FIXED AND PERIODIC POINTS UNDER CONTRACTION MAPPINGS IN METRIC SPACES

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by

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ABSTRACT

The main aim of this thesis is to investigate fixed and periodic points under contraction or distance shrinking mappings in metric spaces.

Various situations are explored where the notion of contraction is relaxed and suitable modifications made on the metric space to ensure fixed or periodic points for the contraction.

During the course of these investigations a few new results which guarantee fixed or periodic points for contractions under suitably weak conditions have been given for metric spaces.

A few fixed point theorems have been also given in generalized complete metric spaces. Some of these are generalizations of well known results in this space.

Convergence of a sequence of contractions and their fixed points have been studied briefly and a few new theorems have been added.

In the end an attempt is made to apply the contraction mapping principle to the theory of differential and integral equations.

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CHAPTER I

INTRODUCTION

In 1922 S. Banach formulated his classical theorem commonly called the Banach Contraction Principle which may be stated as follows:-

"A contraction mapping of a complete metric space into itself has a unique fixed point."

This theorem was first applied to the proof of an existence theorem by Cacciopoli in 1930 and still remains the most fruitful means for proving and analysing the convergence of iterative processes.

The contraction mapping principle has moviated a great deal of research in the field of functional analysis. Extensions of the theorem are of continuing interest and have been given by such mathematicians as Rakotch (1962) and Chu and Diaz (1965).

Others such as Luxemburg (1958), Monna (1961), Edelstein (1964), Margolis (1967), and Diaz and Margolis (1967) have generalized the contraction mapping principle to generalized metric spaces.

These results have been further generalized to uniform spaces by Davis (1963), Kammerer and Kasriel (1964), Naimpally (1965), Edelstein (1967), and many others.

Still more results have been given on fixed point theorems in other spaces by such leading mathematicians as Brouwer (1912), Schauder(1927) and Tychonoff (1935).

Many of these fixed point theorems have been used to guarantee existence and uniqueness to solutions of differential and integral equations.

The main aim of this thesis is to study fixed and periodic points under different contraction mappings restricted to metric and generalized metric spaces.

Various fixed point theorems are developed by relaxing the concept of contraction and at the same time modifying the metric space.

In Chapter II the Banach contraction mapping principle with various modifications and generalizations is discussed. A survey is made of various types of contractive mappings which under suitable conditions have fixed or periodic points.

In Chapter III some new fixed point theorems have been given for generalized complete metric spaces. These theorems include generalizations of fixed point theorems of Luxemburg, Monna, Edelstein and Margolis.

A brief study has been made of the convergence of sequences of contraction mappings and their fixed points in Chapter IV. Sequences of contraction and contractive mappings have been studied in complete metric, compact, and generalized complete metric spaces. A few new theorems have been included.

In the final chapter, Chapter V, the contraction mapping principle is applied to the theory of differential and integral equations.

CHAPTER II

CONTRACTION AND CONTRACTIVE MAPPINGS IN METRIC SPACES

2.1 Preliminary Definitions

Definition 2.1.1 Let X be any set and let R^+ denote the positive reals. We define a distance function $d: X \times X \to R^+$ to be a metric if the following conditions are satisfied:

(i) $d(x, y) \geq 0 \quad \forall\, x, y \in X$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

(i) and (ii) guarantee that the distance between any two points of X is always positive and only zero when the points coincide.

(iii) assures that the order of measurement of distance between two points is insignificant.

(iv) is a statement of the familiar triangular inequality.

(X, d) with d defined as above is called a metric space.

Actually with (ii) modified namely

$d(x, y) = 0$ if $x = y$ (ii)*

we define a more general space called Pseudo-metric Space.

Definition 2.1.2 A function f is said to satisfy Lipschitz condition if $d(f(x), f(y)) \leq K\, d(x, y) \quad \forall\, x, y \in X$.

In the special case when $0 \leq K < 1$ or $K \in [0, 1)$ f is said to be a contraction mapping.

<u>Remark 2.1.3</u> Every contraction mapping is clearly continuous.

<u>Proof:</u> To show continuity we need to justify continuity at each point $x_o \in X$. Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{k} > 0$.

$$\text{Then } d(x, x_o) < \delta \Rightarrow d(f(x), f(x_o)) \leq kd(x, x_o))$$
$$< k\frac{\varepsilon}{k}$$
$$< \varepsilon \qquad (0 < k < 1).$$

Hence f is continuous at x_o which is arbitrary.

Therefore f is everywhere continuous.

<u>Definition 2.1.4</u> A sequence $\{x_n\}$ is said to be a Cauchy sequence if for $\varepsilon > 0$ $\exists$ a number $N(\varepsilon)$ such that $\forall$ m, n > N $d(x_m, x_n) < \varepsilon$. In other words $\lim_{n,m \to \infty} d(x_m, x_n) = 0$.

<u>Definition 2.1.5</u> If for every $\varepsilon > 0$, $\exists$ N such that $n \geq N \Rightarrow d(x_n, x) < \varepsilon$, then $\{x_n\}$ is a convergent sequence and converges to x.

i.e. $$\lim_{n \to \infty} x_n = x .$$

It is well known that every convergent sequence is a Cauchy sequence but not conversely.

<u>Definition 2.1.6</u> A metric space is said to be complete if every Cauchy sequence converges in that space.

With the notion of complete metric space and contraction already defined we are now ready to give the main theorem on contraction mappings. In fact, all of the work that follows was motivated by this well known result of S. Banach. More precisely most of the work the intermediate results are developed by imposing certain restrictions on the contraction mapping or by modifying the space.

2.2 Banach Contraction Theorem

Theorem 2.2.1 Let X be a complete metric space. If $f: X \to X$ is a contraction then f has a unique fixed point. In particular $f(x) = x$ has a unique solution.

Proof: Choose x_o, any arbitrary point, in X .

$$
\begin{aligned}
\text{Let } x_1 &= f(x_o) \\
x_2 &= f(x_1) = f^2(x_o) \\
&\vdots \\
x_n &= f(x_{n-1}) = f^n(x_o).
\end{aligned}
$$

We now show that $\{x_n\}$ is a Cauchy Sequence.

i.e. $$\lim_{n,m \to \infty} d(x_n, x_m) = 0.$$

From the definition of a contraction f,

$d(f(x)\ f(y)) \leq k\, d(x, y) \quad \forall\, x, y \in X.$

$$
\begin{aligned}
\text{Therefore } d(x_n, x_m) &= d(fx_{n-1}\ fx_{m-1}) \\
&\leq kd(x_{n-1}\ x_{m-1}) \\
&\leq k^2 d(x_{n-2}\ x_{m-2}) \\
&\vdots \\
&\leq k^n d(x_o, x_{m-n}).
\end{aligned}
$$

But $d(x_o, x_{m-n}) \leq d(x_o, x_1) + d(x_1, x_2) + \dots\dots\dots + d(x_{m-n-1}, x_{m-n})$.

Also $d(x_1, x_2) = d(f(x_o), f(x_1)) \leq kd(x_o, x_1)$,

and $d(x_2, x_3) = d(f(x_1)\ f(x_2)) \leq k^2 d(x_o, x_1)$.

Continuing we can demonstrate that

$$d(x_o, x_{m-n}) \leq d(x_o, x_1) + kd(x_o, x_1) + k^2d(x_o, x_1) + \ldots\ldots + k^{m-n-1}d(x_o, x_1)$$

$$\leq d(x_o, x_1)\ [1 + k + k^2 + \ldots\ldots\ldots + k^{m-n-1}].$$

Hence $d(x_n, x_m) \leq k^n d(x_o, x_{m-n})$,

$$\leq k^n d(x_o, x_1)\ [1 + k + k^2 + \ldots\ldots\ldots + k^{m-n-1} + \ldots\ldots]$$

$$\leq k^n d(x_o, x_1) \quad \left(\frac{1}{1-k}\right)$$

$$\to o \quad (o \leq k < 1).$$

Hence $\{x_n\}$ is a Cauchy Sequence.

Since X is complete $\{x_n\}$ converges to a point $x \in X$.

Therefore $\lim\limits_{n \to \infty} x_n = x$.

Now $f(x) = f(\lim\limits_{n \to \infty} x_n) = \lim\limits_{n \to \infty} f(x_n)$

(f is continuous)

$= \lim\limits_{n \to \infty} (x_{n+1})$

$= x$.

To show uniqueness choose x, x^1 two fixed points of f, $x \neq x^1$.

Then $f(x) = x$.

$f(x^1) = x^1$.

Now $d(f(x), f(x^1)) \leq kd(x, x^1)$.

But $d(f(x)\ \ f(x^1)) = d(x, x^1)$.

Hence $d(x, x^1) \leq kd(x, x^1)$,

and $1 \leq k$ (contradiction).

Hence $x = x^1$.

Remarks 2.2.3

1) Both conditions of the previous theorem are necessary. e.g.

(i) the map $f:(0,1] \to (0,1]$ defined by $f(x) = \frac{x}{2}$ is a contraction but has no fixed point since $(0, 1]$ is not a complete metric space.

(ii) the map $f : R \to R$ defined by $f(x) = x + 1$ is not a contraction and has no fixed point although R is complete.

2) The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximations.

3) The contraction theorem has the advantages of being constructive, its error of approximation can be estimated, and it guarantees a unique fixed point.

2.3 Some Modifications of Banach's Theorem

By retaining the notion of a contraction in the sense of Banach and merely modifying the space we can obtain many modifications of the contraction principle. The theorems that follow deal with such spaces as Pseudo-metric, and ε-chainable metric spaces. A more general form of the Banach theorem, obtained by generalizing the contraction constant, is also discussed.

Theorem 2.3.1 If f is a contraction self mapping on a complete Pseudo metric space X then f has a fixed point, not unique.

Proof: Existance of a fixed point for f may be justified similiarly to the previous theorem. If X is a Pseudo-metric space then from

(ii*) $d(x, y) = 0$ if $x = y$.

Hence $d(x, y) = 0 \not\Rightarrow x = y$.

Now referring to the discussion on uniqueness in the previous theorem,

$$d(x, x^1) \leq kd(x, x^1) \Rightarrow d(x, x^1) - kd(x, x^1) \leq 0.$$

Suppose $d(x, x^1) - kd(x, x^1) = 0$.

Then $(1 - k)\, d(x, x^1) = 0$. But $1 - k \neq 0$ for $k \in [0, 1)$.

Therefore $d(x, x^1) = 0$.

But this does not necessarily imply that $x = x^1$.

The following generalization of the Banach Theorem is very useful for certain applications. This theorem is due to Chu and Diaz [5].

<u>Theorem 2.3.2</u> If f^p (p positive integer) is a contraction self mapping on a complete metric space X then f has a unique fixed point.

<u>Proof:</u> Let $f^p = g : X \to X$.

Then $f^p = g$ has a unique fixed point x_o.

Now $f^{p+1}(x_o) = f^p(f(x_o)) = g(f(x_o)) = f(g(x_o)) = f(x_o)$

i.e. $f(x_o)$ is a fixed point of g.

Since g has unique fixed point x_o,

$$f(x_o) = x_o \text{ is unique.}$$

<u>Remarks 2.3.3</u> (i) The assumption that f^p is a contraction is not strictly necessary. In fact if X is any non empty set of elements and f is a single valued function on X into itself, the assumption that f^p has a unique fixed point will guarantee a unique fixed point for f.

(ii) Another result due to Chu and Diaz [4] may be formulated in the following way:-

Let f be a function defined on a non empty set X into itself.

Let g be another function defined on X into itself such that $gg^{-1} = I$ where I is the identity function of X. Then f has a fixed point if and only if $g^{-1}fg$ has a fixed point.

Observe (i) The previous case (Theorem 2.3.2) when $p = 1$ will give Banach's Theorem.

(ii) In the case where X is complete Pseudo-metric space the previous theorem won't give a result similiar to that of Theorem 2.3.1.

Definition 2.3.4: Let X be a metric space and $\varepsilon > 0$. Then a finite sequence $x_o, x_1, \ldots, x_n$ of points of X is called an ε-chain joining x_o and x_n if $d(x_{i-1}, x_i) < \varepsilon$ $(i = 1, 2, \ldots, n)$.
The space X is called ε-chainable if for every pair (x, y) of its points there exists an ε-chain joining x , y.

Theorem 2.3.5 Let X be a complete ε-chainable metric space. Suppose $f^p : X \to X$ is any mapping satisfying the condition that $d(x, y) < \varepsilon \Rightarrow d(f^p x, f^p y) \leq \lambda\, d(x, y)$ where $\lambda \in (0, 1)$. Then f has a unique fixed point.

Proof: We have to show that f^p is a contraction with respect to the metric d_ε; then the proof follows by Theorem 2.3.2. Since (X, d) is ε-chainable, define $d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i)$ $\forall$ ε-chains $x = x_o, x_1, \ldots\ldots\ldots\ldots, x_n = y$.
Now d_ε is a distance function on X satisfying

(1) $d(x, y) \leq d_\varepsilon(x, y)$

(2) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$.

From (2) it follows that a Cauchy sequence $\{x_n\}$ in X is a Cauchy sequence with respect to d_ε if it is a Cauchy sequence with respect to d, and is convergent with respect to d_ε if it converges with respect to d. Hence (X, d_ε) is complete whenever (X, d) is complete. Moreover, the mapping f^p is a contraction with respect to d_ε. Given $x, y \in X$

and any ε-chain $x_o, \ldots, x_n$ with $x_o = x$, $x_n = y$ we get

$$d(x_{i-1}, x_i) < \varepsilon, \quad (i = 1, 2, \ldots n).$$

So that $d(f^p x_{i-1}, f^p x_i) \leq \lambda\, d(x_{i-1}, x_i) < \varepsilon, (i = 1, 2, \ldots, n)$.

Hence $f^p x_o, \ldots, f^p x_n$ is an ε-chain joining $f^p x$ and $f^p y$ and

$d_\varepsilon(f^p x, f^p y) \leq \sum_{i=1}^{n} d(f^p x_{i-1}, f^p x_i) \leq \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i)$.

Since $x_o, \ldots, x_n$ is an arbitrary ε-chain we get

$$d(f^p x, f^p y) \leq \lambda\, d_\varepsilon(x, y)$$

and hence f^p has a unique fixed point $x \in X$.

It follows from theorem 2.3.2 that f also has a unique fixed point x.

<u>Remark 2.3.6</u>: If in the previous theorem p = 1, we get Edelstein's theorem [7]. A more general version of the Banach Theorem can be constructed as shown by Rakotch in [23], if λ as previously defined is replaced by λ(x, y), a member of the family of functions $F = \{\lambda(x, y)\}$, satisfying the following conditions:

(i) $\lambda(x, y) = \lambda(d(x, y))$, λ depends only on the distance between x and y.

(ii) $0 \leq \lambda(d) < 1$, for all $d > 0$.

(iii) λ(d) is a monotonically decreasing function of d.

He gives the following theorem:

<u>Theorem 2.3.7</u>: Let f be a contraction self mapping on a complete metric space X such that $d(f(x), f(y)) \leq \lambda(x, y)\, d(x, y)$ for every $x, y \in X$, where $\lambda(x, y) \in F$.

Then f has a unique fixed point.

A similiar result can now be proved for an ε-chainable complete metric space [23]. This theorem has been proved by Singh.

<u>Theorem 2.3.8</u>: Let X be a complete ε-chainable metric space. Suppose $f : X \to X$ is any mapping satisfying the condition that $d(x, y) < \varepsilon \Rightarrow d(f(x), f(y) \leq \lambda(x, y)\, d(x, y)$ for all $x, y \in X$ and $\lambda(x, y) \in F$ then f has a unique fixed point.

<u>Proof</u>: Using exactly the same technique as in theorem 2.3.5 we can show that (X, d_ε) is complete.

Now
$$\begin{aligned} d(fx_{i-1}, fx_i) &\leq \lambda(x_{i-1}, x_i)\, d(x_{i-1}, x_i) \\ &= \lambda(d(x_{i-1}, x_i))\, d(x_{i-1}, x_i) \\ &< \lambda(\varepsilon)\, \varepsilon, \quad (i = 1, 2, \ldots n). \end{aligned}$$

But by definition $\lambda(\varepsilon) < 1$.

Therefore $d(fx_{i-1}, fx_i) < \varepsilon$.

Hence $f(x) = f(x_o), f(x_1), \ldots\ldots, f(x_n) = f(y)$ is an ε-chain for $f(x), f(y)$.

And
$$\begin{aligned} d_\varepsilon(f(x), f(y) &\leq \sum_{i=1}^{n} d(fx_{i-1}, fx_i) \\ &\leq \sum_{i=1}^{n} \lambda(d(x_{i-1}, x_i))\, d(x_{i-1}, x_i) \\ &\leq \lambda\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big) \sum_{i=1}^{n} d(x_{i-1}, x_i). \end{aligned}$$

Hence $d_\varepsilon(fx, fy) \leq \lambda(d_\varepsilon(x, y)\, d_\varepsilon(x, y)$.

The proof now follows by theorem 2.3.7.

2.4 <u>Contractive Mappings</u>

<u>Definition 2.4.1</u>: A mapping $f : X \to X$ is contractive if $d(f(x), f(y)) < d(x, y)$, for all $x, y \in X$ where $x \neq y$.

A contractive mapping on a complete metric space need not have a fixed point as the following example demonstrates.

<u>Example 2.4.2</u>: The map $f : R \to R$ defined by $f(x) = x + \frac{\pi}{2} - \arctan x$ is clearly contractive but has no fixed point.

The following theorem is due to Edelstein [8].

__Theorem 2.4.3__ Let f be a self contractive map on a metric space X and let $x \in X$ be such that the sequence of iterates $\{f^n(x)\}$ has a subsequence $\{f^{n_i}(x)\}$ convergent to a point $x_o \in X$. Then x_o is a unique fixed point of f.

__Corollary 2.4.4__ If f is a contractive mapping of a metric space X into a compact metric space $Y \subset X$, then f has a unique fixed point.

__Definition 2.4.5__ A mapping f of a metric space X into itself is an ε-contractive map if $0 < d(x, y) < \varepsilon \Rightarrow d(f(x), f(y)) < d(x, y)$ where $\varepsilon > 0$.

Two theorems due to Edelstein [8] for such a map are the following:

__Theorem 2.4.6__ An ε-contractive self mapping f on a compact metric space X has at least one periodic point.

__Theorem 2.4.7__ An ε-contractive self mapping f on an ε-chainable compact metric space X has a unique fixed point.

__Definition 2.4.8__ A mapping f of a metric space X into itself is said to be __locally__ contractive if for every $x \in X$ there exist ε and λ ($\varepsilon > 0$, $0 \leq \lambda < 1$) which may depend on x, such that

$$p, q \in S(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\} \Rightarrow d(f(p), f(q)) < \lambda\, d(p, q)\ ,\ p \neq q.$$

__Definition 2.4.9__ A mapping f of a metric space X into itself is said to be (ε, λ) uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Sufficient conditions for a fixed point under mappings such as the above are given by the following theorem due to Edelstein [7].

<u>Theorem 2.4.10</u> If $f : X \to X$ is an (ε, λ) uniform locally contractive self mapping on an ε-chainable complete metric space X, then f has a unique fixed point x_0. i.e. $f(x_0) = x_0$.

<u>Definition 2.4.11</u> A self mapping f on a metric space X is said to be globally contractive if the condition $d(f(x), f(y)) < \lambda\, d(x, y)$ with $\lambda \in [0, 1)$ holds for every $x, y \in X$.

<u>Remark 2.4.12</u> Edelstein [7] has shown that an $(\varepsilon - \lambda)$ uniformly contractive self mapping on a convex complete metric space X is also globally contractive. This suggests the following theorem:

<u>Theorem 2.4.13</u> If f^n, where n is any positive integer, is an (ε, λ) uniformly contractive self mapping of a closed convex subset A of a complete metric space X, then f has a unique fixed point.

<u>Proof:</u> Since A is a closed subset of a complete metric space X it follows that A is complete. Furthermore, since f^n is (ε, λ) uniformly contractive on A, f^n is also globally contractive on A (Edelstein). It now follows from Banach's theorem that f^n has a unique fixed point in A. Hence f also has a unique fixed point in A.

Bailey[1] has studied two very general contractive mappings on compact metric spaces. Using the notions of proximality, distality and asymptoticness he has given suitable conditions under which these mappings have fixed or periodic points. The two mappings investigated were these:-

(1) f is continuous and $0 < d(x, y) \Rightarrow$ there exists $n(x, y) \in I^+$, the positive integers, such that $d(f^n(x), f^n(y)) < d(x, y)$, $x \neq y$.

(2) f is continuous and there exists $\varepsilon > 0$ such that $0 < d(x, y) < \varepsilon \Rightarrow$ there exists $n(x, y) \in I^+$ such that $d(f^n(x), f^n(y)) < d(x, y)$, $x \neq y$.

He gives the following results:-

a) If a mapping f on a compact metric space X satisfies (1), then f has a unique fixed point.

b) If a mapping f on a compact metric space X satisfies (2), then f has periodic points. i.e. there exists $K > 0$ such that $f^K(U) = U$.

Using a result of Chu and Diaz [4] we can now give two simply proven theorems which modify results a) and b):

<u>Theorem 2.4.14</u> Let f be any self mapping of a compact metric space X. Suppose $K : X \to X$ is any mapping with a right inverse K^{-1} such that $KK^{-1} = I$, $K^{-1}fK$ is continuous. If $0 < d(x, y) \Rightarrow d(K^{-1}f^nK(x), K^{-1}f^nK(y)) < d(x, y)$ where $x \neq y$ and $n(x, y) \in I^+$, then f has a unique fixed point in X.

<u>Proof</u> $d(K^{-1}f^nK(x), K^{-1}f^nK(y)) = d((K^{-1}fK)^n x, (K^{-1}fK)^n y)$

$< d(x, y)$, $x \neq y$, $n(x, y) \in I^+$.

Hence by Bailey [1], $K^{-1}fK$ has a unique fixed point in X say η.

i.e. $K^{-1}fK(n) = n$.

Hence $KK^{-1}fK(n) = Kn$,

and $f(Kn) = Kn$.

Therefore Kn is a fixed point of f.

Clearly Kn is unique.

<u>Theorem 2.4.15</u> Let $f : X \to X$ be any self mapping of a compact metric space X. Suppose $K : X \to X$ is any mapping with a right inverse K^{-1} such that $K^{-1}fK$ is continuous. If $0 < d(x, y) < \varepsilon \Rightarrow d(K^{-1}f^nK(x), K^{-1}f^nK(y)) < d(x, y)$ where $x \neq y$ and $n(x, y) \in I^+$, then f has a periodic point in X.

<u>Proof</u> $d(K^{-1}f^nK(x), K^{-1}f^nK(y)) = d((K^{-1}fK)^n x, (K^{-1}fK)^n y)$

$$< d(x, y) \;,\; x \neq y \;,\; n(x, y) \in I^+$$

Hence by Bailey [1], $K^{-1}fK$ has a periodic point U say.

i.e. $(K^{-1}fK)^P U = U$ for some positive integer P.

Hence $K^{-1}f^PK(U) = U$,

and $KK^{-1}f^PK(U) = KU$.

i.e. $f^P(KU) = KU$,

and KU is therefore a periodic point of f.

Replacing conditions (1) and (2) by either a contractive or an ε-contractive mapping f^n on the space X we obtain the following two theorems:

<u>Theorem 2.4.16</u> If f^n, where $n \in I^+$, is a contractive self mapping on a compact metric space X, then any mapping $f : X \to X$ has a unique fixed point.

Proof: Since f^n is contractive on a compact metric space X, then f^n has a unique fixed point (Edelstein)

i.e. $f^n(U) = U$ is unique.

Hence $ff^n(U) = f^n(f(U)) = f(U)$.

Since f^n has a unique fixed point U, then $f(U) = U$ is unique since each fixed point of f is also a fixed point of f^n.

Theorem 2.4.17 If f^n, where $n \in I^+$, is an ε-contractive self mapping of a compact metric space X then any mapping $f : X \to X$ will have a periodic point i.e. $f^r(U) = U$ for some $r \in I^+$.

Proof: Since f^n is ε-contractive, then f^n has a periodic point U (Edelstein). i.e. $(f^n)^K(U) = U$, for some integer $K \in I^+$.

Since $n, K \in I^+$, then $nK \in I^+$ also.

Let $nK = r \in I^+$.

Then $f^r(U) = U$.

Remark 2.4.18: Converses to the Banach Contraction theorem have been provided by Bessaga [2], Meyers [17], Janos [11] and Wong [25].

CHAPTER THREE

Contraction Mappings on a Generalized Complete Metric Space

The concept of a generalized complete metric space, first introduced by W.A. Luxemburg, has been of continuing interest in recent years.

Two contraction mapping theorems were given by Luxemburg on such a space and then applied to the theory of ordinary differential equations. These theorems have since been generalized to a family of contractions by such mathematicians as Monna, Edelstein, and Margolis. Further modifications and generalizations will be given in this chapter.

<u>Definition 3.1</u> Let X be a non empty set. If there is defined on X x X a distance function $d(x, y)$ $(0 \leq d(x, y) \leq \infty)$ satisfying the following conditions:

(D1) $d(x, y) = 0$ iff $x = y$

(D2) $d(x, y) = d(y, x)$ (Symmetry)

(D3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

(D4) $\lim_{n,m \to \infty} d(x_n, x_m) = 0 \Rightarrow \lim_{n \to \infty} (x, x_n) \doteq 0$
where $x_n \in X (n = 1, 2, \ldots.)$ and x is unique .

Then X with the metric d, i.e. (X, d), is called a generalized complete metric space. Examples of such a space would be the extended real line and the extended complex plane with the usual metric.

<u>Theorem 3.2</u> Let f^p (p is any positive integer) be a mapping of the generalized complete metric space X into itself satisfying the following conditions:

(C1) There exists a contant $0 \leq q < 1$ such that $d(f^p x, f^p y) \leq qd(x, y)$ for all (x, y) such that $d(x, y) < \infty$.

(C2) For every sequence of successive approximations $x_n = f^p x_{n-1}$, $n = 1, 2, \ldots$ where x_o is an arbitrary element of X, there exists an index $N(x_o)$ such that $d(x_N, x_{N+\ell}) < \infty$ for all $\ell = 1, 2, \ldots$

(C3) If x and y are two fixed points of f^p i.e. $f^p(x) = x$ and $f^p y = y$, then $d(x, y) < \infty$.

Then f has a unique fixed point $x = \lim_{n \to \infty} x_n$

<u>Proof</u>: Let $x_o \in X$ and form the sequence $x_n = f^p x_{n-1}$ $(n = 1, 2, \ldots)$
By (C2) there exists an index $N(x_o)$ such that
$d(x_N, x_{N+\ell}) < \infty$, $\ell = 1, 2, \ldots$
Hence by (D3) we have $d(x_n, x_{n+\ell}) < \infty$ for $n \geq N$ and $\ell = 1, 2, \ldots$.
Then (C1) implies $d(x_{N+1}, x_{N+2}) \leq qd(x_N, f^p x_N)$ and generally
$d(x_n, x_{n+\ell}) \leq q^{n-N} d(x_N, f^p x_N)$ for $n \geq N$.
Since by (D3) we have $d(x_n, x_{n+\ell}) \leq \sum_{i=1}^{\ell} d(x_{n+i}, x_{n+i-1})$,
we obtain by the above inequality
$d(x_n, x_{n+\ell}) \leq \{q^{n-N}(1 - q^{\ell}) | (1 - q)\}\, d(x_N, f^p x_N)$, $n \geq N$ and $\ell = 1, 2, \ldots$
Hence x_n is a d-Cauchy sequence. From (D4) it follows then that there exists an element $x \in X$ such that $\lim_{n \to \infty} d(x_n, x) = 0$. For this element x we conclude by (D3) that

$$d(x, f^p x) \leq d(f^p x, x_n) + d(x_n, x) \leq qd(x, x_{n-1}) + d(x_n, x) \quad \text{for } n \geq N .$$

Hence $d(x, f^p x) = 0$ and by (D1) $f^p x = x$.

So x is a fixed point of f^p. Assume now that $f^p y = y$ with $x \neq y$.

Then by (D3) $d(x, y) < \infty$ and by (C1) we obtain

$0 \leq d(x, y) = d(f^p x, f^p y) \leq q d(x, y)$.

This implies that $d(x, y) = 0$ and hence $x = y$.

Therefore x is a unique fixed point of f^p, and hence a unique fixed point of f.

Remark 3.3 If p = 1, we get a well-known theorem of Luxemburg [14]. We generalize a theorem due to Luxemburg [14] for a family $\{f_i\}$ of contractions in the following way:

Theorem 3.4 Suppose $\{f_i\}$ $(i = 1, 2, \ldots)$ is a sequence of self mappings of a generalized complete metric space X satisfying the following conditions:

(1) There exists a constant $0 < \rho < 1$ such that $d(f_i x, f_i y) \leq \rho d(x, y)$ for all (x, y) with $d(x, y) < \infty$.

(2) $f_i f_j = f_j f_i$ i.e. any two mappings commute.

(3) For every sequence $x_n = f_i x_{n-1}$, $n = 1, 2, \ldots$ where x_o is an arbitrary element of X, there exists an index $N(x_o)$ such that $d(x_N, x_{N+\ell}) < \infty$ for all $\ell = 1, 2, \ldots$, and $i = 1, 2, \ldots$.

(4) If x, y are two fixed points of the mapping f_i then $d(x, y) < \infty$ $(i = 1, 2, \ldots)$.

Then the sequence $\{f_i\}$ has a common unique fixed point.

Proof: Conditions (1), (3) and (4) ensure that each mapping f_i $(i = 1, 2, \ldots)$ will have a unique fixed point.

Assume now $f_i(x_i) = x_i$

$$f_j(x_j) = x_j \qquad x_i \neq x_j \quad .$$

Since the family $\{f_i\}$ commutes we now have $f_i(f_j(x_i)) = f_j f_i(x_i)$

$$= f_j(x_i) .$$

Hence $f_j(x_i)$ is a fixed point of f_i.

But f_i has a unique fixed point x_i.

Therefore $f_j(x_i) = x_i$ and x_i is a fixed point of f_j.

But f_j has a unique fixed point x_j.

Hence $x_i = x_j$.

Thus $x_1 = x_2 = \ldots\ldots = x$ is a common unique fixed point for all f_i.

We prove the following theorem which is more general in nature than the previous result.

<u>Theorem 3.5</u> Let (X, d) be a generalized complete metric space and f_i $(i = 1, 2, \ldots)$ a family of self mappings of X closed under composition such that

(1) There exist a constant $0 < \rho < 1$ and an integer $m \geq 1$ such that if $(x, y) \in X$ and $d(x, y) < \infty$ then

$d(f_i^m(x) , f_i^m(y)) \leq \rho \cdot d(x, y)$, $i = 1, 2, \ldots$.

(2) $f_i \cdot f_j = f_j \cdot f_i$, $(i, j = 1, 2, \ldots)$.

(3) For every sequence of successive approximations

$x_n = f_i^m x_{n-1}$, $n = 1, 2, \ldots$ where x_o is an arbitrary element of X, there exists an index $N(x_o)$ such that

$d(x_N, x_{N+\ell}) < \infty$.

(4) If x, y are two fixed points of f_i^m $(i = 1, 2, \ldots)$,

i.e. $f_i^m(x) = x$, $f_i^m(y) = y \Rightarrow d(x, y) < \infty$.

Then there exists a unique $y \in X$ such that

$f_i(y) = y$ for all $i = 1, 2, \ldots$.

<u>Proof:</u> Following the procedure of Luxemburg and using conditions (1) and (3), it can be shown that each f_i^m has a fixed point U_i $(i = 1, 2, \ldots)$.
Assume now that $f_i^m(U_i) = U_i$

$$f_i^m(U_j) = U_j \qquad U_i \neq U_j$$

Since by (4) $d(U_i\ , U_j) < \infty$ $(i = 1, 2, \ldots)$, we have

$0 \leq d(U_i\ , U_j) = d(f_i^m U_i\ , f_i^m U_j) \leq \rho d(U_i, U_j)$.

Hence $d(U_i\ , U_j) = 0$, which contradicts $U_i \neq U_j$

Thus the family $\{f_i^m\}$ have unique fixed points $\{U_i\}$, $(i = 1, 2, \ldots)$.
Since each unique fixed point of f_i^m is also a unique fixed point of f_i, it follows that the family $\{f_i\}$ have unique fixed points $\{U_i\}$.
Using (2) we can show as in the previous theorem that $U_1 = U_2 = \ldots = y$ is a common unique fixed point for all $\{f_i\}$.

<u>Remark 3.6</u> If the family $\{f_i\}$ reduces to f with $m = 1$ we get a theorem of Luxemburg.

The following is a "localized" version of theorem 3.2:

<u>Theorem 3.7</u> Let f^p be a mapping of the generalized complete metric space X into itself satisfying:

(C1) There exists a constant $C > 0$ such that for all (x, y) with $d(x, y) \leq C$ we have $d(f^p x, f^p y) \leq \rho\, d(x, y)$ where $0 < \rho < 1$.

(C2) For every sequence $x_n = f^p x_{n-1}, n = 1, 2, \ldots$, where x_o is an arbitrary element of X, there exists an index $N(x_o)$ such that $d(x_n, x_{n+\ell}) \leq C$ for all $n \geq N$ and $\ell = 1, 2, \ldots$.

(C3) If $f^p x = x$, $f^p y = y$ then $d(x, y) \leq C$.

Then f has a unique fixed point $x = \lim_{n \to \infty} x_n$.

Remark 3.8 If p = 1 we get a "local" theorem of Luxemburg [15]. Luxemburg's "local" theorem has been generalized by A.F. Monna [18] to a suitable family of operators as follows:

Theorem 3.9 Suppose $\{f_i\}$, $(i = 1, 2, \ldots)$ is a sequence of mappings of a complete generalized metric space X into itself satisfying the following conditions:

(1) There exist $C > 0$ and ρ $(0 < \rho < 1)$ such that

$d(f_i x , f_i y) \leq \rho d(x, y)$, $(i = 1, 2, \ldots)$, whenever

$d(x, y) \leq C$.

(2) $f_i f_j = f_j f_i$, $(i, j = 1, 2, \ldots)$ i.e. any two mappings commute.

(3) If $x_o \in X$ then a positive integer $N(x_o)$ exists such that

$n \geq N(x_o) \Rightarrow d(f_{n+K}(x_n), x_n) \leq C$, $(K = 1, 2, \ldots)$.

Then the sequence x_n where $x_n = f_i x_{n-1}$, $(n = 1, 2, \ldots)$ converges and if $y_o = \lim_{n \to \infty} x_n$, $\lim_{K \to \infty} f_K y_o = y_o$.

Edelstein [8] has shown that the assumptions of Monna's theorem imply a much stronger theorem in the following way:

Theorem 3.10 If all of the assumptions of Monna's theorem hold then a point y exists with the property that

$f_{n+K}(y) = y$. $(K = 1, 2, \ldots)$

Assumptions (1) and (2) of Monna's theorem can be relaxed to obtain the following "existence" theorem:

<u>Theorem 3.11</u> Let (X, d) be a generalized complete metric space and $\{f_i\}$ where $(i = 1, 2, \ldots)$ a family of self mappings of X closed under composition such that

(1) There exist constants $C > 0$, $0 \leq \rho < 1$ such that if $(x, y) \in X$ and $d(x, y) \leq C$ then $d(f_K(x), f_K(y)) \leq \rho d(x, y)$ for a fixed $K \geq 1$.

(2) $f_i f_K = f_K f_i$ where K is fixed and $i = 1, 2, \ldots$.

(3) For arbitrary $x_o \in X$ and every sequence of successive approximations $x_n = f_i x_{n-1}$ $(n = 1, 2, \ldots)$ there exists an index $N(x_o)$ such that $d(f_{n+j}\, x_n, x_n) \leq C$ for $n \geq N$, $j = 1, 2, \ldots$

Then there exists a $\eta \in X$ such that $f_{n+i}(\eta) = \eta$ for all $i = 1, 2, \ldots$.

<u>Proof</u>: Consider an arbitrary fixed point $x_o \in X$. Let $n \geq N(x_o)$ be fixed.

Suppose Y is the set of all $y \in X$ such that a sequence $C(y, x_n) \subset X$ exists with the property that

$C(y, x_n) = \{y = p_o, p_1, \ldots, p_\ell = x_n\}$

with $d(p_i, p_{i-1}) \leq C$, $(i = 1, 2, \ldots, \ell)$.

Now Y is a closed metric subspace of X and $f_{n+j}(Y) \subset Y$. Also Y is complete.

Thus Y and f_{n+j}, for a fixed j, satify the assumptions of Edelstein's proposition [9].

Hence f_{n+j} has a unique fixed point in Y, say η_j.

i.e. $f_{n+j}(\eta_j) = \eta_j$.

Using condition 2) with j fixed we have

$f_{n+j}\, f_{n+i}(\eta_j) = f_{n+i}\, f_{n+j}(\eta_j) = f_{n+i}(\eta_j)$.

Thus f_{n+j} has a fixed point $f_{n+i}(\eta_j)$.

But f_{n+j} has a unique fixed point η_j.

Hence $f_{n+i}(\eta_j) = \eta_j$ and η_j is a fixed point of f_{n+i}.

Continuing we can show that all of the family $\{f_{n+i}\}$ $(i = 1, 2, \ldots)$ share the common fixed point η_j.

With hypothesis 1) of Monna's theorem relaxed we have the following more general theorem:

<u>Theorem 3.12</u> Let (X, d) be a generalized complete metric space and $\{f_i\}$ where $i = 1, 2, \ldots$ a family of self mappings of X closed under composition such that

(1) There exist constants $C > 0$, $0 \leq \rho < 1$ and an integer $m \geq 1$ such that if $x, y \in X$ and $d(x, y) \leq C$ then
$d(f^m_K(x), f^m_K(y)) \leq \rho d(x, y)$, $K = 1, 2, \ldots$

(2) $f_i f_j = f_j f_i$, $i, j = 1, 2, \ldots$

(3) Let $x_o \in X$ be arbitrary and define
$x_n = f^m_i x_{n-1}$ $(n = 1, 2, \ldots)$. Then there exists $N(x_o)$ such that
$d(f^m_{n+K} x_n, x_n) \leq C$ for $n \geq N$, $K = 1, 2, \ldots$

Then there exists a unique $\eta \in X$ such that $f_{n+i}(\eta) = \eta$, $i = 1, 2, \ldots$

<u>Proof</u>: Let x_o be an arbitrary fixed point of X.

Let $n \geq N(x_o)$ be fixed.

Using the procedure of the previous theorem it can be shown that f^m_{n+K} is a "local" contraction on a closed metric subspace Y of X.

Using a proposition of Edelstein [9] it follows that f^m_{n+K} has a unique fixed point η_K in Y.

i.e. $f^m_{n+K}(\eta_K) = \eta_K$.

Using an earlier result (theorem 2.3.2) it follows that

$f_{n+K}(\eta_K) = \eta_K$.

Since the family $\{f_i\}$ commutes we have

$f_{n+K}f_{n+j}(\eta_K) = f_{n+j}f_{n+K}(\eta_K) = f_{n+j}(\eta_K)$.

By the uniqueness of η_K it follows that η_K is a fixed point of f_{n+j}.

Since f_{n+j} is a "local" contraction on Y then η_K is a unique fixed point of f_{n+j}.

Continuing we can show that $f_{n+i}(\eta_K) = \eta_K$ for all $i = 1, 2, \ldots$.

Moreover since each f_{n+i} is a "local" contraction, η_K is unique.

Remarks:3.13

(1) B. Margolis [16] has given a similiar result to the above using an extra condition. The above theorem shows that this condition is not essential.

(2) If $m = 1$ in the above theorem we get a stronger form of Monna's theorem with Edelstein's complètion of it.

(3) If $m = 1$ and the family $\{f_i\}$ reduces to a single element f we get a theorem of W.A. Luxemburg.

An alternative for any contraction mapping on a generalized complete metric space X is provided by the following theorem due to Diaz and Margolis [6].

Theorem 3.14 Let (X, d) be a generalized complete metric space. Suppose that the function $f : X \to X$ is a contraction in the sense that f satisfies the condition : (c). There exists a constant λ with $0 < \lambda < 1$ such that whenever $d(x, y) < \infty$ then $d(f(x), f(y)) \leq \lambda d(x, y)$. Let $x_0 \in X$ and consider the sequence of successive approximations with initial element x_0:

x_0 ; fx_0 , f^2x_0 , ... f^Kx_0 , ...

The following alternative holds: either

(A) for every integer $K = 0, 1, 2, \ldots$ one has

$d(f^K x_o , f^{K+1} x_o) = \infty$ or

(B) f has a fixed point in X .

Remarks 3.15

1) Banach's contraction mapping theorem is a special case of the previous theorem since if X is a complete metric space alternative (A) is automatically excluded and hence f has a fixed point in X. Uniqueness is evident.

2) Luxemburg's theorem [14] is also a special case of the above theorem. Observe that (C2) of [14] excludes alternative (A) and (C3) implies that the fixed point of f is unique.

We would like to give the following :

Theorem 3.16 Let X be a generalized complete metric space and $K : X \to X$ any mapping with a right inverse (i.e. $KK^{-1} = I$, the identity).- Suppose $g = K^{-1}fK$ is a contraction in the sense that it satisfies the following condition: (f is any self mapping of X).

C) There exists a constant λ with $0 < \lambda < 1$ such that whenever $d(x, y) < \infty$ then $d(g(x), g(y)) \leq \lambda d(x, y)$.

Let $x_o \in X$; then the following alternative holds : either

A) for every integer $\ell = 0, 1, 2, \ldots$ one has

$d(g^\ell x_o , g^{\ell+1} x_o) = \infty$ or

B) f has a fixed point in X,

<u>Proof:</u> Consider the sequence of numbers

$d(x, gx_o), d(gx_o, g^2x_o), \dots, d(g^\ell x_o, g^{\ell+1}x_o), \dots$

There are two mutually exclusive possibilities:

either A') for every integer $\ell = 0, 1, 2, \dots$

$$d(g^\ell x_o , g^{\ell+1}x_o) = \infty$$

which is precisely alternative A)

or B') for some integer $\ell = 0, 1, 2, \dots$

$$d(g^\ell x_o , g^{\ell+1}x_o) < \infty \quad .$$

It now remains to show that B') implies B).

Suppose B') holds:

Let $N = N(x_o)$ denote a particular integer of the set of integers $\ell = 0, 1, 2, \dots$ such that $d(g^\ell x_o , g^{\ell+1}x_o) < \infty$.

Then by C) since $d(g^N x_o , g^{N+1}x_o) < \infty$ it follows that

$$\begin{aligned} d(g^{N+1}x_o , g^{N+2}x_o) &= d(gg^N x_o , gg^{N+1}x_o) \\ &\leq \lambda d(g^N x_o , g^{N+1}x_o) \\ &< \infty \quad . \end{aligned}$$

By induction it can be shown that

$$\begin{aligned} d(g^{N+1}x_o , g^{N+\ell+1}x_o) &\leq \lambda^\ell d(g^N x_o , g^{N+1}x_o) \\ &< \infty \quad \text{for all } \ell = 0, 1, 2, \dots . \end{aligned}$$

In other words for any integer $n > N$:

$$\begin{aligned} d(g^N x_o , g^{N+1}x_o) &\leq \lambda^{n-N} d(g^N x_o , g^{N+1}x_o) \\ &< \infty \quad . \end{aligned}$$

Using the triangular inequality it follows that for $n > N$, $d(g^n x_o , g^{n+\ell} x_o) \leq \sum_{i=1}^{\ell} d(g^{n+i-1} x_o , g^{n+i} x_o)$

$$\leq \sum_{i=1}^{\ell} \lambda^{n+i-1-N} d(g^N x_o , g^{N+1} x_o)$$

$$\leq \lambda^{n-N} \frac{1 - \lambda^{\ell}}{1 - \lambda} d(g^N x_o , g^{N+1} x_o)$$

where $\ell = 1, 2, \ldots$.

Since $0 < \lambda < 1$, the sequence of successive approximations $x_o, gx_o, g^2x_o , \ldots , g^n x_o , \ldots$, is a d-Cauchy sequence, and since X is a generalized complete metric space, is d-convergent.

i.e. $\lim_{n \to \infty} d(g^n x_o , x) = 0$ for some $x \in X$.

We now show x is a fixed point of g.

Whenever $n > N$ it follows from (C) and the triangular inequality that $0 \leq d(x, gx) \leq d(x, g^n x_o) + d(g^n x_o , gx)$

$$\leq d(x, g^n x_o) + \lambda d(g^{n-1} x_o , x) .$$

Taking the limit as $n \to \infty$ it follows that $d(x, gx) = 0$.

Thus $g(x) = x$ and x is a fixed point of g .

Hence $K^{-1}fK(x) = x$,

and $KK^{-1}f(Kx) = Kx = f(Kx)$.

So Kx is a fixed point of f.

<u>Remark 3.17</u> A theorem of Chu and Diaz [4] is a special case of the above theorem since if X is a complete metric space alternative A) is excluded and hence $K^{-1}fK$ has a fixed point in X which is obviously unique. Hence f has a unique fixed point in X.

A "local" version of the above theorem is the following:

<u>Theorem 3.18</u> Suppose X is a complete generalized metric space with a "local" contraction $g = K^{-1}fK : X \to X$ (i.e.

$d(g(x), g(y)) \leq \lambda d(x, y)$ whenever $d(x, y) \leq C$ where C is a positive constant and $\lambda \in (0, 1)$). Then the following alternative holds:

either

A) $d(g^{\ell}x_o, g^{\ell+1}x_o) > C$ $\ell = 0, 1, 2, \ldots$

or B) f has a fixed point in X.

Proof: Assume possibility A) does not exist, follow the proof of the above theorem and we get the following inequality.

For any integer $n > N = N(x_o)$ such that $d(g^n x_o, g^{n+K}x_o) \leq C$,

$K = 1, 2, \ldots$.

$$d(g^n x_o, g^{n+K}x_o) \leq \sum_{i=1}^{K} d(g^{n+i-1}(x_o), g^{n+i}(x_o))$$

$$\leq \sum_{i=1}^{K} \lambda^{n+i-1-N} d(g^N x_o, g^{N+1}x_o)$$

$$\leq \lambda^{n-N} \frac{1-\lambda^K}{1-K} d(g^N x_o, g^{N+1}x_o)$$

Hence since $0 < \lambda < 1$, $x_n = g^n x_o$ is a d-Cauchy sequence in X

i.e. $\lim_{n \to \infty} d(g^n x_o, x) = 0$

Now $d(g^n x_o, x) \leq d(g^n x_o, g^{n+K}x_o) + d(g^{n+K}x_o, x)$

$\leq C + d(g^{n+K}x_o, x)$

It follows by letting $K \to \infty$ that $d(x, g^n x_o) \leq C$ for all $n > N$

Then $d(gx, x) \leq d(gx, g^n x_o) + d(g^n x_o, x)$

$\leq \lambda d(x, g^{n-1}x_o) + d(g^n x_o, x)$ for all $n \geq N$

Hence $d(gx, x) = 0$. i.e. $gx = x$

So x is a fixed point of $g = K^{-1}fK$

Hence x is a fixed point of f.

The only remaining possibility is A).

Remark 3.19 A "localized" version of a theorem of Chu and Diaz [4] is a special case of the above theorem.

CHAPTER IV

Sequences of Contraction Mappings

The main objective of this chapter is to study the convergence of a sequence of contractions in metric space. More specifically we investigate the following question:

"If a sequence of contractions $\{f_n\}$ with fixed points U_n $(n = 1, 2, \ldots)$ converges to a mapping f with fixed point U, under what conditions will the sequence U_n converge to U?"

F.F. Bonsall [3 p.6] has provided a partial answer in the following theorem:

Theorem 4.1 Let X be a complete metric space and f and f_n $(n = 1, 2, \ldots)$ contraction mappings of X into itself with the same Lipschitz constant $K < 1$ and with fixed points U and U_n respectively. Suppose that $\lim_{n \to \infty} f_n x = fx$ for all $x \in X$.
Then $\lim_{n \to \infty} U_n = U$.

This result has been improved by Russell and Singh [21] who showed in the following theorem that the condition that f be a contraction is not essential.

Theorem 4.2 Suppose f_n $(n = 1, 2, \ldots)$ is a family of contractions of a complete metric space X into itself with the same Lipschitz constant $K < 1$ and with fixed points U_n $(n = 1, 2, \ldots)$. Let $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$ where f is any self mapping of X.
Then f has a unique fixed point $U = \lim_{n \to \infty} U_n$.

Proof: Since $K < 1$ is the same Lipschitz constant for all n,

$|fx - fy| = \lim_{n \to \infty} |f_n x - f_n y| \leq K|x - y|$.

Thus f is a contraction mapping and hence has a unique fixed point U.

By the contraction mapping inequality (Theorem 2.2) we have for each n = 1, 2, ...

$d(U_n , f_n{}^m x_o) \leq \frac{K^m}{1-K} d(f_n x_o , x_o), \quad x_o \in X$.

Putting $m = 0$, $x_o = U$ we get

$d(U_n, U) \leq \frac{1}{1-K} d(f_n U, U) = \frac{1}{1-K} d(f_n U, f_U)$.

But $d(f_n U, fU) \to 0$ as $n \to \infty$.

Therefore $\lim_{n \to \infty} d(U_n, U) = 0$.

The above theorem can be generalized to a generalized complete metric space X making use of an inequality of Luxemburg [14].

<u>Theorem 4.3</u> Suppose $f_n (n = 1, 2, ...)$ is a family of self mappings of a generalized complete metric space X satisfying the following:

(1) $d(f_n(x), f_n(y)) \leq \rho d(x, y) (0 \leq \rho < 1)$ for all (x, y) in X with $d(x, y) < \infty$.

(2) The family of contractions f_n have fixed points U_n $(n = 1, 2, ...)$,

(3) $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$ where f is any self mapping of X.

(4) Let $x_o \in X$ be arbitrary and define $x_n = fx_{n-1}$. Then there exists an index $N(x_o)$ such that $d(fx_N, x_N) < \infty$.

Then f has fixed point $U = \lim_{n \to \infty} U_n$;

<u>Proof:</u> Since $\rho < 1$ is the same Lipschitz constant for all f_n,

$d(f x , fy) = \lim_{n \to \infty} d(f_n(x), f_n(y)) \leq \rho d(x, y)$

for all (x, y) in X with $d(x, y) < \infty$.

Hence f is a contraction on X.

Using property (4) we can show f has fixed point U say.

By the inequality of Luxemburg [14], we have for each r = 1, 2, ...

$$d(U_r, f_r^n x_o) \leq \frac{\rho^{n-N}}{1-\rho} d(f_r^N x_o, f_r^{N+1} x_o)$$

where $N(x_o)$ is an index, $n \geq N$.

Put $n = N = 0$ and $x_o = U$.

Then $d(U_r, U) \leq \frac{1}{1-\rho} d(U, f_r U) = \frac{1}{1-\rho} d(fU, f_r U)$.

But as $r \to \infty$ $d(fU, f_r U) \to 0$.

Hence $\lim_{r \to \infty} (U_r, U) = 0$

Remarks 4.4

(1) A local version of the above theorem can be similiarly proven using Luxemburg's "local" theorem [15].

(2) The condition that all of the mappings have the same Lipschitz constant is rather stringent. For example, one can easily construct a sequence of contraction mappings from the reals into the reals which converges uniformly to the zero mapping but whose Lipschitz constants tend to one.

The following theorem has been given by Russell and Singh [21]:

Theorem 4.5 Let $f_n (n = 1, 2, ...)$ be a family of contraction self mappings of a complete metric space X with Lipschitz constants K_n $(n = 1, 2, ...)$ such that $K_{n+1} \leq K_n$ for each n and with fixed points U_n $(n = 1, 2, ...)$. Suppose that $\lim_{n \to \infty} f_n x = fx \; \forall x \in X$ where f is any self mapping of X. Then f has a unique fixed point $U = \lim_{n \to \infty} U_n$.

Proof: Since $|f_n x - f_n y| \leq K_n |x - y|$,

$$\lim_{n \to \infty} |f_n x - f_n y| \leq \lim_{n \to \infty} K_n |x - y| .$$

Since $K_{n+1} \leq K_n$ for each n, it follows that $\lim_{n \to \infty} K_n < 1$. Hence

$\lim_{n \to \infty} f_n x = f x$ is a contraction mapping on X, and hence has unique fixed point U. Moreover K_1 will serve the purpose of a Lipschitz constant for f_n (n = 1, 2, ...).

By the contraction mapping inequality of Theorem 2.2.1,

$$d(U_n, f_n^m x_o) \leq \frac{K_1^m}{1 - K_1} d(f_n x_o, x_o) , \ x_o \in X \text{ for each } n = 1, 2, \ldots$$

Putting m = 0, x_o = U we get

$$d(U_n, U) \leq \frac{1}{1 - K_1} d(f_n U, U) = \frac{1}{1 - K_1} d(f_n U, fU) .$$

Since $d(f_n U, fU) \to 0$ as $n \to \infty$, we have

$$\lim_{n \to \infty} d(U_n, U) = 0.$$

Example 4.6 Let $f_n : [0, 2] \to [0, 2]$ such that

$$f_n x = 1 + \frac{nx}{n+1} , \ (n = 1, 2, \ldots) .$$

Now $\lim_{n \to \infty} f_n(x) = 1 = f(x) \ \forall \ x \in [0, 2]$.

The Lipschitz constants are $K_n = \frac{1}{n+1}$, n = 1, 2, ...

Thus $K_1 = \frac{1}{2}$ will make all mappings contractions .

The fixed points are $U_n = \frac{n+1}{n}$, (n = 1, 2, ...) .

Now $\lim_{n \to \infty} U_n = 1$ and 1 is the unique fixed point for f.

A result analog ous to the above theorem can be proved using a theorem of Chu and Diaz [4].

Theorem 4.7 Suppose the following conditions hold for a complete metric space X:

(1) $K : X \to X$ is any mapping such that $KK^{-1} = I$, the identity.

(2) $\{K^{-1} f_i K\}_{i=1}^{\infty}$ is a sequence of contractions on X with

Lipschitz constants K_i such that $K_i \to K < 1$ and with fixed points U_i $(i = 1, 2, \ldots)$.

(3) $\{f_i\}_{i=1}^{\infty}$ converges uniformly to any mapping $f : X \to X$.

Then f has a unique fixed point U.

Also $\{U_i\}_{i=1}^{\infty}$ converges uniformly to U.

<u>Proof</u>: Since $f_i \to f$, $K^{-1}f_iK \to K^{-1}fK$.

Now $d(K^{-1}f_iK(x), K^{-1}f_iK(y)) \leq K_i d(x, y)$ for any i and $x, y \in X$.

Hence $\lim_{i \to \infty} d(K^{-1}f_iK(x), K^{-1}f_iK(y)) \leq \lim_{i \to \infty} K_i d(x, y)$.

Thus $d(K^{-1}fK(x), K^{-1}fK(y)) \leq Kd(x, y)$, $K < 1$.

i.e. $K^{-1}fK$ is a contraction on X and hence has unique fixed point U.

It follows by a theorem of Chu and Diaz [4] that f has unique fixed point U.

Since $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f, then

$\{K^{-1}f_iK\}_{i=1}^{\infty}$ converges uniformly to $K^{-1}fK$.

Now given $\varepsilon > 0$ there exists a positive integer N such that $i \geq N$ implies that

$d(K^{-1}fK(U), K^{-1}f_iK(U)) < (1 - K_i) \cdot \varepsilon$.

Thus for $i \geq N$

$$\begin{aligned} d(U, U_i) &= d(K^{-1}fK(U), K^{-1}f_iK(U_i)) \\ &\leq d(K^{-1}fK(U), K^{-1}f_iK(U)) \\ &\quad + d(K^{-1}f_iK(U), K^{-1}f_iK(U_i)) \\ &< (1 - K_i) \cdot \varepsilon + K_i \cdot d(U, U_i). \end{aligned}$$

i.e. $(1 - K_i)\, d(U, U_i) < (1 - K_i) \cdot \varepsilon$.

Since $0 < K_i < 1$ we have $d(U, U_i) < \varepsilon$ for $i \geq N$.

i.e. $\lim_{n \to \infty} U_i = U$

We give the following two theorems for sequences of mappings in metric spaces:

<u>Theorem 4.8</u> Suppose X is a metric space and (1) f_n: $X \to X$ is a sequence of continuous mappings with fixed points u_n $(n = 1, 2, ---)$.

(2) $\{f_n\}$ converges uniformly to $f : X \to X$, where f is any self mapping of X.

(3) $\{u_n\}$ has a convergent subsequence $\{u_{n_i}\}$ whose limit is u.

Then u is a fixed point of f.

<u>Proof:</u> Since f_n converges uniformly to f, therefore, $d(f_{n_i} u_{n_i}, fu_{n_i})$ $< \frac{\varepsilon}{2}$, and $d(u_{n_i}, u) < \varepsilon/2$; $i \geq N$.

$$\begin{aligned} \text{Now } d(fu, u_{n_i}) &= d(fu, f_{n_i} u_{n_i}). \\ &\leq d(fu, f_{n_i} u) + d(f_{n_i}u, f_{n_i} u_{n_i}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $fu = \lim_{i \to \infty} u_{n_i}$. Hence $fu = u$.

<u>REMARK 4.9</u> In case the sequence $\{f_n\}$ is not continuous then the continuity of f will serve the purpose of the theorem.

Unlike the previous theorems, the above theorem does not assume that the family $\{f_n\}$ be contraction or that the space X be complete.

<u>Theorem 4.10</u>: Let (X,d) be a metric space and $f_n: X \to X$ a family of continuous mappings with fixed points u_n $(n = 1,2, ---)$.
Let $f: X \to X$ be a map with fixed point u. If the sequence $\{f_n\}$ converges pointwise to f and if $\{u_n\}$ has a convergent subsequence $\{u_{n_i}\}$ with limit u_o, then $u_o = u$.

<u>Proof</u>: Let $\varepsilon > o$. Then there exists a positive integer N such that $i \geq N$ implies $d(u_{n_i}, u_o) < \frac{\varepsilon}{2}$

and $d(f_{n_i}u_o, fu_o) < \frac{\varepsilon}{2}$.

Therefore,

$$d(u_{n_i}, fu_o) = d(f_{n_i}u_{n_i}, fu_o)$$
$$\leq d(f_{n_i}u_{n_i}, f_{n_i}u_o)$$
$$+ d(f_{n_i}u_o, fu_o)$$
$$< \varepsilon \text{ for all } i \geq N .$$

This proves that the sequence $\{u_{n_i}\}$ converges to $f(u_o)$. Hence $fu_o = u_o$, and it follows that $u_o = u$.

In the next two theorems we will consider the question posed at the beginning of this chapter as applied to a sequence of contractive mappings on compact and locally compact metric spaces.

<u>Theorem 4.11</u>:

Let (X,d) be a compact metric space and $f_i: X \to X$ a sequence of contractive self mappings of X. Suppose the sequence $\{f_i\}$ converges uniformly to f, a contraction self mapping of X. Then the sequence $\{f_i\}$ has unique fixed points $\{u_i\}$ $(i = 1,2,...)$ and the sequence u_i converges to u, a unique fixed point of f.

Proof: Since f_i contractive for each $i = 1,2,\ldots$ and X is compact, each f_i has unique fixed point u_i (Edelstein).

Also since f is a contraction and X is complete then f has unique fixed point u.

Let f have contraction constant $K < 1$. Since $\{f_i\}$ converges uniformly to f then for $\varepsilon > 0$ $\exists$ N such that $n > N$ implies

$d(f_i x, f_x) < \varepsilon.(1 - K) \; \forall \; x \in X$.

$$\text{Now } d(u_i,u) = d(f_i u_i, fu)$$

$$\leq d(f_i u_i, fu_i) + d(fu_i, fu)$$

$$< \varepsilon\,(1 - K) + Kd(u_i,u).$$

i.e. $(1 - K)\; d(u_i,u) < (1 - K) \cdot \varepsilon$, $K < 1$.

Hence $d(u_i,u) < \varepsilon$.

i.e. $\lim\limits_{i \to \infty} u_i = u$.

The following theorem for a locally compact space is due to Singh [24]:

Theorem 4.12

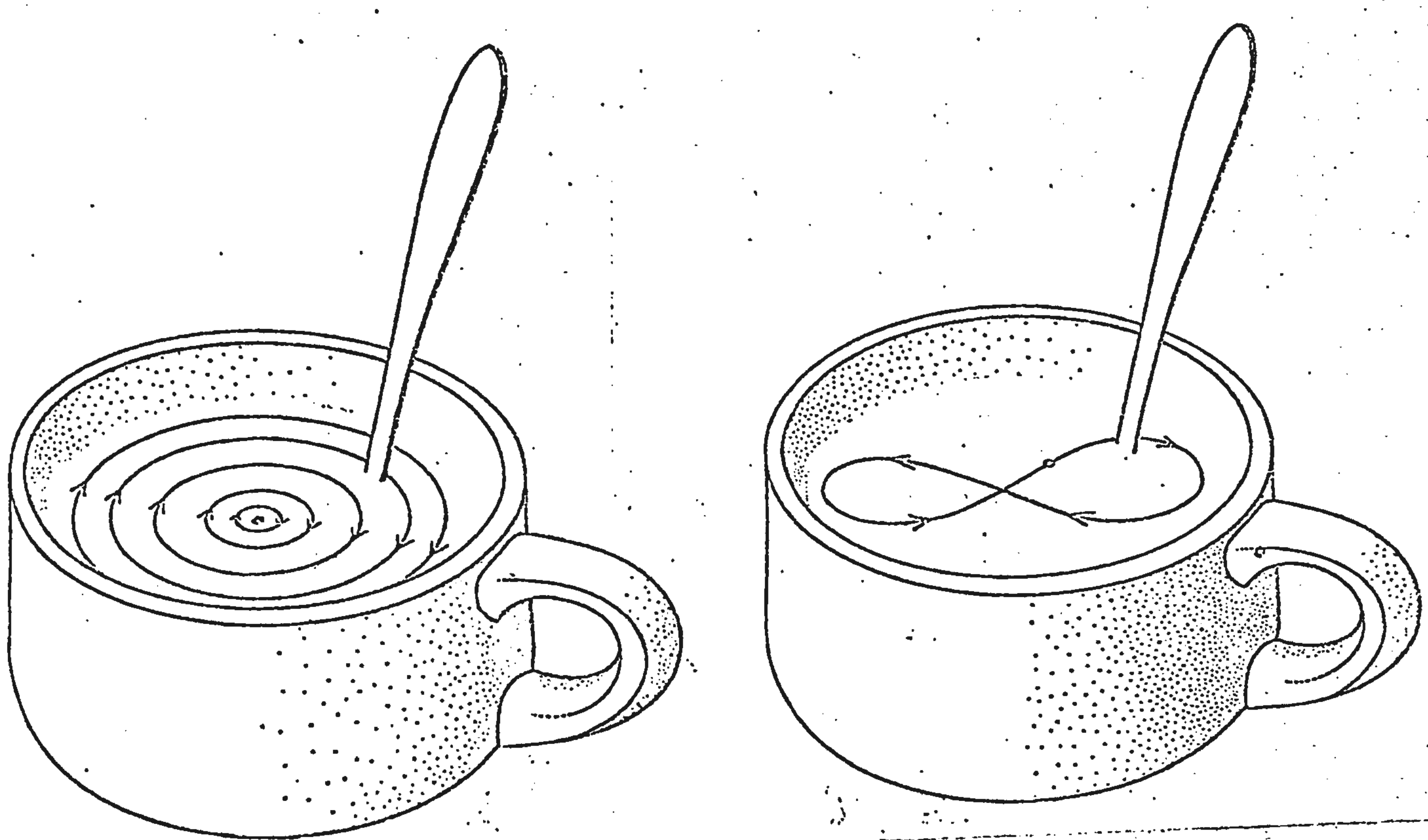
Let (X,d) be a locally compact metric space. Let $f_i : X \to X$. be a sequence of contractive mappings with fixed points u_i $(i = 1, 2, \ldots)$. Let $f : X \to X$ be a contraction self mapping of X with fixed point u. If the sequence $\{f_i\}$ converges pointwise to f, then $\{u_i\}$ converges pointwise to u.

Remark: One might conjecture that if the mapping f of the last two theorems is contractive the conclusions of the theorems are still valid.

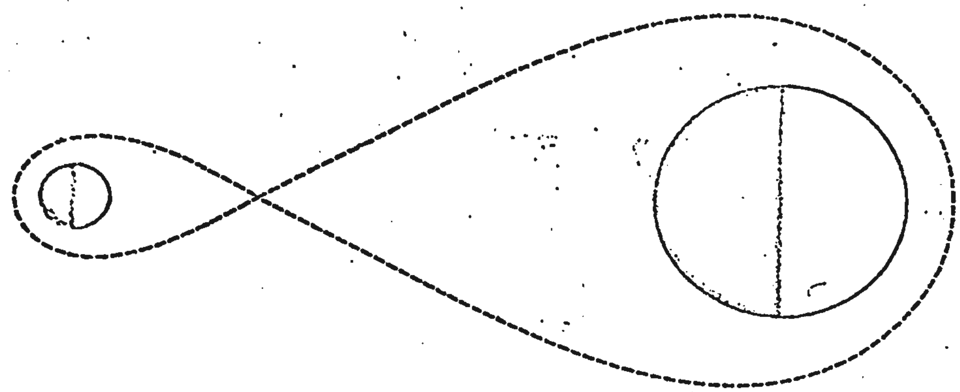
CHAPTER FIVE

Applications of the Contraction Mapping Principle.

The earlier pages of this work have been devoted to a discussion of some fixed point theorems in metric spaces. The following illustrations will serve to give some intuitive idea of how these fixed point theorems can be applied to various everyday situations. These illustrations are due to Shinbrot [22].



FIXED POINT THEOREM states that no matter how the surface of the coffee is continuously deformed, there will always be a point on the surface in the position it occupied at the start. This theorem does not stipulate which point is fixed at any instant in time.



FEASIBILITY OF AN ORBIT by which a satellite would revolve around earth and moon is the type of question to which mathematicians apply fixed-point theorems for infinite-dimensional surfaces. The element of time in any equation for the orbit makes the problem infinite-dimensional, rendering such simple theorems as Brouwer's theorem inapplicable.

The applications that follow, apply the contraction mapping principle to test existence and uniqueness of solutions to algebraic, differential and integral equations using the method of successive approximations.

5.1 Simple applications in one dimensional space

Suppose $y = f(x)$ is a given mapping of the closed interval $[a, b]$ into itself satisfying the Lipschitz condition that $|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|$ where $0 \leq \lambda < 1$. Now f is a contraction mapping and hence the sequence x_o, $x_1 = f(x_o)$, $x_2 = f(x_1)$ ....., $x_n = f(x_{n-1})$ are the successive approximations of the root of the equation $f(x) = x$ and will converge to the one and only one root.

Since $|f(x_2) - f(x_1)| \leq \lambda | x_2 - x_1 |$

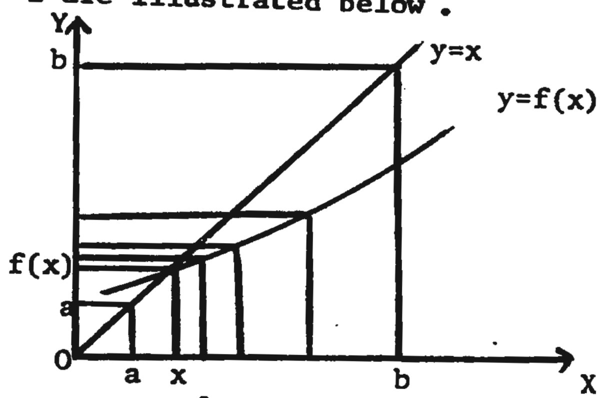
then it follows that $\dfrac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq \lambda$

i.e. $|f'(x)| \leq \lambda$ will guarantee f to be a contraction on [a,b].

There are thus two possibilities for a contraction:

$0 < f'(x) < 1$ and $-1 < f'(x) < 0$.

The successive approximations of the true root of $f(x) = x$ in case $0 < f'(x) < 1$ are illustrated below :



__Example 5.1.1__ Suppose $f : [-1, 1] \Rightarrow [-1, 1]$ is defined by

$$f(x) = \frac{x^2 + 1}{3} .$$

Now f is contraction if $|f'(x)| \leq \lambda < 1 \ \forall \ x \in [-1, 1]$.

Clearly $|f'(x)| = |\frac{2x}{3}|$.

Now $\max.|f'(x)| = \frac{2}{3}$

$\min.|f'(x)| = 0$.

Therefore $0 \leq |f(x)| \leq \frac{2}{3} < 1$.

Therefore $|f'(x)| \leq \lambda < 1$.

Therefore f is contraction .

Therefore $f(x) = \frac{x^2 + 1}{3}$ - (1)

$y = x$ - (2)

have unique solution in [-1, 1] .

Suppose $x_o = -1$ is the initial approximation of the true sulution.

Then $x_1 = f(x_o) = \frac{2}{3}$ is the second approximation.

$x_2 = f(x_1) = \frac{13}{27}$ is the third approximation.

Continuing the process these successive approximations will eventually approach $\frac{1}{3}$ which is the approximate root of the equation $f(x) = \frac{x^2 + 1}{3} = x$.

Suppose we require to find the roots of the polynomial $F(x) = 0$ where $F(a) < 0$, $F(b) > 0$ and $0 < \alpha_1 \leq F'(x) \leq \alpha_2$ on $[a, b]$. One widely used method for finding the roots of $F(x) = 0$ is to put $f(x) = x - \beta F(x)$ and we get the required result by solving the equation $f(x) = 0$. Since $f'(x) = 1 - \beta F'(x)$, $1 - \beta \alpha_1 \leq f'(x) \leq 1 - \beta \alpha_2$. We can apply the method of successive approximations again for the appropriate choice of β .

5.2 Solution of a System of Linear Algebraic Equations by the Method of Successive Approximations

Consider the mapping $f : R^n \to R^n$ given by the system of linear algebraic equations

$$y_i = \sum_{j=1}^{n} m_{ij} x_j + b_i \quad , \quad i = 1, 2, \ldots\ldots, n \ .$$

If f is a contraction mapping it follows that the equation $f(x) = x$ may be solved by the method of successive approximations and we have a unique solution.

Let $\quad x = (x_1, x_2, \ldots\ldots, x_n)$.

$\quad\quad y = (y_1, y_2, \ldots\ldots, y_n)$.

Set $\quad d(x, y) = \max_i |x_i - y_i|$.

We can show that R^n with the above metric is a complete metric space.

Now $d(y_1, y_2) = \max_i |y_{1_i} - y_{2_i}|$

$$\leq \max_i |\sum_j m_{ij} (x_{1_j} - x_{2_j}|$$

$$\leq \max_i \sum_j |m_{ij}| \, |(x_{1_j} - x_{2_j})|$$

$$\leq \max_i \sum_j |m_{ij}| \max_j | x_{1_j} - x_{2_j} |$$

$$= \max_i \sum_j |m_{ij}| \, d(x_1, x_2) .$$

The assumption $\sum_{j=1}^{n} |m_{ij}| \leq \alpha < 1$ where α is contraction mapping constant is sufficient to show that f has exactly one fixed point.

<u>Theorem 5.2.1</u> If $\sum_{j=1}^{n} |m_{ij}| \leq \alpha < 1$ holds for a matrix (a_{ij}) $\forall$ i then the system of equations

$$y_i - \sum_{j=1}^{n} m_{ij} x_i = b_i \quad i = 1, 2, \ldots\ldots, n$$ has exactly one

solution $x_o = (x_1^{o}, x_2^{o}, \ldots, x_n^{o})$ for arbitrary $b_1, b_2, \ldots, b_n$.

The solution can be found by successive approximations beginning with $x = (x_1, x_2, \ldots, x_n)$.

If $x_1 = f(x_o)$, $x_2 = f(x_1)$, ... , $x_K = f(x_{K-1})$,

and $x_K = (x_1^{(K)}, x_2^{(K)} \ldots\ x_n^{(K)})$,

then $x_o = \lim_{n \to \infty} x_n$.

i.e. $x_i = \lim_{n \to \infty} x_i^{(n)}$.

By the proof of the contraction mapping theorem we have

$$\max_i |x_i^{(K)} - x_i| = d(x_K, x_o) \leq \frac{\lambda^K}{1 - \lambda} d(x, f(x)) .$$

The condition of the theorem is now sufficient to establish convergence.

<u>Example 5.2.2</u> Consider the two linear equations

$$y = m_1 x + b_1$$

$$y = m_2 x + b_2$$

A unique solution of the system now exists if $|m_1| < 1$

and $|m_2| < 1$

In particular if $m_1 = \frac{1}{2}$, $b_1 = 2$, $m_2 = \frac{1}{3}$, $b_2 = 3$ we have a unique solution given by $x = 6$, $y = 5$.

5.3 Ordinary Differential Equations (Picard's Theorem)

As an example of the applications of the contraction mapping principle to ordinary differential equations we shall quote Picard's Theorem.

Theorem 5.3.1 Let $f'(x) = f(x, y)$ (i) be a given differential equation with initial condition $y(x_o) = y_o$ (ii). Suppose G is an open region of R containing the point (x_o, y_o) and satisfying the Lipschitz condition $|f(x_1, y_1) - f(x_1, y_2)| \leq M|y_1 - y_2|$. Then there exists a $t > 0$ and a function $g(x)$ continuous and differentiable in $[x_o - t, x_o + t]$ such that $y = g(x)$ is a unique solution of equation (i) with initial condition (ii).

Remark 5.3.2 The successive approximations of the above solution have the form

$$y_n(x) = y_o + \int_{x_o}^{x} f(t, y_{n-1}(t))\, dt.$$

Example 5.3.3 Consider the boundary value problem

$\frac{dy}{dx} = \frac{y}{x} = f(x, y)$ with boundary condition $x = 1$, when $y = 1$

(i.e. $x_o = y_o = 1$).

The successive approximations of its solutions are given by

$$y_1(x) = y_o + \int_{x_o}^{x} f(t, y_o)dt$$

$$= 1 + \int_{1}^{x} \frac{1}{t}\, dt$$

$$= 1 + \ln\, x$$

$$y_2(x) = y_o + \int_{x_o}^{x} f(t, y_1)dt$$

$$= 1 + \int_{1}^{x} \frac{1 + \ln\, t}{t}\, dt$$

$$= 1 + \ln\, x + \frac{(\ln\, x)^2}{2!}$$

.
.
.
.
.
.

$$y_n(x) = 1 + \sum_{K=1}^{n} \frac{(\ln\, x)^K}{K!} = e^{\ln\, x}$$

Hence the true solution is given by $y = e^{\ln\, x} = x$

Remark 5.3.4 The above method can be employed in a more general sense to justify the existance of solutions of a system of ordinary differential equations of the form

$f_i'(x) = f_i(x, y_1, \ldots, y_n)$, $(i = 1, 2, \ldots, n)$ with initial condition $y_i(x_o) = y_o$, where the functions $f_i(x, y_1, \ldots, y_n)$ are defined and continuous over the region G of the space R^{N+1} such that G contains the point $(x_o, y_{o_1}, \ldots, y_{o_n})$ and satisfies a Lipschitz condition

$$|f_i(x, y_1^{(1)}, \ldots, y_n^{(1)}) - f_i(x, y_1^{(2)}, \ldots, y_n^{(2)})|$$

$$\leq M \max\{|y_i^{(1)} - y_i^{(2)}| \,;\, 1 \leq i \leq n\}.$$

It may be proved that on some closed interval $|x - x_o| < d$ there exists a unique system of solutions $y_i = \Phi_i(x)$ to the above equations.

5.4 Integral Equations

An integral equation is one of the form

$$f(x) = \phi(x) + \int_a^b K(x, y)\, f(y)dy$$

where $\phi(x)$ and $K(x, y)$ (called the Kernel) are known and a, b are either constants or functions of x; the function $f(y)$ appearing under the integral sign is to be determined.

If a, b are constants the equation above is called a Fredholm integral equation. If a is constant and $b = x$, it is called a Volterra integral equation.

5.5 Fredholm Integral Equations

Consider the Fredholm non homogeneous linear integral equation of the second kind

$$f(x) = \lambda \int_a^b K(x, y)\, f(y)dy + \phi(x)$$

where λ is any arbitrary constant. The contraction mapping theorem can again be applied in the case of small values of λ to test existance and uniqueness of a solution.

Assume $K(x, y)$ and $\phi(x)$ are continuous functions for $x, y \in [a, b]$ and consequently $|K(x, y)| \leq M$.

Denote the space of all continuous functions on the closed interval $[a, b]$ by $C[a, b]$ with the metric $d(g_1, g_2) = \max |g_2(x) - g_2(x)|$.

$C[a, b]$ with the above metric, is a complete metric space.

Consider the self mapping $g = Tf$ of $C[a, b]$ defined by

$$g(x) = \lambda \int_a^b K(x, y)\, f(y)\, dy + \phi(x)\,.$$

Now $d(g_1, g_2) = \max |g_1(x) - g_2(x)|$

$$\leq |\lambda|\, M\, (b - a) \max|f_1 - f_2|\,.$$

Providing $|\lambda| < \frac{1}{M(b-a)}$, the mapping T is a contraction

Hence the Fredholm equation has a unique solution for every

$|\lambda| < \frac{1}{M(b-a)}$ whose successive approximations are given by

$$f_n(x) = \lambda \int_a^b K(x, y)\, f_{n-1}(y)dy + \phi(x) .$$

Example 5.5.1 Consider the values of λ for which the Fredholm equation $f(x) = x^2 + \lambda \int_0^1 \sin(x - y)\, f(y)dy$ has solution. According to the foregoing, above equation has solution for all λ such that

$|\lambda| < \frac{1}{M(b-a)}$.

Since $|K(x, y)| < M$ we have $|\sin(x - y)| \leq M$, so that

$0 \leq |K(x, y)| \leq 1$.

Hence $M = 1$.

Also $b = 1$, $a = 0$.

Hence $\frac{1}{M(b-a)} = 1$.

Therefore above equation has solution if $|\lambda| < 1$.

5.6 Volterra Type Integral Equation

We shall reiterate for clarity a more general form of the contraction mapping principle stated earlier which will be utilized in the work that follows:-

"If T^n is a contraction self mapping of a complete metric space X then the continuous mapping $T : X \to X$ has a unique fixed point".

It was earlier declared that the contraction mapping principle in the above form is very useful for certain applications. One such application is to test existence and uniqueness of a solution of the Volterra type integral equation.

$$f(x) = \lambda \int_a^x K(x, y)\, f(y)dy + \phi(x) .$$

Actually a unique solution exists to the above equation for all values of the parameter λ .

Consider the mapping $h(x) = \lambda \int_a^x K(x, y)f(y)dy + \phi(x) = Tf(x)$.

If f_1, f_2 are two continuous functions defined on [a, b], then

$$|h_1(x) - h_2(x)| = |Tf_1(x) - Tf_2(x)| = |\lambda \int_a^x K(x, y)f_1(y) - f_2(y)dy|.$$

If $d(f_1, f_2) = \max |f_1(x) - f_2(x)|$ defines the metric on C [a, b], then

C[a, b] is again a complete metric space and

$$|Tf_1(x) - Tf_2(x)| \leq |\lambda| \cdot \int_a^x |(K(x, y))(f_1(y) - f_2(y))|dy$$

$$\leq |\lambda| M \cdot m(x - a)$$

where $M = \max |K(x, y)|$

$m = \max |f_1 - f_2|$.

$$|T^2 f_1(x) - T^2 f_2(x)| \leq \lambda^2 Mm \frac{(x - a)^2}{2!},$$

and $|T^n f_1(x) - T^n f_2(x)| \leq \lambda^n \frac{Mm(x - a)^n}{n!} \leq \lambda^n \frac{Mm(b - a)^n}{n!}$.

For any arbitrary λ we can choose n in such a way that

$$\lambda^n \frac{(b - a)^n}{n!} < 1,$$

and hence the mapping T^n is a contraction. So Tf = h has a unique solution.

Hence the Volterra type equation above has unique solution for all λ...

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