THE STANDARD ALGEBRAS

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by

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ABSTRACT

We discuss the tensor, symmetric, and exterior algebras of a vector space.

Chapter 0 contains algebraic preliminaries.

In Chapter I we define the tensor product $V \otimes W$ of two vector spaces and then the tensor product of a finite number of vector spaces. A theorem concerning the existence and uniqueness of the tensor product is proved. Let V^r denote $V \otimes \ldots \otimes V$ (r times). We define an operation called multiplication of tensors which pairs an element of V^r and an element of V^s with an element of V^{r+s}. This defines a multiplicative structure on the (weak) direct sum $\otimes V = R + V + V^2 + V \otimes V + V^2 \otimes V^2 + \ldots$ We call $\otimes V$ the tensor algebra of the vector space V and prove a theorem concerning its existence and uniqueness. Let V^* denote the dual space of V and $(V^*)^r$ denote $V^* \otimes \ldots \otimes V^*$ (r times). We show that $(V^*)^r$ can be identified with $(V^r)^*$, the dual space of V^r. This identification establishes the pseudo product for the pair V^r, $(V^r)^*$:

$$\langle v_1 \otimes \cdots \otimes v_r, \bar{v}_1 \otimes \cdots \otimes \bar{v}_r \rangle = \langle v_1, \bar{v}_1 \rangle \cdots \langle v_r, \bar{v}_r \rangle .$$

In the final section we discuss the induced covariant and contravariant homomorphisms.

We give parallel discussions for the symmetric and exterior algebras. In Chapter II we give constructual and conceptual definitions of $V^{(r)}$, the space of symmetric contravariant tensors of degree r, and show the existence and uniqueness of $V^{(r)}$. We define an operation called symmetric multiplication which pairs an element of $V^{(r)}$ and an element of $V^{(s)}$ with an element of $V^{(r+s)}$. We then have a multiplicative structure on the direct sum

$$\bigcirc V = \mathbb{R} + V^{(1)} + V^{(2)} + \cdots + V^{(r)}$$

and we call $\bigcirc V$ the symmetric algebra of V. We prove its existence and uniqueness. We discuss the duality in the symmetric algebra and show that $(V^{(r)})^*$ can be identified with $(V^*)^{(r)}$. This establishes the pseudo product for the pair $V^{(r)}, (V^{(r)})^*$. In fact, we prove the formula

$$\langle v_1 \circ \cdots \circ v_r, \bar{v}_1 \circ \cdots \circ \bar{v}_r \rangle = \frac{1}{r!} \sum_{\sigma} \langle v_{\sigma_1}, \bar{v}_1 \rangle \cdots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

and show the relationship between this pseudo product and the permanent function.

In Chapter III we define $V^{[r]}$, the space of antisymmetric (alternate) tensors of degree r . We proceed as in Chapter II. Having defined exterior multiplication, we have a multiplicative structure on the direct sum

$$\wedge V = \mathbb{R} + V^{[1]} + V^{[2]} + \dots + V^{[n]}$$

and we call $\wedge V$ the exterior algebra of V . We show that $(V^{[r]})^*$ can be identified with $(V^*)^{[r]}$

We prove that

$$\langle v_1 \wedge \dots \wedge v_r, \bar{v}_1 \wedge \dots \wedge \bar{v}_r \rangle = \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn} \sigma) \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

and show the relationship between this pseudo product and the determinant.

CONTENTS

PREFACE

Tensors are by no means new. Tensors, as well as vectors, were first discovered and introduced by physicists. In classical mechanics the "tension" of an elastic or quasi-elastic body can be defined and turns out to be a "tensor" - hence the choice of the term. Beside the stress and strain tensors, the theory of relativity works with the tensor of gravitation, the tensor of momentum energy and of the electro-magnetic field. In differential geometry the curvature and torsion of higher dimensional differentiable manifolds are tensors. In topology homology - cohomology theory works with tensors.

The physicist still defines his tensors in the fashion of the nineteenth century when there was great over-indulgence in coordinates and matrix computation. The trend in the twentieth century, however, thanks to the now famous Bourbaki group, is to give an invariant treatment which does not concern itself with an irrelevant choice of a coordinate system. We attempt to give such an invariant treatment since the invariant approach in linear algebra not only results in greater economy but also preserves the geometric insight which is all but lost in the maze of indices and coordinates of the physicist.

In Chapter 0 we collect the various algebraic facts that will be needed and fix our terminology and notation. In Chapter I we discuss the tensor algebra of a vector space. The space of symmetric tensors discussed in Chapter II and the space of antisymmetric tensors discussed in Chapter III are subspaces of the space of contravariant tensors.

In differential geometry the exterior forms of Chapter III are used when we consider submanifolds of a differentiable manifold. We use the symmetric forms of Chapter II when we consider higher order contact osculating surfaces or "supermanifolds" of a differentiable manifold.

The exterior algebra was discovered by Grassman in the nineteenth century but it was Élie Cartan (1869-1951) who rediscovered Grassman's work and applied it to analysis. Cartan is the father of contemporary differential geometry and introduced many new important tools into mathematical research, for example, the exterior derivative of alternating differential forms. Much of his work is still not fully appreciated.

I acknowledge with deep gratitude my debt to Mr. A.E. Fekete for his unfailing help and guidance in the past four years and especially in the preparation of this manuscript. I am also deeply indebted to various members of the Department of Mathematics for their encouragement and to the Bank of Montreal for its financial support.

0

ALGEBRAIC PRELIMINARIES

In this chapter we will collect the various algebraic facts that will be needed and fix our terminology and notation.

We shall be considering further dimensional vector spaces over the field of the real numbers; this field will be denoted by

A transformation f of a vector space V into a vector space Z is called <u>linear</u> if

0.1 $$f(av_1 + bv_2) = af(v_1) + bf(v_2) \quad ; \quad \begin{matrix} v_1, v_2 \in V \\ a, b \in R \end{matrix}$$

A linear transformation from one vector space to another is also called a <u>homomorphism</u> and we have the following classification of homomorphisms:

An injective (resp: a surjective, bijective) homomorphism is called a <u>monomorphism</u> (resp: an <u>epimorphism</u>, <u>isomorphism</u>).

A homomorphism $f: V \longrightarrow Z$ is called <u>canonical</u>, or more commonly, <u>natural</u>, if it depends only on the properties of V, Z as vector spaces and not on some further choice such as bases, etc.

Addition and scalar multiplication of homomorphisms are defined in the following way:

$$\text{0.2} \qquad (f+g)\,v = fv + gv$$
$$(cf)\,v = cf(v) \qquad ; \quad v \in V, \quad c \in R$$

With this definition of addition and scalar multiplication then, the set $Hom(V, Z)$ of homomorphisms from a vector space V into a vector space Z is a vector space.

Let V, W, Z be vector spaces. A transformation $f: V \times W \longrightarrow Z$ is called <u>bilinear</u> if it is linear in each variable separately, i.e.

$$\text{0.3} \qquad f(a_1 v_1 + a_2 v_2,\ b_1 w_1 + b_2 w_2) = a_1 b_1 f(v_1, w_1) + a_1 b_2 f(v_1, w_2) + a_2 b_1 f(v_2, w_1) + a_2 b_2 f(v_2, w_2)$$

$$v_1, v_2 \in V; \quad w_1, w_2 \in W; \quad a_1, a_2, b_1, b_2 \in R$$

With addition and scalar multiplication defined similarly to 0.2 the set $Hom(V, W ; Z)$ of all bilinear transformations from $V \times W$ into Z is a vector space.

We have the following natural isomorphisms:

$$\text{0.4} \quad Hom(V, W; Z) \cong Hom(V; Hom(W, Z)) \cong Hom(W, Hom(V, Z))$$

More generally let $V_1, V_2, \ldots, V_k$ be vector spaces. A transformation f of $V_1, V_2, \ldots, V_k$ into a vector space Z is called <u>multilinear</u> if it is linear in each of its variables separately. The set $Hom(V_1, V_2, \ldots, V_k ; Z)$

of all multilinear transformations from $V_1 \times V_2 \times \ldots \times V_k$ into Z is a vector space.

Let V, W, Z be vector spaces and let $f: V \to W$ be linear. The induced function

$$0.5 \qquad f^*: Hom(W, Z) \longrightarrow Hom(V, Z)$$

defined by $f^* g = gf$ for every $g \in Hom(W, Z)$ is linear.

Similarly, if $g: W \to Z$ is linear, the the induced function

$$0.6 \qquad g_*: Hom(V, W) \longrightarrow Hom(V, Z)$$

defined by $g_* f = gf$ for every $f \in Hom(V, W)$ is linear.

Let S be a set of vectors in V. We say that is a <u>linearly independent</u> set of vectors if for every positive integer n, the relation

$$0 = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad ; \quad v_i \in S, \; c_i \in R$$

implies $c_1 = c_2 = \ldots = c_n = 0$

Otherwise S is said to be <u>linearly dependent</u>.

We denote the set of all linear conbinations by $L(S)$. We say that a set of linearly independent vectors on V is <u>maximal</u> if $L(S) = V$ We define a <u>basis</u> to be a maximal set of linearly independent vectors.

If V is a finite dimensional vector space then V has a basis. Any basis for V is a finite set and any two bases have the same number of elements. This number is called the dimension of V and is denoted by dim V.

A vector space V is called metric if a scalar product or, more commonly, a dot product is defined in in the following way:

A dot product in V is a function which assigns to each pair of vectors $x, y \in V$ a real number denoted by $x.y$ having the following properties:

i) $x.y = y.x$ for all $x, y \in V$

ii) $(cx).y = c(x.y)$; $x, y \in V$, $c \in \mathbb{R}$

iii) $(x+y).z = x.z + y.z$

$x.(y+z) = x.y + x.z$; $x, y, z \in V$

iv) $x.x \geq 0$ but $x.x = 0$ if, and only if, $x = 0$

Every finite dimensional vector space has at least one dot product. Since a basis can be chosen in many ways, it is reasonable to expect that a finite dimensional vector space will have many dot products. This is the case even though different basis can lead to the same dot product.

If $x \in V$, $x.x$ has a unique square root which is ≥ 0 since $x.x \geq 0$ this root is denoted by $|x|$ and is called the length of x. The length is sometimes called the norm or absolute value. A vector x in V is called <u>normal</u> if its length is 1 or, equivalently, if $x.x = 1$

Let V be a vector space with a fixed dot product. Two vectors x, y in V are <u>orthogonal</u> if $x.y = 0$ A <u>basis</u> in V is <u>orthogonal</u> if each two distract basis vectors are othogonal. A basis in V is <u>orthonormal</u> if it is orthogonal and each basis vector is normal. We remark that every metric vector space has an orthonormal basis.

If $x. y$ are non-zero vectors in V, the <u>Schwarz inequality</u> gives

$$0.7 \qquad -1 \leq \frac{x.y}{|x||y|} \leq 1$$

We may therefore define the angle θ between x and y by

$$\cos\theta = \frac{x.y}{|x||y|} , \quad 0 \leq \theta \leq \pi$$

This gives the formula

$$0.8 \qquad x.y = |x|\,|y| \cos\theta$$

For any vector space V the vector space $Hom(V, \mathbb{R})$ is called the <u>dual space</u> of V and is denoted by V^*. The elements of V^* are called <u>linear forms</u> on V and we denote them by $\bar{v}$. We adopt a standard notation

$$0.9 \qquad \bar{v}(v) = \langle v, \bar{v} \rangle \quad ; \quad v \in V, \ \bar{v} \in V^*$$

and refer to this as the "pseudo-dot product" of the covariant vector v and the contravariant vector $\bar{v}$.

For any vector space, the vector space $Hom(V^*, \mathbb{R})$ is called the bidual space of V and is denoted by V^{**}.

Let $\Omega : V \longrightarrow V^{**}$ be defined as follows: Ωv is the linear form on V^* determined by

0.10 $$\langle \bar{v}, \Omega v \rangle = \langle v, \bar{v} \rangle \quad , \quad \bar{v} \in V^*$$

Then, if V is finite dimensional, Ω is a natural isomorphism. We shall use this natural isomorphism for the purposes of identification

0.11 $$V^{**} = V$$

and this will be manifested by the fact that the pseudo product is commutative, i.e.

0.12 $$\langle v, \bar{v} \rangle = \langle \bar{v}, v \rangle$$

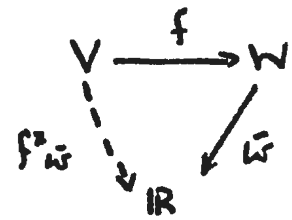
The function $\langle v, \bar{v} \rangle$ is then defined in the Cartesian product $V \times V^*$ taking scalar values and is bilinear; i.e.

0.13 $$\langle a v_1 + b v_2 , \bar{v}_1 \rangle = a \langle v_1, \bar{v}_1 \rangle + b \langle v_2, \bar{v}_1 \rangle$$
$$\langle v_1, a \bar{v}_1 + b \bar{v}_2 \rangle = a \langle v_1, \bar{v}_1 \rangle + b \langle v_1, \bar{v}_2 \rangle$$
$$v_1, v_2 \in V \; ; \; \bar{v}_1, \bar{v}_2 \in V^* \; ; \; a, b \in \mathbb{R}$$

Taking $Z = \mathbb{R}$ in 0.5 we have the following: Any linear transformation $f : V \longrightarrow W$ induces a linear transformation $f^*: V^* \longleftarrow W^*$ defined such that for $\bar{w} \in W^*$,

$$f^* \bar{w} = \bar{w} f$$

f^* is unique, i.e. the diagram



is commutative. In other words, for all $v \in V$, $\bar{w} \in W^*$.

$$\langle fv, \bar{w} \rangle = \langle v, f^* \bar{w} \rangle \tag{0.14}$$

Let V be finite dimensional and let $\{e_1, \ldots, e_n\}$ be a basis for V. Then there is a uniquely defined base $\{\bar{e}_1, \ldots, \bar{e}_n\}$ in V^* which satisfies the conditions

$$\langle e_i, \bar{e}_j \rangle = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{0.15}$$

This basis of V^* is called the dual basis of V^* relative to the given basis for V.

If V is a metric vector space the relation $\Phi: V \to V^*$ defined such that $\langle v, \Phi w \rangle = v \cdot w = \langle \Phi^* v, w \rangle$ is a natural isomorphism which is always used to identify V^* with V ; then the pseudo dot product is identified with the dot product and a self-dual basis with an orthonormal basis.

We remark that if V is a finite dimensional vector space, then

$$\dim V = \dim V^* = \dim V^{**} \tag{0.16}$$

We make only two further remarks in this chapter.

We recall that the coinage of a linear transformation f, written $\mathrm{coim}\, f$, is defined by $\mathrm{coim}\, f = V/K$ where $K = \ker f$. For every homomorphism $f \in \mathrm{Hom}(V, W)$ we have a natural isomorphism

$$\varphi : \mathrm{coim}\, f \longrightarrow \mathrm{im}\, f$$

defined by $$\varphi(v + K) = fv, \quad \text{for all } v \in V.$$

This natural isomorphism is always used as an identification:

$$0.17 \qquad \mathrm{coim}\, f = \mathrm{im}\, f$$

An <u>algebra</u> is a vector space V on which we define a further operation called multiplication which assigns to every pair of elements $x, y \in V$ an element $xy \in V$ called their product. This multiplication is associative and there exists a (unique) unit element $1 \in V$ such that $1x = x1 = x$ for all $x \in V$.

Let V and W be algebras. A linear transformation $f : V \longrightarrow W$ is called an algebra homomorphism if

$$f(xy) = f(x)\, f(y)$$

for all $x, y \in V$ and if $f1$ is the unit element of W; i.e. f preserves multiplication and units as well as addition and multiplication by scalars.

1

THE TENSOR ALGEBRA OF A VECTOR SPACE

1. The tensor product

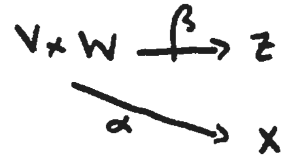
<u>Definition 1.1.1.</u> Let V, W be vector spaces and suppose that a vector space Z with a given bilinear map $\beta: V \times W \longrightarrow Z$ exists satisfying the following universal factorization property:

For an arbitrary vector space X and for any bilinear map $\alpha: V \times W \longrightarrow X$ there corresponds one, and only one, linear map

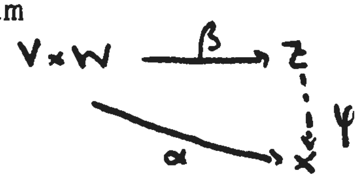
$$\varphi: Z \longrightarrow X$$

such that $\alpha = \varphi\beta$

i.e. we say that every diagram



can be uniquely embedded in the diagram



which is commutative.

If such is the case, Z is called the <u>tensor product</u> of the vector spaces V and W ; in symbols, $Z = V \otimes W$

The bilinear map will be denoted by

$$\beta(v, w) = v \otimes w \qquad \text{for every } v \in V,\ w \in W$$

The elements $v \otimes w$ of $V \otimes W$ are called <u>tensors</u>.

<u>Remark 1.1.2.</u>:

The function β establishes a <u>natural isomorphism</u> between the vector spaces $\mathrm{Hom}(V, W; X)$ and $\mathrm{Hom}(V \otimes W, X)$ The identification $\mathrm{Hom}(V, W; X) = \mathrm{Hom}(V \otimes W; X)$ can be regarded as the implicit definition of the tensor product

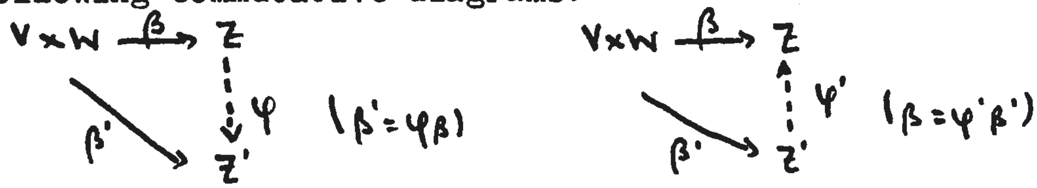
<u>Theorem 1.1.3.</u>:

For every choice of V and W, their tensor product $V \otimes W$ exists and is uniquely determined to within isomorphism.

<u>Proof</u>: (i) The proof of the existence is a construction yielding an isomorphic copy of the tensor product. Such a construction is discussed in [3] pages 204-5.

(ii) Uniqueness:

Suppose two vector spaces Z, Z' satisfy the universal factorization property above. Then we consider the two following commutative diagrams:



Hence $1_{Z'} \beta' = \beta' = \varphi\beta = \varphi(\varphi'\beta') = (\varphi\varphi')\beta'$; i.e. $\varphi\varphi' = 1_{Z'}$

Similarly $\varphi'\varphi = 1_Z$

Hence φ is an isomorphism and Z and Z' are isomorphic; in fact they are naturally isomorphic and the uniqueness of the tensor product $Z = V \otimes W$ (to within isomorphism) is proved.

Definition 1.1.1. gives the tensor product of two vector spaces. We now define the tensor product of a finite number of vector spaces.

<u>Definition 1.1.4.</u>:

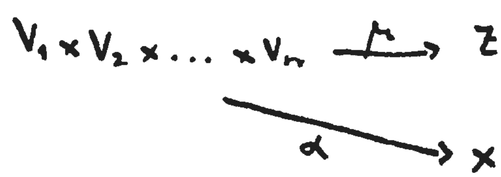
Let $V_1, V_2, \ldots, V_n$ be vector spaces and suppose that a vector space Z together with a multilinear transformation $\mu : V_1 \times V_2 \times \ldots \times V_n \longrightarrow Z$ exists satisfying the following universal factorization property. For every vector space X and an arbitrary multilinear transformation $\alpha : V_1 \times V_2 \times \ldots \times V_n \longrightarrow X$ there exists one, and only one, linear transformation

$$\varphi : Z \longrightarrow X$$

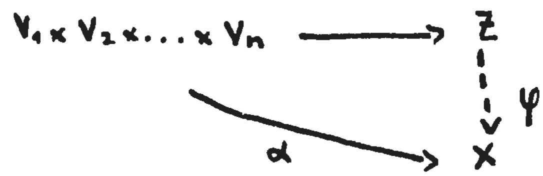
such that $\alpha = \varphi\mu$

i.e.

the diagram



can be uniquely embedded in the diagram



which is commutative.

If such is the case we call Z the <u>tensor product</u> of $V_1, V_2, \ldots, V_n$ and write $Z = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ and $\mu(v_1, v_2, \ldots, v_n) = v_1 \otimes v_2 \otimes \ldots \otimes v_n$

<u>Theorem 1.1.5.</u>:

The tensor product of any finite number n of vector spaces exists and is unique (to within isomorphism).

Proof: (i) Existence: The proof of the existance is again a construction yielding an isomorphic copy of the tensor product. We omit the details but such a construction can be found in [3] (p. 219).

(ii) Uniqueness:

We show that if two vector spaces Z and Z' satisfying the universal factorization property exist then Z and Z' are isomorphic. The proof is exactly similar to the proof of 1.1.3. and so we omit the details.

Theorem 1.1.6.:

The tensor products $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ are naturally isomorphic.

Proof: $\mathrm{Hom}((U \otimes V) \otimes W, Z) = \mathrm{Hom}(U \otimes V, \mathrm{Hom}(W, Z))$

$= \mathrm{Hom}(U, V; \mathrm{Hom}(W, Z)) = \mathrm{Hom}(U, \mathrm{Hom}(V, W; Z))$

$= \mathrm{Hom}(U, \mathrm{Hom}(V \otimes W, Z)) = \mathrm{Hom}(U \otimes (V \otimes W), Z)$

Hence $(U \otimes V) \otimes W = U \otimes (V \otimes W)$

Thus from 1.1.6., the tensor product is associative.

Before we develop the tensor algebra we shall state and prove a theorem concerning the dimension of

Theorem 1.1.7.:

If V, W are vector spaces of dimension m and n respectively then $V \otimes W$ is of dimension mn. If

$\{e_1, \dots, e_m\}$ and $\{f_1, \dots, f_n\}$ are bases of V and W respectively, then $\{e_i \otimes f_j\}$; $i = 1, \dots, m$; $j = 1, \dots, n$ form a base of $V \otimes W$

<u>Proof</u>: Let $\{e_1, \dots, e_m\}$, $\{f_1, \dots, f_n\}$ be bases of V and W . Define the bilinear map $\varphi_{ij} : V \times W \longrightarrow \mathbb{R}$ by $\varphi_{ij}(e_k, f_l) = \delta^i_k \, \delta^j_l$ $\quad i, k = 1, \dots, m$; $j, l = 1, \dots, n$

By bilinearity φ_{ij} is defined for any element of $V \times W$. These bilinear maps are linearly independent in $\mathrm{Hom}(V, W; \mathbb{R})$ For, if

$$\sum_{i,j} a_{ij} \varphi_{ij} = 0 \, , \qquad a_{ij} \in \mathbb{R}$$

we have

$$0 = \sum_{i,j} a_{ij} \, \varphi_{ij}(e_k, f_l) = \sum_{i,j} a_{ij} \, \delta^i_k \, \delta^j_l = a_{kl}$$

Hence the vector space $\mathrm{Hom}(V, W ; \mathbb{R})$ is of dimension $\geq mn$; i.e. $(V \otimes W)^*$ is of dimension $\geq mn$. But $(V \otimes W)^*$ has the same dimension as $V \otimes W$ which is $\leq mn$.

It follows that the dimension of $V \otimes W$ is mn.

2. The tensor algebra

Tensor products of copies of V and V^* are of particular interest. We consider spaces of the form $V_1 \otimes \ldots \otimes V_k$ where each V_i is either V or V^*.

If there are r copies of V and s copies of V^* then the space is called a tensor space of type (r,s) ; r is called the <u>contravariant</u> degree and s the <u>covariant</u> degree. We write

$$1.2.1 \qquad V^r_s = \underbrace{V \otimes \ldots \otimes V}_{r \text{ Copies}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s \text{ Copies}}$$

Observe, however, that the integers (r,s) do not determine the space. The order in which the spaces occur matters; we distinguish between $V \otimes V^*$ and $V^* \otimes V$
In particular, V^r_0 is called a <u>contravariant tensor of degree r</u>
V^0_s is called a <u>covariant tensor of degree s</u>
We also make the convention that: $V^0_0 = \mathbb{R}$

Given two tensor spaces $V_1 \otimes \ldots \otimes V_k$ of type (r,s) and $V'_1 \otimes \ldots \otimes V'_k$ of type (r',s'), the associative law for tensors defines a bilinear map

$$\varphi : (V_1 \otimes \ldots \otimes V_k) \times (V'_1 \otimes \ldots \otimes V'_k) \longrightarrow (V_1 \otimes \ldots \otimes V_k) \otimes (V'_1 \otimes \ldots \otimes V'_k)$$

where $V_1 \otimes \ldots \otimes V_k \otimes V'_1 \otimes \ldots \otimes V'_k$ is a tensor space of type $(r+r', s+s')$.

This operation is called <u>multiplication of tensors</u>. The product of a tensor of type (r,s) with a tensor of type

(r', s') is a tensor of type $(r+r', s+s')$ i.e. this bilinear map defines a multiplicative structure on the (weak) direct sum of all tensor products of V and V^*. We denote this space by $\otimes V$ i.e.

$$1.2.2. \qquad \otimes V = \mathbb{R} + V + V^* + V\otimes V + V^*\otimes V^* + V\otimes V^* + \dots$$

We note that in general $\otimes V$ is infinite dimensional.

<u>Definition 1.2.3.</u>:

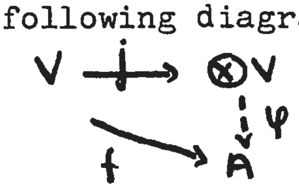
The space $\otimes V$ together with its multiplicative structure is called the <u>tensor algebra of the vector space</u>.

We may also state the definition of the tensor algebra of a vector space in terms of a universal factorization property. We call this definition the conceptual definition of the tensor algebra of a vector space:

<u>Definition 1.2.4.</u>:

Suppose $\otimes V$ is an algebra and $j : V \rightarrow \otimes V$ is an algebra homomorphism satisfying the following universal factorization property:

Given any algebra A and an algebra homomorphism $f : V \longrightarrow A$, there exists one and only one algebra homomorphism $\varphi : \otimes V \longrightarrow A$ such that $f = \varphi j$ i.e. φ makes the following diagram commutative,



then $\otimes V$ is called the <u>tensor algebra</u> of the vector space V.

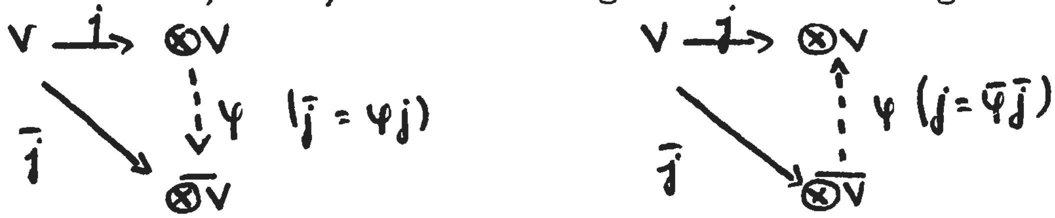
<u>Theorem 1.2.5.</u>:

The tensor algebra $\otimes V$ of a vector space V exists and is uniquely determined to within isomorphism.

<u>Proof</u>: (i) Existence of $\otimes V$ can be shown by construction. In fact $\otimes V = \mathbb{R} + V + V^* + V \otimes V + V^* \otimes V^* + V \otimes V^* + \cdots$ together with j the inclusion map, satisfies the universal factorization property given in 1.2.4.

(ii) Uniqueness:

Suppose that two tensor algebras $\otimes V$ and $\overline{\otimes V}$ satisfying the universal factorization property of 1.2.4. exist. Consider, then, the following commutative diagrams:



Then $1_{\overline{\otimes V}}\, \bar{j} = \bar{j} = \varphi j = \varphi(\bar{\varphi}\bar{j}) = (\varphi\bar{\varphi})\, \bar{j}$

Hence $\varphi\bar{\varphi} = 1_{\overline{\otimes V}}$

Similarly $\bar{\varphi}\varphi = 1_{\otimes V}$

It follows that φ is an isomorphism and so $\otimes V$ and $\overline{\otimes V}$ are isomorphic; in fact they are naturally isomorphic.

3. Duality in the tensor algebra

From now on we shall be concerned with only pure (covariant or contravariant) tensor algebra. First we shall discuss problems arising from duality. We shall show that $(V^r)^*$ can be identified with $(V^*)^r$ and we adopt a neutral notation V_r for it.

We recall that by 1.1.2. we have a natural isomorphism between the vector spaces $\mathrm{Hom}(V, W; X)$ and $\mathrm{Hom}(V \otimes W, X)$. In particular, then, we can identify $\mathrm{Hom}(V, W; \mathbb{R})$ and $\mathrm{Hom}(V \otimes W, \mathbb{R})$ by the definition of the dual space $(V \otimes W)^* = \mathrm{Hom}(V \otimes W; \mathbb{R})$

It follows that we have a natural isomorphism between $(V \otimes W)^*$ and $\mathrm{Hom}(V, W; \mathbb{R})$ which we shall always use for the purpose of identification.

<u>Theorem 1.3.1.</u>:

There is a natural isomorphism

$\varphi : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ defined by

$\varphi(\bar{v} \otimes w) = \langle \bar{v}, v \rangle w$ for all $v \in V$.

<u>Proof</u>: Let the natural transformation

$\varphi : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ be defined by

$\varphi(\bar{v} \otimes w) = \langle \bar{v}, v \rangle w$ for all $v \in V$.

Clearly this defines a linear form and φ is linear. If $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ are bases of V

and W and $\{\bar{e}_1, \ldots, \bar{e}_m\}$ is the dual bases of V^* we know that a basis for $V \otimes W$ is given by

$\{e_i \otimes f_j\}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$ and a basis for $(V^* \otimes W)$ is given by

$$\{\bar{e}_i \otimes f_j\} \quad i = 1, \ldots, m ; \; j = 1, \ldots, n$$

Using this basis we verify that φ is an isomorphism.

<u>Theorem 1.3.2.</u>:

There is a natural isomorphism between $(V \otimes W)^*$ and $V^* \otimes W^*$

<u>Proof</u>: $(V \otimes W)^* = Hom(V, W; \mathbb{R}) = Hom(V; Hom(W, \mathbb{R}))$

$$= Hom(V, W^*) = V^* \otimes W^*$$

We shall always use this isomorphism for the purpose of identification:

$$(V \otimes W)^* = V^* \otimes W^*$$

Thus we can write

1.3.3 $\langle v \otimes w, \bar{v} \otimes \bar{w} \rangle = \langle v, \bar{v} \rangle \langle w, \bar{w} \rangle$;

$$v \in V, \; \bar{v} \in V^*, \; w \in W, \; \bar{w} \in W^*$$

Clearly this can be extended to any further number of factors by linearity. In particular, we can identify $(V^r)^*$ and $(V^*)^r$. The fundamental bilinear relation between $(V^r)^*$ and $(V^*)^r$ is given by

1.3.4 $\langle v_1 \otimes \ldots \otimes v_r, \bar{v}_1 \otimes \ldots \otimes \bar{v}_r \rangle = \langle v_1, \bar{v}_1 \rangle \ldots \langle v_r, \bar{v}_r \rangle$

We identify $(V^r)^*$ and $(V^*)^r$ and adopt a neutral notation V_r for it.

4. The induced homomorphism.

For $f \in \mathrm{Hom}(V, V')$ and $g \in \mathrm{Hom}(W, W')$ we define the induced function $f \otimes g : V \otimes W \to V' \otimes W'$ by

1.4.1. $(f \otimes g)(v \otimes w) = fv \otimes gw$

where $v \otimes w$ is a generator of $V \otimes W$

$f \otimes g$ is clearly a homomorphism.

<u>Proposition 1.4.2.</u>:

If $f \otimes g : V \otimes W \to V' \otimes W'$ and $f' \otimes g' : V' \otimes W' \to V'' \otimes W''$

then $(f' \otimes g')(f \otimes g) = f'f \otimes g'g$

<u>Proof</u>: $[(f' \otimes g')(f \otimes g)]\, v \otimes w = (f' \otimes g')[(f \otimes g)(v \otimes w)]$

$= (f' \otimes g')(fv \otimes gw) = f'(fv) \otimes g'(gw) = (f'f)v \otimes (g'g)w$

$= (f'f \otimes g'g)(v \otimes w)$

We may now define the induced contravariant and the induced covariant homomorphisms.

We write $V^p = \underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}}$ and $V_q = \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ copies}}$ as before.

<u>Definition 1.4.3.</u>:

For $f \in \mathrm{Hom}(V, W)$ we define the induced contravariant homomorphism of degree p, in symbols f^p to be $f^p : V^p \longrightarrow W^p$ where $f^p = \underbrace{f \otimes \cdots \otimes f}_{p \text{ copies}}$

For $f \in \mathrm{Hom}(V, W)$ we define the induced covariant homomorphism of degree q, in symbols f_q to be $f_q : V_q \longleftarrow W_q$ where $f_q = \underbrace{f^* \otimes \cdots \otimes f^*}_{q \text{ copies}}$

Properties: For $f^p : V^p \longrightarrow W^p$, $g^p : W^p \longrightarrow U^p$,

$f_q : V_q \longleftarrow W_q$, $g_q : W_q \longleftarrow U_q$,

$1\!\!1_V : V \longrightarrow V$

we have the following formulae:

(i) $(gf)^p = g^p f^p$

(ii) $(1\!\!1_V)^p = 1\!\!1_{V^p}$

(iii) $(gf)_q = f_q g_q$

(iv) $(1\!\!1_V)_q = 1\!\!1_{V_q}$

Proof:

(i) $$(gf)^p = \underbrace{gf \otimes \ldots \otimes gf}_{p \text{ copies}} = (\underbrace{g \otimes \ldots \otimes g}_{p \text{ copies}})(\underbrace{f \otimes \ldots \otimes f}_{p \text{ copies}}) = g^p f^p$$

(ii) $$(1\!\!1_V)^p = \underbrace{1\!\!1_V \otimes \ldots \otimes 1\!\!1_V}_{p \text{ copies}} = 1\!\!1_{\underbrace{V \otimes \ldots \otimes V}_{p \text{ copies}}} = 1\!\!1_{V^p}$$

(iii) $$(gf)_q = \underbrace{(gf)^* \otimes \ldots \otimes (gf)^*}_{q \text{ copies}} = (\underbrace{f^* \otimes \ldots \otimes f^*}_{q \text{ copies}})(\underbrace{g^* \otimes \ldots \otimes g^*}_{q \text{ copies}}) = f_q g_q$$

(iv) $$(1\!\!1_V)_q = \underbrace{1\!\!1_V^* \otimes \ldots \otimes 1\!\!1_V^*}_{q \text{ copies}} = 1\!\!1_{\underbrace{V^* \otimes \ldots \otimes V^*}_{q \text{ copies}}} = 1\!\!1_{V_q}$$

2

THE SYMMETRIC ALGEBRA OF A VECTOR SPACE

0. The permutation group on r letters acting on V^r_0

Let V be a vector space of dimension n. We consider the space V^r_0 of contravariant tensors of order r; for simpler notation we write V^r for V^r_0.

The permutation group on r letters, Π_r, acts on the space V^r: Given any permutation $\sigma \in \Pi_r$, and any tensor of the form $x_1 \otimes \ldots \otimes x_r$, we define

2.0.1. $$\sigma(x_1 \otimes \ldots \otimes x_r) = x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(r)}$$

and extend by linearity to all of V^r. Thus we have a representation of Π_r on V^r.

Let $GL(n) \subset \mathrm{Hom}(V, V)$ denote the group of automorphisms of the n-dimensional vector space V. It is easy to see that the above representation of Π_r on V^r commutes with the tensor product representation of i.e. for any $t \in V^r$, $\sigma \in \Pi_r$ and $g \in GL(n)$

2.0.2. $$g\sigma t = \sigma g t$$

where gt is $g \otimes \ldots \otimes g$ (r times) acting on t.

It follows from 2.0.2. that any simultaneous eigenvector of Π_r i.e. any tensor satisfying

2.0.3. $$\sigma t = \rho(\sigma)\, t$$

where ρ is some numerical function on Π_r is taken by g into another tensor satisfying the same equation:

$$\sigma g t = g \sigma t = g \rho(\sigma) t = \rho(\sigma) g t$$

The two important cases are

2.0.4. $\rho(\sigma) = 1$

and

2.0.5. $\rho(\sigma) = \operatorname{sgn} \sigma = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$

1. The space of symmetric (contravariant) tensors of degree

We consider the case given by 2.0.4. i.e. $\rho(\sigma) = 1$ Then 2.0.3. becomes $\sigma t = t$ In this case we say that t is symmetric.

Definition 2.1.1.:

The subspace of V^r consisting of those tensors satisfying $\sigma t = t$ will be denoted by $V^{(r)}$ and will be called the space of symmetric (contravariant) tensors of degree r.

There is a natural projection S_r of V^r onto $V^{(r)}$ given by

2.1.2. $$S_r(t) = \frac{1}{r!} \sum_{\sigma \in \Pi_r} \sigma t \qquad ; \ t \in V^r$$

Theorem 2.1.3.:

For every $t \in V^r$, $S_r(t)$ is symmetric i.e. for any $\sigma' \in \Pi_r$, $\sigma' S_r(t) = S_r(t)$. Moreover, S_r is a section i.e. $S_r S_r = S_r$

Proof: For any permutation $\sigma' \in \Pi_r$ we have

$$\sigma' S_r(t) = \frac{1}{r!} \sum_{\sigma} \sigma' \sigma t = \frac{1}{r!} \sum_{\sigma} \sigma t = S_r(t) \ ;$$

$$S_r S_r = \left(\frac{1}{r!} \sum_{\sigma} \sigma\right)\left(\frac{1}{r!} \sum_{\sigma} \sigma\right) = \frac{\sum_{k=1}^{r!} \left(\sum_{\sigma} \sigma_k \sigma\right)}{r!\, r!} = \frac{r! \sum_{\sigma} \sigma}{r!\, r!} = S_r .$$

We may regard 2.1.1. as the constructual definition of $V^{(r)}$. We now give a conceptual definition of $V^{(r)}$

Definition 2.1.4.:

Suppose that a vector space Z together with a symmetric transformation

$$\tau : V^r \longrightarrow Z$$

exists satisfying the following universal factorization property:

For every vector space X and every symmetric transformation

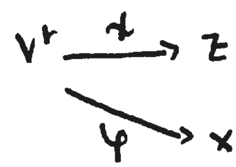
$$\varphi : V^r \longrightarrow X$$

there corresponds one, and only one, linear transformation

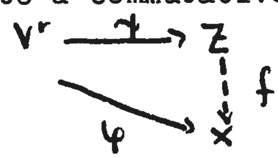
$$f : Z \longrightarrow X$$

such that $\varphi = f\tau$

i.e. every diagram



can be uniquely embedded into a commutative diagram



If such is the case, we write :

$$Z = V^{(r)}$$

Remark 2.1.5.:

The function τ establishes a natural isomorphism between the space of all symmetric linear transformations $\varphi : V^r \longrightarrow X$ and the space of all linear transformations $f: V^{(r)} \longrightarrow X$. We denote the space of all symmetric linear transformations $\varphi : V^r \longrightarrow X$ by $Sym(V^r, X)$. Then the identification $Sym(V^r, X) = Hom(V^{(r)}, X)$ can be regarded as the implicit definition of $V^{(r)}$.

Theorem 2.1.6.:

For any vector space V, $V^{(r)}$ exists and is unique (to within isomorphism).

Proof: (i) Existence· We have shown that $V^{(r)}$ exists by actually constructing it. It is clear that the $V^{(r)}$ defined in 2.1.1. together with the natural projection $\psi : V^r \longrightarrow V^{(r)}$ given by 2.1.2. actually satisfies the universal factorization property 2.1.4.

(ii) Uniqueness: Suppose two vector spaces $V^{(r)}$, $\overline{V^{(r)}}$ satisfy this universal factorization property. Consider, then, the following commutative diagrams:

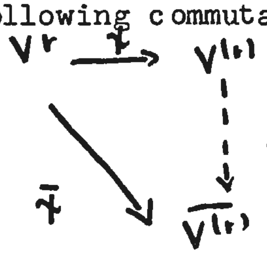


fig. (i)

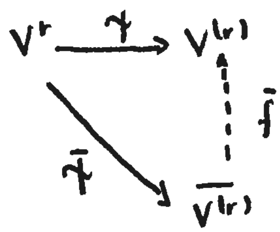


fig. (ii)

From fig. (i) $\bar\psi = f\psi$; from fig. (ii) $\psi = \bar f \bar\psi$

Then $1_{\overline{V^{(r)}}}\bar\psi = \bar\psi = f\psi = f(\bar f \bar\psi) = (f\bar f)\bar\psi$

follows that $f\bar f = 1_{\overline{V^{(r)}}}$

Similarly

$\bar f f = 1_{V^{(r)}}$

Hence f is an isomorphism and $V^{(r)}$ and $\overline{V^{(r)}}$ are naturally isomorphic.

Remark 2.1.7.:

If the dimension of V is n, then the dimension of $V^{(r)}$ is $\binom{n+r-1}{r}$: namely, if $\{e_1, \ldots, e_n\}$ is a basis for V, then a basis for $V^{(r)}$ is given by $\{e_{i_1} \circ e_{i_2} \circ \ldots \circ e_{i_r} : 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$

where $e_{i_1} \circ \ldots \circ e_{i_r} = \psi(e_{i_1} \otimes \ldots \otimes e_{i_r})$

2. The symmetric algebra

We will define an operation called symmetric multiplication which pairs an element of $V^{(r)}$ and an element of $V^{(s)}$ with an element of $V^{(r+s)}$

We recall the natural projection S_r of V^r into $V^{(r)}$ given by $S_r(t) = \frac{1}{r!} \sum_{\sigma \in \pi_r} \sigma t \quad ; t \in V^r$

Also we recall that if $x \in V^r$ and $y \in V^s$, then from the definition of the tensor multiplication $x \otimes y \in V^{r+s}$

Definition 2.2.1.:

We define the symmetric product $x \circ y$ by the formula that

$$x \circ y = S_r(x \otimes y) \quad ; x \in V^r, \ y \in V^s$$

Remark 2.2.2.:

The symmetric product $x \circ y$ is clearly an element of $V^{(r+s)}$ if $x \in V^r$, $y \in V^s$

Remark 2.2.3.: $x \circ y = y \circ x$.

Thus this operation, symmetric multiplication, pairs an element of $V^{(r)}$ and an element of $V^{(s)}$ with an element of $V^{(r+s)}$. Then we have a multiplicative structure on the direct sum

$$V^{(0)} + V^{(1)} + V^{(2)} + \ldots + V^{(r)}$$

We write $\bigcirc V = V^{(0)} + V^{(1)} + V^{(2)} + \ldots + V^{(r)}$ note that, in general $\bigcirc V$ is infinite dimensional.

Definition 2.2.4.:

We call OV (as defined above) the symmetric algebra of the vector space V .

We have also the following conceptual definition of the symmetric algebra.

Definition 2.2.5.:

Suppose OV is an algebra and $j: V \longrightarrow OV$ is a symmetric algebra homomorphism satisfying the following universal factorization property:

Given any algebra A and an algebra homomorphism $f: V \longrightarrow A$, there exists one, and only one, algebra homomorphism $\varphi: OV \longrightarrow A$ such that $f = \varphi j$; i.e. φ makes the following diagram commutative

$$\begin{array}{ccc} V & \xrightarrow{j} & OV \\ & \searrow^{f} & \downarrow{\scriptstyle \varphi} \\ & & A \end{array}$$

Then OV is called the symmetric algebra of the vector space.

Theorem 2.2.6.:

For every vector space V , OV exists and is unique (to within isomorphism).

Proof: (i) Existence: It can be shown that a choice of $OV = V^{(0)} + V^{(1)} + \dots + V^{(r)}$ together with the inclusion $j: V \longrightarrow OV$ satisfies the universal factorization property of 2.2.5.

(ii) Uniqueness: Suppose OV and $\overline{OV}$ both satisfy the universal factorization property of 2.2.5. Then we have the following commutative diagrams:

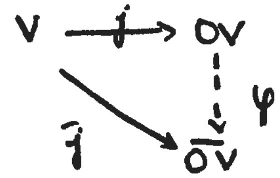


fig. (i)

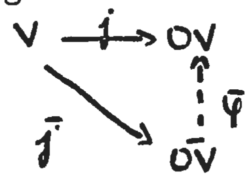


fig. (ii)

Then from fig. (i) $\bar{j} = \varphi j$; from fig. (ii) $j = \bar{\varphi}\bar{j}$

Hence $1_{\bar{O}V}\, \bar{j} = \bar{j} = \varphi j = \varphi(\bar{\varphi}\bar{j}) = (\varphi\bar{\varphi})\bar{j}$

it follows that $\varphi\bar{\varphi} = 1_{\bar{O}V}$

Similarly $\bar{\varphi}\varphi = 1_{OV}$

Hence φ is an isomorphism and so OV and $\bar{O}V$ are naturally isomorphic.

3. Duality in the symmetric algebra

In this section we shall show that $(V^{(r)})^* = (V^*)^{(r)}$ and we shall adopt the neutral notation $V_{(r)}$ for it.

We may construct $V^{(r)}$ in a different way. Actually we may put $V^{(r)} = V^r \,|\, \ker S_r$ where S_r is the symmetric linear transformation $S_r : V^r \longrightarrow V^r$ i.e. $V^{(r)} = V^r | \ker S_r = \operatorname{Coim} S_r$. This construction of $V^{(r)}$ satisfies the universal factorization property of 2.1.4.

For simpler notation we write S instead of S_r

From 0.17 we have a natural isomorphism between the $\operatorname{Coim} S$ and $\operatorname{im} S$, we can identify them. Then we have a natural isomorphism

2.3.1. $\gamma : V^{(r)} \longrightarrow \operatorname{im} S$

Also the linear transformation $S : V \longrightarrow V^{(r)}$ induces a linear transformation

$$S^* : (V^*)^r \longrightarrow (V^*)^{(r)}$$

so that we have a natural isomorphism

2.3.2. $\gamma^* : (V^*)^{(r)} \longrightarrow \operatorname{im} S^*$

Furthermore from the definition of the dual space

$$(V^{(r)})^* = \operatorname{Hom}(V^{(r)}; \mathbb{R})$$

and by Remark 2.1.5. we may identify $\operatorname{Hom}(V^{(r)}; \mathbb{R})$ and $\operatorname{Sym}(V^r; \mathbb{R})$ where $\operatorname{Sym}(V^r, \mathbb{R})$ is the vector space of all symmetric linear

linear transformations $S : V^{(r)} \longrightarrow \mathbb{R}$

Hence we have a natural isomorphism

2.3.3. $\Phi^* : Sym(V^r, \mathbb{R}) \longrightarrow (V^{(r)})^*$

Since we wish to show that we have a natural isomorphism between $(V^{(r)})^*$ and $(V^*)^{(r)}$, we now need only to show that a natural isomorphism exists between $im\, S^*$ and $Sym(V^r, \mathbb{R})$

We define

2.3.4. $\Gamma : im\, S^* \longrightarrow Sym(V^r, \mathbb{R})$

which is defined for every $t \in im\, S^*$ such that

$$\Gamma t(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, t \rangle$$

where $v_1 \otimes \ldots \otimes v_r$ is a generator of V^r.

Extend this definition by linearity and then clearly
$\Gamma t \in Hom(V^r, \mathbb{R})$

In fact, $\Gamma t \in Sym(V^r, \mathbb{R})$ for any $\sigma \in \Pi_r$,

$(\Gamma t)\sigma(v_1 \otimes \ldots \otimes v_r) = \Gamma t(\sigma(v_1 \otimes \ldots \otimes v_r)) = \langle \sigma(v_1 \otimes \ldots \otimes v_r), t \rangle$
$= \langle v_1 \otimes \ldots \otimes v_r, \sigma^* t \rangle = \langle v_1 \otimes \ldots \otimes v_r, t \rangle = \Gamma t(v_1 \otimes \ldots \otimes v_r)$

i.e. $(\Gamma t)\sigma = \Gamma t$ for every σ

i.e. Γt is symmetric, $\Gamma t \in Sym(V^r, \mathbb{R})$

Moreover, Γ is a homomorphism:

$\Gamma(ct + c't')(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, ct + c't' \rangle$
$= c\langle v_1 \otimes \ldots \otimes v_r, t \rangle + c'\langle v_1 \otimes \ldots \otimes v_r, t' \rangle$
$= c(\Gamma t)(v_1 \otimes \ldots \otimes v_r) + c'(\Gamma t')(v_1 \otimes \ldots \otimes v_r) = (c(\Gamma t) + c'(\Gamma t'))\, v_1 \otimes \ldots \otimes v_r$;

$c, c' \in \mathbb{R}$

i.e. $\Gamma(ct+c't') = c(\Gamma t)+c'(\Gamma t')$ so that Γ is a homomorphism.

To show that Γ is bijective (and hence an isomorphism) we define its inverse function

$$B : \mathrm{Sym}(V^r, \mathbb{R}) \longrightarrow \mathrm{im}\, S^*$$

for every $\tau \in \mathrm{Sym}(V^r, \mathbb{R})$ by the formula

$$\langle v_1 \otimes \ldots \otimes v_r, B\tau \rangle = \tau(v_1 \otimes \ldots \otimes v_r)$$

It is clear from 0.10 that such a covariant tensor exists; $B\tau \in \mathrm{Hom}(V^r, \mathbb{R})$; in fact, it is symmetric. $B\tau \in \mathrm{im}\, S^*$ for

$$\langle v_1 \otimes \ldots \otimes v_r, B\tau \rangle = \langle \sigma(v_1 \otimes \ldots \otimes v_r), B\tau \rangle = \tau(\sigma(v_1 \otimes \ldots \otimes v_r))$$
$$= (\tau\sigma)(v_1 \otimes \ldots \otimes v_r) = \tau(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, B\tau \rangle$$

Now $B\Gamma$ is the identity of $\mathrm{im}\, S^*$; for every t,

$$\langle v_1 \otimes \ldots \otimes v_r, B\Gamma t \rangle = \Gamma t(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, t \rangle$$

i.e. $B\Gamma = 1_{\mathrm{im}\, S^*}$.

Similarly, ΓB is the identity of $\mathrm{Sym}(V^r, \mathbb{R})$; for every $\tau \in \mathrm{Sym}(V^r, \mathbb{R})$,

$$(\Gamma B)\tau(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, B\tau \rangle = \tau(v_1 \otimes \ldots \otimes v_r)$$

i.e. $\Gamma B = 1_{\mathrm{Sym}(V^r, \mathbb{R})}$

This means that Γ is bijective, hence a natural isomorphism. We then have a product of three natural isomorphisms

2.3.5. $$\Phi^* \Gamma \psi^* : (V^*)^{(r)} \longrightarrow (V^{(r)})^*$$

$$(V^*)^{(r)} \xrightarrow{\gamma^*} \operatorname{im} s^* \xrightarrow{\Gamma} \operatorname{Sym}(V^r, \mathbb{R}) \xrightarrow{\Phi^*} (V^{(r)})^*$$

We shall use this product for the purpose of identification:

$\Phi^* \Gamma \gamma^* = 1$ Then we write

$$2.3.6. \quad (V^{(r)})^* = (V^*)^{(r)} = V_{(r)}$$

The identification 2.3.6. will be used in order to establish the pseudo product for the pair $V^{(r)}$, $(V^{(r)})^*$. In doing so we shall also have occasion to use the natural isomorphism $\gamma : \operatorname{coim} S \longrightarrow \operatorname{im} S$ with

$$\gamma(v_1 \circ \dots \circ v_r) = S(v_1 \otimes \dots \otimes v_r)$$

$$\langle v_1 \circ \dots \circ v_r, \bar{v}_1 \circ \dots \circ \bar{v}_r \rangle = \langle v_1 \circ \dots \circ v_r, \Phi^* \Gamma \gamma^* (\bar{v}_1 \circ \dots \circ \bar{v}_r) \rangle$$
$$= \langle v_1 \circ \dots \circ v_r, \Phi^*(\Gamma \gamma^*(\bar{v}_1 \circ \dots \circ \bar{v}_r)) \rangle = \langle \Gamma \gamma^*(\bar{v}_1 \circ \dots \circ \bar{v}_r), \Phi(v_1 \circ \dots \circ v_r) \rangle$$
$$= \langle \Gamma(\gamma^*(\bar{v}_1 \circ \dots \circ \bar{v}_r)), S(v_1 \otimes \dots \otimes v_r) \rangle = \langle \Gamma(S^*(\bar{v}_1 \otimes \dots \otimes \bar{v}_r)), S(v_1 \otimes \dots \otimes v_r) \rangle$$
$$= \langle S(v_1 \otimes \dots \otimes v_r), S^*(\bar{v}_1 \otimes \dots \otimes \bar{v}_r) \rangle = \langle S^2(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle$$
$$= \langle S(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle = \frac{1}{r!} \langle \sum_\sigma \sigma(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle$$
$$= \frac{1}{r!} \sum_\sigma \langle v_{\sigma_1} \otimes \dots \otimes v_{\sigma_r}, \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle = \frac{1}{r!} \sum_\sigma \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

wherein various steps have various justifications.

We have proved the formula

$$2.3.7. \quad \langle v_1 \circ \dots \circ v_r, \bar{v}_1 \circ \dots \circ \bar{v}_r \rangle = \frac{1}{r!} \sum_\sigma \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

It is often more convenient, however, to use this pairing without the factor $\frac{1}{r!}$ so that we replace the pairing $\langle\ ,\ \rangle$ by the pairing $\langle\ |\ \rangle$; this new pairing is defined by

2.3.8. $$\langle\ |\ \rangle = r!\ \langle\ ,\ \rangle$$

i.e.

2.3.9. $$\langle v_1 \circ \ldots \circ v_r \mid \bar{v}_1 \circ \ldots \circ \bar{v}_r \rangle = \sum_{\sigma} \langle v_{\sigma_1}, \bar{v}_1 \rangle \cdots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

Or, we may write

2.3.10. $$\langle v_1 \circ \ldots \circ v_r \mid \bar{v}_1 \circ \ldots \circ \bar{v}_r \rangle = \begin{vmatrix} \langle v_{\sigma_1}, \bar{v}_1 \rangle & \langle v_{\sigma_2}, \bar{v}_1 \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_1 \rangle \\ \langle v_{\sigma_1}, \bar{v}_2 \rangle & \langle v_{\sigma_2}, \bar{v}_2 \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_{\sigma_1}, \bar{v}_r \rangle & \langle v_{\sigma_2}, \bar{v}_r \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_r \rangle \end{vmatrix}^{+}$$

where $|\ |^{+}$ denotes the <u>permanent</u>.

The permanent is obtained by making all the negative signs in the determinant positive.

Thus $\langle v_1 \circ \ldots \circ v_r \mid \bar{v}_1 \circ \ldots \circ \bar{v}_r \rangle = \text{per}\ \langle v_i, \bar{v}_j \rangle$

4. The induced homomorphism

To every homomorphism $f : V \longrightarrow W$ there is an induced homomorphism $f^{(r)} : V^{(r)} \longrightarrow W^{(r)}$ defined by

2.4.1. $$f^{(r)} = f^r S_r$$

Given homomorphisms $f : V \longrightarrow W$, $g : W \longrightarrow U$ the induced homomorphism $(gf)^{(r)} : V^{(r)} \longrightarrow U^{(r)}$ has the property that

2.4.2. $$(gf)^{(r)} = g^{(r)} f^{(r)}$$

For,
$$\begin{aligned} g^{(r)} f^{(r)} &= (g^r S_r)(f^r S_r) \\ &= (g^r f^r)(S_r S_r) \\ &= (g^r f^r) S_r \\ &= (gf)^r S_r \\ &= (gf)^{(r)} \end{aligned}$$

Also given $1_V : V \longrightarrow V$, the induced homomorphism $(1_V)^{(r)} : V^{(r)} \longrightarrow V^{(r)}$ has the property that

2.4.3. $$(1_V)^{(r)} = 1_{V^{(r)}}$$

3

THE EXTERIOR ALGEBRA OF A VECTOR SPACE

1. The space of antisymmetric (alternate) tensors of degree

We will now study the subspace of V^r corresponding to the second simultaneous eigen value

2.0.5. i.e. $\rho(\sigma) = \operatorname{sgn}\sigma = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$

Then 2.0.3. becomes $\sigma t = (\operatorname{sgn}\sigma)\, t$

In this case we say that t is <u>anti-symmetric</u> or <u>alternate</u>.

<u>Definition 3.1.1.</u>:

The subspace of V^r consisting of those tensors satisfying $\sigma t = (\operatorname{sgn}\sigma)\, t$ will be denoted by $V^{[r]}$ and will be called the <u>space of anti-symmetric tensors of degree</u> r or <u>the space of alternate tensors of degree</u> r.

There is a natural projection A_r of V^r onto $V^{[r]}$ given by

3.1.2. $$A_r(t) = \frac{1}{r!} \sum_{\sigma \in \Pi_r} (\operatorname{sgn}\sigma)\, \sigma(t) \quad ; \quad t \in V^r$$

<u>Theorem 3.1.3.</u>:

For every $t \in V^r$, $A_r(t)$ is alternate; i.e. for any $\sigma' \in \Pi_r$, $\sigma' A_r(t) = (\operatorname{sgn}\sigma')\, A_r(t)$

Moreover, A_r is a section, i.e. $A_r A_r = A_r$

Proof: For any $\sigma' \in \Pi_r$ we have

$$\sigma' A_r(t) = \sigma'\Big(\frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma(t)\Big) = \frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma'\sigma(t)$$

$$= \frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)(\sigma\sigma')t = \frac{1}{r!}(\operatorname{sgn}\sigma')\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma(t) = \operatorname{sgn}\sigma' A_r(t).$$

$$A_r A_r = \Big(\frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma\Big)\Big(\frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma\Big) = \frac{\sum_{k=1}^{r!}\sum_{\sigma}(\operatorname{sgn}\sigma_k)\sigma_k(\operatorname{sgn}\sigma)\sigma}{r!\,r!}$$

$$= \frac{r!\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma}{r!\,r!} = \frac{1}{r!}\sum_{\sigma}(\operatorname{sgn}\sigma)\sigma = A_r$$

We may regard 3.1.1. as the constructual definition of $V^{[r]}$

We now give a conceptual definition of $V^{[r]}$

Definition 3.1.4.:

Suppose that a vector space Z together with an alternate transformation

$$\psi : V^r \longrightarrow Z$$

exists and satisfies the following universal factorization property:

For every vector space X and every alternate transformation $\varphi : V^r \longrightarrow X$

there corresponds one, and only one, linear transformation

$$f : Z \longrightarrow X$$

such that $\varphi = f\psi$

i.e. every diagram

$$\begin{array}{ccc} V^r & \xrightarrow{\psi} & Z \\ & \searrow^{\varphi} & \\ & & X \end{array}$$

can be uniquely embedded into a commutative diagram

$$\begin{array}{ccc} V^r & \xrightarrow{\psi} & Z \\ & \searrow^{\varphi} & \vdots f \\ & & X \end{array}$$

If such is the case we write $z = V^{[r]}$

Remark 3.1.5.:

The function ψ establishes a natural isomorphism between the space of all alternate linear transformations $\varphi : V^r \longrightarrow X$ and the space of all linear transformations $f : V^{[r]} \longrightarrow X$. We denote the space of all alternate linear transformations $\varphi : V^r \longrightarrow X$ by $Alt(V^r, X)$. Then the identification

$$Alt(V^r, X) = Hom(V^{[r]}, X)$$

can be regarded as the implicit definition of $V^{[r]}$

Theorem 3.1.6.:

For any vector space V, $V^{[r]}$ exists and is unique (to within isomorphism).

Proof: (i) Existence: We have shown that $V^{[r]}$ exists by actually constructing it . It is clear that the $V^{[r]}$ defined in 3.1.1. together with the natural projection $A_r : V^r \longrightarrow V^{[r]}$ given by 3.1.2. actually satisfies the universal factorization property 3.1.4.

(ii) Uniqueness: Suppose two vector spaces $V^{[r]}$, $\overline{V^{[r]}}$ satisfy this universal factorization property. Consider, then, the following commutative diagrams:

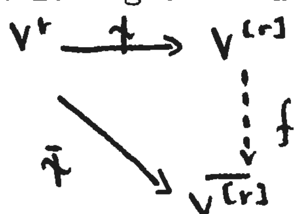


fig. (i)

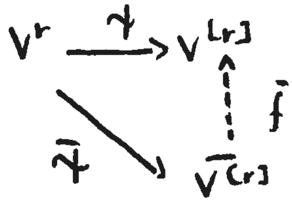


fig. (ii)

From fig. (i) $\bar{\tau} = f\tau$; from fig. (ii) $\tau = \bar{f}\bar{\tau}$

Then $1_{\bar{V}^{[r]}}\bar{\tau} = \bar{\tau} = f\tau = f(\bar{f}\bar{\tau}) = (f\bar{f})\bar{\tau}$;

it follows that $f\bar{f} = 1_{\bar{V}^{[r]}}$

Similarly $\bar{f}f = 1_{V^{[r]}}$

Hence f is an isomorphism and so $V^{[r]}$ and $\bar{V}^{[r]}$ are naturally isomorphic.

Remark 3.1.7.:

If V is a vector space of dimension n then the dimension of $V^{[r]}$ is $\binom{n}{r}$. Notably, if $\{e_1, e_2, \ldots, e_n\}$ is a basis for V^r then $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} : 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$ where

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} = \tau(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r})$$

is a basis for $V^{[r]}$.

2. The exterior algebra

We will define an operation called exterior multiplication which pairs an element of $V^{[r]}$ and an element of $V^{[s]}$ with an element of $V^{[r+s]}$

We recall the natural projection $A_r : V^r \longrightarrow V^{[r]}$ given by $A_r(t) = \frac{1}{r!} \sum_{\sigma \in \Pi_r} (\text{sgn}\,\sigma)\, \sigma(t)$; $t \in V^r$

Also, if $x \in V^r$, $y \in V^s$ then $x \otimes y \in V^{r+s}$ by the definition of the tensor multiplication.

Definition 3.2.1.:

We define the alternate product $x \wedge y$ by the formula that

$$x \wedge y = A_r(x \otimes y) \ ; \ x \in V^r,\ y \in V^s$$

Remark 3.2.2.:

The alternate product $x \wedge y$ is clearly an element of $V^{[r+s]}$

Remark 3.2.3.:

The sign of the permutation on $(r+s)$ letters which moves the first r letters past the last s is $(-1)^{rs}$ (the motion is obtained by moving each of the r letters starting with the last of the r's through the s's , thus by $r.s$ interchanges of adjoining letters). Thus

$$A_r(y \otimes x) = (-1)^{rs} A_r(x \otimes y)$$ so that

$$x \wedge y = (-1)^{rs}\ y \wedge x$$

The operation, exterior multiplication pairs an element of $V^{[r]}$ and an element of $V^{[s]}$ with an element of $V^{[r+s]}$. Then we have a multiplicative structure on the direct sum

$$V^{[0]}(=\mathbb{R}) + V^{[1]} + V^{[2]} + \dots + V^{[r]}$$

We write $$\Lambda V = \mathbb{R} + V^{[1]} + V^{[2]} + \dots + V^{[r]}$$

<u>Definition 3.2.4.</u>:

We call ΛV the <u>exterior (or Grassman) algebra of the vector space V</u> .

We have also the following conceptual definition of the exterior algebra.

<u>Definition 3.2.5.</u>:

Suppose ΛV is an algebra and $j: V \longrightarrow \Lambda V$ is an alternating algebra homomorphism satisfying the following universal factorization property:

Given any algebra A and an algebra homomorphism $f: V \longrightarrow A$ there exists one, and only one, algebra homomorphism $\varphi: \Lambda V \longrightarrow A$ such that $f = \varphi j$

i.e. φ makes the following diagram commutative

$$\begin{array}{ccc} V & \xrightarrow{j} & \Lambda V \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & A \end{array}$$

Then ΛV is called the exterior (Grassman) algebra of the vector space V .

<u>Theorem 3.2.6.</u>:

For every vector space V, ΛV exists and is unique (to within isomorphism).

<u>Proof</u>: (i) Existence: It can be shown that a choice of $\Lambda V = \mathbb{R} + V^{[1]} + V^{[2]} + \dots + V^{[r]}$ together with the inclusion $j : V \longrightarrow V^{[r]}$ actually satisfies the universal factorization property of 3.2.5.

(ii) Uniqueness: Suppose ΛV, $\overline{\Lambda V}$ both satisfy the universal factorization property of 3.2.5. then we have the following commutative diagrams.

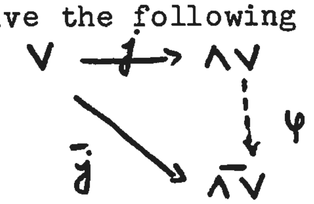


fig. (i)

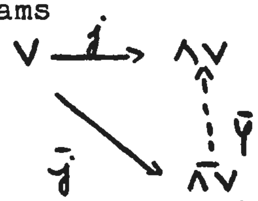


fig. (ii)

Then, from fig. (i) $\bar{j} = \varphi j$; from fig. (ii) $j = \bar{\varphi}\bar{j}$

Hence $1_{\overline{\Lambda V}}\,\bar{j} = \bar{j} = \varphi j = \varphi(\bar{\varphi}\bar{j}) = (\varphi\bar{\varphi})\,\bar{j}$

it follows that $\varphi\bar{\varphi} = 1_{\overline{\Lambda V}}$

Similarly, $\bar{\varphi}\varphi = 1_{\Lambda V}$

Thus φ is an isomorphism and ΛV and $\overline{\Lambda V}$ are naturally isomorphic.

<u>Remark 3.2.7.</u>:

If V is dimension n, then ΛV has dimension 2^n since $\Lambda V = \mathbb{R} + V^{[1]} + \dots + V^{[r]}$ and $\dim V^{[0]} = \binom{n}{0}$, $\dim V^{[1]} = \binom{n}{1}, \dots, \dim V^{[r]} = \binom{n}{r}$ so that

$$\dim \Lambda V = 2^n.$$

<u>Theorem 3.2.8.</u>:

The exterior algebra ΛV of an n -dimensional vector space V is of dimension 2^n . If $\{e_1, \ldots, e_n\}$ constitute a basis of V , then a base of ΛV is given by 1 and the elements.

3.2.9. $e_{i_1} \wedge \ldots \wedge e_{i_r}$; $1 \leq i_1 < \ldots < i_r \leq n$, $r = 1, \ldots, n$

<u>Proof</u>: The elements $A(e_{i_1} \otimes \ldots \otimes e_{i_r}) = e_{i_1} \wedge \ldots \wedge e_{i_r}$ obviously span $V^{[r]}$. By the anticommutativity relation 3.2.3. it follows that $e_{i_1} \wedge \ldots \wedge e_{i_r}$ with $i_1 < \ldots < i_r$ span $V^{[r]}$ and $V^{[r]} = 0$ for $r > n$. All we must show is that the elements 3.2.9. are linearly independent. Since terms corresponding to different values of r are obviously independent we need consider only a fixed r .

For $r = n$, $e_1 \wedge \ldots \wedge e_n = A(e_1 \otimes \ldots \otimes e_n) \neq 0$ since $e_{i_1} \otimes \ldots \otimes e_{i_r}$ are independent.

For $r < n$ suppose there is a linear relation of the form

3.2.10. $$\sum_{i_1 < \ldots < i_r} a_{i_1 \ldots i_r} \, e_{i_1} \wedge \ldots \wedge e_{i_r} = 0$$

where the summation is extended over all combinations $i_1, \ldots, i_r$ of $1, \ldots, n$

For a fixed set of indices $i_1, \ldots, i_r$, let $j_{r+1}, \ldots, j_n$ be the complementary set.

Multiply **3.2.10.** by $e_{j_{r+1}} \wedge \ldots \wedge e_{j_n}$

With the exception of $e_{i_1} \wedge \ldots \wedge e_{i_r}$ all terms in the product will have repeated factors and will therefore vanish by anticommutativity.

Thus $a_{i_1 \dots i_r} \; e_{i_1} \wedge \dots \wedge e_{i_r} \wedge e_{j_{r+1}} \wedge \dots \wedge e_{j_n}$

$$= \pm \, a_{i_1 \dots i_r} \; e_1 \wedge \dots \wedge e_n = 0$$

so $a_{i_1 \dots i_r} = 0$

This proves the theorem.

Theorem 3.2.11.:

A necessary and sufficient condition that the vectors $x_1, \dots, x_r$ be linearly dependent is that

$$x_1 \wedge \dots \wedge x_r = 0$$

Proof: If the vectors are dependent then we can express one in terms of the others, say

$$x_r = \sum_{k=1}^{r-1} a_k x_k$$

thus

$$x_1 \wedge \dots \wedge x_r = x_1 \wedge \dots \wedge \sum_{k=1}^{r-1} a_k x_k$$

$$= \sum_{k=1}^{r-1} a_k \, x_1 \wedge \dots \wedge x_{r-1} \wedge x_k$$

Each summand of the last sum has two repeated factors and is therefore zero. Hence $x_1 \wedge \dots \wedge x_r = 0$

On the other hand, if $x_1, \dots, x_r$ are linearly independent, we can always find $x_{r+1}, \dots, x_n$ so that $x_1, \dots, x_n$ form a base for V. By 3.2.8 then,

$x_1 \wedge \dots \wedge x_n \neq 0$ so $x_1 \wedge \dots \wedge x_r \neq 0$.

3. Duality in the exterior algebra.

In this section we shall show that $(V^{[r]})^* = (V^*)^{[r]}$ $= V_{[r]}$ We may construct $V^{[r]}$ in a different way. Actually we may set

$$V^{[r]} = V^r | \ker A_r$$

where A_r is the alternate linear transformation $A_r : V^r \longrightarrow V^{[r]}$ i.e. $V^{[r]} = V^r | \ker A_r = \operatorname{coim} A_r$. This construction of $V^{[r]}$ will satisfy the universal factorization property of 3.1.4.

For simpler notation let us write A instead of A_r.

Since from 0.17 we have a natural isomorphism between $\operatorname{coim} A$ and $\operatorname{im} A$ we can identify $\operatorname{coim} A$ and $\operatorname{im} A$. Then we have a natural isomorphism

3.3.1. $\quad \psi : V^{[r]} \longrightarrow \operatorname{im} A$

Also the linear transformation $A : V^r \longrightarrow V^{[r]}$ induces a linear transformation $A^* : (V^*)^r \longrightarrow (V^*)^{[r]}$ so that we have a natural isomorphism

3.3.2. $\quad \psi^* : (V^*)^{[r]} \longrightarrow \operatorname{im} A^*$

Furthermore from the definition of the dual space

$$(V^{[r]})^* = \operatorname{Hom}(V^{[r]}, R)$$

and by Remark 3.1.5. we may identify $\operatorname{Hom}(V^{[r]}, R)$ and $\operatorname{Alt}(V^r, R)$ where $\operatorname{Alt}(V^r, R)$ is the vector space of all alternate linear transformations $A : V^{[r]} \longrightarrow \mathbb{R}$

Hence we have a natural isomorphism

3.3.3. $\Phi^* : \mathrm{Alt}(V^r, \mathbb{R}) \longrightarrow (V^{[r]})^*$

Since we wish to show that we have a natural isomorphism between $(V^{[r]})^*$ and $(V^*)^{[r]}$ we need only to show that a natural isomorphism exists between $\mathrm{im}\, A^*$ and $\mathrm{Alt}(V^r, \mathbb{R})$

We define

3.3.4. $\Delta : \mathrm{im}\, A^* \longrightarrow \mathrm{Alt}(V^r, \mathbb{R})$

which is defined for every t in $\mathrm{im}\, A^*$ such that

$$\Delta t(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, t \rangle$$

where $v_1 \otimes \ldots \otimes v_r$ is a generator of V^r.

By linearity this definition is extended and then $\Delta t \in \mathrm{Hom}(V^r, \mathbb{R})$. In fact $\Delta t \in \mathrm{Alt}(V^r, \mathbb{R})$: for any $\sigma \in \Pi_r$, $\Delta t\, \sigma(v_1 \otimes \ldots \otimes v_r) = \Delta t(\sigma(v_1 \otimes \ldots \otimes v_r))$

$$= \mathrm{sgn}\,\sigma \langle \sigma(v_1 \otimes \ldots \otimes v_r), t \rangle = (\mathrm{sgn}\,\sigma) \langle v_1 \otimes \ldots \otimes v_r, \sigma^* t \rangle$$

$$= \mathrm{sgn}\,\sigma \langle v_1 \otimes \ldots \otimes v_r, t \rangle = \mathrm{sgn}\,\sigma\, \Delta t\,(v_1 \otimes \ldots \otimes v_r)$$

i.e. $\Delta t\, \sigma = (\mathrm{sgn}\,\sigma) \Delta t$; hence $\Delta t \in \mathrm{Alt}(V^r, \mathbb{R})$

Moreover, Δ is a homomorphism: for $c, c' \in \mathbb{R}$

$$\Delta(ct + c't')(v_1 \otimes \ldots \otimes v_r) = \langle v_1 \otimes \ldots \otimes v_r, ct + c't' \rangle$$

$$= c \langle v_1 \otimes \ldots \otimes v_r, t \rangle + c' \langle v_1 \otimes \ldots \otimes v_r, t' \rangle$$

$$= c\Delta t(v_1 \otimes \ldots \otimes v_r) + c' \Delta t'(v_1 \otimes \ldots \otimes v_r) = (c\Delta t + c'\Delta t')(v_1 \otimes \ldots \otimes v_r)$$

i.e. $\Delta(ct + c't') = c\,\Delta t + c'\,\Delta t'$ so that Δ is a homomorphism.

To show that Δ is bijective we define its inverse function $E : Alt(V^r, \mathbb{R}) \longrightarrow \text{im}\, A^*$ for every $\tau \in Alt(V^r, \mathbb{R})$ by the formula

$$\langle v_1 \otimes \dots \otimes v_r,\ E\tau \rangle = \tau(v_1 \otimes \dots \otimes v_r)$$

It is clear from 0.10 that such a covariant tensor exists; $E\tau \in Hom(V^r, \mathbb{R})$; in fact it is alternate.

$E\tau \in \text{im}\, A^*$ for

$$\langle v_1 \otimes \dots \otimes v_r,\ E\tau \rangle = (\text{sgn}\,\sigma) \langle \sigma(v_1 \otimes \dots \otimes v_r),\ E\tau \rangle = (\text{sgn}\,\sigma)\tau(\sigma(v_1 \otimes \dots \otimes v_r))$$

$$= \text{sgn}\,\sigma(\tau\sigma)(v_1 \otimes \dots \otimes v_r) = \text{sgn}\,\sigma\ \tau(v_1 \otimes \dots \otimes v_r) = \text{sgn}\,\sigma \langle v_1 \otimes \dots \otimes v_r,\ E\tau \rangle$$

Now ΔE is the identity of $Alt(V^r, \mathbb{R})$;

for every τ

$$(\Delta E)\tau(v_1 \otimes \dots \otimes v_r) = \Delta(E\tau)(v_1 \otimes \dots \otimes v_r) = \langle v_1 \otimes \dots \otimes v_r,\ E\tau \rangle$$
$$= \tau(v_1 \otimes \dots \otimes v_r)$$

i.e. $\Delta E = 1_{Alt(V^r, \mathbb{R})}$

Similarly, $E\Delta$ is the identity of $\text{im}\, A^*$

for every t

$$\langle v_1 \otimes \dots \otimes v_r,\ (E\Delta)t \rangle = \Delta(t(v_1 \otimes \dots \otimes v_r)) = \langle v_1 \otimes \dots \otimes v_r,\ t \rangle$$

i.e. $E\Delta = 1_{\text{im}\, A^*}$

Thus Δ is bijective, hence a natural isomorphism.

We then have a product of three natural isomorphisms:

3.3.5. $$\Phi^* \Delta \chi^* : (V^*)^{[r]} \longrightarrow (V^{[r]})^*$$

$$(V^*)^{[r]} \xrightarrow{\chi^*} \text{im}\, A^* \xrightarrow{\Delta} Alt(V^r, \mathbb{R}) \xrightarrow{\Phi^*} (V^{[r]})^*$$

This product will always be used as an identification:

$\Phi^* \Delta \psi^* = 1$ Then we write

3.3.6. $$(V^{[r]})^* = (V^*)^{[r]} = V_{[r]}$$

This identification 3.3.6. will be used to establish the pseudo-dot product for $V^{[r]}, (V^{[r]})^*$. In the following we shall also use the natural isomorphism

$\psi : \mathrm{coim}\, A \longrightarrow \mathrm{im}\, A$ with $\psi(v_1 \wedge \dots \wedge v_r) = A(v_1 \otimes \dots \otimes v_r)$

$$\langle v_1 \wedge \dots \wedge v_r, \bar{v}_1 \wedge \dots \wedge \bar{v}_r \rangle = \langle v_1 \wedge \dots \wedge v_r, \Phi^* \Delta \psi^* (\bar{v}_1 \wedge \dots \wedge \bar{v}_r) \rangle$$

$$= \langle v_1 \wedge \dots \wedge v_r, \Phi^*(\Delta \psi^*)(\bar{v}_1 \wedge \dots \wedge \bar{v}_r) \rangle = \langle \Delta \psi^* (\bar{v}_1 \wedge \dots \wedge \bar{v}_r), \Phi(v_1 \wedge \dots \wedge v_r) \rangle$$

$$= \langle \Delta(\psi^*(\bar{v}_1 \wedge \dots \wedge \bar{v}_r)), A(v_1 \otimes \dots \otimes v_r) \rangle = \langle \Delta(A^*(\bar{v}_1 \otimes \dots \otimes \bar{v}_r)), A(v_1 \otimes \dots \otimes v_r) \rangle$$

$$= \langle A(v_1 \otimes \dots \otimes v_r), A^*(\bar{v}_1 \otimes \dots \otimes \bar{v}_r) \rangle = \langle A^2(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle$$

$$= \langle A(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle = \frac{1}{r!} \left\langle \sum_\sigma (\mathrm{sgn}\, \sigma) \sigma(v_1 \otimes \dots \otimes v_r), \bar{v}_1 \otimes \dots \otimes \bar{v}_r \right\rangle$$

$$= \frac{1}{r!} \sum_\sigma (\mathrm{sgn}\, \sigma) \langle v_{\sigma_1} \otimes \dots \otimes v_{\sigma_r}, \bar{v}_1 \otimes \dots \otimes \bar{v}_r \rangle$$

$$= \frac{1}{r!} \sum_\sigma (\mathrm{sgn}\, \sigma) \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

wherein various steps have various justifications.

We have proved the formula:

3.3.7. $$\langle v_1 \wedge \dots \wedge v_r, \bar{v}_1 \wedge \dots \wedge \bar{v}_r \rangle = \frac{1}{r!} \sum_\sigma (\mathrm{sgn}\, \sigma) \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

If we replace the pairing $\langle \, , \, \rangle$ by a new pairing $\langle \, | \, \rangle$ defined by the forumla

3.3.8. $$\langle \, | \, \rangle = r! \, \langle \, , \, \rangle$$

we have

$$3.3.9.\ \langle v_1 \wedge \dots \wedge v_r \mid \bar{v}_1 \wedge \dots \wedge \bar{v}_r \rangle = \sum_{\sigma} (\operatorname{sgn} \sigma) \langle v_{\sigma_1}, \bar{v}_1 \rangle \dots \langle v_{\sigma_r}, \bar{v}_r \rangle$$

or, we have the following:

3.3.10

$$\langle v_1 \wedge \dots \wedge v_r \mid \bar{v}_1 \wedge \dots \wedge \bar{v}_r \rangle = \begin{vmatrix} \langle v_{\sigma_1}, \bar{v}_1 \rangle & \langle v_{\sigma_2}, \bar{v}_1 \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_1 \rangle \\ \langle v_{\sigma_1}, \bar{v}_2 \rangle & \langle v_{\sigma_2}, \bar{v}_2 \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_{\sigma_1}, \bar{v}_r \rangle & \langle v_{\sigma_r}, \bar{v}_r \rangle & \cdots & \langle v_{\sigma_r}, \bar{v}_r \rangle \end{vmatrix}$$

Remark 3.3.11.:

The algebra $\wedge V^*$ is called the algebra exterior forms over V .

Remark 3.3.12.:

We will define the pairing $\langle \mid \rangle$ of $\wedge V$ with $\wedge V^*$ by $\langle x \mid \bar{y} \rangle = 0$; $x \in V^{[r]}, V^{[s]}$, $\bar{y} \in (V^*)^{[r]}, (V^*)^{[s]}$, $r \neq s$

and by 3.3.10 if $r = s$.

4. The induced homomorphism.

To every homomorphism $f: V \longrightarrow W$ there is an induced homomorphism $f^{[r]}: V^{[r]} \longrightarrow W^{[r]}$ defined by

3.4.1. $$f^{[r]} = f^r A_r$$

Given homomorphisms $f: V \longrightarrow W$, $g: W \longrightarrow U$ the induced homomorphism $(gf)^{[r]}: V^{[r]} \longrightarrow U^{[r]}$ has the property that

3.4.2. $$(gf)^{[r]} = g^{[r]} f^{[r]}$$

For,
$$\begin{aligned} g^{[r]} f^{[r]} &= (g^r A_r)(f^r A_r) \\ &= (g^r f^r)(A_r A_r) \\ &= (g^r f^r) A_r \\ &= (gf)^r A_r \\ &= (gf)^{[r]} \end{aligned}$$

Also, given $1_V: V \longrightarrow V$ the induced homomorphism $(1_V)^{[r]}: V^{[r]} \longrightarrow V^{[r]}$ has the property that

3.4.3. $$(1_V)^{[r]} = 1_{V^{[r]}}$$

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