SOME PROBLEMS RELATED TO RAMSAY'S THEOREM

by

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ABSTRACT

In 1930, F.P. Ramsay published a paper containing a combinatorial theorem which has since then become very well known and has given rise to an extensive literature. Most of the research which has arisen from Ramsay's Theorem, has dealt with the problem of finding upper and lower bounds for the so called Ramsay numbers. In addition, some exact values of these numbers have been determined and some applications of Ramsay's Theorem have been given.

In this thesis, we survey some of the research which has been done. In addition, some new results have been obtained. These results yield a better lower bound for certain classes of Ramsay numbers, than any of those that have been obtained up to the present time.
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CHAPTER I.

INTRODUCTION

A very significant theorem in combinatorial analysis appeared in 1930 in a paper [7] by the English logician F. P. Ramsay. Before stating Ramsay's Theorem, the following terminology used in formulating the theorem and throughout most of this thesis, is explained: By an s-set of a set the number of whose elements is s. By a t-subset of a set S is meant a subset of S with t elements. The set of all t-subsets of a set S shall be denoted by \( P_t(S) \).

Ramsay's Theorem in its most general form can now be formulated as follows:

**Theorem 1.1**

Let \( k_1, k_2, \ldots, k_n \) and t be positive integers such that each \( k_i \geq t \). Then there exists a least positive integer \( R = R(k_1, k_2, \ldots, k_n; t) \) such that if S is an s-set, \( s \geq R \), and if \( P_t(S) \) is partitioned into n classes \( C_1, C_2, \ldots, C_n \), then for some \( i, 1 \leq i \leq n \), there exists a \( k_i \)-subset \( K_i \subseteq S \) such that \( P_t(K_i) \subseteq C_i \).

The integers \( R = R(k_1, k_2, \ldots, k_n; t) \) are referred to as the Ramsay Numbers. If \( k_1 = k_2 = \ldots = k_n = k \) in Theorem 1.1, then we shall denote Ramsay numbers of this type of \( R_n(k,t) \).
Perhaps the most interesting special cases of Ramsay's Theorem are the cases \( t = 1 \) and \( t = 2 \). A moment's reflection shows that in the case \( t = 1 \), Ramsay's Theorem reduces to the well known pigeon hole principle. In fact \( R(k_1, k_2, \ldots, k_n; 1) = k_1 + k_2 + \ldots + k_n - n + 1 \).

In the case \( t = 2 \), Ramsay's Theorem can be formulated in the language of Graph Theory. If \( S \) is an \( s \)-set, then we can think of the elements of \( S \) as the vertices of a complete graph on \( s \) vertices, the 2-subsets of \( S \) as the edges of this graph and the partitioning of the 2-subsets of \( S \) into \( n \) classes as coloring the edges of the graph in \( n \) colors. The Theorem of Ramsay can thus be formulated as follows:

If \( G \) is a complete graph on \( s \) vertices, \( s \geq R(k_1, k_2, \ldots, k_n; 2) \), and if the edges of \( G \) are colored in any way in \( n \) colors \( c_1, c_2, \ldots, c_n \), then for some \( i, 1 \leq i \leq n \), there results a complete sub-graph of \( G \) with \( k_i \) vertices all of whose edges are colored \( c_i \).

In much of what follows, the language of graph theory shall be used. For notational convenience, a complete graph on \( k \) vertices or "\( k \)-gon" shall be denoted by the symbol \( <k> \). If a \( <k> \) is such that
all of its edges have the same color, then we shall refer to it as a monochromatic $<k>$ or M.C.$<k>$.

Since the appearance of Ramsay's Theorem in 1930, several well known mathematicians have worked on problems arising from it. Most of this research has dealt with finding bounds for the Ramsay Numbers $R(k_1, k_2, \ldots, k_n; t)$ or $R_n(k, t)$. Still, very little is known as to what is the order of magnitude of $R_n(k, t)$ for $t \geq 2$. Also, some exact values for the Ramsay numbers have been given for small values of $n$ and $k$, and $t = 2$. However, the values of $R_n(k, 2)$ are only known for $n \leq 4$ and small values of $k$. In addition, other papers have been devoted to the applications of Ramsay's Theorem.

In this thesis, we give a survey of the research which has gone into some of the above-mentioned problems. Also some new results are obtained.

In Chapter II, we shall develop a proof of the most general formulation of Ramsay's Theorem.

In Chapter III, we shall discuss some of the existing recurrence inequalities and lower bounds for the Ramsay Numbers. In addition, we shall prove a new result which yields for fixed $k$ and large $n$, a better lower bound for $R_n(k, 2)$ than any of those that have been obtained
up to the present time.

In Chapter IV, we shall discuss some of the known exact values of the Ramsay Numbers.

Finally, in Chapter V, some of the applications of Ramsey's Theorem are discussed.
CHAPTER II

PROOF OF RAMSAY'S THEOREM

An exposition of the proof of Ramsay's Theorem is given in the book by Ryser [8]. The proof there is essentially due to G. Szekeres [11]. However, in this Chapter, we shall approach the proof of Theorem 1.1 from a different point of view, showing that the main idea in Szekeres' argument is really contained in the evaluation of the simplest non-trivial Ramsay number $R(3,3;2)$ or $R_2(3,2)$. We evaluate $R(3,3;2)$ and then proceed to generalize the argument until we finally reach the proof of the most general form of Ramsay's Theorem.

Theorem 2.1: $R(3,3;2) = 6$.

Proof: Let $v$ be a vertex of a $<6>$, and let three of the five edges terminating at $v$ have color $c_1$. Consider the three edges joining their farther ends in pairs. If neither of these three edges is colored $c_1$, then all three must be colored $c_2$. In either case, there does exist a monochromatic triangle. Thus $R(3,3;2) \leq 6$.

To show that this result is best possible, we show that $R(3,3;2) > 5$. Color the edges of a $<6>$ in two colors $c_1$ and $c_2$ as follows: The interior diagonals of the
pentagon are colored $c_1$ and the remaining edges are colored $c_2$. Clearly, this coloring scheme does not force the appearance of a M.C. $<3>$. Thus $R(3,3;2) = 6$.

We now proceed to establish the existence of $R(k_1,k_2;2)$. If we assume that $R(k_1,k_2;2)$ exists, then it is clear from the symmetry of the problem that $R(k_1,k_2;2) = R(k_2,k_1;2)$. It is also clear that $R(2,k_2;2) = k_2$ for all $k_2 \geq 2$ and $R(k_1,2;2) = k_1$ for all $k_1 \geq 2$. The existence of $R(k_1,k_2;2)$ can now be proved by induction. We take as our induction hypothesis the existence of $R(k_1-1,k_2;2)$ and $R(l,k_2-1;2)$ for all $l$. In particular, the induction hypothesis insures the existence of $R(k_1-1,k_2;2)$ and $R(k_1,k_2-1;2)$. Let $s = R(k_1-1,k_2;2) + R(k_1,k_2-1;2)$ be a positive integer and color the edges of $<s>$ in two colors $c_1$ and $c_2$. Following the idea used in the proof of theorem 2.1, we select an arbitrary vertex $v$ of $<s>$ and let $n_1$ of the edges incident with $v$ be colored $c_1$ and $n_2$ colored $c_2$. ($n_1 + n_2 = s-1$).

Suppose $n_1 \geq R(k_1-1,k_2;2)$. Consider the edges joining in pairs the farther ends of the $n_1$ edges incident with $v$. Since $n_1 \geq R(k_1-1,k_2;2)$, the coloring of these edges in two colors $c_1$ and $c_2$ forces the appearance of either a M.C. $<k_1-1>$ of color $c_1$ or a M.C. $<k_2>$ of color $c_2$, and hence in $G$, either a M.C. $<k_1>$ of color $c_1$ or a M.C. $<k_2>$ of color $c_2$. Hence we may assume that $n_1 < R(k_1-1,k_2;2)$. 

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...
Then \( n_2 \geq R(k_1, k_2 - 1; 2) \), and the same argument applies. We have therefore proved the following theorem:

**Theorem 2.2**

\[ R(k_1, k_2; 2) \text{ exists and satisfies} \]

\[ (2.1) \quad R(k_1, k_2; 2) \leq R(k_1 - 1, k_2; 2) + R(k_1, k_2 - 1; 2) \]

**Corollary:** \( R(k_1, k_2; 2) \leq \binom{k_1 + k_2 - 2}{k_1 - 1} \)

**Proof:** Let \( T(k_1, k_2) = \binom{k_1 + k_2 - 2}{k_1 - 1} \). Clearly \( T(k_1, k_2) \) satisfies the same recurrence and same boundary conditions as \( R(k_1, k_2; 2) \).

If \( k_1 = k_2 \), then we have the special case

\[ R(k_1, k_1; 2) = R_2(k_2, 2) \leq \binom{2k_1 - 2}{k_1 - 1} \sim \frac{4}{\sqrt{k_1}} \]

The same type of argument can be used to establish the existence of \( R(k_1, k_2, \ldots, k_n; 2) \). We have

**Theorem 2.3**

\[ R(k_1, k_2, \ldots, k_n; 2) \text{ exists and satisfies} \]

\[ R(k_1, k_2, \ldots, k_n; 2) \leq \sum_{i=1}^{n} R(k_1, k_2, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n; 2) \]

Also since \( \binom{k_1 + k_2 + \ldots + k_n - n}{k_1 - 1, k_2 - 1, \ldots, k_n - 1} \) satisfies the same recurrence and boundary conditions as \( R(k_1, k_2, \ldots, k_n; 2) \) we have
It follows from (2.2) that

\[(2.3) \quad R_n(k,2) \leq \frac{(nk-n)!}{((k-1)!)^n}.\]

We mention in passing that in Ramsay's original paper [7], it was proved that \(R_n(k,2)\) exists and that

\[R_n(k,2) \leq f(n,k)\]

where \(f(1,k) = k\) by definition and

\[f(\ell,k) = (f(\ell-1,k))!\]

It is not difficult to check that this upper bound is much larger than (2.3). We also mention for the sake of completeness, that another proof of the existence of \(R_n(k,2)\) was given by T. Skolem in [10]. He obtained the following upper bound.

\[(2.4) \quad R_n(k,2) \leq \frac{kn-n+2}{n-1} \cdot \frac{n}{n-1}\]

It is not difficult to check that the upper bound given by (2.4) is roughly the same as that given by (2.3).

The generalization of the argument to the case \(t > 2\) is somewhat more involved. We consider first the problem of establishing the existence of \(R(k_1, k_2; t)\). By theorem 2.2, we know that \(R(\ell, k; 2)\) exists for all \(\ell, k > 2\). Also it is easy to see that \(R(\ell, t; t) = R(t, \ell; t) = \ell\) for all \(\ell > t\). We may therefore take as our induction hypothesis
the existance of

(1) \( R(k, t; t-1) \) for all \( k, t \geq t-1 \)
(2) \( R(k, k_2-1; t) \) for all \( k > t \)
(3) \( R(k_1-1, t; t) \) for all \( k > t \).

In particular, the induction hypothesis assures the existance of

\[ R = R(R(k_1-1, k_2; t), R(k_1, k_2-1; t); t-1) + 1 \]

Let \( S \) be an \( s \)-set, \( s > R \). Let \( P_t(S) = C_1 \cup C_2 \) be a partition of \( P_t(S) \) into two classes \( C_1 \) and \( C_2 \). If we can show that there exists a \( k_1 \)-subset \( K_1 \subseteq S \) such that \( P_t(K_1) \subseteq C_1 \) or a \( k_2 \)-subset \( K_2 \subseteq S \) such that \( P_t(K_2) \subseteq C_2 \), then the existence of \( R(k_1, k_2; t) \) will follow.

The argument used in the earlier theorems suggests that we select an arbitrary element \( a \in S \) and consider the partition of the \((t-1)\)-subsets of \( S^* = S - \{a\} \) which is induced by the above partition of \( P_t(S) \) in the following natural way:

Partition \( P_{t-1}(S^*) \) into two classes \( B_1 \) and \( B_2 \) by placing a member \( T \) of \( P_{t-1}(S^*) \) in \( B_1 \) if \( T \cup \{a\} \in C_1 \) and in \( B_2 \) if \( T \cup \{a\} \in C_2 \).

\( S^* \) is an \((s-1)\)-set. Since \( s-1 > R-1 \), it follows from the definition of \( R \) that either there is an

\[ l_1 = R(k_1-1, k_2; t) \]-subset \( L_1 \) of \( S^* \) such that \( P_{t-1}(L_1) \subseteq B_1 \) or...
there is an \( L_2 = R(k_1, k_2; t) \) subset of \( S^* \) such that 
\[ P_{t-1}(L_2) \subseteq B_2. \]

If the first alternative holds, then since \( L_1 \) has \( R(k_1-1, k_2; t) \) elements, either there is a \((k_1-1)\)-subset \( K_1^* \) such that \( P_t(K_1^*) \subseteq C_1 \) or there is a \( k_2 \)-subset \( K_2 \) such that \( P_t(K_2) \subseteq C_2 \), in which case we have finished. Hence we assume that there is a \((k_1-1)\)-subset \( K_1^* \) such that 
\[ P_t(K_1^*) \subseteq C_1. \]

Let \( K_1 = K_1^* \cap [a] \). Let \( T \) be any \( t \)-subset of \( K_1 \). If \( T \subseteq K_1^* \), then \( T \subseteq C_1 \). If \( T \not\subseteq K_1^* \), then 
\[ T = T^* \cup [a], \]
where \( T^* \) is a \((t-1)\)-subset of \( K_1^* \). Hence \( T^* \) is a \((t-1)\)-subset of \( L_1 \) and hence \( T^* \subseteq B_1 \). But by the manner in which \( B_1 \) was constructed, \( T^* \cup [a] \subseteq C_1 \), i.e., \( T \subseteq C_1 \). Hence \( P_t(K_1) \subseteq C_1 \).

If the first alternative does not hold, then the second must, and the same argument applies. We have therefore proved the following theorem:

**Theorem 2.4:** \( R(k_1, k_2; t) \) exists and satisfies the following recurrence inequality:

\[ R(k_1, k_2; t) \leq R(R(k_1-1, k_2; t), R(k_1, k_2-1; t); t-1)+1. \]

It is now easy to complete the proof of Theorem 1.1 by induction on \( n \). We have just established the existence of \( R(k_1, k_2; t) \). We take as our induction hypothesis the
existence of $R(k_1, k_2, \ldots, k_{n-1}; t)$, $n > 2$. Let $R = R(R(k_1, k_2, \ldots, k_{n-1}; t), k_n; t)$. Let $S$ be an $s$-set, $s \geq R$, and let $P_t(S) = C_1 \cup C_2 \cup \ldots \cup C_n$ be an arbitrary partition of $P_t(S)$ into $n$ classes. Then either there exists a $k_n$ subset $K_n$ of $S$ such that $P_t(K_n) \subseteq C_n$ in which case we have finished, or there exists an $R(k_1, k_2, \ldots, k_{n-1}; t)$-subset $L$ of $S$ such that $P_t(L) \subseteq C_1 \cup C_2 \cup \ldots \cup C_{n-1}$. The induction hypothesis then implies that for some $i$, $1 \leq i \leq n-1$, there is a $k_i$-subset $K_i$ of $L$ (and consequently of $S$) such that $P_t(K_i) \subseteq C_i$. This completes the proof of Theorem 1.1.

We have as a corollary

$$R(k_1, k_2, \ldots, k_n; t) \leq R(R(k_1, k_2, \ldots, k_{n-1}; t), k_n; t) + 1.$$  

We note in conclusion that if $k_1 = k_2 = \ldots = k_n = 3$ in Theorem 2.3, then this implies that $R_n(3, 2) \leq n R_{n-1}(3, 2)$ and this leads to $R_n(3, 2) \leq 3(n!)$ . A slight improvement was obtained by Greenwood and Gleason in [5]. They proved that

$$(2.5) \quad R_n(3, 2) \leq [n!e] + 1.$$  

Their argument is as follows:

Let $T_n$ be the sequence defined by $T_1 = 2$; $T_n = n T_{n-1} + 1$, for $n \geq 2$. By induction it is easy to prove that

$$T_n = n! \sum_{k=0}^{n} \frac{1}{k!}.$$
From this it follows that

\[ T_n = [n!e] \]

We now want to show that if the edges of a

\( < T_n + 1 > \) are colored in any way in \( n \) colors \( c_1, c_2, \ldots, c_n \),

then there results a monochromatic triangle. This is clearly true when \( n = 1 \). We assume that it holds for \( n - 1 \). Let \( v \) be a vertex of the \( < T_n + 1 > \). There are \( T_n = n T_{n-1} + 1 \) edges incident with \( v \) and there are \( \ell \geq T_{n-1} + 1 \) of these edges with the same color \( c_n \) (say). Consider the \( \binom{\ell}{2} \) edges joining in pairs the farther ends of the \( \ell \) edges incident with \( v \) and colored \( c_n \). If one of these edges is colored \( c_n \), we have finished. If none of these edges are colored \( c_n \), the induction hypothesis implies that there is a monochromatic triangle colored one of \( c_1, c_2, \ldots, c_{n-1} \). This proves (2.5).
Chapter III

Lower Bounds for the Ramsey Numbers

As we mentioned in the introduction, very little is known as to what is the order of magnitude of \( R_n(k,t) \) and all existing upper and lower bounds are quite far apart. In this chapter we discuss the problem of finding lower bounds for \( R_n(k,t) \). Some of the known results are presented and some new results are obtained.

§3.1 Recurrence Inequalities

In this section we prove some recurrence inequalities and from these derive lower bounds for \( R_n(k,t) \). In [1], it is proved that

\[
R_{n+m}(k,2) > (R_n(k,2)-1)(R_m(k,2)-1) + 1.
\]

**Theorem 3.1.1:** For all positive integers \( n \) and \( m \) and fixed \( k \)

\[
(3.1.1)
\]

**Proof:** For notational convenience, let \( p = R_{n+m}(k,2)-1 \), \( q = R_n(k,2)-1 \) and \( r = R_m(k,2)-1 \). We now have to show that \( p > qr \). Let \( P_1, P_2, \ldots, P_q \) be the vertices of a \( \langle q \rangle \). Color the edges of the \( \langle q \rangle \) in \( n \) colors \( c_1, c_2, \ldots, c_n \) in such a way that there does not result a M.C. \( \langle k \rangle \). This can be done by the definition of \( q \). Let \( \langle r \rangle, 1 \leq i \leq q \) have vertices \( P_{i1}, P_{i2}, \ldots, P_{ir} \). Color the edges of each \( \langle r \rangle \) in \( m \).
colors $c_{n+1}, c_{n+2}, \ldots, c_{n+m}$ without forcing the appearance of a M.C. $<k>$. Let $<qr>$ be the graph with vertices $P_{ij}$, $1 \leq i \leq q$, $1 \leq j \leq r$. Let $E = (P_{st}, P_{uv})$ be an edge of $<qr>$. If $s \neq u$, color $E$ the same color as edge $(P_s, P_u)$ in $<q>$. If $s = u$, color $E$ the same as it is colored in $<r>$. Suppose $P_{i_1 j_1}, P_{i_2 j_2}, \ldots, P_{i_k j_k}$ are the vertices of a M.C. $<k>$.

**Case I:** If $i_1 = i_2 = \ldots = i_k$, then $P_{i_1 j_1}, P_{i_2 j_2}, \ldots, P_{i_k j_k}$ are the vertices of a M.C. $<k>$ in $<r>_{i_1}$, which is a contradiction.

**Case II:** If $i_1, i_2, \ldots, i_k$ are all different, then the edges of the $<k>$ are colored the same as those of the $<k>$ in $<q>$ whose vertices are $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$. This again is a contradiction.

**Case III:** If $i_s = i_t \neq i_u$. Then the edge $(P_{i_s j_s}, P_{i_t j_t})$ is colored one of $c_{n+1}, c_{n+2}, \ldots, c_{n+m}$ while the edge $(P_{i_s j_s}, P_{i_t j_t})$ is colored one of $c_1, c_2, \ldots, c_n$. This is also impossible. Hence $p > qr$.

It follows easily from (3.1.1) and the fact that $R_1(k,2) = k$, that

\[(3.1.2) \quad R_n(k,2) \geq (k-1)^n.\]

This lower bound for $R_n(k,2)$ is substantially smaller than the upper bound given by (2.4). We shall obtain some better
lower bounds in sections 2 and 3 of this chapter. However, we can use theorem 3.1.1 to gain a little more insight into the behaviour of $R_n(k,2)$. We prove that for every fixed $k$,

$$\lim_{n \to \infty} R_n(k,2)^{1/n}$$

exists.

To prove this, let $h(n) = R_n(k,2) - 1$. Then we have by theorem 3.1.1,

$$h(n+m) > h(n)h(m).$$

This implies

$$(3.1.3) \quad h(ab) \geq h(b)^a.$$

Let $\alpha = \liminf_{n \to \infty} h(n)^{1/n} \leq \limsup_{n \to \infty} h(n)^{1/n} = \beta$. Suppose first $\beta < \infty$. Let $\varepsilon > 0$ be given and let $b$ be the least integer for which

$$(3.1.4) \quad h(b)^{1/b} > \beta - \varepsilon.$$

If $n = ab$, we have

$$h(n)^{1/n} = h(ab)^{1/ab} \geq (h(b)^a)^{1/ab} = h(b)^{1/b} > \beta - \varepsilon,$$

where we used (3.1.3) and (3.1.4). Let $n = ab + r$ where $1 \leq r \leq b-1$. Then

$$h(n) = h(ab + r) \geq h(ab)h(r) \geq h(ab).$$
and hence

\[ h(n)^{1/n} \geq h(ab)^{1/ab+r} = h(ab)^{\frac{1}{ab}} \left( 1 + \frac{1}{ab} \right) > (\beta - \varepsilon)^{\frac{1}{1+r/ab}}. \]

Hence \( \alpha > \beta - \varepsilon \). Since \( \varepsilon \) is arbitrary, \( \alpha = \beta \). The case \( \beta = \infty \) can be disposed of in the same way. Let \( N \) be a positive number and let \( b \) be the least integer such that \( h(b)^{1/b} > N \). The argument used above then shows that \( \alpha > N \) and hence \( \alpha = \infty \).

We cannot decide whether the above limit is finite or infinite.

One can now ask whether \( R_n(k, t) \) satisfies the same recurrence inequality as \( R_n(k, 2) \). We cannot decide this, but we prove:

**Theorem 3.1.2**: For all positive integers \( n \) and \( m \)

\[ R_{n+m+1}(k, t) \geq (R_n(k, t)-1)(R_m(k, t)-1) + 1 \]

provided \( k \geq (t - 1)^2 + 1 \).

**Proof**: For notational convenience, set \( R_n(k, t)-1 = h(n) \). We now have to prove that \( h(n+m+1) \geq h(n)h(m) \). Let \( S \) be an \( h(n) \)-set with elements \( a_1, a_2, \ldots, a_{h(n)} \). Partition \( P_t(S) \) into \( n \) classes \( C_1, C_2, \ldots, C_n \) in such a way that if \( K \) is a \( k \)-subset of \( S \), then not all \( t \)-subsets of \( K \) belong to the same class. This is possible by the
definition of $h(n)$. For $j = 1, 2, \ldots, h(n)$, let $S_j$ be an $h(m)$-set with elements $a_{j_1}, a_{j_2}, \ldots, a_{j_{h(m)}}$.

Partition each $P_t(S_j)$ into $m$ classes $C_{j_1}, C_{j_2}, \ldots, C_{j_m}$ in such a way that if $K$ is a $k$-subset of $S_j$, then not all $t$-subsets of $K$ belong to the same class. Let

$$R = \bigcup_{j=1}^{h(n)} S_j.$$ Then $R$ has $h(n)h(m)$ elements. Partition $P_t(R)$ into $n+m+1$ classes $B_1, B_2, \ldots, B_n, B_{n+1}, \ldots, B_{n+m}, B_{n+m+1}$ as follows: Let $T$ be a member of $P_t(R)$.

If $T \subseteq S_j$, put $T$ in $B_{n+1}$ if $T \subseteq C_{j_k}$. If $T$ is distributed over exactly $t$ of the $S_j$, say $S_{j_1}, S_{j_2}, \ldots, S_{j_t}$, then $T = \{a_{j_{11}}, a_{j_{12}}, \ldots, a_{j_{1t}}\}$ say. Put $T$ in $B_{n+1}$ if the set $T' = \{a_{j_{21}}, a_{j_{22}}, \ldots, a_{j_{2t}}\} \subseteq C_{j_{2}}$. If $T$ is distributed over $r(l < r < t-1)$ of the $S_j$, put $T$ in $B_{n+m+1}$. The proof will be complete if we show that there does not exist a $k$-subset $K$ of $R$ such that

$$P_t(K) \subseteq B_i \text{ for some } i, 1 \leq i \leq n+m+1.$$

**Case I:** $K \subseteq S_j$. It is then obvious that the desired result holds.

**Case II:** If $K$ is distributed over exactly $t$ of the $S_j$, then there is at least one $t$-subset $T$ of $K$ distributed over exactly $t$ of the $S_j$. Then $T \subseteq B_{j_k}$ for some $k \leq n$. Since $k > t$, there is a $t$-subset $T_1$ of $K$ which has at least two elements in one of the $S_j$ and at least one element in some $S_t$, $t \neq j$. Hence $T_1 \subseteq B_{n+m+1}$. 
Case III: If $K$ is distributed over $r (l < r < t - 1)$ of the $S_j$, then since $k > (t-1)^2 + 1$, there must be at least $k/r > (t-1)^2/r + 1/r > (t-1) + 1/r$, and hence at least $t$ elements of $K$ in one of the $S_j$. Thus there is a $t$-subset $T$ of $K$ such that $T \subseteq S_j$. Then $T \in B_{n+\ell}$, for some $\ell$, $1 \leq \ell \leq m$. However, there must be another $t$-subset of $K$ which belongs to $B_{n+m+1}$. This completes the proof of theorem 3.1.2.

In [1], it is proved that

$$(3.1.5) \quad R_n(k\ell - k-\ell+2,2) > (R_n(k,2)-1)(R_n(\ell,2)-1) + 1.$$ 

One can now ask whether this result can be generalized to the case $t > 2$. We have not been able to do this. However, we prove:

Theorem 3.1.3:

$$(3.1.6) \quad R_{n+1}(k\ell - k-\ell+2,t) > (R_n(k,t)-1)(R_n(\ell,t)-1) + 1.$$ 

Proof: For notational convenience, put $h_n(k) = R_n(k,t)-1$. We then have to prove

$$(3.1.7) \quad h_{n+1}(k\ell - k-\ell+2) > h_n(k)h_n(\ell).$$

Let $S$ be an $h_n(\ell)$-set with elements $a_1, a_2, \ldots, a_{h_n(\ell)}$. Partition $P_t(S)$ into classes $C_1, C_2, \ldots, C_n$ so that if $L$ is an $\ell$-subset of $S$, then not all
t-subsets of $L$ belong to the same class. For $j = 1, 2, \ldots, h_n(k)$, let $S_j$ be an $h_n(k)$-set with elements $a_{j_1}, a_{j_2}, \ldots, a_{j_{h_n(k)}}$. Partition each $P_t(S_j)$ into $n$ classes $C_{1_j}, C_{2_j}, \ldots, C_{n_j}$, so that if $K_j$ is a $k$-subset of $S_j$, then not all $t$-subsets of $K_j$ belong to the same class. Let $W = \bigcup_{j=1}^{h_n(k)} S_j$. Note that $W$ is an $h_n(\ell)h_n(k)$-set. Partition $P_t(W)$ into $n+1$ classes $B_1, B_2, \ldots, B_{n+1}$ as follows: Let $T \in P_t(W)$. Firstly, if $T \subseteq S_j$ for some $j$, then $T \subseteq C_{s_j}$ for some $s$, $1 \leq s \leq n$. Put $T$ in $B_s$. Secondly, if $T$ is distributed over exactly $t$ of the $S_j$, say $S_{j_1}', S_{j_2}', \ldots, S_{j_t}'$, then $T = \{a_{j_1}', \ldots, a_{j_t}'\}$ say. If the set $T' = \{a_{j_1}, a_{j_2}, \ldots, a_{j_t}\}$ belongs to $C_s$, put $T$ in $B_s$. Finally, if $T$ is distributed over $r(l < r < t)$ of the $S_j$, put $T$ in $B_{n+1}$.

The proof of (3.1.7) will be complete if we show that if $M$ is a subset of $W$ with $k\ell - k - l + 2$ elements, then $P_t(M) \subseteq B_i$ is false for all $i$, $1 \leq i \leq n+1$.

**Case I:** $M$ is distributed over $r \geq \ell$ of the $S_j$. Suppose $P_t(M) \subseteq B_s$ for some $s$, $1 \leq s \leq n$. Then $M$ must contain an $\ell$-subset $L$ which is distributed over exactly $\ell$ of the $S_j$, say $S_{j_1}', S_{j_2}', \ldots, S_{j_\ell}'$. Let $L = \{a_{j_1}', a_{j_2}', \ldots, a_{j_\ell}'\}$. Then the set $L' = \{a_{j_1}, a_{j_2}, \ldots, a_{j_\ell}\}$ is a subset of $S$ and the condition $P_t(L) \subseteq B_s$ implies $P_t(L') \subseteq C_s$. 

This is impossible. It is also clear that $P_{t}(M) \subseteq B_{n+1}$ cannot occur since there is at least one $t$-subset of $M$ which is distributed over at least $t$ of the $S_j$.

**Case II:** $M$ is distributed over $r \leq k-1$ of the $S_j$, say $S_{j_1}, S_{j_2}, \ldots, S_{j_r}$. Then there is a subset $K$ of $M$ with at least $k$ elements such that $K \subseteq S_{j_s}$ for some $s$, $1 \leq s \leq r$, since otherwise the number of elements of $M$ would not exceed $r(k-1) \leq (k-1)(k-1) < kk-2$. Suppose $P_{t}(M) \subseteq B_i$, for some $i$, $1 \leq i \leq n$. Then $P_{t}(K) \subseteq B_i$. But this implies $P_{t}(K) \subseteq C_{i_{j_s}}$. This is a contradiction. Also it cannot occur that $P_{t}(M) \subseteq B_{n+1}$, since this would imply $P_{t}(K) \subseteq B_{n+1}$. This is obviously impossible since it indicates that every $t$-subset of $K$ is distributed over at least two of the $S_j$, contradicting the fact that $K \subseteq S_{j_s}$.

This completes the proof of (3.1.7).

§3.2. Probabilistic Arguments

A lower bound for $R_n(k,t)$ was obtained by Erdős in [2] who proved by a probabilistic argument that:

**Theorem 3.2.1:**

\[
R_n(k,t) \geq \binom{k}{t} - 1
\]

**Proof:** Let $S$ be an $s$-set and let $f(s)$ be the number of ways of partitioning $P_t(S)$ into $n$ classes such that for
each such partitioning, there exists a $k$-subset $K \subseteq S$ such that $P_t(K)$ is contained in one of the $n$ classes. The total number of ways of partitioning $P_t(S)$ into $n$ classes is $n^{(t)}$. Hence the probability that for a given partitioning there exists a $k$-subset $K \subseteq S$ such that $P_t(K)$ is contained in one of the $n$ classes is $f(s)/n^{(t)}$. We need therefore

\[(3.2.2) \quad f(s) < \binom{n}{t}^{(s)}\]

Since the number of $k$-subsets of an $s$-set is $\binom{s}{k}$, we have

\[f(s) \leq n^{(s)} k^{(s) - (k)} n^{(t)}\]

where $n^{(s)} k^{(s) - (k)}$ is the number of ways of partitioning the remaining $t$-subsets. Now (3.2.2) will be satisfied if

\[n^{(s) k^{(s) - (k)}} n^{(s)} k^{(s)} < n^{(t)}\]

or if

\[(3.2.3) \quad \binom{k}{t}^{(k) - 1} \binom{s}{k} < n^{(t)}\]

If $s$ is any integer satisfying (3.2.3), then there does exist some way of partitioning $P_t(S)$ into $n$ classes such that no $k$-subset of $S$ has all of its $t$-subsets in one of the $n$ classes. This completes the proof of (3.2.1).
If \( n = t = 2 \) in (3.2.1), then we have

\[
(3.2.4) \quad \binom{R_2(k, 2)}{k} \geq 2^{k-1}.
\]

(3.2.4) yields

\[
(3.2.5) \quad R_2(k, 2) \geq ck^{k/2}
\]

for some constant \( c \) and all sufficiently large \( k \).

It also follows from Theorem 3.1.1 that

\[
(3.2.6) \quad R_{2n}(k, 2) - 1 \geq (R_2(k, 2) - 1)^n
\]

and

\[
R_{2n+1}(k, 2) - 1 \geq (R_1(k, 2) - 1)(R_2(k, 2) - 1)^n.
\]

It is not difficult to see that (3.2.6) and (3.2.5) yield a better lower bound for \( R(n, k, 2) \) than that given by (3.2.1).

Probability arguments have been used by Erdős to obtain lower bounds for other classes of Ramsay numbers, especially the numbers \( R(3, k; 2) \).

The best result that has been obtained up to the present time is

\[
R(3, k; 2) \geq ck^2/(\log k)^2
\]

for some constant \( c \) and all sufficiently large \( k \). For the proof of this result and further references to the literature see [12].
§3.3. An Algebraic Approach

In this section we obtain by an algebraic method a lower bound for \( R_n(k,2) \) which is better than that given by (3.1.2), and also better than that given by (3.2.1) provided \( k \) is small and \( n \) is large compared to \( k \).

Consider the following system \((S)\) of \( \binom{k-1}{2} \) equations in \( \binom{k}{2} \) unknowns:

\[
x_{i,j} + x_{j,i+1} = x_{i,j+1}, \quad 1 \leq i < j \leq k-1.
\]

Suppose there exists some way of partitioning the numbers \( 1, 2, \ldots, m \) into \( n \) sets \( A_1, A_2, \ldots, A_n \), no set containing a solution of \((S)\). Let \( G \) be the complete graph with vertices \( P_0, P_1, P_2, \ldots, P_m \). Color the edges of \( G \) in \( n \) colors \( c_1, c_2, \ldots, c_n \) by coloring the edge \( P_i P_j \) color \( c_r \) if \( |i - j| \in A_r \).

In order to see that \( G \) contains no M.C. \( <k> \), let

\[
P_{i_1}, P_{i_2}, \ldots, P_{i_k}, \quad i_1 > i_2 > \ldots > i_k
\]

be the vertices of a \( <k> \) in \( G \), and suppose all interconnecting edges are colored \( c_r \). Then \( i_t - i_s \in A_r \) for \( 1 \leq t < s \leq k \).

But \( (i_t - i_s) + (i_s - i_{s+1}) = (i_t - i_{s+1}), \quad 1 \leq t < s \leq k-1 \).

Hence we have a solution to system \((S)\) in \( A_r \). This is a contradiction. Hence \( G \) contains no complete M.C. \( <k> \).

It follows from the above argument that

\[
(3.3.1) \quad R_n(k,2) \geq m + 2.
\]
If we define \( t(n,k) \) to be the largest integer for which there exists some way of partitioning the numbers \( 1, 2, \ldots, t(n,k) \) into \( n \) sets, no set containing a solution of \( (S) \), then by (3.3.1) we have

\[
(3.3.2) \quad R_{k,2}^{(n)} \geq t(n,k) + 2.
\]

We have thus translated the problem of finding lower bounds for \( R_{n}^{(k,2)} \) into the problem of finding lower bounds for \( t(n,k) \).

We define a function \( g \) as follows: If \( t(n-1,k) < m \leq t(n,k) \), then \( g(m,k) = n \). \( g(m,k) \) is thus the smallest number of sets into which the integers \( 1, 2, \ldots, m \) can be partitioned, no class containing a solution of \( (S) \).

In [1] it is proved that \( g(m,3) < \log m \) for all sufficiently large \( m \). Since \( g \) is a decreasing function of \( k \), we have

\[
(3.3.3) \quad g(m,k) < \log m.
\]

In fact, one can show that \( g(m,k) < (1 + \varepsilon) \frac{\log m}{\log k} \) for every \( \varepsilon > 0, m \geq m_{0}(\varepsilon) \), but (3.3.3) is sufficient in what follows.
Now we prove

**Theorem 3.3.1.** For all positive integers \( p \) and \( q \)

\[(3.3.4) \quad t(pq + g(pt(q,k),k),k) \geq (2t(q,k) + 1)^p - 1\]

**Proof:** For notational convenience, let \( X = 2t(q,k) + 1 \).

Write the numbers 1, 2, ..., \( x^{p-1} \) in base \( X \). We distinguish these numbers as follows: The set of numbers each of whose digits \( \leq t(q,k) \) is denoted by \( N_1 \). The set of numbers, each of which has at least one of its digits at least \( t(q,k) + 1 \) is denoted by \( N_2 \). We shall split the set \( N_1 \) into \( g(pt(q,k),k) \) sets and the set \( N_2 \) into \( pq \) sets, no set containing a solution of \((S)\). The proof of the theorem shall then be complete.

Let \( C_1, C_2, \ldots, C_{g(pt(q,k),k)} \) be sets containing 1, 2, ..., \( pt(q,k) \), no set containing a solution of \((S)\). We partition the set \( N_1 \) into sets \( A_1, A_2, \ldots, A_{g(pt(q,k),k)} \) by putting a number in \( A_j \) if the sum of its digits belongs to \( C_j \), i.e. put \( a = a_1 + a_2X + a_3X^2 + \ldots + a_xX^{p-1} \) in \( A_j \) if \( \sum_{i=1}^{p} a_i \epsilon C_j \). This can be done since \( \sum_{i=1}^{p} a_i \leq pt(q,k) \). Then \( A_j \) contains no solution of \((S)\) because \( C_j \) does not.

For \( 1 \leq r \leq p \) let \( B_r \) be the set of all numbers \( a = a_1 + a_2X + \ldots + a_rX^{r-1} + \ldots + a_pX^{p-1} \) satisfying \( a_i \leq t(q,k) \) for \( i = 1, 2, \ldots, r-1 \) and \( a_r \geq t(q,k) + 1 \).

The set \( N_2 \) has thus been partitioned into sets.
We now partition each $B_r$ into $q$ sets as follows: Let $D_1, D_2, \ldots, D_q$ be disjoint sets containing $1, 2, \ldots, t(q,k)$, no $D_i$ containing a solution to (S). Let $a \in B_r$. Then $a_r = X - (a_r)_1^1$ where $1 \leq (a_r)_1 \leq t(q,k)$. Put $a \in E_r$ if $(a_r)_1 \in D_m$. Then $B_r$ is partitioned into $q$ sets $E_{r_1}, E_{r_2}, \ldots, E_{r_q}$. Hence the set $N_2$ has now been partitioned into $pq$ sets.

Suppose $E_{r_m}$ contains a solution of (S), i.e. there are numbers $Z_{i,j}$ in $E_{r_m}$ such that

\[
(3.3.5) \quad Z_{i,j} + Z_{j,j+1} = Z_{i,j+1}, \quad 1 \leq i \leq j \leq k-1.
\]

Let

\[
Z_{i,j} = (a_{i,j})_1^1 + (a_{i,j})_2^1 X + \ldots + (a_{i,j})_r^1 X^{r-1} + \ldots + (a_{i,j})_p^1 X^{p-1}.
\]

(3.3.5) implies that

\[
(a_{i,j})_r^1 + (a_{j,j+1})_r^1 = (a_{i,j+1})_r^1 + X
\]

and this in turn implies

\[
(3.3.6) \quad X - (a_{i,j})_r^1 + X - (a_{j,j+1})_r^1 = X - (a_{i,j+1})_r^1 + X,
\]

where $(a_{i,j})_r^1, (a_{j,j+1})_r^1, (a_{i,j+1})_r^1 \in D_m$. But from

(3.3.6) we have

\[
(a_{i,j})_r^1 + (a_{j,j+1})_r^1 = (a_{i,j+1})_r^1.
\]

That is, we have a solution of (S) in $D_m$. This is a
contradiction. Hence $E_r$ does not contain a solution of $(S)$. This completes the proof of Theorem 3.3.1.

If we set $q = 1$ in (3.3.4) and use the easily established fact that $t(1,k) = k - 2$ we get

$$t(p + g(p(k-2),k),k) \geq (2k - 3)p - 1.$$  \hspace{1cm} (3.3.7)

Let $k$ be arbitrary but fixed. Then it follows from (3.3.7) (3.3.3) and (3.3.2) that

$$R_n(k,2) > (2k - 3)^n(1 - \varepsilon)$$  \hspace{1cm} (3.3.8)

for every $\varepsilon > 0$ and $n \geq n_0(k,\varepsilon)$. This result is clearly better than (3.1.2).

In the immediately preceding argument we chose $q = 1$. However there is nothing to prevent us from choosing larger values of $q$ to get still better results for certain values of $k$. We illustrate this in the cases $k = 3, 4$.

If $k = 3$ in (3.3.8) we get

$$R_n(3,2) > 3^n(1 - \varepsilon).$$  \hspace{1cm} (3.3.9)

Let $k = 3$ and $q = 4$ in (3.3.4). This gives

$$t(4p + g(pt(4,3),3),3) \geq (2t(4,3) + 1)p - 1.$$  \hspace{1cm} (3.3.10)

It is known (L. D. Baumert, unpublished, see [1]) that $t(4,3) = 44$. This with (3.3.10), (3.3.3) and (3.3.2) yields
for every \( \epsilon > 0 \) and \( n \geq n_0(\epsilon) \).

If \( k = 4 \) in (3.3.8) we get

\[
R_n(4,2) > 5^n(1-\epsilon) .
\]

This can be improved by taking \( k = 4 \) and \( q = 2 \) in (3.3.4).

We observe first that \( t(2,4) \geq 16 \). This follows from the fact that the numbers 1, 2, ..., 16 can be split into two sets

\[
C_1 = \{1, 2, 4, 8, 9, 13, 15, 16\} \\
C_2 = \{3, 5, 6, 7, 10, 11, 12, 14\}
\]

neither of the sets containing a solution of the system

\[
x_{12} + x_{23} = x_{13} \\
x_{13} + x_{34} = x_{14} \\
x_{23} + x_{34} = x_{24} .
\]

Thus from (3.3.4), (3.3.3) and (3.3.2) we get

\[
R_n(4,2) > 33^{n/2}(1-\epsilon)
\]

for every \( \epsilon > 0 \) provided \( n \geq n_0(\epsilon) \).

Other results of this type can be obtained but we do not discuss these any further here.
CHAPTER IV

SOME EXACT VALUES FOR THE RAMSAY NUMBERS

The problem of determining $R = R(k_1, k_2, \ldots, k_n; t)$ appears to be a very difficult one. No value of $R$ is known for $t > 2$. In fact, the values of $R_1 = R(k_1, k_2, \ldots, k_n; t)$ are not known for $n \geq 4$. Even for $n < 4$, the values of $R_1$ have only been established for small values of $k_i$.

In this chapter, we shall give some of the techniques used in evaluating the known values of the Ramsay numbers.

For notational convenience, in this chapter we shall denote a monochromatic $<k>$ of color $c_i$ by the symbol $c_i<k>$.

The first evaluation we give is the following:

**Theorem 4.1:**

$$R(3, 4; 2) = 9$$

**Proof:** We prove first that if the edges of a $<9>$ are colored arbitrarily in two colors $c_1$ and $c_2$, then there will result either a $c_1<k_3>$ or a $c_2<k_4>$. This will show that $R(3, 4; 2) < 9$.

Let $v$ be a vertex of the $<9>$. Let $n_1$ of the
edges incident with v have color \( c_1 \) and \( n_2 \) of these edges have color \( c_2 \). (\( n_1 + n_2 = 8 \)). We suppose first that \( n_1 \leq 4 \). If one of the edges joining in pairs the farther ends of these \( n_1 \) edges incident with v, is colored \( c_1 \), then we have a \( c_1 < 3 \). Otherwise, we have a \( c_2 < 4 \).

Hence we may assume that \( n_1 \leq 3 \).

Suppose that \( n_1 = 3 \) at every vertex of the \( <9> \). Then the number of edges colored \( c_1 \) is \((9)(3)/2 \) which is impossible. Hence without loss of generality we may assume that \( n_1 \leq 2 \). Hence \( n_2 \geq 6 \). Consider now the edges joining in pairs the farther end of these \( n_2 \geq 6 \) edges incident with v.

By Theorem 2.1, these must yield either a \( c_1 < 3 \) or a \( c_2 < 3 \). Hence in the \( <9> \), there is either a \( c_1 < 3 \) or a \( c_2 < 4 \). Therefore, \( R(3,4;2) \leq 9 \).

That \( R(3,4;2) \geq 8 \), follows from the fact that in the graph sketched below, there is no \( c_1 < 3 \) and no \( c_2 < 4 \).

Hence \( R(3,4;2) = 9 \).
We now prove

**Theorem 4.2** \( R(3,5;2)=14 \).

**Proof:** From (2.1) and (4.1), and the fact that \( R(2,k;2)=k \) for all \( k \geq 2 \), we have

\[
(4.2) \quad R(3,5;2) \leq R(2,5;2) + R(3,4;2) = 14,
\]

Hence we need to show that \( R(3,5;2) > 13 \). To do this we must show how to color the edges of a \( <13> \) in two colors \( c_1 \) and \( c_2 \) without forcing the appearance of \( c_1 <3> \) or a \( c_2 <5> \).

Consider the field \( F \) of residue classes modulo 13. \( F=\{0,1,2,\ldots,12\} \). Let \( H=\{1,5,8,12\} \). \( H \) is a subgroup of the multiplicative group of \( F \). The cosets of \( H \) are \( H_1=\{2,3,10,11\} \) and \( H_2=\{4,6,7,9\} \). Let the vertices of the \( <13> \) be \( P_0, P_1, \ldots, P_{12} \). The edge \( P_iP_j \) is colored \( c_1 \) if \( i-j \in H \), and colored \( c_2 \) if \( i-j \notin H_1 \cup H_2 \). Suppose there results a \( c_1 <3> \), with vertices \( P_i, P_j, P_k \). Then \( i-j, j-k, i-k \in H \). But \( (i-j)+(j-k)=i-k \). This is a contradiction since the sum of any two elements of \( H \) is not in \( H \). Hence there is no \( c_1 <3> \).

Suppose there results a \( c_2 <5> \) with vertices

\[
P_{v_1}, P_{v_2}, \ldots, P_{v_5}.
\]

Then \( v_1 - v_j, 1 \leq i < j \leq 5 \), are all in \( H_1 \cup H_2 \).

Suppose \( v_5 \neq 0 \). Set \( w_1 = v_1 - v_5 \). Then \( w_1 - w_j = v_1 - v_j, w_5 = 0 \), and
\( w_i - w_j \) are all in \( H_1 \cup H_2 \). Consider \( w_1, w_2, w_3, w_4 \). The 3-subsets of \( H_1 \) are \((2,3,10), (2,3,11), (2,10,11), (3,10,11)\). In each of these there is a difference equal to 8, i.e. there is a difference in \( H \). Hence at most two of \( w_1, w_2, w_3, w_4 \) belong to \( H_1 \).

Similarly, at most two of \( w_1, w_2, w_3, w_4 \) belong to \( H_2 \). Hence exactly two (say) \( w_1, w_2 \) belong to \( H_1 \) and the other two, \( w_3, w_4 \) belong to \( H_2 \). Now \((w_1, w_2) \neq (2,3), (2,10), (3,11) \) or \((10,11)\) since \( 3-2=10-11=1 \in H \) and \( 10-2=11-3=8 \in H \). Hence \((w_1, w_2) = (2,11) \) or \((3,10)\). Suppose \((w_1, w_2) = (2,11)\). Then \( w_3 \) and \( w_4 \) are different from 6 since \( 11-6=5 \in H \); also \( w_3 \) and \( w_4 \) are different from 7 since \( 7-2=5 \in H \). Hence \((w_3, w_4) = (4,9)\). But this contradicts \( 9-4=5 \in H \). Hence \((w_1, w_2) \neq (2,11)\). The same argument shows \((w_1, w_2) \neq (3,10)\).

Hence there is no \( c_2 < 5 \). Hence \( R(3,5;2) > 13 \) and this with (4.2) completes the proof of the theorem.

Another evaluation is given by the following theorem:

**Theorem 4.3:** \( R(4,4;2) = 18 \).

**Proof:** From (2.1) and (4.1) we have

\[
(4.3) \quad R(4,4;2) \leq R(3,4;2) + R(4,3;2) = 18.
\]

To show that \( R(4,4;2) > 17 \) we must show how to color the edges of a \( <17> \) in two colors \( c_1 \) and \( c_2 \), without forcing
the appearance of either a $c_1 <4>$ or a $c_2 <4>$. Consider
the field $F$ of residue classes modulo 17. Let
$H = \{1, 2, 4, 8, 9, 13, 15, 16\}$. $H$ is a subgroup of the multiplicative
group of $F$. ($H$ is in fact the set of quadratic residues of
17.) The coset of $H$ is $H_1 = \{3, 5, 6, 7, 10, 11, 12, 14\}$. Let the
vertices of the $<17>$ be labeled $P_0, P_1, \ldots, P_{16}$.

Let $P_i P_j$ be colored $c_1$ if $i-j \in H$ and colored $c_2$ if $i-j \in H_1$.

Suppose there results a $c_1 <4>$ with vertices $P_{u_1}, P_{u_2}, P_{u_3}, P_{u_4}$.
Then $u_i - u_j \in H, 1 \leq i < j \leq 4$. Set $v_i = u_i - u_4$.
Then $v_i - v_j = u_i - u_j$, $v_i - v_j \in H$, and $v_4 = 0$. Set $x_1 = (1/v_3)v_1$.
Then $x_3 = 1$ and $x_1-x_2, x_1-1, x_1, x_2-1, x_2, 1 \in H$. Now $x_1, x_2 \notin H$ implies that
$x_1 \neq 1, 4, 8, 13, 15$, and $x_2, x_2-1 \in H$ implies that $x_2 \neq 1, 4, 8, 13, 15$.
Hence $\{x_1, x_2\} \subset \{2, 9, 16\}$. But since $9-2=7$, $16-2=14$ and
$16-9=7$, this contradicts the fact that $x_1-x_2 \notin H$. Hence there
is no $c_1 <4>$.

Now we assume that there exists a $c_2 <4>$, with
vertices $P_{u_1}, P_{u_2}, P_{u_3}, P_{u_4}$. Then $u_i - u_j \in H_1, 1 \leq i < j \leq 4$. Let $a \in H_1$
and let $v_i = a u_i$. Then $v_i - v_j = a(u_i - u_j) \in H$. Hence $P_{v_1}, P_{v_2}, P_{v_3}, P_{v_4}$
are the vertices of a $c_1 <4>$. But this contradicts the first
part of the proof and hence there is no $c_2 <4>$. Hence
$R(4, 4; 2) > 17$ and this with (4.3) proves the theorem.

Using an argument similar to that used in the proof of
theorem 4.3, we have obtained the following:
Theorem 4.4: \( R(5,5;2) > 38; R(6,6;2) > 90 \) and \( R(7,7;2) > 110 \).

Proof: To show that \( R(5,5;2) > 38 \), we let \( G \) be a complete graph with vertices \( P_0, P_1, \ldots, P_{36} \). Color the edges of \( G \) in two colors \( c_1 \) and \( c_2 \) by coloring the edge \( P_iP_j \) color \( c_1 \) if \( i-j \) is a quadratic residue of 37, and color \( c_2 \) if \( i-j \) is a quadratic non-residue of 37. Then it is not difficult to check that \( G \) contains no monochromatic \( <5> \). The same type of argument using the primes 89 and 109 can be used to show \( R(6,6;2) > 90 \) and \( R(7,7;2) > 110 \) but the details are naturally somewhat more involved.

Finally we prove

Theorem 4.5: \( R_3(3,2) = 17 \)

Proof: From (2.5) we have \( R(3,2) \leq \lfloor n/e \rfloor + 1 \). Hence \( R_3(3,2) \leq \lfloor 3/e \rfloor + 1 = \lfloor 6/e \rfloor + 1 = 17 \). To show that \( R_3(3,2) > 16 \), we use the following argument: Let \( F \) be the field of residue classes modulo 2. Adjoin to \( F \) the indeterminate \( t \) satisfying the equation \( t^4 = t+1 \). This yields the field \( F[t] \) consisting of the elements:

\[
\{0, 1, t, t+1, t^2, t^2+1, t^2+t, t^2+t+1, t^3, t^3+1, t^3+t, t^3+t+1, t^3+t^2, t^3+t^2+1, t^3+t^2+t, t^3+t^2+t+1\}.
\]

Let \( H \) be the multiplicative group of \( F[t] \). \( H_1 = \{1, t^3, t^3+t^2, t^3+t, t^3+t^2+t+1\} \) is a subgroup of \( H \).
\( H_2 = \{t, t+1, t^3+t+1, t^2+t+1, t^3+t^2+1\} \) and

\( H_3 = \{t^2, t^2+t, t^2+1, t^3+t^2+t, t^3+1\} \) are the cosets of \( H_1 \) in \( H \).

Consider a \( <16> \) with vertices labeled \( v_1, v_2, \ldots, v_{16} \)
where \( v_i \in H \). Color the edge \((v_i v_j)\) color \( c_\ell \) if \( v_i + v_j \in H_\ell \), \( \ell = 1, 2, 3 \). If \( v_i, v_j, v_k \) are the vertices of a M.C \( <3> \),
then \( v_i + v_j, v_i + v_k, v_j + v_k \) all belong to one of \( H_1, H_2 \) or \( H_3 \).

But \((v_i + v_j) + (v_j + v_k) = v_i + v_k \). This is a contradiction
since the sum of any two elements in either \( H_1, H_2 \) or \( H_3 \)
is not in the same set. Hence there does not exist a
M.C. \( <3> \). Hence \( R_3(3,2) >16 \) and the theorem is complete.

All of the above results were obtained by Greenwood
and Gleason [5]. Other values of the Ramsay numbers
have been obtained by Kalbfleisch [6]. His arguments do
not involve finite fields, and it seems unlikely that any
new results can be obtained using the methods used above.
CHAPTER V

SOME APPLICATIONS OF RAMSAY'S THEOREM

In this Chapter we discuss some of the applications of Ramsey's Theorem to various problems, in particular, to a problem in Set Theory; to a problem in Geometry, and to a problem in Matrix Theory.

§5.1 An application to a problem in Set Theory.

A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}$ if for every $F \in \mathcal{F}$, there exists a set $B \subseteq F$ such that $F \cap B \neq \emptyset$ and $F \cap B \neq F$.

Erdős and Hajnal in [4] asked the following question: What is the smallest integer $m(n)$ for which there exists a family $\mathcal{F}_n$ of sets $A_1, A_2, \ldots, A_{m(n)}$ such that $|A_i| = n$ for $1 \leq i \leq m(n)$ and which does not possess property $\mathcal{B}$? They observed that $m(1)=1$, $m(2)=3$, $m(3)=7$ and that $m(n) \leq \binom{2n-1}{n}$. The value of $m(n)$ is not known for $n \geq 4$, and the problem of determining $m(n)$, even for $n=4$ appears to be difficult.

Erdős proved in [3], that for all $n \geq 2$

(5.1.1) $m(n) > 2^{n-1}$

Various improvements in the upper and lower bounds for
m(n) have been given, but we do not discuss these here. We mention only that the best known lower bound for m(n) is

\[ m(n) > 2^n \binom{n}{n+4} \]

which was obtained by Schmidt [9].

In [4], Erdos and Hajnal also asked: Does there exist for every positive integer \( k \geq 2 \) a finite family \( \mathcal{F}_k \) of finite sets satisfying:

1. \( |F| = k \) for each \( F \in \mathcal{F}_k \).
2. \( |F \cap G| \leq 1 \) for \( F, G \in \mathcal{F}_k, F \neq G \).
3. \( \mathcal{F}_k \) does not possess property \( \mathcal{B} \).

They observed that such families do exist for \( k=2,3 \).

Abbott proved in [1] that such families exist for every positive integer \( k \), by making use of a special case of Ramsay's Theorem.

**Theorem 5.1.1:** Let \( S \) be an \( s \)-set, \( s \geq R_2(k,t) \) and let \( K \) be a \( k \)-subset of \( S \). Let \( F \) denote the set of all \( t \)-subsets of \( K \) and let \( \mathcal{F}_{k,t} \) denote the family of all possible sets constructed in this way. Then \( \mathcal{F}_{k,t} \) does not possess property \( \mathcal{B} \).

**Proof:** Assume that \( \mathcal{F}_{k,t} \) does possess property \( \mathcal{B} \). Then there exists a set \( B \subset \bigcup \mathcal{F}_{k,t} \) such that \( B \cap F \neq \emptyset \)
and \( F \notin B \) for each \( F \in \mathcal{F} \). Partition \( P_t(S) \) into two classes \( A_1, A_2 \) by placing a \( t \)-subset \( T \) of \( S \) in \( A_1 \) if \( T \notin B \) and in \( A_2 \) if \( T \notin B \). Let \( K \) be any \( k \)-subset of \( S \) and let \( F \) be the corresponding member of \( \mathcal{F}_{k,t} \). Then since \( B \cap F \neq \emptyset \), there is a \( t \)-subset of \( K \) which belongs to \( B \) and hence to \( A_1 \), and since \( F \notin B \), there is a \( t \)-subset of \( K \) which does not belong to \( B \) and hence belongs to \( A_2 \). However, since \( s \geq R_2(k, t) \), there must exist some \( k \)-subset of \( S \) all of whose \( t \)-subsets belong to either \( A_1 \) or \( A_2 \). This is a contradiction and the proof of the theorem is complete.

Since \( \mathcal{F} \) satisfies conditions (1), (2) and (3), therefore the question of Erdős & Hajnal is settled.

If we choose \( s = R_2(k, t) \) in the above theorem, then the number of sets in the family \( \mathcal{F}_{k,t} \) is \( \left( \begin{array}{c} R_2(k, t) \\ k \end{array} \right) \), and the number of elements in each set is \( \left( \begin{array}{c} k \\ t \end{array} \right) \). Therefore we must have:

\[
\left( \begin{array}{c} R_2(k, t) \\ k \end{array} \right) \geq m \left( \begin{array}{c} k \\ t \end{array} \right).
\]

Hence by (5.1.1) we have:

\[
(5.1.2) \quad \left( \begin{array}{c} R_2(k, t) \\ k \end{array} \right) > 2^{(k-1) t - 1}
\]

This is the same result as was obtained in Theorem 3.2.1 for
the case \( n=2 \).

§ 5.2  

An application to a problem in Geometry

In this section, we show that Ramsey's Theorem can be used to settle a problem in geometry. The problem can be formulated as follows: Let \( k \geq 3 \) be a positive integer. Does there exist a least integer \( f(k) \) with the property that among every set of \( f(k) \) points in a plane, no three collinear, there are \( k \) points which form the vertices of a convex \( k \)-gon?

We prove;

Theorem 5.2.1: \( f(k) \) exists and satisfies

\[
f(k) \leq R(5,k;4).
\]

Before we prove theorem 5.2.1, we introduce the following lemmas, the proof of the first of which is not difficult.

Lemma 5.2.1: Among any five points in the plane, no three collinear, there are four points which are the vertices of a convex quadrilateral.

Lemma 5.2.2: If the \( \binom{k}{4} \) quadrilaterals formed from \( k \) points in the plane, no three collinear, are all convex, then the \( k \) points form the vertices of a convex \( k \)-gon.
Proof: Let $P_1, P_2, \ldots, P_k$ be the vertices of the convex cover of the set of $k$ points. If $k=k$, we have finished. Suppose $k<k$. Then there is a point $P$ which must lie in the interior of some triangle, say $P_1, P_{i-1}, P_i$. Then $P_1, P_{i-1}, P_i, P$ is non convex, and this is a contradiction. Hence $k=k$.

Proof of Theorem 5.2.1: Let $R=R(5,k;4)$. Let $S$ be a set of $R$ points in the plane. Partition $P_4(S)$ into two classes $C_1$ and $C_2$ by placing a 4-subset in $C_1$ if these points form a non-convex quadrilateral, and in $C_2$ if these points form a convex quadrilateral. By Ramsey's Theorem, either there is a 5-subset of $S$ all of whose 4-subsets belong to $C_1$, or a $k$-subset all of whose 4-subsets belong to $C_2$. By lemma 5.2.1, the first alternative is impossible. Hence the second alternative must hold. But by lemma 5.2.2, the $k$ points form the vertices of a convex $k$-gon. Hence $f(k)\leq R(5,k;4)$.

It is easy to show that $f(3)=3$, $f(4)=5$ and it is known that $f(5)=9$. The values of $f(k)$ are not known for $k \geq 6$. However, it is conjectured that $f(k)=2^{k-2}+1$.

§5.3 An application to a problem in Matrix Theory:

Another application of Ramsey's Theorem is to matrices of zeroes and ones. We define an $A_{x,y}$ matrix to
be a matrix in which all entries above the diagonal are \( x \) and all entries below the diagonal are \( y \). The diagonal may have both \( x \) and \( y \) entries.

**Theorem 5.3.1:** Let \( k \) be a positive integer. Then there exists a least integer \( g(k) \) such that if \( A = [a_{ij}] \) is an arbitrary \( g(k) \times g(k) \) matrix of zeroes and ones, then \( A \) contains either an \( A_{0,0}, A_{0,1}, A_{1,0} \) or \( A_{1,1} \) \( k \times k \) submatrix.

**Proof:** Let \( R = R_4(k, 2) \) and let \( A = [a_{ij}] \) be an \( R \times R \) matrix of zeroes and ones. Let \( S \) be the set whose elements are the rows of \( A \). Let \( R_i, R_j \) be elements of \( S \), \( i < j \).

We now associate with \( \{R_i, R_j\} \) the vector \( (a_{ij}, a_{ji}) \). Hence \( \{R_i, R_j\} \) corresponds to one of \( (0,0), (1,0), (0,1) \) or \( (1,1) \).

Partition \( P_2(S) \) into four classes \( C_1, C_2, C_3, C_4 \) by putting \( \{R_i, R_j\} \) in \( C_1 \) if \( \{R_i, R_j\} \) corresponds to \( (0,0) \); in \( C_2 \) if it corresponds to \( (1,0) \); in \( C_3 \) if it corresponds to \( (0,1) \) and in \( C_4 \) if it corresponds to \( (1,1) \). However, since \( S \) is an \( R \)-set, then by Ramsay's Theorem, there exists a \( k \)-subset \( K \) of \( S \) such that \( P_2(K) \subseteq C_i \) for some \( i, 1 \leq i \leq 4 \). Hence there exists a \( k \times k \)-submatrix in \( A \), which is of the form \( A_{0,0}, A_{0,1}, A_{1,0} \) or \( A_{1,1} \).

The above argument can be generalized to the case where \( A \) is a matrix whose elements are \( 0,1,2,\ldots,r-1 \). It
can be shown that if \( k \) is a positive integer, then there exists a least integer \( g(k,r) \) such that if \( A \) is an arbitrary \( g(k,r) \times g(k,r) \) matrix whose elements are \( 0, 1, 2, \ldots, r-1 \), then \( A \) contains a \( k \times k \) submatrix \( A \) for some \( p \) and \( q \), \( 0 \leq p \leq r-1 \), \( 0 \leq q \leq r-1 \). Moreover, \( g(k,r) \) satisfies

\[
g(k,r) \leq R_2(k,2).
\]

Using an argument similar to that used in Theorem 3.2.1, we have obtained the following.

**Theorem 5.3.2:** \( g(k) > c k 2^{k^2} \) for some constant \( c \) and all sufficiently large \( k \).

**Proof:** Let \( S(N) \) be the number of \( N \times N \) matrices \( A \) of \( N \) zeroes and ones which contain at least one \( k \times k \) submatrix of the form \( A_{0,0}, A_{0,1}, A_{1,0}, \) or \( A_{1,1} \). The probability that a matrix \( A \) chosen at random contains a \( A_{0,0}, A_{0,1}, A_{1,0}, \) or \( A_{1,1} \) \( k \times k \) submatrix is \( \frac{S(N)}{2^{N^2}} \), since the number of \( N \times N \) matrices of zeroes and ones is \( 2^{N^2} \). Hence the probability that a matrix \( A \) chosen at random does not contain a \( k \times k \) submatrix of the desired type is \( 1 - \frac{S(N)}{2^{N^2}} \). Now since the number of ways of choosing \( k \) rows and columns from \( N \) rows and columns is \( \binom{N}{k}^2 \) and since the remaining entries can be filled in \( 2^{N^2-(k^2-k)} \) ways, we have
\[ S(N) \leq 4 \left( \frac{N}{k} \right)^2 2N^2 - (k^2 - k) < \frac{N^{2k}}{(k!)^2} 2N^2 - (k^2 - k) + 2 \]

Hence, \( \frac{1 - S(N)}{2N^2} > 0 \) or \( S(N) < 2N^2 \) will hold if

\[ \frac{N^{2k}}{(k!)^2} 2N^2 - (k^2 - k) + 2 < 2N^2 \]

or if

\[ N < \frac{k}{c} \]

for some constant \( c \) and all sufficiently large \( k \).

Hence \( g(k) > \frac{k}{c} 2^2 \).

The same type of argument yields, for each fixed \( r > 2 \),

\[ g(k, r) > Ckr^2 \]

for some constant \( C \) and all sufficiently large \( k \).
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