MULTI-VALUED CONTRACTION MAPPINGS AND FIXED POINTS IN METRIC SPACES

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BY

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Abstract

The purpose of this thesis is to set forth some fixed point theorems in (complete) metric spaces for single-valued and multi-valued contraction mappings; with emphasis on multi-valued contraction mappings.

In Chapter I, we discuss Banach's Contraction Mapping Principle and present some fixed point theorems in metric and complete metric spaces which extend and generalize Banach's result. Some results on contractive and non-expansive mappings are also given.

In Chapter II, in the main, we shall consider multi-valued contraction mappings. In this respect, we have the following main definition and theorem due to Nadler Jr.[21].

Definition. Let (X,d) be a complete metric space, let $CB(X)$ denote the nonempty closed and bounded subsets of X, and let H be the Hausdorff metric for $CB(X)$. A function $F : X \to CB(X)$ is said to be a multi-valued contraction mapping if and only if there is a real number α, $0 \leq \alpha < 1$, such that $H(F(x), F(y)) \leq \alpha d(x,y)$, for all $x,y \in X$.

Theorem. If (X,d) is a complete metric space and $F : X \to CB(X)$ is a multi-valued contraction mapping, then F has a fixed point (i.e., there is an $x_0 \in X$ such that $x_0 \in F(x_0)$).

Other results due to Nadler Jr. will also be given for metric spaces and generalized metric spaces, and these results will be extended to the following mappings $F : X \to CB(X)$ such that

(A) $H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y))]$ for all $x,y \in X$, $0 \leq \alpha < 1/2$;

(B) $H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y)) + d(x,y)]$ for all $x,y \in X$, $0 \leq \alpha < 1/3$.

Some fixed point theorems for single-valued mappings will be given which are analogous to those for multi-valued mappings.

In Chapter III, we consider sequences of single-valued and multi-valued contraction mappings and fixed points.

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INTRODUCTION

Many generalizations and extensions of the Banach Contraction Principle have been given since its original formulation in 1922. Edelstein has generalized the Contraction Principle to mappings satisfying a less restrictive Lipschitz inequality such as local contractions [11] and contractive mappings [12]. Knill [19] and others have considered contraction mappings in the more general setting of Uniform Spaces.

Since 1941, much work has been done on fixed points of multi-valued functions. Among those who have worked in this area are such mathematicians as Kakutani [16], Eilenberg and Montgomery [14], Strother [32], Plunkett [23], and Ward [35] to mention a few. The theory of fixed points of multi-valued mappings can be used to prove the existence of periodic solutions of ordinary differential equations. Kakutani showed that this implied the mini-max theorem for finite games. Kakutani's theorem was further extended to convex linear topological spaces and implies the mini-max theorem for continuous games with continuous payoff as well as the existence of Nash equilibrium points.

The concept of a multi-valued contraction mapping is due to Nadler Jr. [21]. It is a combination of the ideas of set-valued mappings and Lipschitz mappings. The resulting fixed point theorems place no severe restrictions on the images of points and, in general, all that is required of the space is that it be complete metric.

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CHAPTER I

Contraction Mappings and Fixed Points

The purpose of this chapter is to introduce some preliminary definitions and to discuss some of the known theorems in connection with contraction, contractive, and nonexpansive mappings of a metric space into itself.

1.1. Preliminary Definitions:

Definition 1.1.1. A metric space is the pair of two things: A set X, whose elements are called points, and a distance, i.e. a single-valued, nonnegative, real function $d : X \times X \to R^+$ (R^+ denotes the positive reals) defined for arbitrary x and y in X and satisfying the following conditions:

(i) $d(x,y) \geq 0 \quad x,y \in X$.

(ii) $d(x,y) = 0$ if and only if $x = y$.

(iii) $d(x,y) = d(y,x)$ (axiom of symmetry)

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

An immediate consequence of this definition is the property

(v) $|d(x,y) - d(x,z)| \leq d(y,z)$.

In cases where no misunderstanding can arise we shall sometimes denote the metric space by the same symbol X which is used for the set of points itself.

Definition 1.1.2 . A metric space (X,d) is called a pseudo-metric space or semi-metric space if condition (ii) in Definition 1.1.1. is replaced by

(ii*) $\quad d(x,y) = 0 \quad$ if $\quad x = y$.

Definition 1.1.3. A sequence $\{x_n\}$ of points of a metric space X is said to converge to a point x_o belonging to X if given $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that

$$d(x_n,x_o) < \varepsilon \quad \text{whenever} \quad n \geq N(\varepsilon),$$

or we write

$$\lim_{n\to\infty} d(x_n,x_o) = 0.$$

One can easily verify that a convergent sequence has a unique limit.

Definition 1.1.4. A sequence $\{x_n\}$ of points of a metric space X is said to be a Cauchy sequence or a fundamental sequence if it satisfies the Cauchy criterion, i.e. if for arbitrary $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that

$$d(x_n,x_m) < \varepsilon \quad \text{for all} \quad n,m \geq N(\varepsilon).$$

It follows directly from the triangle inequality that every convergent sequence is Cauchy.

Definition 1.1.5. A metric space X is said to be complete if every Cauchy sequence of points of X is convergent in X.

Definition 1.1.6. Let T be a mapping of a set X into itself. A point y belonging to X is said to be a fixed point of T if $T(y) = y$.

Definition 1.1.7. A mapping T of a metric space X into itself is said to satisfy Lipschitz condition if there exists a real number α such that

$$d(T(x),\ T(y)) \leq \alpha d(x,y) \quad \text{for all} \quad x,y \in X.$$

(i.e. for all x,y belonging to X).

In the special case when $0 \leq \alpha < 1$, T is said to be a contraction mapping.

<u>Remark</u> 1.1.8. If T is a contraction mapping on a metric space X, then T is continuous on X.

Proof. Let $\varepsilon > 0$ be given and x be any point in X. Since T is a contraction mapping, we have

$$d(T(x),\ T(y)) \leq \alpha d(x,y) \quad \text{for all} \quad x,y \in X.$$

and $0 \leq \alpha < 1$.

If $\alpha = 0$, we have

$$d(T(x),\ T(y)) = 0 < \varepsilon \quad \text{for all} \quad y \in X, \text{ and } T \text{ is continuous}$$

at x.

Otherwise, let $\delta = \varepsilon/\alpha$ and y be any other point in X such that $d(x,y) < \delta$.

We have

$$\begin{aligned} d(T(x),\ T(y)) &\leq \alpha d(x,y) \\ &< \alpha \cdot \varepsilon/\alpha = \varepsilon \ . \end{aligned}$$

Hence T is continuous at x which is an arbitrary point, therefore, the contraction mapping T is continuous everywhere.

<u>Remark</u> 1.1.9. The converse of the above statement is easily shown to be false. We need only consider a translation mapping which is continuous but is not a contraction.

1.2 Contraction Mappings.

In 1922 S. Banach (1892-1945), a famous Polish mathematician and one of the founders of Functional Analysis, formulated the "Principle of Contraction Mappings" which is widely used to prove the existence and uniqueness of solutions to differential and integral equations.
It will be stated as a theorem.

<u>Theorem</u> 1.2.1. Every contraction mapping defined in a complete metric space X to itself has one and only one fixed point (i.e. the equation $T(x) = x$ has one and only one solution).

Proof. Let x_o be an arbitrary point in X and let

$$\begin{aligned} x_1 &= T(x_o) \\ x_2 &= T(x_1) = T^2(x_o) \\ &\cdots\cdots\cdots\cdots \\ x_n &= T(x_{n-1}) = T^n(x_o). \end{aligned}$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.
From the definition of a contraction mapping, we have

$$d(T(x),\ T(y)) \leq \alpha d(x,y) \quad \text{for all } x,y \in X.$$

Therefore

$$\begin{aligned} d(x_n,\ x_m) = d(T^n(x_o),\ T^m(x_o)) &\leq \alpha^n d(x_o,\ x_{m-n}) \quad , \qquad m > n \\ &\leq \alpha^n[d(x_o,\ x_1) + d(x_1,\ x_2) + \dots + d(x_{m-n-1},\ x_{m-n})] \\ &\leq \alpha^n d(x_o,\ x_1)[1 + \alpha + \alpha^2 + \dots + \alpha^{m-n-1}] \\ &\leq \alpha^n d(x_o,\ x_1)[\frac{1}{1 - \alpha}]. \end{aligned}$$

Since $\alpha < 1$, this quantity is arbitrarily small for sufficiently large n.

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, $\lim_{n\to\infty} x_n$ exists. We set $\lim_{n\to\infty} x_n = x$.

Then by virtüe of the continuity of the mapping T, we get

$$T(x) = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Thus, the existence of a fixed point is proved.

We shall now prove its uniqueness.

Let x and y be two fixed points of T, where $x \neq y$.

Then $\quad d(T(x), T(y)) \leq \alpha d(x,y)$,

but $\quad d(T(x), T(y)) = d(x,y)$.

Hence $\quad d(x,y) \leq \alpha d(x,y)$

and $\quad 1 \leq \alpha$.

This is a contradiction and so $x = y$.

Remark 1.2.2. Both conditions of the above theorem are necessary.

(a) The map $T : (0,1] \to (0,1]$ defined by $T(x) = x/2$ is a contraction but has no fixed point. We note that in this case the condition of completeness of the space is violated.

(b) The map $T : R \to R$, where R denotes the set of real numbers, defined by $T(x) = x + 1$ is not a contraction and has no fixed point although R is a complete metric space.

If we replace metric space in Theorem 1.2.1 by pseudo-metric space, we get the following theorem.

Theorem 1.2.3. If T is a contraction self-mapping on a complete pseudo-metric space X then T has a fixed point but not necessarily unique.

Proof. Existence may be justified as in the previous theorem. Now referring to the discussion on uniqueness in Theorem 1.2.1.

$$d(x,y) \leq \alpha d(x,y) \Rightarrow d(x,y) - \alpha d(x,y) \leq 0.$$

Suppose,

$$d(x,y) - \alpha d(x,y) = 0.$$

Then $(1 - \alpha)d(x,y) = 0.$

but $(1 - \alpha) \neq 0$ for $\alpha \in [0,1)$.

Therefore,

$d(x,y) = 0$ but this does not imply that $x = y$.

The Banach Contraction Principle remains the most fruitful means for proving and analysing the convergence of iterative processes. For this reason extensions of the theorem are of continuing interest. The following extensions are worth mentioning.

The following two theorems have been given by Chu and Diaz [6].

Theorem 1.2.4. If T maps a complete metric space X into itself and if T^n (n is a positive integer) is a contraction mapping in X, then T has a unique fixed point.

Proof. Since T^n is a contraction in X, therefore, by the Banach Contraction Principle, it has a unique fixed point, say x.

That is to say $T^n(x) = x$.

$T(x) = T\,T^n(x) = T^nT(x)$, thus $T(x)$ is a fixed point of T^n.

But T^n has a unique fixed point, therefore, $T(x) = x$.

Clearly T has a unique fixed point, for if $T(x_o) = x_o$, then $T^n(x_o) = x_o$, hence $x = x_o$ since T^n has a unique fixed point.

Theorem 1.2.5. Let E be any nonempty set of elements and T be a map of E into itself. If for some positive integer n, T^n has a unique fixed point, then T also has a unique fixed point.

Remark. The above theorem has been improved, under different conditions, by Chu and Diaz [7].

Theorem 1.2.6. Let T be a mapping defined on a nonempty set E into itself, K be another function defined on X mapping it into itself such that $KK^{-1} = I$, where I is the identity function of X. Then T has a unique fixed point if and only if $K^{-1}T\,K$ has a unique fixed point.

Proof. Let x be the unique fixed point of $K^{-1}TK$.

$$K^{-1}TK(x) = x$$

Premultiply by K, we have

$$KK^{-1}TK(x) = K(x),$$

$$TK(x) = K(x).$$

Therefore, $K(x)$ is a fixed point of T. Uniqueness is obvious.

Conversely, let x_o be a fixed point of T, then

$$T(x_o) = x_o.$$

Premultiply by K^{-1}.

$$K^{-1}T(x_o) = K^{-1}(x_o) \tag{1}$$

Since $KK^{-1} = I$, therefore, we write (1) as

$K^{-1}T\,K(K^{-1}x_o) = K^{-1}(x_o)$ i.e. $K^{-1}(x_o)$ is a fixed point of $K^{-1}TK$.

Uniqueness is obvious.

The following is an immediate corollary to the above theorem.

Corollary 1.2.7. Let X be a complete metric space, $T : X \to X$, and $K : X \to X$ be such that $KK^{-1} = I$, the identity function. If $K^{-1}TK$ is a contraction in X, then T has a unique fixed point.
The proof of the corollary follows directly from Theorem 1.2.6. and the Banach Contraction Principle.

The following results have been given by R. Kannan [17].

Theorem 1.2.8. If T_1 and T_2 are two mappings of a complete metric space X into itself and if

$$d(T_1(x), T_2(y)) \leq \alpha[d(x, T_1(x)) + d(y, T_2(y))]$$

for all $x,y \in X$ and $0 \leq \alpha < \frac{1}{2}$, then T_1 and T_2 have a unique common fixed point.

If $T_1 = T_2 = T$, then we obtain the following theorem.

Theorem 1.2.9. If T is a mapping of a complete metric space X into itself and if

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] \quad \text{where}$$

$x,y \in X$ and $0 \leq \alpha < 1/2$, then T has a unique fixed point.
He has also obtained the conclusions of Theorem 1.2.8. under different assumptions.

Theorem 1.2.10. Let T_1 and T_2 be mappings of a complete metric space X into itself and if

(i) $d(T_1(x), T_2(y)) \leq \alpha d(x,y)$, $0 < \alpha < 1$, $x,y \in X$, $x \neq y$,

(ii) T_2 is a contraction mapping, i.e. there exists α , $0 < \alpha < 1$, such that

$$d(T_2(x), T_2(y)) \leq \alpha d(x,y) \quad \text{for all} \quad x,y \in X,$$

(iii) there exists an $x \in X$ such that if $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$ and so on, then $x_r \neq x_s$ if $r \neq s$, then T_1 and T_2 have a unique common fixed point.

Theorem 1.2.10. has been generalized by Singh [27] in the following way.

Theorem 1.2.11. If T_1 and T_2 are two mappings of a complete metric space X into itself and if

(i) $d(T_1(x), T_2(y)) \leq \alpha d(x,y)$; $0 < \alpha < 1$, $x,y \in X$, $x \neq y$.

(ii) T_2 has a unique fixed point,

(iii) there exists an $x \in X$ such that if $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$, and so on, $x_r \neq x_s$ if $r \neq s$, then T_1 and T_2 have a common unique fixed point.

Proof. It can be easily seen that T_1 and T_2 cannot have distinct fixed points, for example, if $y = T_1(y)$ and $z = T_2(z)$, $y \neq z$, then

$$\begin{aligned} d(y,z) &= d(T_1(y), T_2(z)) \\ &\leq \alpha d(y,z), \end{aligned}$$

a contradiction, since $0 < \alpha < 1$.

Since T_2 has a unique fixed point and therefore if T_1 has a fixed point it must be unique. We need to prove that T_1 has a fixed point.

We have

$$x_1 = T_1(x),$$

$$x_2 = T_2(x_1),$$

$$x_3 = T_1(x_2), \quad \text{and so on.}$$

Now,

$$d(x_1,x_2) = d(T_1(x),\ T_2(x_1)) \le \alpha d(x,x_1)$$

and

$$d(x_2,x_3) = d(T_2(x_1),\ T_1(x_2)) \le \alpha d(x_1,x_2) \le \alpha^2 d(x,x_1).$$

In this way we get, $d(x_n,\ x_{n+1}) \le \alpha^n d(x,x_1)$.

Now we show that $\{x_n\}$ is a Cauchy sequence.

$$\begin{aligned} d(x_n,\ x_{n+p}) &\le d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2}) + \dots + d(x_{n+p-1},\ x_{n+p}) \\ &\le \alpha^n d(x,x_1) + \alpha^{n+1} d(x,x_1) + \dots + \alpha^{n+p-1} d(x,x_1) \\ &= d(x,x_1)\ [\alpha^n + \alpha^{n+1} + \dots + \alpha^{n+p-1}] \\ &< \frac{\alpha^n}{1-\alpha}\, d(x,x_1) \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ since } \alpha < 1. \end{aligned}$$

Since X is a complete metric space, the sequence $\{x_n\}$ converges to y in X.

We want to show that $T_1(y) = y$.

$$\begin{aligned} d(y,\ T_1(y)) &\le d(y,\ x_n) + d(x_n,\ T_1(y)) \\ &= d(y,\ x_n) + d(T_2(x_{n-1}),\ T_1(y)) \end{aligned}$$

where n is even positive integer.

Thus,

$$\begin{aligned} d(y,\, T_1(y)) &\le d(y,\, x_n) + \alpha d(x_{n-1},\, y) \\ &< d(y,\, x_n) + d(x_{n-1},\, y), \quad \text{since } \alpha < 1. \end{aligned}$$

Since $\{x_n\}$ converges to y, we have $d(y,\, T_1(y)) = 0$. i.e. $T_1(y) = y$. Hence, $T_1(y) = y = T_2(y)$.

The above theorem is more general than Theorem 1.2.10 since one can easily find a mapping that has a fixed point but is not a contraction mapping. Singh [27] has given the following example.

Example 1.2.12. Let $T_2 : [0,1] \to [0,1]$ defined by

$$\begin{aligned} T_2(x) &= x/4 \quad , \quad x \in [0,\, 1/2) \\ &= x/5 \quad , \quad x \in [1/2,\, 1]. \end{aligned}$$

Clearly, T_2 is discontinuous at $x = 1/2$ and therefore T_2 is not a contraction mapping, but T_2 has a unique fixed point, say $x = 0$.

The following theorem is also due to Singh [27].

Theorem 1.2.13. If T_1 and T_2 are two mappings of a complete metric space X into itself and if

(i) $d(T_1(x),\, T_2(y)) \le \alpha d(x,y)$, $0 \le \alpha < 1$, $x,y \in X$, $x \neq y$,

(ii) there exists an $x \in X$ such that if $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$ and so on, $x_p \neq x_q$ if $p \neq q$;

then T_1 and T_2 have a common fixed point.

Remark. In case T_1 and T_2 are identical, then we get the well-known Banach Contraction Principle. In this case it is not necessary that $x \neq y$ and therefore condition (ii) is not required.

The following theorems are due to Kannan [18] in which he has omitted the completeness of the space.

Theorem 1.2.14. Let X be a metric space. Let T be a map of X into itself such that

(i) $d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))]$, $0 < \alpha < 1/2$, $x,y \in X$.

(ii) T is continuous at a point $z \in X$.

(iii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a sequence $\{T^{n_i}(x)\}$ converging to z.

Then z is the unique fixed point of T.

Theorem 1.2.15. Let X be a metric space and T be a continuous map of X into itself such that

(i) $d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))]$, $0 < \alpha < 1/2$, x,y belong to an everywhere dense subset M of X,

(ii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to a point $z \in X$.

Then z is the unique fixed point of T.

Theorem 1.2.16. Let X be a metric space and T be a map of X into itself. Suppose T is continuous at a point $x_o \in X$. If there exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ converges to x_o, then $T(x_o) = x_o$. If in addition

$$d(T(x_o), T(z)) \leq \alpha d(x_o,z), \quad z \in X, \quad 0 < \alpha < 1,$$

then x_o is the unique fixed point of T.

Singh [26] has given the following extensions to Theorem 1.2.9 by taking $\alpha = 1/2$.

Theorem 1.2.17. Let X be a metric space and let $T : X \to X$ be a continuous mapping such that

$$d(T(x), T(y)) < \frac{1}{2} [d(x, T(x)) + d(y, T(y))]$$

for $x \neq y$.

If for some $x_o \in X$, the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to x, then $\{T^n(x_o)\}$ converges to x and x is the unique fixed point of T.

Proof. The sequence $\{d(T^n(x_o), T^{n+1}(x_o))\}$ is non-increasing.

Since T is continuous we get

$$\lim_{i\to\infty} T^{n_i+1}(x_o) = T(x)$$

and

$$\lim_{i\to\infty} T^{n_i+2}(x_o) = T^2(x)$$

Therefore,

$$\begin{aligned} d(x, T(x)) &= \lim_{i\to\infty} d(T^{n_i}(x_o), T^{n_i+1}(x_o)) \\ &= \lim_{i\to\infty} d(T^{n_i+1}(x_o), T^{n_i+2}(x_o)) \\ &= d(T(x), T^2(x)). \end{aligned}$$

If $x \neq T(x)$, then $d(T(x), T^2(x)) < d(x, T(x))$. This implies $d(x, T(x)) < d(x, T(x))$, impossible, therefore $d(x, T(x)) = 0$ i.e. x is a fixed point of T.

Uniqueness follows immediately. Let x and y be two fixed points, $x \neq y$. Then $x = T(x)$ and $y = T(y)$ imply

$$d(T(x), T(y)) < \frac{1}{2} [d(x, T(x)) + d(y, T(y))] = 0$$

a contradiction to the fact that $d(T(x), T(y)) \geq 0$. Thus the proof.

Theorem 1.2.18 . Let T be a continuous map of a metric space X into itself such that

(i) $d(T(x), T(y)) \leq \frac{1}{2} [d(x, T(x)) + d(y, T(y))]$,

(ii) if $x \neq T(x)$, then $d(T(x), T^2(x)) < d(x, T(x))$,

(iii) for some $x_o \in X$, the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to x. Then the sequence $\{T^n(x_o)\}$ converges to x and x is the unique fixed point.

Theorem 1.2.19. Let T be a mapping of a metric space X into itself such that

(i) $d(T(x), T(y)) \leq d(x,y)$ for all $x,y \in X$,

(ii) $d(T(x), T(y)) < \frac{1}{2} [d(x, T(x)) + d(y, T(y))]$ for all $x,y \in X$,

(iii) if for some $x_o \in X$, the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to x then $\{T^n(x_o)\} \to x$ and x is the unique fixed point.

Definition 1.2.20. Let X be a metric space. A mapping T of X into itself is said to be locally contractive if for every $x \in X$, there exists ε and λ ($\varepsilon > 0$, $0 \leq \lambda < 1$), which may depend on x, such that

$$p, q \in S(x,\varepsilon) = \{y \in X \text{ such that } d(x,y) < \varepsilon\}$$

implies $d(T(p), T(q)) < \lambda d(p,q)$.

Definition 1.2.21. A mapping T of a metric space X into itself is said to be (ε,λ) - uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Definition 1.2.22. A metric space (X,d) is said to be ε-chainable (where $\varepsilon > 0$ is fixed) if and only if given $x,y \in X$ there is an ε-chain from x to y (that is, a finite set of points $x_o, x_1, \ldots, x_n \in X$ such that $x_o = x$, $x_n = y$ and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$).

In this respect, Edelstein [11] has given the following extension of Banach's Contraction Principle.

Theorem 1.2.23. Let T be a complete ε-chainable metric space and T a mapping of X into itself which is (ε,λ)-uniformly locally contractive. Then T has a unique fixed point.

For the following two definitions and theorems we shall assume that (X,d) is a complete metric space, T is a continuous mapping of X into itself, and ϕ is a real-valued function defined over the nonnegative reals which is upper semi-continuous from the right and satisfies $\phi(0) = 0$.

Definition 1.2.24. T is said to be locally contractive at $u \in X$ if there exists a positive integer $n(u)$ such that

$$d(T^{n(u)}(x), \quad T^{n(u)}(y)) \leq \phi(d(x,y)). \tag{1}$$

for all $x,y \in I(u ; T) = \{T^K(u) : k = 0,1,2, \ldots\}$, where ϕ satisfies

$$\phi(r) < r \qquad \text{for} \qquad r > 0. \tag{2}$$

Definition 1.2.25. T is said to be locally iteratively contractive at $u \in X$ if there exists a positive integer $n(u)$ such that (1) holds for all $x,y \in I(u\ ;\ T)$, where ϕ satisfies

$$\phi(s) \leq \phi(t) \quad \text{whenever} \quad s \leq t,$$

and

$$\sum_{j=0}^{\infty} \phi^j(r) < \infty \qquad \text{for all} \quad r > 0.$$

$$(\text{Here } \phi^j(r) = \phi(\phi^{j-1}(r)).)$$

T is called a local contraction on X if T is locally contractive at u for all $u \in X$, and define a local iterative contraction similarly.

In view of Definition 1.2.24. and Definition 1.2.25., Wong [36] has given the following two extensions of the Banach Contraction Principle.

Theorem 1.2.26. Let T be a local contraction on X and ϕ satisfies in addition the following condition:

$$\liminf_{t \to \infty} (t - \phi(t)) = \alpha > 0,$$

then T has a unique fixed point in X. Moreover, for each $x \in X$, the sequence $x_n = T^{n(x)}(x_{n-1})$, $x_o = x$ converges in metric to the fixed point of T.

Theorem 1.2.27. Let T be a local iterative contraction on X. Then T has a unique fixed point in X and the successive approximations $x_n = T^{n(x)}(x_{n-1})$, $x_o = x$ for arbitrary $x \in X$ converge in metric to the fixed point of T.

1.3. Contractive Mappings:

Definition 1.3.1. A mapping T of a metric space X into itself is said to be contractive if

$$d(T(x),\ T(y)) < d(x,y) \quad \text{where} \quad x,y \in X, \quad x \neq y,$$

and is said to be ε-contractive if

$$d(T(x),\ T(y)) < d(x,y) \quad \text{where} \quad 0 < d(x,y) < \varepsilon \ .$$

Remark. A contractive map is clearly continuous and if such a mapping has a fixed point, then this fixed point is unique. However, a contractive mapping of a complete metric space into itself need not have a fixed point. Consider the following example:

Example 1.3.2. Let $T : R \to R$ be defined by

$$T(x) = x + \pi/2 - \text{arc tan } x.$$

Since $\text{arc tan } x < \pi/2$ for every x, the mapping T has no fixed point although it is a contractive map.

The following theorem due to Edelstein [12] states the sufficient conditions for the existence of a fixed point for a contractive mapping.

Theorem 1.3.3. Let T be a contractive mapping of a metric space X into itself and $x_o \in X$ be such that the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to a point $z \in X$. Then z is a unique fixed point of T.

In [15] a proof has been given of the above theorem which is similar to that of Cheney and Goldstein [5].

The following corollary is due to Edelstein.

Corollary 1.3.4. If T is a contraction mapping of a metric space X into a compact metric space $Y \subset X$, then T has a unique fixed point.

The following two theorems have been given by Edelstein [12].

Theorem 1.3.5. If T is ε-contractive and there exists an $x_o \in X$ such that $\{T^{n_i}(x_o)\}$ converges to z in X, then there exists at least one periodic point under T.

Remark. u is a periodic point of T means there is a positive integer p such that

$$T^p u = u.$$

Theorem 1.3.6 . If X is compact and ε-chainable metric space and if T is ε-contractive, then T has a unique fixed point.

The following theorem was given by Bailey [1] for a compact metric space.

Theorem 1.3.7. If T is a continuous mapping of a compact metric space X into itself and $0 < d(x,y)$ implies that there exists $n(x,y) \in I^+$ (the positive integers) such that

$$d(T^n(x), T^n(y) < d(x,y), \quad x,y \in X,$$

then T has a unique fixed point.

V.M. Sehgal [25] has proved the following theorem for a mapping with a contractive iterate.

Theorem 1.3.8. Let (X,d) be a complete metric space and $T : X \to X$ a continuous mapping; there exists an $\alpha < 1$ and for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$

$$d(T^{n(x)}(x),\ T^{n(x)}(y)) \leq \alpha d(x,y).$$

Then T has a unique fixed point.

Rakotch [24] has given the following theorem.

Theorem 1.3.9. If T is a contractive mapping of a metric space X into itself, and there exists a subset $M \subset X$ and a point $x_0 \in M$ such that

$$d(x,x_0) - d(T(x),\ T(x_0)) \geq 2d(x_0,\ T(x_0)) \quad \text{for every} \quad x \in X - M,$$

and T maps M into a compact subset of X, then there exists a unique fixed point.

1.4. Nonexpansive Mappings.

Definition 1.4.1. A mapping T of a metric space X into itself is said to be nonexpansive (ε-nonexpansive) if

$$d(T(x),\ T(y)) \leq d(x,y) \quad \text{for all} \quad x,y \in X \text{ (for all } x,y \in X \text{ with } d(x,y) < \varepsilon).$$

Isometry, i.e. $|T(x) - T(y)| = |x - y|$ for all $x,y \in X$, is a simple example of a nonexpansive mapping.

Definition 1.4.2. A point $y \in Y \subset X$ is said to belong to the T-closure of Y, $y \in Y^T$, if $T(Y) \subset Y$ and there exists a point $\delta \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < n_3 < \ldots < n_i < \ldots)$, so that the sequence $\{T^{n_i}(\delta)\}$ converges to y.

Definition 1.4.3. A sequence $\{x_i\}$ is said to be an isometric (ε-isometric) sequence if the condition

$$d(x_m, x_n) = d(x_{m+k}, x_{n+k})$$

holds for all $k,m,n = 1, 2, \ldots$ (for all $k,m,n = 1, 2, \ldots$ with $d(x_m, x_n) < \varepsilon$). A point x in X is said to generate such an isometric (ε-isometric) sequence under T if $\{T^n(x)\}$ is such a sequence.

The following theorems are given by Edelstein [13] on nonexpansive and ε-nonexpansive mappings on a metric space X.

Theorem 1.4.4. If $T : X \to X$ is an ε-expansive mapping and x in X^T, then the sequence $\{m_j\}$, $(m_1 < m_2 < \ldots)$, of positive integers exists so that $\lim_{j\to\infty} T^{m_j}(x) = x$.

Theorem 1.4.5. If $T : X \to X$ is nonexpansive (ε-nonexpansive) mapping, then each x in X^T generates an isometric (ε-isometric) sequence.

Remark. It can be easily proved that if $T : X \to X$ is a nonexpansive mapping and x is in X^T, then T has a fixed point.

In [5] Cheney and Goldstein have proved the following theorem.

Theorem 1.4.6. Let T be a map of a metric space X into itself such that

(i) $d(T(x), T(y)) \leq d(x,y)$

(ii) if $x \neq T(x)$, then $d(T(x), T^2(x)) < d(x, T(x))$

(iii) for each x, the sequence $\{T^n(x)\}$ has a cluster point.

Then for each x, the sequence $\{T^n(x)\}$ converges to a fixed point of T.

K.L. Singh [31] has proved the above theorem by relaxing conditions (ii) and (iii) in the following way.

Let T be a map of a compact metric space X into itself such that $d(T(x), T(y)) \leq d(x,y)$, equality holds when $x = y$. Then T has a fixed point.

CHAPTER II

Multi-valued Contraction Mappings

During the period 1941-45 several extensions of known fixed point theorems in which the transformation T takes each point of a compact metric space X into a closed subset of X were developed. These extensions first occurred in Von Neumann's work on the theory of games. Kakutani [16], and Wallace [34] also gave theorems in this direction. As well as their applicability to the theory of games, multi-valued mappings, also have applications in Functional Analysis. The concept of a multi-valued contraction mapping was first introduced by Nadler Jr. [21] in 1967. He combined the ideas of multi-valued mappings and Lipschitz mappings, and proved some fixed point theorems in this respect. In this chapter it is our aim to investigate some of the results which he has given and to add a few new results.

The following definition and results are due to Nadler Jr. [21].

Definition 2.1.1. If (X,d) is a metric space, then

(a) $CB(X) = \{C \mid C$ is a nonempty closed and bounded subset of $X\}$,

(b) $2^X = \{C \mid C$ is a nonempty compact subset of $X\}$,

(c) $N(\varepsilon,C) = \{x \in X \mid d(x,c) < \varepsilon$ for some $c \in C\}$ if $\varepsilon > 0$ and $C \in CB(X)$, and

(d) $H(A,B) = \inf\{\varepsilon \mid A \subset N(\varepsilon,B)$ and $B \subset N(\varepsilon,A)\}$ if $A,B \in CB(X)$.

The function H is a metric for $CB(X)$ called the Hausdorff metric.

Remark. We note that H actually depends on the metric for X. However, such dependency shall not be indicated except where confusion may arise.

Definition 2.1.2. Let (X,d_1) and (Y,d_2) be metric spaces. A function $F : X \to CB(Y)$ is said to be a multi-valued Lipschitz mapping (abbreviated m.v.l.m.) of X into Y if and only if

$$H(F(x), F(y)) \leq \alpha d_1(x,y) \quad \text{for all } x,y \in X,$$

where $\alpha \geq 0$ is a fixed real number.

The constant α is called a Lipschitz constant for F.

If F has a Lipschitz constant $\alpha < 1$, then F is called a multi-valued contraction mapping (abbreviated m.v.c.m.).

Remark. A multi-valued Lipschitz mapping is continuous.

Definition 2.1.3. Let F be a multi-valued mapping of a metric space X into a metric space Y. A point $x \in X$ is said to be a fixed point of F provided $x \in F(x)$.

Example 2.1.4. Let $I = [0,1]$ denote the unit interval of real numbers (with the usual metric) and let $f : I \to I$ be given by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & , \quad 0 \leq x \leq \frac{1}{2} \\ -\frac{1}{2}x + 1 & , \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define $F : I \to 2^I$ by $f(x) = \{0\} \cup \{f(x)\}$ for each $x \in I$. We observe that:

(1) F is a multi-valued contraction mapping,

(2) the set of fixed points of F is $\{0, 2/3\}$.

Remark 2.1.5. If $F : X \to 2^Y$ is a m.v.l.m. and $K \in 2^X$, then $\cup\{F(x) \mid x \in K\} \in 2^Y$.

We give the following results due to Nadler Jr. [21] without proof.

Theorem 2.1.6. Let $F : X \to 2^Y$ be a m.v.l.m. with Lipschitz constant α . If $A,B \in 2^X$, then

$$H(\bigcup\{F(a)|a \in A\}\ ,\ \bigcup\{F(b)|b \in B\}) \leq \alpha H(A,B).$$

Theorem 2.1.7. Let $F : X \to 2^Y$ be a m.v.l.m. with Lipschitz constant α and let $G : Y \to 2^Z$ be a m.v.l.m. with Lipschitz constant β . If $G \circ F : X \to 2^Z$ is defined by

$(G \circ F)(x) = \bigcup\{G(y)|y \in F(x)\}$ for all $x \in X$, then $G \circ F$ is a m.v.l.m. with Lipschitz constant $\alpha \cdot \beta$.

Theorem 2.1.8. Let $F : X \to 2^Y$ be a m.v.l.m. with Lipschitz constant α and $\hat{F} : 2^X \to 2^Y$ be given by

$$\hat{F}(A) = \bigcup\{F(a)|a \in A\} \quad \text{for all} \quad A \in 2^X.$$

Then $\hat{F}$ is a Lipschitz mapping with Lipschitz constant α .

We now state and prove the following main theorem due to Nadler Jr. [21].

Theorem 2.1.9. Let (X,d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued contraction mapping, then F has a fixed point.

Proof. Let $\alpha < 1$ be a Lipschitz constant for F, (we may assume $\alpha > 0$) and let $p_o \in X$. Choose $p_1 \in F(p_o)$. Since $F(p_o)$, $F(p_1) \in CB(X)$ and $p_1 \in F(p_o)$, there is a point $p_2 \in F(p_1)$ such that

$$d(p_1,p_2) \leq H(F(p_o),\ F(p_1)) + \alpha \quad .$$

Now, since

$$F(p_1),\ F(p_2) \in CB(X) \quad \text{and} \quad p_2 \in F(p_1),$$

there is a point $p_3 \in F(p_2)$ such that

$$d(p_2,p_3) \leq H(F(p_1), F(p_2)) + \alpha^2 .$$

Continuing in this fashion we produce a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X such that $p_{i+1} \in F(p_i)$ and

$$d(p_i, p_{i+1}) \leq H(F(p_{i-1}), F(p_i)) + \alpha^i \quad \text{for all } i \geq 1.$$

We note that

$$\begin{aligned} d(p_i, p_{i+1}) &\leq H(F(p_{i-1}), F(p_i)) + \alpha^i \leq \alpha d(p_{i-1}, p_i) + \alpha^i \\ &\leq \alpha[H(F(p_{i-2}), F(p_{i-1})) + \alpha^{i-1}] + \alpha^i \\ &\leq \alpha^2 d(p_{i-2}, p_{i-1}) + 2\alpha^i \\ &\leq \\ &\quad \vdots \\ &\leq \alpha^i d(p_o,p_1) + i\alpha^i \quad \text{for all } i \geq 1. \quad \text{Hence} \end{aligned}$$

$$\begin{aligned} d(p_i, p_{i+j}) &\leq d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + \cdots + d(p_{i+j-1}, p_{i+j}) \\ &\leq \alpha^i d(p_o,p_1) + i\alpha^i + \alpha^{i+1} d(p_o,p_1) + (i+1)\alpha^{i+1} + \cdots \\ &\quad + \alpha^{i+j-1} d(p_o,p_1) + (i+j-1)\alpha^{i+j-1} \\ &= \left(\sum_{n=i}^{i+j-1} \alpha^n\right) d(p_o,p_1) + \sum_{n=i}^{i+j-1} n\alpha^n \quad \text{for all } i,j \geq 1. \end{aligned}$$

It follows that the sequence $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since (X,d) is complete, the sequence $\{p_i\}_{i=1}^{\infty}$ converges to some point $x_o \in X$. Therefore, the sequence $\{F(p_i)\}_{i=1}^{\infty}$ converges to $F(x_o)$ and, since $p_i \in F(p_{i-1})$ for all i, it follows that $x_o \in F(x_o)$. That is, F has a fixed point.

We now wish to extend Theorem 1.2.8. to the multi-valued case in the following way.

Theorem 2.1.10. Let (X,d) be a complete metric space and $F_1 : X \to CB(X)$, $F_2 : X \to CB(X)$ be multi-valued mappings such that

$$H(F_1(x), F_2(y)) \leq \alpha[D(x, F_1(x)) + D(y, F_2(y))]$$

for all $x,y \in X$ where $0 \leq \alpha < 1/2$.

Then F_1 and F_2 have a common fixed point.

Proof. Let $x \in X$ and choose $x_1 \in F_1(x)$. Now since $F_1(x)$ and $F_2(x_1) \in CB(X)$ and $x_1 \in F_1(x)$, there is a point $x_2 \in F_2(x_1)$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H(F_1(x), F_2(x_1)) + \frac{\alpha}{(1-\alpha)^0} \\ &\leq \alpha[D(x, F_1(x)) + D(x_1,F_2(x_1))] + \frac{\alpha}{(1-\alpha)^0} \\ &\leq \alpha[d(x,x_1) + d(x_1,x_2)] + \frac{\alpha}{(1-\alpha)^0} \end{aligned}$$

Therefore,

$$d(x_1,x_2) \leq \frac{\alpha}{1-\alpha} d(x,x_1) + \frac{\alpha}{1-\alpha}$$

Similarly since $F_2(x_1)$ and $F_1(x_2) \in CB(X)$ and $x_2 \in F_2(x_1)$, there is a point $x_3 \in F_1(x_2)$ such that

$$\begin{aligned} d(x_2,x_3) &\leq H(F_2(x_1), F_1(x_2)) + \frac{\alpha^2}{1-\alpha} \\ &\leq \alpha[D(x_1, F_2(x_1)) + D(x_2, F_1(x_2))] + \frac{\alpha^2}{1-\alpha} \\ &\leq \alpha[d(x_1,x_2) + d(x_2,x_3)] + \frac{\alpha^2}{1-\alpha} \end{aligned}$$

Therefore,

$$d(x_2,x_3) \leq \frac{\alpha}{1-\alpha}\, d(x_1,x_2) + \left(\frac{\alpha}{1-\alpha}\right)^2$$

$$\leq \left(\frac{\alpha}{1-\alpha}\right)^2 d(x,x_1) + 2\left(\frac{\alpha}{1-\alpha}\right)^2$$

In general, we have that

$$d(x_i, x_{i+1}) \leq \left(\frac{\alpha}{1-\alpha}\right)^i d(x,x_1) + i\left(\frac{\alpha}{1-\alpha}\right)^i \quad \text{for all} \quad i \geq 1.$$

Now, we have

$$d(x_i, x_{i+j}) \leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \dots + d(x_{i+j-1}, x_{i+j}).$$

For the sake of convenience we put $\beta = \frac{\alpha}{1-\alpha}$. Hence,

$$d(x_i, x_{i+j}) \leq \beta^i d(x,x_1) + i\beta^i + \beta^{i+1} d(x,x_1) + (i+1)\beta^{i+1} + \dots +$$

$$+ \beta^{i+j-1} d(x,x_1) + (i+j-1)\beta^{i+j-1}.$$

$$= \left(\sum_{n=i}^{i+j-1} \beta^n\right) d(x,x_1) + \sum_{n=i}^{i+j-1} n\beta^n \quad \text{for all} \quad i,j \geq 1.$$

Therefore $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and, since (X,d) is complete, it must converge to some point $x_o \in X$.

We now show that $x_o \in F_1(x_o)$ and $x_o \in F_2(x_o)$ (i.e. x_o is a common fixed point of F_1 and F_2).

We have,

$$D(x_o, F_1(x_o)) \leq d(x_o,x_n) + D(x_n, F_1(x_o))$$

$$\leq d(x_o,x_n) + H(F_2(x_{n-1}), F_1(x_o))$$

where we have taken n to be an even positive integer.

Hence,

$$D(x_o, F_1(x_o)) \leq d(x_o,x_n) + \alpha[D(x_{n-1}, F_2(x_{n-1})) + D(x_o, F_1(x_o))]$$

$$\leq d(x_o,x_n) + \alpha[d(x_{n-1}, x_n) + D(x_o, F_1(x_o))].$$

So, $$D(x_o, F_1(x_o)) \leq \frac{d(x_o,x_n) + \alpha d(x_{n-1},x_n)}{1 - \alpha}$$

For sufficiently large n and $F_1(x_o)$ being closed we have that $x_o \in F_1(x_o)$.

In a similar way it can be shown that $x_o \in F_2(x_o)$. So, in fact, F_1 and F_2 have a common fixed point.

If we suppose that F_1 is identical with F_2 then we obtain the following theorem due to Dube and Singh [10].

Theorem 2.1.11. Let (X,d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued mapping such that

$$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y))] \quad \text{for all } x,y \in X$$

and $0 \leq \alpha < 1/2$,

then F has a fixed point.

We also have the following theorem.

Theorem 2.1.12. Let (X,d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued mapping such that

$$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y)) + d(x,y)] \quad \text{for all}$$

$x,y \in X$ and $0 \leq \alpha < 1/3$,

then F has a fixed point.

Proof. Let $x_o \in X$ and choose $x_1 \in F(x_o)$. Since $F(x_o), F(x_1) \in CB(X)$ and $x_1 \in F(x_o)$, there is a point $x_2 \in F(x_1)$ such that

$$d(x_1,x_2) \leq H(F(x_o),F(x_1)) + 2\alpha$$

We may assume $\alpha > 0$, since if $\alpha = 0$ we immediately have a fixed point of F.

Now since $x_2 \in F(x_1)$ and $F(x_1), F(x_2) \in CB(X)$, there is a point $x_3 \in F(x_2)$ such that

$$d(x_2,x_3) \leq H(F(x_1), F(x_2)) + (2\alpha)^2/(1 - \alpha).$$

Continuing in this manner we produce a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X such that $x_{i+1} \in F(x_i)$ and

$$d(x_i,x_{i+1}) \leq H(F(x_{i-1}), F(x_i)) + \frac{(2\alpha)^i}{(1 - \alpha)^{i-1}} \quad \text{for all } i \geq 1.$$

Now

$$d(x_i,x_{i+1}) \leq H(F(x_{i-1}, F(x_i)) + \frac{(2\alpha)^i}{(1 - \alpha)^{i-1}}$$

$$\leq \alpha[D(x_{i-1}, F(x_{i-1})) + D(x_i, F(x_i)) + d(x_i,x_{i-1})] + \frac{(2\alpha)^i}{(1 - \alpha)^{i-1}}$$

$$\leq \alpha[d(x_{i-1},x_i) + d(x_i,x_{i+1}) + d(x_i,x_{i-1})] + \frac{(2\alpha)^i}{(1 - \alpha)^{i-1}}$$

That is,

$$d(x_i,x_{i+1}) \leq \frac{2\alpha}{1 - \alpha} d(x_{i-1},x_i) + \left(\frac{2\alpha}{1 - \alpha}\right)^i$$

For the sake of convenience we put $\beta = \frac{2\alpha}{1 - \alpha}$

So,

$$\begin{aligned} d(x_i,x_{i+1}) &\leq \beta d(x_i,x_{i-1}) + \beta^i \\ &\leq \beta[\beta d(x_{i-2},x_{i-1}) + \beta^{i-1}] + \beta^i \\ &= \beta^2 d(x_{i-2},x_{i-1}) + 2\beta^i \\ &\leq \\ &\vdots \\ &\leq \beta^i d(x_o,x_1) + i\beta^i \qquad \text{for all } i \geq 1. \end{aligned}$$

Hence

$$d(x_i,x_{i+j}) \leq d(x_i,x_{i+1}) + d(x_{i+1},x_{i+2}) + \ldots + d(x_{i+j-1},x_{i+j})$$

$$\leq \beta^i d(x_o,x_1) + i\beta^i + \beta^{i+1} d(x_o,x_1) + (i+1)\beta^{i+1} + \ldots$$

$$+ \beta^{i+j-1} d(x_o,x_1) + (i+j-1)\beta^{i+j-1}$$

So,

$$d(x_i,x_{i+j}) \leq \left(\sum_{p=i}^{i+j-1} \beta^p\right) d(x_o,x_1) + \sum_{p=i}^{i+j-1} p\beta^p \quad \text{for all } i,j \geq 1.$$

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since (X,d) is complete, the sequence $\{x_i\}_{i=1}^{\infty}$ converges to some point $y \in X$.

It remains to show that y is a fixed point of F.

We consider,

$$D(y, F(y)) \leq D(y, x_n) + D(x_n, F(y))$$

$$\leq d(y, x_n) + H(F(x_{n-1}), F(y))$$

$$\leq d(y, x_n) + \alpha[D(x_{n-1}, F(x_{n-1})) + D(y, F(y)) + d(x_{n-1},y)]$$

$$\leq d(y, x_n) + \alpha[d(x_{n-1}, x_n) + D(y, F(y)) + d(x_{n-1}, y)]$$

That is,

$$D(y, F(y)) \leq \frac{d(y, x_n) + \alpha d(x_{n-1}, x_n) + \alpha d(x_{n-1},y)}{1 - \alpha} \to 0$$

as $n \to \infty$. Since $F(x_o)$ is closed this means that $x_o \in F(x_o)$ as required.

The following are also due to Nadler Jr. [21].

Definition 2.1.13. Let (X,d) be a metric space. A function $F : X \to CB(X)$ is said to be an (ε,λ)-uniformly locally contractive multi-valued mapping (where $\varepsilon > 0$ and $0 \leq \lambda < 1$) provided that, if $x,y \in X$ and $d(x,y) < \varepsilon$, then

$$H(F(x), F(y)) \leq \lambda d(x,y).$$

Theorem 2.1.14. Let (X,d) be a complete ε-chainable metric space. If $F : X \to 2^X$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then F has a fixed point.

Proof. If $(x,y) \in X \times X$, then let $d_\varepsilon(x,y) = \inf\{\sum_{i=1}^{n} d(x_{i-1},x_i) \mid x_o = x, x_1, \ldots, x_n = y$ is an ε-chain from x to $y\}$. Now d_ε is a metric for X satisfying

(1) $d(x,y) \leq d_\varepsilon(x,y)$ for all $x,y \in X$.

and (2) $d(x,y) = d_\varepsilon(x,y)$ for all $x,y \in X$ such that $d(x,y) < \varepsilon$.

From (1) and (2) and the completeness of (X,d) it follows that (X,d_ε) is complete. Let H_ε be the Hausdorff metric for 2^X obtained from d_ε. Note that if $A,B \in 2^X$ and $H(A,B) < \varepsilon$, then $H_\varepsilon(A,B) = H(A,B)$. We now show that $F : X \to 2^X$ is a multi-valued contraction mapping with respect to d_ε and H_ε. Let $x,y \in X$ and let $x_o = x, x_1, \ldots, x_n = y$ be an ε-chain from x to y. Since $d(x_{i-1},x_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$,

$$H(F(x_{i-1}), F(x_i)) \leq \lambda d(x_{i-1},x_i) < \varepsilon \quad \text{for all } i = 1, 2, \ldots, n.$$

Therefore,

$$\begin{aligned} H_\varepsilon(F(x), F(y)) &\leq \sum_{i=1}^{n} H_\varepsilon(F(x_{i-1}), F(x_i)) \\ &= \sum_{i=1}^{n} H(F(x_{i-1}), F(x_i)) \leq \lambda\sum_{i=1}^{n} d(x_{i-1}, x_i), \end{aligned}$$

i.e. $H_\varepsilon(F(x), F(y)) \leq \lambda\sum_{i=1}^{n} d(x_{i-1},x_i)$. Since $x_o = x, x_1, \ldots, x_n = y$

was an arbitrary ε-chain from x to y, it follows that

$$H_\varepsilon(F(x), F(y)) \leq \lambda d_\varepsilon(x,y).$$

This proves that F is a m.v.c.m. with respect to d_ε and H_ε.

By Theorem 2.1.9., F has a fixed point.

Theorem 2.1.15. If (X,d) is a complete convex (in the sense of Menger) metric space and $F : X \to CB(X)$ is (ε,λ)-uniformly locally contractive, then F is actually a m.v.c.m.

Proof. A metric space (X,d) is said to be convex (in the sense of Menger) provided that, if $x,y \in X$, $x \neq y$, then there exists a point $Z \in X$, $Z \neq x$, $Z \neq y$, such that $d(x,y) = d(x,Z) + d(Z,y)$.
A Theorem by Menger ([3], p. 41) states that a convex and complete metric space contains together with x and y also a metric segment whose extremities are x and y; that is a subset isometric to an interval of length $d(x,y)$. Using this fact we see that if $x,y \in X$ then there are points $x_o = x, x_1, \ldots, x_n = y$ such that $d(x,y) = \sum_{i=1}^{n} d(x_{i-1},x_i)$ and $d(x_{i-1},x_i) < \varepsilon$.

Hence
$$\begin{aligned} H(F(x), F(y)) &\leq \sum_{i=1}^{n} H(F(x_{i-1}), F(x_i)) \\ &\leq \lambda \sum_{i=1}^{n} d(x_{i-1},x_i) \\ &= \lambda d(x,y) \end{aligned}$$

i.e. $H(F(x), F(y)) \leq \lambda d(x,y)$ for all $x,y \in X$ and $0 \leq \lambda < 1$. Hence F is a multi-valued contraction mapping.

As an immediate consequence of the above theorem we have:

Corollary 2.1.16. Let (X,d) be a complete convex (in the sense of

Menger) metric space. If $F : X \to CB(X)$ is an (ε,λ) -uniformly locally contractive multi-valued mapping, then F has a fixed point.

We now give a theorem for a single-valued mapping which is similar to Theorem 1.2.8.

Theorem 2.2.1. If T_1 and T_2 are two mappings of a complete metric space (X,d) into itself and if

$$d(T_1(x),\ T_2(y)) \le \alpha[d(x,\ T_1(x)) + d(y,\ T_2(y)) + d(x,y)]$$

where $x,y \in X$ and $0 \le \alpha < 1/3$,
then T_1 and T_2 have a unique common fixed point.

Proof. We define a sequence of elements $\{x_n\}$ of X as follows:
Let x be any element of X. Let $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$ and so on.
Then,

$$\begin{aligned} d(x_1,x_2) = d(T_1(x),\ T_2(x_1)) &\le \alpha[d(x,\ T_1(x)) + d(x_1,\ T_2(x_1)) + d(x,x_1)] \\ &\le \alpha[d(x,x_1) + d(x_1,x_2) + d(x,x_1)] \end{aligned}$$

Hence,

$$d(x_1,x_2) \le \frac{2\alpha}{1-\alpha}\ d(x,x_1)$$

Similarly,

$$\begin{aligned} d(x_2,x_3) = d(T_2(x_1),\ T_1(x_2)) &\le \alpha[d(x_1,\ T_2(x_1)) + d(x_2,\ T_1(x_2)) \\ &\qquad + d(x_1,x_2)] \\ &\le \alpha[d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_2)] \end{aligned}$$

That is,

$$\begin{aligned} d(x_2,x_3) &\le \frac{2\alpha}{1-\alpha}\, d(x_1,x_2) \\ &\le \left(\frac{2\alpha}{1-\alpha}\right)^2 d(x,x_1) \end{aligned}$$

In general

$$d(x_n,x_{n+1}) \leq \left(\frac{2\alpha}{1-\alpha}\right)^n d(x,x_1)$$

Therefore,

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})$$
$$\leq [r^n + r^{n+1} + \ldots + r^{n+p-1}]d(x,x_1)$$

where $r = \dfrac{2\alpha}{1-\alpha}$

So $\quad d(x_n, x_{n+p}) \leq \dfrac{r^n}{1-r} d(x,x_1) \to 0 \quad$ as $\quad n \to \infty$.

Therefore the sequence $\{x_n\}$ is a Cauchy sequence. Since the space X is complete, there is a point $x_o \in X$ which is the limit of this sequence, i.e. $x_o = \lim\limits_{n\to\infty} x_n$. We now show $T_1(x_o) = x_o = T_2(x_o)$.

Now $\quad d(x_o, T_1(x_o)) \leq d(x_o, x_n) + d(x_n, T(x_o))$

$$= d(x_o, x_n) + d(T_2(x_{n-1}), T_1(x_o))$$

where we have taken n to be even positive integer.

$$\therefore d(x_o, T_1(x_o)) \leq d(x_o,x_n) + \alpha[d(x_{n-1}, T_2(x_{n-1})) + d(x_o, T_1(x_o)) + d(x_{n-1}, x_o)]$$

$$\leq d(x_o,x_n) + \alpha[d(x_{n-1}, x_n) + d(x_o, T_1(x_o)) + d(x_{n-1}, x_o)]$$

So $d(x_o, T_1(x_o))[1-\alpha] \leq d(x_o,x_n) + \alpha d(x_{n-1}, x_n) + \alpha d(x_{n-1}, x_o)$.

If $\varepsilon > 0$ is arbitrary then for all sufficiently large values of n we have $d(x_o, x_n) < \varepsilon \cdot \dfrac{1-\alpha}{1+2\alpha}$, $d(x_{n-1}, x_n) < \varepsilon \cdot \dfrac{1-\alpha}{1+2\alpha}$, and

$$d(x_{n-1}, x_o) < \varepsilon \cdot \frac{1-\alpha}{1+2\alpha} .$$

Hence $\quad d(x_o, T_1(x_o)) \leq \dfrac{d(x_o,x_n) + \alpha d(x_{n-1}, x_n) + \alpha d(x_{n-1}, x_o)}{1-\alpha}$

$$< \frac{\varepsilon}{1+2\alpha} + \frac{\alpha\varepsilon}{1+2\alpha} + \frac{\alpha\varepsilon}{1+2\alpha} = \varepsilon$$

$\therefore \quad x_o = T_1(x_o)$ because $\varepsilon > 0$ is arbitrary.

In a similar way it can be shown that $x_o = T_2(x_o)$.

$\therefore \quad T_1(x_o) = T_2(x_o) = x_o$.

It remains to show that x_o is the unique common fixed point of T_1 and T_2.

Let y_o be a point of X such that $y_o = T_1(y_o) = T_2(y_o)$.

Then

$$d(x_o,y_o) = d(T_1(x_o),\ T_2(y_o)) \leq \alpha[d(x_o,\ T_1(x_o)) + d(y_o,\ T_2(y_o)) + d(x_o,y_o)]$$

i.e. $$d(x_o,y_o) \leq \frac{\alpha[d(x_o,\ T_1(x_o)) + d(y_o,\ T_2(y_o))]}{1-\alpha} = 0$$

$\therefore \quad x_o = y_o$ and this proves the theorem.

Remark.

We observe that under the conditions of the theorem T_1 and T_2 have only one fixed point namely x_o.

For if z_o is any point of X with $T_1(z_o) = z_o$ then

$$d(z_o,x_o) = d(T_1(z_o),\ T_2(x_o)) \leq \alpha[d(z_o,\ T_1(z_o)) + d(x_o,\ T_2(x_o)) + d(z_o,x_o)]$$

i.e. $$d(z_o,x_o) \leq \frac{\alpha[d(z_o,\ T_1(z_o)) + d(x_o,\ T_2(x_o))]}{1-\alpha} = 0$$

$\therefore \quad z_o = x_o$. Similarly for T_2.

If we assume that T_1 is identical with T_2 then we obtain the following theorem.

Theorem 2.2.2. If T is a mapping of a complete metric space (X,d) into itself and if

$$d(T(x), T(y)) \leq \alpha [d(x, T(x)) + d(y, T(y)) + d(x,y)] \quad \text{where} \quad x,y \in X$$

and $0 \leq \alpha < 1/3$,

then T has a unique fixed point in X.

The requirement that the space be complete may be relaxed in the following way.

<u>Theorem</u> 2.2.3. Let X be a metric space. Let T be a map of X into itself such that

(i) $d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y)) + d(x,y)]$, $0 \leq \alpha < 1/3$, $x,y \in X$.

(ii) T is continuous at a point $z \in X$.

(iii) There exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to z.

Then z is a unique fixed point of T.

<u>Definition</u> 2.2.4.. Denote by $\mathfrak{F}$ the family of functions $\alpha(x,y)$ satisfying the following conditions.

(1) $\alpha(x,y) = \alpha(d(x,y))$ i.e. α depends on the distance between x and y only.

(2) $0 \leq \alpha(d) < 1/2$ for every $d > 0$

(3) $\alpha(d)$ is a monotonically increasing function of d.

<u>Theorem</u> 2.2.5. Let (X,d) be a complete metric space and let T be a map of X into itself such that

$$d(T(x), T(y)) \leq \alpha(x,y)[d(x, T(x)) + d(y, T(y))] \quad \text{for every}$$

$x,y \in X$ where $\alpha(x,y) \in F$, then there exists a unique fixed point of T in X.

Proof. Let x_o be any point in X and let

$$x_1 = T(x_o)$$

$$x_2 = T(x_1) = T^2(x_o)$$

$$\cdots\cdots\cdots\cdots$$

$$x_n = T(x_{n-1}) = T^n(x_o)$$

We show that for $x_o \neq x_1$ we have

$$d(x_o,x_1) > d(x_1,x_2) > d(x_2,x_3) > \ldots > d(x_{n-1}, x_n) > \ldots$$

Now,

$$d(x_1,x_2) = d(T(x_o), T(x_1)) \leq \alpha(x_o,x_1)[d(x_o, T(x_o)) + d(x_1, T(x_1))]$$

$$= \alpha(x_o,x_1)[d(x_o,x_1) + d(x_1,x_2)]$$

Hence

$$d(x_1,x_2) \leq \frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)} d(x_o,x_1) < d(x_o,x_1)$$

Similarly,

$$d(x_2,x_3) = d(T(x_1), T(x_2)) \leq \alpha(x_1,x_2)[d(x_1, T(x_1)) + d(x_2, T(x_2))]$$

$$= \alpha(x_1,x_2)[d(x_1,x_2) + d(x_2,x_3)]$$

i.e. $$d(x_2,x_3) \leq \frac{\alpha(x_1,x_2)}{1 - \alpha(x_1,x_2)} d(x_1,x_2)$$

$$< d(x_1,x_2) < d(x_o,x_1)$$

Generally,

$$d(x_n, x_{n-1}) = d(T(x_{n-1}), T(x_{n-2})) \leq \alpha(x_{n-1}, x_{n-2})[d(x_{n-1}, T(x_{n-1})) + d(x_{n-2}, T(x_{n-2}))]$$

$$= \alpha(x_{n-1},x_{n-2})[d(x_{n-1},x_n) + d(x_{n-2},x_{n-1})]$$

i.e. $$d(x_n, x_{n-1}) \leq \frac{\alpha(x_{n-1}, x_{n-2})}{1 - \alpha(x_{n-1},x_{n-2})} d(x_{n-1}, x_{n-2})$$

$$< d(x_{n-1}, x_{n-2})$$

So,

$$d(x_o,x_1) > d(x_1,x_2) > \ldots > d(x_{n-1}, x_n) > \ldots$$

Therefore, by monotonically increasing property of $\alpha(x,y)$ we have

$$\alpha(x_o,x_1) > \alpha(x_1,x_2) > \ldots > \alpha(x_{n-1}, x_n) > \ldots$$

We now show that $\{x_n\}$ is a Cauchy sequence.

$$d(x_1,x_2) \leq \frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)} d(x_o,x_1)$$

Also $$d(x_2,x_3) \leq \frac{\alpha(x_1,x_2)}{1 - \alpha(x_1,x_2)} d(x_1,x_2)$$

$$\leq \left(\frac{\alpha(x_1,x_2)}{1 - \alpha(x_1,x_2)}\right)\left(\frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)}\right) d(x_o,x_1)$$

but $$\frac{\alpha(x_1,x_2)}{1 - \alpha(x_1,x_2)} < \frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)}$$

Hence $$d(x_2,x_3) < \left(\frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)}\right)^2 d(x_o,x_1)$$

In general, we have

$$d(x_{n+1}, x_n) < \left(\frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)}\right)^n d(x_o,x_1)$$

Now since $0 \leq \frac{\alpha(x_o,x_1)}{1 - \alpha(x_o,x_1)} < 1$, the sequence $\{x_n\}$ is a Cauchy sequence and since (X,d) is complete, it must converge to a point $z \in X$. It remains to show that z is a fixed point of T. We have that

$$d(z, T(z)) \leq d(z, x_n) + d(x_n, T(z))$$
$$= d(z, x_n) + d(T(x_{n-1}), T(z))$$
$$\leq d(z, x_n) + \alpha(x_{n-1}, z)[d(x_{n-1}, T(x_{n-1})) + d(z, T(z))]$$
$$\leq d(z, x_n) + \alpha(x_{n-1}, z)[d(x_{n-1}, x_n) + d(z, T(z))]$$

i.e. $$d(z, T(z)) \leq \frac{d(z, x_n) + \alpha(x_{n-1}, z)d(x_{n-1}, x_n)}{1 - \alpha} \to 0$$

as $n \to \infty$.

Therefore $T(z) = z$ as required.

Let z and y $(z \neq y)$ be two distinct fixed points of T. Then

$$d(z,y) = d(T(z), T(y)) \leq \alpha(z,y)[d(z, T(z)) + d(y, T(y))] = 0$$

That is, $d(z,y) = 0$ which implies $z = y$ and uniqueness is established.

If we replace condition (2) in Definition 2.2.4. by

(2*) $0 \leq \alpha(d) < 1/3$, then the following theorem holds:

Theorem 2.2.6. Let X be a complete metric space and T be a map of X into itself such that

$d(T(x), T(y)) \leq \alpha(x,y)[d(x, T(x)) + d(y, T(y)) + d(x,y)]$ for every $x,y \in X$ and $\alpha(x,y)$ satisfies conditions (1) and (3) of Definition 2.2.4 and (2*). Then T has a unique fixed point in X.

For the remainder of this discussion we shall consider a more general setting than a metric space, namely that of a Generalized Metric Space.

Definition 2.3.1. Let X be an abstract set with elements $x,y, \ldots,$ and let $d(x,y)$ be a distance function $(0 \leq d(x,y) \leq \infty)$ defined on $X \times X$ satisfying the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Then X with metric d is called a generalized metric space.

<u>Definition</u> 2.3.2. A generalized metric space (X,d) is said to be complete if and only if every d-Cauchy sequence in X (i.e. $\{x_n\}_{n=1}^{\infty}$ is a d-Cauchy sequence in X if and only if $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$) is d-convergent to a point in X.

We present the following definitions and results due to Covitz and Nadler Jr. [8].

<u>Definition</u> 2.3.3. If (X,d) is a generalized metric space, then

(a) $CL(X) = \{C | C$ is a nonempty closed subset of $X\}$,

(b) $N(\varepsilon,C) = \{x \in X | d(x,c) < \varepsilon$ for some $c \in C\}$ if $\varepsilon > 0$ and $C \in CL(X)$, and

(c) $$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 | A \subset N(\varepsilon,B) \text{ and } B \subset N(\varepsilon,A)\} & \text{if the infimum exists.} \\ \infty \quad , & \text{otherwise,} \end{cases}$$

if $A,B \in CL(X)$.

The pair $(CL(X),H)$ is a generalized metric space and H is called the generalized Hausdorff distance induced by d.

<u>Definition</u> 2.3.4. A function $F : X \to CL(X)$ is called a multi-valued contraction mapping if and only if there exists a fixed real number $\lambda < 1$ such that

$$H(F(x), F(y)) \leq \lambda d(x,y) \quad \text{for all } x,y \in X \text{ such that}$$

$d(x,y) < \infty$.

Definition 2.3.5. A function $F : X \to CL(X)$ is called an (ε,λ)-uniformly locally contractive multi-valued mapping (where $\varepsilon > 0$ and $0 \leq \lambda < 1$) if and only if

$$H(F(x), F(y)) \leq \lambda d(x,y) \quad \text{for all } x,y \in X \text{ such that}$$

$d(x,y) < \varepsilon$.

Definition 2.3.6. Let (X,d) be a generalized metric space, let $x_o \in X$, and let $F : X \to CL(X)$ be a function. A sequence $\{ x_i\}_{i=1}^{\infty}$ of points of X is said to be an iterative sequence of F at x_o if and only if $x_i \in F(x_{i-1})$ for each $i = 1, 2, \ldots$.

We now give a main theorem of Covitz and Nadler Jr. [8] and subsequent corollaries.

Theorem 2.3.7. Let (X,d) be a generalized complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds : either

(a) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o , $d(x_{i-1}, x_i) \geq \varepsilon$ for each $i = 1, 2, \ldots$, or

(b) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Proof. Suppose (a) does not hold. Then there is a choice of $x_1 \in F(x_o)$, $x_2 \in F(x_1)$, $\ldots$, $x_N \in F(x_{N-1})$ such that $d(x_N, x_{N-1}) < \varepsilon$ for some fixed integer $N \geq 1$. This implies

$$H(F(x_{N-1}),F(x_N)) \leq \lambda d(x_{N-1}, x_N) < \lambda \cdot \varepsilon .$$

Therefore, since $x_N \in F(x_{N-1})$, there exists $x_{N+1} \in F(x_N)$ such that $d(x_N, x_{N+1}) < \cdot \varepsilon(< \varepsilon)$. Now

$H(F(x_N), F(x_{N+1})) \leq \lambda d(x_N, x_{N+1}) < \lambda^2 \cdot \varepsilon$ and hence, since $x_{N+1} \in F(x_N)$, there exists $x_{N+2} \in F(x_{N+1})$ such that $d(x_{N+1}, x_{N+2}) < \lambda^2 \cdot \varepsilon$. Continuing in this fashion we produce a sequence $\{x_{N+i}\}_{i=1}^{\infty}$ of points of X such that $x_{N+i+1} \in F(x_{N+i})$ and $d(x_{N+i+1}, x_{N+i}) < \lambda^{i+1} \cdot \varepsilon$ for all $i \geq 1$.

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which, by the completeness of (X,d), converges to some point $p \in X$. Hence, the sequence $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(p)$ and, since $x_{i+1} \in F(x_i)$ for all i and $F(p)$ is closed, $p \in F(p)$. This proves F has a fixed point. Furthermore, the sequence $\{x_i\}_{i=1}^{\infty}$ satisfies the conditions in (b) of the alternative.

<u>Corollary</u> 2.3.8. Let (X,d) be a generalized complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is a m.v.c.m., then the following alternative holds: either

(1) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_{i-1},x_i) = \infty$ for each $i = 1, 2, \ldots,$

or

(2) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Proof. Suppose (1) does not hold. Then there is an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $d(x_{N-1},x_N) < \infty$ for some fixed integer $N \geq 1$. Let $\varepsilon < \infty$ be given such that $d(x_{N-1},x_N) < \varepsilon$. Clearly F is an (ε,λ)-uniformly locally contractive multi-valued mapping which, since (a) of Theorem 2.3.7. is violated by the iterative sequence above, must satisfy (b) of Theorem 2.3.7. But this is (2) of alternative in this corollary.

Definition 2.3.9. A generalized metric space (X,d) is said to be ε-chainable (where $\varepsilon > 0$ is a fixed real number) if and only if given $x,y \in X$ such that $d(x,y) < \infty$ there is an ε-chain from x to y (that is, a finite set of points $z_o = x$, z_1, ..., $z_n = y$ such that $d(z_{i-1}, z_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$).

Corollary 2.3.10. Let (X,d) be a complete ε-chainable generalized metric space and let $x_o \in X$. If $F : X \to CL(X)$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either

(1) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_{i-1}, x_i) = \infty$ for each $i = 1, 2, \ldots$; or

(2) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Proof. We define a new generalized metric $d_\varepsilon : X \times X \to [0,\infty]$ by $d_\varepsilon(x,y) = \inf \{\sum_{i=1}^{n} d(z_{i-1}, z_i) | z_o = x, z_1, \ldots, z_n = y$ is an ε-chain from x to $y\}$ if $d(x,y) < \infty$ and $d_\varepsilon(x,y) = \infty$ if $d(x,y) = \infty$. It is easy to verify that (X,d_ε) is a generalized complete metric space. Let H_ε be the generalized Hausdorff metric on $CL(X)$ obtained from d_ε (note that, since $d(x,y) < \varepsilon$ implies $d_\varepsilon(x,y) = d(x,y)$, $CL(X)$ with respect to d is the same set as $CL(X)$ with respect to d_ε). We now show that F is a m.v.c.m. with respect to d_ε and H_ε. First note that if $A,B \in CL(X)$ and $H(A,B) < \varepsilon$, then $H_\varepsilon(A,B) = H(A,B)$ (where H is the generalized Hausdorff metric obtained from d). Now let $x,y \in X$ such that $d(x,y) < \infty$. Let $z_o = x$, z_1, ..., $z_n = y$ be an ε-chain from x to y. Then

$$H_\varepsilon(F(x), F(y)) \le \sum_{i=1}^{n} H_\varepsilon(F(z_{i-1}), F(z_i))$$
$$= \sum_{i=1}^{n} H(F(z_{i-1}), F(z_i))$$
$$\le \sum_{i=1}^{n} \lambda d(z_{i-1}, z_i) = \lambda \cdot \sum_{i=1}^{n} d(z_{i-1}, z_i),$$

i.e. $$H_\varepsilon(F(x), F(y)) \le \lambda \sum_{i=1}^{n} d(z_{i-1}, z_i).$$

Since $z_o = x, z_1, \ldots, z_n = y$ was an arbitrary ε-chain from x to y, it follows that

$$H_\varepsilon(F(x), F(y)) \le \lambda \cdot d_\varepsilon(x,y).$$

This proves F is a multi-valued contraction mapping with respect to d_ε and H_ε. Now, since d_ε is equivalent to d, Corollary 2.3.8. may be applied to complete the proof.

The next two corollaries by Covitz and Nadler Jr. [8] follow immediately from Corollary 2.3.8. and Corollary 2.3.10. respectively.

<u>Corollary</u> 2.3.11. Let (X,d) be a complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is a multi-valued contraction mapping, then there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

<u>Corollary</u> 2.3.12. Let (X,d) be a complete ε-chainable metric space and let $x_o \in X$. If $F : X \to CL(X)$ is an (ε,λ)-uniformly locally contractive multi-valued mapping, then there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

We now give an extension of Theorem 2.1.11. in the following way.

<u>Theorem</u> 2.3.13. Let (X,d) be a generalized complete metric space and let $x_o \in X$. Let $F : X \to CL(X)$ be a multi-valued mapping such that for all $x,y \in X$ with $d(x,y) < \varepsilon$ we have

$$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y))] \quad \text{where} \quad 0 \leq \alpha < 1/2,$$

then one of the following alternatives holds: either

(a) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_{i-1}, x_i) \geq \varepsilon$ for each $i = 1, 2, \ldots,$ or

(b) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Proof. Suppose (a) fails to hold. Then there is a choice of $x_1 \in F(x_o)$, $x_2 \in F(x_1)$, $\ldots,$ and $x_N \in F(x_{N-1})$ such that $d(x_{N-1}, x_N) < \varepsilon$ for some fixed integer $N \geq 1$. This implies

$$\begin{aligned} H(F(x_{N-1}), F(x_N)) &\leq \alpha[D(x_{N-1}, F(x_{N-1})) + D(x_N, F(x_N))] \\ &\leq \alpha[d(x_{N-1}, x_N) + d(x_N, x_{N+1})] \end{aligned}$$

Now since $x_N \in F(x_{N-1})$ and $x_{N+1} \in F(x_N)$ we have

$$d(x_N,x_{N+1}) \leq \alpha[d(x_{N-1}, x_N) + d(x_N, x_{N+1})]$$

Hence,

$$d(x_N, x_{N+1}) \leq \frac{\alpha}{1 - \alpha} d(x_{N-1}, x_N) < \frac{\alpha}{1 - \alpha} \cdot \varepsilon(< \varepsilon).$$

In the same way we have

$$\begin{aligned} H(F(x_N), F(x_{N+1})) &\leq \alpha[D(x_N, F(x_N)) + D(x_{N+1}, F(x_{N+1}))] \\ &\leq \alpha[d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2})] \end{aligned}$$

and by virtue of $x_{N+1} \in F(x_N)$ and $x_{N+2} \in F(x_{N+1})$ we have

$$d(x_{N+1}, x_{N+2}) \leq \alpha[d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2})]$$

Thus,

$$d(x_{N+1}, x_{N+2}) \leq \frac{\alpha}{1-\alpha} d(x_N, x_{N+1})$$
$$\leq \left(\frac{\alpha}{1-\alpha}\right)^2 d(x_{N-1}, x_N)$$
$$< \left(\frac{\alpha}{1-\alpha}\right)^2 \cdot \varepsilon (< \varepsilon).$$

Continuing in this way, we produce a sequence $\{x_{N+i}\}_{i=1}^{\infty}$ of points of X such that x_{N+i+1} $F(x_{N+i})$ and

$$d(x_{N+i+1}, x_{N+i}) < \left(\frac{\alpha}{1-\alpha}\right)^{i+1} \cdot \varepsilon \quad \text{for all} \quad i \geq 1.$$

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which, by the completeness of (X,d), converges to some point $p \in X$.

It now remains to show that p is a fixed point of F, i.e. $p \in F(p)$.

Consider,

$$D(p, F(p)) \leq d(p, x_{n+1}) + D(x_{n+1}, F(p))$$
$$\leq d(p, x_{n+1}) + H(F(x_n), F(p))$$
$$\leq d(p, x_{n+1}) + \alpha[D(x_n, F(x_n)) + D(p, F(p))]$$
$$\leq d(p, x_{n+1}) + \alpha[d(x_n, x_{n+1}) + d(p, F(p))]$$

Thus

$$D(p, F(p)) \leq \frac{d(p, x_{n+1}) + \alpha d(x_n, x_{n+1})}{1-\alpha} \to 0$$

as $n \to \infty$.

Since $F(p)$ is closed it follows that $p \in F(p)$ as required. Furthermore, the sequence $\{x_i\}_{i=1}^{\infty}$ satisfies condition (b) of the alternative.

Immediately we have:

Theorem 2.3.14. Let (X,d) be a generalized complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is a multi-valued mapping such that

$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y)) + d(x,y)]$ for all $x,y \in X$ with $d(x,y) < \varepsilon$ and $0 \leq \alpha < 1/3$, then the following alternative holds: either

(1) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_{i-1}, x_i) \geq \varepsilon$ for each $i = 1, 2, \ldots$, or

(2) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

The following corollaries follow from Theorem 2.3.13.

Corollary 2.3.15. Let (X,d) be a generalized complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is a multi-valued mapping such that for all $x,y \in X$ and $d(x,y) < \infty$,

$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y))]$ where $0 \leq \alpha < 1/2$, then the following alternative holds : either

(a) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_{i-1}, x_i) = \infty$ for each $i = 1, 2, \ldots$, or

(b) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Corollary 2.3.16. Let (X,d) be a complete metric space and let $x_o \in X$. If $F : X \to CL(X)$ is a multi-valued mapping satisfying

$H(F(x), F(y)) \leq \alpha[D(x, F(x)) + D(y, F(y))]$ for all $x,y \in X$ and $0 \leq \alpha < 1/2$,

then there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Remark. Corollaries analogous to the above two corollaries hold for Theorem 2.3.14.

Definition 2.3.17. Let $\mathcal{F}$ be as defined in Definition 2.2.4. satisfying (1), (2)', and (3) where (2)' is the following condition:

$$0 \leq \alpha(d) < 1 \quad \text{for every} \quad d > 0.$$

Then we have the following result.

Theorem 2.3.18. Let (X,d) be a generalized complete metric space and let $x_o \in X$. Let $F : X \to CL(X)$ be a multi-valued mapping satisfying:

$$H(F(x), F(y)) \leq \alpha(x,y)d(x,y) \quad \text{for all} \quad x,y \in X \quad \text{where}$$

$\alpha(x,y) \in \mathcal{F}$ and $d(x,y) < \varepsilon$.

Then the following alternative holds:

(1) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o, $d(x_i, x_{i-1}) \geq \varepsilon$ for each $i = 1, 2, \ldots,$ or

(2) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_o such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of F.

Proof. Suppose (1) does not hold. Then there is a choice of $x_1 \in F(x_o)$, $x_2 \in F(x_1), \ldots,$ and $x_N \in F(x_{N-1})$ such that $d(x_{N-1}, x_N) < \varepsilon$ for some fixed integer $N \geq 1$.

We show that

$$d(x_N, x_{N-1}) > d(x_N, x_{N+1}) > \ldots > d(x_{N+i}, x_{N+i+1}) > \ldots$$

since $d(x_{N-1}, x_N) < \varepsilon$ we have,

$$H(F(x_{N-1}), F(x_N)) \leq \alpha(x_{N-1}, x_N)d(x_{N-1}, x_N)$$

but since $x_N \in F(x_{N-1})$, there exists $x_{N+1} \in F(x_N)$ such that

$$d(x_N, x_{N+1}) \leq \alpha(x_{N-1}, x_N)d(x_{N-1}, x_N)$$

$$< \alpha(x_{N-1}, x_N) \cdot \varepsilon \ (<\varepsilon)$$

and $\quad d(x_N, x_{N+1}) < d(x_{N-1}, x_N)$.

Again, since $d(x_N, x_{N+1}) < \varepsilon$

$$H(F(x_N), F(x_{N+1})) \leq \alpha(x_N, x_{N+1})d(x_N, x_{N+1})$$

and since $x_{N+1} \in F(x_N)$, there exists $x_{N+2} \in F(x_{N+1})$ such that

$$d(x_{N+1}, x_{N+2}) \leq \alpha(x_N, x_{N+1})d(x_N, x_{N+1})$$

$$< \alpha(x_N, x_{N+1}) \cdot \varepsilon \ (<\varepsilon)$$

and $\quad d(x_{N+1}, x_{N+2}) < d(x_N, x_{N+1})$.

Continuing in this way we get in general

$$d(x_{N+i}, x_{N+i+1}) < d(x_{N+i-1}, x_{N+i})$$

i.e. $d(x_{N-1}, x_N) > d(x_N, x_{N+1}) > \dots > d(x_{N+i}, x_{N+i+1}) > \dots$.

Therefore, by the monotonically increasing property of $\alpha(x,y)$,

$$\alpha(x_{N-1}, x_N) > \alpha(x_N, x_{N+1}) > \dots > \alpha(x_{N+i}, x_{N+i+1}) > \dots .$$

Again, from the choice made at the beginning, since $d(x_N, x_{N-1}) < \varepsilon$

$H(F(x_{N-1}), F(x_N)) \leq \alpha(x_{N-1}, x_N)d(x_{N-1}, x_N)$ and there exists $x_{N+1} \in F(x_N)$ such that $d(x_N, x_{N+1}) < \alpha(x_{N-1}, x_N) \cdot \varepsilon \ (<\varepsilon)$.

Also, $\quad H(F(x_N), F(x_{N+1})) \leq \alpha(x_N, x_{N+1})d(x_N, x_{N+1})$

$$\leq (\alpha(x_N, x_{N+1}))(\alpha(x_{N-1}, x_N))d(x_{N-1}, x_N)$$

$$< (\alpha(x_{N-1}, x_N))^2 \, d(x_{N-1}, x_N)$$

$$< (\alpha(x_{N-1}, x_N))^2 \cdot \varepsilon \quad (<\varepsilon)$$

and since $x_{N+1} \in F(x_N)$, there exists $x_{N+2} \in F(x_{N+1})$ such that $d(x_{N+1}, x_{N+2}) < (\alpha(x_{N-1}, x_N))^2 \cdot \varepsilon$.

Continuing in this fashion we provide a sequence $\{x_{N+i}\}_{i=1}^{\infty}$ of points of X such that $x_{N+i+1} \in F(x_{N+i})$ and

$d(x_{N+i}, x_{N+i+1}) < (\alpha(x_{N-1}, x_N))^{i+1} \cdot \varepsilon$ for all $i \geq 1$.

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which, by the completeness of (X,d), converges to some point $y \in X$. Hence, the sequence $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(y)$ and, since $x_{i+1} \in F(x_i)$ for all i and $F(y)$ is closed, $y \in F(y)$. This proves F has a fixed point. Furthermore, the sequence $\{x_i\}_{i=1}^{\infty}$ satisfies the conditions in (2) of the alternative.

We now give a theorem for a single-valued mapping which is modeled after a theorem due to Diaz and Margolis [9].

<u>Theorem</u> 2.3.19. Suppose that (X,d) is a generalized complete metric space, and $T : X \to X$ is a mapping satisfying

(C1) There exists a constant α with $0 \leq \alpha < 1/2$ such that whenever $d(x,y) < \infty$

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))].$$

Let $x_o \in X$ and consider the "sequence of successive approximations with initial element x_o" ; $x_o, T(x_o), \ldots, T^{\ell}(x_o), \ldots$.

Then the following alternative holds : either

(A) for every integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^{\ell}(x_o), T^{\ell+1}(x_o)) = \infty \quad \text{or}$$

(B) the sequence of successive approximations $x_o, T(x_o), T^2(x_o), \ldots,$

$T^{\ell}(x_o)$, ..., is d-convergent to a fixed point of T.

Proof. Consider the sequence of numbers $d(x_o, T(x_o))$, $d(T(x_o), T^2(x_o))$, ..., $d(T^{\ell}(x_o), T^{\ell+1}(x_o))$, ..., the "sequence of distances between consecutive neighbors" of the sequence of successive approximations with initial element x_o. There are two mutually exclusive possibilities : either

(a) for every integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^{\ell}(x_o), T^{\ell+1}(x_o)) = \infty \quad ,$$

(which is precisely the alternative (A) of the conclusion of the theorem, or else,

(b) for some integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^{\ell}(x_o), T^{\ell+1}(x_o)) < \infty \quad .$$

In order to complete the proof it only remains to show that (b) implies alternative (B) of the conclusion of the theorem.

In case (b) holds, let $N + N(x_o)$ denote a particular one (for definiteness one could choose the smallest) of all the integers $\ell = 0, 1, 2, \ldots$ such that $d(T^{\ell}(x_o), T^{\ell+1}(x_o)) < \infty$.

Then by (C1) since $d(T^N(x_o), T^{N+1}(x_o)) < \infty$, it follows that

$$d(T^{N+1}(x_o), T^{N+2}(x_o)) = d(TT^N(x_o), TT^{N+1}(x_o))$$

$$\leq \alpha[d(T^N(x_o), T^{N+1}(x_o) + d(T^{N+1}(x_o)), T^{N+2}(x_o))]$$

i.e. $d(T^{N+1}(x_o), T^{N+2}(x_o)) \leq \frac{\alpha}{1-\alpha} d(T^N(x_o), T^{N+1}(x_o)) < \infty$.

Similarly, $d(T^{N+2}(x_o), T^{N+3}(x_o)) = d(TT^{N+1}(x_o), TT^{N+2}(x_o))$

$$\leq \alpha[d(T^{N+1}(x_o), T^{N+2}(x_o)) + d(T^{N+2}(x_o, T^{N+3}(x_o))]$$

i.e. $$d(T^{N+2}(x_o), T^{N+3}(x_o)) \leq \frac{\alpha}{1-\alpha} d(T^{N+1}(x_o), T^{N+2}(x_o))$$

$$\leq \left(\frac{\alpha}{1-\alpha}\right)^2 d(T^N(x_o), T^{N+1}(x_o)) < \infty$$

In general,

$$d(T^{N+\ell}(x_o), T^{N+\ell+1}(x_o)) \leq \left(\frac{\alpha}{1-\alpha}\right)^{\ell} d(T^N(x_o), T^{N+1}(x_o)) < \infty$$

for every integer $\ell = 0, 1, 2, \ldots$. In other words if n is any integer such that $n > N$, then

$$d(T^n(x_o), T^{n+1}(x_o)) \leq \left(\frac{\alpha}{1-\alpha}\right)^{n-N} d(T^N(x_o), T^{N+1}(x_o)) < \infty .$$

Whenever $n > N$, one has, for $\ell = 1, 2, \ldots,$ that

$$d(T^n(x_o), T^{n+\ell}(x_o)) \leq \sum_{i=1}^{\ell} d(T^{n+i-1}(x_o), T^{n+i}(x_o))$$

$$\leq \sum_{i=1}^{\ell} \left(\frac{\alpha}{1-\alpha}\right)^{n+i-1-N} d(T^N(x_o), T^{N+1}(x_o)).$$

For the sake of convenience we put $K = \frac{\alpha}{1-\alpha}$

i.e. $$d(T^n(x_o), T^{n+\ell}(x_o)) \leq \sum_{i=1}^{\ell} K^{n+i-1-N} d(T^N(x_o), T^{N+1}(x_o))$$

$$\leq K^{n-N} \cdot \frac{1-K^{\ell}}{1-K} d(T^N(x_o), T^{N+1}(x_o)).$$

Therefore, since $0 \leq k < 1$, the sequence of successive approximations x_o, $T(x_o)$, $T^2(x_o)$, $\ldots$, $T^n(x_o)$, $\ldots$, is a d-Cauchy sequence; and hence is d-convergent. That is to say, there exists an element x in X such that $\lim\limits_{n\to\infty} d(T^n(x_o), x) = 0$.

We now show that x is a fixed point of T. For $n > N$ we have

$$d(x, T(x)) \leq d(x, T^n(x_o)) + d(T^n(x_o), T(x))$$
$$\leq d(x, T^n(x_o)) + \alpha[d(T^{n-1}(x_o), T^n(x_o)) + d(x, T(x))]$$

Hence $$d(x, T(x)) \leq \frac{\alpha d(T^{n-1}(x_o), T^n(x_o)) + d(x, T^n(x_o))}{1 - \alpha}$$

and taking $\lim_{n\to\infty}$, it follows that $d(x, T(x)) = 0$

That is, $x = T(x)$ as required.

Another theorem we could get in the following way.

Theorem 2.3.20. Suppose that (X,d) is a generalized complete metric space, and that the mapping $T : X \to X$ satisfies

(C1)' There exists a constant α with $0 \leq \alpha < 1/3$ such that whenever $d(x,y) < \infty$ one has

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y)) + d(x,y)]$$

Let $x_o \in X$ and consider the "sequence of successive approximations with initial element x_o" : x_o, $T^2(x_o)$, ..., $T^\ell(x_o)$, ... Then the following alternative holds : either

(A) for every integer $\ell = 0, 1, 2, \ldots,$ one has

$$d(T^\ell(x_o), T^{\ell+1}(x_o)) = \infty \qquad \text{or}$$

(B) the sequence of successive approximations x_o, $T(x_o)$, $T^2(x_o)$, ..., $T^\ell(x_o)$, ... is d-convergent to a fixed point of T.

We also have a local form of Theorem 2.3.19.

Theorem 2.3.21. Suppose that (X,d) is a generalized complete metric space, and $T : X \to X$ is a function satisfying the following condition.

(C)" There exists a constant α , with $0 \le \alpha < 1/2$, and a positive constant C, such that whenever $d(x,y) \le C$ one has

$$d(T(x), T(y)) \le \alpha[d(x, T(x)) + d(y, T(y))].$$

Let $x_o \in X$, and consider the sequence of successive approximations with initial element x_o; $x_o, T(x_o), T^2(x_o), \ldots, T^\ell(x_o), \ldots$.

Then the following alternative holds : either

(A) for every integer $\ell = 0, 1, 2, \ldots$, one has $d(T^\ell(x_o), T^{\ell+1}(x_o) > C$

or

(B) the sequence of successive approximations $x_o, T(x_o), T^2(x_o), \ldots, T^\ell(x_o), \ldots$ is d-convergent to a fixed point of T.

Remark. If we replace (C1)" by (C1)*, i.e. there exists a constant α , $0 \le \alpha < 1/3$, and a positive constant C, such that whenever $d(x,y) \le C$ one has $d(T(x), T(y)) \le \alpha[d(x, T(x)) + d(y, T(y)) + d(x,y)]$, and again consider the sequence of successive approximations, then the same alternative holds as in the above theorem.

CHAPTER III

Sequences of Contraction Mappings

In this Chapter we consider the following situation. Suppose (X,d) is a complete metric space and $T_i : X \to X$ is a single-valued contraction mapping with a unique fixed point x_i for each $i = 1, 2, \ldots,$ and $T_o : X \to X$ is a single-valued contraction mapping. The following question arises. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges to T_o, does this imply the convergence of $\{x_i\}_{i=1}^{\infty}$ to the fixed point x_o of T_o? Again, suppose (X,d) is a complete metric space and $F_i : X \to CB(X)$ is a multi-valued contraction mapping with a fixed point x_i for each $i = 1, 2, \ldots,$ and $F_o : X \to CB(X)$ is a multi-valued contraction mapping. Then, if the sequence $\{F_i\}_{i=1}^{\infty}$ converges to F_o, does some subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ converge to a fixed point of F_o?

Nadler Jr. [21] has investigated this question in the case of multi-valued mappings. The first solution for the single-valued case was offered by Bonsall [4] in the following way:

Theorem 3.1. Let X be a complete metric space. Let T_o and T_i be contraction mappings of X into itself for $i = 1, 2, \ldots,$ with the same Lipschitz constant $\alpha < 1$, and with fixed points x_o, x_i $(i = 1, 2, \ldots,)$ respectively. Suppose that $\lim_{i\to\infty} T_i(x) = T_o(x)$ for every $x \in X$. Then $\lim_{i\to\infty} x_i = x_o$.

Remark. The requirement that T_o be a contraction mapping can be omitted since it can be deduced from the remaining statement of the theorem.

In the following, modified form of Theorem 3.1., Singh [28] has given a simpler proof than that of Bonsall [4].

Theorem 3.2. Let (X,d) be a complete metric space and let $\{T_i, i = 1, 2, \ldots\}$ be a sequence of contraction mappings with the same Lipschitz constant $\alpha < 1$, and with fixed points x_i $(i = 1, 2, \ldots)$. Suppose that $\lim\limits_{i\to\infty} T_i(x) = T_o(x)$ for every $x \in X$, where T_o is a mapping from X into itself. Then T_o has a unique fixed point x_o and $\lim\limits_{i\to\infty} x_i = x_o$.

Proof. Since $\alpha < 1$ is the same Lipschitz constant for all i,

$$|T_o(x) - T_o(y)| = \lim_{i\to\infty} |T_i(x) - T_i(y)| \leq \alpha|x - y| .$$

Thus T_o is a contraction mapping with contraction constant α, and as such has a unique fixed point.

Since the sequence of contraction mappings converges to T_o, therefore for a given $\varepsilon > 0$, there exists an N such that $i \geq N$ implies

$$d(T_i(x_o), T_o(x_o)) \leq (1 - \alpha)\varepsilon ,$$

where α is the contractive constant. Now for $i \geq N$,

$$\begin{aligned} d(x_o, x_i) &= d(T_o(x_o), T_i(x_i)) \\ &\leq d(T_o(x_o), T_i(x_o)) + d(T_i(x_o), T_i(x_i)) \\ &\leq (1 - \alpha)\varepsilon + \alpha d(x_o, x_i) \end{aligned}$$

Thus, $(1 - \alpha)d(x_o, x_i) \leq (1 - \alpha)\varepsilon$. Now since $0 \leq \alpha < 1$, we have $d(x_o, x_i) < \varepsilon$, $i \geq N$ and so $\lim\limits_{i\to\infty} x_i = x_o$.

A generalization of Theorem 3.2. has been given by Nadler Jr. [22].

Theorem 3.3. Let (X,d) be a locally compact metric space, let $T_i : X \to X$ be a contraction with fixed point x_i for each $i = 1, 2, \ldots,$

and let $T_o : X \to X$ be a contraction with fixed point x_o. If the sequence $\{T_i, i = 1, 2, \ldots\}$ converges pointwise to T_o, then the sequence $\{x_i, i = 1, 2, \ldots\}$ converges to x_o.

Proof. Let $\varepsilon > 0$ and assume ε is sufficiently small so that $K(x_o, \varepsilon) = \{x \in X \mid d(x_o, x) \leq \varepsilon\}$ is a compact subset of X. Then, since $\{T_i\}_{i=1}^{\infty}$ is an equicontinuous sequence of functions converging pointwise to T_o and since $K(x_o,\varepsilon)$ is compact, the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly on $K(x_o,\varepsilon)$ to T_o.

Choose N such that if $i \geq N$, then

$$d(T_i(x), T_o(x)) < (1 - \alpha_o)\varepsilon \quad \text{for all} \quad x \in K(x_o,\varepsilon),$$

where $\alpha_o < 1$ is a Lipschitz constant for T_o.
Then if $i \geq N$ and $x \in K(x_o,\varepsilon)$,

$$\begin{aligned} d(T_i(x), x_o) &\leq d(T_i(x), T_o(x)) + d(T_o(x), T_o(x_o)) \\ &< (1 - \alpha_o)\varepsilon + \alpha_o d(x, x_o) \\ &\leq (1 - \alpha_o)\varepsilon + \alpha_o \varepsilon \\ &= \varepsilon \end{aligned}$$

This proves that if $i \geq N$, then T_i maps $K(x_o,\varepsilon)$ into itself. Letting f_i be the restriction of T_i to $K(x_o,\varepsilon)$ for each $i \geq N$ we see that each f_i is a contraction mapping of $K(x_o,\varepsilon)$ into itself. Since $K(x_o,\varepsilon)$ is a complete metric space, f_i has a fixed point for each $i \geq N$ which must, from the definition of f_i and the fact that T_i has only one fixed point, be x_i. Hence, $x_i \in K(x_o,\varepsilon)$ for each $i \geq N$. It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ of fixed points converges to x_o.

The following theorem and lemma has been given by Nadler Jr. [22].

Theorem 3.4. Let (X,d) be a metric space, let $T_i : X \to X$ be a function with at least one fixed point x_i, for each $i = 1, 2, \ldots$, and let $T_o : X \to X$ be a contraction mapping with fixed point x_o. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_o, then the sequence $\{x_i\}_{i=1}^{\infty}$ converges to x_o.

Proof. Let $\varepsilon > 0$ and choose a natural number N such that $i \geq N$ implies

$d(T_i(x), T_o(x)) < (1 - \alpha_o)\varepsilon$ for all $x \in X$, where $\alpha_o < 1$ is a Lipschitz constant for T_o.

Then, if $i \geq N$,

$$\begin{aligned} d(x_i, x_o) &= d(T_i(x_i), T_o(x_o)) \\ &\leq d(T_i(x_i), T_o(x_i)) + d(T_o(x_i), T_o(x_o)) \\ &< (1 - \alpha_o)\varepsilon + \alpha_o d(x_i, x_o) \end{aligned}$$

Hence, $d(x_i, x_o) < \varepsilon$ for all $i \geq N$. This proves $\{x_i\}_{i=1}^{\infty}$ converges to x_o and completes the proof of the theorem.

Lemma 3.5. Let (X,d) be a metric space, let $T_i : X \to X$ be a contraction mapping with fixed point x_i, for each $i = 1, 2, \ldots,$ and let $T_o : X \to X$ be a contraction mapping with fixed point x_o. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_o and if a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ converges to a point $a_o \in X$, then $x_o = a_o$.

In [21] Nadler Jr. has generalized Lemma 3.5 as follows:

Lemma 3.6. Let (X,d) be a metric space, let $F_i : X \to CB(X)$ be a multi-valued contraction mapping with fixed point x_i for each $i = 1, 2, \ldots,$ and let $F_o : X \to CB(X)$ be a multi-valued contraction mapping. If the sequence $\{F_i\}_{i=1}^{\infty}$ converges pointwise to F_o and if $\{x_{i_j}\}_{j=1}^{\infty}$ is a convergent subsequence of $\{x_i\}_{i=1}^{\infty}$, then $\{x_{i_j}\}_{j=1}^{\infty}$ converges to a fixed point of F_o.

Proof. Let $x_o = \lim_{j\to\infty} x_{i_j}$ and let $\varepsilon > 0$. Choose an integer N such that

$$H(F_{i_j}(x_o), F_o(x_o)) < \varepsilon/2 \quad \text{and} \quad d(x_{i_j}, x_o) < \varepsilon/2 \quad \text{for all } j \geq N.$$

Then, if $j \geq N$,

$$\begin{aligned} H(F_{i_j}(x_{i_j}), F_o(x_o)) &\leq H(F_{i_j}(x_{i_j}), F_{i_j}(x_o)) + H(F_{i_j}(x_o), F_o(x_o)) \\ &< d(x_{i_j}, x_o) + H(F_{i_j}(x_o), F_o(x_o)) < \varepsilon . \end{aligned}$$

This proves that $\lim_{j\to\infty} F_{i_j}(x_{i_j}) = F_o(x_o)$. Therefore, since $x_{i_j} \in F_{i_j}(x_{i_j})$ for each $j = 1, 2, \ldots,$ it follows $x_o \in F(x_o)$.

This proves the lemma.

The question, posed at the beginning, for multi-valued mappings has been answered by Nadler Jr. [21] in the following way.

Theorem 3.7. Let (X,d) be a complete metric space, let $F_i : X \to 2^X$ be a multi-valued contraction mapping with fixed point x_i, for each $i = 1, 2, \ldots,$ and let $F_o : X \to 2^X$ be a m.v.c.m. If any one of the following holds:

(We assume $F_i(x)$ is compact for all i and for all x)

(i) each of the mappings $F_1, F_2, \ldots$ has the same Lipschitz constant $\alpha < 1$ and the sequence $\{F_i\}_{i=1}^{\infty}$ converges pointwise to F_o;

or

(ii) the sequence $\{F_i\}_{i=1}^{\infty}$ converges uniformly to F_o ;

or

(iii) the space (X,d) is locally compact and the sequence $\{F_i\}_{i=1}^{\infty}$ converges pointwise to F_o ;

then there is a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that $\{x_{i_j}\}_{j=1}^{\infty}$ converges to a fixed point of F_o.

Remark. If F_o has only one fixed point x_o, then (with the hypothesis of the above theorem) the sequence $\{x_i\}_{i=1}^{\infty}$ itself converges to x_o. To see this suppose $\{x_i\}_{i=1}^{\infty}$ does not converge to x_o. Then there is a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that no subsequence of $\{x_{i_k}\}_{k=1}^{\infty}$ converges to x_o. Applying the above theorem in the context of the two sequences $\{F_{i_k}\}_{k=1}^{\infty}$ and $\{x_{i_k}\}_{k=1}^{\infty}$, we see that there is a subsequence of $\{x_{i_k}\}_{k=1}^{\infty}$ which converges to a point of F_o. This establishes a contradiction.

Edelstein [12] has given the following theorem.

Theorem 3.8. Let T be a mapping of a complete ε-chainable metric space (X,d) into itself, and suppose that there is a real number α with $0 \leq \alpha < 1$ such that

$$d(x,y) < \varepsilon \Rightarrow d(T(x), T(y)) \leq \alpha d(x,y).$$

Then T has a unique fixed point $x \in X$, and $x = \lim_{i\to\infty} T^i(x_o)$, where x_o is an arbitrary element of X.

In [30] Singh and Russell have proved a theorem by considering a sequence of such mappings.

<u>Theorem</u> 3.9. Let (X,d) be a complete ε-chainable metric space, and let $T_i (i = 1, 2, \ldots)$ be mappings of X into itself, and suppose that there is a real number α with $0 \leq \alpha < 1$ such that

$$d(x,y) < \varepsilon \Rightarrow d(T_i(x), T_i(y)) \leq \alpha d(x,y) \quad \text{for all } i.$$

If x_i $(i = 1, 2, \ldots)$ are the fixed points for T_i and $\lim_{i\to\infty} T_i(x) = T(x)$ for every $x \in X$, then T has a unique fixed point x_o and $\lim_{i\to\infty} x_i = x_o$.

Proof. (X,d) being ε-chainable, we define for $x,y \in X$,

$$d_\varepsilon(x,y) = \inf \sum_{j=1}^{p} d(z_{j-1}, z_j)$$

where the infimum is taken over all ε-chains $z_o, z_1, \ldots, z_p$ joining $z_o = x$ and $z_p = y$. Then d_ε is a distance function on X satisfying

(i) $d(x,y) \leq d_\varepsilon(x,y)$

(ii) $d(x,y) = d_\varepsilon(x,y)$ for $d(x,y) < \varepsilon$.

From (ii) it follows that a sequence $\{x_n\}$, $x_n \in X$ is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect to d. Hence, (X,d) being complete, (X,d_ε) is also a complete metric space. Moreover, T_i $(i = 1, 2, \ldots)$ are contraction mappings with respect to d_ε. Given $x,y \in X$, and any ε-chain $z_o, z_1, \ldots, z_p$ with $z_o = x$, $z_p = y$, we have $d(z_{j-1}, z_j) < \varepsilon$ $(j = 1, 2, \ldots, p)$, so that

$$d(T_i(z_{j-1}), T_i(z_j)) \leq \alpha d(z_{j-1}, z_j) < \varepsilon \quad (j = 1, 2, \ldots, p).$$

Hence $T_i(z_o), \ldots, T_i(z_p)$ is an ε-chain joining $T_i(x)$ and $T_i(y)$ and

$$d\,(T_i(x),\,T_i(y) \le \sum_{j=1}^{p} d(T_i(z_{j-1}),\,T_i(z_j))$$
$$\le \alpha\sum_{j=1}^{p} d(z_{j-1},\,z_j).$$

$z_o, z_1, \ldots, z_p$ being an arbitrary ε-chain, we have

$$d_\varepsilon(T_i(x),\,T_i(y)) \le \alpha d_\varepsilon(x,y).$$

Now since $T_i (i = 1, 2, \ldots)$ are contraction mappings with respect to d_ε and (X,d_ε) is a complete metric space, then $T(x) = \lim_{i\to\infty} T_i(x)$ is a contraction mapping with respect to d_ε. Moreover T has a unique fixed point x_o and $\lim_{i\to\infty} x_i = x_o$ by Theorem 3.2. This unique fixed point is given by

$$\lim_{n\to\infty} d_\varepsilon(T^n(y_o),\,x_o) = 0 \quad \text{for} \quad y_o \in X.$$

But (i) at the beginning of the proof implies

$$\lim_{n\to\infty} d(T^n(y_o),\,x_o) = 0.$$

We now give the following theorem.

Theorem 3.10. Suppose

(i) $T : X \to X$ is a Banach Contraction with fixed point x_o.

(ii) $T_n : X \to X$ is a sequence of nonexpansive mappings with fixed points x_n $(n = 1, 2, \ldots)$.

(iii) $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the fixed point of T.

Proof. Let α be the contractive constant of T, then $0 \leq \alpha < 1$ and $d(T(x), T(y)) \leq \alpha d(x,y)$ for all $x,y \in X$.

Since $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T, given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(T(x_n), T_n(x_n)) < (1 - \alpha)\varepsilon.$$

Thus for $n \geq N$,

$$\begin{aligned} d(x_n, x_o) &= d(T_n(x_n), T(x_o)) \\ &\leq d(T_n(x_n), T(x_n)) + d(T(x_n), T(x_o)) \\ &< (1 - \alpha)\varepsilon + \alpha d(x_n, x_o) \end{aligned}$$

i.e. $(1 - \alpha)d(x_n, x_o) < (1 - \alpha)\varepsilon$.

Since $0 \leq \alpha < 1$, we have $d(x_n, x_o) < \varepsilon$, so that $\lim_{n\to\infty} x_n = x_o$.

As a concluding remark we give the following theorem.

<u>Theorem</u> 3.11. Suppose

(i) $T : X \to X$ is a mapping such that

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] \quad 0 \leq \alpha < 1/2$$

with fixed point x_o.

(ii) $T_n : X \to X$ is a sequence of nonexpansive mappings with fixed points x_n $(n = 1, 2, \ldots)$,

(iii) $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point x_o.

Proof. Since $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T, given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(T(x_n), T_n(x_n)) < \varepsilon/2 .$$

Thus for $n \geq N$,

$$\begin{aligned} d(x_n, x_o) &= d(T_n(x_n), T(x_o)) \\ &\leq d(T_n(x_n), T(x_n)) + d(T(x_n), T(x_o)) \\ &\leq d(T_n(x_n), T(x_n)) + \alpha[d(x_n, T(x_n)) + d(x_o, T(x_o)] \\ &\leq d(T_n(x_n), T(x_n)) + \alpha d(T_n(x_n), T(x_n)) \\ &< 2d(T_n(x_n), T(x_n)) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

i.e. $d(x_n, x_o) < \varepsilon$, so that $\lim\limits_{n\to\infty} x_n = x_o$.

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