A SURVEY OF MODERN DIMENSION THEORY

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ABSTRACT

The main objective of this thesis is to give an up-to-date account of several dimension functions and relations that exist between them for various spaces.

Chapter 1 contains a brief history of the development of the subject as we know it today. In Chapter 2 we give definitions of the three basic dimension functions 'ind', 'dim', and 'Ind' and a detailed survey of the known properties of these functions. Included is a proof of the famous Čech Sum Theorem for dimension 'dim'.

In Chapter 3 we investigate relations between 'ind', 'dim', and 'Ind' and mention some of the latest examples that have been given to illustrate the gaps that exist between the various dimensions. Finally, Chapter 4 contains a brief account of a relatively new dimension function "Dim'.

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Chapter One

Introduction

1. The Modern Concept of dimension.

In order to present the motivation behind the modern concepts of dimension theory we can hardly do better than quote from Hurewicz and Wallman [1]:

"... to divide spaces cuts that are called surfaces are necessary; to divide surfaces, cuts that are called lines are necessary; to divide lines, cuts that are called points are necessary; we can go no further and a point can not be divided, a point not being a continuum. Then lines, which can be divided by cuts which are not continua, will be continua of one dimension; surfaces which can be divided by continuous cuts of one dimension, will be continua of two dimensions; and finally space, which can be divided by continuous cuts of two dimensions, will be a continuum of three dimensions."

These words were written by Henri Poincaré in 1912. Writing in a philosophical journal [Revue de métaphysique et de morale], Poincaré was concerned only with putting forth an intuitive concept of dimension and not an exact mathematical formulation. Poincaré had, however, penetrated very deeply in stressing the inductive nature of the geometric meaning of dimension and the possibility of disconnecting a space by subsets of lower dimension. One year later Brouwer constructed on Poincaré's foundation a precise and topologically invariant definition of dimension which is essentially as follows:

(a) the empty set ϕ has dimension -1.

(b) the dimension of the topological space X is the least integer n such that for any pair of disjoint closed subsets C_1 and C_2, there is a closed subset K separating C_1 and C_2 , where the dimension of K is less than n."

Brouwer's paper remained practically unnoticed for almost a decade. Then in 1922, independently of Brouwer, and of each other, Urysohn and Menger recreated Brouwer's concept in the following formulation:

(a) the empty set ϕ has dimension -1,

(b) the dimension of a space is the least integer n for which every point has arbitrarily small neighbourhoods whose boundaries have dimension less than n.

2. Previous concepts of dimension.

Before the advent of set theory mathematicians used dimension in only the vaguest sense. A configuration was said to be n-dimensional if the least number of real parameters required to describe its points, in some unspecified way, was n. The dangers and inconsistencies in this approach were vividly brought into view by two celebrated discoveries of the last part of the nineteenth century: Cantor's one-to-one correspondence between the points of a line and the points of a plane, and Peano's continuous mapping of an interval on the whole of a square. The first exploded the feeling that a plane is richer in points than a line, and showed that dimension can be raised by a one-valued continuous transformation.

An extremely important question was left open (and not answered until 1911, by Brouwer): Is it possible to establish a correspondence between Euclidean n-space (the ordinary space of n real variables) and Euclidean m-space combining the features of both Cantor's and Peano's constructions, i.e. a correspondence which is both one-to-one and continuous? The question is crucial since the existence of a transformation of the stated type between Euclidean n-space and Euclidean m-space would signify that dimension (in the natural sense that Euclidean n space has dimension n) has no topological meaning whatsoever! The class of topological transformations would in consequence be much too wide to be of any real geometric use.

3. Topological invariance of the dimension of Euclidean Spaces.

The first proof that Euclidean n-space and Euclidean m-space are not homeomorphic unless n equals m was given by Brouwer in his famous paper: Beweis der Invarianz der Dimensionenzahl (Math.Ann.(1911) pp. 161-165). However, this proof did not explicitly reveal any simple topological property of Euclidean n-space distinguishing it from Euclidean m-space and responsible for the non-existence of a homeomorphism between the two. More penetrating, therefore, was Brouwer's procedure in 1913 when he introduced his "Dimensionsgrad" and integer-valued function of a space which was topologically invariant by its very definition. Brouwer showed that the "Dimensionsgrad" of Euclidean n-space is precisely n (thereby justifying its name).

Meanwhile Lebesgue had approached in another way the proof that the dimension of a Euclidean space is topologically invariant. He had observed [1] that a square can be covered by arbitrarily small "bricks" in such a way that no point of the square is contained in more than three of these bricks; but that if the bricks are sufficiently small, at least three have a point in common. In a similar way a cube in Euclidean n-space can be decomposed into arbitrarily small bricks so that not more than $n + 1$ of these bricks meet. Lebesgue conjectured that this number $n + 1$ could not be reduced further, i.e. for any decomposition in sufficiently small bricks there must be a point in common to at least $n + 1$ of the bricks. The first proof of this theorem was given by Brouwer in the paper already cited. Lebesgue's theorem also displays a topological property of Euclidean n-space distinguishing it from Euclidean m-space and therefore it also implies the topological invariance of the dimension of Euclidean spaces.

Lebesgue's covering theorem thus motivates the following definition of dimension:

(a) the empty set has dimension -1.

(b) a topological space X has dimension $\leq n$ if given any finite open covering $\mathcal{U}$ of X there exists a refinement $\mathcal{V}$ [$\mathcal{V}$ is a refinement of the open covering $\mathcal{U}$ of X if
 (i) each member $V_\alpha \in \mathcal{V}$ is an open subset of X;
 (ii) each member $V_\alpha \in \mathcal{V}$ is contained in some member $U_i \in \mathcal{U}$;
 (iii) $\cup V_\alpha = X$]
 such that at most $n + 1$ sets of this refinement have a non-empty intersection.

(c) the dimension of the space X is equal to n if (b) is true and it is false that the dimension is less than n.

The formulation of Brouwer given above has the following equivalent form, introduced by Edward Čech [1]:

(a) the empty set ϕ has dimension -1;

(b) the dimension of the topological space X is the least integer n such that for any pair of a closed set F and an open set G such that $F \subset G \subset X$ there exists an open set V with $F \subset V \subset G$ where the dimension of the boundary $b(v) = \overline{V} \setminus V$ of V is less than n.

The definition of Urysohn-Menger, Brouwer-Čech, and Lebesgue are respectively termed the 'weak inductive dimension', 'the strong inductive dimension', and the 'covering dimension' and are denoted by 'ind', 'Ind', and 'dim' respectively.

Katetov [1], Morita [1], and Dowker and Hurewicz [2] have all published different proofs that for any metric space X
Ind X = dim X.

The notion of weak inductive dimension (or Urysohn-Menger dimension) is no longer so important as the other two notions, because P. Roy [1] has recently constructed a complete metric space Δ such that $\mathrm{ind}\,\Delta = 0$ but $\mathrm{Ind}\,\Delta = \dim \Delta = 1$. However the weak inductive dimension still has its uses, and we consider it in some detail. The remainder of Chapter 1 involves a detailed treatment of the dimension functions of Brouwer-Cech and Lebesgue.

In Chapter 3 we examine relations that exist between the functions 'ind', 'dim', and 'Ind' for various classes of spaces. We mention several of the latest examples illustrating the gaps that exist between these functions, including Nagami's example of a normal space Z with $\mathrm{ind}\, Z = 0$, $\dim Z = 1$, $\mathrm{Ind}\, Z = 2$.

Chapter 4 contains a brief account of the relatively new dimension function 'Dim'. Attempts were made without success to extend certain of the theorems given for perfectly normal spaces to more general spaces called totally normal. It appears that great difficulties are encountered in trying to extend the theory to more general spaces.

CHAPTER TWO

In this chapter the main results concerning the weak inductive dimension 'ind', the covering dimension 'dim', and the strong inductive dimension 'Ind' are presented together with relations between them for a given class of topological spaces. The discussion in sections (i) and (ii) on the weak inductive dimension 'ind' is similar to that given by Hurewicz and Wallman [1], and unless otherwise stated in these sections, all spaces referred to are separable metric.

(i) Weak inductive dimension 0

Definition 2.1. A space is connected if it is not the union of two non-empty disjoint open sets. Equivalently a space is connected if, except for the empty set and the whole space, there are no sets whose boundaries are empty.

In this section we are concerned with spaces which are disconnected in an exceedingly strong sense, i.e. have so many open sets whose boundaries are empty that every point may be enclosed in arbitrarily small set of this type.

Definition 2.2. A space X has weak inductive dimension 0 at a point $p \in X$, $\mathrm{ind}_p X = 0$, if p has arbitrarily small (open) neighbourhoods with empty boundaries, i.e. if for any neighbourhood U of p there exists a neighbourhood V of p such that

$$p \in V \subseteq U \quad , \quad b(V) = \phi \ .$$

A non-empty space X has weak inductive dimension 0, $\mathrm{ind}\, X = 0$ if $\mathrm{ind}_p X = 0$ for each $p \in X$.

Clearly the property of being 0-dimensional at a point p is a topological invariant. Also, for any topological space X, saying that $\operatorname{ind} X = 0$ is equivalent to saying that $X \neq \phi$ and that there is a basis for the open sets of X which consists of sets which are both open and closed.

EXAMPLES

Example 2.1. If X is any non-empty finite or countable metrizable space, then $\operatorname{ind} X = 0$.

Provide X with the metric ρ. For any given neighbourhood U of any point p let r be a positive real number such that the spherical neighbourhood of radius r about p is contained in U. Let $x_1, x_2, \ldots$ be an enumeration of the points of X; then $\rho(x_i, p)$ is the distance from x_i to p. There exists a positive real number $r' < r$ and different from all the $\rho(x_i, p)$. The spherical neighbourhood of radius r' about p is contained in U and its boundary is empty. Hence $\operatorname{ind} X = 0$.

In particular the space R of rational numbers has $\operatorname{ind} R = 0$.

Example 2.2. If I is the space of irrational numbers, $\operatorname{ind} I = 0$.

For any given neighbourhood U of an irrational point p there exist rational numbers ρ and σ such that $\rho < p < \sigma$ and the set V of irrational numbers between ρ and σ is contained in U. In the space of irrational numbers V is open and has an empty boundary because every irrational point which is an accumulation point of V is between ρ and σ and hence belongs to V.

Example 2.3. The Cantor discontinuum C, the subspace of all real numbers expressible in the form

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad \text{where} \quad a_n = 0 \text{ or } 2$$

has $\operatorname{ind} C = 0$.

Example 2.4. If A is any subspace of the space of real numbers which contains no interval, then $\operatorname{ind} A = 0$. (Example 2.3 above is a specific case of this.)

Example 2.5. If I_2 is the subspace of points in the plane both of whose coordinates are irrational, then $\operatorname{ind} I_2 = 0$. Clearly any such point is contained in arbitrarily small rectangles bounded by lines having rational intercepts with the coordinate axes and intersecting them at right angles, and the boundaries of such rectangles contain no points of I_2.

Example 2.6. $\operatorname{ind} R_2^1 = 0$, where R_2^1 is the subspace consisting of points of the plane exactly one of whose coordinates is rational. Clearly any such point is contained in arbitrarily small rectangles bounded by lines having rational intercepts with the coordinate axes and intersecting them at 45°, and the boundaries of such rectangles do not intersect R_2^1.

Example 2.7. The set R_n of points of Euclidean n-space all of whose coordinates are rational and metrized by the usual Euclidean metric has $\operatorname{ind} R_n = 0$ (R_n is countable.)

Example 2.8. If I_n is the subspace of points of E_n all of whose coordinates are irrational, $\operatorname{ind} I_n = 0$. (A simple generalization of Ex. 2.5)

<u>Remark</u>. Suppose $0 \leq m < n$. Denote by $\mathcal{R}_n^m$ the subspace of E_n exactly m of whose coordinates are rational. Then $\operatorname{ind} \mathcal{R}_n^m = 0$. Examples 2.6, 2.7, and 2.8 above are specific cases of this, but the proof of this more general result depends on the "Sum Theorem for ind = 0" (Theorem 2.3).

<u>Example 2.9</u>. $\operatorname{ind} \mathcal{R}_\omega' = 0$, where $\mathcal{R}_\omega'$ is the subspace of points of the Hilbert cube I_ω all of whose coordinates are rational. (For the proof see Hurewicz and Wallman).

<u>Example 2.10</u>. $\operatorname{ind} I_\omega' = 0$, where I_ω' is the subspace of points of the Hilbert cube I_ω all of whose coordinates are irrational.

<u>Example 2.11</u>. $\operatorname{ind} \mathcal{R}_\omega = 1$, where $\mathcal{R}_\omega$ is the subspace of points of Hilbert space E_ω all of whose coordinates are rational. (For the proof see Hurewicz and Wallman [1] or P. Erdos [1].)

<u>Theorem 2.1</u>. A non empty subset X' of a 0-dimensional space is 0-dimensional.

<u>Proof</u>: Let $p \in X'$ and U' any neighbourhood of p open in X'. Then there exists a neighbourhood U in X of p such that

$$U' = U \cap X'.$$

Since $\operatorname{ind} X = 0$, there exists V open and closed in X such that

$$p \in V \subseteq U.$$

Let $V' = V \cap X'$.

Then V' is both open and closed in X'

and $$p \in V' \subseteq U'$$

so that $$\operatorname{ind} X' = 0.$$

<u>Definition 2.3</u>. If A_1, A_2, and B are mutually disjoint subsets of a space X, we say that A_1 and A_2 are separated in X by B if $X \setminus B$ can be split into two disjoint sets, open in $X \setminus B$ and containing A_1 and A_2 respectively, i.e. if there exists A_1' and A_2' for which

$$X \setminus B = A_1' \cup A_2'$$

$$A_1 \subset A_1' \quad , \quad A_2 \subseteq A_2'$$

$$A_1' \cap A_2' = \phi$$

with A_1' and A_2' both open in $X \setminus B$ (or what is the same, both closed in $X \setminus B$).

If A_1 and A_2 are separated by the empty set we say they are separated in X.

A_1 and A_2 are separated if and only if there exists a set A_1' such that

$$A_1 \subseteq A_1'$$

$$\text{and} \qquad A_1' \cap A_2 = \phi \quad .$$

where A_1' is both open and closed, i.e. $b(A_1') = \phi$.

Then $A_2' = X \setminus A_1'$.

<u>Definition 2.2'</u>. Let X be a non-empty space. Then $\operatorname{ind} X = 0$ if every point $p \in X$ and every closed subset C of X with $p \notin C$ can be separated.

It is trivial to show that Definition 2.2. and Definition 2.2' are equivalent.

Remarks

1) A connected space X with $\operatorname{ind} X = 0$ consists of only one point.

2) If $\operatorname{ind} X = 0$ then X is totally disconnected.

3) It is obvious from Definition 2.2' that if X is a T_1-space and any two disjoint closed subsets can be separated then $\operatorname{ind} X = 0$. We now prove conversely that if X has a countable base and $\operatorname{ind} X = 0$, then any two disjoint closed subsets can be separated.

Theorem 2.2. Let X be a topological space with a countable basis and $\operatorname{ind} X = 0$. Then any two disjoint closed subsets of X can be separated.

Proof. Since $\operatorname{ind} X = 0$, by Definition 2.2' any point $p \in X$ can be separated from any closed set not containing p. Let C and K be two disjoint closed subsets of X. We have to demonstrate a separation of C and K in X.

For each $p \in X$ either $p \notin C$ or $p \notin K$. Hence there exist neighbourhoods $U(p)$ for each point p which are both open and closed and such that either $U(p) \cap C = \phi$ or $U(p) \cap K = \phi$. Since X has a countable basis there exists a sequence $U_1, U_2, \ldots$ of these $U(p)$ whose union is X (Lindelöf's theorem - see Kelley[1], p. 49). We now define a new sequence of sets V_i as follows:

$$V_1 = U_1$$

$$V_i = U_i \setminus \bigcup_{k=1}^{i-1} U_k = U_i \cap (X \setminus \bigcup_{k=1}^{i-1} U_k) \qquad i = 2, 3, \ldots$$

Then we have

(1) $$X = \bigcup_{i=1}^{\infty} V_i$$

(2) $$V_i \cap V_j = \phi \quad \text{if } i \neq j.$$

(3) V_i is open

(4) either $V_i \cap C = \phi$ or $V_i \cap K = \phi$.

(1), (2), and (4) are obvious. To prove (3) we note that

$$\bigcup_{k=1}^{i-1} U_k \text{ is closed, so that}$$

$$X \setminus \bigcup_{k=1}^{i-1} U_k \text{ is open ;}$$

hence $V_i = U_i \cap (X \setminus \bigcup_{k=1}^{i-1} U_k)$ is open.

Let $C' = \bigcup V_i$ such that $V_i \cap K = \phi$.

K' = union of remaining V_i .

Then $X = C' \cup K'$ by (1)

$C' \cap K' = \phi$ by (2)

C' and K' are open by (3)

and $(C' \cap K) \cup (C \cap K') = \phi$ by (4).

It follows that $C \subseteq C'$ and $K \subseteq K'$.

The desired separation is thus given by C' and K' .

The sum or union of zero-dimensional sets need not be zero-dimensional as we see from the decomposition of the real line into the rational numbers and irrational numbers or into its distinct points. We have the following theorem:

<u>Theorem 2.3.</u> (Sum Theorem for zero-dimensional sets).

A separable metric space X which is the countable union of zero-dimensional closed subsets is itself zero-dimensional,

i.e. if $X = \bigcup_{i=1}^{\infty} X_i$

where each X_i is a closed subspace and $\operatorname{ind} X_i = 0$, then $\operatorname{ind} X = 0$.

Proof: Let K and L be two disjoint closed subsets of X. We show that K and L can be separated.

Clearly $K \cap X_1$ and $L \cap X_1$ are disjoint closed subsets of the space X_1, where $\operatorname{ind} X_1 = 0$. Hence by Theorem 2.3, there exist subsets A_1 and B_1 of X_1, closed in X_1 and therefore in X, such that

$$K \cap X_1 \subseteq A_1 \quad , \quad L \cap X_1 \subseteq B_1$$
$$A_1 \cup B_1 = X_1 \quad , \quad A_1 \cap B_1 = \phi \ .$$

The sets $K \cup A_1$ and $L \cup B_1$ are closed and disjoint in X. By the normality of X there exist open sets G_1 and H_1 for which

$$K \cup A_1 \subseteq G_1 \quad , \quad L \cup B_1 \subseteq H_1$$
$$\overline{G}_1 \cap \overline{H}_1 = \phi \ .$$

Therefore

$$G_1 \cup H_1 \supseteq X_1$$
$$K \subseteq G_1 \quad , \quad L \subseteq H_1$$
$$\overline{G}_1 \cap \overline{H}_1 = \phi \ .$$

Now repeat this process replacing K and L by $\overline{G}_1$ and $\overline{H}_1$ and X_1 by X_2. This yeilds open sets G_2 and H_2 for which

$$G_2 \cup H_2 \supseteq X_2$$
$$\overline{G}_1 \subseteq G_2 \quad , \quad \overline{H}_1 \subseteq H_2$$
$$\overline{G}_2 \cap \overline{H}_2 = \phi \ .$$

By induction we construct sequences $\{G_i\}$ and $\{H_i\}$ of sets open in X for which

$$G_i \cup H_i \supseteq X_i$$
$$\overline{G}_{i-1} \subseteq G_i \quad , \quad \overline{H}_{i-1} \subseteq H_i$$
$$\overline{G}_i \cap \overline{H}_i = \phi$$

Let $G = \bigcup_{i=1}^{\infty} G_i$ and $H = \bigcup_{i=1}^{\infty} H_i$.

Then G and H are disjoint open sets,

$$G \cup H \supseteq \bigcup_{i=1}^{\infty} X_i = X$$

and $$K \subseteq G \ , \ L \subseteq H \ ;$$

this is the desired separation.

Definition 2.4. By an F_σ set in a space X we mean any countable union of closed subsets of X. It can be shown that in a metric space any open set is F_σ .

Corollary 1 to Theorem 2.3. A separable metric space which is the countable union of 0-dimensional F_σ sets is 0-dimensional.

Corollary 2. The union of two 0-dimensional subsets of a separable metric space X, at least one of which is closed, is 0-dimensional.

Proof. Suppose $\operatorname{ind} A = \operatorname{ind} B = 0$ and B is closed.
Then $A \cup B \setminus B$ is open in $A \cup B$. As an open set in a metric space it is F_σ in $A \cup B$. The result then follows from Corollary 1 and

$$A \cup B = [A \cup B \setminus B] \cup B.$$

Corollary 3. A 0-dimensional space remains 0-dimensional after the adjunction of a single point (assuming that the enlarged space is separable metric).

Example 2.12. Suppose $0 \leq m < n$. Denote by R_n^m the subspace of points in Euclidean n-space E_n exactly m of whose coordinates are rational. Then $\operatorname{ind} R_n^m = 0$.

For each selection of m induces $i_1, \ldots, i_m$ out of the range $1, 2, \ldots, n$, and each selection of m rational numbers $r_1, r_2, \ldots, r_m$ we have an $(n - m)$ - dimensional linear subspace $E^{(i)}_{n-m}$ of E_n determined by the equations

$$x_{i_1} = r_1 \quad , \quad x_{i_2} = r_2, \ldots, x_{i_m} = r_m$$

The subspace of this space made up of points none of whose remaining coordinates is rational we denote by C_i. Each C_i is congruent to I_{n-m} and is therefore 0-dimensional (Exmaple 2.8). It is clear that each C_i is closed in R^m_n since

$$C_i = R^m_n \cap E^{(i)}_{n-m} \text{ and each } E^{(i)}_{n-m} \text{ is closed in } E_n.$$

The union of the C_i just fills out R^m_n . Since the collection of the C_i is countable the sum theorem implies that $\operatorname{ind} R^m_n = 0$.

Example 2.13. Suppose $0 \leq m$. Denote by R^m_ω the set of points in the Hilbert cube exactly m of whose coordinates are rational. Then $\operatorname{ind} R^m_\omega = 0$.

Let $i = \{i_1, i_2, \ldots, i_m\}$ be a selection of m different integers chosen from the set $\{1, 2, 3, \ldots, n, \ldots\}$. Such a selection can be made in

$$\underbrace{\chi_0 \cdot \chi_0 \cdots\cdots \chi_0}_{m \text{ factors}} = \chi_0 \text{ ways} \qquad (1)$$

Again for a given i_k of this selection the number of different rationals r_{i_k} such that

$$-\frac{1}{i_k} \leq r_{i_k} \leq \frac{1}{i_k}$$

is χ_0. Thus for a given selection i we have

$$\underbrace{\aleph_0 \cdot \aleph_0 \cdots\cdots \aleph_0}_{m \text{ factors}} = \aleph_0 \tag{2}$$

sets of rationals.

By (1) and (2) the number of different subsets C_i of R^m_ω is $\aleph_0 \cdot \aleph_0 = \aleph_0$. Thus the set of C_i is countable and just fills out R^m_ω. Moreover each C_i is congruent to a subspace X' of I'_ω in which exactly m of the rational coordinates are fixed. Hence $\text{ind } C_i = \text{ind } X' = 0$ (because $\phi \neq X' \subsetneq I'_\omega$ and $\text{ind } I'_\omega = 0$ implies $\text{ind } X' \leq 0$ while $X' \neq \phi$ implies $\text{ind } X' \geq 0$).

We now prove that each C_i is closed in R^m_ω.

Let $y \in C_i$ where

$y_{i_1} = r_1 \quad , \quad y_{i_2} = r_2 \, , \, \ldots\ldots , \, y_{i_m} = r_m$

(the r's being rational, the rest of the coordinates of y being irrational) define a C_i in the space of the Hilbert cube and let

$x = (x_1, x_2, \ldots\ldots, x_j, \ldots.)$ be a point of I_ω.

Then if $y \in C_i$ we have

$\rho(x,y)^2 = (x_{i_1} - r_1)^2 + (x_{i_2} - r_2)^2 + \ldots\ldots + (x_{i_m} - r_m)^2 + \sum'_k (x_k - y_k)^2$

where $\sum'_k$ denotes that k takes all values $1, 2, 3, \ldots.$ except $i_1, i_2, \ldots\ldots, i_m$.

We can always make $\sum'_k$ zero by choosing $x_k = y_k$.

Choosing $\epsilon = \sqrt{(x_{i_1} - r_1)^2 + \ldots + (x_{i_m} - r_m)^2}$ we see that $x \in I_\omega$ does not belong to $\overline{C}_i$ unless $x_{i_1} = r_1$, i.e. $x \notin C_i$ implies $x \notin \overline{C}_i$.

$\therefore$ C_i is closed.

Consider the following four properties of a space X :

(0) X is totally disconnected.

(1) Any two distinct points of X can be separated.

(2) Any point can be separated from any closed set C to which it does not belong, i.e. $\operatorname{ind} X = 0$ (Definition 2.2').

(3) Any two disjoint closed sets can be separated.

For T_1-spaces (3) => (2) , (2) => (1) , (1) => 0 . Conversely, for separable metric spaces (2) => (3); for spaces without countable basis (2) does not imply (3) as the 'Tychonov Plank' shows. (See Appendix). Properties (0), (1), and (2) however are not equivalent, even for separable metric spaces.

Example 2.14. Sierpinski [1] gives an example of a subset of the plane satisfying (0) but not (1).

Example 2.15. By example 2.11, $\operatorname{ind} R_\omega = 1$, so that R_ω does not satisfy (2). On the other hand it does satisfy (1). For, let p and q be two points of R_ω and let i be an index such that the i^{th} coordinate p_i of p differs from the i^{th} coordinate q_i of q ; p_i and q_i are, of course, rational. Let ρ be any irrational number between p_i and q_i. The decomposition of R_ω into the closed and disjoint subsets determined by $x_i \leq \rho$, $x_i \geq \rho$ gives the desired separation of p and q.

For compact spaces the conditions (0) - (3) are equivalent (see Hurewicz and Wallman [1].)

Remark. It is not true, as is seen in the following two examples, that if a space has properties (0) or (1) it will retain that property upon the adjunction of a single point; compare with Corollary 3 to Theorem 2.3.

Hence the Sum Theorem would not be true for a theory of dimension in which dimension 0 were either defined by total-disconnectedness or the separation of pairs of points.

Example 2.16. Knaster and Kuratowski [1] give an example of a subset of the plane which is totally disconnected but becomes connected upon the adjunction of a single point.

Example 2.17. J.H. Roberts [1] gives another example of this phenomenon.

Example 2.18. Sierpinski [1] gives an example of a space which has property (1) but loses it upon the adjunction of a single point.

Denoting the spaces investigated by Erdos, Sierpinski, Knaster and Kuratowski, and Roberts by E, K, S, and R. T.H. Walton [1] has given the following summary of the relationships of these various spaces to one another:

	E	K	$K \setminus \{a\}$	S	$S \setminus \{p\}$	R	$R \setminus \{q\}$
(0)	+	-	+	+	+	+	-
(1)	+	-	-	+	-	+	-
(2)	-	-	-	-	-	-	-
(3)	-	-	-	-	-	-	-

A detailed analysis of these spaces shows that

$$\begin{aligned} \mathrm{ind}\, E &= \mathrm{ind}\, K = \mathrm{ind}\, (K \setminus \{a\}) = \mathrm{ind}\, S \\ &= \mathrm{ind}\, (S \cup \{p\}) = \mathrm{ind}\, R = \mathrm{ind}\, (R \cup \{q\}) = 1. \end{aligned}$$

Thus E, $K \setminus \{a\}$, S, R all have weak inductive dimension equal to unity although they are totally disconnected.

Moreover, Mazurkiewicz [1] has shown that for each finite n there exists a space X for which $\mathrm{ind}\, X = n$ and X is totally disconnected.

Thus the fact that a space has positive weak inductive dimension implies very little about its connectedness.

Finally Mazurkiewicz [1] proves the existence of a plane connected set containing no bounded connected subset.

(ii) Weak Inductive Dimension n.

Roughly speaking, we may say that a space has dimension $\leq n$ if an arbitrarily small piece of the space surrounding each point may be delimited by subsets of dimension $\leq n - 1$. This method of definition is inductive, and an elegant starting point for the induction is provided by prescribing the empty set ϕ as the (-1)-dimensional space.

<u>Definition 2.4</u>. (a) $\mathrm{ind}\, X = -1$ if and only if $X = \phi$.

(b) A space X has <u>weak inductive dimension</u> $\leq n$ $(n \geq 0)$ at a point p if p has arbitrarily small (open) neighbourhoods whose boundaries have weak inductive dimension $\leq n - 1$, denoted by $\mathrm{ind}_p X \leq n$.

(c) X has weak inductive dimension $\leq n$, $\mathrm{ind}\, X \leq n$ if $\mathrm{ind}_p X \leq n$ for each $p \in X$.

(d) $\text{ind}_p X = n$ if it is true that $\text{ind}_p X \leq n$ and it is false that $\text{ind}_p X \leq n - 1$.

(e) $\text{ind } X = n$ if it is true that $\text{ind } X \leq n$ and it is false that $\text{ind } X \leq n - 1$.

(f) $\text{ind } X = \infty$ if $\text{ind } X \leq n$ is false for each n.

It is obvious that the property of having weak inductive dimension n (or having dimension n at a point p) is topologically invariant. Dimension is not, however, an invariant of continuous transformations. Projection of a plane into a line is an illustration of a transformation which lowers dimension and Peano's mapping of an interval onto the whole of a square is an illustration of a continuous transformation which raises dimension.

Equivalent to the condition that $\text{ind } X \leq n$ is the existence of a basis for the topology of X made up of open sets whose boundaries have dimension $\leq n - 1$.

It is clear that definitions 2.2 and 2.4 are equivalent for $n = 0$.

<u>Proposition 2.1</u>. If $\text{ind } X = n$, then X contains an m-dimensional subset for every $m \leq n$.

<u>Proof</u>. Since $\text{ind } X > n - 1$ there exists a point $p_0 \in X$ and an open neighbourhood U_0 of P_0 with the property that if V is any open set satisfying

$$p_0 \in V \subseteq U_0 \quad \text{then}$$
$$\text{ind } b(V) \geq n - 1.$$

On the other hand, because $\text{ind } X \leq n$, there exists an open set V_0 satisfying $p_0 \in V_0 \subseteq U_0$ for which

$$\text{ind } b(V_o) \leq n - 1.$$

Hence $b(V_o)$ is a subset of X of the precise dimension $n - 1$. The rest of proposition 2.1 is now evident.

Remark. The statement of the above proposition cannot be extended to infinite dimensional spaces. Indeed, under the hypothesis of the continuum there even exist infinite dimensional spaces whose only finite dimensional subspaces are countable sets. See Hurewicz [1].

Example 2.19. ind $E_1 = 1$ and the dimension of any interval of the Euclidean line E_1 is 1.

Example 2.20. Any polygon has dimension 1.

Example 2.21. Any 2-manifold has dimension ≤ 2.

Example 2.22. ind $E_n \leq n$ (proof by induction).

The proof that the dimension of E_n is precisely n is given by Hurewicz and Wallman [1].

Theorem 2.4. A subspace of any topological space of dimension $\leq n$ has dimension $\leq n$.

Proof. Obvious for $n = -1$. Assume true for $n - 1$.
Let ind $X \leq n$ and X' be a subspace of X, p a point of X'. Let U' be any open neighbourhood of p in X'. Then

$$U' = U \cap X', \text{ where } U \text{ is open in } X.$$

Since ind $X \leq n$ there exists V open in X such that

$$p \in V \subseteq U$$

$$\text{ind } b(V) \leq n - 1.$$

Let $V' = V \cap X'$. Then V' is open in X' and

$$p \in V' \subseteq U'$$

Let $b_X(V)$ = boundary of V in X

and $b_{X'}(V')$ = boundary of V' in X'.

Then

$$\begin{aligned} b_{X'}(V') &= \overline{V}' \setminus V' \quad \text{(where } \overline{V}' \text{ is the closure of } V' \text{ in } X'\text{)} \\ &= (\overline{V}' \cap X') \setminus V' \quad (\overline{V}' \text{ is the closure of } V' \text{ in } X) \\ &= \overline{V}' \cap X' \setminus V \cap X' \\ &\subseteq \overline{V} \cap X' \setminus V \cap X' \\ &= (\overline{V} \setminus V) \cap X' \\ &= b_X(V) \cap X' \end{aligned}$$

i.e. $b_{X'}(V') \subseteq b_X(V) \cap X'$

$\therefore \quad b_{X'}(V') \subseteq b_X(V)$

We are given that $\operatorname{ind} b_X(V) \leq n - 1$.

By the hypothesis of the theorem, $\operatorname{ind} b_{X'}(V) \leq n - 1$, and the theorem is proved.

i.e. $\operatorname{ind} X' \leq n$.

<u>Definition 2.4'</u>. $\operatorname{ind} X \leq n$ if every point p can be separated by a closed set of dimension $\leq n - 1$ from any closed set not containing p.

Definitions 2.4 and 2.4' are equivalent.
(see Hurewicz and Wallman [1].)

<u>Definition 2.5</u>. A space X is hereditarily normal if every subspace of X is normal.

Lemma. A space X is hereditarily normal if and only if given any pair of disjoint subsets X_1 and X_2 satisfying

$$X_1 \cap \overline{X}_2 = \phi = \overline{X}_1 \cap X_2$$

there exist open sets W_1 and W_2 such that

$$X_1 \subseteq W_1 \ , \ X_2 \subseteq W_2 \ \text{ and } \ W_1 \cap W_2 = \phi \ .$$

Moreover it is clear that $\overline{W}_1 \cap W_2 = \phi = W_1 \cap \overline{W}_2$.

Proof. See Walton [1].

Theorem 2.5. A subspace X' of an hereditarily normal space X has dimension $\leq n$ if and only if every point of X' has arbitrarily small neighbourhoods open in X whose boundaries have intersections with X' of dimension $\leq n - 1$.

Proof. (Sufficiency). Let $p \in X'$ and $U'(p) = U'$ be open in X'. Then there exists $U(p) = U$ open in X such that $U' = U \cap X'$. Hence there exists, by hypothesis, V open in X such that

$$p \in V \subseteq U$$

$$\text{and} \quad \text{ind } (X' \cap b(V)) \leq n - 1.$$

Let $V' = V \cap X'$. Then V' is open in X', $p \in V' \subseteq U'$ and $b_{X'}(V') \subseteq b_X(V) \cap X'$. Hence ind $b_{X'}(V') \leq n - 1$, so that ind $X' \leq n$.

(Necessity). Suppose ind $X' \leq n$. Let $p \in X'$ and $U(p) = U$ open in X.

Then $U' = U \cap X'$ is a neighbourhood of p open in X'.

Hence there exists V' open in X' such that

$$p \in V' \subseteq U' \quad \text{and}$$

$$\text{ind } b_{X'}(V') \leq n - 1.$$

Neither of the disjoint sets V' and $X' \setminus \overline{V'}$ contains an accumulation point of the other, so by the hereditary normality of X there exists an open set W satisfying $V' \subseteq W$ and $\overline{W} \cap (X' \setminus \overline{V'}) = \phi$.

Replacing W if necessary by $W \cap U$ we may assume that

$$W \subseteq U.$$

The set $\overline{W} \setminus W = b(W)$ contains no point of $X' \setminus \overline{V'}$ and no point of V'. It follows that

$$b(W) \cap X' \subseteq b_{X'}(V').$$

Hence by theorem 2.4,

$$\text{ind } (b(W) \cap X') \leq n - 1 \quad \text{as required.}$$

<u>Theorem 2.6</u>. For any two subspaces A and B of an hereditary normal space X

$$\text{ind}(A \cup B) \leq \text{ind } A + \text{ind } B + 1.$$

<u>Proof</u>. The theorem is true for

$$\text{ind } A = \text{ind } B = -1.$$

Suppose $\text{ind } A = m$, $\text{ind } B = n$, and assume true for the cases

(i) $\quad \text{ind } A \leq m \quad , \quad \text{ind } B \leq n - 1$

(ii) $\quad \text{ind } A \leq m - 1 \quad , \quad \text{ind } B \leq n$

Let $p \in A \cup B$. As a matter of notation take $p \in A$.
Let U be a neighbourhood of p open in X.

By Theorem 2.5 there exists an open set V such that

$$p \in V \subseteq U \quad \text{and}$$

$$\text{ind } (b(V) \cap A) \leq m - 1.$$

But $\quad b(V) \cap B \subseteq B$

$\therefore \quad \text{ind } (b(V) \cap B) \leq n$

Hence by hypotheses (i) and (ii) of the induction

$$\operatorname{ind}[b(V) \cap (A \cup B)] \leq m + n.$$

Hence by Theorem 5

$$\operatorname{ind}(A \cup B) \leq m + n + 1 \quad , \text{ as required}$$

<u>Corollary</u>. Let $X = \bigcup_{i=o}^{n} X_i$ where $\operatorname{ind} X_i \leq 0$, $i = 0, 1, \ldots, n$.

Then $\operatorname{ind} X \leq n$.

<u>Example 2.23</u>. Suppose $0 \leq m \leq n$. Denote by M_n^m the set of points of E_n at most m of whose coordinates are rational and by L_n^m the set of points of E_n at least m of whose coordinates are rational.

Then $\operatorname{ind} M_n^m \leq m$, and

$\operatorname{ind} L_n^m \leq n - m$.

Evidently $M_n^m = R_n^o \cup R_n' \cup \ldots\ldots \cup R_n^m$

and $L_n^m = R_n^m \cup R_n^{m+1} \cup \ldots\ldots \cup R_n^n$.

The assertion then follows from the Corollary to Theorem 2.6 and the fact that each summand is 0-dimensional. (Example 2.12).

<u>Example 2.24</u>. Suppose $0 \leq m$. Denote by M_ω^m the set of points in the Hilbert cube I_ω at most m of whose coordinates are rational.

Then

$$\operatorname{ind} M_\omega^m \leq m.$$

For $M_\omega^m = R_\omega^o \cup R_\omega' \cup \ldots\ldots \cup R_\omega^m$.

The assertion then follows from the Corollary to Theorem 2.6 and the fact that each summand is 0-dimensional.(Example 13).

The Sum and Decomposition Theorems for n-dimensional Sets.

Theorem 2.7. (Sum Theorem for Dimension n).

A separable metric space which is the countable union of closed subsets of dimension $\leq n$ has dimension $\leq n$, i.e.

if $X = \bigcup_{1}^{\infty} X_i$ where each X_i is a closed subset of X and $\operatorname{ind} X_i \leq n$, then

$$\operatorname{ind} X \leq n.$$

Proof. (by induction). Denote the sum theorem for dimension n by $\sum_n$. Clearly $\sum_n$ is equivalent to the statement that for any space which is the countable union of F_σ sets of dimension $\leq n$ has dimension $\leq n$. $\sum_{-1}$ is trivial. We now deduce $\sum_n$ from $\sum_{n-1}$ by making use of $\sum_o$, which is Theorem 2.3.

We first prove that $\sum_{n-1}$ implies the following proposition Δ_n: Any space of dimension $\leq n$ is the union of a subspace of dimension $\leq n - 1$ and a subspace of dimension ≤ 0.

Proof of Δ_n: Let X be a separable metric space of dimension $\leq n$. Then by a condition equivalent to $\operatorname{ind} X \leq n$ (p.) there exists a basis for the open sets of X made up of sets whose boundaries have dimension $\leq n - 1$. Since X is separable metric there exists a countable basis $\{U_i\}$, $i = 1, 2, \ldots.$ made up of sets whose boundaries $\{B_i\}$ have dimension $\leq n - 1$. From $\sum_{n-1}$ it follows that

$$B = \bigcup_{i=1}^{\infty} B_i \text{ has dimension } \leq n - 1,$$

i.e. $\operatorname{ind} B \leq n - 1$.

We assert that (1) $\operatorname{ind} (X \setminus B) \leq 0$.

For obviously the boundaries of the sets U_i do not meet $X \setminus B$ and hence the condition of Theorem 2.5 (with $n = 0$ and X' replaced by $X \setminus B$) is satisfied.

Δ_n then follows from the equation

$$X = B \cup (X \setminus B).$$

We now combine $\sum_{n-1}$ and Δ_n to prove $\sum_n$.

Suppose $X = X_1 \cup X_2 \cup \ldots \cup X_i \cup \ldots . = \bigcup_1^\infty X_i$

$$\text{ind } X_i \leq n \qquad i = 1, 2, 3, \ldots .$$

and each X_i is closed.

We wish to prove that $\text{ind } X \leq n$.

Let $K_1 = X_1$

$$K_i = X_i \setminus \bigcup_{j=1}^{i-1} X_j = X_i \cap (X \setminus \bigcup_{j=1}^{i-1} X_j) \qquad i = 1, 2, 3, \ldots .$$

Then (2) $X = \bigcup_1^\infty K_i$.

(3) $K_i \cap K_j = \phi$ if $i \neq j$.

(4) K_i is an F_σ in X.

(5) $\text{ind } K_i \leq n$

(2) and (3) are obvious. To prove (4) note that

$$\bigcup_{j=1}^{i-1} X_j \text{ is closed.}$$

Hence $X \setminus \bigcup_{j=1}^{i-1} X_j$ is open, and any open set in a metric space is F_σ.

K_i, as the intersection of this F_σ with the closed set X_i is thus also F_σ.

(5) holds because K_i is a subset of X_i (Theorem 2.4).

(5) enables us to apply Δ_n to each K_i ; we have

$$K_i = M_i \cup N_i$$

where $\text{ind } M_i \leq n - 1$ and $N_i \leq 0$.

Denote $\bigcup M_i$ by M and $\bigcup N_i$ by N.

From (2) $X = M \cup N$.

Each M_i is an F_σ set in M, since

$$\begin{aligned} M_i &= M_i \cap K_i \\ &= (M_1 \cup M_2 \cup \ldots\ldots \cup M_i \cup \ldots) \cap K_i \\ &= M \cap K_i \end{aligned}$$

(since $M_i \subseteq K_i$ and $K_i \cap K_j = \phi$ for $i \neq j$ by (3)).

Hence M_i, as the intersection of M with K_i, which is an F_σ set by (4), is itself F_σ in M. Therefore we may apply $\sum_{n-1}$ to conclude that

$$\text{ind } M \leq n - 1.$$

By a similar argument each N_i is an F_σ set in N and therefore

$$\text{ind } N \leq 0 \text{ by } \textstyle\sum_0.$$

Thus we have

$$X = M \cup N \quad \text{with}$$

$$\text{ind } M \leq n - 1 \text{ and } \text{ind } N \leq 0.$$

From Theorem 2.6 we conclude that $\text{ind } X \leq n$.

Corollary 1. The union of two subspaces each of which has dimension $\leq n$ and one of which is closed has dimension $\leq n$.

Proof. As in Corollary 2 to Theorem 2.3.

Corollary 2. The dimension of a non-empty space cannot be increased by the adjunction of a single point.

Proof. Obvious from Corollary 1.

Corollary 3. If a space X' of dimension $\leq n$ is contained in an arbitrary space X, then every point of the containing space has arbitrarily small neighbourhoods (in X) whose boundaries have intersections with X' of dimension $\leq n - 1$. (Compare with Theorem 5 and observe that Theorem 5 imposes a condition on the neighbourhoods of points of X' only.)

Proof. For each point $p \in X$, $X' \cup \{p\}$ has dimension $\leq n$ by Corollary 2; the proof then follows from Theorem 5.

Corollary 4. If a space has dimension $\leq n$ it is the union of a subspace of dimension $\leq n - 1$ and a subspace of dimension ≤ 0.

Proof. This is Δ_n which in the proof of Theorem 2.7 is shown to be a consequence of $\sum_{n-1}$.

Theorem 2.8. (The Decomposition Theorem for Dimension n). A space has dimension $\leq n$, n finite if and only if it is the union of $n + 1$ subspaces of dimension ≤ 0.

$$\text{i.e.} \qquad \text{ind } X \leq n \iff X = \bigcup_{i=0}^{n} X_i$$

where $\text{ind } X_i \leq 0$ for $i = 0, 1, \ldots\ldots, n$.

Proof. Follows from repeated application of Corollary 4 above and the Corollary to Theorem 2.6.

<u>Corollary</u>. If $\operatorname{ind} X = n$ and p, q are two integers ≥ -1 such that

$$p + q + 1 = n \quad , \quad \text{then}$$

$$X = P \cup Q$$

where $\operatorname{ind} P = p$ and $\operatorname{ind} Q = q$.

<u>Proof</u>. Directly from Theorem 2.8.

<u>Theorem 2.9</u>. (Dimension of a topological product)

If $A \times B$ denotes the topological productof two spaces A and B, at least one of which is non-empty, then

$$\operatorname{ind} (A \times B) \leq \operatorname{ind} A + \operatorname{ind} B.$$

<u>Proof</u>. (by induction). The proposition is obvious if either $\operatorname{ind} A = -1$ or $\operatorname{ind} B = -1$.

Let $\operatorname{ind} A = m$, $\operatorname{ind} B = n$ and assume the proposition for the cases

(1) $\operatorname{ind} A \leq m$, $\operatorname{ind} B \leq n - 1$ and

(2) $\operatorname{ind} A \leq m - 1$, $\operatorname{ind} B \leq n$.

Each point $p = (a,b)$ in $A \times B$ has arbitrarily small neighbourhoods of the form $U \times V$, U being a neighbourhood of a in A and V a neighbourhood of b in B and we may assume that

$$\operatorname{ind} b(U) \leq m - 1, \quad \operatorname{ind} b(V) \leq n - 1.$$

Now $b(U \times V) = (\overline{U} \times b(V)) \cup (b(U) \times \overline{V})$

(Kelly [1], p. 103)

Each summand is closed and by hypotheses (1) and (2) of the induction has dimension $\leq m + n - 1$.

Hence by the Sum Theorem,

$$\text{ind } b(U \times V) \leq m + n - 1$$

whence ind $(A \times B)$ $m + n =$ ind A + ind B

<u>Corollary</u>. If ind $B = 0$ then

$$\text{ind } (A \times B) = \text{ind } A + \text{ind } B$$

<u>Proof</u>. Since $B \neq \phi$; $A \times B$ contains homeomorphs of A.

Therefore ind $(A \times B) \geq$ ind A = ind A + ind B.

Combining this with Theorem 2.9 gives the Corollary.

<u>Remark</u>. One might expect that the logarithmic law in the above Corollary be true in general. Unfortunately, this is not so, for R_ω , the set of points in Hilbert space all of whose coordinates are rational, is homeomorphic to $R_\omega \times R_\omega$, while Example 2.11 shows that ind $R_\omega = 1$. The result of the Corollary does not even hold if both A and B are compact. This is shown by Pontryagin's example of two compact 2-dimensional spaces whose product is 3-dimensional. It can be shown that the result does hold if B is one-dimensional provided A is compact. It is an open problem to characterize the spaces B for which the result holds for arbitrary A.

(iii) The Lebesque Covering dimension 'dim'.

<u>Definition 2.6</u>. A <u>covering</u> of a space X is a collection $\mathcal{U}$ of subsets of X satisfying the condition

$$\bigcup \{U \mid U \in \mathcal{U}\} = X$$

<u>Definition 2.7</u>. A collection $\mathcal{V}$ is a <u>refinement</u> of a collection $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ for which $V \subseteq U$, and $\bigcup \{V \mid V \in \mathcal{V}\} = X$.

<u>Definition 2.8</u>. A covering $\mathcal{U}$ has order $\leq n$ if and only if at most $n + 1$ members of $\mathcal{V}$ have a non-empty intersection.

<u>Definition 2.9</u>. A space X has covering dimension $\leq n$, denoted by $\dim X \leq n$, if given any finite open covering $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ whose order is $\leq n$. If $\dim X \leq n$ and the statement $\dim X \leq n - 1$ is false, we say that $\dim X = n$. For the empty set, $\dim \phi = -1$. If the statement $\dim X \leq n$ is false for all n, then $\dim X = \infty$.

Clearly if Y is a topological space homeomorphic to X, then $\dim Y = \dim X$.

<u>Theorem 2.10</u>. For any space X, $\dim X \leq n$ if and only if for each finite open covering $\mathcal{U} = \{U_1, U_2, U_3, \ldots\ldots, U_k\}$ there exists an open refinement $\mathcal{V} = \{V_1, V_2, \ldots\ldots, V_k\}$ of order $\leq n$ satisfying

$$V_i \subseteq U_i \qquad \text{for each } i = 1, 2, \ldots\ldots, k.$$

<u>Proof</u>. The condition is clearly sufficient.

Conversely, suppose $\dim X \leq n$ and $\mathcal{U} = \{U_1, U_2, \ldots U_k\}$ is any finite open covering of X. Then there exists an open refinement

$$\mathcal{W} = \{W_\alpha\} \qquad \text{of order } \leq n \text{ such that}$$
$$W_\alpha \subseteq U_i \qquad \text{for some } i\ ,\ 1 \leq i \leq k.$$

One obtains the desired <u>finite</u> refinement $\mathcal{V}$ of $\mathcal{U}$ by defining

$$V_j = \bigcup \{W_\alpha | W_\alpha \subseteq U_j \text{ and } W_\alpha \nsubseteq U_i \text{ for } i < j\}\ .$$

It is easy to show that the order of $\mathcal{V}$ is $\leq n$.

Theorem 2.11. If $\dim X \leq n$ and X' is a closed subspace of X then $\dim X' \leq n$.

Proof. Let $\mathcal{U}'$ be a finite open covering of X'. For each $U' \in \mathcal{U}'$ we have

$$U' = U \cap X' \quad \text{for some } U \text{ open in } X.$$

These open sets U, together with $X \setminus X'$ form an open covering of X. Since $\dim X \leq n$, there exists a system $\mathcal{V}$ of sets open in X satisfying the conditions in Definition 2.9.

Let $V' = V \cap X'$ where $V \in \mathcal{V}$, giving the desired refinement $\mathcal{V}'$ of the open covering $\mathcal{U}'$ of X'.

Hence $\dim X' \leq n$.

Note: The above theorem is not true if X' is not closed; see Tychonov Plank (Appendix).

The following theorems (Theorem 2.12 - 2.19) are stated without proof. For proofs, the reader is referred to Walton [1].

Theorem 2.12. Let X be a normal space and $F_1, F_2, \ldots F_m$ be closed subsets of X, finite in number. Then each F_i $(1 \leq i \leq m)$ can be associated with $U_i \supseteq F_i$, U_i open in X such that for any arbitrary combination $(i_1, \ldots, i_k)$, where $1 \leq k \leq m$, of indices $1, 2, \ldots, m$ if

$$\bigcap_{r=1}^{k} \overline{U}_{i_r} \neq \phi$$

then

$$\bigcap_{r=1}^{k} F_{i_r} \neq \phi .$$

<u>Theorem 2.13</u>. Let X be a normal space. Let $U_1, U_2, \ldots, U_m$ be finite in number, open in X, and cover X. Then for each U_i $(1 \leq i \leq m)$ we can find a V_i open in X such that

(1) $\overline{V}_i \subseteq U_i$ for $1 \leq i \leq m$.

(2) $V_1, V_2, \ldots, V_m$ cover X.

<u>Theorem 2.14</u>. Let the topological space X have the following property: Each finite open cover $\mathcal{U}$ of X can be associated with a finite closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is a subset of some $U \in \mathcal{U}$. Then X is normal.

<u>Theorem 2.15</u>. Let X be a normal space and A,B be F_σ sets in X satisfying

$$A \cap \overline{B} = \overline{A} \cap B = \phi$$

Then there exist U,V open in X such that

$$A \subseteq U \text{ , } B \subseteq V \text{ , } U \cap V = \phi.$$

<u>Theorem 2.16</u>. Let X be a normal space and Y a subspace of X, where Y is an F_σ-set in X. Then Y is normal.

<u>Theorem 2.17</u>. Let X be a normal space and $\dim X \leq n$. Let $\{U_1, U_2, \ldots U_n\}$ be a finite open cover of X. Then there exists another finite open cover $\{V_1, V_2, \ldots, V_n\}$ of X such that

(1) $\overline{V}_i \subseteq U_i$ for $1 \leq i \leq m$

(2) order of $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_m\}$ is $\leq n$.

Theorem 2.18. Let $n = -1, 0, 1, 2, \ldots$ and let the topological space X have the following property: For each finite open cover $\mathcal{U}$ of X there exists a finite closed cover $\mathcal{F}$ of X such that

(1) each $F \in \mathcal{F}$ is a subset of some $U \in \mathcal{U}$

(2) $\mathcal{F}$ has order $\leq n$.

Then X is normal and $\dim X \leq n$.

Theorem 2.19. Let X be a normal space, A a closed subset of X and $\dim A \leq n$. Let $U_1, U_2, \ldots, U_m$ be finite in number, open in X, and cover A. Then there exist $V_1, V_2, \ldots, V_m$ open in X such that

(1) $\overline{V}_i \subsetneq U_i$

(2) $V_1, V_2, \ldots, V_m$ cover A

(3) order of $\{\overline{V}_1, \overline{V}_2, \ldots\} \leq n$.

The following theorem was first proved by E. Čech in 1933. The theorem is not true if the space X is not normal; see 'Tychanov plank (Appendix).

Theorem 2.20. Let X be a normal space and

$$X = \bigcup_{\nu=1}^{\infty} A_\nu$$

where A_ν is closed in X and $\dim A_\nu \leq n$ for $\nu = 1, 2, 3, \ldots$.

Then $\dim X \leq n$.

Proof. Let U_i $(1 \leq j \leq m)$ be a finite open cover of X. We construct V_i $(1 \leq i \leq m)$ open in X such that

(i) the system $\mathcal{V} = \{V_1, V_2, \ldots\ldots, V_m\}$ covers X and has order $\leq$ n.

(ii) $V_i \subseteq U_i \quad (1 \leq i \leq m)$.

By Theorem 2.19 there exists a system $\mathcal{V}'$ of V_i $(1 \leq i \leq m)$ open in X such that

(i) $\overline{V}_i \subseteq U_i$

(ii) the system $\mathcal{V}'$ covers A

(iii) the system $\overline{\mathcal{V}}'$ of sets $\overline{V}_i$ has order $\leq n$.

Assume now that the system $\mathcal{V}^\nu$ of V_i^ν $(1 \leq i \leq m)$ open in X has been constructed such that

(i) $\overline{V}_i^\nu \subseteq U_i$

(ii) the system $\mathcal{V}^\nu$ covers $\bigcup_{\lambda=1}^{\nu} A_\nu$

(iii) the system $\overline{\mathcal{V}}^\nu$ of $\overline{V}_i^\nu$ has order $\leq n$.

We now show that the system $\mathcal{V}^{\nu+1}$ can be constructed.

Since $\overline{\mathcal{V}}^\nu$ is a finite system of order $\leq n$ of sets closed in X ; then by Theorem 2.12 there exist S_i $(1 \leq i \leq m)$ open in X such that

(i) $\overline{V}_i^\nu \subseteq S_i$

(ii) the system $S_1, S_2, \ldots\ldots, S_m$ has order $\leq n$.

Since $\overline{V}_i^\nu \subseteq U_i$, by the normality of X there exist T_i $(1 \leq i \leq m)$ open in X such that

$$\overline{V}_i^\nu \subseteq T_i \subseteq \overline{T}_i \subseteq U_i \ .$$

Let $$W_i = S_i \cap T_i$$

Then the system $\mathcal{W}$ of sets $W_1, W_2, \ldots\ldots W_m$ is a finite system of order $\leq n$ of open sets of X and

$$\overline{V}_i^\nu \subseteq W_i \subseteq \overline{W}_i \subseteq U_i \ .$$

By normality there exists P_i $(1 \leq i \leq m)$ open in X such that

$$\overline{V}_i^{\nu} \subseteq P_i \subseteq \overline{P}_i \subseteq W_i$$

For $1 \leq i \leq m$ let:

1) $\mathcal{M}_i$ mean the system of binary open coverings

$$\{P_i \, , \, X \setminus \overline{V}_i^{\nu}\} \, .$$

2) $\mathcal{N}_i$ the system of binary open coverings

$$\{W_i \, , \, X \setminus \overline{P}_i\} \, .$$

Clearly each of the $2m$ systems $\mathcal{M}_i$, $\mathcal{N}_i$ cover X. Let $\mathcal{H}$ denote the system of $m.4^m$ open sets of X formed by

$$U_i \cap \bigcap_{j=1}^{m} M_j \cap \bigcap_{k=1}^{m} N_k \qquad (1 \leq i, j,k \leq m)$$

$$M_j \in \mathcal{M}_j \; ; \; N_k \in \mathcal{M}_k$$

Then $\mathcal{H}$ is a finite system of open sets covering X. Since $A_{\nu+1}$ is closed in X and since $\dim A_{\nu+1} \leq n$, by Theorem 19 there exists a finite system Z of open sets Z_r $(1 \leq r \leq t = m.4^m)$ of X such that

(i) each $\overline{Z}_r \subset$ some element of $\mathcal{H}$;

(ii) Z covers $A_{\nu+1}$;

(iii) Z has order $\leq n$.

From (i) it follows by definition of the systems $\mathcal{M}_i$, $\mathcal{N}_i$, $\mathcal{H}$ that

(iv) $\overline{Z}_r \cap \overline{V}_i^{\nu} \neq \phi \Rightarrow \overline{Z}_r \subseteq P_i$;

(v) $\overline{Z}_r \cap \overline{P}_i \neq \phi \Rightarrow \overline{Z}_r \subseteq W_i$.

We divide the sets Z_r $(1 \leq r \leq t)$ into three kinds, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$;

$Z_r \in \mathcal{A}$ if there exists i $(1 \leq i \leq m)$ such that $\overline{Z}_r \cap \overline{V}_i^{\nu} \neq \phi$.

$Z_r \in \mathcal{B}$ if $Z_r \notin \mathcal{A}$ and if there exists i $(1 \leq i \leq m)$ such that $\overline{Z}_r \cap \overline{P}_i \neq \phi$.

$$Z_r \in \mathcal{C} \text{ if } Z_r \notin \mathcal{A} \cup \mathcal{B} .$$

With each $Z_r \in \mathcal{A}$ we associate each index i such that $Z_r \cap \overline{V}_i^{\nu} \neq \phi$.

With each $Z_r \in \mathcal{B}$ we associate a single index i so chosen that $Z_r \cap \overline{P}_i \neq \phi$.

With each $Z_r \in \mathcal{C}$ we associate a single index i so chosen that $\overline{Z}_r \subsetneq U_i$. (This is possible by definition of $\mathcal{H}$ and condition (1) for the system $\mathcal{Z}$.)

For $1 \leq i \leq m$ let $W_i' = V_i^{\nu} \cup Z_r$ summed over all $Z_r \in \mathcal{A} \cup \mathcal{B}$ associated with index i.

Clearly $V_i^{\nu} \subsetneq W_i'$ and by (5), $\overline{W}_i' \subsetneq W_i$. Further let $V_i^{\nu+1} = W_i' \cup Z_r$ summed over all $Z_r \in \mathcal{C}$ associated with the index i.

Clearly $V_i^{\nu+1}$ is open in X and

$$V_i^{\nu} \subseteq V_i^{\nu+1} \subsetneq \overline{V}_i^{\nu+1} \subsetneq U_i .$$

Since $\mathcal{V}^{\nu}$ covers $\bigcup_{\lambda=1}^{\nu} A_{\lambda}$ and since $\mathcal{Z}$ covers $A_{\nu+1}$, it is clear that the system $\mathcal{V}^{\nu+1}$ of sets $V^{\nu+1}, \ldots\ldots, V_m^{\nu+1}$ covers $\bigcup_{\lambda=1}^{\nu+1} A_{\lambda}$.

It is still to be shown that $\mathcal{V}^{\nu+1}$ has order $\leq n$, i.e. that each point $a \in X$ belongs to at most $n + 1$ elements of the system $\mathcal{V}^{\nu+1}$. We distinguish two cases.

(a) Suppose there exists an index j $(1 \leq j \leq m)$ such that $a \in \overline{P}_j$. Then $a \in Z_r \Rightarrow Z_r \cap \overline{P}_j \neq \phi$, i.e. $Z_r \notin \mathcal{C}$. Hence $a \in V_i^{\nu+1} \Rightarrow a \in W_i' \subsetneq W_i$. Since $\mathcal{W}$ is of order $\leq n$, the point a belongs to at most $n + 1$ of the $V_i^{\nu+1}$.

(b) Suppose $a \notin P_i$, $i = 1, 2, \ldots\ldots, m$.

By (4) $a \in Z_r \Rightarrow Z_r \cap \overline{V}_i^{\nu} = \phi$.

Hence on the one hand $a \notin$ any V_i^ν and on the other hand $a \notin$ any $Z_r \in A$. Hence $a \in W_i' \Rightarrow a \in Z_r$ where $Z_r \in \mathcal{B}$ and is associated with the index i.

Since each $Z_r \in \mathcal{B}$ is associated with a single index i, the point a is in at most as many W_i' as in $Z_r \in \mathcal{B}$. Similarly a is in at most as many $V_i^{\nu+1} \setminus W_i'$ as in $Z_r \in \mathcal{C}$. Hence the point a is in at most as many $V_i^{\nu+1}$ as Z_r ; since Z is of order $\leq n$ the point a belongs to at most $n + 1$ of the sets $V_i^{\nu+1}$.

Thus it is shown by recurrence that we can construct systems $\mathcal{V}^\nu$ $(\nu = 1, 2, 3, \ldots)$ of open sets V_i^ν $(1 \leq i \leq m)$ in X such that

(1) $V_i^\nu \subseteq V_i^{\nu+1}$,

(2) $V_i^\nu \subseteq U_i$,

(3) $\mathcal{V}^\nu$ covers A_ν.

(4) $\mathcal{V}^\nu$ has order $\leq n$.

Let $V_i = \bigcup_{\nu=1}^{\infty} U_i^\nu$ $\qquad (1 \leq i \leq m)$

Then the V_i are open sets of X. By (2) $V_i \subseteq U_i$. Since $X = \bigcup_{\nu=1}^{\infty} A_\nu$, by (3) the system $\mathcal{V}$ of sets $V_1, V_2, \ldots, V_m$ covers X. Suppose, on the contrary, that there were $n + 2$ different induces $i_1, i_2, \ldots\ldots, i_{n+2}$ and a point a X such that

$$a \in V_{i_s} \qquad (1 \leq s \leq n+2).$$

But this contradicts (4).

Corollary. Let X be a normal space. Let A_ν $(\nu = 1,2,3,\ldots\ldots)$ be F_σ-sets in X and let $\dim A_\nu \leq n$. Then $\dim \bigcup_{\nu=1}^{\infty} A_\nu \leq n$.

Definition 2.10. A topological space X is perfectly normal if

(i) X is normal,

(ii) each open set of X is an F_σ-set in X, or equivalently, each closed set of X is a G_δ-set in X (i.e. the countable intersection of open sets).

Theorem 2.21. (Čech's Subset Theorem). Let X be a perfectly normal space and let $\dim X \leq n$. Then if Y is any subspace of X, $\dim Y \leq n$.

C.H. Dowker has extended Čech's subset theorem to a class of spaces called totally normal.

Definition 2.11. A family $\mathcal{U}$ of subsets of a topological space is locally finite if each point of the space has a neighbourhood which intersects only finitely many members of $\mathcal{U}$.

Definition 2.12. A normal space X is totally normal if each open set G of X has a locally finite covering by subsets each of which is an open F_σ-set of X. Totally normal spaces include hereditarily paracompact Hausdorff as well as perfectly normal spaces.

Theorem 2.22. (Dowker). If Y is a subspace of a totally normal space X, then $\dim Y \leq \dim X$.

Definition 2.13. A collection $\mathcal{U}$ of subsets of a space X is called star-finite if every element of $\mathcal{U}$ intersects at most a finite number of other elements of $\mathcal{U}$. (A star-finite open covering is of course locally finite).

Definition 2.14. A space is said to have the star-finite property if every open covering can be refined by a star-finite open covering.

Definition 2.15. A topological space is paracompact if each open cover of the space has an open locally finite refinement.

Definition 2.16. If $\mathcal{U}$ is a family of subsets of a set X and $x \in X$, then the star of $\mathcal{U}$ is the union of the members of $\mathcal{U}$ to which x belongs. A cover $\mathcal{B}$ is a star-refinement of $\mathcal{U}$ (or Δ-refinement) if the family of stars of $\mathcal{U}$ at points of X is a refinement of $\mathcal{U}$. A topological space is fully normal if each open cover has an open star refinement.

A.H. Stone [1] has proved the following important result:

Theorem 2.23. A Hausdorff space is fully normal if and only if it is paracompact.

Corollary. Every metric space is paracompact.

We now state without proof several results about the covering dimension of a topological product.

In 1946 Hemmingsen showed that if X and Y are both compact Hausdorff spaces then

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

This result was sharpened by Miyazaki in 1951.

Theorem 2.24. If X is compact normal and Y is paracompact normal, then

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Morita [2] proved the following three theorems:

<u>Theorem 2.25</u>. Let X and Y be Hausdorff spaces such that $X \times Y$ has the star-finite property (S-space) then

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

<u>Theorem 2.26</u>. If X is a fully normal space and Y is a locally compact fully normal space, then

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

<u>Theorem 2.27</u>. If X is a countably paracompact normal space and Y is a locally compact metric space, then

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

In the same paper Morita proves the following stronger relation between the covering dimension of $X \times Y$ and those of X and Y.

<u>Theorem 2.28</u>. The relation

$$\dim(X \times Y) = \dim X + \dim Y$$

holds for the following cases:

(i) X is locally compact fully normal space of dimension ≥ 0 and Y is a fully normal space of dimension 1.

(ii) X is a fully normal space of dimension ≥ 0 and Y is a locally finite polytype of dimension ≥ 0.

We end this section on the covering dimension by quoting the following theorem, analogous to Theorem 2.6.

<u>Theorem 2.29</u>. If $X = Y \cup Z$ is a normal space and $\dim Y \leq m$, $\dim Z \leq n$, then $\dim X \leq m + n + 1$.

(iv) The Čech strong inductive dimension 'Ind' .

Definition 2.17. If $X = \phi$, Ind $X = -1$. For $n = 0,1,2,\ldots,$ a space X has strong inductive dimension Ind $X \leq n$ if for every pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set U such that

$$F \subseteq U \subseteq G \quad , \quad \text{Ind } b(U) \leq n - 1.$$

Ind $X = n$ if it is true that Ind $X \leq n$ and false that Ind $X \leq n - 1$. If Ind $X \leq n$ is false for $n = -1,0,1,2, \ldots,$ then we say that Ind $X = \infty$.

Clearly if X^* is any homeomorph of X, then Ind X^* = Ind X.

C.H. Dowker [1] has established the following results:

Theorem 2.30. If X' is a closed subset of any space X, Ind $X' \leq$ Ind X.

Note: The theorem is not true if X' is not closed; see the Tychonov Plank. (Appendix).

Theorem 2.31. Ind $X \leq n$ is equivalent to the following condition on X: If $F \subseteq G \subseteq X$ with F closed and G open, then X is the union of three disjoint sets U, V, C with U, V open, $F \subseteq U \subseteq G$ and Ind $C \leq n - 1$.

Theorem 2.32. If X is normal, Ind $X \leq n$ is equivalent to the following condition: If E and F are disjoint closed subsets of X, then X is the union of disjoint sets U, V, and C with U and V open, $E \subseteq U$, $F \subseteq V$, and Ind $C \leq n - 1$.

Theorem 2.33. Let Y_i $(i = 1,2, \ldots.)$ be open sets in a hereditarily normal space Y such that

$$Y = Y_1 \supseteq Y_2 \supseteq \ldots\ldots, \quad \bigcap_{i=1}^{\infty} Y_i = \phi .$$

and, for each i, $\operatorname{Ind}(Y_i \setminus Y_{i+1}) \leq n$. Then $\operatorname{Ind} Y \leq n$.

A particular case of the above theorem is obtained by putting $Y_1 = Y$, $Y_2 = Y \setminus A$, $Y_3 = Y_4 = \ldots = \phi$. If A is a closed subset of a hereditarily normal space Y and if $\operatorname{Ind} A \leq n$ and $\operatorname{Ind}(Y \setminus A) \leq n$, then $\operatorname{Ind} Y \leq n$. Lokucievskiǐ has produced an example to show that this special case of the theorem does not hold for arbitrary normal spaces.

We consider the following conditions which a space X may satisfy:

(a_n) If $B \subseteq A \subseteq X$ and $\operatorname{Ind} A \leq n$, then $\operatorname{Ind} B \leq n$.

(b_n) If $G \subseteq A \subseteq X$ with G open in A and $\operatorname{Ind} A \leq n$, then $\operatorname{Ind} G \leq n$.

(c_n) If $A = B \cup C \subseteq X$ with B closed in A, $\operatorname{Ind} B \leq n$ and $\operatorname{Ind} C \leq n$, then $\operatorname{Ind} A \leq n$.

(d_n) If $A = \bigcup_{i=1}^{\infty} A_i \subseteq X$ with each A_i closed in A and $\operatorname{Ind} A_i \leq n$, then $\operatorname{Ind} A \leq n$.

Dowker [1] has proved the following theorem:

<u>Theorem 2.34</u>. If X is a hereditarily normal space satisfying condition (b_n) for all n, then X also satisfies (a_n), (c_n), (d_n) for all n.

Dowker concludes the paper with proofs of the following theorems:

<u>Theorem 2.35</u>. Let $A \subseteq X$ with X totally normal and $\operatorname{Ind} X \leq n$. Then $\operatorname{Ind} A \leq n$.

Theorem 2.36. Let a totally normal space X be the union of two sets A and B with A closed and $\operatorname{Ind} A \leq n$ and $\operatorname{Ind} B \leq n$. Then $\operatorname{Ind} X \leq n$.

Theorem 2.37. Let $\{A_i\}$ be a sequence of closed sets in a totally normal space and let $\operatorname{Ind} A_i \leq n$. Then

$$\operatorname{Ind} \bigcup_{i=1}^{\infty} A_i \leq n.$$

These last two theorems are extensions of theorems given earlier by Čech for the case of perfectly normal spaces.

Product Theorems.

Katetov [2] and Morita [1] gave different proofs of the following theorem:

Theorem 2.38. If X and Y are metric spaces, at least one of which is nonempty, then

$$\operatorname{Ind} (X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

Nagami [1] extended the result as follows:

Theorem 2.39. Let X be a perfectly normal space and Y a metric space. If at least one of X and Y is nonempty, then

$$\operatorname{Ind} (X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

We now state a Theorem analogous to Theorems 2.6 and 2.29.

Theorem 2.40. Let $X = Y \cup Z$ where X is totally normal and $\operatorname{Ind} Y \leq m$, $\operatorname{Ind} Z \leq n$, then $\operatorname{Ind} X \leq m + n + 1$.

CHAPTER 3

Relations between the dimension functions 'ind', 'Ind', and 'dim'

In this Chapter we summarize various relationships between the three dimension functions introduced in Chapter 2.

Hurewicz and Wallman [1] completed the theory for separable metric spaces and proved the equivalence of the three ideas for such spaces. P. Roy [1] in 1962 showed that the three dimension concepts are not equivalent for general metric spaces by producing a metric space X for which $\text{ind } X = 0$ but $\text{Ind } X = \dim X = 1$. Katětov [1], Morita [1] and Dowker and Hurewicz [2] have all published different proofs that for any metric space X, $\text{Ind } X = \dim X$.

We now present several results on relationships between the dimension functions for spaces subject to various conditions.

Theorem 3.1. If X is a T_1-space then $\text{ind } X \leq \text{Ind } X$, i.e. $\text{Ind } X \leq n \Rightarrow \text{ind } X \leq n$.

Proof. The theorem follows immediately from the definitions of the two dimension functions and the fact that singleton sets are closed in T_1-spaces.

Theorem 3.2. If X is a compact normal space,

$$\text{ind } X \leq 0 \Rightarrow \text{Ind } X \leq 0$$

Proof. Let F and G be closed and open subsets respectively in X and let $F \subseteq G$. Since $\text{ind } X \leq 0$, for each point $p \in F$ there exists an open and closed set V in X such that

$$p \in V \subseteq G.$$

Since F is a closed subset of a compact space X, it also is compact, and a finite collection

$$\{V_1, V_2, \ldots\ldots, V_k\} \quad \text{covers} \quad F.$$

Then $F \subseteq \bigcup_{i=1}^{k} V_i \subseteq G$, where $\bigcup_{i=1}^{k} V_i$ is closed and open.

Hence $\operatorname{Ind} X \leq 0$.

We have the following counter-example to show that the implication in Theorem 3.2 cannot be reversed.

<u>Counter-example</u>. Let $X = \{a,b\}$ and the open sets be $\{a\}$, X, ϕ; then the closed sets are X, $\{b\}$, ϕ. X is trivially compact and normal but not T_1 and $\operatorname{Ind} X = 0$, while $\operatorname{ind} X = 1$.

Theorems 3.1 and 3.2 together imply

<u>Theorem 3.3</u>. If X is a compact Hausdorff space,

$$\operatorname{ind} X \leq 0 \iff \operatorname{Ind} X \leq 0.$$

<u>Theorem 3.4</u>. $\operatorname{Ind} X \leq 0 \Rightarrow X$ is normal.

<u>Proof</u>. This follows immediately from Definition 2.17.

<u>Theorem 3.5</u>. For any space X, $\dim X \leq 0 \iff \operatorname{Ind} X \leq 0$.

The proof of Theorem 3.5 follows readily from the definitions of the two dimension functions.

<u>Theorem 3.6</u>. $\dim X \leq 0 \iff \operatorname{Ind} X \leq 0 \Rightarrow X$ is normal.

<u>Proof</u>. This is an immediate consequence of the previous two theorems.

However, $\dim X = n$ does not imply that X is normal if $n \leq 1$, as is illustrated by the following example.

Example. Let $X = \{a,b,c\}$ with open sets ϕ, $\{b\}$, $\{a,b\}$, $\{b,c\}$, and X.

Given the open covering $\{\{a,b\}, \{b,c\}\}$, both $\{a,b\}$ and $\{b,c\}$ must occur in any refinemant and they intersect in the point b; hence the order of this covering is 1. Every covering of X by open sets has a refinement of order ≤ 1. Hence $\dim X = 1$. However X is not normal because $\{a\}$, $\{b\}$ are disjoint closed sets and every two open sets both contain the point b.

Theorem 3.7. If X is a compact Hausdorff space, $\operatorname{ind} X \leq 0 \iff \operatorname{Ind} X \leq 0 \iff \dim X \leq 0$.

Proof. Combine Theorems 3.1, 3.2, and 3.5.

N. Vedenissoff [1] has proved the following theorems.

Theorem 3.8 . If X is a normal space, $\dim X \leq \operatorname{Ind} X$.

Theorem 3.9 . If X is a compact normal space, then $\dim X \leq \operatorname{ind} X$.

Lokucievskiǐ [1] has given an example of a compact Hausdorff space S with

$$\dim S = 1 \quad , \quad \operatorname{ind} S = \operatorname{Ind} S = 2.$$

This shows that strict inequality can occur in the above theorems 3.8 and 3.9.

Theorem 3.10. If X is a compact totally normal space, then $\operatorname{Ind} X \leq \operatorname{ind} X$.

Proof. follows easily by induction.

We state without proof the following theorems on relations between the various dimension functions for metric spaces.

Theorem 3.11. If X is a separable metric space, then $\operatorname{ind} X = \operatorname{Ind} X$.

Theorem 3.12. For any metric space X, $\dim X = \operatorname{Ind} X$.

As was mentioned earlier, Katětov, Morita, and Dowker gave independent proofs of this theorem.

Theorem 3.13. For any separable metric space X

$$\dim X = \operatorname{Ind} X = \operatorname{ind} X.$$

P. Roy [1] showed that Theorem 3.13 is not true for arbitrarily metric spaces by constructing a space complete metric space S for which

$$\operatorname{ind} S = 0, \quad \text{but} \quad \dim S = \operatorname{Ind} S = 1.$$

Dowker [2] has also given an example of a normal space M with $\operatorname{ind} M = 0$ and $\dim M = 1$.

In Chapter Two we proved that the sum theorem for dim holds for normal spaces. Lokucievskiĭ [1] constructs a compact space R for which the sum theorems for ind and Ind are not true.

In the same book Nagami gives an example of a normal space with $\operatorname{ind} = 0$, $\dim = 1$, and $\operatorname{Ind} = 2$.

Thus, we see that basic gaps exist between the various dimension functions, and that normality and even compactness do not effect the equalities between the dimensions.

CHAPTER 4

The Dimension Dim

Definition 4.1. The dimension Dim X of a topological space X is the least integer $n (n = -1, 0, 1, 2 \ldots)$ for which

$$X = \bigcup_{i=0}^{n} X_i$$

where each subspace X_i of X has $\dim X_i \leq 0$.

$\mathrm{Dim}\, \phi = -1$.

Theorem 4.1. If X is a topological space with $\mathrm{Dim}\, X \leq n$ and A is a closed subset of X, then $\mathrm{Dim}\, A \leq n$.

Proof. The theorem is clearly true if $n = -1$.

We consider two cases : (1) $n = 0$, (ii) $n > 0$.

Case (i) $n = 0$. Let $\{V_1, V_2, \ldots, V_k\}$ be any finite open covering of A. Then $V_i = A \cap U_i$, where U_i is open in X. The sets $U_1, \ldots, U_k$, together with $X \setminus A$ form an open covering of X. Since $\dim X \leq 0$ there is a disjoint refinement $\{W_1, \ldots, W_n\}$ where $W_j \subseteq U_i$ for some i or $W_i \subset X \setminus A$. The system $\{W_j \cap A\}$ forms a disjoint open covering of A which is a refinement of $\{V_i\}$. Hence $\dim A \leq 0$. Since $\dim A \leq 0 \iff \mathrm{Dim}\, A \leq 0$, we have $\mathrm{Dim}\, A \leq 0$.

Case (ii) $n > 0$. Let $X = \bigcup_{i=0}^{n} X_i$, where $\dim X_i \leq 0$.

Then $A = \bigcup_{i=0}^{n} A_i$, where $A_i = A \cap X_i$ $(i = 0, 1, \ldots, n)$.

(Some of the A_i may be empty).

Since A is a closed subset of X, each A_i is a closed subset of X_i.

Therefore by Case (i) $\dim A_i \leq 0$.

Since $A = \bigcup_{i=0}^{n} A_i$ and $\dim A_i \leq 0$, $\mathrm{Dim}\, A \leq 0$.

Note: Theorem 4.1 is not true if A is not closed; see Tychonov Plank (Appendix).

Theorem 4.2. If A and B are subspaces of a topological space X and $\operatorname{Dim} A \leq m$, $\operatorname{Dim} B \leq n$, then

$$\operatorname{Dim}(A \cup B) \leq \operatorname{Dim} A + \operatorname{Dim} B + 1.$$

Proof. By definition

$$A = \bigcup_{i=0}^{m} A_i \quad \text{and} \quad B = \bigcup_{j=0}^{n} B_j$$

where $\dim A_i \leq 0$ and $\dim B_j \leq 0$.

Define $C_i = A_i \qquad (i = 0, 1, 2, \dots m)$

$C_{m+1+j} = B_j \qquad (j = 0, 1, \dots, n)$

Then $A \cup B = \bigcup_{k=0}^{m+n+1} C_k$, where $\dim C_k \leq 0$.

Hence by definition

$$\operatorname{Dim}(A \cup B) \leq m + n + 1$$

i.e. $\operatorname{Dim}(A \cup B) \leq \operatorname{Dim} A + \operatorname{Dim} B + 1$.

Lemma. Let A be any subset of a topological space X where $\dim A \leq 0$. Then if $\{U_1, \dots, U_r\}$ is any finite open covering of A by open sets of X, there exists a system $\{V_1, V_2, \dots, V_r\}$ of open subsets of X such that

(i) $V_i \subseteq U_i$

(ii) $\{V_i\}$ covers A

(iii) $V_i \cap V_j \cap A^- = \phi$ if $i \neq j$.

Proof. Since $\dim A \leq 0$ the covering $\{A \cap U_i\}$ of A has an open refinement $W_i \subset A \cap U_i$ with $W_i \cap W_j = \phi$ if $i \neq j$ where $W_i = A \cap Z_i$ for some Z_i

open in X. Let $V_i = Z_i \cap U_i$. Then $V_i \subseteq U_i$ and $V_i \cap V_j \cap A = \phi$. Since $V_i \cap V_j$ is open, no point of it is an accumulation point of A; hence $V_i \cap V_j \cap A^- = \phi$ if $i \neq j$.

Theorem 4.3. If X is a normal space, $\dim X \leq \operatorname{Dim} X$.

Proof. We show that if X is normal then $\operatorname{Dim} X \leq n$ implies $\dim X \leq n$. The result is true for $n = -1, 0$. Assume it true for $n = m - 1$. Let $U_1, \ldots, U_k$ be any finite open covering of X_o^-, where $X = \bigcup_{i=0}^{m} X_i$, $\dim X_i \leq 0$.

Since $\dim X_o \leq 0$, by the lemma there exists a system $\{V_1, \ldots, V_k\}$ open in X such that $V_i \subseteq U_i$ and $\{V_i\}$ covers X_o and $V_i \cap V_j \cap X_o^- = \phi$ $(i \neq j)$. Let $V = \bigcup_{i=1}^{k} V_i$. Put $S = X_o^- \setminus V$. Then S is closed in X and $S \subseteq X \setminus X_o$. Hence $\dim S \leq m - 1$ by the hypothesis of induction.

Since X_o^- is closed in X, X_o^- is normal, so that by Theorem 2.19 (Čech) there exists $\{W_1, \ldots, W_k\}$ open in X_o^- with $W_i^- \subseteq U_i$, $W = \bigcup W_i \supset S$ and order of $\{W_i^-\}$ is $\leq m - 1$. Thus $\{V_i, W_i\}$ is an open covering of X_o^- which forms a refinement of $\{U_i\}$ and is of order $\leq m$. Hence $\dim X_o^- \leq m$. Similarly $\dim X_i^- \leq m$, where $i = 1, 2, \ldots, m$.

Then $X \bigcup_{i=0}^{m} X_i^-$, and by the Čech Sum Theorem,

$$\dim X \leq m.$$

Theorem 4.4. Let X be an hereditarily normal space. Then if $A_1, A_2, \ldots, A_k$ are closed subsets of X for which $\operatorname{Dim} A_i \leq n$, then $\operatorname{Dim} A \leq n$, where $A = \bigcup_{i=1}^{k} A_i$.

Proof. We first prove the theorem for the case when $k = 2$.

Let L, M be closed subsets of the hereditarily normal space X, and let

$$\text{Dim } L \leq n \quad , \quad \text{Dim } M \leq m.$$

i.e. $L = \bigcup_{i=0}^{n} L_i$, $M = \bigcup_{i=0}^{n} M_i$ where $\dim L_i \leq 0$ and $\dim M_i \leq 0$.

Let $P_o = L_o \cup (M_o \setminus L)$.

Then P_o is normal and $L_o = P_o \cap L$, i.e. L_o is closed in P_o.

Let F_o be a closed subset of P_o such that $F_o \cap L_o = \phi$.

Then $F_o = F \cap M_o$ where F is closed in X, i.e. F_o is a closed subset of M_o, so that $\dim F_o \leq 0$.

By a lemma of Dowker ("If H is closed in the normal space S and if $\dim H \leq n$ and $\dim F \leq n$ for each closed set F such that $F \cap H = \phi$, then $\dim S \leq n$") for the case $n = 0$, we have $\dim P_o \leq 0$.

Let $P_1 = L_1 \cup (M_1 \setminus L)$. Then, as before, $\dim P_1 \leq 0$, etc.
Now $L \cup M = \bigcup_{i-0}^{n} P_i$ where $\dim P_i \leq 0$. Hence $\text{Dim } (L \cup M) \leq n$.

The theorem now follows directly for the union of a finite number of closed subsets of X.

<u>Theorem 4.5</u>. If X is an hereditarily normal space, then $\text{Ind } X \leq \text{Dim } X$.

<u>Proof</u>. We show that $\text{Dim } X \leq n$ implies $\text{Ind } X \leq n$. The result is true for $n = -1, 0$; assume true for $n = m - 1$. Let $X = \bigcup_{i=0}^{m} X_i$, where X is hereditarily normal and $\dim X_i \leq 0$. By a theorem of Walton [1], p. 81, given F closed in X and G open in X with $F \subsetneq G$ there exists a set V open in X with $F \in V \subset G$ such that $b(V)$ does not meet X_m ,

$$\text{i.e.} \quad b(V) \subsetneq \bigcup_{i=1}^{m-1} X_i.$$

Hence by the inductive hypothesis,

$$\text{Ind } b(V) \leq m - 1, \quad \text{and} \quad \text{Ind } X \leq m.$$

<u>Corollary</u>. For an hereditarily normal T_1 space X, $\text{ind } X \leq \text{Dim } X$.

However, if X is hereditrarily normal, but not T_1, the result is not true, by an example of Dowker [1].

<u>Theorem 4.6</u>. Let X be a hereditrarily normal space, A a closed subset of X, and $\text{Dim } A \leq n$, $\text{Dim } (X \setminus A) \leq n$. Then $\text{Dim } X \leq n$.

<u>Proof</u>. If a normal space X is the union of two sets A and B with A closed and $\dim A \leq n$ and $\dim B \leq n$, then $\dim X \leq n$. Hence the result is true for $n = -1, 0$. Assume it is true for the case $n = m - 1$. Let $A = A_o \cup P_{n-1}$, $B = X_m \setminus A = B_o \cup Q_{n-1}$ where $\dim A_o \leq 0$, $\dim B_o \leq 0$, and $P_{m-1} = \bigcup_{i=1}^{m} A_i$, $Q_{m-1} = \bigcup_{i=1}^{m} B_i$, where $\text{Dim } P_{m-1} \leq m - 1$, $\text{Dim } Q_{m-1} \leq m - 1$.

Let $C_o = A_o \cup B_o$.

and $C_{m-1} = P_{m-1} \cup Q_{m-1}$.

Then $A_o = A \cap (A_o \cup B_o)$, i.e. A_o is closed in $A_o \cup B_o = C_o$ and C_o is normal so $\dim C_o \leq 0$.

Similarly $P_{m-1} = A \cap C_{m-1}$ is closed in C_{m-1} and C_{m-1} is hereditarily normal. Hence $\text{Dim } C_{m-1} \leq m - 1$. Therefore since $X = C_o \cup C_{m-1}$, X is the union of at most $m + 1$ subsets each of which has covering dimension ≤ 0, i.e. $\text{Dim } X \leq m$.

Corollary. If A is a closed subset of a hereditarily normal space X, then

$$\operatorname{Dim} X \leq \max(\operatorname{Dim} A, \operatorname{Dim} X \setminus A).$$

Theorem 4.7. If A is any subset of a totally normal space X, then $\operatorname{Dim} A \leq \operatorname{Dim} X$.

Proof. Let $\operatorname{Dim} X \leq n$. Then

$$X = \bigcup_{i=0}^{n} X_i \text{ , where } \dim X_i \leq 0.$$

Now $$A = \bigcup_{i=0}^{n} A_i \text{ , where } A_i = A \cap X_i .$$

Then by Theorem 2.22, $\dim A_i \leq 0$, since total normality is a hereditary property.

Hence $\operatorname{Dim} A \leq n$.

Theorem 4.8. Let X be a perfectly normal space and $Y = \bigcup_{i=1}^{\infty} F_i$, where each F_i is a closed subset of X and $\operatorname{Dim} F_i \leq n$. Then $\operatorname{Dim} Y \leq n$.

Proof. Since $\operatorname{Dim} F_i \leq n$

$$F_i = \bigcup_{j=0}^{n} F_{ij} \quad \text{where } \dim F_{ij} \leq 0$$

for $j = 0,1,\ldots,n$ for each $i = 1,2,\ldots$

and where without loss of generality we can take $F_{ij} \cap F_{ik} = \phi$ for each i if $j \neq k$.

Let $K_1 = F_1$, $K_2 = F_2 \setminus F_1$, $K_m = F_m \setminus \bigcup_{i=1}^{m-1} F_i$.

Then $K_m = F_m \cap F_1^c \cap \ldots \cap F_{m-1}^c$ is an F_σ-set in X and $Y = \bigcup_{i=1}^{\infty} K_i$.

Thus $K_i = \bigcup_{j=0}^{n} K_{ij}$, where $K_{ij} \subseteq F_{ij}$.

Since X is perfectly normal $\dim K_{ij} \leq 0$.

Hence $\text{Dim } K_i \leq n$.

Let $H_o = \bigcup_{i=1}^{\infty} K_{io}$. Then $K_{io} = H_o \cap F_1$.

Now $X \setminus F_1$ is an F_σ-set.

Let $D_2 = (X \setminus F_1) \cap F_2$.

Then $D_2 \cap H_o = K_{2o}$.

Let $D_3 = (X \setminus F_1 \setminus F_2) \cap F_3$. Then $D_3 \cap H_o = K_{3o}$, etc.

Hence each K_{io} is an F_σ-set in H_o and since $\dim K_{io} \leq 0$ we have by Theorem 2.20 (Čech) that

$$\dim H_o \leq 0.$$

Similarly $\dim H_i \leq 0 \qquad i = 1,\ldots,n$

and since $$Y = \bigcup_{i=0}^{h} H_i$$

we have $\text{Dim } Y \leq n$ as required.

An interesting question is whether Theorem 4.8 holds for totally normal spaces as well.

APPENDIX

The Tychonov Plank

Let ω_o be the first infinite ordinal and ω_1 the first uncountable ordinal. Provide each of the sets

$$N = \{k | k \text{ is an ordinal, } 0 \leq k \leq \omega_o\}$$

$$P = \{\alpha | \alpha \text{ is an ordinal, } 0 \leq \alpha \leq \omega_1\}$$

with the order topology.

Then both N and P are compact Hausdorff spaces. Hence the topological product

$$X = P \times N$$

is a compact Hausdorff space and consequently a normal space, called the 'Tychonov Plank'. X, however, is not hereditarily normal, since the subspace

$$Y = X \setminus \{(\omega_1, \omega_o)\} \quad \text{is not normal.}$$

Let $\quad A = \{(\alpha, \omega_o) | \ 0 \leq \alpha < \omega_1\}$

$\qquad\ B = \{(\omega_1, k) | \ 0 \leq k < \omega_o\}$.

Then A and B are disjoint closed subsets of Y. Hence $U = Y \setminus A$ is an open subset of Y containing the closed subset B. Now let V be any open set in Y containing B. Each point (ω_1, k) of B has a neighbourhood contained in V. This means that for each k there exists an ordinal $\alpha_k < \omega_1$ such that $x > \alpha_k$ implies $(x,k) \in V$.

But a countable collection of ordinals each of which is less than ω_1 has its supremum less than ω_1 , i.e. there exists an ordinal $\beta < \omega_1$ such that $\alpha_k \leq \beta$ so that for each $k = 0,1,2,...$ the point $(\beta,k) \in V$; therefore $\overline{V}$ must contain the point (β,ω_o). But $(\beta,\omega_o) \in A$, proving there there exists no open set V satisfying

$$B \subset V \subset \overline{V} \subset U$$

i.e. Y is not normal (and not closed).

It can be shown that $\text{ind } X = \text{Ind } X = \dim X = \text{Dim } X = 0$.

Also $$Y = \bigcup_{k=0}^{\infty} Y_k$$

where $Y_k = \{(\alpha,k) \mid 0 \leq \alpha \leq \omega_1\} \cap Y$.

Then each Y_k is closed in the completely regular non-normal Hausdorff space Y, and $\dim Y_k = 0$, yet $\dim Y > 0$. This shows that the sum theorem is not true for 'dim' for all completely regular Hausdorff spaces.

Now $\dim A = 0$, where

$$A = \{(\alpha,\omega_o) \mid 0 \leq \alpha < \omega_1\}.$$

Let $C = Y \setminus A$.

Then $\dim C = 0$, for every finite open cover of C has a disjoint refinement. Hence $\text{Dim } Y \leq 1$. But also $\text{Dim } Y \geq 1$. Hence $\text{Dim } Y = 1$.

It can also be shown that $\dim Y = 1$.

To show that $\text{Ind } Y = 1$, let F, G be closed and open subsets of Y respectively, with $F \subset G$.

Consider the statements:

A : "there exists an ordinal $\alpha_o < \omega_1$ such that $(\alpha,\omega_o) \notin F$ for $\alpha \geq \alpha_o$".

B : "there exists an ordinal $k_o < \omega_o$ such that $(k,\omega_1) \notin F$ for $k \geq k_o$".

We now distinguish four cases:

(i) A,B both true.

(ii) A,B both false.

(iii) A false, B true.

(iv) A true, B false.

In cases (i), (ii) it is not difficult to see that there exists an open set U satisfying

$$F \subset U \subset G \quad , \quad b(U) = \phi.$$

In case (iii) for every open set $U \supset F$, $b(u) \supset \{(k,\omega_1) \mid k_o \leq k \leq \omega_o\}$ for some $k_o < \omega_o$. Hence $\operatorname{Ind} b(u) \geq 0$ in this case. However, we can build up U from basic open sets so that $b(U) \subset \{(\omega_1,k) \mid 0 \leq k \leq \omega_o\} = B$.

Hence $\operatorname{Ind} b(U) = 0$.

The analysis of the 'Tychonov Plank' to show that Y is not normal shows that in this case there exists an open set U such that $F \subset U \subset G$ where

$$\phi \neq b(U) \subseteq A = \{(\alpha,\omega_o) \mid 0 \leq \alpha < \omega_1\}.$$

Thus again $\operatorname{Ind} b(U) = 0$. Hence in all cases $\operatorname{Ind} b(U) \leq 0$ and $\operatorname{Ind} Y \leq 1$.

Since Y is not normal, Theorem 3.6 => $\operatorname{Ind} Y \geq 1$.

Hence $\operatorname{Ind} Y = 1$.

Then $\operatorname{ind} X = \operatorname{ind} Y = \dim X = \operatorname{Dim} X = \operatorname{Ind} X = 0$, but $\dim Y = \operatorname{Dim} Y = \operatorname{Ind} Y = 1$.

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