let $T : \bar{B} \to X$ be densifying. Suppose

$(T(x), K(x) \leq (x, K(x)) - a||K(x)||$, for all x in $\dot{B}$,

where $a \geq 0$ is some constant.

Then for any fixed f in X and any real number λ satisfying the condition $\lambda||f|| \leq a$ there exists x in $\bar{B}$ such that $x - T(x) = \lambda f$.

Proof. Consider the mapping $G = T + \lambda f$, where $0 \leq \lambda \leq 1$ is such that $\lambda||f|| \leq a$.

Suppose $G(x) = \beta x$ for some x in $\dot{B}$, then it is easy to show that the above equality implies $\beta \leq 1$, hence by Petryshyn's fixed point theorem (see Theorem 4.3.2) it follows that there exists an element x in $\bar{B}$ such that $G(x) = x$, i.e. $x - T(x) = \lambda f$.

4.4. Condensing Mappings in Hilbert Space.

In this section we will give a few results of Edmunds and Webb [30] on condensing mappings in Hilbert spaces. (See Section 2.3 for a discussion of condensing mappings). B, $\dot{B}$, $\bar{B}$ will be as in Section 4.3.

Theorem 4.4.1. Let $T : H \to H$ be a condensing mapping such that for some $r > 0$, $||x|| = r$, $x \in B$, implies $T(x) \neq \lambda x$ for any $\lambda > 1$. Then T has a fixed point in $\bar{B}$, the closed ball of radius r and centre 0.

Proof. Let $R : H \to \bar{B}$ be the radial retraction of H onto $\bar{B}$.
It is known [22] that $||R(x) - R(y)|| \leq ||x - y||$ for all x,y in the Hilbert space H.
A routine calculation shows that $R \circ T$ is a condensing mapping which maps $\bar{B}$ into itself (see Section 4.3). So by Sadovskii's fixed point theorem (see Theorem 2.3.3) there exists x_o in $\bar{B}$ such that $x_o = R \circ T(x_o)$. There are two possibilities: either x_o is a boundary point of $\bar{B}$ (i.e. $x_o \in \dot{B}$) or x_o is an interior point (i.e., $x_o \in B$).
If $||x_o|| < r$, then $||R \circ T(x_o)|| < r$ so that $x_o = R \circ T(x_o) = T(x_o)$.
Alternately, if $||x_o|| = r$, then

$$x_o = R \circ T(x_o) = \frac{r\,T(x_o)}{T(x_o)} ,$$

which gives $T(x_o) = \lambda x_o$ where

$$\lambda = \frac{||T(x_o)||}{r} \geq 1.$$

However, $\lambda > 1$ is excluded by hypothesis so that $x_o = T(x_o)$.

Corollary 4.4.2. Let $T : H \to H$ be a condensing mapping such that, for some $r > 0$,

$$(T(x), x) \leq ||x||^2 \quad \text{for all } ||x|| = r.$$

Then T has a fixed point in $\bar{B}$.

Proof. The situation $T(x) = \lambda x$ implies $(\lambda x, x) \leq ||x||^2$ for $||x|| = r$, so that $\lambda < 1$. Hence Theorem 4.4.1 applies.

Corollary 4.4.3. Let $T : H \to H$ be a condensing mapping such that for some $r > 0$, T maps $\dot{B}$ into $\bar{B}$. Then T has a fixed point in $\bar{B}$.

Note. In our discussion of α-measure of concompactness, we studied k-set-contractions and 1-set-contractions. To avoid confusion in our terminology, we will denote a k-set-contraction in χ-measure of concompactness by k_χ-set-contraction and a 1-set-contraction by 1_χ-set-contraction. This terminology will prove useful in the next result.

Corollary 4.4.4. Let $T : H \to H$ be a 1_χ-set-contraction such that $(I - T)$ maps closed balls into closed sets. Suppose that for some $r > 0$, $||x|| = r$ and $\lambda > 1$ imply $T(x) \neq \lambda x$.
Then T has a fixed point in $\bar{B}$.

Proof. For $0 < t < 1$, let $T_t = tT$.
Suppose $||x|| = r$; then if $\lambda > 1$, $tT(x) \neq r \lambda x$, so that if $\mu > 1$ and t satisfies $0 < t < 1$, we have $T_t(x) \neq \mu x$.
By Theorem 4.4.1, since T_t is a condensing mapping, there exists x_t in $\bar{B}$ such that $T_t(x_t) = x_t$.
Then $(I - T)(x_t) = T_t(x_t) - T(x_t) = (t - 1)T(x_t) \to 0$ as $t \to 1$, since $\{x_t\}$ is bounded and T, being a 1_χ-set-contraction, maps bounded sets into bounded sets. Thus 0 belongs to the closure of $(I - T)(\bar{B})$. But $(I - T)(\bar{B})$ is closed by hypothesis; hence there exists x in $\bar{B}$ such that $0 = x - T(x)$, i.e., $x = T(x)$.

Remarks.

1. Corollary 4.4.3 is, in fact, true in a strengthened form; i.e. let C be a closed bounded convex subset of H and $T : H \to H$ be a condensing mapping which maps the boundary of C into C. Then T has a fixed point in C.
   To see this, define a map $L : H \to C$ by associating with every x in H the point $L(x)$ in C nearest to x. Such a map is a contraction in Hilbert space and the desired conclusion follows from an argument similar to that used in Theorem 4.4.1.

2. An examination of the proof of Sadovskii's fixed point theorem (Theorem 2.3.3) shows that its conclusion remains true if the condition $\chi(T(A)) < \chi(A)$ imposed on T is replaced by $\alpha(T(A)) < \alpha(A)$ for every bounded noncompact set A. It follows that, Theorem 4.4.1 together with Corollaries 4.4.2 and 4.4.3, hold for maps T densifying, while Corollary 4.4.4 holds for 1-set-contractions. Moreover, the strengthened form of Corollary 4.4.3 given in Remark 1 above, may similarily be varied to read as follows:
   Let $T : H \to H$ be densifying. Suppose C is a closed bounded convex subset of H, and let $T(C) \subset C$. Then T has a fixed point in C.

3. The set $F = F(T)$ of fixed points of T, in Theorem 4.4.1 and in its corollaries, is compact. For, F is a closed subset of $\bar{B}$ and hence complete. Now, by the condensing property of T we have $\chi(F) = 0$ and hence F is totally bounded. Thus F is compact.

4. The radial retraction R is a contraction. This is valid only in Hilbert spaces; in Banach spaces which are not Hilbert spaces the corresponding result is $||R(x) - R(y)|| \leq k||x - y||$ for $1 < k \leq 2$. In some Banach spaces (e.g. ℓ') the constant 2 is necessary, but in uniformly convex spaces the inequality holds with $k < 2$. Karlovitz and de Figuerido [21] have obtained the estimate $k \leq 2 - 2\delta$ where δ is the modulus of convexity corresponding to $\varepsilon = 1$. Theorem 4.4.1 is valid in any Banach space provided that the condition on T is strengthened by requiring that T be a k_X-set contraction with $k < \frac{1}{2}$; in uniformly convex spaces $k < 1/(2 - 2\delta)$ will suffice.

CHAPTER V

Applications

5.1 Applications

Many applications of fixed point theory occur in differential and integral equations, optimal control theory, nonlinear optimization, nonlinear approximation and many other fields. Here we give only some applications in optimization theory.

Theorems 5.12 and 5.13 proved in [35] give a characterization of well posed minimum problems and well posed minimum problems in the generalized sense respectively.

Let J be a lower bounded, lower semicontinuous functional defined on a metric space X. Consider the problem of minimum for the functional J. Suppose J has a unique minimum x_o in X. This problem is said to be well posed (or correctly posed) if any minimining sequence $\{x_n\}$ of J converges to x_o. We will now look at a somewhat more general definition of well posed problems which does not require the uniqueness of the minimum point.

Definition 5.1.1. Let X be a metric space and $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous real functional. The problem

$$J(x) = \inf J(X) \qquad (5.1.1)$$

is said to be well posed if any minimizing sequence $\{x_n\}$ of J is compact.

Note that, if $\{x_n\}$ is a compact minimizing sequence of J, then any limit point of $\{x_n\}$ is a solution of problem (5.1.1), i.e. is a minimum point of J.

Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional on a metric space X. Let

$$\Omega_\rho = \{x \in X : J(x) \leq \inf J(X) + \rho\}, \quad \rho > 0, \text{ and}$$
$$\Omega = \{x \in X : J(x) = \inf J(X)\}.$$

Then the following theorems hold.

Theorem 5.1.2. Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional defined on a complete metric space X. Then, the minimum problem for J is well posed if and only if $\delta(\Omega_\rho) \to 0$ as $\rho \to 0$, where $\delta(\Omega_\rho)$ is the diameter of Ω_ρ.

Theorem 5.1.3. Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional defined on a complete metric space X. Then the problem $J(x) = \inf J(X)$ is well posed in the generalized sense if and only if $\lim_{\rho\to 0} \alpha(\Omega_\rho) = 0$.

In the case of the uniqueness of the minimum point for J, we have the following result.

Theorem 5.1.4. Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional defined on a complete metric space X. Then any minimizing sequence for J is convergent if and only if $\delta(\Omega_\rho) \to 0$ as $\rho \to 0$, where $\delta(\Omega_\rho)$ is the diameter of the set Ω_ρ.

Furi and Vignoli [35] have given the following example of a well posed problem where $\alpha(\Omega_\rho) > 0$ for $\rho > 0$ and $\delta(\Omega_\rho)$ does not vanish as $\rho \to 0$.

Example 5.1.5. Let X be the closed unit ball of $C[0,1]$. Define $J : X \to \mathbb{R}$ by $J(x) = OSC(x)$, where $OSC(x)$ is the oscillation of the function x in $[0,1]$. Clearly, $\Omega = \{x \in X : J(x) = 0\}$ is the set of all constant functions belonging to X. It is easily seen that $\alpha(\Omega_\rho) > 0$ for $\rho > 0$ and $\Omega_\rho \subset \beta(\Omega,\rho)$. Hence $\alpha(\Omega_\rho) \leq \alpha(\Omega) + 2\rho$. Therefore, $\alpha(\Omega_\rho) \to 0$ as $\rho \to 0$, since Ω is homeomorphic to the compact interval $[-1,1]$.

Definition 5.1.6. Let $\sigma : D \to \bar{\mathbb{R}}_+$ be an extended real function defined on a subset D of $\bar{\mathbb{R}}_+ = [0, +\infty]$. Let $\bar{\sigma}(A) = \{r \in D : \sigma(r) \in A\}$, where $A \subset \bar{\mathbb{R}}_+$.

We define

$$\text{(i)} \quad \sigma^+(\rho) = \operatorname{Sup} \bar{\sigma}([0, \rho]) \quad \text{if} \quad \bar{\sigma}([0, \rho]) \neq \phi$$
$$= 0 \quad \text{if} \quad \bar{\sigma}([0, \rho]) = \phi$$

$$\text{(ii)} \quad \sigma^-(\rho) = \inf \bar{\sigma}([0, \rho]) \quad \text{if} \quad \bar{\sigma}([\rho, +\infty]) \neq \phi$$
$$= +\infty \quad \text{if} \quad \bar{\sigma}([\rho, +\infty]) = \phi .$$

It can be shown that σ^+ and σ^- are nondecreasing extended real functions defined on $\bar{\mathbb{R}}_+$ such that $\sigma^+(\sigma(r)) \geq r$, $\sigma^-(\sigma(r)) \leq r$, for any r in D.

We introduce the following definition given by Furi and Vignoli in [36].

Definition 5.1.7. A lower bounded, lower semicontinuous functional $J \to \mathbb{R}$ defined on a metric space X is called a funnel functional if

$$\inf J(X) \leq \max\{J(x), J(y)\} - \sigma(d(x,y)), \quad \text{for all } x,y \text{ in } X,$$

where $\sigma : D \to \bar{\mathbb{R}}_+$ is such that

$$\sigma^+(\rho) \to 0 \quad \text{as} \quad \rho \to 0 \quad \text{and} \quad D \supset \{d(x,y) : x, y \in X\} .$$

We have the following theorem.

Theorem 5.1.8. Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional defined on a metric space X. Then J is a funnel functional if and only if $\delta(\Omega_\rho) \to 0$ as $\rho \to 0$, where

$$\Omega_\rho = \{x \in X : J(x) \leq \inf J(X) + \rho\} , \quad \rho > 0.$$

As a consequence of Theorem 5.1.2 and Theorem 5.1.8, we have the following characterization of well posed problems.

Theorem 5.1.9. The minimum problem for a functional $J : X \to \mathbb{R}$, X a complete metric space X is well posed if and only if J is a funnel functional.

If J is a funnel functional defined on a complete metric space, then the following theorem gives an error estimate for the unique minimum point for J.

Theorem 5.1.10. Let $J : X \to \mathbb{R}$ be a funnel functional defined on a complete metric space X, and let z in X be the unique minimum point for J. Then, the following error estimate holds:

$$d(x,z) \leq \sigma^+ (J(x) - \inf J(X)) \quad , \quad x \text{ in } X,$$

where $\sigma^+(\rho) \to 0$ as $\rho \to 0$.

The following examples are given by Furi and Vignoli [36] to illustrate the concept of a funnel functional.

Example 5.1.11. Let $(E^n, |\cdot|)$ be the n-dimensional Euclidean space, and let $C_n[a,b]$ be the Banach space of all continuous mappings $Q : [a,b] \to E^n$ of th compact interval [a,b] into E^n, with norm

$$||x|| = \text{Sup } \{|x(t)| : a \leq t \leq b\} .$$

Let $f : [a,b] \times E^n \to E^n$ be a continuous mapping which satisfies a Lipschitz condition

$$|f(t,x) - f(t,y)| \leq K (x - y), \text{ for all } x,y \text{ in } E^n.$$

Consider the integral equation

$$Q(t) = x_o + \int_a^t f(s, Q(s))ds$$

where x_o is a given point of E^n. The solutions of this equation are fixed points of the mapping

$$T : C_n[a,b] \to C_n[a,b]$$

such that

$$T(Q)[t] = x_o + \int_a^t f(s, Q(s))ds.$$

The functional

$$J : C_n[a,b] \to \mathbb{R} : Q \longmapsto ||Q - T(Q)||$$

is a funnel functional.

Example 5.1.12. Let $T : X \to X$ be a contraction mapping defined on a metric space X. Then, the functional $J(x) = d(x, T(x))$ is a funnel functional. We have

$$d(x,y) \leq d(x, T(x)) + d(T(x), T(y)) + d(T(y), y)$$
$$\leq 2 \max \{J(x), J(y)\} + kd(x,y).$$

But it is well known that $\inf J(X) = 0$.

Hence,

$$0 = \inf J(x) \leq \max \{J(x), J(y)\} - \frac{1}{2} (1 - k)d(x,y).$$

Example 5.1.13. Let X be a complete metric space and $T : X \to X$ be a function satisfying

$$d(T(x), T(y)) \leq d(x,y) - \Delta(d(x,y)), \quad \text{for all } x,y \text{ in } X,$$

where Δ is nondecreasing, right continuous, such that $\Delta(r) > 0$ for $r > 0$. Then, the functional $J(x) = d(x, T(x))$ is a funnel functional.

Example 5.1.14. Let $T : X \to X$ be defined on a bounded complete metric space X and satisfy

$$d(T(x), T(y)) \leq Q(d(x,y)), \quad \text{for all } x,y \text{ in } X,$$

where Q is a nondecreasing, right continuous, real function such that $Q(r) < r$ for $r > 0$ (see Browder [11]). Then, $J(x) = d(x, T(x))$ is a funnel functional.

Example 5.1.15. Let x_o be a given point of a metric space X. Then, the functional $J(x) = d(x, x_o)$ is a funnel functional.

Indeed, we have

$$d(x,y) \leq d(x, x_o) + d(x_o, y),$$

i.e. $$0 \leq J(x) + J(y) - d(x,y).$$

Then, $$0 \leq \max\{J(x), J(y)\} - \frac{1}{2} d(x,y).$$

The concept of funnel functionals have been extended by Furi and Vignoli [36] to tank functionals.

Definition 5.1.16. A lower bounded, lower semicontinuous functional $J : X \to \mathbb{R}$ defined on a metric space X is called a tank functional if

$$\inf J(x) \leq \operatorname{Sup} J(A) - \sigma(\alpha(A))$$

for any set $A \subset X$ such that $\operatorname{Sup} J(A) < +\infty$, where $\sigma : D \to \bar{\mathbb{R}}_+$ is such that $D \supset \{\alpha(A) : A \subset X\}$ and $\sigma^+(\rho) \to 0$ as $\rho \to 0$.

We have the following results.

Theorem 5.1.17. Let $J : X \to \mathbb{R}$ be a lower bounded, lower semicontinuous functional defined on a metric space X. Then J is a tank functional if and only if $\alpha(\Omega_\rho) \to 0$ as $\rho \to 0$, where

$$\Omega_\rho = \{x \in X : J(x) \leq \inf J(X) + \rho\} , \quad \rho > 0.$$

As a consequence of Theorem 5.1.3 and 5.1.17, we have the following characterization of well posed problems in the general sense.

Theorem 5.1.18. The minimum problem for a functional $J : X \to \mathbb{R}$ defined on a complete metric space X is well posed in the generalized sense if and only if J is a tank functional.

The following consequence of Theorems 5.1.9 and 5.1.18 shows the connection between tank functionals and funnel functionals.

Theorem 5.1.19. Let J be a functional defined on a complete metric space X. Then, J is a funnel functional if and only if it is a tank functional with a unique minimum point.

Some examples of tank functionals are now given.

Example 5.1.20. Let $T : X \to X$ be a continuous mapping defined on a bounded complete metric space X, such that $\alpha(T(A)) < \alpha(A)$ for any $A \subset X$, $\alpha(A) > 0$, i.e. T is densifying. If $\inf\{d(x, T(x)) : x \in X\} = 0$, then $J(x) = d(x, T(x))$ is a tank functional (see [32], Corollary 3.3.4).

Example 5.1.21. Let $T : C \to C$ be a continuous mapping defined on a closed bounded convex subset C of a Banach space X such that T is densifying, i.e. $\alpha(T(A)) < \alpha(A)$ for any $A \subset C$, $\alpha(A) > 0$. Then, the functional $J(x) = ||x - T(x)||$ is a tank functional. Indeed, in [33] (see Theorem 4.2.4), it was proved that T has at least a fixed point; therefore

$$\inf \{||x - T(x)|| : x \in C\} = 0.$$

Hence, by Example 5.1.20, the functional J is a tank functional.

Example 5.1.22. Let $J : X \to \mathbb{R}$ be a lower semicontinuous functional defined on a compact metric space X. Then, J is a tank functional.

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