SOME PROBLEMS RELATED TO RAMSAY'S THEOREM

EDGAR R. WILLIAMS

SOME PROBLEMS RELATED TO RAMSAY'S THEOREM

by

EDGAR ROLAND WILLIAMS

A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF ARTS.

DEPARTMENT OF MATHEMATICS

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

JUNE, 1967.

ABSTRACT

In 1930, F.P. Ramsay published a paper containing a combinatorial theorem which has since then become very well known and has given rise to an extensive literature. Most of the research which has arisen from Ramsay's Theorem, has dealt with the problem of finding upper and lower bounds for the so called Ramsay numbers. In addition, some exact values of these numbers have been determined and some applications of Ramsay's Theorem have been given.

In this thesis, we survey some of the research which has been done. In addition, some new results have been obtained. These results yield a better lower bound for certain classes of Ramsay numbers, than any of those that have been obtained up to the present time.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my sincere thanks to my Supervisor, Dr. H.L. Abbott, for his invaluable assistance throughout the preparation of this thesis; and to Mrs. Buchanan and Mrs. Abbott for typing the manuscript.

TABLE OF CONTENTS

CHAPTER I.

INTRODUCTION

A very significant theorem in combinatorial analysis appeared in 1930 in a paper [7] by the English logician F. P. Ramsay. Before stating Ramsay's Theorem, the following terminology used in formulating the theorem and throughout most of this thesis, is explained: By an s-set is meant a set the number of whose elements is s. By a t-subset of a set S is meant a subset of S with t elements. The set of all t-subsets of a set S shall be denoted by $P_t(S)$.

Ramsay's Theorem in its most general form can now be formulated as follows:

Theorem 1.1 Let $k_1, k_2, \ldots, k_n$ and t be positive integers such that each $k_i \geq t$. Then there exists a least positive integer $R = R(k_1, k_2, \ldots, k_n; t)$ such that if S is an s-set, $s \geq R$, and if $P_t(S)$ is partitioned into n classes $C_1, C_2, \ldots, C_n$, then for some i, $1 \leq i \leq n$, there exists a k_i-subset $K_i \subseteq S$ such that $P_t(K_i) \subseteq C_i$.

The integers $R = R(k_1, k_2, \ldots, k_n; t)$ are referred to as the Ramsay Numbers. If $k_1 = k_2 = \ldots = k_n = k$ in Theorem 1.1, then we shall denote Ramsay numbers of this type of $R_n(k,t)$.

Perhaps the most interesting special cases of Ramsay's Theorem are the cases $t = 1$ and $t = 2$. A moments reflection shows that in the case $t = 1$, Ramsay's Theorem reduces to the well known pigeon hole principle. In fact $R(k_1, k_2, \ldots, k_n;1) = k_1 + k_2 + \ldots + k_n - n + 1$.

In the case $t = 2$, Ramsay's Theorem can be formulated in the language of Graph Theory. If S is an s-set, then we can think of the elements of S as the vertices of a complete graph on s vertices, the 2-subsets of S as the edges of this graph and the partitioning of the 2-subsets of S into n classes as coloring the edges of the graph in n colors. The Theorem of Ramsay can thus be formulated as follows:

If G is a complete graph on s vertices, $s \geq R(k_1, k_2, \ldots, k_n;2)$, and if the edges of G are colored in any way in n colors $c_1, c_2, \ldots, c_n$, then for some i, $1 \leq i \leq n$, there results a complete sub-graph of G with k_i vertices all of whose edges are colored c_i.

In much of what follows, the language of graph theory shall be used. For notational convenience, a complete graph on k vertices or "k-gon" shall be denoted by the symbol $<k>$. If a $<k>$ is such that all of its edges have the same color, then we shall refer to it as a monochromatic <k> or M.C.<k>.

Since the appearance of Ramsay's Theorem in 1930, several well known mathematicians have worked on problems arising from it. Most of this research has dealt with finding bounds for the Ramsay Numbers $R(k_1, k_2, \ldots, k_n; t)$ or $R_n(k,t)$. Still, very little is known as to what is the order of magnitude of $R_n(k,t)$ for $t \geq 2$. Also, some exact values for the Ramsay numbers have been given for small values of n and k, and $t = 2$. However, the values of $R_n(k,2)$ are only known for $n \leq 4$ and small values of k. In addition, other papers have been devoted to the applications of Ramsay's Theorem.

In this thesis, we give a survey of the research which has gone into some of the above mentioned problems. Also some new results are obtained.

In Chapter II, we shall develop a proof of the most general formulation of Ramsay's Theorem.

In Chapter III, we shall discuss some of the existing recurrence inequalities and lower bounds for the Ramsay Numbers. In addition, we shall prove a new result which yields for fixed k and large n, a better lower bound for $R_n(k,2)$ than any of those that have been obtained up to the present time.

In Chapter IV, we shall discuss some of the known exact values of the Ramsay Numbers.

Finally, in Chapter V, some of the applications of Ramsay's Theorem are discussed.

CHAPTER II

PROOF OF RAMSAY'S THEOREM

An exposition of the proof of Ramsay's Theorem is given in the book by Ryser [8] . The proof there is essentially due to G. Szekeres [11]. However, in this Chapter, we shall approach the proof of Theorem 1.1 from a different point of view, showing that the main idea in Szekeres' argument is really contained in the evaluation of the simplest non-trivial Ramsay number $R(3,3;2)$ or $R_2(3,2)$. We evaluate $R(3,3;2)$ and then proceed to generalize the argument until we finally reach the proof of the most general form of Ramsay's Theorem.

Theorem 2.1: $R(3,3;2) = 6$.

Proof: Let v be a vertex of a <6>, and let three of the five edges terminating at v have color c_1. Consider the three edges joining their farther ends in pairs. If neither of these three edges is colored c_1, then all three must be colored c_2. In either case, there does exist a monochromatic triangle. Thus $R(3,3;2) \leq 6$.

To show that this result is best possible, we show that $R(3,3;2) > 5$. Color the edges of a <5> in two colors c_1 and c_2 as follows: The interior diagonals of the pentagon are colored c_1 and the remaining edges are colored c_2 . Clearly, this coloring scheme does not force the appearance of a M.C.$<3>$. Thus $R(3,3;2) = 6$.

We now proceed to establish the existance of $R(k_1,k_2;2)$. If we assume that $R(k_1,k_2;2)$ exists, then it is clear from the symmetry of the problem that $R(k_1,k_2;2) = R(k_2,k_1;2)$. It is also clear that $R(2,k_2;2) = k_2$ for all $k_2 \geq 2$ and $R(k_1,2;2) = k_1$ for all $k_1 \geq 2$. The existance of $R(k_1, k_2; 2)$ can now be proved by induction. We take as our induction hypothesis the existance of $R(k_1-1,\ell;2)$ and $R(\ell,k_2-1;2)$ for all ℓ. In particular, the induction hypothesis insures the existance of $R(k_1-1,k_2;2)$ and $R(k_1,k_2-1;2)$. Let $s \geq R(k_1-1,k_2;2) + R(k_1,k_2-1;2)$ be a positive integer and color the edges of $<s>$ in two colors c_1 and c_2. Following the idea used in the proof of theorem 2.1, we select an arbitrary vertex v of $<s>$ and let n_1 of the edges incident with v be colored c_1 and n_2 colored c_2. $(n_1+n_2 = s-1)$.

Suppose $n_1 \geq R(k_1-1,k_2;2)$. Consider the edges joining in pairs the farther ends of the n_1 edges incident with v. Since $n_1 \geq R(k_1-1,k_2;2)$, the coloring of these edges in two colors c_1 and c_2 forces the appearance of either a M.C. $<k_1-1>$ of color c_1 or a M.C. $<k_2>$ of color c_2, and hence in G, either a M.C. $<k_1>$ of color c_1 or a M.C. $<k_2>$ of color c_2 . Hence we may assume that $n_1 < R(k_1-1,k_2;2)$.

Then $n_2 \geq R(k_1, k_2-1; 2)$, and the same argument applies. We have therefore proved the following theorem:

Theorem 2.2 $R(k_1, k_2; 2)$ exists and satisfies

(2.1) $R(k_1,k_2;2) \leq R(k_1-1, k_2;2) + R(k_1, k_2-1;2)$

Corollary: $R(k_1,k_2;2) \leq \binom{k_1+k_2-2}{k_1-1}$

Proof: Let $T(k_1,k_2) = \binom{k_1+k_2-2}{k_1-1}$. Clearly $T(k_1,k_2)$ satisfies the same recurrence and same boundary conditions as $R(k_1,k_2;2)$

If $k_1 = k_2$, then we have the special case

$$R(k_1,k_1;2) = R_2(k_2,2) \leq \binom{2k_1-2}{k_1-1} \sim c\frac{4^{k_1}}{\sqrt{k_1}}$$

The same type of argument can be used to establish the existance of $R(k_1, k_2, \ldots, k_n; 2)$. We have

Theorem 2.3 $R(k_1, k_2, \ldots, k_n; 2)$ exists and satisfies

$$R(k_1,k_2,\ldots,k_n;2) \leq \sum_{i=1}^{n} R(k_1,k_2,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_n;2)$$

Also since $\binom{k_1+k_2+\ldots+k_n-n}{k_1-1,k_2-1,\ldots,k_n-1}$ satisfies the same recurrence

and boundary conditions as $R(k_1,k_2,\ldots,k_n;2)$ we have

(2.2) $R(k_1, k_2, \ldots, k_n; 2) \le \binom{k_1+k_2+\ldots+k_n-n}{k_1-1, k_2-1, \ldots, k_n-1}$.

It follows from (2.2) that

(2.3) $$R_n(k,2) \le \frac{(nk-n)!}{((k-1)!)^n} .$$

We mention in passing that in Ramsay's original paper [7] , it was proved that $R_n(k,2)$ exists and that $R_n(k,2) \le f(n,k)$ where $f(1,k) = k$ by definition and $f(\ell,k) = (f(\ell-1,k))!$ It is not difficult to check that this upper bound is much larger than (2.3). We also mention for the sake of completeness, that another proof of the existance of $R_n(k,2)$ was given by T. Skolem in [10]. He obtained the following upper bound.

(2.4) $$R_n(k,2) \le \frac{n^{kn-n+2}-1}{n-1}$$

It is not difficult to check that the upper bound given by (2.4) is roughly the same as that given by (2.3).

The generalization of the argument to the case $t > 2$ is somewhat more involved. We consider first the problem of establishing the existance of $R(k_1, k_2; t)$. By theorem 2.2, we know that $R(\ell,k;2)$ exists for all $\ell,k \ge 2$. Also it is easy to see that $R(\ell,t;t)=R(t,\ell;t)=\ell$ for <u>all</u> $\ell \ge t$. We may therefore take as our induction hypothesis the existance of

(1) $R(k,\ell;t-1)$ for <u>all</u> $k,\ell \geq t-1$

(2) $R(k,k_2-1;t)$ for <u>all</u> $k \geq t$

(3) $R(k_1-1,\ell;t)$ for <u>all</u> $\ell \geq t$.

In particular, the induction hypothesis assures the existance of

$$R=R(R(k_1-1,k_2;t),R(k_1,k_2-1;t);t-1)+1$$

Let S be an s-set, $s \geq R$. Let $P_t(S)=C_1 \cup C_2$ be a partition of $P_t(S)$ into two clases C_1 and C_2. If we can show that there exists a k_1-subset $K_1 \subseteq S$ such that $P_t(K_1) \subseteq C_1$ or a k_2-subset $K_2 \subseteq S$ such that $P_t(K_2) \subseteq C_2$, then the existence of $R(k_1,k_2;t)$ will follow.

The argument used in the earlier theorems suggests that we select an arbitrary element $a \in S$ and consider the partition of the $(t-1)$-subsets of $S^*=S-[a]$ which is induced by the above partition of $P_t(S)$ in the following natural way: Partition $P_{t-1}(S^*)$ into two classes B_1 and B_2 by placing a member T of $P_{t-1}(S^*)$ in B_1 if $T \cup [a] \in C_1$ and in B_2 if $T \cup [a] \in C_2$.

S^* is an $(s-1)$-set. Since $s-1 \geq R-1$, it follows from the definition of R that <u>either</u> there is an $\ell_1 = R(k_1-1,k_2;t)$-subset L_1 of S^* such that $P_{t-1}(L_1) \subseteq B_1$ <u>or</u>

there is an $\ell_2 = R(k_1,k_2-1;t)$- subset L_2 of S^* such that $P_{t-1}(L_2) \subseteq B_2$.

If the first alternative holds, then since L_1 has $R(k_1-1,k_2;t)$ elements, either there is a (k_1-1)-subset K_1^* such that $P_t(K_1^*) \subseteq C_1$ or there is a k_2-subset K_2 such that $P_t(K_2) \subseteq C_2$, in which case we have finished. Hence we assume that there is a (k_1-1)-subset K_1^* such that $P_t(K_1^*) \subseteq C_1$. Let $K_1 = K_1^* \cup [a]$. Let T be any t-subset of K_1 . If $T \subset K_1^*$, then $T \in C_1$. If $T \not\subset K_1^*$, then $T = T^* \cup [a]$, where T^* is a (t-1)-subset of K_1^* . Hence T^* is a (t-1)-subset of L_1 and hence $T^* \in B_1$. But by the manner in which B_1 was constructed, $T^* \cup [a] \in C_1$, i.e. $T \in C_1$. Hence $P_t(K_1) \subseteq C_1$.

If the first alternative does not hold, then the second must, and the same argument applies. We have therefore proved the following theorem:

<u>Theorem 2.4</u>: $R(k_1,k_2;t)$ exists and satisfies the following recurrence inequality:

$$R(k_1,k_2;t) \leq R(R(k_1-1,k_2;t),\ R(k_1,k_2-1;t);t-1)+1 .$$

It is now easy to complete the proof of Theorem 1.1 by induction on n. We have just established the existence of $R(k_1,k_2;t)$. We take as our induction hypothesis the existence of $R(k_1,k_2,\ldots,k_{n-1};t)$, $n > 2$. Let $R = R(R(k_1,k_2,\ldots,k_{n-1};t),k_n;t)$. Let S be an s-set, $s \geq R$, and let $P_t(S) = C_1 \cup C_2 \cup \ldots \cup C_n$ be an arbitrary partition of $P_t(S)$ into n classes. Then either there exists a k_n subset K_n of S such that $P_t(K_n) \subseteq C_n$ in which case we have finished, or there exists an $R(k_1,k_2,\ldots,k_{n-1};t)$-subset L of S such that $P_t(L) \subseteq C_1 \cup C_2 \cup \ldots \cup C_{n-1}$. The induction hypothesis then implies that for some i, $1 \leq i \leq n-1$, there is a k_i-subset K_i of L (and consequently of S) such that $P_t(K_i) \subseteq C_i$. This completes the proof of Theorem 1.1.

We have as a corollary

$$R(k_1,k_2,\ldots,k_n;t) \leq R(R(k_1,k_2,\ldots,k_{n-1};t),k_n;t)+1 .$$

We note in conclusion that if $k_1 = k_2 = \ldots = k_n = 3$ in Theorem 2.3, then this implies that $R_n(3,2) \leq nR_{n-1}(3,2)$ and this leads to $R_n(3,2) \leq 3(n!)$. A slight improvement was obtained by Greenwood and Gleason in [5]. They proved that

$$R_n(3,2) \leq [n!e] + 1 . \tag{2.5}$$

Their argument is as follows:

Let T_n be the sequence defined by $T_1 = 2$; $T_n = n\, T_{n-1} + 1$, for $n \geq 2$. By induction it is easy to prove that

$$T_n = n! \sum_{k=0}^{n} \frac{1}{k!}$$

From this it follows that

$$T_n = [n!e] .$$

We now want to show that if the edges of a $< T_n + 1 >$ are colored in any way in n colors $c_1, c_2, \ldots, c_n$, then there results a monochromatic triangle. This is clearly true when $n = 1$. We assume that it holds for $n - 1$. Let v be a vertex of the $< T_n + 1 >$. There are $T_n = n\, T_{n-1} + 1$ edges incident with v and there are $\ell \geq T_{n-1} + 1$ of these edges with the same color c_n (say). Consider the $\binom{\ell}{2}$ edges joining in pairs the farther ends of the ℓ edges incident with v and colored c_n. If one of these edges is colored c_n, we have finished. If none of these edges are colored c_n, the induction hypothesis implies that there is a monochromatic triangle colored one of $c_1, c_2, \ldots, c_{n-1}$. This proves (2.5).

CHAPTER III

LOWER BOUNDS FOR THE RAMSAY NUMBERS

As we mentioned in the introduction, very little is known as to what is the order of magnitude of $R_n(k,t)$ and all existing upper and lower bounds are quite far apart. In this chapter we discuss the problem of finding lower bounds for $R_n(k,t)$. Some of the known results are presented and some new results are obtained.

§3.1 Recurrence Inequalities

In this section we prove some recurrence inequalities and from these derive lower bounds for $R_n(k,t)$. In [1], it is proved that

Theorem 3.1.1: For all positive integers n and m and fixed k

$$(3.1.1) \qquad R_{n+m}(k,2) \geq (R_n(k,2)-1)(R_m(k,2)-1) + 1 .$$

Proof: For notational convenience, let $p = R_{n+m}(k,2)-1$, $q = R_n(k,2)-1$ and $r = R_m(k,2)-1$. We now have to show that $p \geq qr$. Let $P_1, P_2, \ldots, P_q$ be the vertices of a $<q>$. Color the edges of the $<q>$ in n colors $c_1, c_2, \ldots, c_n$ in such a way that there does not result a M.C. $<k>$. This can be done by the definition of q. Let $<r>_i$, $1 \leq i \leq q$ have vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_r}$. Color the edges of each $<r>_i$ in m

colors $c_{n+1}, c_{n+2}, \ldots, c_{n+m}$ without forcing the appearance of a M.C. $\langle k\rangle$. Let $\langle qr\rangle$ be the graph with vertices P_{ij}, $1 \leq i \leq q$, $1 \leq j \leq r$. Let $E = (P_{st}, P_{uv})$ be an edge of $\langle qr\rangle$. If $s \neq u$, color E the same color as edge (P_s, P_u) in $\langle q\rangle$. If $s = u$, color E the same as it is colored in $\langle r\rangle_s$. Suppose $P_{i_1 j_1}, P_{i_2 j_2}, \ldots, P_{i_k j_k}$ are the vertices of a M.C. $\langle k\rangle$.

<u>Case I</u>: If $i_1 = i_2 = \ldots = i_k$, then $P_{i_1 j_1}, P_{i_2 j_2} \ldots P_{i_k j_k}$ are the vertices of a M.C. $\langle k\rangle$ in $\langle r\rangle_{i_1}$, which is a contradiction.

<u>Case II</u>: If $i_1, i_2, \ldots, i_k$ are all different, then the edges of the $\langle k\rangle$ are colored the same as those of the $\langle k\rangle$ in $\langle q\rangle$ whose vertices are $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$. This again is a contradiction.

<u>Case III</u>: If $i_s = i_t \neq i_u$. Then the edge $(P_{i_s j_s}, P_{i_t j_t})$ is colored one of $c_{n+1}, c_{n+2}, \ldots, c_{n+m}$ while the edge $(P_{i_s j_s}, P_{i_u j_u})$ is colored one of $c_1, c_2, \ldots, c_n$. This is also impossible. Hence $p \geq qr$.

It follows easily from (3.1.1) and the fact that $R_1(k,2) = k$, that

$$R_n(k,2) \geq (k-1)^n . \tag{3.1.2}$$

This lower bound for $R_n(k,2)$ is substantially smaller than the upper bound given by (2.4). We shall obtain some better lower bounds in sections 2 and 3 of this chapter. However, we can use theorem 3.1.1 to gain a little more insight into the behaviour of $R_n(k,2)$. We prove that for every fixed k ,

$$\lim_{n \to \infty} R_n(k,2)^{1/n} .$$

exists.

To prove this, let $h(n) = R_n(k,2)-1$. Then we have by theorem 3.1.1,

$$h(n+m) \geq h(n)h(m) .$$

This implies

(3.1.3) $$h(ab) \geq h(b)^a.$$

Let $\alpha = \liminf_{n \to \infty} h(n)^{1/n} \leq \limsup_{n \to \infty} h(n)^{1/n} = \beta$. Suppose first $\beta < \infty$. Let $\varepsilon > 0$ be given and let b be the least integer for which

(3.1.4) $$h(b)^{1/b} > \beta - \varepsilon .$$

If $n = ab$, we have

$$h(n)^{1/n} = h(ab)^{1/ab} \geq (h(b)^a)^{1/ab} = h(b)^{1/b} > \beta - \varepsilon ,$$

where we used (3.1.3) and (3.1.4). Let $n = ab + r$ where $1 \leq r \leq b-1$. Then

$$h(n) = h(ab + r) \geq h(ab)h(r) \geq h(ab)$$

and hence

$$h(n)^{1/n} \geq h(ab)^{1/ab+r} = h(ab)^{\frac{1}{ab}\left(\frac{1}{1+r/ab}\right)} > (\beta - \varepsilon)^{\frac{1}{1+r/ab}} .$$

Hence $\alpha \geq \beta - \varepsilon$. Since ε is arbitrary , $\alpha = \beta$. The case $\beta = \infty$ can be disposed of in the same way. Let N be a positive number and let b be the least integer such that $h(b)^{1/b} > N$. The argument used above then shows that $\alpha \geq N$ and hence $\alpha = \infty$.

We cannot decide whether the above limit is finite or infinite.

One can now ask whether $R_n(k,t)$ satisfies the same recurrence inequality as $R_n(k,2)$. We cannot decide this, but we prove:

<u>Theorem 3.1.2</u>: For all positive integers n and m

$$R_{n+m+1}(k,t) \geq (R_n(k,t)-1)(R_m(k,t)-1) + 1$$

provided $k \geq (t - 1)^2 + 1$.

<u>Proof</u>: For notational convenience, set $R_n(k,t)-1 = h(n)$. We now have to prove that $h(n+m+1) \geq h(n)h(m)$. Let S be an $h(n)$-set with elements $a_1, a_2, \cdots, a_{h(n)}$. Partition $P_t(S)$ into n classes $C_1, C_2, \cdots, C_n$ in such a way that if K is a k-subset of S , then not all t-subsets of K belong to the same class. This is possible by the

definition of $h(n)$. For $j = 1,2,\ldots,h(n)$, let S_j be an $h(m)$-set with elements $a_{j_1}, a_{j_2}, \ldots, a_{j_{h(m)}}$. Partition each $P_t(S_j)$ into m classes $C_{1_j}, C_{2_j}, \ldots, C_{m_j}$ in such a way that if K is a k-subset of S_j, then not all t-subsets of K belong to the same class. Let $R = \bigcup_{j=1}^{h(n)} S_j$. Then R has $h(n)h(m)$ elements. Partition $P_t(R)$ into $n+m+1$ classes $B_1, B_2, \ldots, B_n, B_{n+1}, \ldots, B_{n+m}, B_{n+m+1}$ as follows: Let T be a member of $P_t(R)$. If $T \subset S_j$, put T in $B_{n+\ell}$ if $T \in C_{\ell_j}$. If T is distributed over exactly t of the S_j, say $S_{j_1}, S_{j_2}, \ldots, S_{j_t}$, then $T = \{a_{j_1 i_1}, a_{j_2 i_2}, \ldots, a_{j_t i_t}\}$ say. Put T in B_ℓ if the set $T' = \{a_{j_1}, a_{j_2}, \ldots, a_{j_t}\} \in C_\ell$. If T is distributed over $r (1 < r \leq t-1)$ of the S_j, put T in B_{n+m+1}. The proof will be complete if we show that there does not exist a k-subet K of R such that $P_t(K) \subseteq B_i$ for some i, $1 \leq i \leq n+m+1$.

<u>Case I</u>: $K \subset S_j$. It is then obvious that the desired result holds.

<u>Case II</u>: If K is distributed over exactly t of the S_j, then there is at least one t-subset T of K distributed over exactly t of the S_j. Then $T \in B_\ell$ for some $\ell \leq n$. Since $k > t$, there is a t-subset T_1 of K which has at least two elements in one of the S_j and at least one element in some S_t, $t \neq j$. Hence $T_1 \in B_{n+m+1}$.

<u>Case III</u>: If K is distributed over $r (1 < r \le t-1)$ of the S_j, then since $k \ge (t-1)^2 + 1$, there must be at least $k/r > (t-1)^2/r + 1/r \ge (t-1) + 1/r$, and hence at least t elements of K in one of the S_j. Thus there is a t-subset T of K such that $T \subset S_j$. Then $T \in B_{n+\ell}$, for some ℓ, $1 \le \ell \le m$. However, there must be another t-subset of K which belongs to B_{n+m+1}. This completes the proof of theorem 3.1.2.

In [1], it is proved that

$$(3.1.5) \qquad R_n(k\ell-k-\ell+2,2) \ge (R_n(k,2)-1)(R_n(\ell,2)-1) + 1 .$$

One can now ask whether this result can be generalized to the case $t > 2$. We have not been able to do this. However, we prove:

<u>Theorem 3.1.3</u>:

$$(3.1.6) \qquad R_{n+1}(k\ell-k-\ell+2,t) \ge (R_n(k,t)-1)(R_n(\ell,t)-1) + 1 .$$

<u>Proof</u>: For notational convenience, put $h_n(k) = R_n(k,t)-1$. We then have to prove

$$(3.1.7) \qquad h_{n+1}(k\ell-k-\ell+2) \ge h_n(k)h_n(\ell) .$$

Let S be an $h_n(\ell)$-set with elements $a_1, a_2, \ldots, a_{h_n(\ell)}$. Partition $P_t(S)$ into classes $C_1, C_2, \ldots, C_n$ so that if L is an ℓ-subset of S, then not all t-subsets of L belong to the same class. For $j = 1,2,\ldots,h_n(\ell)$, let S_j be an $h_n(k)$-set with elements $a_{j_1}, a_{j_2}, \ldots, a_{j_{h_n(k)}}$. Partition each $P_t(S_j)$ into n classes $C_{1_j}, C_{2_j}, \ldots, C_{n_j}$, so that if K_j is a k-subset of S_j, then not all t-subsets of K_j belong to the same class. Let $W = \bigcup_{j=1}^{h_n(\ell)} S_j$. Note that W is an $h_n(\ell)h_n(k)$-set. Partition $P_t(W)$ into $n+1$ classes $B_1, B_2, \ldots, B_{n+1}$ as follows: Let $T \in P_t(W)$. Firstly, if $T \subset S_j$ for some j, then $T \in C_{s_j}$ for some s, $1 \leq s \leq n$. Put T in B_s. Secondly, if T is distributed over exactly t of the S_j, say $S_{j_1}, S_{j_2}, \ldots, S_{j_t}$, then $T = \{a_{j_1 i_1}, \ldots, a_{j_t i_t}\}$ say. If the set $T' = \{a_{j_1}, a_{j_2}, \ldots, a_{j_t}\}$ belongs to C_s, put T in B_s. Finally, if T is distributed over $r (1 < r < t)$ of the S_j, put T in B_{n+1}.

The proof of (3.1.7) will be complete if we show that if M is a subset of W with $k\ell-k-\ell+2$ elements, then $P_t(M) \subseteq B_i$ is false for all i, $1 \leq i \leq n+1$.

<u>Case I</u>: M is distributed over $r \geq \ell$ of the S_j. Suppose $P_t(M) \subseteq B_s$ for some s, $1 \leq s \leq n$. Then M must contain an ℓ-subset L which is distributed over exactly ℓ of the S_j, say $S_{j_1}, S_{j_2}, \ldots, S_{j_\ell}$. Let $L = \{a_{j_1 i_1}, a_{j_2 i_2}, \ldots, a_{j_\ell i_\ell}\}$. Then the set $L' = \{a_{j_1}, a_{j_2}, \ldots, a_{j_\ell}\}$ is a subset of S and the condition $P_t(L) \subseteq B_s$ implies $P_t(L') \subseteq C_s$.

This is impossible. It is also clear that $P_t(M) \subseteq B_{n+1}$ cannot occur since there is at least one t-subset of M which is distributed over at least t of the S_j.

<u>Case II</u>: M is distributed over $r \leq \ell-1$ of the S_j, say $S_{j_1}, S_{j_2}, \ldots, S_{j_r}$. Then there is a subset K of M with at least k elements such that $K \subset S_{j_s}$ for some s, $1 \leq s \leq r$, since otherwise the number of elements of M would not exceed $r(k-1) \leq (\ell-1)(k-1) < k\ell-k-\ell+2$. Suppose $P_t(M) \subseteq B_i$, for some i, $1 \leq i \leq n$. Then $P_t(K) \subseteq B_i$. But this implies $P_t(K) \subseteq C_{i_{j_s}}$. This is a contradiction. Also it cannot occur that $P_t(M) \subseteq B_{n+1}$, since this would imply $P_t(K) \subseteq B_{n+1}$. This is obviously impossible since it indicates that every t-subset of K is distributed over at least two of the S_j, contradicting the fact that $K \subset S_{j_s}$. This completes the proof of (3.1.7).

§3.2. <u>Probabilistic Arguments</u>

A lower bound for $R_n(k,t)$ was obtained by Erdös in [2] who proved by a probabilistic argument that:

<u>Theorem 3.2.1</u>:

(3.2.1) $$\binom{R_n(k,t)}{k} \geq n^{\binom{k}{t}-1}$$

<u>Proof</u>: Let S be an s-set and let f(s) be the number of ways of partitioning $P_t(S)$ into n classes such that for

each such partitioning, there exists a k-subset $K \subseteq S$ such that $P_t(K)$ is contained in one of the n classes. The total number of ways of partitioning $P_t(S)$ into n classes is $n^{\binom{s}{t}}$. Hence the probability that for a given partitioning there exists a k-subset $K \subset S$ such that $P_t(K)$ is contained in one of the n classes is $f(s)/n^{\binom{s}{t}}$. We need therefore

$$(3.2.2) \qquad f(s) < n^{\binom{s}{t}} .$$

Since the number of k-subsets of an s-set is $\binom{s}{k}$, we have

$$f(s) \leq n\binom{s}{k} n^{\binom{s}{t}-\binom{k}{t}}$$

where $n^{\binom{s}{t}-\binom{k}{t}}$ is the number of ways of partitioning the remaining t-subsets. Now (3.2.2) will be satisfied if

$$n\binom{s}{k} n^{\binom{s}{t}-\binom{k}{t}} < n^{\binom{s}{t}}$$

or if

$$(3.2.3) \qquad \binom{s}{k} < n^{\binom{k}{t}-1} .$$

If s is any integer satisfying (3.2.3), then there does exist some way of partitioning $P_t(S)$ into n classes such that no k-subset of S has all of its t-subsets in one of the n classes. This completes the proof of (3.2.1).

If $n = t = 2$ in (3.2.1), then we have

(3.2.4) $$\binom{R_2(k,2)}{k} \geq 2^{\binom{k}{2}-1} .$$

(3.2.4) yields

(3.2.5) $$R_2(k,2) \geq ck2^{k/2}$$

for some constant c and all sufficiently large k .

It also follows from Theorem 3.1.1 that

(3.2.6) $$\begin{aligned} R_{2n}(k,2)-1 &\geq (R_2(k,2)-1)^n \\ R_{2n+1}(k,2)-1 &\geq (R_1(k,2)-1)(R_2(k,2)-1)^n . \end{aligned}$$

It is not difficult to see that (3.2.6) and (3.2.5) yield a better lower bound for $R_n(k,2)$ than that given by (3.2.1).

Probability arguments have been used by Erdős to obtain lower bounds for other classes of Ramsay numbers, especially the numbers $R(3,k;\ 2)$.

The best result that has been obtained up to the present time is

$$R(3,k;\ 2) \geq ck^2/(\log k)^2$$

for some constant c and all sufficiently large k . For the proof of this result and further references to the literature see [12].

§3.3. An Algebraic Approach

In this section we obtain by an algebraic method a lower bound for $R_n(k,2)$ which is better than that given by (3.1.2), and also better than that given by (3.2.1) provided k is small and n is large compared to k.

Consider the following system (S) of $\binom{k-1}{2}$ equations in $\binom{k}{2}$ unknowns:

$$x_{i,j} + x_{j,j+1} = x_{i,j+1}, \quad 1 \leq i < j \leq k-1 .$$

Suppose there exists some way of partitioning the numbers $1, 2, \ldots, m$ into n sets $A_1, A_2, \ldots, A_n$, no set containing a solution of (S). Let G be the complete graph with vertices $P_0, P_1, P_2, \ldots, P_m$. Color the edges of G in n colors $c_1, c_2, \ldots, c_n$ by coloring the edge P_iP_j color c_r if $|i - j| \in A_r$. In order to see that G contains no M.C. <k>, let $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$, $i_1 > i_2 > \ldots > i_k$, be the vertices of a <k> in G, and suppose all interconnecting edges are colored c_r. Then $i_t - i_s \in A_r$ for $1 \leq t < s \leq k$. But $(i_t - i_s) + (i_s - i_{s+1}) = (i_t - i_{s+1})$, $1 \leq t < s \leq k-1$. Hence we have a solution to system (S) in A_r. This is a contradiction. Hence G contains no complete M.C. <k>.

It follows from the above argument that

(3.3.1) $$R_n(k,2) \geq m + 2 .$$

If we define $t(n,k)$ to be the largest integer for which there exists some way of partitioning the numbers $1, 2, \ldots, t(n,k)$ into n sets, no set containing a solution of (S), then by (3.3.1) we have

$$(3.3.2) \qquad R_n(k,2) \geq t(n,k) + 2 .$$

We have thus translated the problem of finding lower bounds for $R_n(k,2)$ into the problem of finding lower bounds for $t(n,k)$.

We define a function g as follows: If $t(n-1,k) < m \leq t(n,k)$, then $g(m,k) = n$. $g(m,k)$ is thus the smallest number of sets into which the integers $1, 2, \ldots, m$ can be partitioned, no class containing a solution of (S).

In [1] it is proved that $g(m,3) < \log m$ for all sufficiently large m. Since g is a decreasing function of k, we have

$$(3.3.3) \qquad g(m,k) < \log m .$$

In fact, one can show that $g(m,k) < (1 + \varepsilon) \dfrac{\log m}{\log k}$ for every $\varepsilon > 0$, $m \geq m_0(\varepsilon)$, but (3.3.3) is sufficient in what follows.

Now we prove

<u>Theorem 3.3.1</u>. For all positive integers p and q

(3.3.4) $\quad t(pq + g(pt(q,k),k),k) \geq (2t(q,k) + 1)^p - 1$

<u>Proof</u>: For notational convenience, let $X = 2t(q,k) + 1$. Write the numbers $1, 2, \ldots, X^p-1$ in base X. We distinguish these numbers as follows: The set of numbers each of whose digits $\leq t(q,k)$ is denoted by N_1. The set of numbers, each of which has at least one of its digits at least $t(q,k) + 1$ is denoted by N_2. We shall split the set N_1 into $g(pt(q,k),k)$ sets and the set N_2 into pq sets, no set containing a solution of (S). The proof of the theorem shall then be complete.

Let $C_1, C_2, \ldots, C_{g(pt(q,k),k)}$ be sets containing $1, 2, \ldots, pt(q,k)$, no set containing a solution of (S). We partition the set N_1 into sets $A_1, A_2, \ldots, A_{g(pt(q,k),k)}$ by putting a number in A_j if the sum of its digits belongs to C_j, i.e. put $a = a_1 + a_2X + a_3X^2 + \ldots + a_pX^{p-1}$ in A_j if $\sum_{i=1}^{p} a_i \in C_j$. This can be done since $\sum_{i=1}^{p} a_i \leq pt(q,k)$. Then A_j contains no solution of (S) because C_j does not.

For $1 \leq r \leq p$ let B_r be the set of all numbers $a = a_1 + a_2X + \ldots + a_rX^{r-1} + \ldots + a_pX^{p-1}$ satisfying $a_i \leq t(q,k)$ for $i = 1, 2, \ldots, r-1$ and $a_r \geq t(q,k) + 1$. The set N_2 has thus been partitioned into sets

$B_1, B_2, \ldots, B_p$. We now partition each B_r into q sets as follows: Let $D_1, D_2, \ldots, D_q$ be disjoint sets containing $1, 2, \ldots, t(q,k)$, no D_i containing a solution to (S). Let $a \in B_r$. Then $a_r = X - (a_r)^1$ where $1 \le (a_r)^1 \le t(q,k)$. Put $a \in E_{r_m}$ if $(a_r)^1 \in D_m$. Then B_r is partitioned into q sets $E_{r_1}, E_{r_2}, \ldots, E_{r_q}$. Hence the set N_2 has now been partitioned into pq sets.

Suppose E_{r_m} contains a solution of (S), i.e. there are numbers $Z_{i,j}$ in E_{r_m} such that

$$(3.3.5) \qquad Z_{i,j} + Z_{j,j+1} = Z_{i,j+1}, \quad 1 \le i \le j \le k-1.$$

Let

$$Z_{i,j} = (a_{i,j})_1 + (a_{i,j})_2 X + \ldots + (a_{i,j})_r X^{r-1} + \ldots + (a_{i,j})_p X^{p-1}.$$

(3.3.5) implies that

$$(a_{i,j})_r + (a_{j,j+1})_r = (a_{i,j+1})_r + X$$

and this in turn implies

$$(3.3.6) \qquad X - (a_{i,j})_r^1 + X - (a_{j,j+1})_r^1 = X - (a_{i,j+1})_r^1 + X,$$

where $(a_{i,j})_r^1, (a_{j,j+1})_r^1, (a_{i,j+1})_r^1 \in D_m$. But from (3.3.6) we have

$$(a_{i,j})_r^1 + (a_{j,j+1})_r^1 = (a_{i,j+1})_r^1 .$$

That is, we have a solution of (S) in D_m. This is a contradiction. Hence E_{r_m} does not contain a solution of (S). This completes the proof of Theorem 3.3.1.

If we set $q = 1$ in (3.3.4) and use the easily established fact that $t(1,k) = k - 2$ we get

$$(3.3.7) \qquad t(p + g(p(k-2),k),k) \geq (2k - 3)^p - 1.$$

Let k be arbitrary but fixed. Then it follows from (3.3.7) (3.3.3) and (3.3.2) that

$$(3.3.8) \qquad R_n(k,2) > (2k - 3)^{n(1-\varepsilon)}$$

for every $\varepsilon > 0$ and $n \geq n_0(k,\varepsilon)$. This result is clearly better than (3.1.2).

In the immediately preceding argument we chose $q = 1$. However there is nothing to prevent us from choosing larger values of q to get still better results for certain values of k . We illustrate this in the cases $k = 3, 4$.

If $k = 3$ in (3.3.8) we get

$$(3.3.9) \qquad R_n(3,2) > 3^{n(1-\varepsilon)} .$$

Let $k = 3$ and $q = 4$ in (3.3.4). This gives

$$(3.3.10) \qquad t(4p + g(pt(4,3),3),3) \geq (2t(4,3) + 1)^p - 1.$$

It is known (L. D. Baumert, unpublished, see [1]) that $t(4,3) = 44$. This with (3.3.10), (3.3.3) and (3.3.2) yields

$$R_n(3,2) > 89^{n/4(1-\varepsilon)}$$

for every $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$.

If $k = 4$ in (3.3.8) we get

$$R_n(4,2) > 5^{n(1-\varepsilon)} . \tag{3.3.11}$$

This can be improved by taking $k = 4$ and $q = 2$ in (3.3.4).

We observe first that $t(2,4) \geq 16$. This follows from the fact that the numbers 1, 2, ..., 16 can be split into two sets

$$C_1 = \{1, 2, 4, 8, 9, 13, 15, 16\}$$
$$C_2 = \{3, 5, 6, 7, 10, 11, 12, 14\}$$

neither of the sets containing a solution of the system

$$x_{12} + x_{23} = x_{13}$$

$$x_{13} + x_{34} = x_{14}$$

$$x_{23} + x_{34} = x_{24} .$$

Thus from (3.3.4), (3.3.3) and (3.3.2) we get

$$R_n(4,2) > 33^{n/2(1-\varepsilon)}$$

for every $\varepsilon > 0$ provided $n \geq n_0(\varepsilon)$.

Other results of this type can be obtained but we do not discuss these any further here.

CHAPTER IV

SOME EXACT VALUES FOR THE RAMSAY NUMBERS

The problem of determining $R=R(k_1,k_2,\ldots,k_n;t)$ appears to be a very difficult one. No value of R is known for $t>2$. In fact, the values of $R_1=R(k_1,k_2,\ldots,k_n;t)$ are not known for $n\geq 4$. Even for $n<4$, the values of R_1 have only been established for small values of k_i.

In this chapter, we shall give some of the techniques used in evaluating the known values of the Ramsay numbers.

For notational convenience, in this chapter we shall denote a monochromatic $<k>$ of color c_i, by the symbol $c_i<k>$.

The first evaluation we give is the following:

Theorem 4.1:

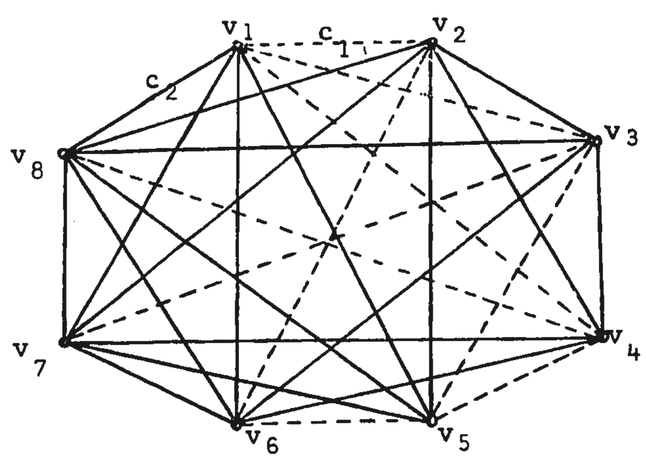
$$(4.1) \qquad R(3,4;2)=9$$

Proof: We prove first that if the edges of a $<9>$ are colored arbitrarily in two colors c_1 and c_2, then there will result either a $c_1<3>$ or a $c_2<4>$. This will show that $R(3,4;2)\leq 9$.

Let v be a vertex of the $<9>$. Let n_1 of the edges incident with v have color c_1 and n_2 of these edges have color c_2. $(n_1+n_2=8)$. We suppose first that $n_1 \geq 4$. If one of the edges joining in pairs the farther ends of these n_1 edges incident with v, is colored c_1, then we have a $c_1<3>$. Otherwise, we have a $c_2<4>$. Hence we may assume that $n_1 \leq 3$.

Suppose that $n_i = 3$ at every vertex of the $<9>$. Then the number of edges colored c_i is $(9)(3)/2$ which is impossible. Hence without loss of generality we may assume that $n_1 \leq 2$. Hence $n_2 \geq 6$. Consider now the edges jcining in pairs the farther end of these $n_2 \geq 6$ edges incident with v. By Theorem 2.1, these must yield either a $c_1<3>$ or a $c_2<3>$. Hence in the $<9>$, there is either a $c_1<3>$ or a $c_2<4>$. Therefore, $R(3,4;2) \leq 9$.

That $R(3,4;2) > 8$, follows from the fact that in the graph sketched below, there is no $c_1<3>$ and no $c_2<4>$.



Hence $R(3,4;2)=9$.

We now prove

Theorem 4.2 R(3,5;2)=14.

Proof: From (2.1) and (4.1), and the fact that R(2,k;2)=k for all $k \geq 2$, we have

(4.2) $R(3,5;2) \leq R(2,5;2) + R(3,4;2) = 14,$

Hence we need to show that R(3,5;2)>13. To do this we must show how to color the edges of a <13> in two colors c_1 and c_2 without forcing the appearance of c_1<3> or a c_2<5> .

Consider the field F of residue classes modulo 13. F={0,1,2,...,12}. Let H={1,5,8,12}. H is a subgroup of the multiplicative group of F. The cosets of H are H_1={2,3,10,11} and H_2={4,6,7,9}. Let the vertices of the <13> be P_0, P_1, ..., P_{12}. The edge P_iP_j is colored c_1 if $i-j \in H$, and colored c_2 if $i-j \in H_1 \cup H_2$. Suppose there results a c_1<3>, with vertices P_i, P_j, P_k. Then $i-j, j-k, i-k \in H$. But $(i-j)+(j-k)=i-k$. This is a contradiction since the sum of any two elements of H is not in H. Hence there is no c_1<3>.

Suppose there results a c_2<5> with vertices $P_{v_1}, P_{v_2}, \ldots, P_{v_5}$. Then $v_i - v_j$, $1 \leq i < j \leq 5$, are all in $H_1 \cup H_2$. Suppose $v_5 \neq 0$. Set $w_i = v_i - v_5$. Then $w_i - w_j = v_i - v_j$, $w_5 = 0$, and

$w_i - w_j$ are all in $H_1 \cup H_2$. Consider w_1, w_2, w_3, w_4. The 3-subsets of H_1 are (2,3,10),(2,3,11),(2,10,11),(3,10,11). In each of these there is a difference equal to 8, i.e. there is a difference in H. Hence at most two of w_1, w_2, w_3, w_4 belong to H_1.

Similiarly, at most two of w_1, w_2, w_3, w_4 belong to H_2. Hence exactly two (say) w_1, w_2 belong to H_1 and the other two, w_3, w_4 belong to H_2. Now $(w_1, w_2) \neq (2,3), (2,10), (3,11)$ or $(10,11)$ since $3-2=10-11=1 \varepsilon H$ and $10-2=11-3=8 \varepsilon H$. Hence $(w_1, w_2)=(2,11)$ or $(3,10)$. Suppose $(w_1, w_2)=(2,11)$. Then w_3 and w_4 are different from 6 since $11-6=5 \varepsilon H$; also w_3 and w_4 are different from 7 since $7-2=5 \varepsilon H$. Hence $(w_3, w_4)=(4,9)$. But this contradicts $9-4=5 \varepsilon H$. Hence $(w_1, w_2) \neq (2,11)$. The same argument shows $(w_1, w_2) \neq (3,10)$. Hence there is no c_2<5>. Hence $R(3,5;2)>13$ and this with (4.2) completes the proof of the theorem.

Another evaluation is given by the following theorem:

<u>Theorem 4.3</u>: $R(4,4;2)=18$.

<u>Proof</u>: From (2.1) and (4.1) we have

(4.3) $$R(4,4;2) \leq R(3,4;2) + R(4,3;2) = 18.$$

To show that $R(4,4;2)>17$ we must show how to color the edges of a <17> in two colors c_1 and c_2, without forcing the appearance of either a $c_1\langle 4\rangle$ or a $c_2\langle 4\rangle$. Consider the field F of residue classes modulo 17. Let $H=\{1,2,4,8,9,13,15,16\}$. H is a subgroup of the multiplicative group of F. (H is in fact the set of quadratic residues of 17.) The coset of H is $H_1=\{3,5,6,7,10,11,12,14\}$. Let the vertices of the $\langle 17\rangle$ be labeled $P_0,P_1,\ldots,P_{16}$. The edge P_iP_j is colored c_1 if $i-j\epsilon H$ and colored c_2 if $i-j\epsilon H_1$. Suppose there results a $c_1\langle 4\rangle$ with vertices $P_{u_1},P_{u_2},P_{u_3},P_{u_4}$. Then $u_i-u_j\epsilon H, 1\leq i<j\leq 4$. Set $v_i=u_i-u_4$. Then $v_i-v_j=u_i-u_j$, $v_i-v_j\epsilon H$, and $v_4=0$. Set $x_i=(1/v_3)v_i$. Then $x_3=1$ and $x_1-x_2, x_1-1, x_1, x_2-1, x_2, 1\epsilon H$. Now $x_1, x_1-1\epsilon H$ implies that $x_1\neq 1,4,8,13,15$, and $x_2, x_2-1\epsilon H$ implies that $x_2\neq 1,4,8,13,15$. Hence $\{x_1,x_2\}\subset\{2,9,16\}$. But since $9-2=7$, $16-2=14$ and $16-9=7$, this contradicts the fact that $x_1-x_2\epsilon H$. Hence there is no $c_1\langle 4\rangle$.

Now we assume that there exists a $c_2\langle 4\rangle$, with vertices $P_{u_1},P_{u_2},P_{u_3},P_{u_4}$. Then $u_i-u_j\epsilon H_1, 1\leq i<j\leq 4$. Let $a\epsilon H_1$ and let $v_i=a\,u_i$. Then $v_i-v_j=a(u_i-u_j)\epsilon H$. Hence $P_{v_1},P_{v_2},P_{v_3},P_{v_4}$ are the vertices of a $c_1\langle 4\rangle$. But this contradicts the first part of the proof and hence there is no $c_2\langle 4\rangle$. Hence $R(4,4;2)>17$ and this with (4.3) proves the theorem.

Using an argument similar to that used in the proof of theorem 4.3, we have obtained the following:

<u>Theorem 4.4</u>: $R(5,5;2) \geq 38$; $R(6,6;2) \geq 90$ and $R(7,7;2) \geq 110$.

<u>Proof</u>: To show that $R(5,5;2) \geq 38$, we let G be a complete graph with vertices $P_0, P_1, \ldots, P_{36}$. Color the edges of G in two colors c_1 and c_2 by coloring the edge P_iP_j color c_1 if $i-j$ is a quadratic residue of 37, and color c_2 if $i-j$ is a quadratic non-residue of 37. Then it is not difficult to check that G contains no monochromatic <5>. The same type of argument using the primes 89 and 109 can be used to show $R(6,6;2) \geq 90$ and $R(7,7;2) \geq 110$ but the details are naturally somewhat more involved.

Finally we prove

<u>Theorem 4.5</u>: $R_3(3,2) = 17$

<u>Proof:</u> From (2.5) we have $R_n(3,2) \leq [n!e] + 1$. Hence $R_3(3,2) \leq [3!e] + 1 = [6e] + 1 = 17$. To show that $R_3(3,2) > 16$, we use the following argument: Let F be the field of residue classes modulo 2. Adjoint to F the indeterminate t satisfying the equation $t^4 = t+1$. This yields the field $F[t]$ consisting of the elements:

$$\{0,1,t,t+1,t^2,t^2+1,t^2+t,t^2+t+1,t^3,t^3+1,t^3+t,t^3+t+1,$$
$$t^3+t^2,t^3+t^2+1,t^3+t^2+t,t^3+t^2+t+1\}.$$

Let H be the multiplicative group of $F[t]$.

$H_1 = \{1,t^3,t^3+t^2,t^3+t,t^3+t^2+t+1\}$ is a subgroup of H.

$H_2 = \{t, t+1, t^3+t+1, t^2+t+1, t^3+t^2+1\}$ and

$H_3 = \{t^2, t^2+t, t^2+1, t^3+t^2+t, t^3+1\}$ are the cosets of H_1 in H. Consider a <16> with vertices labeled $v_1, v_2, \ldots, v_{16}$ where $v_i \in H$. Color the edge $(v_i v_j)$ color c_ℓ if $v_i + v_j \in H_\ell$, $\ell = 1,2,3$. If v_i, v_j, v_k are the vertices of a M.C <3>, then $v_i+v_j, v_i+v_k, v_j+v_k$ all belong to one of H_1, H_2 or H_3. But $(v_i+v_j) + (v_j+v_k) = v_i + v_k$. This is a contradiction since the sum of any two elements in either H_1, H_2 or H_3 is not in the same set. Hence there does not exist a M.C. <3>. Hence $R_3(3,2) > 16$ and the theorem is complete.

All of the above results were obtained by Greenwood and Gleason [5]. Other values of the Ramsay numbers have been obtained by Kalbfleisch [6]. His arguments do not involve finite fields, and it seems unlikely that any new results can be obtained using the methods used above.

CHAPTER V

SOME APPLICATIONS OF RAMSAY'S THEOREM

In this Chapter we discuss some of the applications of Ramsay's Theorem to various problems, in particular, to a problem in Set Theory; to a problem in Geometry, and to a problem in Matrix Theory.

§5.1 An application to a problem in Set Theory.

A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}$ if for every $F \in \mathcal{F}$, there exists a set $B \subset \cup F$ such that $F \not\subset B$ and $F \cap B \neq \phi$.

Erdös and Hajnal in [4] asked the following question: What is the smallest integer m(n) for which there exists a family $\mathcal{F}_n$ of sets $A_1, A_2, \ldots, A_{m(n)}$ such that $|A_i| = n$ for $1 \leq i \leq m(n)$ and which does not possess property $\mathcal{B}$? They observed that m(1)=1, m(2)=3, m(3)=7 and that $m(n) \leq \binom{2n-1}{n}$. The value of m(n) is not known for $n \geq 4$, and the problem of determining m(n), even for n=4 appears to be difficult.

Erdös proved in [3], that for all $n \geq 2$

$$m(n) > 2^{n-1} \qquad (5.1.1)$$

Various improvements in the upper and lower bounds for m(n) have been given, but we do not discuss these here. We mention only that the best known lower bound for m(n) is

$$m(n) > 2^n \left(\frac{n}{n+4}\right)$$

which was obtained by Schmidt [9].

In [4], Erdös and Hajnal also asked: Does there exist for every positive integer $k \geq 2$ a finite family $\mathcal{F}_k$ of finite sets satisfying:

(1) $|F| = k$ for each $F \in \mathcal{F}_k$.

(2) $|F \cap G| \leq 1$ for $F, G \in \mathcal{F}_k$, $F \neq G$,

(3) $\mathcal{F}_k$ does not possess property $\mathcal{B}$.?

They observed that such families do exist for k=2,3. Abbott proved in [1] that such families exist for every positive integer k, by making use of a special case of Ramsay's Theorem.

<u>Theorem 5.1.1</u>: Let S be an s-set, $s \geq R_2(k,t)$ and let K be a k-subset of S. Let F denote the set of all t-subsets of K and let $\mathcal{F}_{k,t}$ denote the family of all possible sets constructed in this way. Then $\mathcal{F}_{k,t}$ does not possess property $\mathcal{B}$.

<u>Proof:</u> Assume that $\mathcal{F}_{k,t}$ does possess property $\mathcal{B}$. Then there exists a set $B \subset \bigcup \mathcal{F}_{k,t}$ such that $B \cap F \neq \phi$

and $F \not\subset B$ for each $F \in \mathcal{F}_{k,t}$. Partition $P_t(S)$ into two classes A_1, A_2 by placing a t-subset T of S in A_1 if $T \in B$ and in A_2 if $T \notin B$. Let K be any k-subset of S and let F be the corresponding member of $\mathcal{F}_{k,t}$. Then since $B \cap F \neq \phi$, there is a t-subset of K which belongs to B and hence to A_1, and since $F \not\subset B$, there is a t-subset of K which does not belong to B and hence belongs to A_2. However, since $s \geq R_2(k,t)$, there must exist some k-subset of S all of whose t-subsets belong to either A_1 or A_2. This is a contradiction and the proof of the theorem is complete.

Since $\mathcal{F}_{k,k-1}$ satisfies conditions (1),(2) and (3), therefore the question of Erdös & Hajnal is settled.

If we choose $s = R_2(k,t)$ in the above theorem, then the number of sets in the family $\mathcal{F}_{k,t}$ is $\binom{R_2(k,t)}{k}$ and the number of elements in each set is $\binom{k}{t}$. Therefore we must have:

$$\binom{R_2(k,t)}{k} \geq m\left(\binom{k}{t}\right).$$

Hence by (5.1.1) we have:

$$\binom{R_2(k,t)}{k} > 2^{\binom{k}{t}-1} \tag{5.1.2}$$

This is the same result as was obtained in Theorem 3.2.1 for the case n=2.

§5.2 An application to a problem in Geometry

In this section, we show that Ramsay's Theorem can be used to settle a problem in geometry. The problem can be formulated as follows: Let $k \geq 3$ be a positive integer. Does there exist a least integer f(k) with the property that among every set of f(k) points in a plane, no three collinear, there are k points which form the vertices of a convex k-gon?

We prove;

Theorem 5.2.1: f(k) exists and satisfies

$$f(k) \leq R(5,k;4).$$

Before we prove theorem 5.2.1, we introduce the following lemmas, the proof of the first of which is not difficult.

Lemma 5.2.1: Among any five points in the plane, no three collinear, there are four points which are the vertices of a convex quadrilateral.

Lemma 5.2.2: If the $\binom{k}{4}$ quadrilaterals formed from k points in the plane, no three collinear, are all convex, then the k points form the vertices of a convex k-gon.

Proof: Let P_1, P_2, ..., P_ℓ be the vertices of the convex cover of the set of k points. If $\ell=k$, we have finished. Suppose $\ell<k$. Then there is a point P which must lie in the interior of some triangle, say P_1, P_{i-1}, P_i. Then P_1, P_{i-1}, P_i, P is non convex, and this is a contradiction. Hence $\ell=k$.

Proof of Theorem 5.2.1: Let $R=R(5,k;4)$. Let S be a set of R points in the plane. Partition $P_4(S)$ into two classes C_1 and C_2 by placing a 4-subset in C_1 if these points form a non-convex quadrilateral, and in C_2 if these points form a convex quadrilateral. By Ramsay's Theorem, either there is a 5-subset of S all of whose 4-subsets belong to C_1, or a k-subset all of whose 4-subsets belong to C_2. By lemma 5.2.1, the first alternative is impossible. Hence the second alternative must hold. But by lemma 5.2.2, the k points form the vertices of a convex k-gon. Hence $f(k)\leq R(5,k;4)$.

It is easy to show that $f(3)=3$, $f(4)=5$ and it is known that $f(5)=9$. The values of $f(k)$ are not known for $k\geq 6$. However, it is conjectured that $f(k)=2^{k-2}+1$.

§5.3 An application to a problem in Matrix Theory:

Another application of Ramsay's Theorem is to matrices of zeroes and ones. We define an $A_{x,y}$ matrix to be a matrix in which all entries above the diagonal are x and all entries below the diagonal are y. The diagonal may have both x and y entries.

Theorem 5.3.1: Let k be a positive integer. Then there exists a least integer g(k) such that if $A=[a_{ij}]$ is an arbitrary g(k) x g(k) matrix of zeroes and ones, then A contains either an $A_{0,0}$, $A_{0,1}$, $A_{1,0}$ or $A_{1,1}$, k x k submatrix.

Proof: Let $R=R_4(k,2)$ and let $A=[a_{ij}]$ be an RxR matrix of zeroes and ones. Let S be the set whose elements are the rows of A. Let R_i, R_j be elements of S, $i<j$. We now associate with $\{R_i,R_j\}$ the vector (a_{ji},a_{ij}). Hence $\{R_i,R_j\}$ corresponds to one of (0,0),(1,0),(0,1) or (1,1). Partition $P_2(S)$ into four classes C_1,C_2,C_3,C_4 by putting $\{R_i,R_j\}$ in C_1 if $\{R_i,R_j\}$ corresponds to (0,0); in C_2 if it corresponds to (1,0); in C_3 if it corresponds to (0,1) and in C_4 if it corresponds to (1,1). However, since S is an R-set, then by Ramsay's Theorem, there exists a k-subset K of S such that $P_2(K)\subseteq C_i$ for some i, $1\leq i\leq 4$. Hence there exists a kxk-submatrix in A, which is of the form $A_{0,0}$, $A_{0,1}$, $A_{1,0}$ or $A_{1,1}$.

The above argument can be generalized to the case where A is a matrix whose elements are 0,1,2,...,r-1. It can be shown that if k is a positive integer, then there exists a least integer g(k,r) such that if A is an arbitrary g(k,r)xg(k,r) matrix whose elements are 0,1,2,...,r-1, then A contains a kxk submatrix A_{pq} for some p and q, $0 \leq p \leq r-1$, $0 \leq q \leq r-1$. Moreover, g(k,r) satisfies

$$g(k,r) \leq R_{r^2}(k,2).$$

Using an argument similar to that used in Theorem 3.2.1, we have obtained the following,

<u>Theorem 5.3.2</u>: $g(k) > c\,k\,2^{\frac{k}{2}}$ for some constant c and all sufficiently large k.

<u>Proof</u>: Let S(N) be the number of NxN matrices A_N of zeroes and ones which contain at least one kxk submatrix of the form $A_{0,0}$, $A_{0,1}$, $A_{1,0}$ or $A_{1,1}$. The probability that a matrix A_N chosen at random contains a $A_{0,0}$, $A_{0,1}$, $A_{1,0}$ or $A_{1,1}$ - kxk submatrix is $\frac{S(N)}{2^{N^2}}$, since the number of NxN matrices of zeroes and ones is 2^{N^2}. Hence the probability that a matrix A_N chosen at random does not contain a kxk-submatrix of the desired type is $1-\frac{S(N)}{2^{N^2}}$. Now since the number of ways of choosing k rows and columns from N rows and columns is $\binom{N}{k}^2$ and since the remaining entries can be filled in $2^{N^2-(k^2-k)}$ ways, we have

$$S(N) \le 4\binom{N}{k}^2 2^{N^2-(k^2-k)} < \frac{N^{2k}}{(k!)^2} 2^{N^2-(k^2-k)+2}$$

Hence, $1-\frac{S(N)}{2^{N^2}} > 0$ or $S(N) < 2^{N^2}$ will hold if

$$\frac{N^{2k}}{(k!)^2} 2^{N^2-(k^2-k)+2} < 2^{N^2}$$

or if

$$N < ck2^{\frac{k}{2}}$$

for some constant c and all sufficiently large k.

Hence $g(k) \ge ck2^{\frac{k}{2}}$.

The same type of argument yields, for each fixed $r \ge 2$.

$$g(k,r) \ge Ckr^{\frac{k}{2}}$$

for some constant C and all sufficiently large k.

BIBLIOGRAPHY

1. Abbott, H.L., Ph.D. Thesis, University of Alberta, (1965).

2. Erdös, P., Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947), p.292-294.

3. Erdös, P., On a combinatorial problem, Nordisk. Mat. Tidski., 2 (1963), p. 5-10.

4. Erdös, P. and A. Hajnal, On a property of family of sets, Acta. Math. Acad. Hung. Sci., 12 (1961), p.87-123.

5. Greenwood, R.E. and A.M. Gleason, Combinatorial Relations and chromatic graphs, Can. Jour. of Math., 7 (1955), p. 1-17.

6. Kalbfleish, J.G., Construction of special edge chromatic graphs, Can. Math. Bull., Vol. 8, No. 5 (1965), p. 575-584.

7. Ramsay, F.P., On a problem in formal logic, Proc. Lon. Math. Soc., 30 (1930), p. 264-286.

8. Ryser, H.J., Combinatorial Mathematics, Rahway, New Jersey: Quinn and Boden Company, Inc., 1963.

9. Schmidt, W.M., Ein kombinatorisches problem Von P. Erdös and A. Hajnal, Acta Math. Acad. Sci. Hungar. 15 (1964), p. 373-374.

10. Skolem, T., Ein kombinatorischer Satz mit Anwendung auf ein logisches entscheidungsproblem, Fund. Math. 20 (1933), p. 254-261.

11. Szekeres, G. and P. Erdös, A combinatorial problem in geometry, Compositio Mathematica, 2 (1935), p. 463-470.

12. Erdös, P., Graph theory and probability II, Can. Jour. of Math., 13 (1961), p. 364-352.