FIXED POINT THEOREMS IN METRIC SPACES AND APPLICATIONS

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FIXED POINT THEOREMS IN METRIC SPACES AND APPLICATIONS

by

PRITAM SINGH CHEEMA

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TABLE OF CONTENTS

Abstract

The main object of this thesis is to study the fixed point theorems under contraction and contractive mappings in metric spaces.

We have discussed the Banach's contraction principle, "A contraction mapping of a complete metric space into itself has a unique fixed point", together with its various generalizations in metric spaces.

A few new results which guarantee the existence and uniqueness of fixed points for contraction, contractive mappings and mappings with a contractive iterate have been given for metric spaces.

An attempt has been made to give more general theorems for mappings of the form $d(Tx,Ty) \leq \psi(d(x,y))$ on metric spaces. A few fixed point theorems on generalized metric space have been obtained.

In the end, some applications of the fixed point theorems are illustrated by taking suitable examples.

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Introduction

Existence theorems in analysis appeared in the nineteenth century when, on the one hand, basic mathematical facts were considered critically. Cauchy was the first mathematician to prove an existence theorem for systems of differential equations with analytic right-hand sides. Picard suggested the method of successive approximations for proving the existence theorems. Birkhoff and Kellogg in 1922 gave a proof of the classical existence theorem for the equation $\frac{dy}{dx} = f(x,y)$ in function spaces. But the most elementary and by far the most fruitful method for proving theorems on the existence and uniqueness of solutions is the principle formulated by S. Banach in 1922 and first applied to the proof of the existence theorem by R. Cacciopoli in 1930. This principle is a result of geometric interpretation of Picard's method of successive approximations. It is stated as

"Let T be a mapping of a complete metric space X into itself. If for every pair of points $x,y \in X$, and some k, $0 \leq k < 1$ and

$$d(Tx,Ty) \leq kd(x,y)$$

Then T has a unique fixed point".

Because of its widespread applicability in proving uniqueness solutions to the differential and integral equations, many extensions of the above principle have been given by such mathematicians as Edelstein (1961), Rokotch (1962), Chu and Diaz (1965) and Wong (1968).

The contraction mapping principle has been generalized to the generalized complete metric space by Luxemburg (1958), Monna (1961), Edelstein (1964), Margolis (1967), and Diaz and Margolis (1967). For detailed results see C.W. Norris' thesis (1969) "Fixed Points and Periodic Points Under Contraction Mappings in Metric Spaces", Memorial University, Newfoundland. Many others such as Davis (1963), Kamerer and Kasriel (1964), Naimpally (1965), and Edelstein (1967) have generalized these results to uniform spaces. For further information one could see "Fixed Point Theorem in Uniform Spaces" thesis written by W. Russell (1970), Memorial University, Newfoundland.

The aim of this thesis is to study existence and uniqueness of fixed points under contraction and contractive mappings in the metric spaces.

In Chapter I, Banach contraction principle, with its various extensions, has been discussed. Also, some results on contractive and nonexpansive mappings with some modifications are given.

In Chapter II, R. Kannan's fixed point theorems in complete metric spaces and metric spaces, F. Bailey's theorem on contractive mappings in a compact metric space, V.M. Sehgal's result on contractive iterate in a complete metric space, sequences of contraction mappings by S. Nadler, S.P. Singh, & W. Luxemburg's theorems in generalized complete metric spaces are discussed. A few theorems under weaker conditions have been proved.

In Chapter III, some applications of the fixed point theorems, have been illustrated. For example, in proving the existence and uniqueness of solutions of linear equations, nonlinear integral equations.

Chapter I

Mapping and Fixed Points

In this chapter, preliminary definitions of some terms and some of the well-known theorems in connection with contraction, contractive, and non-expansive mappings of metric space into itself are discussed.

1.1. Preliminary Definitions:

Definition 1.1.1. Let X be a set and let R^+ denote the positive reals. A distance function $d : X \times X \to R^+$ is defined to be a metric if the following conditions are satisfied for all x,y,z belonging to X:

(i) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

An immediate consequence of this definition is the property

(iv) $|d(x,y) - d(x,z)| \leq d(y,z)$.

The set X with metric d is called a metric space and is denoted by the pair (X,d). Usually the metric space is represented by X when the metric d is understood.

Definition 1.1.2. A sequence $\{x_n\}$ of points of a metric space X is said to converge to a point x_o belonging to X if $\lim_{n\to\infty} d(x_n,x_o) = 0$. In this case we write $x_n \to x_o$.

It can be easily proved that if $x_n \to x_o$ and $x_n \to y_o$, then $x_o = y_o$, that is, a convergent sequence has a unique limit.

Definition 1.1.3. A sequence $\{x_n\}$ of points in a metric space X is called a Cauchy sequence if for any $\varepsilon > 0$, there is an $N(\varepsilon)$ such that

$$d(x_m, x_n) < \varepsilon \quad \text{whenever} \quad m,n \geq N(\varepsilon).$$

Definition 1.1.4. A metric space X is said to be a complete metric space if every Cauchy sequence in X converges to a point in X.

Definition 1.1.5. Let T be a mapping of a set X into itself. A point y belonging to X is said to be a fixed point of T if $Ty = y$, that is, a point which remains invariant under mapping is known as a fixed point.

Definition 1.1.6. A mapping T of a metric space X into itself is said to satisfy Lipschitz condition if there exists a real number k such that

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all} \quad x,y \quad \text{in} \quad X$$

In the special case, when $k \in [0,1)$, T is said to be a contraction mapping.

Remark 1.1.7. If T is a contraction mapping on a metric space X, then T is continuous on X.

Proof. Let $\varepsilon > 0$ be given and x be any point in X. Since T is a contraction mapping, we have

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all} \quad x,y \in X$$

and $k \in [0,1)$. If $k = 0$, we have

$$d(Tx,Ty) = 0 < \varepsilon \quad \text{for all} \quad y \in X, \quad \text{and} \quad T \quad \text{is continuous at} \quad x.$$

Otherwise, let $\delta = \frac{\varepsilon}{k}$ and y be any other point in X such that

$$d(x,y) < \delta$$

we have

$$\begin{aligned} d(Tx,Ty) &\le kd(x,y) \\ &< k \cdot \frac{\varepsilon}{k} = \varepsilon. \end{aligned}$$

Hence T is continuous at x which is an arbitrary point, therefore, the contraction mapping T is continuous everywhere.

1.2. Contraction Mappings.

S. Banach (1892-1945), a famous Polish mathematician, and one of the founders of the Functional Analysis formulated the "Principle of Contraction Mapping" in 1922 which is widely used to prove the existence and uniqueness of solutions to the differential and integral equations. It will be stated as a theorem.

Theorem 1.2.1. If X is a complete metric space and T is a contraction mapping of X into itself, then T has a unique fixed point.

Proof. We first prove uniqueness. Suppose $Tx_o = x_o$ and $Ty_o = y_o$. Then if $x_o \neq y_o$

$$\begin{aligned} d(x_o,y_o) &= d(Tx_o,Ty_o) \\ &\le kd(x_o,y_o) \quad \text{i.e.} \quad 1 \le k \quad \text{which is impossible.} \end{aligned}$$

Hence $x_o = y_o$.

For existence of a fixed point, we may start with any arbitrary point x_o in X and set up the sequence $\{x_n\}$ of points in X as follows:

$$x_1 = Tx_o \ , \ x_2 = Tx_1 = T^2x_o \ , \ x_3 = Tx_2 = T^3x_o,$$
$$x_4 = Tx_3 = T^4x_o, \ \dots \ , \ x_n = Tx_{n-1} = T^nx_o \ .$$

We shall show that $\{x_n\}$ is a Cauchy sequence.

$$d(T^{n+m}x_o,T^nx_o) \leq kd(T^{n+m-1}x_o,T^{n-1}x_o) < \dots < k^n d(T^mx_o,x_o)$$

By triangle inequality

$$d(T^mx_o,x_o) \leq d(T^mx_o,T^{m-1}x_o) + \dots + d(Tx_o,x_o)$$
$$\leq d(Tx_o,x_o)(k^{m-1} + k^{m-2} + \dots + k + 1)$$
$$< \frac{1}{1-k} d(Tx_o,x_o).$$

Therefore , $d(T^{n+m}x_o,T^nx_o) < \dfrac{k^n}{1-k} d(Tx_o,x_o)$.

Since $k < 1$ and $d(Tx_o,x_o)$ is fixed, hence $\{x_n\}$ is a Cauchy sequence.

Since X is a complete metric space, the sequence $\{x_n\}$ converges to a point x_o in X. T, being a contraction mapping, is continuous, therefore

$$Tx_o = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim x_{n+1} = x_o \ .$$

Hence T has a unique fixed point.

Examples: (1) Let $T : R \to R$ be defined by

$Tx = \frac{x}{3}$ for all $x \in R$

Then T is a contraction mapping in R and 0 is the unique fixed point.

(2) Let $T : R \to R$ be defined by

$Tx = \frac{2}{3}x + 2$ for all $x \in R$

Then T is a contraction mapping and $x = 6$ is the unique fixed point.

Various generalizations of the Principle of Contraction Mapping have been given. The following results are worth mentioning.

Chu and Diaz [7] gave the following two useful generalizations which are stated as theorems.

Theorem 1.2.2. If T maps a complete metric space X into itself and if T^n (n is a positive integer) is a contraction mapping in X , then T has a unique fixed point.

Proof. Since T^n is a contraction in X, therefore, by Banach Contraction Principle, it has a unique fixed point, say x. That is to say $T^n x = x$. $Tx = T.\ T^n x = T^n Tx$, thus Tx is a fixed point of T^n. But T^n has a unique fixed point, therefore, $Tx = x$.

Clearly T has a unique fixed point, for if $Tx_o = x_o$, then $T^n x_o = x_o$, hence $x = x_o$ since T^n has a unique fixed point.

Example: Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \frac{1}{2} \text{ if } x \text{ is rational}$$
$$= 0 \text{ if } x \text{ is irrational}$$

Then $T^2x = T^3x = \ldots = \frac{1}{2}$ for all $x \in [0,1]$ is a contraction in $[0,1]$ and has a unique fixed point $x = \frac{1}{2}$. Although T is not continuous, hence not contraction. The unique fixed point of T is $x = \frac{1}{2}$.

Theorem 1.2.3. Let S be any non-empty set of elements and T be a map of S into itself. If for some positive integer n, T^n has a unique fixed point, then T has also a unique fixed point.

Remark. The above theorem has been improved, under different conditions by Chu and Diaz [8].

Theorem 1.2.4. Let T be a mapping defined on a non-empty set S into itself, k be another function defined on X mapping it into itself such that $kk^{-1} = I$, where I is the identity function of x. Then T has a unique fixed point if and only if $k^{-1}Tk$ has a unique fixed point.

Proof. Let x be a unique fixed point of $k^{-1}Tk$

$$k^{-1}Tkx = x$$

Premultiply by k, we have

$$kk^{-1}Tkx = kx$$

$$Tkx = kx$$

Therefore kx is a fixed point of T. Uniqueness is obvious. Conversely, let x_o be a fixed point of T, then

$$Tx_o = x_o$$

Premultiply by k^{-1}

$$k^{-1}Tx_o = k^{-1}x_o \qquad (1)$$

Since $kk^{-1} = I$, therefore, we write (1) as

$$k^{-1}T\,k(k^{-1}x_o) = k^{-1}x_o$$ i.e. $k^{-1}x_o$ is a fixed point of $k^{-1}Tk$. Uniqueness is quite obvious.

The following Corollary can be easily obtained from the above theorem.

Corollary 1.2.5. Let X be a complete metric space, $T : X \to X$, and $k : X \to X$ be such that $kk^{-1} = I$, the identity function. If $k^{-1}Tk$ is a contraction in X, then T has a unique fixed point.

Its proof follows directly from the above theorem and the Banach Contraction Principle.

Definition 1.2.6. Let X be a metric space. A mapping T of X into itself is said to be locally contractive if for every $x \in X$, there exists ε and λ ($\varepsilon > 0$, $0 \leq \lambda < 1$), which may depend on x, such that $p,q \in S(x,\varepsilon) = \{y$ such that $d(x,y) < \varepsilon\}$ implies

$$d(Tp,Tq) < \lambda d(p,q).$$

Definition 1.2.7. A mapping T of a metric space X into itself is said to be (ε,λ) - uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Definition 1.2.8. A metric space X is said to be ε-chainable space if for every $a,b \in X$, there exists an ε-chain, that is a finite set of points $a = x_o, x_1, \ldots, x_n = b$ such that $d(x_{i-1},x_i) < \varepsilon$ $(i = 1, 2, \ldots, n)$.

The following theorem is due to Edelstein [9] as an extension to the Banach Contraction Principle.

Theorem 1.2.9. Let X be a complete metric ε-chainable space T a mapping of X into itself which is (ε,λ) - uniformly locally contractive, then there exists a unique fixed point of T in X.

Rakotch [25] has also given an extension to the Contraction Principle by introducing the following definition.

Definition 1.2.10. Denote by F_1 the family of functions $k(x,y)$ satisfying the following conditions

(1) $k(x,y) = k(d(x,y))$ i.e. k depends on the distance between x and y only.

(2) $0 \leq k(d) < 1$ for every $d > 0$

(3) $k(d)$ is a monotonically decreasing function of d.

The following result has been established by Rakotch [25].

Theorem 1.2.11. Let X be a complete metric space and let T be a map of X into itself such that

$$d(Tx,Ty) \leq k(x,y)d(x,y) \quad \text{for every} \quad x,y \in X$$

where $k(x,y) \in F_1$, then there exists a unique fixed point of T in X.

Remark. The above theorem can also be proved by assuming $k(x,y)$ to be monotonically increasing and $0 \leq k(d) < 1$ for $d > 0$. Let x_o be any point in X and let

$$x_1 = Tx_o$$
$$x_2 = Tx_1 = T^2x_o$$
$$\cdots\cdots$$
$$x_n = Tx_{n-1} = T^nx_o$$

Since $d(Tx,Ty) \leq k(x,y)d(x,y)$ for all $x,y \in X$, it can be easily proved that for $x_o \neq x_1$,

$$d(x_o,x_1) > d(x_1,x_2) > d(x_2,x_3) > \ldots > d(x_{n-1},x_n) > \ldots$$

Therefore, by monotonically increasing property of $k(x,y)$ we have

$$k(x_o,x_1) > k(x_1,x_2) > \ldots > k(x_{n-1},x_n) > \ldots$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence.

$$d(x_1,x_2) = d(Tx_o,Tx_1) \leq k(x_o,x_1)d(x_o,x_1)$$

$$d(x_2,x_3) = d(Tx_1,Tx_2) \leq k(x_1,x_2)d(x_1,x_2)$$
$$< [k(x_o,x_1)]^2 d(x_o,x_1)$$

In general

$$d(x_{n+1},x_n) < [k(x_o,x_1)]^n d(x_o,x_1)$$

Since $0 \leq k(x_o,x_1) < 1$, the sequence $\{x_n\}$ is easily seen to be Cauchy. Since X is complete, the sequence $\{x_n\}$ converges to a point say z in X. By the continuity of T,

$Tz = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$. Thus z is a fixed point of T. Uniqueness follows immediately.

Wong [34] has given two extensions of the Banach Contraction Principle which are stated below after the definitions of locally iteratively contractive and locally contractive mappings.

<u>Definition 1.2.12</u>. T is said to be locally contractive at $u \in X$, where X is a complete metric space, if there exists a positive integer $n(u)$ such that

$$d(T^{n(u)}x,\ T^{n(u)}y) \leq \phi(d(x,y)) \qquad (1)$$

for all $x,y \in I(u;\ T) = \{T^k u : k = 0, 1, 2, \ldots\}$ where ϕ satisfies

$$\phi(r) < r\ , \qquad (2)$$

<u>Definition 1.2.13</u>. T is said to be locally iteratively contractive at $u \in X$, where X is a complete metric space, if there exists a positive integer $n(u)$ such that (1) holds for all $x,y \in I(u;T)$, where ϕ satisfies

$$\phi(s) \leq \phi(t) \quad \text{whenever} \quad s \leq t$$

and $$\sum_{j=0}^{\infty} \phi^j(t) < \infty \quad \text{for all} \quad t > 0.$$

T is called local contraction if it is locally contractive at u for all $u \in X$ and local iterative contraction can be defined similarly.

Theorem 1.2.14. Let T be a local contraction on X satisfying (1), ϕ satisfies (2), and in addition ϕ satisfies the following condition

$$\lim\inf(t - \phi(t)) = k > 0$$

Then T has a unique fixed point.

Theorem 1.2.15. Let T be a local iterative contraction mapping. Then T has a unique fixed point and the successive approximations $x_n = T^{n(x)} x_{n-1}$, $x_o = x$ for all arbitrary $x \in X$ converge in metric to the fixed point of T.

Corollary 1.2.16. Theorem [1.2.11] given by Rakotch [25] directly follows from the theorem [1.2.14] by taking

$$\phi(r) = k(r) \quad \text{and} \quad n(x) = 1.$$

1.3. Contractive Mappings.

Definition 1.3.1. A mapping T of a metric space X into itself is said to be contractive if

$$(1) \ldots\ldots d(Tx,Ty) < d(x,y) \quad \text{for all} \quad x,y \in X , \ x \neq y,$$

Remark. A contractive map is clearly continuous and if such a mapping has a fixed point, then this fixed point is unique. While the condition $d(Tx,Ty) < d(x,y)$ is sufficient to assure that T has a fixed point but it is too weak to guarantee the existence of one as will be seen from the following examples:

Example 1. Let $T : R \to R$ be defined by

$$T(x) = x + \frac{\pi}{2} - \text{arc tan } x ,$$

Since arc tan $x < \frac{\pi}{2}$ for every x, the operator T has no fixed point although T is a contractive map for

$$T'(x) = 1 - \frac{1}{1 + x^2} < 1 .$$

Example 2. Let $T : R \to R$ be defined by

$$T(x) = \ln(1 + e^x) ,$$

Differentiating we obtain

$$T'(x) = \frac{e^x}{1 + e^x} < 1 \quad \text{i.e. } T \text{ is a contractive mapping,}$$

and it is easy to see that T has no fixed point.

Several authors have considered mapping satisfying condition (1) in one form or the other. Some of these are listed in the bibliography.

The following theorem due to Edelstein [10] states the sufficient conditions for the existence of a fixed point for a contractive mapping.

Theorem 1.3.2. Let T be a contractive mapping of a metric space into itself and $x_o \in X$ be such that the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to a point z in X, then z is a unique fixed point of T.

A simple proof of the above theorem than that due to Edelstein can be given as follows. This proof is similar to that of Cheney and Goldstein [6].

Proof. Since T is contractive hence continuous, we may write

$$d(T^n(x_o), T^{n+1}(x_o)) = d(T. T^{n-1}(x_o), T T^n(x_o))$$
$$< d(T^{n-1}(x_o), T^n(x_o))$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$< d(x_o, T(x_o))$$

Thus $\{d(T^n(x_o), T^{n+1}(x_o))\}$ is a decreasing sequence of real numbers bounded below by 0, and therefore has a limit. Since $\{T^{n_i}(x_o)\}$ converges to $z \in X$, therefore, the sequence $\{T^{n_i+1}(x_o)\}$ converges to $T(z)$ and $\{T^{n_i+2}x_o\}$ converges to $T^2(z)$. Now if $z \neq Tz$

$$d(z,T(z)) = \lim_{k\to\infty} d(T^{n_k}(x_o), T^{n_k+1}(x_o))$$
$$= \lim_{k\to\infty} d(T^{n_k}(x_o), T \cdot T^{n_k}(x_o))$$

By continuity of T, we have

$$d(z,Tz) = \lim_{k\to\infty} d(T^{n_k+1}(x_o), T^{n_k+2}(x_o))$$
$$= \lim_{k\to\infty} d(T \cdot T^{n_k}(x_o), T^2 . T^{n_k}(x_o))$$
$$= d(Tz, T^2z)$$, a contradiction to the fact that T is contractive. Hence $Tz = z$. For uniqueness of z, let $y \neq z$ be another fixed point of T. Then

$$d(y,z) = d(Ty,Tz)$$
$$< d(y,z)$$, a contradiction.

Hence z is a unique fixed point of T.

The following Corollary is due to Edelstein.

<u>Corollary 1.3.3.</u> If T is a contractive mapping of a metric space X into a compact metric space $Y \subset X$, then T has a unique fixed point.

In [10] Edelstein has considered local version of the contractive mapping. There exists $\varepsilon > 0$ such that $0 < d(x,y) < \varepsilon$ implies

$$d(Tx,Ty) < d(x,y). \tag{2}$$

The following two theorems have been given by Edelstein [10].

<u>Theorem 1.3.4.</u> If T satisfies (2) and there exists an x_o in X such that $\{T^{n_i}(x_o)\}$ converges to z in X, then there exists at least one periodic point under T.

<u>Theorem 1.3.5.</u> If X is compact and ε-chainable metric space and if T satisfies (2), then T has a unique fixed point.

1.4. <u>Nonexpansive Mapping</u>.

<u>Definition 1.4.1.</u> A mapping T of a metric space X into itself is said to be nonexpansive (ε-nonexpansive) if

$d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$ (for all x,y with $d(x,y) < \varepsilon$).

Isometry i.e. $|Tx - Ty| = |x - y|$ for all x,y in X is a simple example of nonexpansive mapping.

<u>Definition 1.4.2.</u> A point $y \in Y \subset X$ is said to belong to T-closure of Y, $y \in Y^T$, if $T(Y) \subset Y$ and there exists a point $\eta \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < n_3 < \ldots < n_i < \ldots)$, so that the sequence $\{T^{n_i}(\eta)\}$ converges to y.

Definition 1.4.3. A sequence $\{x_i\} \in X$ is said to be an isometric (ε-isometric) sequence if the condition

$$d(x_m,x_n) = d(x_{m+k},x_{n+k})$$

holds for all $k, m, n = 1, 2, \ldots$ (for all $k, m, n = 1, 2, \ldots$ with $d(x_m,x_n) < \varepsilon$). A point x in X is said to generate such an isometric (ε-isometric) sequence under T if $\{T^n(x)\}$ is such a sequence.

The following theorems are given by Edelstein [11] on nonexpansive and ε-nonexpansive mappings on a metric space X.

Theorem 1.4.4. If $T : X \to X$ is an ε-nonexpansive mapping and x in X^T, then the sequence $\{m_j\}$, $(m_1 < m_2 < \ldots)$, of positive integers exists so that $\lim_{j\to\infty} T^{m_j}(x) = x$.

Theorem 1.4.5. If $T : X \to X$ is nonexpansive (ε-nonexpansive) mapping, then each x in X^T generates an isometric (ε-isometric) sequence.

Remark. It can be easily proved that if $T : X \to X$ is nonexpansive mapping and x is in X^T, then T has a fixed point. For by the above theorem, an isometric sequence is generated, therefore,

$$d(x,Tx) = d(Tx,T^2x) \,. \tag{1}$$

But T is nonexpansive, therefore we have

$$d(T^2x,Tx) \leq d(Tx,x) \tag{2}$$

From (1) and (2), $d(x,Tx) = 0$ i.e. x is a fixed point of T.

In [6] Cheney and Goldstein have proved the following theorem.

<u>Theorem 1.4.5</u>. Let T be a map of a metric space X into itself such that

(i) $d(Tx,Ty) \leq d(x,y)$;

(ii) if $x \neq Tx$, then $d(Tx,T^2x) < d(x,Tx)$

(iii) for each x, the sequence $\{T^n(x)\}$ has a cluster point.

Then for each x, the sequence $\{T^n(x)\}$ converges to a fixed point of T.

K.L. Singh [30] has proved the above theorem by relaxing condition (ii) and (iii) in the following way.

Let T be a map of a compact metric space X into itself such that

$$d(Tx,Ty) \leq d(x,y), \quad \text{equality holds when } x = y.$$

Then T has a fixed point.

Then T_1 and T_2 have a unique common fixed point.

The conditions of the above theorem are not necessary could be seen from the following example given by Kannan [17].

Example: Let $T_1(x) = \frac{1}{2} - x$

$$T_2(x) = \frac{3}{8} - \frac{x}{2}$$

for $x \in [0,1]$ and $X = [0,1]$.

Clearly $T_1(\frac{1}{4}) = \frac{1}{4}$, $T_2(\frac{1}{4}) = \frac{1}{4}$.

$$\therefore \quad T_1(\tfrac{1}{4}) = T_2(\tfrac{1}{4}) = \tfrac{1}{4} .$$

It can be easily proved that $\frac{1}{4}$ is the only common fixed point.

$$\begin{aligned} \text{Also } d(T_2(x),T_2(y)) &= \left|\frac{3}{8} - \frac{y}{2} - \frac{3}{8} - \frac{x}{2}\right| \\ &= \frac{|y - x|}{2} \\ &\leq \frac{3}{4}|y - x| \\ &= \frac{3}{4} d(x,y) \end{aligned}$$

Thus T_2 is a contraction mapping. But considering the condition (i), we have

$$\begin{aligned} d(T_1(x),T_2(y)) &= \left|\frac{3}{8} - \frac{y}{2} - \frac{1}{2} + x\right| \\ &= \left|x - \frac{y}{2} - \frac{1}{8}\right| \end{aligned}$$

Now for $x = \frac{1}{8}$, $y = \frac{1}{4}$

$$d(T_1(x),T_2(y)) = \frac{1}{8}$$

$$\text{and} \quad d(x,y) = \frac{1}{8}$$

Hence $d(T_1(x),T_2(y)) = d(x,y)$ for $x = \frac{1}{8}$, $y = \frac{1}{4}$.

Thus the conditions of Theorem 2.1.3 are not necessary.

Theorem 2.1.4. Let T be a continuous mapping of a complete metric space X into itself. Suppose there exists an everywhere dense subset E_1 of X such that for any two points $x,y \in E_1$

$$d(T(x),T(y)) \leq kd(x,y) \quad \text{where} \quad 0 < k < 1$$

Then T has a unique fixed point.

The following theorems are due to Kannan [18] in which he has omitted the completeness of the space.

Theorem 2.1.5. Let X be a metric space. Let T be a map of X into itself such that

(i) $d(T(p),T(q)) \leq k[d(p,T(p)) + d(q,T(q))]$, $0 < k < \frac{1}{2}$, $p, q \in X$.

(ii) T is continuous at a point $z \in X$.

(iii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to z.

Then z is a unique fixed point of T.

Theorem 2.1.6. Let X be a metric space and T be a continuous map of X into itself such that

(i) $d(T(x),T(y)) \leq k[d(x,T(x)) + d(y,T(y))]$, $0 < k < \frac{1}{2}$, x,y belonging to everywhere dense subset M of X,

(ii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to a point $z \in X$.

Then z is a unique fixed point of T.

Theorem 2.1.7. Let X be a metric space and T be a map of X into itself. Suppose T is continuous at a point $x_o \in X$. If there exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ converges to x_o, then $T(x_o) = x_o$. If in addition

$$d(T(x_o),T(z)) \leq kd(x_o,z) , \quad z \in X , \quad 0 < k < 1 ,$$

then x_o is a unique fixed point of T.

In this chapter an attempt has been made to give more general result under less restricted conditions.

Generalization of Theorem [2.1.5].

Theorem 2.2.1. Let X be a metric space and T be a map of X into itself such that

(i) $d(T(x),T(y)) \leq k_1 d(x,T(x)) + k_2 d(y,T(y))$

for $x,y \in X$, $0 < k_1 + k_2 < 1$ and $k_1,k_2 > 0$,

(ii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to x_o.

Then x_o is a unique fixed point.

Proof. Let $x_1 = T(x_o)$

$$x_2 = T(x_1) = T^2 x_o$$

$$\cdots\cdots$$

$$x_n = T(x_{n-1}) = T^n(x_o) .$$

Now
$$\begin{aligned} d(x_1,x_2) &= d(T(x_o),T(x_1)) \\ &\leq k_1 d(x_o,T(x_o)) + k_2 d(x_1,T(x_1)) \\ &= k_1 d(x_o,T(x_o)) + k_2 d(x_1,x_2) , \end{aligned}$$

$$\therefore \quad d(x_1,x_2) \leq \frac{k_1}{1-k_2} d(x_o,x_1) .$$

$$d(x_2,x_3) = d(T(x_1),T(x_2))$$

$$\leq k_1 d(x_1,T(x_1)) + k_2 d(x_2,T(x_2))$$

$$= k_1 d(x_1,T(x_1)) + k_2(x_2,x_3)$$

$$d(x_2,x_3) \leq \frac{k_1}{1-k_2} d(x_1,x_2)$$

$$\leq \left(\frac{k_1}{1-k_2}\right)^2 d(x_o, x_1)$$

In general $d(x_n,x_{n+1}) \leq \left(\frac{k_1}{1-k_2}\right)^n d(x_o,x_1)$.

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{m-1},x_m)$$

$$\leq \left[\left(\frac{k_1}{1-k}\right)^n + \left(\frac{k_1}{1-k_2}\right)^{n+1} + \dots + \left(\frac{k_1}{1-k_2}\right)^{m-1}\right] d(x_o,x_1)$$

$$= (r^n + r^{n+1} + \dots + r^{m-1}) d(x_o,x_1)$$

where $r = \frac{k_1}{1-k_2}$ for $k_1 + k_2 < 1$

$$d(x_n,x_m) \leq \sum_1^\infty r^n \; d(x_o,x_1) \to 0 \quad \text{as} \quad n \to \infty$$

Hence $\{x_n\}$ is a Cauchy sequence. Since the subsequence $\{x_{n_i}\}$ of the Cauchy sequence converges to x_o, therefore, $\{x_n\}$ converges to x_o.

Now we shall prove that $Tx_o = x_o$.

$$d(x_o,Tx_o) \leq d(x_o,x_n) + d(x_n,T(x_o))$$

$$= d(x_o,x_n) + d(T(x_{n-1}),T(x_o))$$

$$\leq d(x_o,x_n) + k_1 d(x_{n-1},T(x_{n-1})) + k_2 d(x_o,T(x_o)) .$$

$$d(x_o,Tx_o) \leq \frac{1}{1-k_2} d(x_o,x_n) + \frac{k_1}{1-k_2} d(x_{n-1},T(x_{n-1})) .$$

If $\varepsilon > 0$ is arbitrary, then for sufficiently large values of n, we have

$$d(x_o,x_n) < \varepsilon \cdot \frac{1-k_2}{1+k_1} \quad \text{and} \quad d(x_{n-1},x_n) < \varepsilon\frac{1-k_2}{1+k_1} .$$

Thus $d(x_o,Tx_o) < \frac{1}{1+k_1}\varepsilon + \frac{k_1}{1+k_1}\varepsilon = \varepsilon$.

Hence $x_o = Tx_o$ because $\varepsilon > 0$ is arbitrary.

It can be easily proved that x_o is a unique fixed point of T. For, if x_o,y_o are two fixed points of T, then for $x_o \neq y_o$

$$\begin{aligned} d(x_o,y_o) &= d(T(x_o),T(y_o)) \\ &\leq k_1 d(x_o,T(x_o)) + k_2 d(y_o,T(y_o)) \\ &= 0 \end{aligned}$$

we get a contradiction, hence T has a unique fixed point.

Example. Let $X = [0,1)$ and $T : [0,1) \to [0,1)$.

Define $T(x) = \frac{x}{8}$ for $x \in [0,\frac{1}{2})$

$T(x) = \frac{x}{9}$ for $x \in [\frac{1}{2},1)$.

Here T is discontinuous at $x = \frac{1}{2}$, hence the hypothesis of Theorem 2.1.5. is not satisfied. But

$$d(T(x),T(y)) \leq k_1 d(x,T(x)) + k_2 d(y,T(y))$$

is satisfied by taking $x = \frac{1}{3}$, $y = \frac{1}{8}$ and $k_1 = \frac{1}{7}$, $k_2 = \frac{3}{7}$

and 0 is the unique fixed point of the mapping T.

Theorem 2.2.3. Let X be a metric space and T be a map of X into itself such that

(i) $d(Tx,Ty) < \frac{1}{2}[d(x,Tx) + d(y,Ty)]$ for $x,y \in X$

(ii) There exists an element $x_o \in X$ such that the sequence $\{T^n(x_o)\}$ contains a subsequence $\{T^{n_i}(x_o)\}$ converging to a point $z \in X$.

Then z is a unique fixed point of T.

Proof. Let $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$

Since $\{T^{n_i}(x_o)\}$ is a convergent sequence, therefore it is Cauchy sequence.

$$d(z,Tz) \leq d(z,T^{n_i}(x_o)) + d(T^{n_i}(x_o),Tz)$$

$$< d(z,T^{n_i}(x_o)) + \frac{1}{2} d(T^{n_i-1}(x_o)) + \frac{1}{2} d(z,Tz)$$

$$\therefore \quad d(z,Tz) < 2d(z,T^{n_i}(x_o)) + d(T^{n_i-1}(x_o), T^{n_i}(x_o)) . \tag{1}$$

For arbitrary $\varepsilon > 0$, we choose n_i so that

$$d(z,T^{n_i}(x_o) < \frac{\varepsilon}{4}$$

and $$d(T^{n_i-1}(x_o), T^{n_i}(x_o)) < \frac{\varepsilon}{2} .$$

Therefore from (1)

$$d(z,Tz) < \varepsilon$$

Since ε is arbitrary, hence $z = Tz$ i.e. z is a fixed point of T.

For uniqueness, let z and u be two fixed points of T,

$$\therefore \quad d(z,u) = d(Tz,Tu) < \frac{1}{2} d(u,Tu) + \frac{1}{2} d(z,Tz) = 0 .$$

Hence $z = u$.

In order to illustrate Theorem 2.2.3, we would like to give the following:

Example. Let $X = [0,2)$ and the map T of X into itself be defined by

$$Tx = \frac{x}{4} \quad \text{for } x \in [0,1)$$

$$= \frac{x}{8} \quad \text{for } x \in [1,2)$$

Here we take $x_o = \frac{1}{2}$ and form the sequence $\{x_n\}$ as

$$x_n = \frac{1}{2^{2n+1}} \qquad n = 1, 2, 3, \ldots$$

and the subsequence $\{x_{n_i}\}$ as

$$x_{n_i} = \frac{1}{2^{4i-3}} \qquad i = 1,2,3, \ldots$$

The subsequence $\{x_{n_i}\}$ converges to 0 and 0 is the unique fixed point.

Theorem [2.1.1] is true for family of maps if they are commutative.

<u>Theorem 2.2.4</u>. If T_i is a family of maps of a complete metric space X into itself such that

(i) $d(T_i x, T_i y) \le k[d(x,T_i x) + d(y,T_i y)]$

for x,y in X and $0 < k < \frac{1}{2}$, for at least one i

(ii) $T_i T_j = T_j T_i \quad ; \quad i \neq j$.

Then T_i has a unique common fixed point.

Proof. Let $i = 1$. Since T_1 satisfies [2.1.2], therefore, it has a unique fixed point say $z \in X$.

By (ii) $T_1 T_j(z) = T_j T_1(z)$

$= T_j(z)$

Therefore $T_j(z)$ is a fixed point of T_1. By uniqueness

$T_j(z) = z$, i.e. z is a fixed point of T_j,

Hence the family of maps T_i has a unique common fixed point.

Example. Let $X = [0,1]$ and

$T_i : [0,1] \to [0,1]$ be defined by

$T_i(x) = x$ for all i except $i = 1$

$T_1(x) = \frac{x}{4}$ when $i = 1$,

Here X is a complete metric space and 0 is the unique common fixed point for the family of maps.

Theorem 2.3.1. Let X be a metric space having two metrics d and δ such that

(1) $d(x,y) \leq \delta(x,y)$ for all x,y in X

(2) X is complete with respect to d

(3) $T : X \to X$ be a mapping continuous with respect to d

(4) $P = \{\delta(x,y) \mid x,y \in X\}$

(5) $\delta(T(x),T(y)) \leq \psi(\delta(x,y))$ for all $x,y \in X$

where $\psi: \overline{P} \to [0,\infty)$ is upper semicontinuous from right on $\overline{P}$ and satisfies $\psi(t) < t$ for all t belonging to $\overline{P} - \{0\}$.

Then T has a unique fixed point.

Proof. Given $x_o \in X$, we define

$$x_1 = Tx_o$$
$$x_2 = Tx_1 = T^2x_o$$
$$\cdots\cdots\cdots$$
$$x_n = Tx_{n-1} = T^nx_o \ .$$

It has been proved by Boyd and Wong [4] that $\{x_n\}$ is a Cauchy sequence with respect to δ. By (1), $\{x_n\}$ is a Cauchy sequence with respect to d. Since X is a complete metric space with respect to d, therefore the limit of this sequence $\{x_n\}$ say x is a fixed point of T i.e. $Tx = x$.

Uniqueness: Let x,y be two fixed points of T. Then

$$\delta(x,y) = \delta(T(x),T(y))$$
$$\leq \psi(\delta(x,y)) \quad \text{which is a contradiction since}$$

$\psi(t) < t$. Hence T has a unique fixed point.

Remark. The above result can be obtained without assuming X to be complete with respect to d. In place of completeness we use condition (5) given below.

Theorem 2.3.2. Let X be a metric space having metrics d and δ such that

(1) $d(x,y) \leq \delta(x,y)$

(2) $P = \{\delta(x,y) \mid x,y \in X\}$

(3) $T : X \to X$ such that

$\delta(T(x),T(y)) \leq \psi(\delta(x,y))$ for all x,y in X

where $\psi : \overline{P} \to [0,\infty)$ is upper semicontinuous from the right on $\overline{P}$ and satisfies $\psi(t) < t$ for all $t \in \overline{P} - \{0\}$

(4) T is continuous at $z \in X$ with respect to d

(5) There exists a point $x_o \in X$ such that the sequence of iterates $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to z in metric d.

Then T has a unique fixed point.

Proof. Given $x_o \in X$, we define $x_{n+1} = Tx_n$, $n = 0,1,2, \ldots$ It follows from the previous theorem that $\{x_n\}$ is a Cauchy sequence with respect to δ, By (1) $\{x_n\}$ is a Cauchy sequence with respect to d. Since the subsequence $\{x_{n_i}\}$ of the Cauchy sequence $\{x_n\}$ converges to z, therefore, the sequence $\{x_n\}$ converges to z under d, i.e. $\lim\limits_{n\to\infty} x_n = z$.

Since T is continuous at z, therefore, we have

$Tz = T \lim x_n = \lim Tx_n = \lim x_{n+1} = z$ i.e. z is a fixed point of T.

Uniqueness can be proved in the same way as in the previous theorem.

Theorem 2.3.3. Let X be a complete metric space and

$T : X \to X$

$k : X \to X$ has a right inverse k^{-1} such that

$k^{-1}Tk : X \to X$

and $d(k^{-1}Tk(x),\ k^{-1}TK(y)) \leq \psi(d(x,y))$ for all $x,y \in X$

where ψ is defined in the same manner as in the previous theorem

and $P = \{d(x,y) | x,y \in X\}$

Then T has a unique fixed point.

Proof. Given $x \in X$, define

$c_n = d(k^{-1}Tk)^n(x)\ ,\ (k^{-1}Tk)^{n-1}(x))$

Arguing in the same way as in Boyd and Wong [4] it can be proved that $\{(k^{-1}Tk)^n(x)\}$ is a Cauchy sequence. Since X is complete, this sequence converges to a point $x_o \in X$ which is a fixed point of $k^{-1}Tk$ i.e. $k^{-1}Tk(x_o) = x_o$.

Premultiply by k

$$kk^{-1}Tk(x_o) = kx_o$$

$Tk(x_o) = kx_o$ i.e. kx_o is a fixed point of T.

Uniqueness can be easily proved.

Definition 2.3.4. A metric space X is said to be metrically convex if for each $x,y \in X$, there is a $z \neq x,y$ for which $d(x,y) = d(x,z) + d(z,y)$.

The following Lemma has been proved by L. Blumenthal [3]

Lemma 1 (Menger). If X is a completely metrically convex space, then for any α, $0 < \alpha < 1$, and any $x,y \in X$, there exists $z \in X$ such that $d(x,z) = \alpha d(x,y)$ and $d(x,z) = (1 - \alpha)\ d(x,y)$.

Lemma 2. Let X be a completely metrically convex metric space and that

$T : X \to X$

$k : X \to X$ which has the right inverse k^{-1} which makes the mapping $k^{-1}Tk$ satisfy

$k^{-1}Tk : X \to X$ and

(1) $d(k^{-1}Tk(x),\ k^{-1}Tk(y)) \leq Ad(x,y)$

for some constant $A < \infty$ and for all $x,y \in X$. Define the function $\phi : [0,b) \to [0,b)$ by

(2) $\phi(t) = \sup\{d(k^{-1}Tk(x),\ (k^{-1}Tk)y \mid x,y \in X,\ d(x,y) = t\}$

Then

(a) $s > 0$, $t > 0$ and $s + t < b$ implies

$\phi(s + t) \leq \phi(s) + \phi(t)$ i.e. ϕ is subadditive.

(b) ϕ is upper semi-continuous from right on $[0,b)$.

Proof. (a) Let $d(x,y) = s + t$ where $x,y \in X$ and let $z \in X$ be such that

$$d(x,z) = s \quad , \quad d(y,z) = t$$

Then

$$d(k^{-1}Tk(x),\ k^{-1}Tk(y)) \leq d(k^{-1}Tk(x),\ k^{-1}Tk(z)) + d(k^{-1}Tk(z),\ k^{-1}Tk(y))$$

(3) $$d(k^{-1}Tk(x),\ k^{-1}Tk(y)) \leq \phi(s) + \phi(t)$$

Take supremum of (3) over all $x,y \in X$ with $d(x,y) = s + t$, we have

$$\phi(s + t) \leq \phi(s) + \phi(t)$$

(b) From (a), if t, t_o are such that $t - t_o < b$ and $t > t_o$

then $\phi(t) \leq \phi(t - t_o) + \phi(t_o)$

$\leq A(t - t_o) + \phi(t_o)$ by (1)

$\therefore \limsup_{t \to t_o +} \phi(t) \leq \phi(t_o)$

Thus ϕ is uppersemicontinuous from right.

Theorem 2.3.4. Suppose that X is a completely metrically convex metric space and

$T : X \to X$

$K : X \to X$ has a right inverse k^{-1} such that

$k^{-1}Tk : X \to X$ and $d(k^{-1}Tk(x), k^{-1}Tk(y)) \leq \psi(d(x,y))$

for all $x,y \in X$ where $\psi : \overline{P} \to [0,\infty)$ satisfies $\psi(t) < t$ for all $t \in \overline{P} - \{0\}$ and $P = \{d(x,y) \mid x,y \in X\}$.

Then T has a unique fixed point.

Proof. Let $\phi(t)$ be defined as in Lemma 2 for $t \in [0,b)$. Then

$\phi(t) \leq \psi(t)$ for all $t \in [0,b)$.

We define $\phi(b) = \psi(b)$ where $b < \infty$. Then

$d(k^{-1}Tkx , k^{-1}Tky) \leq \phi(d(x,y))$

for all $x,y \in X$, and $\phi(t) \leq \psi(t) < t$ for $t \in \overline{P} - \{0\}$.

By Lemma 2, ϕ is upper semicontinuous from right on $[0,b)$, Applying Theorem 2.3.3 replacing ψ by ϕ , it can be proved that T has a unique fixed point.

Theorem 2.3.5. If T is a single-valued function mapping a complete metric space X into itself such that

$$d(T^n x, T^n y) \leq \psi(d(x,y)) \qquad (1)$$

when n is a positive integer, $x,y \in X$, $\psi : \overline{P} \to [0,\infty)$ is upper semicontinuous from the right on $\overline{P}$ and satisfies $\psi(t) < t$ for all $t \in \overline{P} - \{0\}$, then T has a unique fixed point.

Proof. Since T^n satisfy the condition (1), therefore, by [4], T^n has a unique fixed point say x_o such that

$$T^n(x_o) = x_o \,,$$

But $T(x_o) = TT^n(x_o) = T^nT(x_o)$ i.e. Tx_o is a fixed point of T^n. But T^n has a unique fixed point, and therefore, $Tx_o = x_o$. It is easy to prove that x_o is a unique fixed point of T by using (1).

The following theorem was given by D.F. Bailey [1] for a compact metric space.

Theorem 2.4.1. If T is a continuous mapping of a compact metric space into itself and $0 < d(x,y)$ implies that there exists $n(x,y) \in I^+$ (the positive integer) such that

$$d(T^n(x),\ T^n(y)) < d(x,y),$$

then T has a unique fixed point.

The foregoing theorem can be proved under the following weaker assumptions:

Theorem 2.4.2. If T is a continuous mapping of a metric space X into itself such that

(1) for $x \neq y$, there is some n depending on x,y such that

$$d(T^n(x),\ T^n(y)) < d(x,y)$$

(2) there exists $x \in X$ such that the sequence $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to $z \in X$.

(3) $\{x_n\}$ is an isometric sequence at x.

Then T has a unique fixed point.

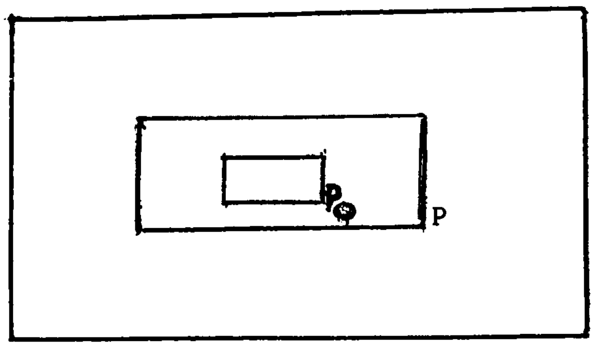
Proof. We have a subsequence $\{T^{n_i}(x)\}$ in X converging to some z in X and for some n depending on $(z,Tz), d(z,Tz) > d(T^n z, T^{n+1}z)$ if $z \neq Tz$. But we also have

$$\begin{aligned} d(z,Tz) &= d(\lim_i T^{n_i}(x),\ \lim_i T^{n_i+1}(x)) \\ &= \lim d(x_{n_i},\ x_{n_i+1}) \\ &= \lim d(x_{n_i+n},\ x_{n_i+n+1}) \\ &= d(T^n z\ ,\ T^n Tz) \end{aligned}$$

giving a contradiction unless $z = Tz$.

Now to prove that T has a unique fixed point, let $y \neq z$ be another fixed point of T. Then $d(y,z) > d(T^n y, T^n z)$ for some n. However, $Tz = z$ and $Ty = y$ contradict this. Hence z is a unique fixed point of T.

Theorem 2.4.3. If A is a contractive mapping which maps the closed set Q into a compact set $P \subset Q$, then A has a unique fixed point.



Proof. Since A is contractive

$$d(A(x),A(y)) < d(x,y) \qquad (1)$$

for $x,y \in Q$ and $x \neq y$. Also $A : Q \to P \subset Q$.

$\therefore \; P_o = A(P) \subset P = A(Q)$.

We shall prove that A is a contraction map of the set P. Suppose A is not a contraction on P, we can find $x_n, y_n \in P$ $(n = 1, 2, \ldots)$ such that

$$d(A(x_n), A(y_n)) = k_n d(x_n, y_n) \qquad (2)$$

where $n = 1, 2, \ldots, k_n \to 1$ and we could assume $x_n \to x \in Q$, $y_n \to y \in Q$. Since P is compact and Q is closed. Taking limits in (2) would give us

$$d(A(x),A(y)) = d(x,y)$$

which contradicts (1). Thus A is a contraction map of P. By Banach Contraction Principle A has a fixed point $x_1 \in P$ i.e. $Ax_1 = x_1$.

Let x_1, y_1 be two fixed points of T. Then

$$d(x_1,y_1) = d(A(x_1), A(y_1))$$

which contradicts (1). Hence A has a unique fixed point.

V.M. Sehgal [28] has proved the following theorem for mapping with a contractive iterate.

Theorem 2.4.4. Let (X,d) be a complete metric space and $T : X \to X$ a continuous mapping; there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all y

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leq kd(x,y) \qquad (1)$$

Then T has a unique fixed point.

Remark. The above theorem can be proved without assuming X to be a complete metric space.

Theorem 2.4.5. Let (X,d) be a metric space and $T : X \to X$ a continuous mapping ; there exists a $\lambda < 1$ such that for each $x \in X$, there is a positive integer $N(x) \in I^+$ such that for all y

$$d(T^{N(x)}(x), T^{N(x)}(y)) \leq \lambda d(x,y) \qquad (1)$$

and $\{T^n(x)\}$ contains a subsequence $\{T^{n_i}(x)\}$ converging to some point $z \in X$.

Then z is the unique fixed point.

Proof. Set $N_1 = N(x)$

and $N_{k+1} = N(T^{N_k}(x))$, $\qquad k = 1, 2, 3 \ldots$

Thus $d(T^{N_k}(x), T^{N_k+1}(x)) \leq \lambda^k d(x, T(x))$.

Let i_r be the smallest integer such that $n_{i_r} \geq N_r$ and $i_r > i_{r-1}$.

Since $T^{n_i r}(x) \to z$ and $d(T^{n_i r}(x), T^{n_i+1\,r}(x)) \leq \lambda^r d(x,T(x))$ where $\lambda < 1$. Hence

$$\begin{aligned} T(z) &= \lim_{r\to\infty} T^{n_i+1\,r}(x) \\ &= \lim_{r\to\infty} T^{n_i r}(x) \\ &= z \quad \text{i.e. } z \text{ is a fixed point of } T. \end{aligned}$$

The uniqueness of the fixed point follows immediately from (1).

Holmes [15] has proved the theorem 2.4.5. by taking $N = N(x,y)$.

Theorem 2.4.5. can be generalized to mappings which are not necessarily continuous at each point in a metric space.

The proof of the following lemma and subsequent theorem follow the line of argument presented in [28].

Lemma: Let (X,d) be a metric space and $T : X \to X$ a mapping. If there exist $x,y \in X$ and a positive integer $n(x,y)$ such that

$$T^{n(x,y)}(z) = z \tag{1}$$

and

$$d(T^{n(x,y)}(x), T^{n(x,y)}(z)) \leq kd(x,z) \tag{2}$$

for some k, $0 < k < 1$, then z is the unique fixed point of T.

Proof. By (1) and (2), z is the unique fixed point of $T^{n(x,y)}$. Then $T(z) = T \cdot T^{n(x,y)}(z) = T^{n(x,y)}T(z)$ i.e. $T(z)$ is a fixed point of $T^{n(x,y)}$. But $T^{n(x,y)}$ has a unique fixed point z, hence z is a unique fixed point of T.

<u>Theorem 2.4.6.</u> Let (X,d) be a metric space and $T : X \to X$ a mapping.

If there exists $x,y \in X$ and a positive integer $n(x,y)$ such that

(i) $d(T^{n(x,y)}(x), T^{n(x,y)}(y)) \le kd(x,y)$, $0 < k < 1$

(ii) for each $x_o \in X$, $T^{n(x,y)}(x_o) \to z$ where $z \in X$

Then T has a unique fixed point.

Proof. It can be easily proved that for $x \in X$, $r(x) = \sup_n d(T^n(x),x)$ is finite. Let $m_o = n(x,Tx)$, $m_i = m_{i-1} + n(T^{m_{i-1}}(x), T^{m_{i-1}}T(x))$ and $x_i = T^{m_{i-1}}(x_{i-1})$ $i = 1, 2, 3, \ldots$

By (ii) $x_i \to z \in X$ and by (i), there is an integer $n(x,y) \ge 1$ such that

$$d(T^{n(x,y)}(x_i), T^{n(x,y)}(z)) \le kd(x_i,z)$$

for each x_i, $z \in X$. It follows that $T^{n(x,y)}(x_i) \to T^{n(x,y)}(z)$. Thus

$$d(T^{n(x,y)}(z), z) = \lim_i d(T^{n(x,y)}(x_i), x_i)$$

But

$$\begin{aligned} d(T^{n(x,y)}(x_i), x_i) &= d(T^{m_{i-1}}T^{n(x,y)}(x_{i-1}), T^{m_{i-1}}(x_{i-1})) \\ &\le kd(T^{n(x,y)}(x_{i-1}), x_{i-1}) \le \cdots \\ &\le k^i d(T^{n(x,y)}(x_o), x_o) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Hence $T^{n(x,y)}(z) = z$.

By the lemma z is the unique fixed point of T.

The following example is due to Guseman [14].

Example. Let $X = [0,1)$ and define $T = [0,1) \to [0,1)$ by

$$\begin{aligned} T(x) &= 0 \quad \text{if } x \text{ is zero or irrational } \in [0,1) \\ &= \bar{2}^n x \quad \text{if } x \in [\bar{2}^n, \bar{2}^{(n-1)}), \ n = 1, 2, \ldots \end{aligned}$$

Then 0 is the unique fixed point of T in $[0,1)$. For each

$x_o \in [0,1)$, $T^n(x_o) \to 0$ and

$$d(T(x_o) , T(0)) \leq \frac{1}{2} d(x_o, 0).$$

It can be easily verified that T has no contractive iterate at any other point $\in [0,1)$.

If a sequence of mappings T_n , with fixed points U_n, converges to T, with a fixed point U, then does the sequence of corresponding fixed points converge to the fixed point of T? Some work has been done on this line by Bonsall [5], Nadler Jr. [25], Singh [31], and Russell and Singh [34]. Nadler Jr. has shown that if the sequence of contraction mappings with different Lipschitz constants converges pointwise to a contraction mapping, then the sequence of their fixed points does not converge to the fixed point of T. He has proved the following theorem.

If $T_n : X \to X$ is a map for $n = 1, 2, \ldots$ with fixed points U_n $(n = 1, 2, \ldots)$ and if T_n converges to T uniformly, where T is a contraction map with fixed point U, then U_n converges to U.

Singh [33] has proved the following:

<u>Theorem 2.5.1.</u> Let X be a complete metric space such that

(1) $T_n : X \to X$ be a map with fixed point

U_n for $n = 1, 2, \ldots$

(2) T_n converges uniformly to T where $T : X \to X$ is a map such that

$$d(T(x),T(y)) \leq k[d(x,T(x)) + d(y,T(y))] \text{ for } x,y \text{ in } X \text{ and}$$

$0 < k < \frac{1}{2}$, with fixed point U.

Then U_n converges to U.

Remark. The above theorem has been proved by Singh [32] relaxing the convergence of T_n to T in the following way.

Theorem 2.5.2. Let X be a metric space such that

(1) $T_n : X \to X$ be a map with fixed point U_n for $n = 1, 2, \ldots$

(2) T_n converges pointwise to T where $T : X \to X$ is a map such that $d(T(x),T(y)) \leq k[d(x,T(x)) + d(y,T(y))]$ for $x,y \in X$ and $0 < k < \frac{1}{2}$, with fixed point U.

Then U_n converges to U.

Proof. Since T_n converges pointwise to T, given $\varepsilon > 0$, there exists a positive integer N such that for $n \geq N(x,\varepsilon)$ implies

$$d(T_n x, Tx) < \frac{\varepsilon}{1 + k}$$

Thus for $n \geq N$

$$\begin{aligned} d(U_n,U) &= d(T_n(U_n),T(U)) \\ &\leq d(T_n(U_n),T(U_n)) + d(T(U_n),T(U)) \\ &\leq d(T_n(U_n),T(U_n)) + k[d(U_n,T(U_n)) + d(U,T(U))] \\ &\leq d(T_n(U_n),T(U_n)) + k[d(U_n,T_n(U_n)) + d(T_n(U_n),T(U_n)] \\ &= (1 + k)d(T_n(U_n),T(U_n)) \\ &< \varepsilon \end{aligned}$$

Hence U_n converges to U.

A more general form of the Theorem 2.5.2. on sequence of mappings can be formulated in the following way:

Theorem 2.5.3. Let X be a metric space and T_i be a sequence of mappings of X into itself such that

(1) $d(T_i x, T_i y) \leq pd(x, T_i x) + qd(y, T_i y)$

$x, y \in X$ and each T_i has a fixed a_i, $0 \leq p + q < 1$, and $p, q > 0$.

(2) $\{T_i\}$ converges pointwise to a mapping $T : X \to X$.

The sequence $\{a_i\}$ of fixed points converges to a where a is a fixed point of T. Also T has a unique fixed point.

Proof. By pointwise convergence of $\{T_i\}$ to T, we get that given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(Ta, T_n a) < \frac{\varepsilon}{1 + p}$$

Hence for $n \geq N$,

$$\begin{aligned} d(a, a_n) &= d(Ta, T_n a_n) \\ &\leq d(Ta, T_n a) + d(T_n a, T_n a_n) \\ &< \frac{\varepsilon}{1 + p} + pd(a, T_n a) + qd(a_n, T_n a_n) \end{aligned}$$

a_n being a fixed point of T_n, we have

$$\begin{aligned} d(a, a_n) &< \frac{\varepsilon}{1 + p} + pd(a, T_n a) \\ &\leq \frac{\varepsilon}{1 + p} + p[d(a, Ta) + d(Ta, T_n a)] \\ &< \frac{\varepsilon}{1 + p} + pd(Ta, T_n a), \text{ } a \text{ being fixed point of } T \\ &< \frac{\varepsilon}{1 + p} + \frac{p\varepsilon}{1 + p} \\ &= \varepsilon \end{aligned}$$

Hence $\lim\limits_{n\to\infty} a_n = a$

For uniqueness, let b be another fixed point of T. It can be easily proved by the above argument that $\lim_{n\to\infty} a_n = b$. Hence, T has a unique fixed point.

Luxemburg [12] has given two contraction mapping theorems. These have since been generalized to a family of contractions by Monna [24], Edelstein [22], and Margolis [23]. An attempt has been made to give further generalizations.

Definition 2.6.1. Let X be an abstract set with elements x, y, ..., and let $d(x,y)$ be a distance function $(0 \leq d(x,y) \leq \infty)$, defined on $X \times X$ satisfying the following conditions.

(1) $d(x,y) = 0$ if and only if $x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Then X with metric d is called generalized metric space. It differs from the usual concept of metric space by the fact that not every pair of elements necessarily has a finite distance.

Theorem 2.6.2. Let T be a mapping of a generalized metric space (X,d) into itself satisfying the following conditions:

(1) There exists a constant $0 < q < 1$, such that

$$d(Tx,Ty) \leq qd(x,y)$$

for all (x,y) such that $d(x,y) < \infty$

(2) For some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to x.

(3) For the sequence of successive approximations

$$x_n = Tx_{n-1}\ ,\quad n = 1, 2, \ldots, \quad \text{there exists an index}$$

$N(x_o)$ such that

$$d(x_N, x_{N+\ell}) < \infty \quad \text{for } \ell = 1, 2, \ldots$$

(4) If x and y are two fixed points of T, then $d(x,y) < \infty$.

Then T has a unique fixed point.

Proof. Let $x_o \in X$ and foam the sequence

$$x_n = Tx_{n-1} \quad (n = 1, 2, \ldots)$$

Then by (3) there exists an index $N(x_o)$ such that $d(x_N, x_{N+\ell}) < \infty$, $\ell = 1, 2, \ldots$ and hence by definition of the generalized metric space we have

$$d(x_n, x_{n+\ell}) < \infty \quad \text{for } n \geq N \text{ and } \ell = 1, 2, \ldots .$$ Then

(1) implies

$$d(x_{N+1}\ , x_{N+2}) \leq qd(x_N\ , Tx_N) \quad \text{and generally}$$

$$d(x_n\ , x_{n+1}) \leq q^{n-N}\, d(x_N, Tx_N) \quad \text{for } n \geq N.$$

By triangle inequality

$$d(x_n, x_{n+\ell}) \leq \sum_{i=1}^{\ell} d(x_{n+i}, x_{n+i-1}) \ .$$

By using the previous inequality, we obtain

$$d(x_n, x_{n+\ell}) \leq q^{n-N} \frac{(1 - q^{\ell})}{1 - q}\ d(x_N, Tx_N)$$

and $\ell = 1, 2, \ldots$ which proves that $\{x_n\}$ is a Cauchy sequence since $0 < q < 1$. Since the subsequence $\{x_{n_i}\}$ converges to x and X is a generalized metric space, hence $\{x_n\}$ converges to x i.e. $\lim_{n\to\infty} d(x_n, x) = 0$. Now for this element x, we have

$$d(x,Tx) \le d(Tx,x_n) + d(x_n,x)$$
$$\le qd(x,x_{n-1}) + d(x_n,x) \quad \text{for} \quad n \ge N \quad \text{and}$$

hence $d(x,Tx) = 0$ i.e. $x = Tx$. This shows that x is a fixed point of T. To show that T has a unique fixed point, we assume that there are two fixed points x and y then

$$0 \le d(x,y) = d(Tx,Ty) \le qd(x,y)$$

hence $d(x,y) = 0$ i.e. $x = y$. This completes the proof.

The following is a localized version of Theorem 2.6.2 .

Theorem 2.6.3. Let T be a mapping of a generalized metric space (X,d) into itself satisfying the following conditions:

(1) There exists a constant $c > 0$ such that for all (x,y) with $d(x,y) \le c$ we have $d(Tx,Ty) \le qd(x,y)$, where $0 < q < 1$

(2) For some $x_o \in X$, the sequence $\{T^n x_o\}$ has a subsequence $\{T^{n_i} x_o\}$ converging to x.

(3) For the sequence of successive approximations

$$x_n = Tx_{n-1} \quad , \quad n = 1, 2, \ldots,$$ there exists an index $N(x_o)$ such that

$$d(x_n,x_{n+\ell}) \le c \quad \text{for all} \quad n \ge N \quad \text{and} \quad \ell = 1, 2, \ldots$$

(4) If x and y are two fixed points of the mapping T, then $d(x,y) \le c$.

Then T has a unique fixed point.

Proof. Let $x_o \in X$ and form the sequence $x_n = Tx_{n-1}$ $(n = 1, 2, \ldots)$

From (3) it follows that there exists an index $N(x_o)$ such that $d(x_n, x_{n+\ell}) \leq c$ for $n \geq N$ and $\ell = 1, 2, \ldots$. It can be easily proved, as in the previous theorem, that $\{x_n\}$ is a Cauchy sequence and it converges to x.

Now
$$d(x_n, x) \leq d(x_n, x_{n+\ell}) + d(x_{n+\ell}, x)$$
$$\leq c + d(x_{n+\ell}, x)$$

By letting $\ell \to \infty$, it follows that

$$d(x_n, x) \leq c \quad \text{for all} \quad n \geq N$$

Then
$$d(x, Tx) \leq d(Tx, x_n) + d(x_n, x)$$
$$\leq qd(x, x_{n-1}) + d(x_n, x) \quad \text{for all} \quad n \geq N$$

Hence $d(Tx, x) = 0$ i.e. $Tx = x$.

For uniqueness of the fixed point, assume there are two fixed points x and y ; then

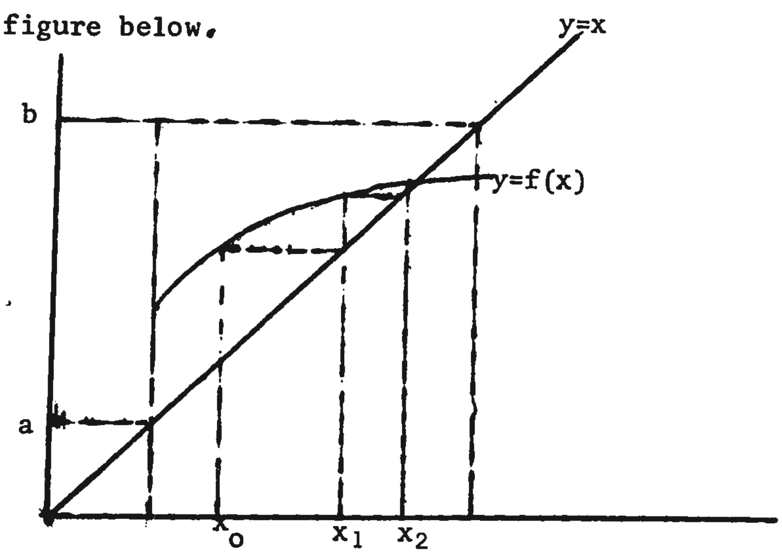
$$0 \leq d(x,y) = d(Tx, Ty) \leq qd(x,y)$$

Hence $d(x,y) = 0$ i.e. $x = y$. This completes the proof.

Chapter III

Applications of Fixed Point Theorems

1. Let $f(x)$ be a real differentiable function, given on the segment $[a,b]$ of the real line, all values of which are also bounded in that segment. Suppose we want to solve $f(x) = x$ i.e. to find the point of intersection of the graph of $f(x)$ and the straight line $y = x$ as shown in the figure below.



We consider the function f as an operator mapping the segment $[a,b]$ which is the closed subset of the complete metric space X into itself. If $|f'(x)| \leq k < 1$, then f is a contraction operator. In fact due to Lagrange formula, for any $x_1, x_2 \in [a,b]$

$$|f(x_1) - f(x_2)| = |f'(\eta)(x_1 - x_2)| \leq k|x_1 - x_2|$$

where η is a point between x_1 and x_2. By Banach Contraction Principle f has a unique fixed point which can be obtained by method of successive approximations, beginning from an arbitrary point $x_o \in [a,b]$. The fixed point of the operator is the solution of the equation $f(x) = x$.

Example. Suppose $f : R \to R$ be defined by $f(x) = \frac{x}{2} + 1$, in which case the fixed point is $x = 2$ which can be obtained by successive approximations. Let $x_1 = 0$, we obtain

$$x_2 = f(0) = 1$$

$$x_3 = f(1) = \frac{3}{2}$$

$$x_4 = f(\tfrac{3}{2}) = \frac{7}{4}$$

$$x_5 = f(\tfrac{7}{4}) = \frac{15}{8} = \frac{2^4 - 1}{2^3}$$

$$\cdots\cdots\cdots\cdots$$

$$x_n = f(x_{n-1}) = \frac{2^{n-1} - 1}{2^{n-2}}$$

$$\cdots\cdots\cdots\cdots$$

The sequence $\{x_n\}$ clearly converges to 2 which is the fixed point of f.

2. Consider the system of n algebraic equations with n unknowns $x_1, x_2, \ldots, x_n$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \qquad (1)$$

We rewrite (1) by transposing all terms on the right side and add x_1 to both sides of the first equation, to the second x_2 and so on, we get the equivalent system

$$
\begin{aligned}
x_1 &= (1 - a_{11})x_1 - a_{12}x_2 - \dots - a_{1n}x_n + b_1 \\
x_2 &= -a_{12}x_1 + (1 - a_{22})x_2 - \dots - a_{2n}x_n + b_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
x_n &= - a_{1n}x_1 - a_{2n}x_2 - \dots + (1 - a_{nn})x_n + b_n \quad .
\end{aligned}
\tag{2}
$$

We introduce new coefficients on the right side of (2), namely we assume that

$$c_{ik} = -a_{ik} + \delta_{ik} \quad \text{where} \quad \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

and (2) can be written in the concise form

$$x_i = \sum_{k=1}^{n} c_{ik}x_k + b_i \dots \tag{3}$$

where $i = 1, 2, \dots, n$.

Taking an arbitrary point $x = (x_1, x_2, \dots x_n)$ of the space, we determine mapping of the space R^n in itself by $y = f(x)$ assuming the coordinates y_i of the point y by the formulas

$$y_i = \sum_{k=1}^{n} c_{ik}x_k + b_i \qquad (i = 1, 2, \dots, n) \, ,$$

The solutions of the system (1) or the equivalent system (3) are fixed points of the operator f.

Sufficient condition for the operator f to be a contraction operator will now be derived. For this purpose, equate the operator f to the operator c of the linear transformation, determined by the matrix (c_{ik}) :

$$y_i = \sum_{k=1}^{n} c_{ik} \, x_k \qquad (i = 1, 2, \dots, n) \quad .$$

Namely, $f(x) = cx + b$, where $b = (b_1, b_2, \ldots, b_n)$.

For any two points $x_1, x_2 \in R^n$, we have

$$
\begin{aligned}
d(fx_1, fx_2) &= ||fx_1 - fx_2|| \\
&= ||cx_1 - cx_2|| \\
&= ||c(x_1 - x_2)|| \\
&\leq ||c||\ ||x_1 - x_2|| \\
&= ||c||\ d(x_1, x_2)
\end{aligned}
$$

Consequently, if $||c|| < 1$, then f is a contraction operator. This inequality holds good if

$$\sum_{i,k}^{n} c_{ik}^2 < 1 , \tag{4}$$

so, if the conditions (4) satisfied, the operator f has a unique fixed point, which can be found by method of successive approximations.

3. The following is the application of the fixed point theorem in economics.

Suppose we have an ecomony with 6000 commodities; the list of commodities include money, stocks, bonds, houses, etc. A typical quantity on the list, say houses, might be item 20. It is assumed that the demand for houses depend on the prices of each of the 6000 items on the market, as well as a few non-commodity type influences such as the stock of cash in existence (which might be listed as item 6001 to keep records straight). Call D(20) the demand for commodity 20, i.e. D(20) is the demand for houses. We are asserting that D(20) depends on P(1), the price of commodity 1, P(2), the price of commodity 2, ..., to P(6000), the price of commodity 6000, P(6001), the price of the stock of cash in existence, and perhaps a few more. Hence, D(20) depends on at least 6000 variables. Similar statements apply to D(1), the demand for commodity 1, ..., D(6000), the demand for commodity 6000.

We have similar set of supply equations. If S(20) denotes the supply of houses, we shall assume S(20) depends on P(1), P(2), P(3) ..., P(6001), where P's have the same meaning as mentioned in the previous paragraph. This gives us a list of equations in which each of S(1), S(2), ..., S(6000) depends on P(1), P(2), ... P(6001), and perhaps a few more.

The economy is said to be in general equilibrium if the supply for each item and its demands are equal i.e. if D(k) = S(k) for each k. But these equations are 6000 in at least 6001 unknown. Using some assumptions from economics, we can reduce the system to as many unknowns as equations. In our particular example, we would have 6000 equations in 6000 unknowns. General equilibrium will be realized if there are numbers P(1), P(2), ..., P(6000) which will satisfy every one of the equations D(1) = S(1), D(2) = S(2), ..., D(6000) = S(6000) simultaneously.

The question of whether such numbers P(1), P(2), ..., P(6000) exist which simultaneously satisfy the 6000 equations D(k) = S(k) is far from trivial.

Consider a simple one-commodity situation, and label this commodity as 1. Suppose at price P(1) = p, the demand D(1) is given by D(1) = 900 - p and the supply S(1) is given by S(1) = D(1), and hence the equilibrium occurs if 900 - p = 1000 - p. But this equation implies 100 = 0 which is absurd and so the assumption that we can have equilibrium is untenable.

We shall next consider equations under which general equilibrium is realized. We shall suppose that we have a two-commodity economy, with

$P(1) = p$, $P(2) = q$, the price of commodity 1 and 2 respectively. We suppose that supply functions, $S(1)$ and $S(2)$ each depends on the prices p and q according to the equations

$$S(1) = 12 - 5p + q$$
$$S(2) = 5 + 2p + q$$

Similarly the demand functions $D(1)$ and $D(2)$ each depend on the same prices p and q according to the equations

$$D(1) = 2 - 3p + 5q$$
$$D(2) = 2 + p + 3q$$

Equilibrium is realized if there exists pieces p and q such that $S(1) = D(1)$ and $S(2) = D(2)$ simultaneously. This leads to the equations

$$12 - 5p + q = 2 - 3p + 5q$$

and

$$5 + 2p + q = 2 + p + 3q$$

or

$$p + 2q = 5$$
$$-p + 2q = 3.$$

Hence, at prices $p = 1$, $q = 2$, general equilibrium is realized.

In general then, more assumptions will be necessary to insure a set $P(1), P(2), \ldots, P(6000)$ which solves the equilibrium equations. An outline of the proof of the existence of a solution to the general equilibrium equations based on a modification of a model displayed by Lionel McKenzie is given below. To simplify the model, we shall restrict our discussion to the one variable case, though the McKenzie model assumes a system of demand equations in many variables.

P_d = the demand price

P_s = the supply price

Y_d = the demand quantity

Y_s = the supply quantity

The economic assumptions are sets of inequalities which assert the following:

(1) Production is proportional to demand using no more than available resources.

(2) 'Perfect competition' economy is assumed, so profits are zero.

Essentially we assume that the demand quantity is a continuous function of the demand price, and write $Y_d = D(P_d)$ and we assume that supply quantity is a continuous function of the supply price, and we write $Y_s = S(P_s)$. Our object is to show that the economy is in general equilibrium; i.e. there is a value $P = P_s = P_d$ such that $S(P) = D(P)$. It is assumed that it is possible to solve $Y_s = S(P_s)$ for P_s to obtain the continuous function $P_s = G(Y_s)$. This leads to the equations

$$Y_d = D(P_d)$$

$$P_s = G(Y_s)$$

Our model guarantees that there are values of Y_d and Y_s that coincide. Then

$$G[D(P_d)] = P_s.$$

Since $G[D(P_d)]$ is continuous, therefore, there exists a value P by the fixed-point theorem 2.7.2.[27] such that $G[D(P)] = P$. At this price, the demand conditions, the production conditions and no profit conditions are satisfied.

4. A Nonlinear Initial Value Problem.

The initial value problem:

$$u_{xx} - u_{tt} - 2Ku_t - \psi u = b(x,t) + \epsilon u^3 \tag{1}$$

$- \infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

where ψ, K , ϵ are constants, has been investigated by Ficken and Fleishman and their result are indicated here.

In order to reduce the problem to that of solving an integral equation, we solve the linear problem first,

$$Lu \equiv u_{xx} - u_{tt} - 2Ku_t - \psi(u) = r(x,t) \qquad (2)$$

The solution to the linear problem (2) can be described in terms of the linear factors H, M, G, and F which are defined as follows

$$H\, w(x,y,v) = \frac{1}{2} e^{-k(t-v)} \int_{x-t+v}^{x+t-v} w(\alpha,v)\, J_o(\nu\beta) d\alpha$$

where $\beta = ((t - v)^2 - (x - \alpha)^2)^{\frac{1}{2}}$, $\nu = (\psi - k^2)^{\frac{1}{2}}$

$$M\, w(x,t) = - \int_o^t H\, w(x,t,v) dv$$

$$G\, g(x,t) = H\, w(x,t,0) \quad \text{where} \quad w(x,0) = g(x)$$

$$F\, f(x,t) = G\, f(x,t) + 2kGf(x,t)$$

More explicitly

$$G\, g(x,t) = \frac{1}{2} e^{-kt} \int_{x-t}^{x+t} g(\alpha)\, J_o(\nu(t^2 - (x - \alpha)^2)^{\frac{1}{2}} d\alpha$$

$$F\, f(x,t) = \frac{1}{2} e^{-kt} (f(x + t) + f(x - t))$$
$$+ k \int_{x-t}^{x+t} f(\alpha)\, J_o(\nu\beta) d\alpha$$
$$+ \int_{x-t}^{x+t} f(\alpha)\, \frac{\partial J_o}{\partial t} (\nu\beta)\, d\alpha$$

where the initial data f,g satisfy $f(x) \in c^1$ (class of bounded functions with bounded continuous first derivatives) and $g(x)$ belongs to c^2 (class of bounded functions with bounded continuous first and second derivatives) and $r(x,t)$ belongs to c^1c (class of functions which are bounded and continuous together with their first derivative with respect to x for all real x and all non-negative t), then the problem (2) is solved by

$$u = Ff + Gg + Mr \tag{3}$$

The sum of first two terms on the right side of (3) is a solution of the homogeneous equation $Lu = 0$ and also satisfy the initial conditions. The last term satisfy the inhomogeneous equation $Lu = r$ with zero initial data.

Thus we have the integral equation

$u = Ff + Gg + M\ (b + \epsilon u^3)$ for the determination of the solution of (1). In solving (3) employ the space of cc (functions of x and t which are bounded and continuous for all real x and $t \geq 0$) with the norm $||w|| = \sup|w(x,t)|$. It is assumed that $k > 0$, $\psi > 0$.

Using contraction mapping theorem, we shall try to solve (3) by successive approximations.

Consider a closed ball

$$S_R = \{w : ||w|| \leq R\}$$

in the space cc such that the operator U maps S_R into itself and such that U satisfies a Lipschitz condition

$$||U(u - z)|| \leq \theta\ ||u - z||$$

with $0 < \theta < 1$ and u,z belong to S_R. First we find the norm of the operators M,G and F

$$||Mw|| \leq m \; ||w||$$

where (i) $m = k^{-2}$ when $0 < k^2 < \psi$

(ii) $m = (k - (k^2 - \psi)^{\frac{1}{2}})^{-2}$ when $0 < \psi < k^2$

Now, let a number θ be chosen between 0 and 1, and let R and ϵ satisfy the inequality

$$3|\epsilon| mR^2 \leq \theta$$

Then if $||u|| \leq R$ and $||z|| \leq R$, we have

$$\begin{aligned} ||U(u - z)|| = ||\epsilon M(u^3 - z^3)|| &\leq |\epsilon|m||u^2 + uz + z^2|| \; ||u - z)|| \\ &\leq 3|\epsilon|mR^2||u - z|| \\ &\leq \theta||u - z|| \end{aligned}$$

so that the Lipschitz condition is satisfied in S_R.

With R and ϵ subject to the same inequality and with $||u|| \leq R$,

$$\begin{aligned} ||Uu|| &\leq ||F|| \; ||f|| + ||G|| \; ||g|| + m||b|| + |\epsilon|mR^3 \\ &\leq ||F|| \; ||f|| + ||G|| \; ||g|| + m||b|| + \frac{\theta R^3}{3} \end{aligned}$$

Thus the operator U will be contraction mapping of S_R into itself providing that the inequalities

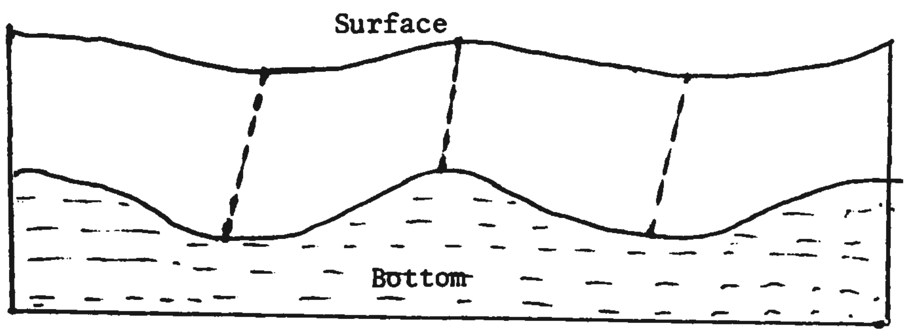
$$||F|| \; ||f|| + ||G|| \; ||g|| + m||b|| \leq (1 - \tfrac{\theta}{3})R$$

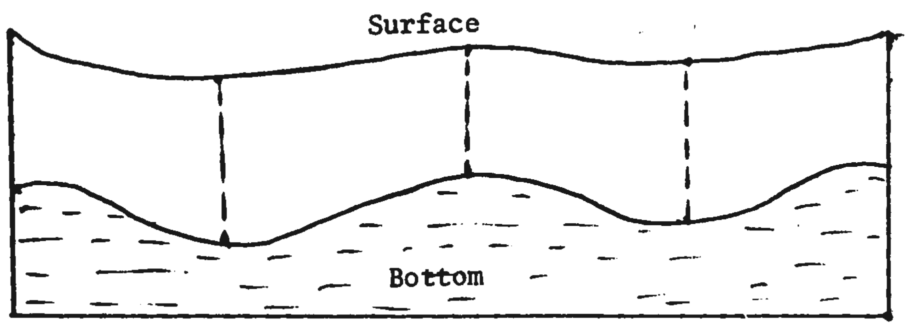
$$3m \; R^2|\epsilon| \leq \theta < 1$$

are both satisfied. Under these conditions the method of successive approximations starting with a function $u_o \in S_R$ will converge to a solution $u(x,t)$ of (3) which in turn is a solution of (1).

5. These illustrations are due to Shinbrot [29].

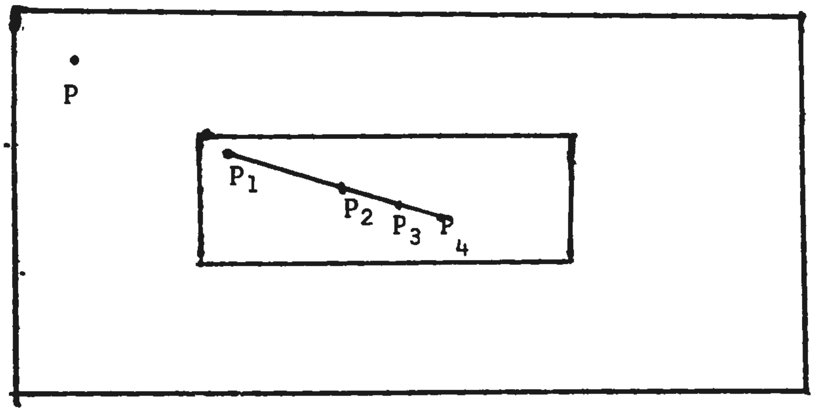
(i)





The feasability of water flow of a certain type over a periodically rising and falling bottom can be demonstrated by use of fixed-point theorems. It has been recently shown that the surface of water could rise and fall according to the same general period as the bottom and that the high and low points can lie directly above the high and low points of the bottom.

(ii)



Contraction of a surface must result in point remaining in the position it occupies before contraction. The larger rectangle represents the original surface, a sheet of rubber stretched taut; smaller rectangle represents the sheet after it has sprung back to its relaxed position. We consider the point, P, near the corner at top left on the original rectangle. After the contraction it assumed a position we designate P_1. The point that was at P_1 originally has moved inward to a new position P_2. The point originally at P_2 has moved to P_3, and so on. The interval between P_2 and P_3 is smaller than the interval between P_1 and P_2. In fact $P_1, P_2, P_3, \ldots$ form a series approaching a limit: the fixed point.

BIBLIOGRAPHY

1. Bailey, D.F. "Some Theorems on Contractive Mappings", Jour. Lond. Math. Soc. 41(1966), p. 101-106.

2. Banach, S. "Surles Operations dans les Ensembles Abstraits et leur applications aux Equations Integrals", Fund. Math. 3(1922), p. 133-181.

3. Blumenthal, L.M. "Theory and applications of distance Geometry", Oxford Clarendon Press (1953).

4. Boyd, D.W. and Wong, J.S.W. "On Nonlinear Contractions", Proc. Amer. Math. Soc. Vol. 20, No. 2(1969), p. 458-464.

5. Bonsall, F.F. "Lectures on Some Fixed Point Theorems of Functional Analysis", Tata Institute of Fundamental Research, Bombay (1962).

6. Cheney, W. and Goldstein, A.A. "Proximity Maps for Convex Sets", Proc. Amer. Math. Soc. Vol. 10(1959) p. 448-450.

7. Chu, S.C. and Diaz, J.B. "Remarks on a Generalization of Banach's Principle of Contraction Mappings", Jour. Math. Anal. Appl. 11(1965) p. 440-446.

8. Chu, S.C. and Diaz, J.B. "A Fixed Point Theorem for << in the large >> Application of the Contraction Principle", Atti. della. Aca. delle.Sc.di Torino. 99(1964-1965) p. 351-363.

9. Edelstein, M. "An extension of Banach's Contraction Principle", Proc. Amer. Math. Soc. 12(1961). p. 7-10.

10. Edelstein, M. "On fixed and Periodic points under Contractive Mappings", Jour. Lond. Math. Soc. Vol.37 (1962) p. 74-79.

11. Edelstein, M. "On Nonexpansive Mappings", Proc. Amer. Math. Soc. 15(1964), p. 689-695.

12. Edelstein, M. "A Remark on a Theorem of A.F. Monna", Indag. Math. 26, No. 1(1964) p. 88-89.

13. Edwards, R.E. "Functional Analysis, Theory and Applications", Holt, Rinehart, Winston, Inc.(1965).

14. Guseman, L.F. Jr. "Fixed Point Theorems for Mapping with a Contractive iterate at a Point", Proc. Amer. Math. Soc. Vol. 24(1970) p. 615-618.

15. Holmes, R.D. "On Fixed and Periodic Points under Certain Sets of Mappings", Can. Math. Bull. Vol 12, No. 6(1969) p. 813-822.

16. Kolmogorov, A.N. and Fomin, S.V. "Elements of the Theory of Functions and Functional Analysis", Vol. 1, Greylock Press(1957), New York.

17. Kannan, R. "Some Results on Fixed Points - I", Bull. Cal. Math. Soc. Vol. 60(1968) p. 71-76.

18. Kannan, R. "Some Results on Fixed Points - II", Amer. Math. Monthly, 76(1969) p. 405-408.

19. Kantorovich, L.V. and Akilov, G.P. "Functional Analysis in Normed Spaces", Pergamon Press (1964).

20. Kreider, D.L., Kuller, R.G., Ostberg,D.R. "Elementary Differential Equations", Addison Wesley Publishing Co. (1968).

21. Lusternik, L. and Sobolev, V.J. "Elements of Functional Analysis", Frederick Ungar Publishing Company, New York (1961).

22. Luxemburg, W.A.J. "On the convergence of successive approximations in Theory of Ordinary Differential Equations II", Indag. Math. 20, No. 5(1958) p. 540-546.

23. Margolis, B. "On Some Fixed Point Theorems in Generalized Complete Metric Spaces", Bull. Amer. Math. Soc. 74(1968) p. 275-282.

24. Monna, A.F. "Sur un Theoreme dem Luxemburg concernant les points fixes dume classed applications dume spaces metrique dans lui meme", Indag. Math. 23(1961) p. 89-96.

25. Nadler Jr., S.B. "Sequences of Contractions and Fixed Points", Pacific Jour. Math. Vol. 27(1968) p. 579-585.

26. Rakotch, E. "A note on Contractive Mappings", Proc. Amer. Math. Soc. Vol. 13(1962) p. 459-465.

27. Saltz, D. "Introduction to Analysis", Prentice-Hall(1965).

28. Sehgal, V.M. "A Fixed Point Theorem for Mappings with a Contractive Iterate", Proc. Amer. Math. Society, Vol. 23(1969) p. 631-634.

29. Shinbrot, M. "Fixed Point Theorems", Math. in the Modern World, Scientific American, (1966) p. 145-150.

30. Singh, K.L. "Some Fixed Point Theorems in Analysis", Master Thesis, Memorial Univ. (1968).

31. Singh, S.P. "On a Fixed Point Theorem in Metric Space", Rend. Semi. Mat. Univ. Padova, Vo. XLIII (1970) p. 229-232.

32. Singh, S.P. "On Fixed Points", Publ. Inst. Math., Nouvelle Serie Vol. 11(25)1971 (to appear).

33. Singh, S.P. "Some Results on Fixed Point Theorems", Yokohoma Math. Jour. Vol. XVII, No. 2(1969) p. 61-64.

34. Singh, S.P. and Russell W. "A Note on Sequence of Contraction Mappings", Can. Math. Bull. Vol.12, No. 4 (1969) p. 513-516.

35. Wong, J.S.W. "Two extensions of Banach Contraction Mapping Principle", Jour. of Math. Anal. and Appl. Vol. 22, No. 2, May (1968). p. 438-443.