NON-EXPANSIVE MAPPINGS AND FIXED POINTS IN METRIC SPACES

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by

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ABSTRACT

The main objective of this thesis is to investigate fixed points under non-expansive and some other types of mappings in metric spaces. During the course of this dissertation, several new results have been obtained. In order to illustrate some of the theorems proved, a few interesting examples have been constructed.

Chapter I includes a brief survey of fixed point theorems for non-expansive mappings. In the end a few theorems, which seem to be new, have been added.

In Chapter II, a mapping considered by Kannan [20]has been observed and by introducing some more general forms of this mapping, a few interesting results on fixed points have been investigated. Furthermore, giving a brief account of multi-valued contraction mappings and their fixed points, a search for extending some fixed point theorems to their multivalued analogues has been made.

Chapter III deals with the convergence of sequence of mappings (contraction and Kannan type) and their fixed points. A few interesting generalizations of some known results have been investigated.

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INTRODUCTION

S. Banach [2], in 1922, formulated a simple but quite striking result known as Banach contraction principle which states:

"A contraction mapping of a complete metric space into itself has a unique fixed point".

This theorem has been extensively used in proving the existence and uniqueness of solutions to various functional equations particularly differential and integral equations. Because of its simplicity and usefulness, Banach contraction theorem has been generalized by several mathematicians such as Chu & Diaz [12,13], Edelstein [14,15], Rakotch [31], Bailey [1], Boyd & Wong [6], Browder [7], Sehgal [34] and others.

A mapping, called "Nonexpansive mapping" which is more general in nature than a contraction mapping has been studied by Cheney & Goldstein [11], Edelstein [16], Belluce & Kirk [3,4] Kirk [4 ,23], KiWang Ng [29] and others. They have tried to obtain fixed points for such mappings in metric spaces. Browder [8, 9], Kirk [24], Edelstein [17] and others have considered nonexpansive mappings in Banach spaces and have concluded the existence of fixed points.

In Chapter 1, we have given a brief survey of the fixed point theorems proven for contraction and nonexpansive mappings in metric spaces

In the later portion of the chapter we have given a generalization of a theorem of . . . [4] for the fixed point of a continuous and assymptotically regular mapping. Also we have proved two more new theorems giving fixed points for a continuous and densifying mapping [42] in complete metric spaces.

In Chapter II, some different types of mappings and their fixed points have been studied. First section of this Chapter begins with the mapping introduced by Kannan [20] i.e. $T : X \to X$ such that for all $x, y \in X$,

$(*)$.......$d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\}$,

where $0 \leq \alpha < \frac{1}{2}$. Some more general forms of this mapping have been introduced and the fixed points are obtained under sufficiently relaxed conditions. We have generalized a theorem of Kannan [20] and have offered a simple example for the illustration. Further, we have extended our own theorem in the lines of Chu & Diaz [13] and have cited an example also for the verification. In the same continuation five more new theorems have been presented out of which the first two extend the results of Singh [39] respectively, and the remaining bear a close similarity with the results of Rakotch [31], Belluce and Kirk [4] and Ki-Wang Ng [29] respectively.

In the second section of Chapter II, we have given a brief account of some results on multi-valued contraction mappings due to Nadler [28]. Also we have extended results of Kannan [20] and Maia [26] respectively to the multivalued mappings.

In Chapter III we have considered the convergence of sequences of mappings and their fixed points. Besides a brief survey of the works of Bonsall [5], Nadler [27], Singh [36] , and Singh & Russell [38], we have extended the result of Bonsall to the sequence of mappings satisfying the condition of Rakotch [31] . Further, a more general theorem for the sequence of mappings satisfying a localized version of the condition of Rakotch has been investigated. In addition, we have also given some new results for the sequence of mappings of the type (⁂) . The last theorem of this Chapter generalizes a theorem due to Singh [37] , and we have constructed an example also to verify our generalization.

CHAPTER I

NON-EXPANSIVE MAPPINGS

1.1. Preliminary Definitions:

Definition 1.1.1: Let X be a set and let R^+ denote the positive reals. We define a distance function $d : X \times X \to R^+$ to be a metric if the following conditions are satisfied:

(i) $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ iff $x = y$.

(iii) $d(x, y) = d(y, x)$.

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The set X with metric d is called a metric space and is denoted by a pair (X, d). We may denote the space by X alone when the metric d is understood with.

Definition 1.1.2: A sequence $\{x_n\}$ of points of a metric space X is said to converge to a point x and we write $x_n \to x$, if corresponding to each $\varepsilon > 0$ there is a positive integer N such that for $n \geq N$, one has $d(x_n, x) < \varepsilon$. In other words $x_n \to x$ if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 1.1.3: A sequence $\{x_n\}$ of points of a metric space X is said to be Cauchy sequence if for each $\varepsilon > 0$ there is a positive integer N such that for $m, n \geq N$ implies $d(x_m, x_n) < \varepsilon$.

Definition 1.1.4: A metric space (X, d) is said to be complete if every Cauchy sequence of points of X converges in X.

Definition 1.1.5: Given a vector space E, a norm on E is a mapping $x \to ||x||$, from E into the set R of positive real numbers which satisfies the following axioms:

(i) $||x|| = 0$ if and only if $x = 0$.

(ii) $||\lambda x|| = |\lambda| \; ||x||$ for all $\lambda \in F$ and $x \in E$, where F is either the field of real numbers or the field of complex numbers.

(iii) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality).

A vector space on which a norm is defined is called a normed vector space, or simply a normed space.

(Every normed space is a metric space with a metric d defined as $d(x, y) = ||x - y||$).

Definition 1.1.6: A normed vector space E is called a Banach space if it is complete as a metric space.

Definition 1.1.7: Let T be a *mapping of a set X into itself. A point $x \in X$ is said to be a fixed point of T if $Tx = x$. In other words, a point which remains invariant under a mapping is known as a fixed point.

Definition 1.1.8: A topological space X is said to have fixed point property (or X is a fixed point space) if each continuous function of

*Some authors have used the word "transformation" in place of "mapping".

X into itself has at least one fixed point.

(Example: The closed interval $[-1, 1]$ has fixed point property for let $T : [-1, 1] \to [-1, 1]$ be a continuous function. Define a new function F as $F(x) = T(x) - x$ for each $x \in [-1, 1]$. We see that $F(-1) \geq 0$ and $F(1) \leq 0$. Therefore by Weierstrass Intermediate-value Theorem there exists a point $x_o \in [-1, 1]$ such that $F(x_o) = 0$. This gives $T(x_o) = x_o$).

<u>Definition 1.1.9</u>: Let X and X' be two metric spaces with the metrics d and d' respectively. Let $T : X \to X'$ be a bijection of X to X'. Then T is called an isometry if and only if

$$d(x, y) = d'(Tx, Ty) \quad \text{for all} \quad x, y \in X.$$

In particular if $X = X'$ and the metrics d and d' are the same then $T : X \to X$ is an isometry if

(1.1A) $\quad d(Tx, Ty) = d(x, y) \quad$ for all $x, y \in X$.

<u>Definition 1.1.10</u>: A mapping T of a metric space X into itself is said to be Non-expansive if for all $x, y \in X$

(1.1B) $\quad d(Tx, Ty) \leq d(x, y)$.

<u>Definition 1.1.11</u>: A mapping T of a metric space X into itself is said to satisfy Lipschitz condition if there exists a real number k (known as Lipschitz constant) such that,

(1.1C) $\quad d(Tx, Ty) \leq kd(x, y) \quad$ for all $x, y \in X$.

In the special case when $0 \leq k \leq 1$, we call as a contraction mapping.

<u>Remark</u>: At a first glance to the mappings defined above one can say that the Isometries (1.1A) and the Contraction mappings (1.1C) as well, fall in the class of Non-expansive mappings (1.1B). Moreover, all of the mappings (1.1A), (1.1B), and (1.1C) are continuous on X.

1.2. It is of great importance in the applications to find out if non-expansive mappings have fixed points.

One of the best known theorems in connection with the fixed point of a mapping in a metric space is that given by Banach [2] and known as Banach's contraction mapping theorem.

The statement and proof of the theorem is given as follows:

<u>Theorem 1.2.1</u>. Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping i.e. there exists a real number k, $0 \leq k < 1$ such that,

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for any two points } x, y \in X.$$

Then T has a unique fixed point (i.e. the equation $Tx = x$ has a unique solution).

<u>Proof</u>: Let x_o be an arbitrary point in X. Set $x_1 = Tx_o$, $x_2 = Tx_1 = T^2x_o$, and in general let $x_n = Tx_{n-1} = T^nx_o$.

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. In fact,

$$
\begin{aligned}
d(x_n, x_m) &= d(T^n x_o, T^m x_o) \\
&\leq kd(T^{n-1} x_o, T^{m-1} x_o) \\
&\leq k^n d(x_o, T^{m-n} x_o) \\
&= k^n d(x_o, x_{m-n}) \\
&\leq k^n \{d(x_o, x_1) + d(x_1, x_2) + \ldots + d(x_{m-n-1})\} \\
&\leq k^n d(x_o, x_1)\{1 + k + k^2 + \ldots + k^{m-n-1}\}. \\
&\leq k^n d(x_o, x_1) \frac{1}{1-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),
\end{aligned}
$$

Since $k < 1$, this quantity is arbitrarily small for sufficiently large n.

Since (X, d) is complete, the sequence $\{x_n\}$ converges in X. Let $\lim_{n\to\infty} x_n = u$. Then by virtue of the continuity of the mapping T, $Tu = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$. Thus, the existence of a fixed point is proved. To prove its uniqueness let v be a point in x such that

$Tv = v$. Then $d(u, v) = d(Tu, Tv) \leq kd(u, v)$, where $k < 1$; this implies $d(u, v) = 0$, i.e. $u = v$.

Hence the theorem.

Remark 1.2.2: (i) The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximations.

(ii) Banach's contraction theorem has been applied to test existence and uniqueness of solutions to differential and integral equations using the method of successive approximations.

(iii) The method of successive approximations can be used not only for the proof of existence of unique fixed point u but also for finding an approximate value. Namely, the points x_n are the successive approximations to u. The error of approximations may be estimated by the inequality,

$$d(x_n, u) \leq \frac{k^n}{1 - k} d(x_o, x_1) \qquad \text{.............................(II)}$$

which is obtained by passing to the limit for $m \to \infty$ in the inequality (I).

Due to its wide spread applicability, the Banach's contraction theorem has been generalized by several Mathematicians. We quote few of these generalizations without going into detail.

Chu & Diaz [13] have given the following:

Theorem 1.2.3. If $T : X \to X$ is a function defined on a complete metric space X into itself such that the function T^n is a contraction for some positive integer n, then T has a unique fixed point.

Remark 1.2.4: The function T in the above theorem is not necessarily contraction or continuous. The following example illustrates the theorem:

Example 1.2.4: Let a function $T : [0, 2] \to [0, 2]$ be defined as,

$$Tx = \begin{cases} 0 , & x \in [0, 1] \\ 1 , & x \in (1, 2] \end{cases} .$$

We see that T is discontinuous at 1 and therefore it is not contraction on $[0, 2]$. But T^2 is contraction as $T^2x = 0$ for all $x \in [0, 2]$. The unique fixed point of T is zero.

Few interesting examples to illustrate the above theorem are also given in [13].

According to a remark due to the same authors, the conclusion of the above theorem may be obtained even without assuming that T^n is contraction and X is a complete metric space. All that is needed is that T^n has exactly one fixed point. Thus one has:

<u>Theorem 1.2.5</u>. Let S be any non empty set of elements and T be a single valued function defined on S and with values in S. Suppose that, for some positive integer n , the function T^n has a unique fixed point x_o . Then T also has a unique fixed point, namely x_o.

The proof is quite simple and short. In fact $T^n x_o = x_o$ gives

$TT^n x_o = Tx_o$

i.e. $T^{n+1} x_o = Tx_o$

i.e. $T^n x_o = Tx_o$. Thus Tx_o is a fixed point of T^n.

But T^n has only one fixed point namely x_o. Hence $Tx_o = x_o$ i.e. x_o is a fixed point of T. For uniqueness of x_o as a fixed point of T, let y be a point such that $Ty = y$, then $T^n y = y$ and hence $y = x_o$, since T^n has only one fixed point.

In another paper [12], Chu and Diaz have also given the following:

Theorem 1.2.6. Let T be a function defined on a non empty set with values in S. Let K be another function also defined on S to S, such that K possesses a right inverse K^{-1} (i.e. a function K^{-1} such that $KK^{-1} = I$, where I is the identity mapping of S). Then the function T has a fixed point if and only if the composite function $K^{-1}TK$ has a fixed point. The theorem gives the following useful corollary.

Corollary 1.2.7: Let (X, d) be a complete metric space and T be a self mapping of X into X. Suppose that there exists a self mapping K of X into X which has a right inverse K^{-1} and which makes the composite function $K^{-1}TK$ a contraction. Then T has a unique fixed point.

This result follows directly from Banach's contraction theorem and the preceeding theorem.

Edelstein [14] has extended Banach's contraction theorem introducing the following definitions.

Definition 1.2.8: A mapping T of a metric space X into itself is said to be locally contractive if for every $x \in X$ there exist ε and λ $(\varepsilon > 0, 0 \leq \lambda < 1)$, which may depend on x such that,

$$p, q \in S(x, \varepsilon) = \{y : d(x, y) < \varepsilon\} \text{ implies}$$
$$d(Tp, Tq) < \lambda d(p, q).$$

Definition 1.2.9: A mapping T of X into itself is said to be $(\varepsilon - \lambda)$-uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Definition 1.2.10: A metric space X will be said to be ε-chainable if for every $a, b \in X$ there exists an ε-chain, that is a finite set of points $a = x_o, x_1, x_2, \ldots, x_n = b$ (n may depend on both a and b) such that $d(x_{i-1}, x_i) < \varepsilon$ $(i = 1, 2, \ldots, n)$.

We state the theorem of Edelstein as follows:

Theorem 1.2.11. Let (X, d) be a complete ε-chainable metric space and T be a mapping of X into itself which is ε-λ uniformly locally contractive then there exists a unique point ξ in X such that $T\xi = \xi$.

Next, we define a mapping introduced by Edelstein [15] which is more general than a contraction mapping.

Definition 1.2.12: A mapping T of a metric space X into itself is said to be contractive if,

(i) $\quad d(Tx, Ty) < d(x, y) \qquad$ for $x, y \in X$, $x \neq y$.

It is to note that a contractive mapping of a complete metric space into itself need not have a fixed point. For example, if $X = \{x : x \in R, x \geq 1\}$ and $T : X \to X$ is defined as $Tx = x + \frac{1}{x}$ then T has no fixed point, although T is contractive and X is complete.

However, if a contractive mapping has a fixed point, it will always be unique.

Edelstein [1] has given the following theorem for the existence of a fixed point for a contractive mapping.

<u>Theorem 1.2.13</u>. Let X be a metric space and let T be a contractive mapping of X into itself. If there exists a point $x_o \in X$ such that its sequence of iterates $\{T^n x_o\}$ has a convergent subsequence $\{T^{n_i} x_o\}$ converging to a point ξ in X, then ξ is a unique fixed point of T.

A simpler proof of this theorem than that due to Edelstein may be given as following:

<u>Proof</u>: Since $\{T^{n_i} x_o\}$ converges to $\xi \in X$ and T, being a contractive mapping, is continuous on X therefore the sequence $\{T^{n_i+1} x_o\}$ converges to $T\xi$ and consequently the sequence $\{T^{n_i+2} x_o\}$ converges to $T^2\xi$.

Consider the sequence $\{d(T^n x_o, T^{n+1} x_o)\}$ of non-negative real numbers. If for any n, $d(T^n x_o, T^{n+1} x_o) = 0$, there remains nothing to prove as $T^n x_o$ comes out to be a fixed point of T. Thus we may assume without loss of generality that each term of this sequence is positive. Since T is contractive therefore for $x_o \neq Tx_o$, we have $d(x_o, Tx_o) > d(Tx_o, T^2 x_o) > \dots > d(T^n x_o, T^{n+1} x_o) > \dots$ i.e. $\{d(T^n x_o, T^{n+1} x_o)\}$ is a decreasing sequence of positive real numbers bounded by $d(x_o, Tx_o)$. Hence it converges together with all its subsequences to some real number α.

Now, assume $\xi \neq T\xi$,

Therefore,

$$\begin{aligned} d(\xi, T\xi) &= d(\lim_i T^{n_i} x_o, \lim_i T^{n_i+1} x_o) \\ &= \lim_i d(T^{n_i} x_o, T^{n_i+1} x_o) \\ &= \alpha \\ &= \lim_i d(T^{n_i+1} x_o, T^{n_i+2} x_o) \\ &= d(\lim_i T^{n_i+1} x_o, \lim_i T^{n_i+2} x_o) \\ &= d(T\xi, T^2\xi) \\ &< d(\xi, T\xi), \quad \text{a contradiction to the assumption.} \end{aligned}$$

Hence $\xi = T\xi$ i.e. ξ is a fixed point of T. For uniqueness of ξ, let $\overline{\xi} \neq \xi$ be a point in X such that $T\overline{\xi} = \overline{\xi}$. Then $d(\xi, \overline{\xi}) = d(T\xi, T\overline{\xi}) < d(\xi, \overline{\xi})$, a contradiction. Thus ξ is a unique fixed point of T.

Hence the theorem.

Corollary 1.2.14: If X is a compact metric space and T is a contractive mapping of X into itself then there exists a unique fixed point.

The proof of this corollary follows from the theorem and the fact that each sequence in a compact metric space has a convergent subsequence.

Remark 1.2.15: As pointed out by the author [15]. an extra conclusion regarding the convergence of the sequence of iterates from the previous theorem may be drawn as follows:

Let all the assumptions of the theorem hold. If $\{T^n p\}$, $p \in X$, contains a convergent subsequence $\{T^{n_i} p\}$ then $\lim_{n\to\infty} T^n p$ exists and coincides with the fixed point ξ .

Proof: By the previous theorem we have, $\lim_{i\to\infty} T^{n_i} p = \xi$. Then for $\varepsilon > 0$ there is a positive integer N such that $i > N$ implies $d(\xi, T^{n_i} p) < \varepsilon$. If $m = n_i + \ell$ (n_i fixed, ℓ variable) is any positive integer $> n_i$ then

$$d(\xi, T^m p) = d(T^\ell \xi, T^{n_i+\ell} p) < d(\xi, T^{n_i} p) < \varepsilon,$$

which proves the assertion.

Rakotch [31] generalized Banach's contraction theorem by allowing contraction constant λ to vary in a restricted way.

He has defined a family F_1 of functions $\lambda(x, y)$ satisfying the following conditions:

(1) $\lambda(x, y) = \lambda(d(x, y))$, i.e. λ is dependent on the distance between x and y only.

(2) $0 \leq \lambda(d) < 1$ for every $d > 0$.

(3) $\lambda(d)$ is a monotonically decreasing function of d.

He gave the following result:

<u>Theorem 1.2.16</u>. Let (X, d) be a complete metric space and let T be a mapping of X into itself such that,

(2) $d(Tx, Ty) \leq \lambda(x, y) d(x, y)$, for all $x, y \in X$, where $\lambda(x, y) \in F_1$. Then T has a unique fixed point.

<u>Remark 1.2.17:</u> The conslusion of the above theorem also holds when the function $\lambda(x, y)$ is assumed to be monotone increasing and $0 \leq \lambda(d) < 1$ for $d \geq 0$. This may be seen as follows:

Let x_o be an arbitrary point in X. Let $x_1 = Tx_o$, $x_2 = Tx_1 = T^2x_o$ and in general $x_n = Tx_{n-1} = T^n x_o$. Now by the condition $d(Tx, Ty) \leq \lambda(x, y)$, $\forall\, x, y \in X$, one can easily infer for $x_o \neq x_1$ that, $d(x_o, x_1) > d(x_1, x_2) > \ldots > d(x_{n-1}, x_n) > \ldots$, and therefore, $\lambda(x_o, x_1) > \lambda(x_1, x_2) > \ldots > \lambda(x_{n-1}, x_n) > \ldots$, as λ is monotone increasing. Thus we have,

$$d(x_1, x_2) = d(Tx_o, Tx_1) \leq \lambda(x_o, x_1)d(x_o, x_1),$$

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leq \lambda(x_1, x_2)d(x_1, x_2)$$

$$< \lambda(x_o, x_1)\lambda(x_o, x_1)d(x_o, x_1)$$

i.e. $d(x_2, x_3) < [\lambda(x_o, x_1)]^2 d(x_o, x_1)$.

In general, $d(x_n, x_{n+1}) < [\lambda(x_o, x_1)]^n d(x_o, x_1)$.

Now, since $0 \leq \lambda(x_o, x_1) < 1$, the sequence $\{x_n\}$ is easily seen to be Cauchy. The rest of the proof goes parallel to that as in the Banach's contraction theorem.

In the end of this section we would like to have a look on the following two general contractive mappings introduced by Bailey [1].

$T : X \to X$,

(i) T is continuous and $0 < d(x, y) \Rightarrow$ there exists $n = n(x, y) \in I^{+}$ (set of positive integers) such that

$$d(T^n x, T^n y) < d(x, y).$$

(ii) T is continuous and there exists $\varepsilon > 0$ such that $0 < d(x, y) < \varepsilon \Rightarrow$ there exists $n(x, y) \in I^{+}$ such that,

$$d(T^n x, T^n y) < d(x, y).$$

He has given the following results:

Th.1.2.18: If a mapping T of a compact metric space X into itself satisfies (i), then T has a unique fixed point.

Th.1.2.19: If a mapping T of a compact metric space X into itself satisfies (ii), then T has finitely many periodic points.

1.3 In this section we will study the various conditions under which a non-expansive mapping (1.1.B) has fixed point in metric spaces.

Cheney and Goldstein [11] have given the following.

<u>Theorem 1.3.1</u>. Let T be a mapping of a metric space X into itself such that,

(i) T is non-expansive i.e. $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.

(ii) if $x \neq Tx$ then $d(Tx, T^2x) < d(x, Tx)$.

and (iii) for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ has a cluster point.

Then for each x, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

In [16] , Edelstein has defined the following terms.

Definition 1.3.2 : Let T be a mapping of a metric space X into itself. A point $y \in Y \subset X$ is said to belong to the T-closure of Y, $y \in Y^T$, if $T(Y) \subset Y$ and there is a point $\eta \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < \ldots < n_i < \ldots)$, so that $\lim_i T^{n_i}(\eta) = y$.

Definition 1.3.3 A mapping $T : X \to X$ of a metric space X into itself is said to be ε-nonexpansive if condition, $d(Tp), Tq) \leq d(p, q)$ holds for all p, q with $0 < d(p, q) < \varepsilon$.

Definition 1.3.4 : A sequence $\{x_i\} \subset X$ is said to be isometric (ε-isometric) if the condition, $d(x_m, x_n) = d(x_{m+k}, x_{n+k})$ holds for all $m, n, k = 1, 2, \ldots$ (for all $m, n, k = 1, 2, \ldots$, with $d(x_m, x_n) < \varepsilon$). A point $x \in X$ is said to generate an isometric (ε-isometric) sequence under T, if $\{T^n x\}$ is such a sequence.

With these definitions, he [16] proved the following theorem.

Theorem 1.3.5. If $T : X \to X$ is a non-expansive (ε-nonexpansive) mapping of a metric space X into itself then each $x \in X^T$ generates an isometric (ε-isometric) sequence.

Remark 1.3.6: Although this theorem does not guarantee the existence of fixed*(periodic) points but it does generalize Theorem 1 [15] (Theorem 2,[15]) for if T is contractive (ε-contractive) and $x \in X^T$ then it is seen that x is a fixed (periodic) point of T.

In the same paper an interesting theorem giving a fixed point of a non-expansive mapping is an Euclidean n-space E^n has been proved. The theorem is stated as follows:

<u>Theorem 1.3.7 :</u> Let $T : E^n \to E^n$ be a nonexpansive mapping and $(E^n)^T \neq \phi$. Then

(a) there is a point $\xi \in E^n$ such that $T\xi = \xi$.

(b) if $x \in (E^n)^T$ and V is the linear variety of smallest dimension containing $\{T^m(x)\}$ then V contains a unique fixed point.

As we have seen in [16] , a point of X^T is fixed if T is a contractive mapping of a metric space X into itself. A corresponding statement for a non-expansive mapping does not hold necessarily unless some further condition is imposed on T. With this motivation, Belluee and Kirk [3] introduced the notion of "diminishing orbital diameters" on T. To define this term they required another term "limiting orbital diameter" of T as described below.

Diameter of a set $A \subset X$ is defined as
$\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Let T be a mapping of a metric space X into itself. For each $x \in X$, let $O(T^n x)$ denote the sequence of iterates of $T^n(x)$ i.e.,

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*A point $x \in X$ is periodic if $T^k x = x$ for some $k \in I^+$.

$O(T^n x) = \bigcup_{i=n}^{\infty} \{T^i x\}$, $n = 0, 1, 2, \ldots$, where $T^0(x) = x$. The diameter $\delta(O(T^n x))$ of the sets $O(T^n x)$, when finite, form a non-increasing sequence of numbers. The limit $r(x) = \lim_{n\to\infty}\delta(O(T^n x))$ is a non-negative real number and is called the limiting orbital diameter of T at the point x.

Definition 1.3.8 : Let $T : X \to X$ be a mapping of a metric space X into itself. If for each $x \in X$ the limiting orbital diameter $r(x)$ of T at x is less than $\delta(O(x))$ when $\delta(O(x)) > 0$, then T is said to have diminishing orbital diameters.

(Contraction and contractive mappings are easy examples of the mappings having diminishing orbital diameters. Another example of such a mapping is the mapping $T : X \to X$ such that for each $x \in X$ there is an $\alpha(x)$, $0 \leq \alpha(x) < 1$ and $d(Tx, Ty) \leq \alpha(x)d(x, y)$ for each $y \in X$.).

Now we are in position to give the theorem of Belluce and Kirk [3] which says:

Theorem 1.3.9 . Let X be a metric space and let T be a nonexpansive mapping of X into itself which has diminishing orbital diameters. Suppose for some $x \in X$ a subsequence of the sequence $\{T^n x\}_{n=1}^{\infty}$ of iterates of T on x has limit z. Then $\{T^n x\}_{n=1}^{\infty}$ has limit z and z is afixed point of T.

Proof: Suppose $\lim_{i\to\infty} T^{n_i}x = z$. Thus $z \in X^T$ and therefore by Theorem 1.3.5 of Edelstein, z generates an isometric sequence. This means that for given positive integers m and n,

$$d(T^m z, T^n z) = d(T^{m+k} z, T^{n+k} z), \quad k = 1, 2, \ldots .$$

Therefore if k is any positive integer,

$$\begin{aligned}\delta(O(Tz)) &= \sup_{n \geq 1} d(Tz, T^n z) \\ &= \sup_{n \geq 1} d(T^k z, T^{n+k-1} z) \\ &= \delta(O(T^k z)) .\end{aligned}$$

This implies, $\lim_{k \to \infty} \delta(O(T^k z)) = r(z) = \delta(O(Tz))$. But $r(z) = r(Tz))$, therefore $r(Tz)) = \delta(O(Tz))$ which gives due to the fact that T has diminishing oribtal diameters, $\delta(O(Tz)) = 0$ and hence Tz is a fixed point of T. Continuity of T implies $\lim_{i \to \infty} T^{n_i+1} x = Tz$. Thus $\varepsilon > 0$ there is a positive integer i such that $d(T^{n_i+1} x, Tz) < \varepsilon$. The fact that $T(z)$ is a fixed point and T is nonexpansive implies,

$d(T^n x, Tz) < \varepsilon$ if $n \geq n_i + 1$. Thus $\lim_{n \to \infty} T^n x = Tz$. But since a subsequence of $\{T^n x\}$ has limit z, $z = Tz$.

Hence the theorem.

A simple corollary to this theorem is as follows:

<u>Corollary 1.3.10</u>: If X is any compact metric space and T is a non expansive mapping of X into itself which has diminishing orbital diameters, then for each $x \in X$ the sequence $\{T^n x\}$ of iterates converges to a fixed point of T.

Further, Kirk [23] proved the following theorem:

<u>Theorem 1.3.11</u>. Suppose X is a compact metric space and $T : X \to X$ is continuous with diminishing orbital diameters. Then for each $x \in X$, some subsequence $\{T^{n_i}x\}_{i=1}^{\infty}$ of the sequence $\{T^n x\}_{n=1}^{\infty}$ of iterates of x has a limit z which is a fixed point of T.

Browder and Petryshyn [10] have introduced the following definition:

<u>Definition 1.3.12:</u> A mapping $T : X \to X$ of a metric space X into itself is said to be assymptotically regular on X if

$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0$ for each $x \in X$.

Belluce & Kirk* have observed that if X is compact and $T : X \to X$ nonexpansive then the conditions, "T has diminishing orbital diameters on X" and "T is assymptotically regular on X" are equivalent. But this equivalence need not be true when T is not nonexpansive (no matter T is continuous and X is compact).

They [4]have given the following theorem along with other results:

<u>Theorem 1.3.13</u>. Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping which is assymptotically regular on X. Then every sequence $\{T^n x\}_{n=1}^{\infty}$ of iterates contains a subsequence which converges to a fixed point of T.

We give a direct rather simple proof of this theorem.

Proof: (X, d) is compact, therefore it is sequentially compact. Thus for any $x \in X$ the sequence $\{T^n x\}$ has a convergent subsequence. Let

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*[4]

$\{T^{n_i}x\}_{i=1}^{\infty}$ be such a subsequence converging to a point $z \in X$. Since T is continuous therefore the sequence $\{T^{n_i+1}x\}_{i=1}^{\infty}$ converges to Tz. Thus,

$$d(z, Tz) = d(\lim_i T^{n_i}x , \lim_i T^{n_i+1}x)$$
$$= \lim d(T^{n_i}x , T^{n_i+1}x) \to 0,$$ since the assymptotic regularity of T on X gives that the sequence $\{d(T^n x, T^{n+1}x)\}_{n=1}^{\infty}$ and consequently one of its subsequence $\{d(T^{n_i}x, T^{n_i+1}x)\}_{i=1}^{\infty}$ converges to zero.

Thus z is a fixed point of T.

Hence the theorem.

As we see below, the existence of a fixed point in the above theorem can also be insured by replacing the compactness of the space with a weaker condition (condition(ii)in the following theorem).

<u>Theorem 1.3.14</u>. Let X be a metric space and $T : X \to X$ be a continuous mapping of X into itself. Suppose,

(i) T is assymptotically regular on X, and

(ii) for some $x_o \in X$, the sequence $\{T^n x_o\}$ of iterates has a convergent subsequence $\{T^{n_i}x_o\}$ converging to some point z .

Then z is a fixed point of T .

The proof follows in the same lines as in the previous theorem.

The theorem emits the following corollary:

Corollary 1.3.15: Let X be a metric space and let $T : X \to X$ be a nonexpansive mapping of X into itself such that conditions (i) and (ii) of the above theorem hold. Then z is a fixed point of T and the sequence $\{T^n x_o\}_{n=1}^{\infty}$ converges to z.

Proof: T is nonexpansive and is therefore continuous. Thus it follows from the theorem that z is a fixed point of T. The convergence of the sequence $\{T^n x_o\}_{n=1}^{\infty}$ to the fixed point z follows easily from condition (ii) and the fact that T is nonexpansive, as has been seen earlier.

Now, we give a few more interesting results which seem to be new.

Let X be a metric space and $A \subseteq X$ be a bounded subset of X. Denote by $\alpha(A)$, the infimum of all $\varepsilon > 0$ such that a finite number of open spheres of diameter less than ε cover A. [25] .

Definition 1.3.16: A mapping $T : X \to X$ of a metric space X into itself is said to be densifying if for each bounded subset $A \subseteq X$ with $\alpha(A) > 0$ we get $\alpha(TA) < \alpha(A)$. [42]

It is easily seen that

1) $0 \leq \alpha(A) \leq \delta(A)$ where $\delta(A)$ is the diameter of A.
2) $\alpha(A) = 0$ and X complete imply A is compact.
3) If A is compact then $\alpha(A) = 0$.
4) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
5) If $\overline{A}$ is the closure of A then

$$\alpha(\overline{A}) = 0 \Leftrightarrow \alpha(A) = 0 .[42] .$$

<u>Theorem 1.3.17</u>. Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping which is continuous, densifying. If for all $x, y \in X$, $x \neq y$ there is some n depending on x, y such that $d(T^n x, T^n y) < d(x, y)$ and for some $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is bounded then T has a unique fixed point.

<u>Proof</u>: Let $A = \bigcup_{n=0}^{\infty} \{T^n x\}$, where $T^0 x = x$ and $\bar{A}$ be the closure of A. Let us assume $\alpha(\bar{A}) > 0$ or equivalently $\alpha(A) > 0$. Then $\alpha(TA) < \alpha(A)$, since T is densifying. But $A = TA \cup \{x\}$, therefore

$$\begin{aligned}\alpha(A) &= \max\{\alpha(TA), \alpha(\{x\})\\ &= \max\{\alpha(TA), 0\}\\ &= \alpha(TA), \text{ a contradiction.}\end{aligned}$$

Thus $\alpha(\bar{A}) = 0$. Since X is complete, it follows from property 2) that $\bar{A}$ is compact. Now, by continuity of T we get $T(\bar{A}) \subseteq \overline{T(A)} \subseteq \bar{A}$. Let F be the restriction of T on $\bar{A}$. Then $F : \bar{A} \to \bar{A}$ satisfies all the assumption of Theorem 1.2.18 of Bailey giving that there is a unique fixed point in $\bar{A}$.

Hence the theorem.

<u>Theorem 1.3.18</u>. Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous, densifying mapping which has dinimishing orbital diameters. If for some $x \in X$, the sequence $\{T^n x\}$ of iterates is bounded then T has a fixed point.

<u>Proof</u>: Denote $\bigcup_{n=0}^{\infty} \{T^n(x)\}$ by the set A. Exactly as in the above theorem it is seen that the closure $\bar{A}$ of A is compact and

$T(\overline{A}) \subseteq \overline{A}$. Thus it follows from the Theorem 1.3.11 of Kirk that T has a fixed point in $\overline{A}$ and hence in X.

CHAPTER II

SOME DIFFERENT TYPES OF MAPPINGS AND THEIR FIXED POINTS

2.1 We recall Banach's contraction theorem which states :

Let (X, d) be a complete metric space and $T:X \to X$ be a mapping of X into itself satisfying,

$$(2.1A) \qquad d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X,$$

where $0 \leq \alpha < 1$.

Then T has a unique fixed point.

Recently, Kannan [20] gave the following:

<u>Theorem 2.1.1</u>: Let (X, d) be a complete metric space and $T:X \to X$ be a mapping of X into itself satisfying,

$$(2.1B) \qquad d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\}, \forall x, y \in X,$$

where α is a real number such that $0 \leq \alpha < \frac{1}{2}$.

Then T has a unique fixed point.

The condition (2.1A) implies the continuity of the mapping in the whole space but the condition (2.1B) does not necessarily.

The following two examples illustrate that conditions (2.1A) and (2.1B) are independent.

<u>Example 2.1.2</u>: Let $X = [0, 1]$. Define $T:X \to X$ by

$$Tx = \begin{cases} \frac{x}{4}, & \text{for } x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The distance function d is defined in the usual way by $d(x, y) = |x - y|$. Here T is discontinuous at $x = \frac{1}{2}$; consequently condition (2.1A) is not satisfied. But it is easily seen that condition (2.1B) is satisfied by taking $\alpha = \frac{4}{9}$ [21].

<u>Example 2.1.3</u>: Let $X = [0, 1]$, $T:X \to X$ be defined by $Tx = \frac{x}{3}$. The distance function is the usual distance. Here condition (2.1A) is satisfied but the condition (2.1B) is not satisfied for $x = \frac{1}{3}$ and $y = 0$ [21].

However, if $\alpha < \frac{1}{3}$ then (2.1A) implies (2.1B).

The condition (2.1B) motivates to give a similar condition for the existence of fixed points of two mappings T_1 and T_2 simultaneously.

The following theorem due to Kannan [20] is worth mentioning.

<u>Theorem 2.1.4</u>: Let (X, d) be a complete metric space. If T_1 and T_2 are two mappings of X into itself satisfying

(2.1C) $$d(T_1x, T_2y) \leq \alpha\{d(x, T_1x) + d(y, T_2y)\}, \quad \forall\, x, y \in X,$$

where T_1, T_2 are two mappings of X into itself and α is a real number such that $0 \leq \alpha < \frac{1}{2}$, then T_1 and T_2 have a unique common fixed point.

The proof follows from successive iteration procedure. (Taking $x_0 \in X$ and setting $x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$, $x_4 = T_2x_3$ and so on, the sequence $\{x_n\}_{n=1}^{\infty}$ so obtained, is shown to be Cauchy, which, due to completeness of (X, d) converges in X, giving the unique common fixed point of T_1 and T_2).

Remark 2.1.5: If in the above theorem, mappings T_1 and T_2 fail to satisfy condition (2.1C) but, however, the condition (2.1D) (appearing in the next theorem) is satisfied, still the conclusion of the theorem holds.

Thus we give a modified version of the preceding theorem as follows:

Theorem 2.1.6: Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself. If there exist two positive real numbers α and β such that $\alpha + \beta < 1$ and,

$$\text{(2.1D)} \qquad d(T_1x, T_2y) \leq \alpha d(x, T_1x) + \beta d(y, T_2y), \quad \forall x, y \in X,$$

then T_1 and T_2 have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Set a sequence $\{x_n\}_{n=1}^{\infty}$ of points in X as $x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$, $x_4 = T_2x_3$ and so on.

Then,

$$\begin{aligned} d(x_1, x_2) &= d(T_1x_0, T_2x_1) \\ &\leq \alpha d(x_0, T_1x_0) + \beta d(x_1, T_2x_1) \\ &= \alpha d(x_0, x_1) + \beta d(x_1, x_2) \end{aligned}$$

$$\therefore d(x_1, x_2) \leq \frac{\alpha}{1-\beta} d(x_0, x_1).$$

$$\begin{aligned} d(x_2, x_3) &= d(T_2x_1, T_1x_2) \\ &\leq \alpha d(x_2, T_1x_2) + \beta d(x_1, T_2x_1) \\ &= \alpha d(x_2, x_3) + \beta d(x_1, x_2) \end{aligned}$$

$$\therefore\ d(x_2, x_3) \le \frac{\beta}{1-\alpha} d(x_1, x_2)$$

$$\le \frac{\beta}{1-\alpha} \cdot \frac{\alpha}{1-\beta} d(x_0, x_1).$$

Similarly, $d(x_3, x_4) \le \frac{\alpha}{1-\beta} \cdot \frac{\beta}{1-\alpha} \cdot \frac{\alpha}{1-\beta} d(x_0, x_1).$

In general,

$d(x_n, x_{n+1}) \le \left(\frac{\alpha}{1-\beta}\right)^{\frac{n}{2}} \cdot \left(\frac{\beta}{1-\alpha}\right)^{\frac{n}{2}} d(x_0, x_1)$, when n is an even positive integer.

and, $d(x_n, x_{n+1}) \le \left(\frac{\alpha}{1-\beta}\right)^{\frac{n+1}{2}} \cdot \left(\frac{\beta}{1-\alpha}\right)^{\frac{n-1}{2}} d(x_0, x_1)$, when n is an odd positive integer.

For simplicity, put $\frac{\alpha}{1-\beta} = k$ and $\frac{\alpha.\beta}{(1-\beta)(1-\alpha)} = \gamma$ and rewrite the above two inequalities as:

$d(x_n, x_{n+1}) \le \gamma^{\frac{n}{2}} d(x_0, x_1)$, when n is an even positive integer....(i)

and $d(x_n, x_{n+1}) \le k\gamma^{\frac{n-1}{2}} d(x_0, x_1)$, when n is an odd positive integer. ..(ii)

Now, for $m > n$; m, n both even, we have

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + \ldots\ldots\ldots\ldots + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m).$$

$$\le \gamma^{\frac{n}{2}} d(x_0, x_1) + k\,\gamma^{\frac{n}{2}} d(x_0, x_1) + \gamma^{\frac{n}{2}+1} d(x_0, x_1) + k\gamma^{\frac{n}{2}+1} d(x_0, x_1) + \ldots\ldots\ldots + \gamma^{\frac{m}{2}-1} d(x_0, x_1) + k\gamma^{\frac{m}{2}-1} d(x_0, x_1).$$

$$= \gamma^{\frac{n}{2}} d(x_0, x_1) \{1 + \gamma + \gamma^2 + \ldots\ldots\ldots + \gamma^{\frac{m-n}{2}-1}\}.$$

$$+ k\,\gamma^{\frac{n}{2}} d(x_0, x_1) \{1+\gamma + \gamma^2 + \ldots\ldots\ldots + \gamma^{\frac{m-n}{2}-1}\}.$$

$\therefore$ For $m > n$ and n-even, we have:

$$d(x_n, x_m) \leq \frac{\gamma^{\frac{n}{2}}}{1-\gamma} d(x_0, x_1) + \frac{k.\gamma^{\frac{n}{2}}}{1-\gamma} d(x_0, x_1) \qquad \ldots(iii)$$

Similarly for $m > n$ and n-odd, we can have:

$$d(x_n, x_m) \leq \frac{k}{1-\gamma} \cdot \gamma^{\frac{n-1}{2}} d(x_0, x_1) + \frac{1}{1-\gamma} \cdot \gamma^{\frac{n+1}{2}} \cdot d(x_0, x_1) \qquad \ldots(iv)$$

Since $\alpha, \beta > 0$ and $\alpha + \beta < 1$, it follows that

$\frac{\alpha}{1-\beta} < 1$, $\frac{\beta}{1-\alpha} < 1$ and consequently $\gamma < 1$.

$\therefore$ for large n the terms on right hand sides of both the inequalities (iii) and (iv) become arbitrarily small. Thus $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since the space X is complete, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some point $u \in X$.

Now, $d(u, T_1u) \leq d(u, x_n) + d(x_n, T_1u)$

$= d(u, x_n) + d(T_2x_{n-1}, T_1u)$,

where n is chosen to be even positive integer.

$\therefore d(u, T_1u) \leq d(u, x_n) + \alpha d(u, T_1u) + \beta d(x_{n-1}, T_2x_{n-1})$

or, $(1-\alpha)d(u, T_1u) \leq d(u, x_n) + \beta d(x_{n-1}, x_n)$.

or, $d(u, T_1u) \leq \frac{1}{1-\alpha} d(u, x_n) + \frac{\beta}{1-\alpha} d(x_{n-1}, x_n)$

$\leq \frac{1}{1-\alpha} d(u, x_n) + \frac{\gamma}{k} k\gamma^{\frac{n-2}{2}} d(x_0, x_1)$,

(By inequality (ii))

$\leq \frac{1}{1-\alpha} d(u, x_n) + \gamma^{\frac{n}{2}} d(x_0, x_1)$.

Therefore $(u, T_1u) \to 0$, as $n \to \infty$, which gives $T_1u = u$ i.e., u is a fixed point of T_1.

In the similar way, taking the triangle inequality $d(u, T_2u) \leq d(u, x_n) + d(x_n, T_2u)$, and n an odd positive integer, we can show that u is a fixed point of T_2.

Thus u is a common fixed point of T_1 and T_2. To show that u is a unique common fixed point of T_1 and T_2, let v be a point in X such that $T_1v = v$ and $T_2v = v$.

Then,
$$d(u, v) = d(T_1u, T_2v)$$
$$\leq \alpha d(u, T_1u) + \beta d(v, T_2v) = 0$$
$$\therefore u = v.$$

Hence the theorem.

To illustrate the above Remark 2.1.5 and Theorem 2.1.6, we give a simple example as follows:

<u>Example 2.1.7</u>: Let $T_1, T_2: [0, 1] \longrightarrow [0, 1]$, be defined respectively as,

$$T_1x = \frac{x}{3}$$

and
$$T_2x = \frac{x}{4}.$$

The distance function d is defined in the usual way as $d(x, y) = |x - y|$. The space $X = [0, 1]$, being a closed subset of a complete space R (set of reals) is complete.

It is easily seen that condition (2.1C) is not satisfied by these mappings for any $\alpha < \frac{1}{2}$, if we take $x = 1$ and $y = 0$. But on taking $\alpha = \frac{5}{8}$ and $\beta = \frac{11}{30}$ so that $\alpha + \beta < 1$, we see that condition (2.1D) is satisfied for all the points in $[0, 1]$, and the unique common fixed point of T_1 and T_2 is seen to be zero.

Remark 2.1.8: The conditions of the above theorem also imply that both T_1 and T_2 have only one fixed point, namely u. For, if $\bar{u}$ is a point in X such that $T_1\bar{u} = \bar{u}$ then,

$$d(\bar{u}, u) = d(T_1\bar{u}, T_2u)$$

$$\leq \alpha d(\bar{u}, T_1\bar{u}) + \beta(u, T_2u) = 0$$

$\therefore \bar{u} = u$, i.e., u is a unique fixed point of T_1. Similarly it can be shown that u is a unique fixed point of T_2.

Thus in the enunciation of the theorem "unique common fixed point" may be replaced by "common unique fixed point".

Remark 2.1.9: In the previous theorem,

(i) If $\alpha = \beta$, we obtain Theorem 2.1.4 as a corollary to our Theorem.

(ii) If $T_1 = T_2 = T$, we get a similar generalization of Theorem 2.1.1.

(iii) If $T_1 = T_2 = T$ and $\alpha = \beta$, we get Theorem 2.1.1 as a simple corollary.

Next, if the condition (2.1D) in the last theorem, is not satisfied by T_1 and T_2, but it is satisfied by some iterates T_1^p and T_2^p (p is a positive integer) of T_1 and T_2 respectively, even then the conclusion of the theorem holds.

Thus we have,

<u>Theorem 2.1.10</u>: Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself. If there exist positive reals α and β, $\alpha + \beta < 1$ and a positive integer p such that,

$$(2.1E) \qquad d(T_1^p x, T_2^p y) \leq \alpha d(x, T_1^p x) + \beta d(y, T_2^p y), \ \forall x, y \in X,$$

where T_1^p and T_2^p stand for p^{th} iterates of T_1 and T_2 respectively, then T_1 and T_2 have a unique common fixed point.

<u>Proof</u>: By the previous Theorem 2.1.6 and the Remark 2.1.8, we conclude that T_1^p and T_2^p have a common unique fixed point. Let u be such a point. It follows from a theorem of Chu and Diaz [13] that u is a unique fixed point of T_1 as well as of T_2. Hence the theorem.

In order to illustrate this theorem we take the following example:

<u>Example 2.1.11</u>: Let $T_1, T_2: [0, 1] \to [0, 1]$ be defined respectively as $T_1x = \frac{x}{3}$ and $T_2x = \frac{x}{2}$ for $x \in [0, 1]$. The metric d is defined as $d(x, y) = |x - y|$. It is easily seen that condition (2.1D) is not satisfied by T_1 and T_2 for $x = 0$ and $y = 1$. But it is satisfied by T_1^2 and T_2^2 for all the points in $[0, 1]$, when $\alpha = \frac{1}{4}$ and $\beta = \frac{2}{5}$. The common unique fixed point of T_1 and T_2 is 0.

A generalization of Theorem 2.1.6 in the lines of Chu and Diaz may be given as follows:

Theorem 2.1.12: Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself. Suppose that T_1 and T_2 are such that there exists a mapping K of X into itself which has a right inverse K^{-1} and that the composite maps $K^{-1}T_1K$ and $K^{-1}T_2K$ satisfy,

$$d(K^{-1}T_1K(x), K^{-1}T_2K(y)) \leq \alpha d(x, K^{-1}T_1K(x)) + \beta(y, K^{-1}T_2K(y))$$

for all $x, y \in X$, where $\alpha > 0, \beta > 0$ and $\alpha + \beta < 1$.

Then T_1 and T_2 have a unique common fixed point.

Proof: That $K^{-1}T_1K$ and $K^{-1}T_2K$ have same unique fixed point, is implied by Theorem (2.1.6) and Remark 2.1.8. If u is such a point then by a corollary to a theorem of Chu and Diaz [12], u is a unique fixed point of T_1 and T_2. Hence the theorem.

We give another extension of Theorem 2.1.4 by permitting α to be equal to $\frac{1}{2}$.

Theorem 2.1.13: Let (X, d) be a metric space and T_1, T_2 be two continuous mappings of X into itself. Suppose,

(i) $d(T_1x, T_2y) < \frac{1}{2}\{d(x, T_1x) + d(y, T_2y)\} \quad \forall x, y \in X$.

and (ii) there is a point $x_0 \in X$ such that the sequence $x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$ and so on with $x_r \neq x_s$ when $r \neq s$, has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, converging to a point ξ in X.

Then T_1 and T_2 have ξ as a unique common fixed point. Moreover, the sequence $\{x_n\}_{n=1}^{\infty}$ also converges to the point ξ.

<u>Proof</u>: We see that,

$$
\begin{aligned}
d(x_1, x_2) &= d(T_1x_0, T_2x_1) \\
&< \frac{1}{2}\{d(x_0, T_1x_0) + d(x_1, T_2x_1)\} \\
&= \frac{1}{2} d(x_0, x_1) + \frac{1}{2}d(x_1, x_2)
\end{aligned}
$$

$$\text{i.e., } \frac{1}{2} d(x_1, x_2) < \frac{1}{2} d(x_0, x_1)$$

$$\therefore \quad d(x_1, x_2) < d(x_0, x_1) .$$

$$
\begin{aligned}
d(x_2, x_3) &= d(T_2x_1, T_1x_2) \\
&< \frac{1}{2}\{d(x_1, T_2x_1) + d(x_2, T_1x_2)\} \\
&= \frac{1}{2} d(x_1, x_2) + \frac{1}{2} d(x_2, x_3)
\end{aligned}
$$

$$\text{i.e., } \quad d(x_2, x_3) < d(x_1, x_2)$$

$$\therefore \quad d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1).$$

Proceeding in the same way we have in general,

$$\ldots\ldots\ldots d(x_n, x_{n+1}) < d(x_{n-1}, x_n) < \ldots\ldots\ldots\ldots < d(x_1, x_2) < d(x_0, x_1).$$

Thus $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ is a monotonic decreasing sequence of non-negative real numbers, moreover it is bounded above by $d(x_0, x_1)$. Therefore the sequence $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ converges to some non-negative real number.

$$\text{Let, } \lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$$

$$\text{Now, } \lim_{k\to\infty} x_{n_k} = \xi \quad \text{(by condition (ii))}$$

and T_1 is continuous.

$\therefore$ $T_1 \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} T_1 x_{n_k} = \lim_{k\to\infty} x_{n_k+1}$, where n_k is chosen to be even positive integer.*

i.e. $T_1\xi = \lim_{k\to\infty} x_{n_k+1}$ ....(I)

T_2 is also given to be continuous,

$\therefore$ $T_2(T_1\xi) = T_2 \lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} T_2 x_{n_k+1} = \lim_{k\to\infty} x_{n_k+2}$ ....(II)

Assume $\xi \neq T_1\xi$ i.e. $d(\xi, T_1\xi) > 0$.

Now, $d(\xi, T_1\xi) = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1})$

$= \lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$

$= \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2})$

$= d(T_1\xi, T_2T_1\xi)$, (By (I) and (II))

$< d(\xi, T_1\xi)$, for

$d(T_1\xi, T_2T_1\xi) < \frac{1}{2}\{d(\xi, T_1\xi) + d(T_1\xi, T_2T_1\xi)\}$

or $\frac{1}{2} d(T_1\xi, T_2T_1\xi) < \frac{1}{2} d(\xi, T_1\xi)$

i.e., $d(T_1\xi, T_2T_1\xi) < d(\xi, T_1\xi)$.

Hence the contradiction to our assumption.

$\therefore$ $d(\xi, T_1\xi) = 0$ i.e., $T_1\xi = \xi$

---

* Had the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ not contained, for large k, the terms x_{n_k}, with n_k even, we would have chosen n_k, an odd integer and have operated x_{n_k} by T_2.

Also, the relation $d(\xi, T_1\xi) = d(T_1\xi, T_2T_1\xi)$ under $T_1\xi = \xi$ gives $0 = d(\xi, T_2\xi)$ i.e., $T_2\xi = \xi$. Thus ξ is a common fixed point of T_1 and T_2. For uniqueness of ξ, let $\bar{\xi}$ be another common fixed point of T_1 and T_2.

Then, $d(\xi, \bar{\xi}) = d(T_1\xi, T_2\bar{\xi})$

$$< \tfrac{1}{2}\{d(\xi, T_1\xi) + d(\bar{\xi}, T_2\bar{\xi})\} = 0, \text{ a contradiction.}$$

Thus ξ is a unique common fixed point of T_1 and T_2.

Next, we have to show that the sequence $\{x_n\}_{n=1}^{\infty}$ $(x_1 = T_1x_0, x_2 = T_2x_1, x_3 = T_1x_2, \ldots\ldots)$ converges to ξ.

Since the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to ξ, given $\varepsilon > 0$, there is a positive integer N such that, for all $k > N$, $d(x_{n_k}, \xi) < \varepsilon$.

If $m = n_k + \ell$ (n_k fixed, ℓ variable), is any positive integer $>n_k$ then,

$$d(x_m, \xi) = d(x_{n_k+\ell}, \xi)$$

$$= d(T_1x_{n_k+\ell-1}, T_2\xi), \text{ if } n_k+\ell \text{ is even.}$$

$$< d(x_{n_k+\ell-1}, \xi), \text{ (By condition (i)}$$

$$= d(T_2x_{n_k+\ell-2}, T_1\xi)$$

$$< d(x_{n_k+\ell-2}, \xi), \text{ (By Condition (i)}$$

$< d(x_{n_k}, \xi) < \epsilon$, which proves that $\{x_n\}_{n=1}^{\infty}$ converges to ξ.

Hence the theorem.

<u>Corollary 2.1.14</u>: Putting $T_1 = T_2 = T$, we get the result due to Singh [39], as follows:

Let (X, d) be a metric space and $T:X \to X$ be a mapping of X into itself. If

(i) $d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$, $\forall x, y \in X$,

and (ii) there is a point $x_0 \in X$ such that a subsequence $\{T^{n_i}(x_0)\}_{i=1}^{\infty}$ of the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ of iterates of T on x_0 converges to a point $\xi \in X$, then $\{T^n(x_0)\}_{n=1}^{\infty}$ converges to ξ and T has ξ as its unique fixed point.

<u>Remark 2.1.15</u>: The mapping satisfying condition (i) and the above Corollary under this mapping may be considered as respective analogues of the contractive mapping and the corresponding result due to Edelstein [15].

For the mappings satisfying condition (i) of the above Corollary a simple and interesting result similar to that due to Rakotch [31, Theorem 1] for the contractive mappings may be given as follows:

<u>Theorem 2.1.16</u>: Let (X, d) be a metric space and $T:X \to X$ be a mapping of X into itself satisfying,

(a) $$d(Tx, Ty) < \tfrac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \forall\, x, y \in X.$$

If there exists a subset $M \subset X$ and a point $x_0 \in M$ such that

(b) $$d(x, x_0) - d(Tx, Tx_0) \geq 2d(x_0, Tx_0) \quad \forall\, x \in X \setminus M,$$

and T maps M into a compact subset of X, then T has a unique fixed point.

Proof: Suppose $Tx_0 \neq x_0$. and let,

$$x_n = T^n x_0, \; n = 1, 2, 3, \ldots\ldots .$$

i.e. $$x_{n+1} = Tx_n, \quad (n = 0, 1, 2, \ldots.) \qquad \text{(I)}$$

Since T maps M into a compact subset of X, it suffices to show that $x_n \in M$, $n = 1, 2, \ldots$, for then, the theorem follows directly from the previous Corollary.

Now, $$d(x_1, x_2) = d(Tx_0, Tx_1) < \tfrac{1}{2}\{d(x_0, Tx_0) + d(x_1, Tx_1)\}$$
$$= \tfrac{1}{2}\, d(x_0, x_1) + \tfrac{1}{2} d(x_1, x_2)$$

i.e., $d(x_1, x_2) < d(x_0, x_1)$, since $x_0 \neq x_1$.

$$d(x_2, x_3) = d(Tx_1, Tx_2) < \tfrac{1}{2}\{d(x_1, Tx_1) + d(x_2, Tx_2)\}$$
$$= \tfrac{1}{2} d(x_1, x_2) + \tfrac{1}{2} d(x_2, x_3)$$

or $$d(x_2, x_3) < d(x_1, x_2)$$

$$\therefore \quad d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1).$$

Continuing in the same way we can show that,

$$d(x_n, x_{n+1}) < d(x_0, x_1), \quad (n = 1, 2, \ldots) \qquad \text{(II)}$$

By triangle inequality we have,

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1})$$

$$< 2d(x_0, x_1) + d(Tx_0, Tx_n); \quad \text{(By (I) and (II))}.$$

i.e. $d(x_0, x_n) - d(Tx_n, Tx_0) < 2d(x_0, Tx_0)$,

which, in light of condition (b) gives that $x_n \in M$ for all $n = 1, 2, \ldots$. Thus the theorem.

The next theorem gives a generalization of Theorem 2.1.13 by relaxing condition (i) to replace the strict inequality '<' by '$\leq$'.

<u>Theorem 2.1.17</u>: If T_1, T_2 are two continuous mappings of a metric space X into itself such that,

(i) $d(T_1x, T_2y) \leq \frac{1}{2}\{d(x, T_1x) + d(y, T_2y)\}, \quad x, y \in X.$

(ii) if $x \neq T_1x$ then, $\quad d(T_1x, T_2T_1x) < d(x, T_1x)$.

and (iii) there exists a point $x_0 \in X$ such that the sequence $x_1 = T_1x_0, \; x_2 = T_2x_1, \; x_3 = T_1x_2, \; \ldots$ with $x_r \neq x_s$ when $r \neq s$, has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, converging to a point ξ in X.

Then ξ is a unique common fixed point of T_1 and T_2 and sequence $\{x_n\}_{n=1}^{\infty}$ converges to ξ.

<u>Proof</u>: As in previous theorem, we can easily show with the help of condition (i) that $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ is a monotonic nonincreasing sequence of non-negative real numbers and is bounded above by $d(x_0, x_1)$. Therefore it converges to some non-negative real number.

Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$.

Since $\lim_{k\to\infty} x_{n_k} = \xi$ and T_1 is continuous,

we have,

$$T_1\xi = T_1 \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} T_1 x_{n_k} = \lim_{k\to\infty} x_{n_k+1} ,$$

(where n_k is chosen to be even).

T_2 is also continuous.

$$\therefore\ T_2(T_1\xi) = T_2 \lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} T_2 x_{n_k+1} = \lim_{k\to\infty} x_{n_k+2}$$

Then,

$$\begin{aligned} d(\xi, T_1\xi) &= \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) \\ &= \lim_{n\to\infty} d(x_n, x_{n+1}) = \eta \\ &= \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) \\ &= d(T_1\xi, T_2T_1\xi) \end{aligned}$$

, which is contrary to condition (ii) unless $\xi = T_1\xi$, and then , $0 = d(\xi, T_2\xi)$ i.e. $\xi = T_2\xi$.

Thus ξ is a common fixed point of T_1 and T_2 .

The uniqueness of ξ follows easily from condition (i). Also the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ to ξ can be easily shown with the help of condition (i) as in Theorem 3.1.13. Hence the Theorem.

<u>Corollary 2.1.18</u>: In case $T_1 = T_2 = T$, we get a result due to Singh[39].

Further, we prove the following:

<u>Theorem 2.1.19</u>: Let (X, d) be a metric space and let T be a continuous mapping of X into itself such that,

(i) $d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$ for all $x, y \in X$.

(ii) T is assymptotic regular on X i.e.,

$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for each $x \in X$.

and (iii) for some $x_o \in X$ the sequence $\{T^n x_o\}_{n=1}^{\infty}$ of iterates has a convergent subsequence $\{T^{n_k} x_o\}_{k=1}^{\infty}$ converging to some point $z \in X$.

Then $\lim_{n\to\infty} T^n x_o = z$ and z is a unique fixed point of T.

<u>Proof</u>: Continuity of T with condition (ii) and (iii) gives that z is a fixed point of T (see Theorem 1.3.14 of previous chapter). The uniqueness of the fixed point z is given by condition (i) for, if $\bar{z}$ is another fixed point of T then,

$$d(z, \bar{z}) = d(Tz, T\bar{z}) \leq \frac{1}{2}\{d(z, Tz) + d(\bar{z}, T\bar{z})\} = 0.$$

Now it remains to show that $\lim_{n\to\infty} T^n x_o = z$.

It is given that $\lim_{i\to\infty} T^{n_i} x_o = z$; therefore for $\epsilon > 0$ there is a positive integer N such that for $i \geq N$, $d(T^{n_i} x_o , z) < \epsilon$.

Let $m = n_i + \ell$ (n_i fixed, ℓ variable) be any positive integer greater than n_i and therefore greater than N (as $n_i \geq i \geq N$) then $d(T^m x_o, z) = d(T^m x_o, Tz) \leq \frac{1}{2}\{d(T^{m-1} x_o, T^m x_o) + d(z, Tz)\}$. (By condition (i)).

But $d(z, Tz) = 0$ as z is a fixed point of T,

$$\therefore \quad d(T^m x_o, z) \leq \frac{1}{2} d(T^{m-1} x_o , T^m x_o)$$

$$\leq \frac{1}{2}\{d(T^{m-1}x_o, z) + d(z, T^m x_o)\} .$$

$$\therefore \; d(T^m x_o, z) \leq d(T^{m-1}x_o , z).$$

Continuing in the same way we have,

$$d(T^m x_o, z) \leq d(T^{m-1}x_o, z) \leq d(T^{m-2}x_o, z) \leq \ldots\ldots \leq d(T^{n_i}x_o, z) < \varepsilon,$$

which proves the assertion.

Remark 2.1.20: Since the condition (i) and the condition of non-expansiveness are independent of each other, the above theorem is different from Corollary 1.3.15, previous chapter (an alternative result to Theorem 1 of Belluce & Kirk [3]).

The indepencency of these conditions is seen by the following examples.

Let $T : [0, 1] \to [0, 1]$ be defined by $Tx = \frac{x}{2}$ for each $x \in [0, 1]$. The metric is the usual distance. We see that T is contraction and therefore it is non-expansive. But it can be easily seen by taking $x = 0$ and $y = \frac{1}{2}$ that T does not satisfy condition (i) of the above theorem.

Next, let $T : [0, 1] \to [0, 1]$ be defined as $Tx = \frac{x}{3}$ for $x \in [0, \frac{1}{3})$
and $Tx = \frac{x}{4}$ for $x \in [\frac{1}{3}, 1]$.

Clearly condition (i) is satisfied for all $x \in [0, 1]$, but T is not non-expansive as it is discontinuous at the point $\frac{1}{3}$.

In the end of this section we give a similar result analogous to a theorem of Ki Wong Ng[29] as follows:

<u>Theorem 2.1.21</u>: Let T be a continuous mapping of a metric space X into itself.

Suppose,

(i) $d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$ for all x, y in X.

(ii) for $x \neq y$, there is some n depending on x, y such that $d(T^n x, T^n y) < d(x, y)$.

(iii) there exists some x_o in X such that the sequence $\{T^n x_o\}_{n=1}^{\infty}$ of iterates has a convergent subsequence $\{T^{n_i} x_o\}_{i=1}^{\infty}$ converging to some point z in X.

Then z is a unique fixed point of T and $\lim_{n\to\infty} T^n x_o = z$.

<u>Proof</u>: We have by condition (i),

$$d(Tx_o , T^2x_o) \leq \frac{1}{2}\{d(x_o, Tx_o) + d(Tx_o , T^2x_o)\} .$$

or $d(Tx_o , T^2x_o) \leq d(x_o , Tx_o)$.

Similarly, $d(T^2x_o , T^3x_o) \leq d(Tx_o , T^2x_o)$,

and in general, $d(T^n x_o , T^{n+1} x_o) \leq d(T^{n-1} x_o , T^n x_o)$.

Therefore $d(x_o , Tx_o) \geq d(Tx_o , T^2x_o) \geq d(T^n x_o , T^{n+1} x_o) \geq \ldots .$

Thus $\{d(T^n x_o , T^{n+1} x_o)\}_{n=1}^{\infty}$ is a monotone nonincreasing sequence of reals and therefore it converges along with all its subsequences to some non negative real α .

Now, for some n depending on (z, Tz) we have by condition (ii),

$d(z, Tz) > d(T^n z, T^{n+1} z)$ if $z \neq Tz$.

Also $\lim_{i\to\infty} T^{n_i} x_o = z$

Thus we have,

$$
\begin{aligned}
d(z, Tz) &= d(\lim_i T^{n_i} x_o, \lim_i T^{n_i+1} x_o) \\
&= \lim_i d(T^{n_i} x_o, T^{n_i+1} x_o) \\
&= \alpha \\
&= \lim_i d(T^{n_i+n} x_o, T^{n_i+n+1} x_o) \\
&= d(T^n z, T^{n+1} z),
\end{aligned}
$$

giving a contradiction unless $z = Tz$.

Thus z is a fixed point of T. Uniqueness of z as a fixed point of T is obvious by condition (i).

The convergence of the sequence $\{T^n x_o\}_{n=1}^{\infty}$ to z can be shown exactly as in the previous theorem.

2.2 Multi-valued contraction mappings.

A multi-valued function $F : X \to Y$ is a correspondence which to each x in X assigns one or more points of Y. For every x in X, $F(x)$ will denote the set of all "images" of x.

A point x is said to be a fixed point of F if $x \in F(x)$.

Several interesting results on fixed points of multi-valued functions have been given by various mathematicians. In 1941, Kakutani [19] proved that if M is a compact convex subset of Euclidean n-space and $F : M \to M$, a continuous multi-valued function such that for every x in M, the set F(x) is convex, then F has a fixed point. This result may be considered as an extension of Brouwer's fixed point theorem for Euclidean n-space from single-valued to multi-valued function. In 1946, Eilenberg and Montgomery [18] generalized Kakutani's result to acyclic absolute neighbourhood retracts* and upper semi-continuous mappings F such that F(x) is non-empty, compact and acyclic** for each x.

Strother [41], in 1953 showed that every continuous multi-valued mapping of the unit interval I into the non-empty compact subset of I has a fixed point but that the analogous result for the square $I \times I$ is false.

Plunkett [30], Ward [43] and others have studied the spaces having fixed point property for continuous, compact set valued mappings.

Recently Nadler Jr. [28] introduced the notion of multi-valued contraction mappings and gave some interesting results on existence of their fixed points. He has used the following notations and definitions:

<u>2.2.1</u>. If (X, d) is a metric space, then

(i) $CB(X) = \{C \mid C$ is a non-empty, closed and bounded subset of $X\}$.

(ii) $2^X = \{C \mid C$ is a non-empty compact subset of $X\}$.

---

* See page 15, T. Van Der Walt, "Fixed and Almost Fixed Points", Mathematical Centre Tracts, 1967.

**As above.

(iii) $N(\varepsilon, C) = \{x \in X | d(x, c) < \varepsilon$ for some c in $C\}$,

where $\varepsilon > 0$ and $C \in CB(X)$.

(iv) $H(A, B) = \inf\{\varepsilon | A \subset N(\varepsilon, B)$ and $B \subset N(\varepsilon, A)\}$,

where $\varepsilon > 0$; A and $B \in CB(X)$.

The function H is a metric (see [12]), called the Hausdorff metric. It is to notice that the metric H depends on the metric for X and that two equivalent metrics for X may not generate equivalent Hausdorff metric for $CB(X)$.

<u>Definition 2.2.2</u>. Let (X, d_1) and (Y, d_2) be two metric spaces. A multi-valued mapping $F : X \to CB(Y)$ is said to be continuous at a point x in X, if a sequence $\{x_n\}_{n=1}^{\infty}$ in X, converging to x (with respect to metric d) implies the convergence of the sequence $\{F(x_n)\}_{n=1}^{\infty}$ to $F(x)$ (with respect to metric H for $CB(Y)$). F is said to be continuous in X if it is continuous at each x in X.

<u>Definition 2.2.3</u>. Let (X, d_1) and (Y, d_2) be metric spaces. A function $F : X \to CB(Y)$ is said to be multi-valued Lipschitz mapping of X into Y iff $H(F(x), F(z)) \leq \alpha d_1(x, z)$, for all x, z in X, where $\alpha \geq 0$ is a fixed real number. If $\alpha < 1$, then F is called a <u>multi-valued contraction mapping</u> (abbreviated as m.v.c.m).

Since the mapping $i : X \to CB(X)$, given by $i(x) = \{x\}$ for each x in X, is an isometry, the fixed point theorem for the multi-valued mappings are generalizations of their single-valued analogues.

Nadler Jr. [28] gave the following theorem as a multi-valued analogue of Banach's contraction theorem.

<u>Theorem 2.2.4</u>. Let (X, d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued contraction mapping, then F has a fixed point.

<u>Proof</u>: Let $\alpha < 1$, a positive real number, be a contraction constant for F and p_o be an arbitrary point in X. Choose a point p_1 in $F(p_o)$. Since $F(p_o), F(p_1) \in CB(X)$ and $p_1 \in F(P_o)$, there is a point p_2 in $F(p_1)$ such that,

$$^* \; d(p_1, p_2) \leq H(F(p_o), F(p_1)) + \alpha \;.$$

Now, since $F(p_1), F(p_2) \in CB(X)$ and $p_2 \in F(p_1)$, there is a point p_3 in $F(p_2)$ such that,

$$d(p_2, p_3) \leq H(F(p_1), F(p_2)) + \alpha^2.$$

Proceeding in the same way we get a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X such that $p_i \in F(p_{i-1})$ and

$$d(p_i, p_{i+1}) \leq H(F(p_{i-1}), F(p_i)) + \alpha^i \quad \text{for } i = 1, 2, \ldots .$$

Now, $$d(p_i, p_{i+1}) \leq H(F(p_{i-1}), F(p_i)) + \alpha^i$$
$$\leq \alpha d(p_{i-1}, p_i) + \alpha^i .$$

---

*If $A, B \in CB(X)$ and $a \in A$, $\eta > 0$, then it is a simple consequence of the definition of $H(A, B)$ that there exists b in B such that $d(a, b) \leq H(A, B) + \eta$. Here α and subsequently α^i play the role of such an η.

$$\leq \alpha[H(F(p_{i-2}), F(p_{i-1})) + \alpha^{i-1}] + \alpha^{i}$$

$$\leq \alpha^2 d(p_{i-2}, p_{i-1}) + 2\alpha^{i} \leq \dots\dots\dots\dots$$

$$\leq \alpha^{i} d(p_o, p_1) + i\alpha^{i}, \text{ for all } i = 1, 2, \dots\dots .$$

$$\therefore\ d(p_i, p_{i+j}) \leq d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + d(p_{i+j-1}, p_{i+j}).$$

$$\leq \alpha^{i} d(p_o, p_1) + i\alpha^{i} + \alpha^{i+1} d(p_o, p_1) + (i+1)\alpha^{i+1} + \dots$$

$$+ \alpha^{i+j-1} d(p_o, p_1) + (i+j-1)\alpha^{i+j-1}.$$

$$= \alpha^{i} d(p_o, p_1)\{1 + \alpha + \alpha^2 + \dots + \alpha^{j-1}\}$$

$$+ i\alpha^{i}\{1 + \alpha + \alpha^2 + \dots + \alpha^{j-1}\}$$

$$+ \alpha^{i+1}\{1 + 2\alpha + 3\alpha^2 + \dots + (j-1)\alpha^{j-2}\} .$$

$$\leq \frac{\alpha^{i}}{1-\alpha} d(p_o, p_1) + \frac{i\alpha^{i}}{1-\alpha} + \frac{\alpha^{i+1}}{(1-\alpha)^2} .$$

For large i, the quantities on right hand side of the above inequality are sufficiently small.

Therefore $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since (X, d) is complete, the sequence $\{p_i\}_{i=1}^{\infty}$ converges to some point x_o in X. F being m.v.c.m. is continouos. Therefore the sequence $\{F(p_i)\}_{i=1}^{\infty}$ converges to $F(x_o)$ and since $p_i \in F(p_{i-1})$ for all $i = 1, 2, \dots,$ it follows that $x_o \in F(x_o)$, i.e. x_o is a fixed point of F.

Hence the theorem.

Remark: The theorem does not guarantee the uniqueness of fixed point.

A localized version of the above theorem, as a generalization of the theorem of Edelstein [14] to multi-valued mappings has also been given by Nadler Jr. [28].

First he gives the following definition which was modeled after Edelstein's definition for single-valued mapping.

Definition 2.2.5. A function $F : X \to CB(X)$ is said to be $(\varepsilon - \lambda)$-uniformly locally contractive multi-valued mapping (where $\varepsilon > 0$ and $0 \leq \lambda < 1$) provided that if, x, y in X and $d(x, y) < \varepsilon$, then, $H(F(x), F(y)) \leq \lambda d(x, y)$.

Theorem 2.2.6. Let (X, d) be a complete ε-chainable metric space. If $F : X \to 2^X$ is an $(\varepsilon - \lambda)$-uniformly locally contractive multi-valued mapping, then F has a fixed point.

Proof: For (x, y) in $X \times X$, define, $d_\varepsilon(x, y) = \inf\{\sum_{i=1}^{n} d(x_{i-1}, x_i)\}$, where infimum is taken over all ε-chains $x = x_o, x_1, x_2, \cdots, x_n = y$, joining x and y.

It is easily seen that d_ε is a metric for X satisfying,

(i) $d(x, y) \leq d_\varepsilon(x, y)$ for all x, y in X

and (ii) $d(x, y) = d_\varepsilon(x, y)$ if $d(x, y) < \varepsilon$.

Since (X, d) is complete, from (i) and (ii) it follows that (X, d_ε) is complete. Let H_ε be the Hausdorff metric for 2^X obtained from d_ε. It can be easily seen that if $A, B \in 2^X$ and $H(A, B) < \varepsilon$, then $H_\varepsilon(A, B) = H(A, B)$.

Now, let $x, y \in X$ and $x = x_o, x_1, x_2, \ldots, x_n = y$ be an ε-chain joining x to y. Since $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$,

$$H(F(x_{i-1}), F(x_i)) \le \lambda d(x_{i-1}, x_i) < \varepsilon \text{ , for all } i = 1, 2, \dots, n.$$

Thus
$$\begin{aligned} H_\varepsilon(F(x), F(y)) &\le \sum_{i=1}^{n} H_\varepsilon(F(x_{i-1}), F(x_i)) \\ &= \sum_{i=1}^{n} H(F(x_{i-1}), F(x_i)) \\ &\le \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) \end{aligned}$$

Since $x = x_0, x_1, x_2, \dots, x_n = y$ is an arbitrary ε-chain joining x to y, it follows that,

$H_\varepsilon(F(x), F(y)) \le \lambda d_\varepsilon(x, y)$. This proves that F is a m.v.c.m. with respect to d_ε and H_ε. Then by previous theorem F has a fixed point.

Thus the proof.

Let us recall a theorem of Kannan [20], which says that if (X, d) is a complete metric space and f, a mapping of X into itself, satisfying,

$$d(f(x), f(y) \le \alpha\{d(x, f(x)) + d(y, f(y))\} , \forall\, x, y \in X,$$

$$\text{where } 0 \le \alpha < \frac{1}{2} ,$$

then f has a unique fixed point.

We have been successful in generalizing this theorem to multi-valued mappings, under the similar notion of $CB(X)$, Hausdorff metric H, etc. as used by Nadler.

We denote the distance of a point $x \in X$, from a set $A \subset X$ by $\delta(x, A)$ which is defined as

$$\delta(x, A) = \inf\{d(x, y) \mid y \in A\} .$$

Thus we give:

<u>Theorem 2.2.7</u>: Let (X, d) be a complete metric space and let $F : X \to CB(X)$ be a continuous multi-valued mapping satisfying,

(a) $$H(F(x), F(y)) \leq \alpha\{\delta(x, F(x)) + \delta(y, F(y))\},$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{2}$.

Then F has a fixed point.

<u>Proof</u>: Let p_o be an arbitrary point in X.

Pick up a point $p_1 \in F(p_o)$. Since $F(p_o)$ $F(p_1) \in CB(X)$ and $p_1 \in F(p_o)$, there exists a point $p_2 \in F(p_1)$ such that,

$d(p_1, p_2) \leq H(F(p_o), F(p_1)) + \frac{\alpha}{1-\alpha}$ (see foot note page 49; we have taken $\frac{\alpha}{1-\alpha}$ and subsequently $(\frac{\alpha}{1-\alpha})^i$ in place of η).

Again since $F(p_1), F(p_2) \in CB(X)$ and $p_2 \in F(p_1)$, there exists a point $p_3 \in F(p_2)$ such that

$$d(p_2, p_3) \leq H(F(p_1), F(p_2)) + (\frac{\alpha}{1-\alpha})^2$$

Continuing in the same way we get a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X such that, $p_i \in F(p_{i-1})$ and,

$$d(p_i, p_{i+1}) \leq H(F(p_{i-1}), F(p_i)) + (\frac{\alpha}{1-\alpha})^i \quad \text{.................(I)},$$

for all $i = 1, 2, \ldots$.

Now for all i, $\delta(p_i, F(p_i)) = \inf\{d(p_i, y) | y \in F(p_i)\}$

$$\leq d(p_i, p_{i+1}) \quad \text{.................(II)},$$

since $p_{i+1} \in F(p_i)$.

The inequality (I) in light of condition (a) gives,

$$d(p_i, p_{i+1}) \leq \alpha\{\delta(p_{i-1}, F(p_{i-1})) + \delta(p_i, F(p_i))\} + (\frac{\alpha}{1-\alpha})^i$$

$$\leq \alpha\{d(p_{i-1}, p_i) + d(p_i, p_{i+1})\} + (\frac{\alpha}{1-\alpha})^i,$$

(By (II)).

or, $\quad (1-\alpha)d(p_i, p_{i+1}) \leq \alpha d(p_{i-1}, p_i) + (\frac{\alpha}{1-\alpha})^i$

i.e. $\quad d(p_i, p_{i+1}) \leq \frac{\alpha}{1-\alpha} d(p_{i-1}, p_i) + \frac{\alpha^i}{(1-\alpha)^{i+1}}$.

Similarly, $\quad d(p_i, p_{i-1}) \leq \frac{\alpha}{1-\alpha} d(p_{i-2}, p_{i-1}) + \frac{\alpha^{i-1}}{(1-\alpha)^i}$

$$\therefore \quad d(p_i, p_{i+1}) \leq (\frac{\alpha}{1-\alpha})^2 d(p_{i-2}, p_{i-1}) + 2\frac{\alpha^i}{(1-\alpha)^{i+1}} .$$

Proceeding in the same way, we have,

$$d(p_i, p_{i+1}) \leq (\frac{\alpha}{1-\alpha})^i d(p_o, p_1) + i\frac{\alpha^i}{(1-\alpha)^{i+1}}$$

i.e. $\quad d(p_i, p_{i+1}) \leq (\frac{\alpha}{1-\alpha})^i d(p_o, p_1) + \frac{i}{\alpha}(\frac{\alpha}{1-\alpha})^{i+1}$.

Now,

$$d(p_i, p_{i+j}) \leq d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + \ldots + d(_{i+j-1}, p_{i+j}).$$

$$\leq (\frac{\alpha}{1-\alpha})^i d(p_o,p_1) + \frac{i}{\alpha}(\frac{\alpha}{1-\alpha})^{i+1} + (\frac{\alpha}{1-\alpha})^{i+1} d(p_o,p_1) + \frac{i+1}{\alpha}(\frac{\alpha}{1-\alpha})^{i+2}$$

$$+ \ldots + (\frac{\alpha}{1-\alpha})^{i+j-1} d(p_o, p_1) + \frac{i+j-1}{\alpha}(\frac{\alpha}{1-\alpha})^{i+j} .$$

Since $\alpha < \frac{1}{2}$, therefore $\frac{\alpha}{1-\alpha} < 1$; and putting for simplicity,

$\frac{\alpha}{1-\alpha} = \gamma(< 1)$ we have,

$$d(p_i, p_{i+j}) \leq \gamma^i d(p_o, p_1) \{1 + \gamma + \gamma^2 + \ldots + \gamma^{j-1}\}$$

$$+ \frac{i}{\alpha}\gamma^{i+1} \{1 + \gamma + \gamma^2 + \ldots + \gamma^{j-1}\}$$

$$+ \frac{1}{\alpha}\gamma^{i+2} \{1 + 2\gamma + 3\gamma^2 + \ldots (j-1)\gamma^{j-2}\}$$

$$\leq \frac{\gamma^i}{1-\gamma} d(p_o, p_1) + \frac{i}{\alpha}\frac{\gamma^{i+1}}{(1-\gamma)} + \frac{1}{\alpha}\frac{\gamma^{i+2}}{(1-\gamma)^2} .$$

Since $\gamma < 1$, the quantities on right hand side of the above inequality are sufficiently small for large i. Therefore $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence.

Since (X, d) is complete, the sequence $\{p_i\}_{i=1}^{\infty}$ converges to some point $x_o \in X$. According to our assumption F is continuous in X, therefore the sequence $\{F(p_i)\}_{i=1}^{\infty}$ converges to $F(x_o)$. Since $p_i \in F(p_{i-1})$ for all $i = 1, 2, \ldots$, it follows that $x_o \in F(x_o)$ i.e. x_o is a fixed point of F.

Hence the theorem.

We generalize the result of Maia [26] also, to multi-valued mappings as follows:

<u>Theorem 2.2.8</u>. Let X be a set and, d_1 and d_2 be two different metrics on X (i.e. X_{d_1} and X_{d_2} be two metric spaces). If,

(i) $F : X_{d_1} \to CB(X_{d_1})$ is a continuous mapping of X_{d_1} into X_{d_1}, and X_{d_1} is complete.

(ii) $F : X_{d_2} \to CB(X_{d_2})$ is a multi-valued contraction mapping of X_{d_2} into X_{d_2} with contraction constant $\alpha (0 \leq \alpha \leq 1)$.

and (iii) $d_1(x, y) \leq d_2(x, y)$, $\forall\ x, y \in X$;

then F has a fixed point.

<u>Proof</u>: Let $x_o \in X$. Pick up any point $x \in F(x_o)$. Since $F(x_o)$ and $F(x_1)$ are non-empty closed and bounded subsets of X_{d_2} and $x_1 \in F(x_o)$, there exists a point $x_2 \in F(x_1)$ such that,

$$d_2(x_1, x_2) \leq H(F(x_o), F(x_1)) + \alpha .$$

Similarly $F(x_1), F(x_2) \in CB(X_{d_2})$ and $x_2 \in F(x_1)$, there is a point $x_3 \in F(x_2)$ such that, $d_2(x_2, x_3) \leq H(F(x_1), F(x_2)) + \alpha^2$.

Continuing in the same way, we get a sequence $\{x_i\}_{i=1}^{\infty}$ in X such that $x_i \in F(x_{i-1})$ and $d_2(x_i, x_{i+1}) \leq H(F(x_{i-1}), F(x_i)) + \alpha^i$.

The sequence $\{x_i\}_{i=1}^{\infty}$ can be shown to be Cauchy with respect to d_2 as in Theorem 3.2.4. By condition (iii) of the theorem it follows that $\{x_i\}_{i=1}^{\infty}$ is also Cauchy sequence in X_{d_1}. Since X_{d_1} is complete, therefore the sequence $\{x_i\}_{i=1}^{\infty}$ is d_1-convergent to a point $a_o \in X$. Since F is continuous on X_{d_1}, the sequence $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(a_o)$ and since $x_i \in F(x_{i-1})$ for all $i = 1, 2, \ldots,$ it follows that $a_o \in F(a_o)$, i.e. a_o is a fixed point of F.

Hence the theorem.

CHAPTER III

SEQUENCES OF MAPPINGS AND FIXED POINTS

3.1 The main objective of this chapter is to investigate the conditions under which the convergence of a sequence of contraction mappings to a mapping T (these mappings may also be of the type (2.1B), Chapter II) of a metric space into itself implies the convergence of their fixed points to the fixed point of T.

A partial solution to this problem has been given by Bonsall [5] as follows:

Theorem 3.1.1: Let (X, d) be a complete metric space. Let $T_n (n = 1, 2, \ldots)$ and T be contraction mappings of X into itself with the same Lipschitz constant $k < 1$, and with fixed points u_n and u respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$. Then $\lim_{n\to\infty} u_n = u$.

As pointed out by Nadler [27], the restriction that all contraction mappings have the "same Lipschitz constant $k < 1$" is very strong for one can easily construct a sequence of contraction mappings from the reals into the reals which converges uniformly to the zero mapping but whose Lipschitz constants tend to one.

Considering separately the uniform convergence and the point-wise convergence of a sequence of contraction mappings, Nadler [27] gave the following two theorems which modify the above result.

<u>Theorem 3.1.2</u>. Let (X, d) be a metric space, let $T_i : X \to X$ be a function with at least one fixed point u_i for each $i = 1, 2, \ldots$, and let $T_o : X \to X$ be a contraction mapping with fixed point u_o. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_o, then the sequence $\{u_i\}_{i=1}^{\infty}$ of fixed points converges to u_o.

<u>Proof</u>: $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_o therefore for $\varepsilon > 0$, there is a positive integer N such that $i \geq N$ implies $d(T_i x, T_o x) < \varepsilon(1-\alpha_o)$ for all $x \in X$, where $\alpha_o < 1$ is a Lipschitz constant for T_o.

We have,

$$\begin{aligned} d(u_i, u_o) &= d(T_i u_i, T_o u_o) \\ &\leq d(T_i u_i, T_o u_i) + d(T_o u_i, T_o u_o) \\ &\leq d(T_i u_i, T_o u_i) + \alpha_o d(u_i, u_o) \end{aligned}$$

i.e. $(1 - \alpha_o) d(u_i, u_o) \leq d(T_i u_i, T_o u_i)$

$\therefore$ $i \geq N$, $(1 - \alpha_o) d(u_i, u_o) < \varepsilon(1 - \alpha_o)$.

i.e. $d(u_i, u_o) < \varepsilon$, since $0 \leq \alpha_o < 1$.

This proves that $\{u_i\}_{i=1}^{\infty}$ converges to u_o.

<u>Theorem 3.1.3</u>. Let (X, d) be a locally compact metric space, let $A_i : X \to X$ be a contraction mapping with fixed point a_i for each $i = 1, 2, \ldots$ and let $A_o : X \to X$ be a contraction mapping with fixed point a_o. If the sequence $\{A_i\}_{i=1}^{\infty}$ converges pointwise to A_o, then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_o.

Proof: Let $\varepsilon > 0$ be a sufficiently small real number so that $K(a_o, \varepsilon) = \{x \in X | d(a_o, x) \leq \varepsilon\}$ is a compact subset of X.

$\{A_i\}_{i=1}^{\infty}$, being a sequence of contraction mappings, is an equicontinuous sequence of functions converging pointwise to A_o, and $K(a_o, \varepsilon)$ is compact therefore the sequence $\{A_i\}_{i=1}^{\infty}$ converges uniformly* on $K(a_o, \varepsilon)$ to A_o. Thus for $\varepsilon > 0$, there is a positive integer N such that $i \geq N$ implies $d(A_i(x), A_o(x)) < (1 - \alpha_o)\varepsilon$ for all $x \in K(a_o, \varepsilon)$, where $\alpha_o < 1$ is Lipschitz constant for A_o. Now, for $i \geq N$ and $x \in K(a_o, \varepsilon)$,

$$\begin{aligned} d(A_i(x), a_o) &= d(A_i(x), A_o(a_o)) \\ &\leq d(A_i(x), A_o(x)) + d(A_o(x), A_o(a_o)) \\ &< \varepsilon(1 - \alpha_o) + \alpha_o d(x, a_o) \\ &\leq \varepsilon(1 - \alpha_o) + \alpha_o \varepsilon = \varepsilon , \end{aligned}$$

which proves that A_i maps $K(a_o, \varepsilon)$ into itself for $i \geq N$. Let B_i be the restriction of A_i to $K(a_o, \varepsilon)$ for each $i \geq N$. Thus B_i is a contraction mapping of $K(a_o, \varepsilon)$ into itself for $i \geq N$. Since $K(a_o, \varepsilon)$ is compact, it is a complete metric space. Therefore B_i has a unique fixed point for each $i \geq N$, which must be a_i because $B_i = A_i$ on $K(a_o, \varepsilon)$ for $i \geq N$ and a_i is a fixed point of A_i. Hence $a_i \in K(a_o, \varepsilon)$ for each $i \geq N$. It follows that the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_o.

Hence the theorem.

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* The pointwise convergence of an equicontinuous sequence of functions on a compact set implies the uniform convergence of the sequence. See Rudin [32].

Note: A similar theorem for a sequence of contractive mappings converging pointwise to a contraction mapping has been proved by Singh [36].

Remark 3.1.4: In the above theorem the condition of locally compactness on the space is necessary. An example has been cited in [27] to show that in non-locally compact spaces, a sequence of contraction mappings may converge pointwise to a contraction mapping without the sequence of their fixed points converging.

Another approach to modify Theorem 3.1.1 of Bonsall is due to Singh [40] where the restriction that all the contractions have the same Lipschitz constant has been relaxed in the following way:

Theorem 3.1.5. Let (X, d) be a complete metric space and let $T_n : X \to X$ be a contraction mapping with Lipschitz constant k_n and with fixed point u_n for each $n = 1, 2, \ldots$. Furthermore, if $k_{n+1} \leq k_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself. Then T has a unique fixed point and sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

Proof: Since T_n is contraction with Lipschitz constant k_n,

$$d(T_n x, T_n y) \leq k_n d(x, y), \quad \text{for all } x, y \in X.$$

and thus $$\lim_{n\to\infty} d(T_n x, T_n y) \leq \lim_{n\to\infty} k_n d(x, y).$$

Since $k_{n+1} \leq k_n < 1$ for each n, it follows that $\lim_{n\to\infty} k_n < 1$.

Hence $\lim_{n\to\infty} T_n x = Tx$ is a contraction mapping. Moreover k_1 serves the purpose of a Lipschitz constant for all $T_n (n = 1, 2, \ldots)$. Thus the proof follows from Theorem 3.1.1 on replacing k by k_1.

The theorem may be illustrated by taking the following example.

Example 3.1.6: Let $T_n : [0, 1] \to [0, 1]$ be defined by,

$T_n x = 1 - \frac{1}{n+1} x$ for all $x \in [0, 1]$; $n = 1, 2, 3, \ldots$.

Obviously T_n is a contraction mapping of $[0, 1]$ into itself, with Lipschitz constant $k_n = \frac{1}{n+1}$ for each $n = 1, 2, \ldots$. As we observe $k_{n+1} \leq k_n < 1$ for each n, $k_1 = \frac{1}{2}$ will serve the purpose of Lipschitz constant for all the mappings. The unique fixed point for T_n is $u_n = \frac{n}{n+1}$ for each $n = 1, 2, \ldots$. The limiting function T is given by,

$$Tx = \lim_{n\to\infty} T_n x = 1 \quad \text{for every } x \in [0, 1] .$$

Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$, where 1 is a unique fixed point for T.

Remark 3.1.7: (i) If the Lipschitz constants are such that $k_{n+1} \geq k_n$ for each n, the theorem is, in general false. Russell [33] has given the following example to justify this remark.

Let $T_n : E^1 \to E^1$ be defined as

$$T_n x = p + \frac{n}{n+1} x \quad (n = 1, 2, \ldots) , \ p > 0,$$

for all $x \in E^1$, where $E^1 = (-\infty , +\infty)$.

We see that T_n is a contraction mapping, with Lipschitz constant $k_n = \frac{n}{n+1}$ and with fixed point $u_n = (n + 1)p$ for each $n = 1, 2, \ldots$.

Now $Tx = \lim_{n\to\infty} T_n x = p + x$ for every $x \in E^1$. Thus under the mapping T, every point of E^1 has been translated by a distance p and therefore T has no fixed point. Moreover,

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (n + 1)p = \infty \notin E^1$.

Remark 3.1.7: (ii) Singh [36] has further modified the last theorem by replacing the condition $k_{n+1} \leq k_n < 1$ by $k_n \to k < 1$.

Further Singh and Russell [38] proved the following theorem for a sequence of uniformly locally contractive mappings.

Theorem 3.1.8. Let (X, d) be a complete ε-chainable metric space and let T_n $(n = 1, 2, \ldots)$ be a mapping of X into itself such that

$$d(x, y) < \varepsilon \Rightarrow d(T_n x, T_n y) \leq kd(x, y),$$

where k is a real number such that $0 \leq k < 1$. If u_n is the fixed point of T_n, for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself, then T has a unique fixed point and sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

Now we want to extend the result of Bonsall to the sequence of mappings satisfying the condition of Rakotch [31].

Theorem 3.1.9. Let (X, d) be a complete metric space and let T_n $(n = 1, 2, \ldots)$ be mappings of X into itself satisfying,

(i) $d(T_n x, T_n y) \leq \lambda(x, y) \cdot d(x, y)$ for all $x, y \in X$.

where $\lambda(x, y) = \lambda(d(x, y))$ is a monotonic decreasing real valued function such that $0 \leq \lambda(x, y) < 1$. If u_n is the fixed point of T_n $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself, then T has a unique fixed point and sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

<u>Proof</u>: Since $\lim_{n\to\infty} T_n x = Tx$ for each $x \in X$ and the function $\lambda(x, y)$ in condition (i) is the same for all n, we have

$$d(Tx, Ty) = d(\lim_n T_n x, \lim_n T_n y) = \lim_n d(T_n x, T_n y) \leq \lambda(x, y) d(x, y)$$

for all $x, y \in X$. Since (X, d) is complete, it follows from a result of Rakotch [31] that T has a unique fixed point u (say).

Now $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, therefore for $\varepsilon > 0$, there is a positive integer N such that $n \geq N$ implies $d(T_n u, Tu) < \varepsilon\, \eta$ where $\eta = \min_{n \geq N} \{(1 - \lambda(u_n, u))\}$, obviously η is a real number such that $0 < \eta \leq 1$.

Now for any n,
$$\begin{aligned} d(u_n, u) &= d(T_n u_n, Tu) \\ &\leq d(T_n u_n, T_n u) + d(T_n u, Tu) \\ &\leq \lambda(u_n, u) d(u_n, u) + d(T_n u, Tu) \end{aligned}$$

i.e. $\quad (1 - \lambda(u_n, u))\, d(u_n, u) \leq d(T_n u, Tu)$

$\therefore$ For $n \geq N$, $(1 - \lambda(u_n, u)) d(u_n, u) < \varepsilon\, \eta$

and therefore $\min_{n \geq N}\{(1 - \lambda(u_n, u))\} \quad d(u_n, u) < \varepsilon\, \eta$

i.e. $\quad \eta d(u_n, u) < \varepsilon\, \eta$

i.e. $\quad d(u_n, u) < \varepsilon$, since $\eta > 0$.

Hence $\lim_{n\to\infty} u_n = u$.

This completes the proof.

Remark 3.1.10: As discussed in Chapter I (Remark 1.2.18), the result of Rakotch holds also when the function $\lambda(x, y)$ is taken to be monotonic increasing on $d(x, y)$ in place of monotonic decreasing. Thus the preceeding theorem will remain true, without loss of generality, when the monotonicity of the function $\lambda(x, y)$ taken is reversed.

The next theorem deals with a sequence of mappings satisfying the localized version of the condition of Rakotch. In this case we will assume the usual function $\lambda(x, y)$ to be monotonic increasing.

The notion of localization and that of ε-chainability of a metric space which we use is the same as mentioned earlier..

Theorem 3.1.11. Let (X, d) be a complete, ε-chainable (ε being a positive real number) metric space and let T_n $(n = 1, 2, \dots)$ be a mapping of X into itself such that,

$$d(x, y) < \varepsilon \Rightarrow d(T_n x, T_n y) \leq \lambda(x, y) d(x, y),$$

where $\lambda(x, y) = \lambda(d(x, y))$ is a real valued monotonic increasing function of the interval $(0, \varepsilon]$ into the interval $[0, 1)$. If u_n is the fixed point of T_n for $n = 1, 2, \dots$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$ where T is a mapping of X into itself, then T has a unique fixed point and the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points of T_n converges to the fixed point of T.

Proof: For $(x, y) \in X \times X$ define,

$$d_\varepsilon(x, y) = \inf \{ \sum_{i=1}^{p} d(x_{i-1}, x_i) \}$$, where infimum is taken over all ε-chains $x = x_o, x_1, x_2, \dots, x_p = y$ joining x and y.

Obviously, d_ε is a metric for X satisfying,

(i) $d(x, y) \leq d_\varepsilon(x, y)$ for all $x, y \in X$, and

(ii) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$.

Since (X, d) is complete, from (i) and (ii) it follows that (X, d_ε) is complete.

Now, for any $x, y \in X$ and any ε-chain $x = x_o, x_1, x_2, \dots, x_p = y$ joining x and y we have,

$$d(x_{i-1}, x_i) < \varepsilon \qquad (i = 1, 2, \dots, p).$$ Therefore for all $n = 1, 2, \dots$,

$$d(T_n x_{i-1}, T_n x_i) \leq \lambda(x_{i-1}, x_i) d(x_{i-1}, x_i) < \varepsilon \qquad (i = 1, 2, \dots, p).$$

Hence $T_n(x_o), T_n(x_1), \dots, T_n(x_p)$ is an ε-chain joining $T_n(x)$ and $T_n(y)$, and

$$\begin{aligned} d_\varepsilon(T_n x, T_n y) &\leq \sum_{i=1}^{p} d(T_n x_{i-1}, T_n x_i) \\ &\leq \sum_{i=1}^{p} \lambda(x_{i-1}, x_i) d(x_{i-1}, x_i) \\ &\leq \lambda(\varepsilon) \sum_{i=1}^{p} d(x_{i-1}, x_i) \end{aligned}$$, (Since $d(x_{i-1}, x_i) < \varepsilon$ implies $\lambda(d(x_{i-1}, x_i)) = \lambda(x_{i-1}, x_i) \leq \lambda(\varepsilon)$, for $i = 1, 2, \dots, p$)

Now since $x_o, x_1, x_2, \ldots, x_p$ is an arbitrary ε-chain, we have

$$d_\varepsilon(T_n x, T_n y) \le \lambda(\varepsilon) d_\varepsilon(x, y) \qquad (n = 1, 2, \ldots),$$

where $\lambda(\varepsilon) < 1$.

Thus $T_n (n = 1, 2, \ldots)$ are contraction mappings with respect to d_ε. (X, d_ε) is a complete metric space. Then $Tx = \lim_{n\to\infty} T_n x$ for every $x \in X$, is a contraction mapping with respect to d_ε and with $\lambda(\varepsilon)$ as Lipschitz constant so that T has a unique fixed point u (say) and $\lim_{n\to\infty} u_n = u$ by Theorem 3.1.1.

Hence the theorem.

3.2. We now investigate few interesting results as a solution to the problem posed in the beginning of this chapter for the mappings of the type:

$$f : X \to X \text{ s.t. } d(f(x), f(y)) \le \alpha\{d(x, f(x)) + d(y, f(y))\}$$

for all $x, y \in X$, where α is a non-negative real number.

Let us call α to be a mapping constant for f.

<u>Theorem 3.2.1</u>. Let (X, d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1, 2, \ldots$. Suppose there is a non-negative real number α such that,

(A) $\quad d(T_n x, T_n y) \le \alpha\{d(x, T_n x) + d(y, T_n y)\}$ for all $x, y \in X$ $(n = 1, 2, \ldots)$.

If the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping

$T : X \to X$, with a fixed point u, then u is a unique fixed point of T and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u.

<u>Proof</u>: $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T, therefore for $\varepsilon > 0$ and for $u \in X$, there is a positive integer N such that $n \geq N$ implies,

$$d(Tu, T_n u) < \frac{\varepsilon}{(1 + \alpha)},$$ where α is the same as in condition (A).

Now we have for any n,

$$\begin{aligned} d(u, u_n) &= d(Tu, T_n u_n) \\ &\leq d(Tu, T_n u) + d(T_n u, T_n u_n) \\ &\leq d(Tu, T_n u) + \alpha\{d(u, T_n u) + d(u_n, T_n u_n)\} \end{aligned}$$

i.e. $d(u, u_n) \leq (1 + \alpha) d(u, T_n u)$, Since u and u_n are fixed-points of T and T_n respectively.

Therefore for $n \geq N$, $d(u, u_n) < (1 + \alpha) \dfrac{\varepsilon}{(1 + \alpha)} = \varepsilon$.

i.e. $\{u_n\}_{n=1}^{\infty}$ converges to u.

To show that u is a unique fixed point of T, let v be another fixed point of T. Then in the similar way $\{u_n\}_{n=1}^{\infty}$ converges to v which implies $u = v$.

Hence the theorem.

A simple corollary to this theorem assuring the existence of fixed points of the mappings T_n together, the convergence of the sequence of the fixed points to the fixed point of the limiting function may be given as follows:

Corollary 3.2.2: Let (X, d) be a complete metric space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of X into itself satisfying condition (A) of the above theorem with $0 \leq \alpha < \frac{1}{2}$. If $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping $T : X \to X$ with a fixed point u, then (i) the fixed point u is unique, (ii) each T_n has a unique fixed point for $n = 1, 2, \ldots$ and (iii) the sequence of fixed points of T_n converges to u.

Proof: The conclusion (ii) is implied directly by a result of Kannan [20] and (i) and (iii) follow from the theorem.

The condition (A) in the preceeding theorem can be relaxed by allowing the mapping constant α to change with T_n in the following way generalizing thereby the above theorem.

Theorem 3.2.3. Let (X, d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n and with mapping constant α_n for each $n = 1, 2, \ldots$, such that,

$$d(T_n x, T_n y) \leq \alpha_n \{d(x, T_n x) + d(y, T_n y)\}, \text{ for all } x, y \in X,$$

where α_n is a non-negative real number.

Suppose $\lim_{n\to\infty} \alpha_n = \alpha \geq 0$ and the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping $T : X \to X$, with a fixed point u. Then u is the unique fixed point of T and the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Proof: $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T, therefore for $\varepsilon > 0$ and $u \in X$, there is a positive integer N such that $n \geq N$ implies,

$d(Tu, T_n u) < \frac{\varepsilon}{\eta}$, where η is a positive real number defined as,

$$\eta = \max_{n \geq N} \{(1 + \alpha_n)\}$$

Now, we have for any n, exactly in the same way as in previous theorem,

$$d(u, u_n) \leq (1 + \alpha_n) d(Tu, T_n u)$$

Therefore for $n \geq N$,

$$d(u, u_n) < (1 + \alpha_n) \frac{\varepsilon}{\eta}$$

$$\leq \max_{n \geq N} \{(1 + \alpha_n)\} \frac{\varepsilon}{\eta} = \eta \frac{\varepsilon}{\eta} = \varepsilon .$$

Thus $\{u_n\}_{n=1}^{\infty}$ converges to u.

The uniqueness of u as a fixed point of T follows as in previous theorem.

Hence the theorem.

Next we give the following interesting result under the uniform convergence of the sequence of mappings.

<u>Theorem 3.2.4</u>. Let (X, d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1, 2, \ldots$. Let $T : X \to X$ be a mapping with a fixed point u such that,

(B) $d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\}$ for all $x, y \in X$, where α is a non-negative real number. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Proof: The condition (B) implies that the given fixed point u of T is unique for let v be another fixed point of T, then

$$d(u, v) = d(Tu, Tv)$$
$$\leq \alpha\{d(u, Tu) + d(v, Tv)\} = 0,$$

which gives that $u = v$.

Since $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, given $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies,

$$d(T_n u, Tu) < \frac{\varepsilon}{1 + \alpha} \quad , \text{ where } \alpha \text{ is the same as in condition (B).}$$

Now for any n,

$$d(u, u_n) = d(Tu, T_n u_n)$$
$$\leq d(Tu, Tu_n) + d(Tu_n, T_n u_n)$$
$$\leq \alpha\{d(u, Tu) + d(u_n, Tu_n)\} + d(Tu_n, T_n u_n)$$
$$= \alpha \cdot 0 + \alpha d(T_n u_n, Tu_n) + d(Tu_n, T_n u_n),$$

since u and u_n are fixed points of T and T_n respectively.

$$\therefore \; d(u, u_n) \leq (1 + \alpha) d(T_n u_n, Tu_n)$$

Thus for $n \geq N$, $d(u, u_n) < (1 + \alpha) \cdot \dfrac{\varepsilon}{(1 + \alpha)} = \varepsilon$.

Hence $\{u_n\}_{n=1}^{\infty}$ converges to u.

This completes the proof.

Remark 3.2.5: Singh [37] has proved the above theorem under a restriction $0 \leq \alpha < \frac{1}{2}$ on 'α' which in fact is not necessary, as is seen in the proof.

We present an example also which justifies this remark.

Example 3.2.6: Let $T_n : [0, 2] \to [0, 2]$ be defined as

$$T_n x = \frac{1}{n} + \frac{n}{3n+1} x \quad \text{for all } x \in [0, 2] .$$
$$(n = 1, 2, \ldots).$$

Clearly the fixed point of T_n is given by,

$$u_n = \frac{3n+1}{2n^2+1} \qquad \text{for each } n = 1, 2, \ldots .$$

Also $\quad Tx = \lim_{n\to\infty} T_n x = \frac{1}{3} x \qquad$ for all $x \in [0, 2]$,

and thus $u = 0$ is the fixed point of T.

It is easily seen by taking $x = 1$ and $y = 0$ in condition (B) that T fails to satisfy this condition for $\alpha < \frac{1}{2}$. But for any real $\alpha \geq \frac{1}{2}$ the condition is satisfied for all the points in $[0, 2]$.

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3n+1}{2n^2+1} = 0 = u.$

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