GENERALIZED TENSOR PRODUCTS

W. WAYNE THISTLE

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by

W. WAYNE THISTLE

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ABSTRACT

If R is a commutative ring and A and B are R-modules then $hom(A,B)$, $Hom(A,B)$ and $A \otimes B$ will denote the set of morphisms $A \to B$, the set of morphisms $A \to B$ regarded as a R-module and the usual algebraic tensor product of A and B, respectively. The R-module, $A \otimes B$ can be defined by any of the following results:

(i) $t: A \times B \to A \otimes B;\ (a,b) \mapsto a \otimes b$ is a universal bilinear function in the sense that any other bilinear function $A \times B \to C$ factors uniquely through t.

(ii) there is a natural isomorphism $hom(A \otimes B, C) \cong hom(A, Hom(B,C))$.

(iii) the functor $- \otimes B$ is a left adjoint to the functor $Hom(B,-)$, i.e. the isomorphism of (ii) is natural in the variables A and C.

(iv) there is a natural isomorphism $Hom(A \otimes B, C) \cong Hom(A, Hom(B,C))$.

$\otimes$ also has the property that:

(v) there exists natural isomorphisms:

$a: A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

$r: A \otimes R \cong A$ where R is regarded as an R-module

$e: R \otimes A \cong A$

$c: A \otimes B \cong B \otimes A$.

The existence of these isomorphisms does not constitute a definition of $\otimes$ since analogous isomorphisms exist for the direct sum $A \oplus B$ of R-modules A and B.

In this thesis we abstract the definitions (i), (ii), (iii) and (iv) from the category of R-modules to a general category $\mathcal{C}$ calling the tensor products so defined the (i) Bimorphism Product

(ii) the Exponential Product (iii) the Adjoint Product (iv) the Strong Exponential Product, respectively. The relation between (ii), (iv) and (v) is discussed in (17); (i) is related to these ideas in (24); (iii) does not seem to have been discussed elsewhere.

Our main purpose is to examine the conditions under which the different products coincide and the extent to which the products satisfy the conditions (v). The value of this theory lies in the number and diversity of the examples.

Chapter I gives the necessary details about category theory and defines many of the terms which occur in the main discussion. In Chapter II the Adjoint Product and the properties of associativity, commutativity and left and right identities are introduced. The "coherence" of the above isomorphisms forms the content of the third chapter which is a survey of the works of MacLane (21) and Kelly (17). Chapter IV gives the definition of the Bimorphism Product as explained in Pumplun (24). In Chapter V it is proved that any Bimorphism Product is an Exponential Product and conversely any commutative Exponential Product is a Bimorphism Product. In Chapter VI (sections 1 to 3) the relationships between the natural isomorphisms a, r, e and c and the Exponential and Strong Exponential Products are discussed; in section 6.4 these natural isomorphisms and the Strong Exponential Product are related to the Bimorphism Product. In Chapter VII the Exponential Product and the Adjoint Product are shown to be distinct, in general; conditions are given for their equivalence. Chapter VIII is devoted to a selection of examples from many branches of mathematics. The Appendix gives necessary and sufficient conditions for the concepts of a morphism and bimorphism to coincide. The lengthy bibliography lists papers related to the thesis.

Our results are proved in full detail (apart from Chapter III which is a survey of results from (18) and (21), detailed proofs being given in these papers, and two proofs involving trimorphisms in section 6.4). Chapter VII and the Appendix are basically original work. The proofs of the theorems in Chapter V are given in greater detail than by the original author.

CONTENTS

CHAPTER I

BASIC CATEGORY THEORY

§1.1 Categories

The definition of a category was first given by Eilenberg and MacLane in 1945.

Definition 1.1.1 Let $\mathcal{C}$ be a class of "objects", A B, C .... (denoted by ob $\mathcal{C}$) together with two functions, as follows:

(i) A function assigning to each pair (A,B) of objects of $\mathcal{C}$ a set $\mathcal{C}(A,B)$. An element $f \in \mathcal{C}(A,B)$ in this set is called a <u>morphism</u> $f: A \to B$ of $\mathcal{C}$ with domain A and codomain B.

(ii) A function assigning to each triple (A,B,C) of objects of $\mathcal{C}$ a function $\mathcal{C}(B,C) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$. For morphisms $g: B \to C$ and $f: A \to B$ this function is written as $(g,f) \mapsto g \circ f$ and the morphism $g \circ f: A \to C$ is called the <u>composite</u> of g with f. The class $\mathcal{C}$ with these two functions is called a <u>category</u> when the following two axioms hold:

(a) Associativity: If $h: C \to D$, $g: B \to C$ and $f: A \to B$ are morphisms of $\mathcal{C}$ with the indicated domains and codomains then $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) Identity: For each object B of $\mathcal{C}$ there exists a morphism $1_B: B \to B$ such that

$$\text{if } f: A \to B \quad \text{then } 1_B \circ f = f$$

$$\text{and if } g: B \to C \quad \text{then } g \circ 1_B = g.$$

Definition 1.1.2 The <u>dual</u> of a category C is denoted C^*. The objects are the same as the objects of C. A morphism $A \to B$ in C^* is a morphism $B \to A$ in C.

Definition 1.1.3 For each pair of categories C, C', there exists a <u>product category</u> $C \times C'$. An object of this product is an ordered pair (C,C') of objects of C and C', respectively; a morphism $(C,C') \to (D,D')$ with the indicated domain and codomain is an ordered pair (f,f') of morphisms $f: C \to D$, $f': C' \to D'$. The composite of morphisms is defined termwise; thus (f,f') as above and a second such ordered pair $(g,g'): (D,D') \to (E,E')$ have the composite $(g,g') \circ (f,f') = (g \circ f',g' \circ f'): (C,C') \to (E,E')$.

Definition 1.1.4 A morphism $f: A \to B$ in a given category is called an <u>isomorphism</u> if there is a morphism $g: B \to A$ in the category such that $f \circ g = 1_A$ and $g \circ f = 1_B$.

1.2 Functors

Definition 1.2.1 Let C and $\mathcal{D}$ be given categories and consider a function $\Theta: C \to \mathcal{D}$ which assigns to each object $A \in C$ an object $\Theta(A) \in \mathcal{D}$ and to each morphism $f \in C$ a morphism $\Theta(f) \in \mathcal{D}$. The function Θ is said to be a <u>covariant functor</u> (or simply a <u>functor</u>) from C to $\mathcal{D}$ if and only if it satisfies the following three conditions:

(i) If $f: A \to B$ then $\Theta(f): \Theta(A) \to \Theta(B)$

(ii) $\Theta(1_A) = 1_{\Theta(A)}$

(iii) If $f \circ g$ is defined then $\Theta(f \circ g) = \Theta(f) \circ \Theta(g)$.

Definition 1.2.2 A <u>contravariant functor</u> $\Phi: \mathcal{D} \to \mathcal{C}$ assigns to each object $A \in \mathcal{D}$ an object $\Phi(A) \in \mathcal{C}$ and to each morphism $f \in \mathcal{D}$ a morphism $\Phi(f) \in \mathcal{C}$ in such a way that:

(i) If $f: A \to B$ then $\Phi(f): \Phi(B) \to \Phi(A)$

(ii) $\Phi(1_A) = 1_{\Phi(A)}$

(iii) If $g \circ f$ is defined then $\Phi(g \circ f) = \Phi(f) \circ \Phi(g)$.

Definition 1.2.3 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $\Psi: \mathcal{C} \to \mathcal{D}$ will be said to be <u>faithful</u> if for each $A, B \in \text{ob}\, \mathcal{C}$ the function $\mathcal{C}(A,B) \to \mathcal{D}(\Psi(A),\Psi(B))$; $f \mapsto \psi(f)$ is injective, i.e. Ψ preserves distinctness of morphisms.

Definition 1.2.4 A functor $\Omega: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ on a product category $\mathcal{C} \times \mathcal{D}$ to another category $\mathcal{E}$ is called a <u>bifunctor</u> on $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$.

If $g: B \to B'$ is a given morphism then we denote the induced function $\mathcal{C}(A,B) \to \mathcal{C}(A,B')$; $h \mapsto g \circ h$, $h \in \mathcal{C}(A,B)$ by $\mathcal{C}(A,g)$.

Similarly, if $f: A \to A'$ is a given morphism then we denote the induced function $\mathcal{C}(A',B) \to \mathcal{C}(A,B)$; $h \mapsto h \circ f$, $h \in \mathcal{C}(A',B)$ by $\mathcal{C}(f,B)$.

If f and g are as given above then $\mathcal{C}(f,g)$ denotes the induced function $\mathcal{C}(A,B) \to \mathcal{C}(A',B')$.

One useful example of a bifunctor is the <u>usual hom functor</u> to sets, i.e. $\mathcal{C}: \mathcal{C}^* \times \mathcal{C} \to \mathcal{S}$; $(A.B) \mapsto \mathcal{C}(A,B)$; $(f,g) \mapsto \mathcal{C}(f,g)$ where $\mathcal{S}$ represents the category of sets.

§1.3 Transformations of Functors

Definition 1.3.1 Let θ and Φ be any two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. By a <u>natural transformation</u> of the functor θ into the functor Φ, we mean a function υ which assigns to each object A of the category $\mathcal{C}$ a morphism $\upsilon(A)$ of the category $\mathcal{D}$ such that the following two conditions are satisfied:

(i) For every object A of $\mathcal{C}$, we have $\upsilon(A): \theta(A) \to \Phi(A)$

(ii) For every morphism $f: A \to B$ of $\mathcal{C}$ we have

$\upsilon(B)\ \theta(f) = \Phi(f)\ \upsilon(A)$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \theta(A) & \xrightarrow{\theta(f)} & \theta(B) \\ \downarrow \upsilon(A) & & \downarrow \upsilon(B) \\ \Phi(A) & \xrightarrow{\Phi(f)} & \Phi(B) \end{array} \qquad (1.3.1)$$

A natural transformation $\upsilon: \theta \to \Phi$ is also called a " morphism of functors ".

Definition 1.3.2 If each $\upsilon(A)$ is an isomorphism in category $\mathcal{D}$ we call $\upsilon: \theta \to \Phi$ a <u>natural isomorphism</u>. We donote a natural isomprphism by the symbol $\cong$.

§1.4 Universal Elements, Representability and Adjoint Functors

Definition 1.4.1 Let $\Theta: \mathcal{C} \to \mathcal{S}$ be a functor to the category of sets. A universal element for Θ is a pair (u,R) consisting of an object R of $\mathcal{C}$ and an element $c \in \Theta(R)$ with the following property: To any object C of $\mathcal{C}$ and any element $c \in \Theta(R)$ there is exactly one morphism $f: R \to C$ with $\Theta(f)u = c$.

$$(1.4.1) \qquad \begin{array}{c} R \\ \Big\downarrow h \\ C \end{array} \qquad\qquad \begin{array}{ccc} u & \varepsilon & \Theta(R) \\ \Big\downarrow & & \Big\downarrow \Theta(h) \\ c & \varepsilon & \Theta(C). \end{array}$$

Definition 1.4.2 A functor $\Theta: \mathcal{C} \to \mathcal{S}$ is representable if there is a distinguished object K in $\mathcal{C}$ such that $\Theta A \cong \mathcal{C}(K,A)$ where A is an arbitrary object of $\mathcal{C}$. K will then be called a representing object for Θ(4, p.524).

Definition 1.4.3 An adjunction of the functor Θ to the functor Φ where $\Theta: \mathcal{C} \to \mathcal{D}$ and $\Phi: \mathcal{D} \to \mathcal{C}$ is a natural bijection $\Psi = \Psi_{A,B}: \mathcal{D}(\Theta A, B) \cong \mathcal{C}(A, \Phi B)$. Given such an adjunction, the functor Θ is called a left adjoint of Φ, while Φ is a right adjoint of Θ.

The following theorem describes the sense in which adjoints are unique.

THEOREM 1.4.1 If a functor $\Theta: \mathcal{C} \to \mathcal{D}$ has a left (right) adjoint Φ then any other functor which has Φ as its left (right) adjoint is naturally equivalent to Θ.

Proof: The proof is immediate from (4, Theorem 9, p.28) and (4, Corollary to Theorem 6, p.533).

From these basic definitions and results, category theory has followed an evolutionary pattern which occurs frequently in mathematics. We begin by observing similarities and recurring arguments in several situations which superficially seem to bear little resemblance to each other. Then by isolating concepts and methods which are common to the various examples we may find a theory containing many or all of our examples, which in itself seems worthy of study. One of the outstanding features of category theory is the unity it brings to mathematics. Familiar, but seemingly quite different constructions turn out to be versions of the same categorical constructions. For example, the following three constructions are just coproducts in different categories:

(i) $X_1 \dot{\cup} X_2$ - disjoint union of the sets X_1 and X_2

(ii) $G_1 * G_2$ - free product of the groups G_1 and G_2

(iii) $A_1 \oplus A_2$ - direct sum of the Abelian groups A_1 and A_2.

The following chapters are devoted to the development of another product in category theory - the <u>tensor product</u>.

CHAPTER II

AN INTRODUCTION TO TENSOR PRODUCTS

§2.1 Categories with Multiplication

To define an abstraction of the usual algebraic tensor product in a more general category, the method is clearly to list some of the properties that the usual tensor product satisfies; the problem then is to select a suitable set of properties.

MacLane (21) has introduced a category with multiplication defining it as a category $\mathcal{C}$ and a covariant bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(A,B) \mapsto A \otimes B$. If $\otimes$ is to be a tensor product then there should exist an object $K \in \mathcal{C}$ such that for all objects A, B and $C \in \mathcal{C}$ there are natural isomorphisms:

(i) $a = a(A,BC): (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ .... associativity

(ii) $e = e(K,A): K \otimes A \to A$ .... left identity

(iii) $r = r\ (A,K): A \otimes K \to A$ ....right identity

(iv) $c = c\ (A,B): A \otimes B \to B \otimes A$ .... commutativity

where $A, B, C, K \in \mathcal{C}$.

We also require that these morphisms are coherent in the sense that any natural isomorphisms defined by their repeated uses such as

$$A \otimes B \xrightarrow{r^{-1} \otimes 1} (A \otimes K) \otimes B \xrightarrow{a} A \otimes (K \otimes B) \xrightarrow{1 \otimes e} A \otimes B$$

is to be the identity. Chapter III deals with this more fully.

Eilenberg and Kelly (11) have defined a monoidal category as a category with a multiplication, the above natural isomorphisms, apart from commutativity, and a set of conditions ensuring coherence. They defined a symmetric monoidal category as a monoidal category with a commutativity isomorphism and a set of conditions ensuring coherence.

§2.2 The Tensor Product in Terms of Adjointness

The question arises as to what properties ⊗ should have in order that we may reasonably call it a " tensor product ". One of the most important properties a tensor product should have is that it be the left adjoint of what is called a Hom functor. This Hom functor is not the usual hom functor to sets but must be defined on $C^* \times C$ and take values in C. This we will denote by H and call an " internal Hom functor " since when composed with some functor $P: C \to S$ we have the usual hom functor to sets. It is the existence of Hom together with P that is important. This functor P is usually a forgetful functor, i.e. an " underlying set " functor. The Hom functor is concerned with " function constructions ". We should note however, that the requirement of merely having a right adjoint is not sufficient since using that definition $A \otimes B = A$ is a tensor product.

Definition 2.2.1 A semi-structured category $(C,H,P,)$ consists of a category C and functors $H: C^* \times C \to C$ and $P: C \to S$ such that $PH = C$.

Definition 2.2.2 If $(C,H,P,)$ is a semi-structured category and $\otimes: C \times C \to C$ is a bifunctor such that $- \otimes B: C \to C$ is a left adjoint to $H(B,-): C \to C$ then $\otimes$ will be called an Adjoint Tensor Product (abbreviated to Adjoint Product for (C,H,P) and

$A \otimes B$ the Adjoint Product of A and B. Hence $\otimes$ is an Adjoint Product if and only if there is an isomorphism $C(A \otimes B, C) \cong C(A, H(B,C))$ natural in the variables A and C.

In Chapter V we require an Adjoint Product with the above isomorphism natural in all three variables

Definition 2.2.3 If (C,H,P) is a semi-structured category then the bifunctor $\otimes: C \times C \to C$ will be called an <u>Exponential Tensor Product</u> (abbreviated to <u>Exponential Product</u>) for (C,H,P) if there is a natural isomorphism $C(A \otimes B, C) \cong C(A,H(B,C))$, i.e. natural in all three variables.

We have chosen the name, Exponential Product from the corresponding result in the category of topological spaces which is usually called the exponential law of spaces.

The relation between the Adjoint Product and the Exponential Product will be further discussed in Chapter VII.

From Theorem 1.4.1 we see that a tensor product defined in either of these senses is unique up to natural isomorphism.

<u>PROPOSITION 2.2.1</u> If (C,H,P) is a semi-structured category with an Adjoint Product $\otimes$ and $\otimes$ has a left identity K then $C(A,B) \cong C(K, H(A,B))$.

Proof: $C(A,B) \cong C(K \otimes A, B)$

$\cong C(K, H(A,B))$.

Before introducing additional theory we should elaborate on the isomorphisms in this chapter and it is for this purpose that we explore the concept of "coherence" and its role in the theory of the tensor product.

CHAPTER III

COHERENCE IN CATEGORIES WITH MULTIPLICATION

§3.1 The Nature of Coherence

In this chapter we give a precise definition of coherence and then give minimal conditions necessary for coherence of the natural isomorphisms of §2.1 .

Definition 3.1.1 An iterate of the bifunctor $\otimes: C \times C \to C$ is any functor formed by repeated applications of $\otimes$ - multiplication.

A functor $\otimes: C^p \to C$ will be said to have multiplicity p. If Θ and Φ are functors of multiplicity p and q, respectively, then $\Theta \otimes \Phi$ will be the functor of multiplicity $p + q$ defined by $(\Theta \otimes \Phi)\ (A,B) = \Theta(A) \otimes \Phi(B)$ where $A \in \text{ob}\ C^p$ and $B \in \text{ob}\ C^q$.

Definition 3.1.2 The set I of iterates of $\otimes$ is the smallest set of functors $C^p \to C$ as the multiplicity p ranges over the positive integer values, satisfying:

(i) the identity functor $i: C \to C$ and the functor $j: C \times C \to C$ defined by $(A,B) \mapsto B \otimes A$ belong to I.

(ii) Θ and $\Phi \in I$ implies $\Theta \otimes \Phi \in I$.

Example 3.1.1 The following functors clearly belong to I :

(i) $i \otimes i: C^2 \to C;\quad (A,B) \mapsto A \otimes B$

(ii) $(i \otimes i) \otimes i: C^3 \to C;\quad (A,B,C) \mapsto (A \otimes B) \otimes C$

(iii) $i \otimes (i \otimes i): C^3 \to C;\quad (A,B,C,) \mapsto A \otimes (B \otimes C)$

(iv) $i \otimes j\colon C^3 \to C;\ (A,B,C) \mapsto A \otimes (C \otimes B)$.

Definition 3.1.3 If Θ and Φ are iterates of $\otimes$ and $\alpha\colon \Theta \to \Phi$ is a natural isomorphism then α will be called an <u>instance</u> of I.

The set of instances $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ determines a category $I = I(\alpha_1, \alpha_2, \ldots, \alpha_n)$ whose objects are the iterates of $\otimes$ and whose morphisms are generated from the instances $\{\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-n}\}$ and every identity natural transformation $1\colon \Theta \to \Theta$. Θ ranges over the iterates of $\otimes$ defined by one of the following methods:

(i) composition, i.e., if $\alpha\colon \Theta \to \Phi$ and $\beta\colon \Phi \to \Theta$ are morphisms in I then $\beta\alpha \in I$.

(ii) forming tensor products, i.e., if $\alpha\colon \Theta \to \Phi$ and $\alpha'\colon \Theta' \to \Phi'$ are morphisms in I then so is $\alpha \otimes \alpha'\colon \Theta \otimes \Theta' \to \Phi \otimes \Phi'$.

Example 3.1.2 If $a_{B,C,D}\colon (B \otimes C) \otimes D \to B \otimes (C \otimes D)$ is a natural isomorphism then the following natural isomorphisms are elements of $I = I(a)$:

(i) $1_A \otimes a_{B,C,D}\colon A \otimes ((B \otimes C) \otimes D) \to A \otimes (B \otimes (C \otimes D))$

(ii) $a^{-1}_{A,B,C \otimes D}\colon A \otimes (B \otimes (C \otimes D)) \to (A \otimes B) \otimes (C \otimes D)$

(iii) $a^{-1}_{A,B,C \otimes D}\, 1_A \otimes a_{B,C,D}\colon A \otimes ((B \otimes C) \otimes D) \to (A \otimes B) \otimes (C \otimes D)$

Definition 3.1.4 If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are instances of I and the category $I(\alpha_1, \alpha_2, \ldots, \alpha_n)$ has the property that for every pair of objects Θ and $\Phi\colon C^p \to C$ there is at most one morphism $\Theta \to \Phi$ in the category, then $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ will be said to be <u>coherent</u>.

§3.2 Conditions Required for Coherence

It may easily be seen that the coherence of a, e, r and c of §2.1 involves an infinite number of conditions. MacLane (21), p. 33) shows that this infinite list could be reduced to a finite sublist without the loss of coherence. For coherence of the natural associativity isomorphism he requires that only one diagram be commutative while for joint coherence of associativity and commutativity, three particular diagrams must be commutative. Ensuring the joint coherence of associativity, commutativity and left and right identities requires that a minimum of eight specific diagrams be commutative.

Kelly (18, p. 401) also did some work on coherence conditions and with regard to the conditions for coherence of the associativity isomorphism and joint coherence of the associativity and commutativity isomorphisms he arrives at the same conditions as required by MacLane. However, for sufficient joint coherence of associativity, commutativity and left and right isomorphisms he is successful in reducing the conditions. We devote this section to the statement of those results.

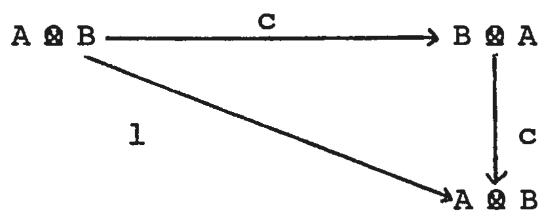
THEOREM 3.2.1 In a category $\mathcal{C}$ with a multiplication, the natural isomorphism $a: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is coherent if and only if the following pentagonal diagram is commutative:

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A \otimes B,C,D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle a_{A,B,C} \otimes 1} & & & & \uparrow{\scriptstyle 1 \otimes a_{B,C,D}} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow{a_{A,B \otimes C,D}} & & A \otimes ((B \otimes C) \otimes D) \end{array} \qquad (3.2.1)$$

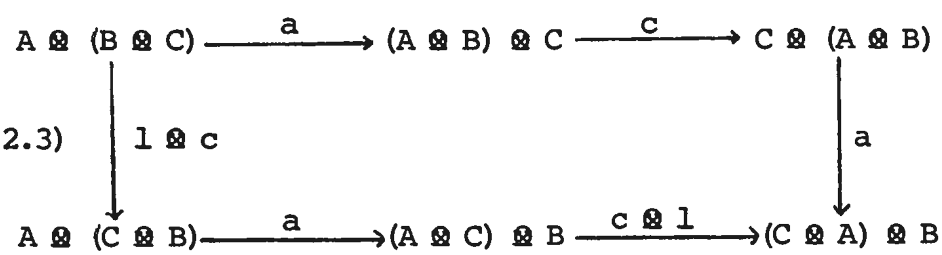
Proof: (21, p.33, Theorem 3.1).

<u>THEOREM 3.2.2</u> The natural isomorphisms $a: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$ and $c: A \otimes B \rightarrow B \otimes A$ are jointly coherent if and only if the pentagonal diagram (3.2.1) and the following diagrams are always commutative:

(3.2.2)



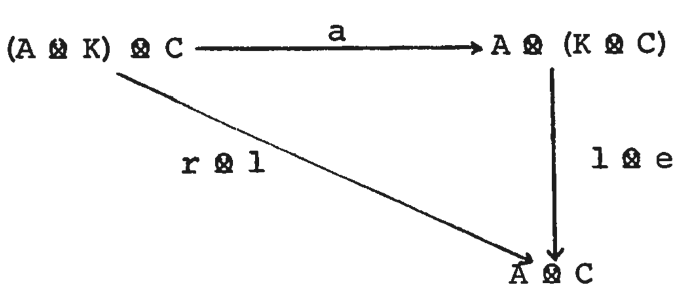
(3.2.3)



Proof: (21, p.38, Theorem 4.2).

<u>THEOREM 3.2.3</u> The natural isomorphisms $a: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$, $c: A \otimes B \rightarrow B \otimes A$, $e: K \otimes C \rightarrow C$ and $r: A \otimes K \rightarrow A$ are jointly coherent if and only if diagrams (3.2.1), (3.2.2) and (3.2.3) with the following diagram are commutative:

(3.2.4)

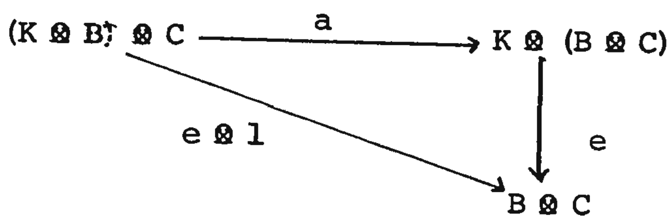


Proof: (18, p.400, Theorem 5')

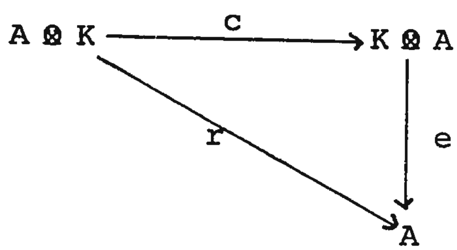
MacLane has required for the previous theorem that the following three commutative diagrams also exist:

(3.2.5) $e = r\colon K \otimes K \longrightarrow K$

(3.2.6)

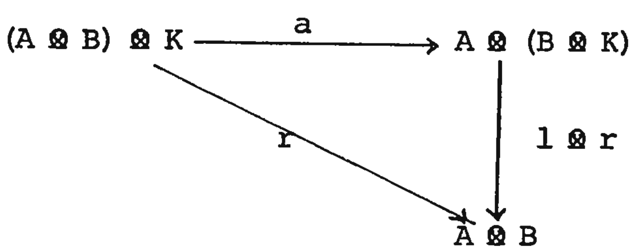


(3.2.7)



Kelly reduces MacLane's conditions by using, in addition, the following diagram:

(3.2.8)



Kelly proves that (3.2.1) and (3.2.4) imply (3.2.6) whence by symmetry (3.2.1) and (3.2.4) imply (3.2.8). Also (3.2.4) and (3.2.6) imply (3.2.5) whence by symmetry (3.2.4) and (3.2.8) imply (3.2.5). Then since (3.2.2), (3.2.3), (3.2.4) and (3.2.8) imply (3.2.5) the condition for joint coherence of a, c, e and r reduce to the commutativity of diagrams (3.2.1), (3.2.2), (3.2.3) and (3.2.4).

CHAPTER IV

TENSOR PRODUCTS - AN ALTERNATIVE APPROACH

§4.1 Structured Categories

In the category of R - modules the tensor product is defined as the Exponential Product in that category. An alternative approach is to define it to be the solution to a universal problem using bilinear maps. An obvious question is whether this definition can also be extended to a general category.

Before discussing this approach, we require the concept of a structured category as defined by Pümplin (24).

Definition 4.1.1 A structured category (C,H,P,K,i) consists of a semi-structured category (C,H,P), a distinguished object K in C and natural isomorphism $i(A): A \to H(K,A)$ such that:

(i) P is faithful

(ii) the function $h_C: C(A,B) \to C(H(B,C), H(A,C))$ defined by $h_C(w) = H(w,C)$, $w \in C(A,B)$, can be lifted to a morphism in C in the sense that there is a morphism $H_C: H(A,B) \to H(H(B,C), H(A,C))$ such that $PH_C = h_C$, for all objects A, B, C in C.

(iii) the function ${}_Ch: C(A,B) \to C(H(C,A), H(C,B))$ defined by ${}_Ch(w) = H(C,w)$, $w \in C(A,B)$, can be lifted to a morphism in C in the sense that there is a morphism ${}_CH: H(A,B) \to H(H(C,A), H(C,B))$ such that $P({}_CH) = {}_Ch$, for all objects A, B, C in C.

The structured category considered by Pumplün does not have the same stringent conditions as the autonomous category defined by Linton (19, p.322). However, the requirement that P be faithful makes it stronger than the semi-structured category of Kelly (18, p.21).

§4.2 Bimorphisms

In a structured category (C,H,P,K,i), Pumplün defines an n-linear map. For this thesis, it will suffice to define a 2-linear or bilinear map which has alredy been referred to as a bimorphism.

Definition 4.2.1 If A, B and C are objects of a structured category (C,H,P,K,i), then a bimorphism $f: A \times B \to C$ is a function $f: PA \times PB \to PC$ such that:

(i) $f(-,b): PA \to PC$ is $P\Phi_1(b)$ for all $b \in PB$ where $\Phi_1: PB \to C(A,C)$ is Pm_1 for some $m_1: B \to H(A,C)$.

(ii) $f(a,-): PB \to PC$ is $P\Phi_2(a)$ for all $a \in PA$ where $\Phi_2: PA \to C(B,C)$ is Pm_2 for some $m_2: A \to H(B,C)$.

The definition may be rephrased in the following more concise form:

Definition 4.2.2 Given a structured category (C,H,P,K,i), a bimorphism $f: A \times B \to C$ is a function $f: PA \times PB \to PC$ where A,B and C are objects of C such that there exists morphisms $m_1: B \to H(A,C)$ and $m_2: A \to H(B,C)$ with $(P(Pm_1)(b))(a) = f(a,b) = (P(Pm_2)(a))(b)$.

<u>LEMMA 4.2.1</u> Let A, B, C and D be objects in a structured category (C,H,P,K,i). If $f: A \times B \to C$ is a bimorphism and $g: C \to D$ is a morphism then $(Pg)\, f: A \times B \to D$ is a bimorphism. This bimorphism $A \times B \to D$ will be denoted by $g\, f$.

Proof: f is a bimorphism if there exists m_1, m_2 such that m_1: $B \to H(A,C)$ and $m_2: A \to H(B,C)$. Also if g is a morphism it induces morphisms $H(B,g): H(B,C) \to H(B,D)$ and $H(A,g): H(A,C) \to H(A,D)$. By composition, $H(B,g)\, m_2: A \to H(B,D)$ and $H(A,g)\, m_1: B \to H(A,D)$, thus giving the morphisms required to prove $(Pg)\, f: A \times B \to D$ is a bimorphism.

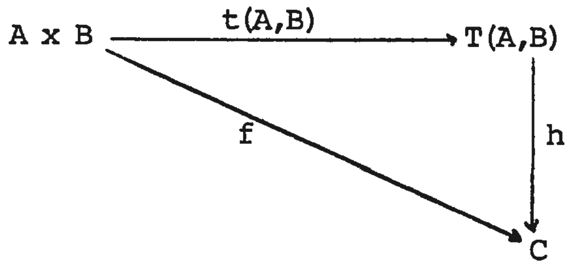
<u>LEMMA 4.2.2</u> Let A, A', B and B' be objects in a structured category (C,H,P,K,i). If $f: A \to A'$ and $g: B \to B'$ are morphisms and $h: A' \times B' \to C$ is a bimorphism then $h\,(Pf \times Pg): A \times B \to C$ is a bimorphism.

Proof: h is a bimorphism if there exists morphisms m_1, m_2, such that $m_1: B' \to H(A',C)$ and $m_2: A' \to H(B',C)$. By composition $H(g,C)\, m_2\, f: A \to H(B,C)$ and $H(f,C)\, m_1\, g: B \to H(A,C)$, giving the morphisms required to prove $h\,(Pf \times Pg): A \times B \to C$ is a bimorphism.

§4.3 <u>The Bimorphism Product</u>

Definition 4.3.1 Let A and B be objects in a structured category (C,H,P,K,i). If T(A,B) is an object of C and $t(A,B): A \times B \to T(A,B)$ is a bimorphism with the property that if C is any object of $\mathcal{C}$ and $f: A \times B \to C$ is any bimorphism, and if there exists a unique morphism $h: T(A,B) \to C$ in $\mathcal{C}$ such that $h\,t(A,B) = f$,

(4.3.1)



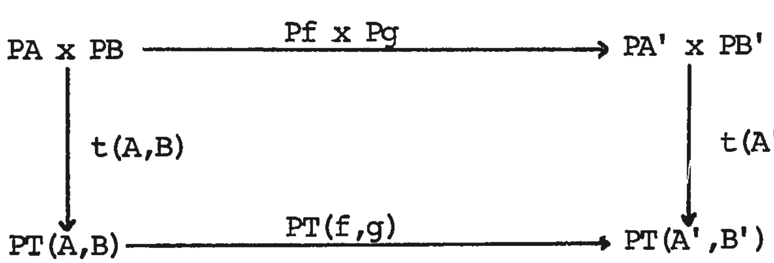
then $t(A,B)$ will be called a universal bimorphism for $A \times B$ and the ordered pair $(t(A,B), T(A,B))$ a *Bimorphism Tensor Product* (abbreviated to *Bimorphism Product*) *of* A *and* B.

This definition of $T(A,B)$ generalizes the standard definition of the tensor product of R-modules by means of a universal bilinear map.

Let $(\mathcal{C},H,P,K,i)$ be a structured category. A *Bimorphism Product for the category* is a rule which associates a *Bimorphism Product* $(t(A,B),\ T(A,B))$ with each ordered pair (A,B) of objects of $\mathcal{C}$.

Let $T(A,B)$ be a Bimorphism Product of A and B and $T(A',B')$ be a Bimorphism Product of A' and B'. If we have morphisms $f: A \to A'$ and $g: B \to B'$ then there exists a unique bimorphism $T(f,g): T(A,B) \to T(A',B')$ such that the following diagram commutes:

(4.3.2)



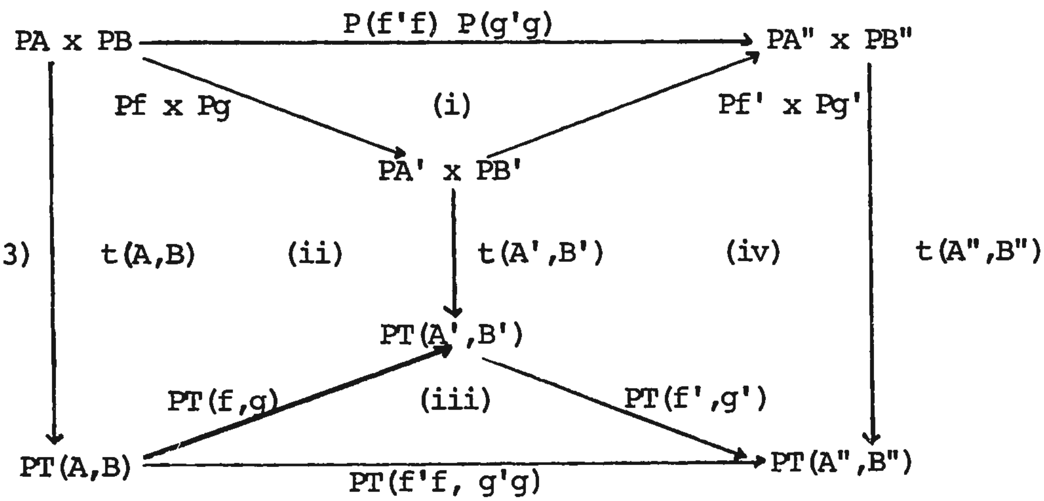
This result follows from Lemma 4.2.2 and the definition of $t(A,B)$.

<u>PROPOSITION 4.3.1</u> If (C,H,P,K,i) is a structured category with a Bimorphism Product then $T: C \times C \to C;\ (A,B) \mapsto T(A,B);\ (f,g) \mapsto T(f,g)$ is a bifunctor.

Proof: We must show that T satisfies the two conditions necessary for a functor.

(i) Consider the following diagram:

(4.3.3)



$T(f,g)$, $T(f'g')$, $T(f'f, g'g)$ are the unique morphisms whose images under P make quadrilateral (ii), quadrilateral (iv) and the outer rectangle commute. Hence $T(f'f, g'g) = T(f', g')\, T(f,g)$.

(ii) $t(A,B)\, 1_{PA \times PB} = P(1_{T(A,B)})\, t(A,B)$. Hence $1_{T(A,B)} = T(1_A, 1_B)$.

Thus T is a bifunctor.

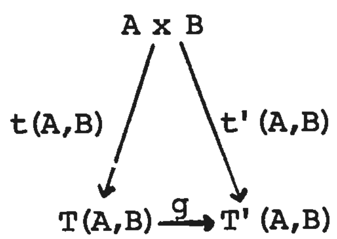
If we let $\text{bimorph}(A \times B, C)$ denote the set of all bimorphisms $h: A \times B \to C$ and if $k: C \to C'$ is a morphism then by Lemma 4.2.1 the composite $k\,h: A \times B \to C'$ is a bimorphism. For fixed A and

B in C, $\Theta(C)$ = bimorph(A x B, C) with $\Theta(k)h = k\ h$, $h \in \Theta(C)$ thus define a functor Θ = bimorph(A x B, -) from $C \to S$.

Our definition of $t(A,B): A \times B \to T(A,B)$ is simply that $(t(A,B), T(A,B))$ is a universal element with respect to bimorph(A x B, -). Hence by uniqueness of universal elements, $t(A,B): A \times B \to T(A,B)$ is unique up to isomorphism, i.e

<u>PROPOSITION 4.3.2</u> If $t(A,B): A \times B \to T(A,B)$ and $t'(A,B): A \times B \to T'(A,B)$ each satisfy Definition 4.3.1, then there is a unique morphism $g: T(A,B) \to T'(A,B)$ such that $g\ t(A,B) = t'(A,B)$.

(4.3.4)



Proof: The proof is immediate from (4, p.28, Theorem 9).

CHAPTER V

THE RELATIONSHIP BETWEEN THE BIMORPHISM AND EXPONENTIAL PRODUCTS

§5.1 The Bimorphism Product is an Exponential Product

Before we proceed to prove the main results of this section we must introduce some preliminary details. We define an evaluation function in a structured category $(\mathcal{C},H,P,K,i)$ and prove that it is a bimorphism.

Definition 5.1.1 For fixed $B \in ob\ \mathcal{C}$ we define for each $C \in ob\ \mathcal{C}$ the evaluation function $\Psi_{B,C}: \mathcal{C}(B,C) \times PB \to PC$ by $\Psi_{B,C}(u,b) = P(u)(b)$ for all $u \in \mathcal{C}(B,C)$, $b \in PB$.

PROPOSITION 5.1.1 The evaluation function $\mathcal{C}(B,C) \times PB \to PC$ is a bimorphism $\Psi_{B,C}: H(B,C) \times B \to C$.

Proof: We must show that $\Psi_{B,C}$ is a function such that:

(i) $\Psi_{B,C}(-,b): \mathcal{C}(B,C) \to PC$ is $P\Phi_1(b)$ for some $\Phi_1(b) \in \mathcal{C}(H(B,C), C)$ for all $b \in PB$ and $\Phi_1: PB \to \mathcal{C}(H(B,C), C)$ is Pm_1 for some $m_1: B \to H(H(B,C), C)$.

(ii) $\Psi_{B,C}(u,-): PB \to PC$ is $P\Phi_2(u)$ for some $\Phi_2(u) \in \mathcal{C}(B,C)$ for all $u \in \mathcal{C}(B,C)$ and $\Phi_2: \mathcal{C}(B,C) \to \mathcal{C}(B,C)$ is Pm_2 for some $m_2: H(B,C) \to H(B,C)$.

From the definition of a structured category, $b \in PB$ can be represented uniquely in the form $b = P(i^{-1}(B))(w)$ with $w \in \mathcal{C}(K,B)$ so that

$$\Psi_{B,C}(u,b) = P(u)\ (P\ i^{-1}(B))(w)$$
$$= P(u\ i^{-1}(B))(w).$$

But $u\, i^{-1}(B) = i^{-1}(C)\ H(K,u)$ and therefore $\Psi_{B,C}(u,b) = P(i^{-1}(C))(u\ w)$

$$= P(i^{-1}(C)\ H(w,C))(u).$$

Hence $\Psi_{B,C}(-,b)$ is $P\Phi_1(b)$ for $\Phi_1(b) = i^{-1}(C)\ H(w,C): H(B,C) \to C$.

If we take $h_C: C(K,B) \to C(H(B,C), H(K,C))$ then this lifts to $H_C: H(K,B) \to H(H(B,C), H(K,C))$ in the sense that $P\ H_C = h_C$ from the definition of a structured category.

We define $m_1 = H(H(B,C), i^{-1}(C))\ H_C\ i(B)$. Then

$$Pm_1 = PH(H(B,C), i^{-1}(C))\ PH_C\ Pi(B)$$
$$= C(H(B,C), i^{-1}(C))\ h_C\ Pi(B).$$

Therefore
$$(Pm_1)(b) = (C(H(B,C), i^{-1}(C))\ h_C\ Pi(B))(b)$$
$$= C(H(B,C), i^{-1}(C))\ h_C(w)$$
$$= i^{-1}(C)\ H(w,C)$$
$$= \Phi_1(b).$$

Condition (ii) for a bimorphism is trivial since m_2 is the identity on $H(B,C)$, i.e. $P(Pm_2)(u)(b) = P(u)(b) = \Psi_{B,C}(u,b)$.

<u>LEMMA 5.1.2</u> Given sets X, Y, Z, X', Y', Z' and functions $\lambda: X \to X'$, $\mu: Y \to Y'$, $\rho: Z \to Z'$; $Fn(Y,Z)$ denotes the set of functions Y into Z. Let $\alpha: X \times Y \to Z$ and $\alpha': X \to Fn(Y,Z)$ be functions related by $\alpha(x,y) = \alpha'(x)(y)$.

(i) If $\beta: X \times Y' \to Z'$ and $\beta': X \to Fn(Y',Z')$ are functions related by $\beta(x',y') = \beta'(x')(y')$ then diagram (5.1.1) commutes if and only if diagram (5.1.2) commutes:

(5.1.1)
$$\begin{array}{ccc} X \times Y & \xrightarrow{X \times \mu} & X \times Y' \\ \downarrow \alpha & & \downarrow \beta \\ Z & \xrightarrow{\rho} & Z' \end{array}$$

(5.1.2)
$$\begin{array}{ccc} X & \xrightarrow{\beta'} & Fn(Y',Z') \\ \downarrow \alpha' & & \downarrow Fn(\mu,Z') \\ Fn(Y,Z) & \xrightarrow{Fn(Y,\rho)} & Fn(Y,Z') \end{array}$$

(ii) If $\gamma: X' \times Y \to Z'$ and $\gamma': X' \to Fn(Y,Z')$ are functions related by $\gamma(x',y) = \gamma'(x')(y)$ then diagram (5.1.3) commutes if and only if diagram (5.1.4) commutes.

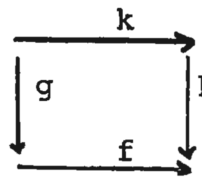
(5.1.3)
$$\begin{array}{ccc} X \times Y & \xrightarrow{\lambda \times Y} & X' \times Y \\ \downarrow \alpha & & \downarrow \gamma \\ Z & \xrightarrow{\rho} & Z' \end{array}$$

(5.1.4)
$$\begin{array}{ccc} X & \xrightarrow{\lambda} & X' \\ \downarrow \alpha' & & \downarrow \gamma' \\ Fn(Y,Z) & \xrightarrow{Fn(Y,\rho)} & Fn(Y,Z') \end{array}$$

Proof: The proof is immediate from the relations between the given functions.

Note: Although we would not normally express the condition $f\,g = h\,k$

in the form that the diagram



commutes, we sometimes find it convenient to do this below as this technique makes clear the domain and codomain of each morphism involved.

<u>THEOREM 5.1.3</u> If the structured category $(\mathcal{C},H,P,K,i)$ has a Bimorphism Product then the Bimorphism Product is an Exponential Product.

Proof: Let us denote the Bimorphism Product of A and B by $(t(A,B), T(A,B))$. We use $m_{1,B}(A)$ to denote the morphism $A \to H(B, T(A,B))$ corresponding to the bimorphism $A \times B \to T(A,B)$. Also since $\Psi_{B,C}: H(B,C) \times B \to C$ is a bimorphism, it factors uniquely through the universal bimorphism $H(B,C) \times B \to T(H(B,C), B)$ thus defining a morphism $m_{0,B}(C): T(H(B,C), B) \to C$ natural in C.

Given $g: T(A,B) \to C$, we define $\Phi(g) = H(B,g)\, m_{1,B}(A)$.

Given $f: A \to H(B,C)$, we define $\Theta(f) = m_{0,B}(C)\, T(f,B)$.

Thus we have the required functions

$$\Theta_{A,B,C}: \mathcal{C}(A, H(B,C)) \to \mathcal{C}(T(A,B), C))$$

$$\Phi_{A,B,C}: \mathcal{C}(T(A,B), C) \to \mathcal{C}(A, H(B,C)).$$

We prove below that:

(i) $\Phi_{A,B,C}$ is natural in A, B and C (Lemmas 5.1.4, 5.1.5 and 5.1.6 respectively).

(ii) $\Theta_{A,B,C}\, \Phi_{A,B,C} = 1$; $\Phi_{A,B,C}\, \Theta_{A,B,C} = 1$ (Proposition 5.1.7).

Therefore $\Phi_{A,B,C}$ is bijective. Hence $\Phi_{A,B,C}$: $C(T(A,B), C) \rightarrow C(A, H(B,C))$ is a natural isomorphism with inverse natural isomorphism $\Theta_{A,B,C}$ (4, p.519) and the theorem is proved.

<u>LEMMA 5.1.4</u> $\Phi_{A,B,C}$ is natural in A.

Proof: We assume $h: A \rightarrow A'$ is a morphism in . Then we must show the following diagram commutes:

$$(5.1.5) \qquad \begin{array}{ccc} C(T(A',B),C) & \xrightarrow{\Phi_{A',B,C}} & C(A', H(B,C)) \\ \downarrow{\scriptstyle C(T(h,B),\, C)} & & \downarrow{\scriptstyle C(h,\, H(B,C))} \\ C(T(A,B), C) & \xrightarrow{\Phi_{A,B,C}} & C(A, H(B,C)) \end{array}$$

i.e. $C(h, H(B,C))\, \Phi_{A',B,C}(g) = \Phi_{A,B,C}\, C(T(h,B), C)(g)$ where $g \in C(T(A',B), C)$. Examining this relation we see that it has

$$\text{L.H.S.} = C(h, H(B,C))\, \Phi_{A',B,C}(g) = H(B,g)\, m_{1,B}(A')\, h \quad \ldots\ldots\ldots \quad (5.1.6)$$

$$\text{R.H.S} = \Phi_{A,B,C}\, C(T(h,B), C)(g) = H(B,g)\, T(h,B)\, m_{1,B}(A)$$
$$= H(B,g)\, H(B, T(h,B))\, m_{1,B}(A) \quad \ldots \quad (5.1.7)$$

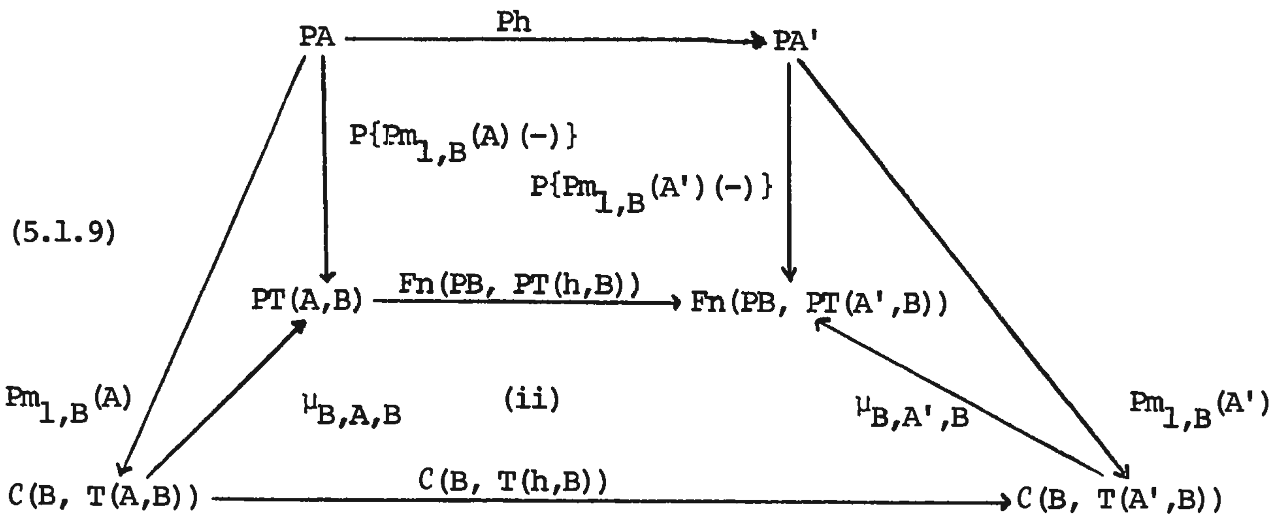
(by definition of H).

Hence it is enough to show the following diagram commutes:

$$(5.1.8) \qquad \begin{array}{ccc} A & \xrightarrow{h} & A' \\ \downarrow{\scriptstyle m_{1,B}(A)} & & \downarrow{\scriptstyle m_{1,B}(A')} \\ H(B, T(A,B)) & \xrightarrow{H(B,\, T(h,B))} & H(B, T(A',B)) \end{array}$$

If we define the natural injection $\mu_{X,Y,Z}$: $C(X, T(Y,Z)) \rightarrow Fn(PX, PT(Y,Z))$ where $X, Y, Z \in ob\ C$ then we may set up the following diagram:

(5.1.9)



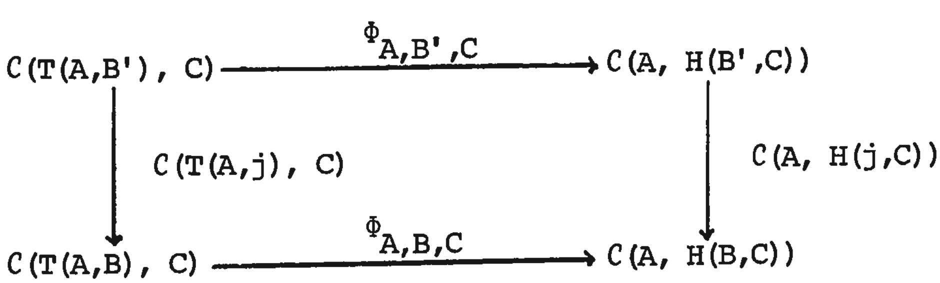
The rectangle is commutative by Lemma 5.1.2 (ii) and components (i), (ii) and (iii) are commutative from the definitions of the functions involved.

It follows that the diagram boundary is commutative and since P is faithful its lifting diagram (5.1.8) is also commutative and the lemma is proved.

<u>Lemma 5.1.5</u> $\Phi_{A,B,C}$ is natural in B.

Proof: We assume j: B $\rightarrow$ B' is a morphism in C. This will induce the following diagram which we must show commutative:

(5.1.10)



i.e. $\mathcal{C}(A, H(j,C))\ \Phi_{A,B',C}(g') = \Phi_{A,B,C}\ \mathcal{C}(T(A,j), C)(g')$ where $g' \in \mathcal{C}(T(A,B'), C)$. Examining this relation we see that it has

$$\begin{aligned} \text{L.H.S.} = \mathcal{C}(A, H(j,C))\ \Phi_{A,B',C}(g') &= H(j,C)\ H(B',g')\ m_{1,B'}(A) \\ &= H(B,g')\ H(j, T(A,B'))\ m_{1,B'}(A) \quad \ldots\ (5.1.11) \end{aligned}$$

(by the naturality of H).

$$\begin{aligned} \text{R.H.S.} = \Phi_{A,B,C}\ \mathcal{C}(T(A,j), C)(g') &= H(B,g')\ T(A,j)\ m_{1,B}(A) \\ &= H(B,g')\ H(B, T(A,j))\ m_{1,B}(A) \quad \ldots\ (5.1.12) \end{aligned}$$

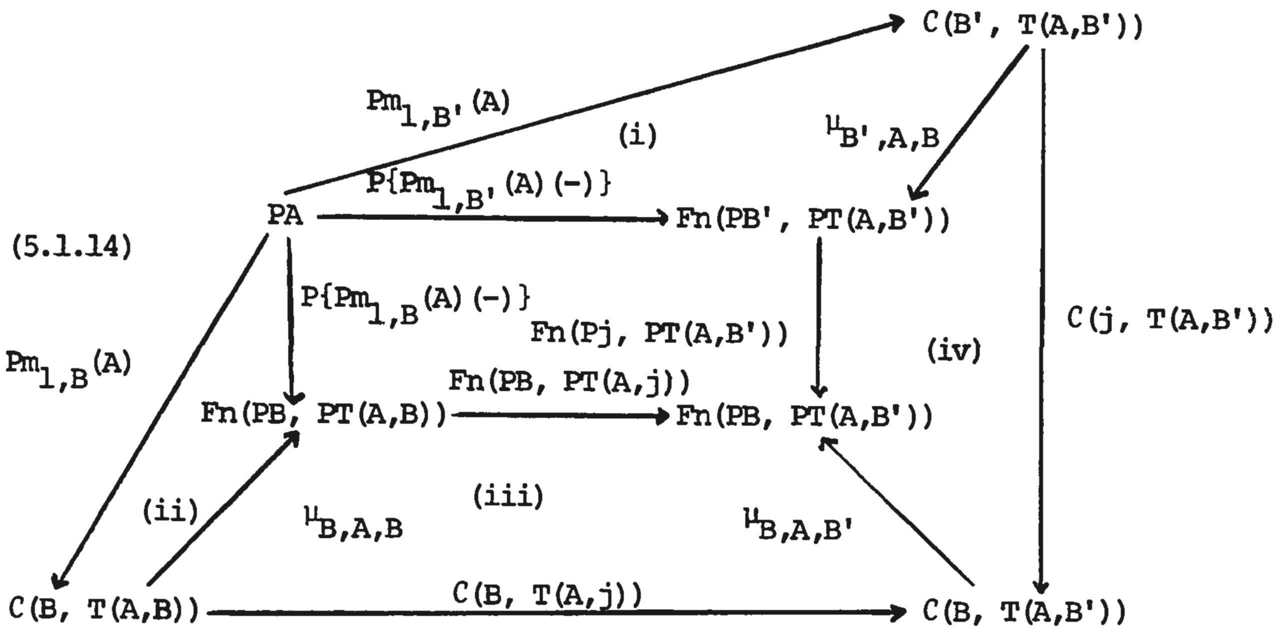
(by the naturality of H).

From (5.1.11) and (5.1.12) it is enough to show the following diagram commutes:

(5.1.13)
$$\begin{array}{ccc} A & \xrightarrow{\quad m_{1,B'}(A) \quad} & H(B', T(A,B')) \\ \Big\downarrow {\scriptstyle m_{1,B}(A)} & & \Big\downarrow {\scriptstyle H(j,\, T(A,B'))} \\ H(B, T(A,B)) & \xrightarrow{\quad H(B,\, T(A,j)) \quad} & H(B, T(A,B')) \end{array}$$

If we define the natural injection $\mu_{X,Y,Z}: \mathcal{C}(X, T(Y,Z)) \to Fn(PX, PT(Y,Z))$ where $X, Y, Z \in \text{ob}\ \mathcal{C}$ then we may set up the following diagram:

(5.1.14)



It follows from the definition of a universal bimorphism that $Pt(A,B')\ PA \times Pj = PT(A,j)\ Pt(A,B)$. Hence by Lemma 5.1.2 the rectangle in the above diagram commutes.

The commutativity of the component diagrams (i), (ii), (iii) and (iv) is immediate from the definitions of the functions involved. Commutativity of the boundary then follows from the commutativity of the rectangle and of the component diagrams. The boundary is the image of diagram (5.1.13) and hence we conclude (5.1.13) is commutative and the lemma is shown.

<u>LEMMA 5.1.6</u> $\Phi_{A,B,C}$ is natural in C.

Proof: We assume $k: C \to C'$ is a morphism in $\mathcal{C}$. Then we must show the following diagram commutes:

$$
(5.1.15) \qquad
\begin{array}{ccc}
C(T(A,B),\ C) & \xrightarrow{\Phi_{A,B,C}} & C(A,\ H(B,C)) \\
\Big\downarrow {\scriptstyle C(T(A,B),\ k)} & & \Big\downarrow {\scriptstyle C(A,\ H(B,k))} \\
C(T(A,B),\ C') & \xrightarrow{\Phi_{A,B,C'}} & C(A,\ H(B,C'))
\end{array}
$$

i.e. $C(A,\ H(B,k))\ \Phi_{A,B,C}(g'') = \Phi_{A,B,C'}\ C(T(A,B),\ k)(g'')$ where $g'' \in C(T(A,B),\ C)$.

Then
$$
\begin{aligned}
C(A,\ H(B,k))\ \Phi_{A,B,C}(g'') &= C(A,\ H(B,k))\ H(B,g'')\ m_{1,B}(A) \\
&= H(B,k)\ H(B,g'')\ m_{1,B}(A) \\
&= H(B,\ kg'')\ m_{1,B}(A) \\
&= \Phi_{A,B,C'}(kg'') \\
&= \Phi_{A,B,C'}\ C(T(A,B),\ k)(g'')
\end{aligned}
$$

Hence $\Phi_{A,B,C}$ is natural in C.

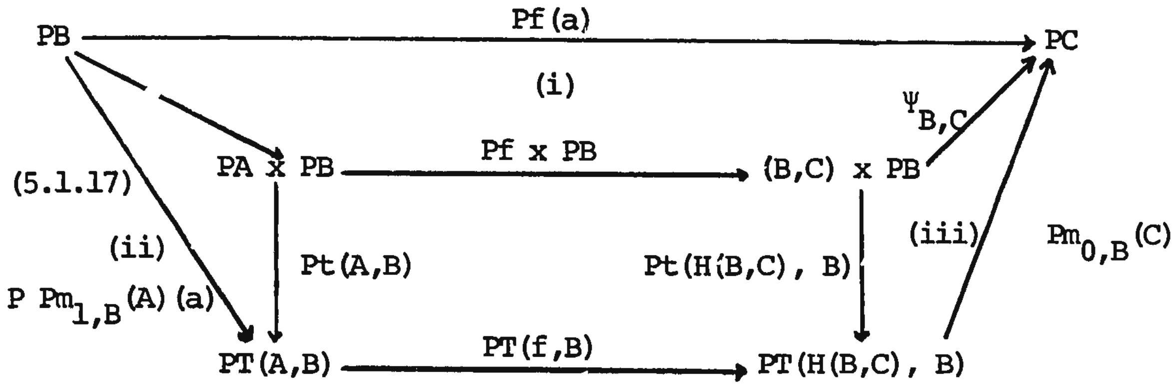
<u>PROPOSITION 5.1.7</u> If $\Theta_{A,B,C}\colon C(A,\ H(B,C)) \to C(T(A,B),\ C)$ and $\Phi_{A,B,C}\colon C(T(A,B),\ C) \to C(A,\ H(B,C))$ are functions as defined in Theorem 5.1.3 then

(i) $\Phi_{A,B,C}\ \Theta_{A,B,C} = 1$

(ii) $\Theta_{A,B,C}\ \Phi_{A,B,C} = 1$ .... (5.1.16)

Proof: If $f\colon A \to H(B,C)$ and $a \in PA$ then we may set up the following diagram:

(5.1.17)



The components (i), (ii) and (iii) and the rectangle commute from the definitions involved (i.e. of $m_{1,B}(A)$ and the Bimorphism Product).

Lifting the outer edges to C we have

$m_{0,B}(C)\ T(f,B)\ (Pm_{1,B}(A))(a) = (Pf)(a)$

i.e. $(PH(B,\ m_{o,B}(C)\ T(f,B))\ m_{1,B}(A))(a) = (Pf)(a)$

Therefore $H(B,\ m_{0,B}(C)\ T(f,B)) = f$

i.e. $\Phi_{A,B,C}\ \Theta_{A,B,C}(f) = f$

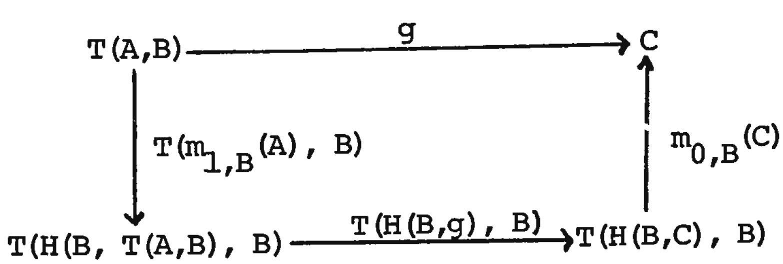
Next if we are given $g: T(A,B) \to C$ then

$\Phi(g) = H(B,g)\ m_{1,B}(A): A \to H(B,C)$ and

$$\begin{aligned}\Theta_{A,B,C}\ \Phi_{A,B,C}(g) &= m_{0,B}(C)\ T(H(B,g)\ m_{1,B}(A),\ B): T(A,B) \to C \\ &= m_{0,B}(C)\ T(H(B,g),\ B)\ T(m_{1,B}(A),\ B)\end{aligned}$$

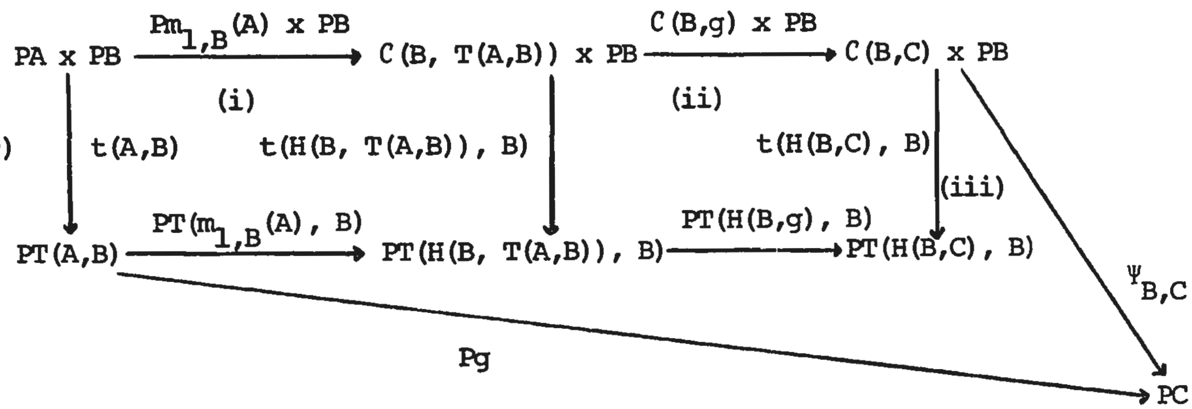
We will show $\Theta_{A,B,C}\ \Phi_{A,B,C}(g) = g$, i.e. the following diagram commutes:

(5.1.18)



Consider the faithful functor P acting on the diagram and then using the definition of the Bimorphism Product, set up the following diagram:

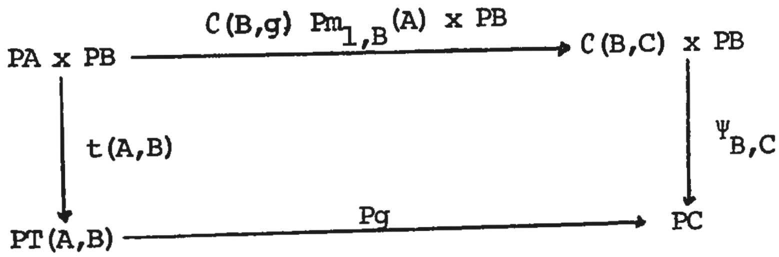
5.1.19)



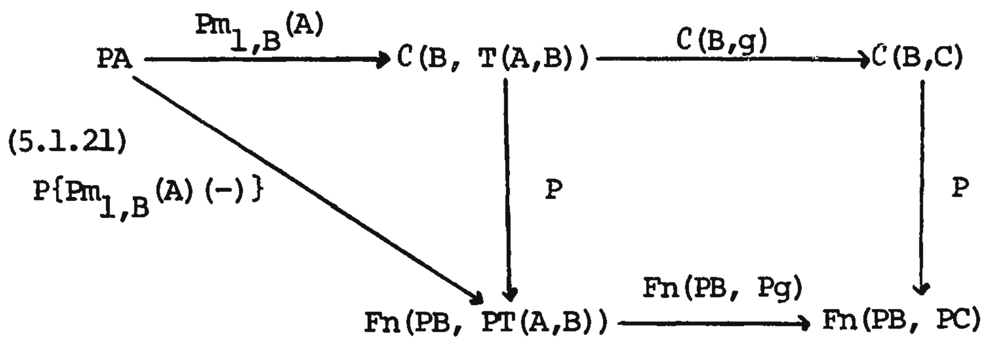
If we let $q = \Psi_{B,C}\ C(B,g) \times PB\ Pm_{1,B}(A) \times PB$ and $s = Pm_{0,B}(C)\ PT(H(B,g),\ B)\ PT(m_{1,B}(A),\ B)$ then q is a bimorphism since the composite of a bimorphism with the product of morphisms is again a bimorphism (Lemma 4.2.2). Also $s\ t(A,B) = q$ since rectangles (i) and (ii) commute from Definition 4.3.1 and triangle (iii) commutes since bimorphism $\Psi_{B,C}$ factors uniquely through the corresponding universal bimorphisms.

We conclude by showing $(Pg)\ t(A,B) = q$. We need the commutativity of the following diagram where $\Psi_{B,C}$ is the evaluation function:

(5.1.20)



By Lemma 5.1.2(ii) this is equivalent to proving the commutativity of:



Since the functor P preserves composition this diagram commutes and we necessarily have $(Pg)\ t(A,B) = q$.

Then we can conclude

$$\Theta_{A,B,C}\ \Phi_{A,B,C}(g) = g.$$

§5.2 A Commutative Exponential Product is a Bimorphism Product

This section will conclude the relationship between the Exponential Product and Bimorphism Product. Certain results must, however be introduced before we can prove the main result.

We will assume our category has an Exponential Product $A \otimes B$ and use it to define morphisms $m_{1,B}(A): A \rightarrow H(B, A \otimes B)$ and $m_{0,A}(C): H(A,C) \otimes A \rightarrow C$. The morphisms of the same name in Chapter IV were defined using the universal bimorphism. Since our category has an Exponential Product there exists natural isomorphisms $C(A, H(B, A \otimes B)) \cong C(A \otimes B, A \otimes B)$ and $C(H(A,C) \otimes A, C) \cong C(H(A,C), H(A,C))$.

Definition 5.2.1 Let (C,H,P) be a semi-structured category with an Exponential Product. We define $m_{1,B}(A): A \rightarrow H(B, A \otimes B)$ as the morphism corresponding to the identity on $A \otimes B$.

Definition 5.2.2 Let (C,H,P) be a semi-structured category with an Exponential Product. We define $m_{0,A}(C): H(A,C) \otimes A \rightarrow C$ as the morphism corresponding to the identity on $H(A,C)$.

Definition 5.2.3 Let (C,H,P) be a semi-structured category with an Exponential Product $A \otimes B$; this product being commutative in the sense that there is a natural isomorphism $c(A,B): A \otimes B \rightarrow B \otimes A$ with the property that $c(A,B)\ c(B,A)$ is the identity morphism on $B \otimes A$ for all choices of objects A and B in C. We define the function $t(A,B)$: $PA \times PB \rightarrow P(A \otimes B)$ by $t(A,B)\ (a,b) = P\{P[H(A,c(B,A))\ m_{1,A}(B)]\ (b)\ \}\ (a)$.

THEOREM 5.2.1 If (C,H,P,K,i) is a structured category with an Exponential Product then $t(A,B): PA \times PB \to P(A \otimes B)$ is a bimorphism.

Proof: We will define $m_1 = H(A, c(B,A))\, m_{1,A}(B): B \to H(A, A \otimes B)$. If $\Phi_1 = Pm_1$ then it is immediate by the definition of $t(A,B)$ that $t(A,B)(a,b) = \{P(\Phi_1(b))\}(a)$, $a \in PA$, $b \in PB$.

In the argument below we use the morphism $H_{A \otimes B}: H(K,A) \to H\{H(A, A \otimes B), H(K, A \otimes B)\}$ of Definition 4.1.1 and the morphism $w: K \to A$ defined by $w = (Pi(A))(a)$ where a is an arbitrary element of PA. Let $m_2 = H\{H(A, c(B,A))\, m_{1,A}(B), i^{-1}(A \otimes B)\}H_{A \otimes B}i(A)$; if $\Phi_2 = Pm_2$ then it is clear that

$$\Phi_2(a) = C\{H(A, c(B,A))\, m_{1,A}(B), i^{-1}(A \otimes B)\}\, h_{A \otimes B}(w) \quad \text{as } P(qrs) = Pq\, Pr\, Ps$$
$$= \{i^{-1}(A \otimes B)\}\{h_{A \otimes B}(w)\}\, \{H(A, c(B,A))\, m_{1,A}(B)\} \quad \text{as } C(f,g)(\alpha) = g\alpha f$$

Hence

$$\begin{aligned}
(P\Phi_2(a))(b) &= P\{i^{-1}(A \otimes B)\}\, P\{h_{A \otimes B}(w)\}\, (\Phi_1(b)) && \text{by the definition of } \Phi_1 \\
&= P\{i^{-1}(A \otimes B)\}\, (\Phi_1(b)\, w) && \text{by the definition of } h_{A \otimes B} \\
&= P\{i^{-1}(A \otimes B)\}\, C(K, \Phi_1(b))(w) && \text{as } C(K,f)(g) = fg \\
&= P\{i^{-1}(A \otimes B)\}H(K, \Phi_1(b))\ (w) && \text{as } P(qr) = Pq\, Pr \\
&= P\{\Phi_1(b)\ i^{-1}(A)\}(w) && \text{by the naturality of } i \\
&= \{P\ \Phi_1(b)\}(a) && \text{by the definition of } w \\
&= t(A,B)(a,b).
\end{aligned}$$

Hence $t(A,B)(a,b): PA \times PB \to P(A \otimes B)$ is a bimorphism.

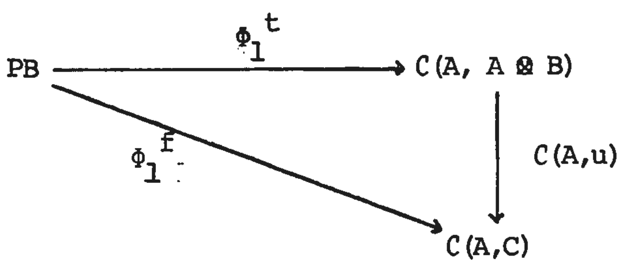
LEMMA 5.2 2 If (C,H,P) is a semi-structured category with an Exponential Product and $f: A \to A'$ and $g: C \to C'$ are morphisms between objects of C, then the morphisms $m_{1,B}(A): A \to H(B, A \otimes B)$

and $m_{0,A}(C): H(A,C) \otimes A \to C$ are natural in A and C respectively.

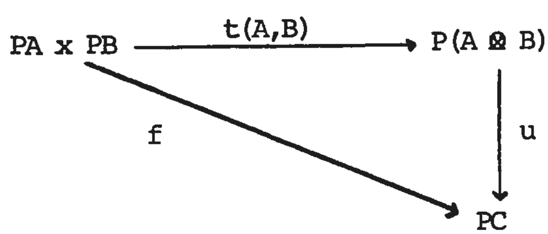
Proof: Immediate from (11, p.477, 3.3).

<u>LEMMA 5.2.3</u> Let $f: A \times B \to C$ be a bimorphism and $u: A \otimes B \to C$ be a morphism in the semi-structured category (C,H,P). If $\Phi_1^f: PB \to C(A,C)$ is the function associated with f by virtue of the definition of a bimorphism and $\Phi_1^t: PB \to C(A, A \otimes B)$ is the analogous morphism for t as defined in Theorem 5.2.1 then diagram (5.2.1) commutes if and only if diagram (5.2.2) commutes.

(5.2.1)



(5.2.2)



Proof: From the definitions involved we have

$P(\Phi_1^f(b))(a) = f(a,b)$ and $P(\Phi_1^t(b))(a) = t(A,B)(a,b)$.

First we assume diagram 5.2.1 commutes, i.e.

$C(A,u)\ P(\Phi_1^t(b))(a) = P(\Phi_1^f(b))(a)$.

L.H.S. $(C(A,u)\ P(\Phi_1^t(b)))(a) = P(H(A,u)\ \Phi_1^t(b))(a)$

$= P(u\ \Phi_1^t(b))(a)$

$= u\ t(A,B)(a,b)$

R.H.S. $(\Phi_1^f(b))(a) = f(a,b)$ and therefore diagram (5.2.2) commutes

The converse follows similarly.

THEOREM 5.2.4 If the structured category (C,H,P,K,i) has an Exponential Product that is commutative in the sense of Definition 5.2.3 then the Exponential Product is a Bimorphism Product.

Proof: We have shown in Lemma 5.2.1 that $t(A,B): A \times B \to A \otimes B$ is a bimorphism. We proceed to show that any bimorphism $f: A \times B \to C$ factors through $t(A,B)$ and the factoring morphism $u: P(A \otimes B) \to PC$ is uniquely determined.

For a bilinear map $f: PA \times PB \to PC$, we have $f(-,b) = P(\Phi_1^f(b)):$ $PA \to PC$ and there exists $m_1^f: B \to H(A,C)$ with $\Phi_1^f = P(m_1^f):$ $PB \to C(A,C)$.

If we assume that for a given bimorphism f, there exists a $u \in C$ with $f = (Pu)\ t(A,B)$ then it follows that $\Phi_1^f = C(A,u)\ \Phi_1^t:$ $PB \to C(A,C)$ and $m_1^f = H(A,u)\ m_1^t$ where $m_1^t: B \to H(A, A \otimes B)$.

Let $q = m_{0,A}(C)\ m_1^f \otimes A\ \ c(A,B)$. Since $m_1^f = H(A,u)\ m_1^t$ then $m_1^f \otimes A = H(A,u)\ m_1^t \otimes A$

$$= (H(A,u) \otimes A)(m_1^t \otimes A)$$

and $q = m_{0,A}(C)\ H(A,u) \otimes A\ \ m_1^t \otimes A\ \ c(A,B)$

But $m_{0,A}(C)\ H(A,u) \otimes A = u\ m_{0,A}(A \otimes B)$ and therefore

$q = u\ \ m_{0,A}(A \otimes B)\ \ m_1^t \otimes A\ \ c(A,B)$.

Also $m_1^t \otimes A = H(A,\ c(B,A)) \otimes A = m_{1,A}(B) \otimes A$ by definition of $t(A,B)$ and m_1^t.

Thus $q = u\ \ m_{0,A}(A \otimes B)\ \ H(A,\ c(B,A)) \otimes A\ \ m_{1,A}(B) \otimes A\ \ c(A,B)$ and since $m_{0,A}(A \otimes B)\ \ H(A,\ c(B,A)) \otimes A = c(B,A)\ \ m_{0,A}(B \otimes A)$ then

$q = u\ \ c(B,A)\ \ m_{0,A}(B \otimes A)\ \ m_{1,A}(B) \otimes A\ \ c(A,B)$

$\ \ = u\ \ c(B,A)\ \ c(A,B)$ by (11, p.478, 3.7) c is commutative and so $q = u$; hence if u exists then it is uniquely determined.

Conversely we must show that given
$u = m_{0,A}(C)\ m_1^{\ f} \otimes A\ \ c(A,B)$ then $f = Pu\ t(A,B)$.

It is enough to show $m_1^{\ f} = H(A,u)\ m_1^{\ t}$ since by applying the functor P we have $\Phi_1^{\ f} = C(A,u)\ \Phi_1^{\ t}$ and Lemma 5.2.3 will give the required result.

From (11, pp. 477-478) we obtain the following commutative diagrams:

(5.2.3)
$$\begin{array}{ccc} B & \xrightarrow{\ m_{1,A}(B)\ } & H(A,\ B \otimes A) \\ & \searrow^{\Phi_1^{\ f}} & \downarrow{\scriptstyle H(A,\ u\ c(B,A))} \\ & & H(A,C) \end{array}$$

(5.2.4)
$$\begin{array}{ccc} B \otimes A & \xrightarrow{\ m_1^{\ f} \otimes A\ } & H(A,C) \otimes A \\ & \searrow^{u\ c(B,A)} & \downarrow{\scriptstyle m_{0,A}(C)} \\ & & C \end{array}$$

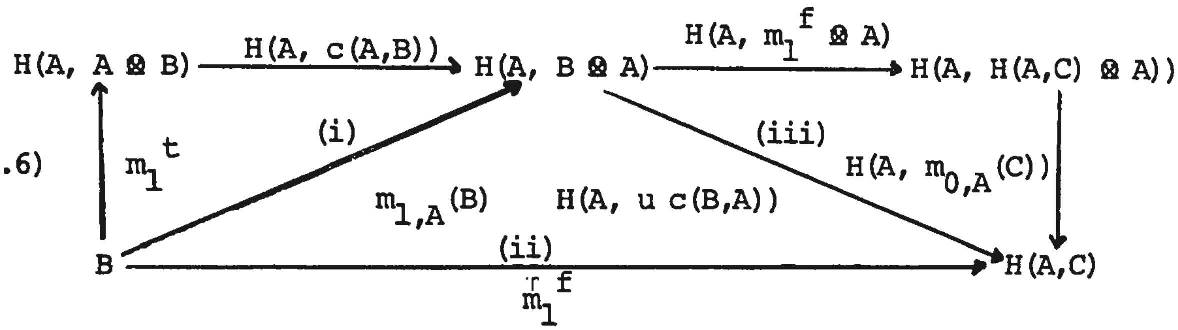
Next we must show that

(5.2.5)
$$\begin{array}{ccc} B & \xrightarrow{\ m_1^{\ t}\ } & H(A,\ A \otimes B) \\ & \searrow^{\Phi_1^{\ f}} & \downarrow{\scriptstyle H(A,u)} \\ & & H(A,C) \end{array}$$

commutes.

This amounts to showing that triangles (i), (ii) and (iii) of the following diagram commute:

(5.2.6)



Triangle (ii) commutes from diagram (5.2.3).

Triangle (iii) commutes from diagram (5.2.4).

To prove triangle (i) commutes we must prove $H(A,\ c(A,B))\ m_1^{\ t} = m_{1,A}(B)$.

Since $m_1^{\ t} = H(A,\ c(B,A))\ m_{1,A}(B)$ we have

$$
\begin{aligned}
H(A,\ c(A,B))\ m_1^{\ t} &= H(A,\ c(A,B))\ H(A,c(B,A))\ m_{1,A}(B) \\
&= H(A,\ c(A,B)\ c(B,A))\ m_{1,A}(B) \\
&= H(A,\ B \otimes A)\ m_{1,A}(B) \\
&= m_{1,A}(B)
\end{aligned}
$$

and hence diagram (5.2.5) commutes and the required result follows from Lemma 5.2.3 .

CHAPTER VI

THE IDENTITY, ASSOCIATIVITY AND COMMUTATIVITY ISOMORPHISMS

§6.1 Identities and Representability

We will assume in sections 6.1 to 6.3 that (C,H,P) is a semi-structured category with an Exponential Product $\otimes$. We will discuss the relationship which exists between properties which have been mentioned previously.

Definition 6.1.1 We will say that $L \in \text{ob } C$ is a left identity for $\otimes$ if there is a natural isomorphism $e: L \otimes A \to A$. In a similar way, $R \in \text{ob } C$ will be called a right identity for $\otimes$ if there is a natural isomorphism $r: A \otimes R \to A$.

THEOREM 6.1.1 If P is representable by means of an object $K \in \text{ob } C$ then K is a left identity for $\otimes$.

Proof: Given $PA \cong C(K,A)$, then
$$C(K \otimes A, B) \cong C(K, H(A,B)) \cong PH(A,B) \cong C(A,B)$$
and by Theorem 1.4.1 we conclude that $K \otimes A \cong A$.

THEOREM 6.1.2 K is a right identity with respect to $\otimes$ if and only if K is a left identity with respect to H, i.e., $A \otimes K \cong A$ if and only if $H(K,A) \cong A$.

Proof: We assume K is a right identity with respect to $\otimes$. Then
$$C(A,B) \cong C(A \otimes K, B) \cong C(A, H(K,B)).$$

Therefore by Theorem 1.4.1, $B \cong H(K,B)$.

Next we assume that K is a left identity with respect to H.

Then $C(A,B) \cong C(A, H(K,B))$

$\cong C(A \otimes K, B)$.

Therefore by Theorem 1.4.1, $A \cong A \otimes K$.

The question of whether or not H has a right identity is not valid since if we assume there exists such an identity J, then we reach a contradictory situation, i.e. $A \mapsto A$ is covariant whereas $A \mapsto H(A,J)$ is contravariant and obviously $A \not\cong H(A,J)$.

<u>THEOREM 6.1.3</u> If $\otimes$ has a right identity K then P is representable by means of the object K.

Proof: There is a natural isomorphism $r: B \otimes K \to B$ which belongs to $C(B \otimes K, B)$. This gives rise, by Theorem 6.1.2, to the natural isomorphism $v: B \to H(K,B)$ which belongs to $C(B, H(K,B))$. From the definition of an Exponential Product we have $C(B, H(K,B)) \cong C(B \otimes K,B)$. Therefore we have the following natural isomorphism where H is an internal Hom functor:

$$PB \cong PH(K,B)$$
$$\cong C(K,B).$$

Thus $PB \cong C(K,B)$ and P is representable.

<u>THEOREM 6.1.4</u> If $\otimes$ has a right identity K then K is a unique right identity and also a unique left identity.

Proof: K is also a left identity from Theorem 6.1.1 and 6.1.3.

If R is also a right identity then $R \cong K \otimes R$

$\cong K.$

Similarly, if L is also a left identity then $L \cong L \otimes K$

$\cong K.$

Hence the identities are unique.

The converse of Theorem 6.1.1 is false. To illustrate this we use the category $\mathcal{L}$ of finite simplicial complexes (17, p.28) understood as finite sets with distinguished (spanning) subsets, any subset of a spanning set spanning but not all the points in the complex necessarily belonging to spanning sets; and simplicial maps, understood as those set maps that take spanning sets to spanning sets. Let F be the forget functor to sets. Two different internal Hom functors on $\mathcal{L}$ are given by $M(A,B) = \mathcal{L}(A,B)$ as a set with the structure of a complex given by $f_1 = f_2 = \ldots\ldots = f_n$. This example has a left identity and even though it is associative the identity is not unique since every connected complex is a left identity for the tensor product.

The existence of a unique left identity even together with associativity does not imply the existence of a right identity. This is demonstrated by again using the example of the category $\mathcal{L}$ with the same set maps. However, in this case there is no condition given and all subsets span.

§6.2 Associativity and the Strong Exponential Product

Definition 6.2.1 If $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor then we say that $\otimes$ is <u>associative</u> if and only if there exists a natural isomorphism $a: (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

It is possible to define $\otimes$ as being an Exponential Product relative to the semi-structured category (C,H,P) in the stronger sense that there is a natural isomorphism $\alpha: H(A \otimes B, C) \cong H(A, H(B,C))$.

Definition 6.2.2 If $\otimes$ is an Exponential Product in this stronger sense we call it a <u>Strong Exponential Product;</u> then $P\alpha: C(A \otimes B, C) \cong C(A, H(B,C))$ and it is an Exponential Product in the usual sense.

<u>THEOREM 6.2.1</u> Let $\otimes$ be an Exponential Product relative to the semi-structured category (C,H,P). $\otimes$ is associative if and only if it is a Strong Exponential Product.

Proof: Given $a: (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. We have

$$C((A \otimes B) \otimes C, D) \cong C(A \otimes B, H(C,D)) \cong C(A, H(B, H(C,D))$$

and $C(A \otimes (B \otimes C), D) \cong C(A, H(B \otimes C, D))$. Since we are given the natural isomorphism a then $C((A \otimes B) \otimes C, D) \cong C(A \otimes (B \otimes C), D)$. Therefore there is a natural isomorphism $C(A, H(B, H(C,D)) \cong C(A, H(B \otimes C, D))$ and by Theorem 1.4.1 we have the required natural isomorphism, $H(B, H(C,D)) \cong H(B \otimes C, D)$.

To prove the converse we are given $H(B, H(C,D)) \cong H(B \otimes C, D)$; then from Theorem 1.4.1 $C(A, H(B, H(C,D)) \cong C(A \otimes B, H(C,D)) \cong C((A \otimes B) \otimes C, D)$ and $C(A, H(B \otimes C, D)) \cong C(A \otimes (B \otimes C), D)$ and again from Theorem 1.4.1 the result follows.

§6.3 Commutativity and the Exponential Product

Definition 6.3.1 A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is commutative if and only if there exists a natural isomorphism $c: A \otimes B \cong B \otimes A$.

THEOREM 6.3.1 Let $(\mathcal{C}, H, P)$ be a semi-structured category with an Exponential Product $\otimes$. Then $\otimes$ is commutative if and only if there exists a natural isomorphism $\mathcal{C}(A, H(B,C)) \cong \mathcal{C}(B, H(A,C))$.

Proof: If we assume $\otimes$ is commutative then

$$\begin{aligned} \mathcal{C}(A, H(B,C)) &\cong \mathcal{C}(A \otimes B, C) \\ &\cong \mathcal{C}(B \otimes A, C) \\ &\cong \mathcal{C}(B, H(A,C)). \end{aligned}$$

Proving the converse, we have

$$\begin{aligned} \mathcal{C}(A \otimes B, C) &\cong \mathcal{C}(A, H(B,C)) \\ &\cong \mathcal{C}(B, H(A,C)) \\ &\cong \mathcal{C}(B \otimes A, C). \end{aligned}$$

Therefore $A \otimes B \cong B \otimes A$ by Theorem 1.4.1.

§6.4 Properties of the Bimorphism Product

In this section we assume $(\mathcal{C}, H, P, K, i)$ is a structured category with a Bimorphism Product $T(A,B)$.

THEOREM 6.4.1 The Bimorphism Product is commutative.

Proof: We define a function $d(A,B): A \times B \to B \times A$; $(a,b) \mapsto (b,a)$.

Then consider the following diagram:

$$(6.4.1) \qquad \begin{array}{ccc} A \times B & \xrightarrow{d(A,B)} & B \times A \\ \downarrow{\scriptstyle t(A,B)} & & \downarrow{\scriptstyle t(B,A)} \\ T(A,B) & \xrightarrow{j(A,B)} & T(B,A) \end{array}$$

$t(B,A)\, d(A,B)$ is a bimorphism by Definition 4.2.2, not Lemma 4.2.2;

hence there exists morphism $j(A,B): T(A,B) \to T(B,A)$ such that $Pj(A,B)\ t(A,B) = t(B,A)\ d(A,B)$. In a similar way there exists a morphism $j(B,A): T(B,A) \to T(A,B)$ such that $Pj(B,A)\ t(B,A) = t(A,B)\ d(B,A)$. We have

$$\begin{aligned} Pj(B,A)\ Pj(A,B)\ t(A,B) &= Pj(B,A)\ t(B,A)\ d(A,B) \\ &= t(A,B)\ d(B,A)\ d(A,B) \\ &= t(A,B). \end{aligned}$$

Now by the uniqueness in the definition of a Bimorphism Product, $P(j(B,A)\ \ j(A,B)) = 1_{PT(A,B)}$. Since P is faithful, $j(B,A)\ j(A,B) = 1_{T(A,B)}$. Similarly $j(A,B)\ j(B,A) = 1_{T(B,A)}$ and therefore $j(A,B)$ is an isomorphism. The naturality of $j(A,B)$ is easily verified. Hence $j(A,B)$ is a natural isomorphism.

<u>THEOREM 6.4.2</u> There are left and right natural identiy isomorphisms $e: T(K,A) \to A$ and $r: T(A,K) \to A$.

Proof: In a structured category we have $A \cong H(K,A)$. Hence K is a right identity by Theorem 6.1.2 and the result follows by Theorem 6.1.4.

<u>THEOREM 6.4.3</u> The Bimorphism Product is associative.

(Note - This is proved on pp. 254-256 of (24); we give an outline of the method).

Proof: The concept of a bimorphism leads to the idea of a <u>trimorphism</u> $A \times B \times C \to D$ as a suitable function $PA \times PB \times PC \to PD$ where A, B, C and D are objects of C. $t(A,\ t(B,C)): PA \times PB \times PC \to PT(A,\ T(B,C))$ and $t(t(A,B),\ C): PA \times PB \times PC \to PT(T(A,B),\ C)$ are trimorphisms with the universal property that all trimorphisms on $A \times B \times C$ factor uniquely through them. The result follows by the theorem for trimorphisms analagous to our Proposition 4.3.2 for bimorphisms.

<u>COROLLARY 6.4.4</u> The Bimorphism Product is a Strong Exponential Product.

Proof: This is immediate from Theorems 6.2.1 and 6.4.3 .

If (C,H,P,K,i) is a structured category with a Bimorphism Product T(A,B), then we have shown in Theorems 6.4.1, 6.4.2 and 6.4.3 that natural isomorphisms c, e, r and a are defined. If we assume that the corresponding diagram (3.2.4) commutes then we may show that c, e, r and a are coherent; for (3.2.1) commutes by the first statement after Theorem 11 in (17), (3.2.2) by an easy extension of Theorem (6.4.1) and (3.2.3) by an argument that defines trimorphisms $A \times B \times C \to T(A, T(B,C))$ and $A \times B \times C \to T(T(C,A), B)$ and shows that the two paths in (3.2.3) are the unique morphism by which the second trimorphism factors uniquely through the first. The details of this argument will not be given in this paper but will be given elsewhere. It follows that C, with the associated structures, is a <u>symmetric monoidal closed category</u> in the sense of (11).

CHAPTER VII

EXPONENTIAL PRODUCTS AND ADJOINT PRODUCTS

The concepts of the Exponential Product and the Adjoint Product were defined in Chapter II. The Exponential Product is often described in the literature as a tensor product defined by adjointness. We discuss the Adjoint Product to illustrate that this last statement is slightly misleading. Clearly any Exponential Product is an Adjoint Product; we show by a counter-example that even in a structured category these two concepts are distinct.

Example 7.1 If we have the ring of integers, $(Z,+,x)$, then we consider a category C with one object Z and morphisms, the group-endomorphisms of $(Z,+)$. Composition in C is simply the usual composition. We define $P(Z,+) = Z$ and if f is an endomorphism of $(Z,+)$ then $Pf\colon Z \to Z$ is the underlying function.

We have bifunctors:

(a) $\otimes\colon C \times C \to C$ such that $Z \otimes Z = Z$ and $f \otimes g = f\,g$.

(b) $H\colon C^* \times C \to C$ such that $H(Z,Z) = Z$ and $H(f,g) = f\,g$

The natural isomorphism $\alpha\colon C(Z \otimes Z, Z) \cong C(Z, H(Z,Z))$ with $\alpha(f) = f$ implies $\otimes$ is an Exponential Product. We define $K = (Z,+)$ and $i_K = H_K = {}_K H$ as the identity endomorphism of $(Z,+)$. (C,H,P,K,i) is then a structured category. We also define the functor $\Theta\colon C \to C$ such that if f_i is a morphism $Z \to Z$; $f_i(x) = ix$, $x \in Z$ then $\Theta(f_i) = f_i 2\colon Z \to Z$ and $\Theta(f_i\, f_j) = \Theta(f_{ij}) = f_{(ij)}2 = f_i 2_j 2 = f_i 2_j 2 = \Theta(f_i)\,\Theta(f_j)$.

Then we require the bijection

$\beta\colon C(Z \otimes \Theta(Z), Z) = C(Z \otimes Z, Z)$ with $\beta(f) = f$

and define $\gamma = \alpha\,\beta\colon C(Z \otimes \Theta(Z),\, Z) \to C(Z,\, H(Z,Z))$.

α, β and therefore γ are natural in the first and third variables; α is also natural in the second variable; therefore γ is natural in the second variable if and only if β is natural in the second variable.

Consider the following diagram:

$$\text{(7.1a)}\qquad \begin{array}{ccc} Z \otimes Z & \xrightarrow{\;Z \otimes \Theta\;} & Z \otimes \Theta(Z) \\ \Big\downarrow{\scriptstyle 1 \otimes f_2} & & \Big\downarrow{\scriptstyle 1 \otimes \Theta(f_2) \,=\, 1 \otimes f_4} \\ Z \otimes Z & \xrightarrow{\;Z \otimes \Theta\;} & Z \otimes \Theta(Z) \end{array} \qquad\qquad \text{(7.1b)}\qquad \begin{array}{ccc} 1 & \longmapsto & 1 \\ \Big\downarrow & & \Big\downarrow \\ 2 & \longmapsto & 2 \neq 4 \end{array}$$

This diagram obviously does not commute and therefore the following induced diagram is not commutative:

$$\text{(7.2)}\qquad \begin{array}{ccc} C(Z \otimes \Theta(Z),\, Z) & \xrightarrow{\quad\beta\quad} & C(Z \otimes Z,\, Z) \\ \Big\downarrow{\scriptstyle C(Z \otimes \Theta(f_2),\, Z)} & & \Big\downarrow{\scriptstyle C(Z \otimes f_2,\, Z)} \\ C(Z \otimes \Theta(Z),\, Z) & \xrightarrow{\quad\beta\quad} & C(Z \otimes Z,\, Z) \end{array}$$

Therefore β is not natural in the second variable and hence γ is not natural in the second variable. Hence the bifunctor $C \times C \to C;\ (Z,Z) \mapsto Z \otimes Z = Z;\ (f,g) \mapsto f \otimes \Theta(g)$ is an Adjoint Product and not an Exponential Product.

Under suitable conditions we can ensure that the naturality of the isomorphism α in the first and third variables is equivalent to naturality in all three variables.

<u>THEOREM 7.1</u> If (C,H,P) is a semi-structured category with a commutative Adjoint Product $\otimes$, then the Adjoint Product is an Exponential Product if there exists a natural isomorphism $\Psi_{A,B,C}: C(A, H(B,C)) \rightarrow C(B, H(A,C))$ natural in all three variables such that the following diagram commutes:

$$
(7.3) \qquad
\begin{array}{ccc}
C(A \otimes B, C) & \xrightarrow{\alpha_{A,B,C}} & C(A,H(B,C)) \\
\downarrow{\scriptstyle C(c,\, C)} & & \downarrow{\scriptstyle \Psi_{A,B,C}} \\
C(B \otimes A, C) & \xrightarrow{\alpha_{B,A,C}} & C(B, H(A,C))
\end{array}
$$

Proof: Since $C(c, C)$ is natural in A, B and C, $\alpha_{B,A,C}$ is natural in B and C, $\Psi_{A,B,C}$ is natural in A, B and C and the diagram is commutative then $\alpha_{A,B,C}$ must be natural in all three variables and the theorem is proved.

CHAPTER VIII

EXAMPLES

In this chapter we have chosen different categories and illustrated the theory of the preceding chapters by the following examples in which the Adjoint Product, the Exponential Product, the Strong Exponential Product and the Bimorphism Product coincide and are simply referred to as the tensor product. In each example we shall define and explain the following:

(a) the functor $H: \mathcal{C}^* \times \mathcal{C} \to \mathcal{C}$

(b) the functor $P: \mathcal{C} \to S$

(c) the distinguished object $K \in \mathcal{C}$

(d) the natural isomorphism i

(e) the mappings h_C, H_C, ${}_Ch$, ${}_CH$

(f) bimorphism $f: A \times B \to C$

(g) the tensor product

(h) the exponential law and the strong exponential law, i.e. the natural isomorphisms that define the Exponential Product and the Strong Exponential Product.

1. Let M be the category of R-modules where R is a commutative ring. Let $A, B, C \in M$.

(a) $H(A,B) = \mathrm{Hom}(A,B)$, i.e. the set $M(A,B)$ of homorphisms of A into B regarded as an R-module in the usual way

(b) $P: M \to S$ is the forgetful functor (or underlying set functor).

(c) K is the ring R regarded as an R-module.

(d) $i(A): A \cong \mathrm{Hom}(R,A)$ such that $\{i(A)(a)\}(r) = ra$, $a \varepsilon R$, $r \varepsilon R$.

(e) $h_C: M(A,B) \to M(\mathrm{Hom}(B,C), \mathrm{Hom}(A,C))$; $h_C(f)(g) = g\, f$ where $f \varepsilon M(A,B)$, $g \varepsilon \mathrm{Hom}(B,C)$.

$H_C: \mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathrm{Hom}(B,C), \mathrm{Hom}(A,C))$. $H_C(f) \cong h_C(f)$, $f \varepsilon \mathrm{Hom}(A,B)$.

${}_Ch: M(A,B) \to M(\mathrm{Hom}(C,A), \mathrm{Hom}(C,B))$; ${}_Ch(f)(g) = f\, g$ where $f \varepsilon M(A,B)$, $g \varepsilon \mathrm{Hom}(C,A)$.

${}_CH: \mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathrm{Hom}(C,A), \mathrm{Hom}(C,B))$. ${}_CH(f) \cong {}_Ch(f)$, $f \varepsilon \mathrm{Hom}(A,B)$.

(f) A bimorphism $f: A \times B \to C$ is a function such that if $m_1: B \to \mathrm{Hom}(A,C)$, $m_2: A \to \mathrm{Hom}(B,C)$, then $m_1(b)(a) = f(a,b)$ and $m_2(a)(b) = f(a,b)$, i.e. a bimorphism is simply a bilinear function.

(g) $T(A,B) = A \otimes B$ and $t(A,B): A \times B \to A \otimes B$; $(a,b) \mapsto a \otimes b$ where $t(A,B)$ is the function on the underlying set.

(h) Exponential law $\alpha: M(A \otimes B, C) \cong M(A, \mathrm{Hom}(B,C))$; $\{\alpha(f)(a)\}(b) = f(a \otimes b)$, $f \varepsilon M(A \otimes B, C)$, $a \varepsilon A$, $b \varepsilon B$.

Strong exponential law $\beta: \mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C))$; $\beta(f) = \alpha(f)$.

2. Let S be the category of sets and $A, B, C \varepsilon S$.

(a) $H(A,B) = S(A,B)$, i.e. the set of all functions $A \to B$.

(b) $P = 1: S \to S$.

(c) $K = \{x\}$, a singleton set.

(d) $i(A): A \stackrel{\sim}{=} S(\{x\}, A)$ such that $i(A)\{(a)\{x\}\} = a, \ a \in A$.

(e) $h_C = H_C: S(A,B) \to S(S(B,C), S(A,C))$; $H_C(f)(g) = g\,f$, $f \in S(A,B)$, $g \in S(B,C)$.

${}_Ch = {}_CH: S(A,B) \to S(S(C,A), S(C,B))$: ${}_CH(f)(g) = f\,g$, $f \in S(A,B)$, $g \in S(C,A)$.

(f) A bimorphism $f: A \times B \to C$ is any function such that $m_1: B \to S(A,C)$, $m_2: A \to S(B,C)$; $m_1(b)(a) = f(a,b)$, $m_2(a)(b) = f(a,b)$ are functions, i.e. a bimorphism is any function $f: A \times B \to C$.

(g) $T(A,B)$ is the Cartesian product $A \times B$ and $t(A,B)$: $A \times B \to A \times B$ is the identity function.

(h) The exponential law and strong exponential law coincide as $\alpha = \beta: S(A \times B, C) \to S(A, S(B,C))$.

3. Let S_* be the category of sets with base points and let $A, B, C \in S_*$.

(a) $H(A,B) = S_*(A,B)$, i.e. the set of base point preserving functions $A \to B$.

(b) $P: S_* \to S$; $(X,*) \mapsto X$.

(c) $K = \{\{x,*\}, *\}$.

(d) $i(A): A \stackrel{\sim}{=} S_*(K,A)$ such that $i(A)\{(a)(*)\} = *$; $i(A)\{(a)(x)\} = a$.

(e) $h_C: S(A,B) \to S(S_*(B,C), S_*(A,C))$; $h_C(f)(g) = g\,f$, $f \in S(A,B)$, $g \in S_*(B,C)$.

$H_C: S_*(A,B) \to S_*(S_*(B,C), S_*(A,C))$; $H_C(f)(g) = g\,f$, $f \in S_*(A,B)$, $g \in S_*(B,C)$.

$_C h: S(A,B) \to S(S_*(C,A), S_*(C,B))$; $_C h(f)(g) = f g$,

$f \in S(A,B)$, $g \in S_*(C,A)$.

$_C H: S_*(A,B) \to S_*(S_*(C,A), S_*(C,B))$; $_C H(f)(g) = f g$,

$f \in S_*(A,B)$, $g \in S_*(C,A)$.

(f) $f: A \times B \to C$ is a bimorphism means that f is a function of $A \times B$ into C such that $f(a,*) = *$, $f(*,b) = *$.

(g) $T(A,B) = \dfrac{A \times B}{A \times * \cup * \times B} = A \# B$ (the smashed product of A and B) and $t(A,B): A \times B \to A \# B$ is simply the bimorphism $(a,b) \mapsto [(a,b)]$.

(h) Exponential law $\alpha: S(A \# B, C) \cong S(A, S_*(B,C))$ where $\alpha(f)(a)(b) = f[(a,b)]$, $a \in A$, $b \in B$, $f \in S(A \# B, C)$.

Strong exponential law $\beta: S_*(A \# B, C) \cong S_*(A, S_*(B,C))$ where $\beta(f)(a)(b) = f[(a,b)]$, $a \in A$, $b \in B$, $f \in S_*(A \# B, C)$.

4. Let $\mathcal{U}$ be the category whose objects are the subsets of a given set U and whose morphisms are all inclusion functions. Hence $\mathcal{U}(A,B)$ is a set of one element if $A \to B$ and is an empty set otherwise.

(a) $H(A,B) = A' \cup B$ where A' is the complement of A.

(b) $PA = \mathcal{U}(U,A) = \begin{cases} \emptyset & \text{if } A \neq U \\ \text{the singleton set whose element is the inclusion } U \subset U, & \text{if } A = U. \end{cases}$

(c) $K = U$.

(d) $i(A): A \cong H(K,A)$ is simply the identity function on A since $H(K,A) = U' \cup A = \emptyset \cup A = A$.

(e) $h_C: \mathcal{U}(A,B) \to \mathcal{U}(H(B,C), H(A,C))$. If $\mathcal{U}(A,B) = \emptyset$ then h_C is simply an empty function. If $\mathcal{U}(A,B)$ has one element then h_C must take this one element to the one element of $\mathcal{U}(H(B,C), H(A,C))$. For this to make sense we require that $\mathcal{U}(A,B)$ has one element shall imply that $\mathcal{U}(H(B,C), H(A,C))$ has one element, i.e. $A \subseteq B$ must imply $B' \cup C \subseteq A' \cup C$, which it clearly does.

$H_C: H(A,B) \to H(H(B,C), H(A,C))$. H_C is a morphism in the category of subsets of U, i.e. it is the inclusion $H(A,B) \subseteq H(H(B,C), H(A,C))$ or $A' \cup B \subseteq (B' \cup C)' \cup (A' \cup C)$.

This is obviously true since

$$\begin{aligned}
&(B' \cup C)' \cup (A' \cup C) \\
&= (B \cap C') \cup (A' \cup C) \\
&= (B \cup (A' \cup C)) \cap (C' \cup (A' \cup C)) \\
&= (B \cup A' \cup C) \cap U \\
&= A' \cup B \cup C \\
&\supseteq A' \cup B.
\end{aligned}$$

${}_C h: \mathcal{U}(A,B) \to \mathcal{U}(H(C,A), H(C,B))$. If $\mathcal{U}(A,B) = \emptyset$ then ${}_C h$ is simply an empty function. If $\mathcal{U}(A,B)$ has one element then ${}_C h$ must take this to the one element of $\mathcal{U}(H(C,A), H(C,B))$.

${}_C H$ is the inclusion $H(A,B) \to H(H(C,A), H(C,B))$.

(f) Let A, B, C be subsets of U satisfying the condition that if $A = B = U$ then $C = U$. Then f is a bimorphism means that if $A = B = C = U$ then f takes the one element of $PA \times PB$, i.e. $\{\lambda\} \times \{\lambda\}$ where $\lambda: U \to U$ is the identity function, over to $\lambda \in PC$. Otherwise $PA \times PB = \emptyset$ and f is an empty function $\emptyset \to PC$.

(g) $T(A,B) = A \cap B$. $t(A,B): PA \times PB \to P(A \cap B)$. If $A = B = U$ then $t(A,B): (\lambda,\lambda) \mapsto \lambda$. Otherwise $t(A,B): \emptyset \mapsto \emptyset$.

(h) Exponential law $\alpha: \mathcal{U}(A \cap B, C) \cong \mathcal{U}(A, B' \cup C)$, i.e. the condition that $A \cap B \subseteq C$ if and only if $A \subseteq B' \cup C$.

Strong exponential law $\beta: (A \cap B)' \cup C \cong A' \cup (B' \cup C)$.

5. Let B and C be topological spaces, map(B,C) will denote the set of maps $B \to C$ and Map(B,C) will denote the set, map(B,C) topologised with the compact open topology. Define the function α: map(A x B, C) $\to$ map(A, Map(B,C)) as $\alpha(f)(a)(b) = f(a,b)$ where $f \in$ map(A x B, C), $a \in A$, $b \in B$. α is a natural bijection if either B is locally compact and Hausdorff or if A and B both satisfy the first axiom of countability (15). It is shown in (25, Lemma 5.5) that there exist spaces such that α is not a homeomorphism. Hence the category of all spaces with H(B,C) = Map(B,C) and T(B,C) = B x C does not have a satisfactory exponential law for the application of the preceding theory.

It is proved in (6, p.240) that the strong exponential law is satisfied if we take the category of Hausdorff spaces and define H(B,C) = Map(B,C) and T(B,C) = $B \times_s C$ where $B \times_s C$ is the set B x C with a certain weak topology (7, p.309). The product $B \times_s C$ is not commutative, i.e. $B \times_s C$ and $C \times_s B$ are not in general homeomorphic, ($C \times_s B$ is homeomorphic to the weak product $C \times_{s*} B$ and it is shown in (7, p.315) that these products are distinct). Hence these last definitions of H(B,C) and T(B,C) do not make the category of Hausdorff spaces into a category in which our full theory can be applied.

There are however, certain categories that are closely related to the ususal category of spaces and maps in which our full theory can be applied, e.g. the category of compactly generated spaces (26) and the category of quasi-topological spaces described in (25). We will assume that we are working in one of these categories which we call C.

In category C, the product space $B \times C$ has the Cartesian product of the sets B and C as its underlying set; map(B,C) is the set of maps of B into C and Map(B,C) is the set, map(B,C) topologised (or quasi-topologised) in a suitable fashion such that there is a natural bijection, $\mathrm{map}(A \times B, C) \to \mathrm{map}(A, \mathrm{Map}(B,C))$.

(a) $H(B,C) \cong \mathrm{Map}(B,C)$.

(b) $P: C \to S$ is the forgetful functor.

(c) K is a singleton space.

(d) $i(A): A \cong \mathrm{Map}(*,A)$ is the continuous map defined by $i(A)(a)(*) = a$.

(e) $h_C: \mathrm{map}(A,B) \to \mathrm{map}(\mathrm{Map}(B,C), \mathrm{Map}(A,C))$ where $h_C(f)(g) = g\,f$ for $f \in \mathrm{map}(A,B)$, $g \in \mathrm{Map}(B,C)$.

H_C is simply the function h_C made into a continuous map by giving its domain and codomain the Map topology.

${}_Ch: \mathrm{map}(A,B) \to \mathrm{map}(\mathrm{Map}(C,A), \mathrm{Map}(C,B))$ where ${}_Ch(f)(g) = f\,g$ for $f \in \mathrm{map}(A,B)$, $g \in \mathrm{Map}(C,A)$.

${}_CH$ is simply the function ${}_Ch$ made into a continuous map by giving its domain and codomain the Map topology.

(f) bimorphism $f: A \times B \to C$ is easily seen to be simply a map $A \times B \to C$.

(g) $T(A,B) = A \times B$ and $t(A,B): A \times B \to T(A,B)$ is simply the identity map.

(h) the exponential law is given above; the strong exponential law asserts the existence of a natural homeomorphism $\mathrm{Map}(A \times B, C) \to \mathrm{Map}(A, \mathrm{Map}(B,C))$.

A well known example in topology is the following: if R is the real line with the usual topology then the function $\beta: R \times R \to R$, defined by $(x,y) = \dfrac{xy}{x^2+y^2}$ is continuous in each variable but is not itself continuous. R is a k - space but this " bicontinuous " function is clearly not a bimorphism in our sense.

6. Let C be a category of quasi - topological spaces with base points and base point preserving quasi - continuous maps in the sense of the latter part of the above example. Then take $H(A,B)$ as the space of base point preserving continuous maps of $A \to B$ and $T(A,B)$ is the smashed product of A and B. Results analoguous to those in the category of sets with base points may be obtained.

7. Let C be the category whose objects are the elements of a chain (sometimes called a simply ordered set or a linearly ordered set) with a greatest element 1, and whose morphisms are the valid $\leqq$, i.e. if a and b are objects and $a \leqq b$ then there is a unique morphism $a \to b$.

Let $a, b, c \in C$. $\min(a,b) \leqq c \iff a \leqq b \pitchfork c = \begin{cases} c \text{ if } b > c. \\ 1 \text{ if } b \leqq c. \end{cases}$

$$\text{i.e. } C(\min(a,b), c) \cong C(a, b \pitchfork c).$$

Therefore $C(a,b)$ is a set of one element if $b \leqq c$ and $\emptyset$ otherwise.

(a) $H(b,c) = b \pitchfork c$.

(b) $Pa = C(1,a) = \begin{cases} \emptyset \text{ if } a \neq 1 \\ \text{the singleton set whose element is the} \\ \text{inequality } 1 \leqq 1 \text{ if } a = 1. \end{cases}$

(c) $K = 1$.

(d) $i(A)$ is the equality $a = a$.

(e) h_C is an empty function if $a > b$ since $C(a,b) = \emptyset$. If $a \leqq b$ then h_C takes the inequality $a \leqq b$ to the inequality $b \pitchfork c \leqq a \pitchfork c$.

H_C is the inequality $a \pitchfork b \leqq (b \pitchfork c) \pitchfork (a \pitchfork c)$.

${}_C h$ is an empty function if $a > b$ since $C(a,b) = \emptyset$. If $a \leqq b$ then ${}_C h$ takes the inequality $a \leqq b$ to the inequality $c \pitchfork a \leqq c \pitchfork b$.

${}_C H$ is the inequality $a \pitchfork b \leqq (c \pitchfork a) \pitchfork (c \pitchfork b)$.

(f) If $\min(a,b) \leqq c$ then the unique function $Pa \times Pb \to Pc$ is a bimorphism; otherwise there are no such bimorphisms.

(g) $T(a,b) = \min(a,b)$.

(h) Exponential law α: $\min(a,b) \leqq c$ if and only if $a \leqq b \pitchfork c$.

Strong exponential law β: $\min(a,b) \pitchfork c = a \pitchfork (b \pitchfork c)$.

8. Let $(A,+)$ be an Abelian group with the structure of a pre-ordered set such that $x \leqq y$ implies $x + z \leqq y + z$, $x, y, z \in A$, e.g. the additive group of all real numbers or the multiplicative group of all positive reals.

Let $\mathcal{A}$ be the category whose objects are the elements of the set A and whose morphisms are the valid $\leqq$. Let $a, b, c \in A$.

If we define $T(a,b) = a + b$ and $H(a,b) = -a + b$ then we have the following explanation of (a) - (h):

(a) $H(a,b) = -a + b$.

(b) $Pa = A(0,a)$.

(c) $K = 0$.

(d) $i(A)$ is the equality $a = a$.

(e) h_C is an empty map if $a > b$. If $a \leqq b$ then h_C takes this inequality to the inequality $-b + c \leqq -a + c$.

H_C is the inequality $-a + b \leqq -(-b + c) + (-a + c)$.

${}_Ch$ is an empty map if $a > b$. If $a \leqq b$ then ${}_Ch$ takes this inequality over to the inequality $-c + a \leqq -c + b$.

${}_CH$ is the inequality $-a + b \leqq -(-c + a) + (-c + b)$.

(f) If $a + b \leqq c$ then the unique function $Pa \times Pb \to Pc$ is a bimorphism, otherwise there are no such bimorphisms.

(g) $T(a,b) = a + b$. $t(a,b): Pa \times Pb \to P(a + b)$.

(h) Exponential law. α: $A(a + b, c) = A(a, -b + c)$, i.e. the assertion that $a + b \leqq c$ if and only if $a \leqq -b + c$.

Strong exponential law β: $-(a + b) + c = -a + (-b + c)$.

APPENDIX

In Examples 2 and 5 of Chapter VIII, the concepts of morphism and bimorphism coincide. Our aim in this appendix is to clarify this relation and give a necessary and sufficient set of conditions for this relation to hold.

The concept of a Cartesian closed category was introduced in (11); it has been pointed out by Lawvere that a category C is Cartesian closed if and only if it has finite direct products (the product of A and B will be written $A \times B$); $A \times B$ is an Exponential Product for C and C has a terminal object.

In a Cartesian closed category we may compare the concepts of a morphism $h: A \times B \to C$ and a bimorphism $A \times B \to C$, i.e. a function $f: PA \times PB \to PC$. We cannot ask whether $Ph = f$ since $Ph: P(A \times B) \to PC$, but if we have a natural isomorphism $\Theta_{A,B}: PA \times PB \to P(A \times B)$ we can ask whether $f = Ph\, \Theta_{A,B}$.

It is now clear that morphism and bimorphism coincide in Examples 4 and 7 as well as 2 and 5.

<u>THEOREM A.1</u> If

(i) (C,H,P) is a semi-structured category with P a faithful functor;

(ii) C has finite direct products (we write $A \times B$ for the direct product of the objects A and B) such that there is a natural isomorphism

$$d = d(A,B): A \times B \rightarrow B \times A$$

and a natural bijection

$$\Theta = \Theta_{A,B}: PA \times PB \rightarrow P(A \times B)$$

such that the composite function

$$PA \times PB \xrightarrow[\Theta_{A,B}]{} P(A \times B) \xrightarrow[Pd(A,B)]{} P(B \times A) \xrightarrow[\Theta^{-1}_{A,B}]{} PB \times PA$$

is simply the function $(a,b) \mapsto (b,a)$, $a \in PA$, $b \in PB$;

and (iii) there is a natural bijection

$$\alpha = \alpha_{A,B,C}: C(A \times B, C) \rightarrow C(A, H(B,C))$$

defined by $((Pf)\ \Theta_{A,B})(a,b) = P\{(P\ \alpha(f))(a)\}(b)$ for all $a \in PA$, $b \in PB$ and $c \in PC$;

then if $g: PA \times PB \rightarrow PC$ is a function then g is a bimorphism if and only if $g = (Pf)\ \Theta_{A,B}$ where $f \in C(A \times B, C)$.

Proof: We first assume that $g = (Pf)\,\Theta_{A,B}$ where $f \in C(A \times B, C)$ and prove that g satisfies the two conditions that are required for it to be a bimorphism.

If $f: A \times B \to C$ and $\underline{d}(B,A): B \times A \to A \times B$ then we define $m_1 = \alpha(f\,\underline{d}(B,A)): B \to H(A,C)$. Then

$$
\begin{aligned}
g(a,b) &= (Pf)\,\Theta_{A,B}(a,b) \\
&= (Pf)(P\underline{d}(B,A))\,\Theta_{B,A}(b,a) && \text{from assumption (v)} \\
&= P(f\,\underline{d}(B,A))\,\Theta_{B,A}(b,a) && \text{from assumption that } P \text{ is a functor} \\
&= P\{P\alpha(f\,\underline{d}(B,A))(b)\}(a) && \text{from assumption (iv)} \\
&= P\{Pm_1(b)\}(a).
\end{aligned}
$$

Also if we define $m_2 = \alpha(f): A \to H(B,C)$ then

$$
\begin{aligned}
g(a,b) &= (Pf)\,\Theta_{A,B}(a,b) \\
&= P\{P\alpha(f)(a)\}(b) && \text{from assumption (iv)} \\
&= P\{Pm_2(a)\}(b).
\end{aligned}
$$

Hence g is a bimorphism.

Next we assume $g: PA \times PB \to PC$ is a bimorphism. Then there exists $m_2: A \to H(B,C)$ such that $g(a,b) = P\{Pm_2(a)\}(b)$ and we define $f: A \times B \to C$ such that $f = \alpha^{-1}(m_2)$. Then

$$
\begin{aligned}
g(a,b) &= P\{Pm_2(a)\}(b) \\
&= (P\,\alpha^{-1}(m_2))\,\Theta_{A,B}(a,b) && \text{from assumption (iv)} \\
&= (Pf)\,\Theta_{A,B} && \text{since } f = \alpha^{-1}(m_2).
\end{aligned}
$$

BIBLIOGRAPHY

1. BÉNABOU, J., Catégories avec Multiplication. C.R. Acad. Sci. Paris 256 (1963), 1887-1890.

2. BÉNABOU, J., Algèbre Élémentaire dans les Catégories avec Multiplication. C.R. Acad. Sci. Paris 258 (1964), 771-774.

3. BÉNABOU, J., Catégories Relatives. C.R. Acad. Sci. Paris 260 (1965), 3824-3827.

4. MACLANE, S. and BIRKHOFF, G., Algebra. rev. ed. The MacMillan Co., New York, 1967.

5. BOOTH, P.I., Distributive Laws in Categories. To Be Published.

6. BROWN, R., Function Spaces and Product Topologies. Quart.J. Math. Oxford (2), 15 (1964), 237-250.

7. BROWN, R., Ten Topologies for X x Y. Quart. J. Math. Oxford (2), 14 (1963), 303-319.

8. BUNGE, M.C., Relative Functor Categories and Categories of Algebras. J. Algebra 11 (1969), 64-101.

9. EHRESMAN, C., Catégories et Structures. Dunod, Paris, 1965.

10. EILENBERG, S. and KELLY, G.M., A Generalization of the Functorial Calculus. J. Algebra 3 (1966), 366-375.

11. EILENBERG, S. and KELLY, G.M., Closed Categories. Proceedings of the Conference on Categorical Algebra - La Jolla, 1965. Springer, Berlin-Heidelberg-New York, 1966.

12. EPSTEIN, D.B.A., Functors Between Tensored Categories. Invent. Math. 1, (1966) 221-228.

13. FISHER, J.L., The Tensor Product of Functors, Satellites and Derived Functors. J. Algebra 8 (1968) 277-294.

14. FOLTZ, F., Produit Tensoriel Généralisé. C.R. Acad. Sci. Paris Ser. A - B 264 (1067), A660-A663.

15. FOX, R.H., On Topologies for Function Spaces. Bull Amer. Math. Soc. 51 (1945) 429-432.

16. FREYD, P., Abelian Categories. Harper and Row, New York, 1964.

17. KELLY, G.M., Tensor Products in Categories. J. Algebra 2 (1965), 15-37.

18. KELLY, G.M., On MacLane's Conditions for Coherence of Natural Associativities, Commutatives, etc. J. Algebra 1 (1964), 397-402.

19. LINTON, F.E.J., Autonomous Categories and Duality of Functors. J. Algebra 2 (1965), 315-349.

20. MACLANE, S., Categorical Algebra. Bull. Amer. Soc. 71 (1965), 40-106.

21. MACLANE, S., Natural Associativity and Commutativity. Rice University Studies 49 (1963), 28-46.

22. MACLANE, S., Possible Programs for Categorists. Category Theory, Homology Theory and their Applications 1. Battelle Institute Conference (1968), Vol. 1.

23. MITCHELL, B., Theory of Categories. Academic Press, New York, 1965.

24. PUMPLÜN, D., Das Tensor Produkt als Universelles Problem. Math. Ann. 171 (1967), 247-262.

25. SPANIER, E., Quasi - Topologies. Duke Math. J. 30 (1963) 1-14.

26. STEENROD, N., A Convenient Category of Topological Spaces. Mich. Math. J. 14 (1967), 133-152.