THE INCONVENIENT CATEGORY OF TOPOLOGICAL SPACES

RONALD S. LEWIS

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by

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Abstract

An extensive classification of initial and final topologies characterizes chapters one through three. The one exception is a generalization of the locally compact and Hausdorff concept which appears at the beginning of chapter three and plays a role of significance later in the thesis. Most of work in the first three chapters is standard material, the general theory is laid out and followed by specific constructions. Features of this treatment include an initial topology in the function space setting, several final topology constructions that satisfy convenient category criteria, and some basic properties of a specific product topology are explored in detail. Modification of the exponential law is the thrust of chapter four. There is a desire for a law which utilizes no assumptions on the spaces involved. A *compact Hausdorff image*-open topology is defined to replace the standard *compact*-open topology on a function space. The χ-open topology coupled with a χ-product topology give life to a χ-exponential law that has the usual exponential law as a consequence. A fifth chapter examines initial and final topologies in regards to their commutativity. Improvements to the current body of knowledge are made in the area of product and identification commutative. In particular, an interesting case of initial and final commutation using fibred mapping spaces is explored.

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Introduction

Little time is spent on preliminaries, but there are certain points to be brought to the reader's attention.

It should be noted that initial and final topologies are commonly referred to as weak and strong topologies in the literature. Be warned that the latter pair are used interchangeably at times, to the point of blatant discrepancy Christenson and Voxman [10].

There is a deliberate attempt to highlight the universal properties of initial and final topologies in general, and in regards to specific constructions. Both Massey [17] and Brown [7] have recognized the significance of universal properties. Indeed, usage of such properties is a running theme of this work thereby making proofs of many results significantly easier, shorter, and more intuitive.

Let us fix our notation on certain fronts. The standard practice of calling a continuous function a map will be employed in this work. $Fn(X,Y)$ and $Map(X,Y)$ denote the set of functions and maps from the space X to the space Y. $X_{\mathcal{T}}$ denotes the topological space with underlying set X and topology $\mathcal{T}$, when there is a possibility of confusion. The symbol $\cong$ will be used exclusively to denote a homeomorphism.

Chapter 1

INITIAL TOPOLOGIES

$$f_i : X \to X_i$$

1.1 Characterizing the initial topology

Let us set the table. Let $\{X_i\}$ be a family of *inducing spaces* and $\{f_i : X \to X_i\}$ be a family of *inducing functions* for each $i \in I$, where I is an indexing set. An *initial topology* can be induced on the arbitrary set X.

Definition 1 *Let X be a set and $\{f_i : X \to X_i\}_{i \in I}$ be as above. The corresponding* **initial topology on** X, *denoted $\mathcal{I}$, has a subbase*

$$\mathcal{S} = \{f_i^{-1}(U) \text{ such that } U \text{ open in } X_i, \text{ for } i \in I\}.$$

The reader is directed to [23] or [10] for alternate descriptions.

Proposition 1.1 *The previous definition describes a well defined topology on the set X.*

Proof: Any collection of open subsets of a set X is a subbasis for a unique topology on X, it follows that the initial topology is a well defined topology on the set X. ■

Proposition 1.2 *If X has the initial topology then the functions $\{f_i\}_{i\in I}$ are continuous.*

Proof: The result follows from the definition of the subbasis for $\mathcal{I}$. ■

Thus it is safe to describe $\{f_i\}$ as a family of mappings.

Proposition 1.3 : Universal Property for Initial Topologies

Given a set X with the initial topology relative to a set of corresponding maps and spaces $\{f_i : X \to X_i\}_{i\in I}$, and a function $g : Z \to X$ for some space Z. Then the function g is continuous if and only if $f_i \circ g$ is continuous for all $i \in I$.

Proof: Assume the data.

Let g be continuous. It follows from 1.2 that the composites $f_i \circ g$ are continuous for each $i \in I$.

Suppose each $f_i \circ g$ is continuous and let there be an open set U in X_i. Then $(f_i \circ g)^{-1}(U)$ is open in Z. Now

$$(f_i \circ g)^{-1}(U) = (g^{-1} \circ f_i^{-1})(U) = g^{-1}(f_i^{-1}(U)).$$

Now $f_i^{-1}(U)$ is a typical member of the subbasis for X and $g^{-1}(f_i^{-1}(U))$ is open in Z for each $f_i^{-1}(U)$. It follows that $g : Z \to X$ is continuous. ■

In proving the next proposition, the reader should bear in mind that the initial topology satisfies the Universal Property for Initial Topologies. It is necessary to show that any topology satisfying this condition is unique.

Proposition 1.4 *The Universal Property for Initial Topologies is a characterization for the initial topology on a set X.*

Proof: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X both satisfying 1.3 for a family of mappings $\{f_i\}_{i\in I}$. Consider the identity functions $1_X : X_{\mathcal{T}_1} \to X_{\mathcal{T}_2}$ and $1'_X : X_{\mathcal{T}_2} \to X_{\mathcal{T}_1}$. Our hypothesis implies that $f_i \circ 1_X = f_i : X_{\mathcal{T}_1} \to X_i$ is continuous for each $i \in I$. Similarly $f_i \circ 1'_X = f_i$ is continuous. It follows by 1.3 that both 1_X and $1'_X$ are continuous. Thus $\mathcal{T}_1 = \mathcal{T}_2$, thereby proving any topology satisfying this property is unique. Moreover, the initial topology satisfies 1.3; so any topology satisfying the Universal Property for Initial Topologies must be the initial topology. ■

This universal property permits proof of the following proposition concerning a *composition rule* or *transitive rule* which can be applied in certain situations.

Proposition 1.5 : Composition Rule for Initial Topologies

Let each set X_i have an initial topology relative to $\{g_k : X_i \to X_k\}_{k\in K_i}$ for all $i \in I$. Then the initial topology on X relative to $\{f_i : X \to X_i\}_{i\in I}$ coincides with the initial topology on X relative to $\{g_k \circ f_i : X \to X_k\}_{i\in I, k\in K_i}$.

Proof: Let $\mathcal{I}_1$ be the initial topology on X relative to the maps $\{f_i : X \to X_i\}_{i\in I}$ and let $\mathcal{I}_2$ be the initial topology on X relative to the maps $\{g_k \circ f_i : X \to X_k\}_{i \in I, k\in K_i}$. Consider the identity functions $1_X : X_{\mathcal{I}_1} \to X_{\mathcal{I}_2}$ and $1'_X : X_{\mathcal{I}_2} \to X_{\mathcal{I}_1}$.

Associativity of function composition ensures $g_k \circ (f_i \circ 1'_X) = (g_k \circ f_i) \circ 1'_X = (g_k \circ f_i) : X_{\mathcal{I}_2} \to X_k$ where $(g_k \circ f_i)$ is continuous for all $i \in I$ and all $k \in K_i$. It follows that $(f_i \circ 1'_X)$ is continuous by way of 1.3 for X_i. Then $1'_X$ is continuous by 1.3 for $\mathcal{I}_1$.

For all maps f_i and g_k every composite $g_k \circ f_i : X_{\mathcal{I}_1} \to X_k$ is continuous. Now $g_k \circ f_i = (g_k \circ f_i) \circ 1_X : X_{\mathcal{I}_1} \to X_k$ is continuous. Thus 1_X is continuous by 1.3 for $\mathcal{I}_2$. Therefore the identity function is continuous in both directions, thus the topologies $\mathcal{I}_1$ and $\mathcal{I}_2$ must coincide. ∎

A *decomposition rule* for initial topologies can be proven. If an inducing family of maps can be factored through a family of spaces, then an initial topology coincides relative to the latter family.

Proposition 1.6 : Decomposition Rule for Initial Topologies

Let X have an initial topology relative to $\{f_i : X \to X_i\}_{i\in I}$. Let there be a family of spaces $\{Y_i\}_{i\in I}$ and families of mappings $\{h_i : Y_i \to X_i\}_{i\in I}$ and $\{k_i : X \to Y_i\}_{i\in I}$ such that $h_i \circ k_i = f_i$ for all $i \in I$. Then X has an initial topology relative to $\{k_i : X \to Y_i\}_{i\in I}$.

Proof: It will be proven that $\{k_i : X \to Y_i\}$ satisfies 1.3. Let $g : Z \to X$ be a function for an arbitrary space Z.

Suppose that g is continuous, then $k_i \circ g : Z \to Y_i$ is continuous for each $i \in I$.

Now suppose that each $k_i \circ g : Z \to Y_i$ is continuous. Then $h_i \circ (k_i \circ g) : Z \to X_i$ is continuous. Moreover,

$$h_i \circ (k_i \circ g) = (h_i \circ k_i) \circ g = f_i \circ g \quad \forall i \in I.$$

Hence, g is continuous by 1.3 for $\{f_i : X \to X_i\}_{i \in I}$. The result follows by 1.4.
■

Proposition 1.7 *The initial topology is the coarsest topology on X ensuring that each $f_i : X \to X_i$ is continuous on X relative to $\{f_i : X \to X_i\}$.*

Proof: Assume some other topology from $\mathcal{I}$, say $\mathcal{T}$, on X such that each f_i is continuous. Universal property 1.3 ensures that the identity function $1 : X_{\mathcal{T}} \to X_{\mathcal{I}}$ is continuous. So for all $V \in \mathcal{I}$, it follows that $V \in \mathcal{T}$, thus $\mathcal{I} \subset \mathcal{T}$. ■

It is possible to characterize several initial topology examples armed with the general results of section one.

1.2 Initial-inverse topology

The most immediate example of an initial topology is the *inverse image topology.* However, to alleviate future confusion it is referred to as the *initial-inverse topology.* Induce an initial topology on X by taking any function f into any arbitrary space Y, $f : X \to Y$. This leads to the characterizing theorem of this example of an initial topology.

Characterization 1.8 *Let Y be a space, let X be a set and let $f : X \to Y$ be a function. The following conditions each determine precisely the same topology on X:*

(i) *X carries the initial topology relative to $f : X \to Y$,*

(ii) **Universal Property:** *Given any space Z and any function $g : Z \to X$, then g is continuous if and only if $f \circ g$ is continuous,*

(iii) *X has the coarsest topology such that f is continuous.*

Proof: It follows from 1.4 that (i)⇔(ii) and 1.7 ensures that (i)⇔(iii). ■

The unique topology satisfying conditions (i)-(iii) of the previous theorem is called the *initial-inverse topology.* The following commutative triangle illustrates the universal property of this last proposition.

$$\begin{tikzcd} Z \arrow[d, "g"'] \arrow[dr, "f \circ g"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

1.3 Subspace topology

A specific case of the initial-inverse topology is the *subspace* or *relative* topology. A topology is induced on any subset A, of an arbitrary topological space X via an inclusion. We have simply $i : A \hookrightarrow X$. Note that in this situation the inducing map is an injection.

Characterization 1.9 *Let X be a space, let A be a subset of X and let $i : A \hookrightarrow X$ be an inclusion. The following conditions each determine precisely the same topology on A:*

(i) A has the initial topology relative to the inclusion $i : A \hookrightarrow X$,

(ii) **Universal Property:** *Given any space Z and any function $g : Z \to A$, then g is continuous if and only if $i \circ g$ is continuous,*

(iii) A has the coarsest topology such that i is continuous.

Proof: Similar to 1.8. ■

The unique topology satisfying (i)-(iii) of the previous theorem is called the *subspace topology.* The commutative diagram illustrates the universal property described in the previous theorem.

$$\begin{tikzcd} Z \arrow[d, "g"'] \arrow[dr, "i \circ g"] & \\ A \arrow[r, hook, "i"'] & X \end{tikzcd}$$

1.4 Product topology

There are different points to be considered when an initial topology is induced on a Cartesian product of spaces.

Let I be a possibly infinite index set with $\iota \in I$ and let $\{X_\iota\}_{\iota \in I}$ be a family of inducing spaces. Take the underlying set to be of the form

$$X = \Pi_\iota X_\iota = \left\{ f : I \to \bigcup_\iota X_\iota \text{ such that } f(\iota) \in X_\iota, \forall \iota \in I \right\}.$$

Then define a family of projections $\{\rho_\iota : X \to X_\iota\}_{\iota \in I}$ by

$$\rho_\iota(f) = f(\iota) \quad \forall \iota \in I, \forall f \in \Pi X_\iota.$$

Now it is possible to induce an initial topology on X.

Characterization 1.10 *Let $\{X_\iota\}_{\iota \in I}$ be a family of spaces, let X be the set above and let $\{\rho_\iota : X \to X_\iota\}_{\iota \in I}$ be the family of projections as above. The following conditions each determine precisely the same topology on X:*

(i) X has the initial topology relative to the projections $\rho_\iota : X \to X_\iota$ for all ι,

(ii) **Universal Property** *Given any space Y and any function $g : Y \to X$, then g is continuous if and only if $\rho_\iota \circ g$ is continuous for all ι,*

(iii) X has the coarsest topology such that each ρ_ι is continuous for all ι.

Proof: Similar to 1.8. ■

The unique topology satisfying the conditions of this last proposition is called the *Tychonoff product topology.* The universal property of this situation is illustrated by the following commutative diagram.

$$\begin{tikzcd}
Y \arrow[d, "g"'] \arrow[dr, "\rho_\iota \circ g"] & \\
X \arrow[r, "\rho_\iota"'] & X_\iota
\end{tikzcd}$$

Suppose I is a finite index set with $i \in I$. The reader will recall that the *box topology* on the set $X = \Pi_i X_i$, where each X_i is a topological space, has as a basis all sets of the form $U = \Pi_i U_i$, where $U_i \subset X_i$ is open for each $i \in I$. The box topology is an initial topology induced by the family of natural projections $\{\pi_i : X \to X_i\}_{i \in I}$. In this case, with I finite, the box topology coincides with the Tychonoff topology. Now suppose that I is an infinite set with $\iota \in I$. It is not possible to express $U = \Pi_\iota U_\iota$, where each $U_\iota \subset X_\iota$ is open, as a union of a finite number of basic open sets. We direct the reader's attention to example 2, page 98 [12]. Hence $U \subset X = \Pi_\iota X_\iota$ is not open in general in the Tychonoff topology, and the box topology on X is not an initial topology relative to the infinite family of natural projections. In practice, such a topology is too fine. It is possible to describe the Tychonoff topology in a box like manner. A basis for the Tychonoff topology can be described in a box like manner by adding further conditions on the basic open sets $U = \Pi_\iota U_\iota$: each U_ι is open in X_ι and all but finitely many $U_\iota = X_\iota$.

1.5 Pullback space topology

For spaces X_1, X_2, and Y a *pullback space* or *fibred product space* is formed in the following manner.

Let $p_1 : X_1 \to Y$ and $p_2 : X_2 \to Y$ be continuous functions. A subset of $X_1 \times X_2$ is the underlying set of this situation.

$$X_1 {}_{p_1}\sqcap_{p_2} X_2 := \{(x_1, x_2) \text{ such that } p_1(x_1) = p_2(x_2)\}.$$

When no confusion can occur this is denoted by $X_1 \sqcap X_2$. Let $j : X_1 \sqcap X_2 \hookrightarrow X_1 \times X_2$ be an inclusion. Then define the projection $\pi_1' : X_1 \sqcap X_2 \to X_1$ by

$$\pi_1'(x_1, x_2) := \pi_1 \circ j(x_1, x_2) = x_1 \quad \forall (x_1, x_2) \in X_1 \sqcap X_2.$$

Similarly, we can define the projection $\pi_2' : X_1 \sqcap X_2 \to X_2$ by

$$\pi_2'(x_1, x_2) := \pi_2 \circ j(x_1, x_2) = x_2 \quad \forall (x_1, x_2) \in X_1 \sqcap X_2.$$

Then the following diagram commutes.

$$\begin{tikzcd}
X_1 \sqcap X_2 \arrow[r, "\pi_2'"'] \arrow[d, "\pi_1'"'] & X_2 \arrow[d, "p_2"'] \\
X_1 \arrow[r, "p_1"'] & Y
\end{tikzcd}$$

An initial topology can be induced on $X_1 \sqcap X_2$ by taking the projections π_1' and π_2' as inducing maps and X_1 and X_2 as inducing spaces. It is also possible to induce an initial topology on $X_1 \sqcap X_2$ relative to the inclusion $j : X_1 \sqcap X_2 \hookrightarrow X_1 \times X_2$.

Proposition 1.11 *The initial topology on $X_1 \sqcap X_2$ relative to π_1' and π_2' coincides with the initial topology on $X_1 \sqcap X_2$ relative to j, i.e. $X_1 \sqcap X_2$ regarded as a subspace of $X_1 \times X_2$.*

Proof: The result follows from 1.5 with $g_1 = \pi_1'$, $g_2 = \pi_2'$ and $f = j$. ∎

Consider the following characterization theorem.

Characterization 1.12 *Let X_1 and X_2 be spaces, let $X = X_1 \sqcap X_2$ and let π_1' and π_2' be the projections as defined above. The following conditions each determine precisely the same topology on X:*

(i) X has the initial topology relative to $\pi_1' : X \to X_1$ and $\pi_2' : X \to X_2$,

(ii) X has the initial topology relative to $j : X \hookrightarrow X_1 \times X_2$,

(iii) **Universal Property:** *Given any space Z and any function $g : Z \to X$, then X has the unique topology such that g is continuous if and only if $\pi_1' \circ g$ and $\pi_2' \circ g$ are continuous,*

(iv) X has the coarsest topology such that π_1' and π_2' are continuous,

(v) X has the coarsest topology such that j is continuous.

Proof: It follows from 1.11 $(i) \Leftrightarrow (ii)$ and from 1.4 that $(i) \Leftrightarrow (iii)$. Proposition 1.7 ensures that $(i) \Leftrightarrow (iv)$ and $(ii) \Leftrightarrow (v)$. ∎

The unique topology satisfying the conditions of this theorem is called the *pullback space topology.* The universal property of this last situation is illustrated by the following commutative diagram.

$$\begin{tikzcd}
Z \arrow[dr, "g"'] \arrow[drr, bend left, "\pi_2' \circ g"] \arrow[ddr, bend right, "\pi_1' \circ g"'] & & \\
& X_1 \sqcap X_2 \arrow[r, "\pi_2'"'] \arrow[d, "\pi_1'"'] & X_2 \arrow[d, "p_2"] \\
& X_1 \arrow[r, "p_1"'] & Y
\end{tikzcd}$$

The topology described above can easily be generalized for any finite index set I and family $\{p_i : X_i \to Y\}_{i \in I}$. Define

$$\sqcap_{p_i} X_i = \{(x_1, x_2, ..., x_n) \text{ such that } p_i(x_i) = p_j(x_j), \forall i, j \in I\}.$$

A topology can be induced on this subset of $\Pi_i X_i$ relative to projections of the form $\pi_i' = \pi_i \circ j$.

Suppose that I is not finite. It is possible to generalize in a manner similar to the product topology by considering sets of functions. For $\iota \in I$ and a family of maps and spaces $\{p_\iota : X_\iota \to Y\}_{\iota \in I}$ define

$$\sqcap_{p_\iota} X_\iota = \left\{ g : I \to \bigcup_\iota X_\iota \text{ such that } g(\iota) \in X_\iota \text{ and } p_\iota \circ g(\iota) = p_{\iota'} \circ g(\iota'), \forall \iota, \iota' \in I \right\}.$$

An initial topology is induced on this subset relative to $\{\rho_\iota \circ j\}_{\iota \in I}$ where j is the inclusion of $\sqcap_{p_\iota} X_\iota \subset \Pi_\iota X_\iota$.

1.6 Free range functional space topology

To complete the initial topology exposition consider an example of greater interplay between initial topologies and function spaces.

For an arbitrary space $Z_{\mathcal{T}}$, define the space $Z^{\omega}_{\mathcal{T}'}$ as having underlying set

$$Z^{\omega} = Z \cup \{\omega\} \text{ for some } \omega \notin Z,$$

and topology

$$\mathcal{T}' = \{\emptyset\} \cup \{U \cup \{\omega\} \text{ such that } U \in \mathcal{T}\}.$$

For simplicity's sake the subscript is disregarded in future discussion. Let Y be an arbitrary space. For C closed in Y and any map $f : C \to Z$ we can define a map $f^{\omega} : Y \to Z^{\omega}$ by

$$f^{\omega}(y) = \begin{cases} f(y) & \text{if } y \in C, \\ \omega & \text{if } y \notin C. \end{cases}$$

For an arbitrary space B and any mapping $q : Y \to B$ define the space

$$Y|b = q^{-1}(b) \quad \forall b \in B,$$

and the set

$$Y!Z = \bigcup_{b \in B} Map(Y|b, Z).$$

If B is a T_1-space, then we can define functions

$$\begin{array}{llll} q!Z : Y!Z \longrightarrow B & \text{by} & (q!Z)(h) = b & \text{where } h : Y|b \to Z, \\ i : Y!Z \longrightarrow Map(Y, Z^{\omega}) & \text{by} & i(f) = f^{\omega} & \text{where } f \in Map(Y|b, Z). \end{array}$$

Note that i is well defined as a result of the T_1 condition on B. It implies that each fibre $Y|b$ is closed in Y.

Characterization 1.13 *Let Y and Z be spaces, let B be a T_1-space, and let Z^{ω} and $Y!Z$ be as defined above. Also, let $q!Z : Y!Z \longrightarrow B$ and $i : Y!Z \longrightarrow Map(Y, Z^{\omega})$ be the maps defined above. The following conditions each determine precisely the same topology on $Y!Z$:*

(i) $Y!Z$ *has the initial topology relative to the functions* $q!Z : Y!Z \longrightarrow B$ *and* $i : Y!Z \longrightarrow Map(Y, Z^\omega)$,

(ii) **Universal Property** *Given any space* W *and any function* $g : W \to Y!Z$, *then* g *is continuous if and only if* $q!Z \circ g$ *and* $i \circ g$ *are continuous,*

(iii) $Y!Z$ *has the coarsest topology such that* $q!Z$ *and* i *are continuous.*

Proof: Similar to 1.8. ■

The unique topology satisfying the conditions of this theorem is the called *free range functional* or *free range fibred mapping* space topology. A more extensive discussion of this example is beyond the scope of this thesis; we refer the reader to [2] for more extensive details.

Chapter 2

FINAL TOPOLOGIES I

$$g_j : X_j \to X$$

2.1 Characterizing the final topology

In the language of Spanier [19], a final topology on X is said to be *coinduced* by the family of functions and spaces indexed by J.

Definition 2 *Let J be a set and $\{X_j\}_{j \in J}$ be a set of spaces indexed by J. Let X be a set and $\{g_j : X_j \to X\}_{j \in J}$ be as above. Then the corresponding* **final topology on** *X, denoted $\mathcal{F}$, consists of all subsets $U \subset X$ such that $g_j^{-1}(U)$ is open in X_j for all $j \in J$.*

Proposition 2.1 *The previous definition describes a well defined topology on the set X.*

Proof: Let $\mathcal{F}$ be the final topology on X with respect to a family of maps and spaces $\{g_j : X_j \to X\}_{j \in J}$. It follows that both $\emptyset$ and $X \in \mathcal{F}$, since $g_j^{-1}(\emptyset) = \emptyset$ and $g_j^{-1}(X) = X_j$ for all $j \in J$.

Let $\{U_i\}$ be a finite family of open sets from $\mathcal{F}$. Now

$$g_j^{-1}\left(\bigcap_i^n U_i\right) = \bigcap_i^n g_j^{-1}(U_i).$$

Each $g_j^{-1}(U_i)$ is open in each X_j, thus their intersection is open in each space.

Lastly, take $\{U_\iota\}$ as any indexed family of open sets in $\mathcal{F}$. Now

$$g_j^{-1}\left(\bigcup_\iota U_\iota\right) = \bigcup_\iota g_j^{-1}(U_\iota).$$

Each $g_j^{-1}(U_\iota)$ is open in each X_j for all j. It follows $\bigcup_\iota g_j^{-1}(U_\iota)$ is open in each X_j. Thus all the necessary conditions are satisfied and the final topology is a well defined topology. ∎

Proposition 2.2 *If X has the final topology, then the functions $\{g_j\}_{j \in J}$ are continuous.*

Proof: The result follows from definition 2. ∎

We can utilize a closed set definition of this topology. Let us denote such by $\mathcal{F}'$. That is, a set $C \subset X$ is closed in $\mathcal{F}'$ only when each $g_j^{-1}(C)$ is closed in the corresponding space X_j. Some useful final topologies discussed in chapter three utilize a closed set definition of a final topology.

Proposition 2.3 *$\mathcal{F}'$ is a well defined topology in the closed set sense.*

Proof: Assume the closed set definition of this topology. Both $\emptyset$ and X_j are closed in X_j for all $j \in J$. Hence, both $\emptyset$ and $X \in \mathcal{F}'$.

Let $\{C_i\}$ be any finite family of non-empty closed sets from $\mathcal{F}'$. Now

$$g_j^{-1}\left(\bigcup_i^n C_i\right) = \bigcup_i^n g_j^{-1}(C_i).$$

Recall that each $g_j^{-1}(C_i)$ is closed in each X_j for all j. Consequently any finite union of such closed sets will be closed in each X_j. Hence $\bigcup_i^n C_i \in \mathcal{F}'$.

As well, for any any family $\{C_\iota\}$

$$g_j^{-1}\left(\bigcap_\iota C_\iota\right) = \bigcap_\iota g_j^{-1}(C_\iota).$$

Now each $g_j^{-1}(C_\iota)$ is closed in each X_j, thus $\bigcap_\iota g_j^{-1}(C_\iota)$ is closed there as well. Thereby ensuring that $\bigcap_\iota C_\iota \in \mathcal{F}'$ for any ι. Thus $\mathcal{F}'$ is closed under arbitrary intersections. Hence $\mathcal{F}'$ is a well defined topology. ■

The following proposition establishes a one-to-one correspondence between the sets of either collection $\mathcal{F}$ and $\mathcal{F}'$.

Proposition 2.4 *The open and closed set definitions for the final topology on X relative to $\{g_j : X_j \to X\}_{j\in J}$ specify the same topology on X.*

Proof: Let X have the final topology relative to $\{g_j : X_j \to X\}_{j\in J}$. Then for any $U \subset X$

$$\begin{aligned}
& U \in \mathcal{F} \\
\Leftrightarrow\ & g_j^{-1}(U) \text{ is open in each } X_j \\
\Leftrightarrow\ & X_j \backslash g_j^{-1}(U) \text{ is closed in each } X_j \\
\Leftrightarrow\ & g_j^{-1}(X \backslash U) \text{ is closed in each } X_j \\
\Leftrightarrow\ & X \backslash U \in \mathcal{F}'.
\end{aligned}$$

Thereby establishing the necessary correspondence between the topologies $\mathcal{F}$ and $\mathcal{F}'$. ■

It is possible to prove several results which characterize the topology of this chapter. Consider a Universal Property for Final Topologies.

Proposition 2.5 : Universal Property of Final Topologies

Given a set X with the final topology relative to a set of corresponding maps and spaces $\{g_j : X_j \to X\}_{j\in J}$, and a function $f : X \to Y$ for some space Y. Then the function f is continuous if and only if $f \circ g_j$ is continuous for all $j \in J$.

Proof: Assume the data.

Let $f : X \to Y$ be continuous. From hypothesis each g_j is continuous, so the composition $f \circ g_j$ is continuous for all $j \in J$.

Now assume each $f \circ g_j : X_j \to Y$ is continuous. Let U be open in Y. Then $(f \circ g_j)^{-1}(U)$ is open in each X_j. Now

$$(f \circ g_j)^{-1}(U) = (g_j^{-1} \circ f^{-1})(U) = g_j^{-1}(f^{-1}(U)).$$

Thus $g_j^{-1}(f^{-1}(U))$ is open in each space X_j. Since X has the final topology relative to each continuous g_j, it follows $f^{-1}(U)$ is open in X by definition 2. Consequently f is continuous. ■

In fact, 2.5 is a characterization of the final topology. Indeed, it has been considered an alternative definition for a final topology [7]. The next proposition verifies this property is a suitable characterization for such topologies.

Proposition 2.6 *The Universal Property for Final Topologies is a characterization for the final topology on a set X.*

Proof: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X both satisfying 2.5 for a family of mappings $\{g_j : X_j \to X\}_{j \in J}$. Consider the identity functions $1_X : X_{\mathcal{T}_1} \to X_{\mathcal{T}_2}$ and $1'_X : X_{\mathcal{T}_2} \to X_{\mathcal{T}_1}$. It follows from $1'_X \circ g_j = g_j$ and $1_X \circ g_j = g_j$ that the identity function is continuous in both directions. Hence $\mathcal{T}_1 = \mathcal{T}_2$, and any such topologies satisfying this definition must coincide. Moreover, the final topology satisfies the Universal Property for Final Topologies, so any topology satisfying 2.5 is coincident with a final topology on X relative to $\{g_j : X_j \to X\}_{j \in J}$. ■

There is a *composition* or *transitive rule* for final topologies. The initial terminology is favoured.

Proposition 2.7 : Composition Rule for Final Topologies

Let each set X_j have a final topology relative to $\{h_k : X_k \to X_j\}_{k \in K_j}$ for all $j \in J$. Then the final topology on X relative to $\{g_j : X_j \to X\}_{j \in J}$ coincides with the final topology on X relative to $\{g_j \circ h_k : X_k \to X\}_{j \in J, k \in K_j}$.

Proof: Let X have a final topology relative to the $\{g_j : X_j \to X\}_{j \in J}$ and let each X_j the final topology relative to the $\{h_k : X_k \to X_j\}_{k \in K_j}$. Let U be an open subset in X. Then $g_j^{-1}(U)$ is open in each X_j. It follows that $h_k^{-1}(g_j^{-1}(U))$ is open in X_k. Now $h_k^{-1}(g_j^{-1}(U)) = (g_j \circ h_k)^{-1}(U)$ for all $j \in J$ and all $k \in K_j$, then U is open in the final topology on X relative to $\{g_j \circ h_k\}_{j \in J, k \in K_j}$. This argument is reversible, thereby ensuring the desired result. ■

A result which mirrors 1.6 can be proven. It is shown that if a coinducing family of maps can be factored through a family of spaces then a final topology coinduced by that family of spaces coincides with the original final topology.

Proposition 2.8 : Decomposition Rule for Final Topologies

Let X have a final topology relative to $\{g_j : X_j \to X\}_{j\in J}$. Let there be a family of spaces $\{Y_j\}_{j\in J}$ and families of mappings $\{h_j : X_j \to Y_j\}_{j\in J}$ and $\{k_j : Y_j \to X\}_{j\in J}$ such that $g_j = k_j \circ h_j$ for all $j \in J$. Then X has a final topology relative to $\{k_j : Y_j \to X\}_{j\in J}$.

Proof: It will be proven that $\{k_j : Y_j \to X\}$ satisfies 2.5. Let $f : X \to Z$ be a function for an arbitrary space Z.

Suppose that f is continuous, then $f \circ k_j : Y_j \to Z$ is continuous for each $j \in J$.

Now suppose that each $f \circ k_j : Y_j \to Z$ is continuous for each $j \in J$. Then $(f \circ k_j) \circ h_j : X_j \to Z$ is continuous. Moreover

$$(f \circ k_j) \circ h_j = f \circ (k_j \circ h_j) = f \circ g_j, \quad \forall j \in J.$$

Hence f is continuous by 2.5. The result follows by 2.6. ■

Finally a namesake property for this type of topology.

Proposition 2.9 *The final topology is the finest topology on X ensuring that each $g_j : X_j \to X$ is continuous on X relative to $\{g_j : X_j \to X\}$*

Proof: Let $\mathcal{T}$ be some other topology on X such that each g_j is continuous. Consider the identity function on X, $1_X : X_{\mathcal{F}} \to X_{\mathcal{T}}$. Each composite $1_X \circ g_j = g_j$ where each g_j is continuous, then 2.5 guarantees the continuity of 1_X. Thus any set open in $\mathcal{T}$ has an open preimage under the identity map in X with the final topology, i.e. $\mathcal{T} \subset \mathcal{F}$. ■

There are several final topology constructions to characterize using these general results. They are examined here and in chapter three. Particular attention is devoted to a non-standard product space topology toward the end of the third chapter. It will be employed in the chapters four and five.

2.2 Final-inverse topology

The *final-inverse topology* is the basic example of a final topology. Given any function $g : X \to Y$, a final topology can be coinduced on Y for any arbitrary space X.

Characterization 2.10 *Let X be a space, let Y be a set and let $g : X \to Y$ be a function. The following conditions each determine precisely the same topology on Y:*

(i) Y carries the final topology relative to $g : X \to Y$, i.e. if $U \subset Y$, then U is open in Y if and only if $g^{-1}(U)$ is open in X,

(ii) Let $C \subset Y$, then C is closed in Y if and only if $g^{-1}(C)$ is closed in X,

(iii) **Universal Property:** *Given any space Z and any function $f : Y \to Z$, then f is continuous if and only if $f \circ g$ is continuous,*

(iv) Y has the finest topology such that g is continuous.

Proof: Proposition 2.4 ensures (i)$\Leftrightarrow$(ii) and 2.6 guarantees (i) $\Leftrightarrow$ (iii). Then (i) $\Leftrightarrow$ (iv) is verified by 2.9. ■

The unique topology of this last theorem satisfying conditions (i)-(iv) is called the *final-inverse topology.* The following commutative diagram illustrates the universal property of this last characterization.

$$\begin{tikzcd} & Z \\ X \arrow[ur, "f \circ g"] \arrow[r, "g"'] & Y \arrow[u, "f"'] \end{tikzcd}$$

2.3 Identification topology

This final topology is coinduced on Y by a surjective function $p : X \to Y$. In this context, p is known as an *identification* and the topology on Y is called the *identification topology.* Note that the identification topology is a specific case of the final-inverse topology, where g is a surjective function.

Characterization 2.11 *Let X be a space, Y be a set and $p : X \to Y$ a surjective function. The following conditions each determine precisely the same topology on Y:*

(i) Y carries the final topology relative to the surjection $p : X \to Y$, i.e. if $U \subset Y$, then U is open in Y if and only if $p^{-1}(U)$ is open in X,

(ii) Let $C \subset Y$, then C is closed in Y if and only if $p^{-1}(C)$ is closed in X,

(iii) **Universal Property:** *Given any space Z and any function $f : Y \to Z$, then f is continuous if and only if $f \circ p$ is continuous,*

(iv) Y has the finest topology such that p is continuous.

Proof: Similar to 2.10. ■

The unique topology of this last theorem satisfying conditions (i)-(iv) is called the *identification topology.* The following commutative diagram illustrates the universal property of the previous characterization.

$$\begin{tikzcd}
 & Z \\
X \arrow[ur, "f \circ p"] \arrow[r, "p"'] & Y \arrow[u, "f"]
\end{tikzcd}$$

A composition rule and a decomposition rule can be readily justified by 2.5 in this context.

Proposition 2.12 *Let W, X and Y be spaces, and let $p : W \to X$ and $q : X \to Y$ be identifications. Then $q \circ p : W \to Y$ is an identification.*

Proof: It is necessary to show that 2.11(iii) is satisfied. Let $f : Y \to Z$ be a function and Z be an arbitrary space.

Suppose that f is continuous, then $f \circ (q \circ p) : W \to Z$ is continuous.

Now suppose that $f \circ (q \circ p) = (f \circ q) \circ p$ is continuous. Then $f \circ q$ is continuous by 2.11(iii); since p is an identification. It follows f is continuous by 2.11(iii); since q is an identification. Hence 2.11(iii) is satisfied for $q \circ p$, i.e. $q \circ p : W \to Y$ is an identification. ■

Proposition 2.13 *Let Y have the identification topology relative to $p : W \to Y$. Suppose there is a space X and maps $g : W \to X$ and $q : X \to Y$ such that $p = q \circ g$. Then Y has the identification topology relative to $q : X \to Y$.*

Proof: For any point $y \in Y$ there is a point $g \circ p^{-1}(y) \in X$ such that $q \circ g \circ p^{-1}(y) = y$. Thus q is surjective. It will be proven that $q : X \to Y$ satisfies 2.11(iii). Let $f : Y \to Z$ be a function for an arbitrary space Z. Suppose that f is continuous, then $f \circ q : X \to Z$ is continuous.

Now suppose that $f \circ q : X \to Z$ is continuous. Then $(f \circ q) \circ g : W \to Z$ is continuous. Now

$$(f \circ q) \circ g = f \circ (q \circ g) = f \circ p,$$

and f is continuous by 2.11(iii) for p. The result follows by 2.11(iii). ■

2.4 Quotient topology

The *quotient topology* is a specific type of identification topology. One must be careful when reading the literature. We need only look to the exposition of Bourbaki [6] concerning identification spaces to verify this last point!

A quotient topology is constructed by defining an equivalence relation on an arbitrary space X. It is then possible to coinduce a final topology relative to the natural projection of X into its equivalence classes. Such a projection is surjective, thus ensuring the validity of our earlier statement relating identifications. We prove this notion formally after characterizing this topology and illustrating its universal property.

Characterization 2.14 *Let X be a space, let $X/\mathcal{R}$ be a set of equivalence classes of X under an equivalence relation $\mathcal{R}$, and let $\pi : X \to X/\mathcal{R}$ be the projection of X into the corresponding equivalence classes. The following conditions each determine precisely the same topology on $X/\mathcal{R}$:*

(i) *$X/\mathcal{R}$ carries the final topology relative to the projection $\pi : X \to X/\mathcal{R}$, i.e. if $U \subset X/\mathcal{R}$, then U is open in $X/\mathcal{R}$ if and only if $\pi^{-1}(U)$ is open in X,*

(ii) *Let $C \subset X/\mathcal{R}$, then C is closed in $X/\mathcal{R}$ if and only if $\pi^{-1}(C)$ is closed in X,*

(iii) **Universal Property:** *Given any space Z and any function $f : X/\mathcal{R} \to Z$, then f is continuous if and only if $f \circ \pi$ is continuous,*

(iv) *$X/\mathcal{R}$ has the finest topology such that π is continuous.*

Proof: This is a special case of 2.11 with $p = \pi$ and $Y = X/\mathcal{R}$. ■

The unique topology of this last theorem satisfying conditions (i)-(iv) is called the *quotient topology.* The following commutative diagram illustrates the universal property of the Characterization 2.14.

$$\begin{tikzcd}
 & Z \\
X \arrow[ur, "f \circ \pi"] \arrow[r, "\pi"'] & X/\mathcal{R} \arrow[u, "f"']
\end{tikzcd}$$

Proposition 2.15 *Let Y have the identification topology coinduced by $p : X \to Y$ for an arbitrary space X and surjection p. Define a relation $\mathcal{R}$ on X by $x_1 \sim x_2$ if and only if $p(x_1) = p(x_2)$. Let $\pi : X \to X/\mathcal{R}$ be the projection of X into classes, where $[x]$ denotes the equivalence class of x. Define $\theta : Y \to X/\mathcal{R}$ by $\theta(y) = [x]$ for $y = p(x)$. Then θ is a homeomorphism such that $\theta \circ p = \pi$.*

Proof: Clearly $\mathcal{R}$ is a well defined equivalence relation, i.e. it is reflexive, symmetric, and transitive. Moreover, θ is bijective.

Now $\theta \circ p(x) = [x] = \pi(x)$, so $\theta \circ p = \pi$. Since π is continuous, it follows $\theta \circ p$ is continuous. Then θ is continuous by 2.11(iii). Since θ is a bijection θ^{-1} exists. Now $p = \theta^{-1} \circ \theta \circ p$, thus $p = \theta^{-1} \circ \pi$. Since p is continuous, the composite $\theta^{-1} \circ \pi$ is continuous. Then θ^{-1} is continuous follows from 2.14(iii). Consequently θ is a homeomorphism such that $\theta \circ p = \pi$. ■

2.5 Sum topology

The sum topology is a final topology coinduced upon the set $X = \bigsqcup_j X_j$ for a family of disjoint sets X_j, where J is an arbitrary indexing set. This final topology is coinduced by a family of inclusions of disjoint topological spaces, $i_j : X_j \hookrightarrow X$.

Characterization 2.16 *Let $\{X_j\}_{j \in J}$ be a family of disjoint spaces, let X be the set given above, and let $\{i_j : X_j \hookrightarrow X\}_{j \in J}$ be a family of inclusions. The following conditions each determine precisely the same topology on X:*

(i) *X carries the final topology relative to the inclusions $\{i_j : X_j \to X\}_{j\in J}$, i.e. if $U \subset X$, then U is open in X if and only if $i_j^{-1}(U)$ is open in X_j for all j,*

(ii) *If $C \subset X$, then C is closed in X if and only if $i_j^{-1}(C)$ is closed in X_j for all j,*

(iii) **Universal Property:** *Given any space Y and any function $f : X \to Y$, then f is continuous if and only if $f \circ i_j : X_j \to Y$ is continuous for all j,*

(iv) *X has the finest topology such that i_j is continuous for all j.*

Proof: Similar to 2.10. ■

The unique topology of this last theorem satisfying conditions (i)-(iv) is called the *sum topology.* In the case of the sum of just two spaces the universal property is illustrated by the following commutative diagram.

$$\begin{tikzcd}
 & Z & \\
X_1 \arrow[ur, "f \circ i_1"] \arrow[r, hook, "i_1"'] & X_1 \sqcup X_2 \arrow[u, "f"] & X_2 \arrow[ul, "f \circ i_2"'] \arrow[l, hook', "i_2"]
\end{tikzcd}$$

2.6 Adjunction space topology

Intuitively, an adjunction space is formed from attaching two disjoint spaces by identifying a closed subset of one space with a subset of the other space.

It is possible to describe this space as a type of identification space [7], or by using a quotient topology [14]. Another approach which utilizes simpler inducing spaces is favoured here. However, the quotient topology approach is highlighted due to its frequency in the literature.

Consider two disjoint spaces X and Y. Suppose there is a map $f : A \to Y$ where A is some closed subspace of X. We can define an relation $\sim$ on $X \sqcup Y$ by $x \sim y$ if and only if $x \in A$ and $f(x) = y$. Now $\sim$ generates the equivalence relation $\mathcal{R}$. Set $X \sqcup_f Y := X \sqcup Y/\mathcal{R}$. The projection $\pi_f : X \sqcup Y \to X \sqcup_f Y$ is surjective. Then it is possible to give $X \sqcup_f Y$ the quotient topology relative to π_f. Alternatively, let us take into account the inclusions $i_X : X \hookrightarrow X \sqcup Y$ and $i_Y : Y \hookrightarrow X \sqcup Y$. It is possible to realize a final topology on $X \sqcup_f Y$ where the coinducing functions are taken as the compositions $\pi_f \circ i_X$ and $\pi_f \circ i_Y$, and X and Y are coinducing spaces. Note that the compositions $\pi_f \circ i_X$ and $\pi_f \circ i_Y$ are each continuous. Moreover, $\pi_f \circ i_Y$ is injective and $\pi_f \circ i_X$ is injective whenever f is injective. The following commutative diagram describes the situation.

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, hook, "i"'] & Y \arrow[d, "\pi_f \circ i_Y"'] \\ X \arrow[r, "\pi_f \circ i_X"'] & X \sqcup_f Y \end{tikzcd}$$

Our approach to adjunction spaces coincides nicely with the quotient topology approach.

Proposition 2.17 *The final topology on $X \sqcup_f Y$ relative to $\pi_f \circ i_X : X \to X \sqcup_f Y$ and $\pi_f \circ i_Y : Y \to X \sqcup_f Y$ coincides with the final topology on $X \sqcup_f Y$ relative to $\pi_f : X \sqcup Y \to X \sqcup_f Y$.*

Proof: The result follows from 2.7 with $h_1 = i_X$, $h_2 = i_Y$ and $g = \pi_f$. ■

Characterization 2.18 *Let X and Y be disjoint spaces, let $f : A \to Y$ be a map for some closed subset A in X and let π_f, i_X and i_Y be the maps defined above. The following conditions each determine precisely the same topology on $X \sqcup_f Y$:*

(i) $X \sqcup_f Y$ carries the final topology relative to $\pi_f \circ i_X : X \sqcup_f Y \to X$ and $\pi_f \circ i_Y : X \sqcup_f Y \to Y$, i.e. if $U \subset X \sqcup_f Y$, then U is open in $X \sqcup_f Y$ if and only if $(\pi_f \circ i_X)^{-1}(U)$ is open in X and $(\pi_f \circ i_Y)^{-1}(U)$ is open in Y,

(ii) Let $C \subset X \sqcup_f Y$, then C is closed in $X \sqcup_f Y$ if and only if $(\pi_f \circ i_X)^{-1}(C)$ is closed in X and $(\pi_f \circ i_Y)^{-1}(C)$ is closed in Y,

(iii) $X \sqcup_f Y$ carries the final topology relative to $\pi_f : X \sqcup Y \to X \sqcup_f Y$, i.e. if $U \subset X \sqcup_f Y$, then U is open in $X \sqcup_f Y$ if and only if $\pi_f^{-1}(U)$ is open in $X \sqcup Y$,

(iv) Let $C \subset X \sqcup_f Y$, then C is closed in $X \sqcup_f Y$ if and only if $\pi_f^{-1}(C)$ is closed in $X \sqcup Y$,

(v) **Universal Property:** *Given any space Z and any function $g : X \sqcup_f Y \to Z$, then g is continuous if and only if $g \circ (\pi_f \circ i_X) : X \to Z$ and $g \circ (\pi_f \circ i_Y) : Y \to Z$ are continuous,*

(vi) $X \sqcup_f Y$ has the finest topology such that $\pi_f \circ i_X$ and $\pi_f \circ i_Y$ are continuous.

Proof: It follows from 2.4 that (i)$\Leftrightarrow$(ii) and (iii)$\Leftrightarrow$(iv). Proposition 2.17 ensures (i)$\Leftrightarrow$(iii), 2.6 guarantees (i)$\Leftrightarrow$(v), and 2.9 provides that (i)$\Leftrightarrow$(vi). ■

The unique topology satisfying conditions (i)-(vi) of this last theorem is called the *adjunction space topology.* The following commutative diagram is illustrative of the universal property in the context of Characterization 2.18.

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, hook, "i"'] & X \sqcup_f Y \arrow[d, "\pi_f \circ i_Y"'] \arrow[ddr, bend left, "g \circ (\pi_f \circ i_Y)"] & \\
X \arrow[r, "\pi_f \circ i_X"'] \arrow[drr, bend right, "g \circ (\pi_f \circ i_X)"'] & Y \arrow[dr, "g"] & \\
 & & Z
\end{tikzcd}$$

A topological sum of two spaces can be described in terms of an adjunction space topology. Take $A = \emptyset \subset X$ and consider the empty map $\phi : \emptyset \to Y$. Then $X \sqcup_\phi Y = X \sqcup Y$.

2.7 Wedge space topology

Let $\{X_j, x_{0_j}\}_{j \in J}$ be an indexed family of disjoint topological spaces with base points. A wedge product or one-point union [14], is formed by identifying each base point x_{0_j} to a single point ω_0 within $\bigsqcup_j X_j$ with the sum topology. In this situation, the underlying set is taken as

$$\bigvee_j X_j = \bigsqcup_j (X_j \backslash \{x_{0_j}\}) \cup \{\omega_0\}.$$

Define a projection $\varphi : \bigsqcup_j X_j \to \bigvee_j X_j$ by

$$\varphi(x) = \begin{cases} x & x \in X_j \backslash \{x_{0_j}\}, \\ \omega_0 & x \notin X_j \backslash \{x_{0_j}\}. \end{cases}$$

The *wedge space topology* is the final topology on $\bigvee_j X_j$ coinduced by composites $\{\varphi \circ i_j : X_j \to \bigvee_j X_j\}_{j\in J}$ where $\{i_j : X_j \hookrightarrow \bigsqcup_j X_j\}_{j\in J}$ is the family of inclusions. However, it is possible to characterize it in a different manner.

Proposition 2.19 *The final topology on $\bigsqcup_j X_j$ relative to $\varphi \circ i_j : X_j \to \bigvee X_j$, for all $j \in J$, coincides with the final topology on $\bigvee X_j$ relative to $\varphi : \bigsqcup_j X_j \to \bigvee_j X_j$.*

Proof: The result follows from 2.7 with $g = \varphi$ and $h_j = i_j$ for all j. ■

Characterization 2.20 *Let $\{(X_j, x_{0_j})\}_{j\in J}$ be a family of disjoint spaces with base points, let $X = \bigvee X_j$, and let $i_j : X_j \to \bigsqcup X_j$ and $\varphi : \bigsqcup X_j \to \bigvee_j X_j$ be as defined above. The following conditions each determine precisely the same topology on X:*

(i) X carries the final topology relative to the composites $\varphi \circ i_j : X_j \to X$, i.e. if $U \subset X$, then U is open in X if and only if $(\varphi \circ i_j)^{-1}(U)$ is open in X_j for all j,

(ii) If $C \subset X$, then C is closed in X if and only if $(\varphi \circ i_j)^{-1}(C)$ is closed in X_j for all j,

(iii) X carries the final topology relative to the composites $\varphi : \bigsqcup_j X_j \to X$, i.e. if $U \subset X$, then U is open in X if and only if $\varphi^{-1}(U)$ is open in $\bigsqcup_j X_j$,

(iv) If $C \subset X$, then C is closed in X if and only if $\varphi^{-1}(C)$ is closed in $\bigsqcup_j X_j$,

(v) **Universal Property:** *Given any space Z and any base point preserving function $f : X \to Z$, then f is continuous if and only if $f \circ (\varphi \circ i_j)$ is continuous for all j,*

(vi) *X has the finest topology such that $\varphi \circ i_j$ is continuous for all j.*

Proof: Proposition 2.19 ensures (i)$\Leftrightarrow$(iii). The remaining details of the proof are similar to 2.18. ■

The unique topology satisfying conditions (i)-(vi) is called the *wedge space topology.* In this new space $\bigvee X_j$, each X_j retains its original topology. In the case of two spaces, the universal property is illustrated by the following commutative diagram. All maps are taken as base point preserving in this situation.

$$\begin{tikzcd}
& Z & \\
X_1 \arrow[ur, "f \circ i_1"] \arrow[r, hook, "i_1"'] & X_1 \vee X_2 \arrow[u, "f"] & X_2 \arrow[ul, "f \circ i_2"'] \arrow[l, hook', "i_2"]
\end{tikzcd}$$

The wedge of two spaces can be expressed in terms of an adjunction space topology. Take $A = \{*\} \subset X$ and the basepoint preserving map $f : \{*\} \to Y$, then $X \vee Y = X \sqcup_f Y$.

2.8 Union of an expanding sequence of subspaces

Let J be a set of non-negative integers and $\{X_j\}$ be a sequence of spaces such that the predecessors of each space in the sequence are its subspaces, i.e. $X_1 \subset X_2 \subset X_3 \subset \ldots$. Take the set X to be $\bigcup_j X_j$, i.e. the union of the underlying sets of the spaces $X_1, X_2, X_3, \ldots$. A final topology on X is realized by taking $\{X_j\}$ as the family of coinducing spaces and coinduce a final topology via inclusions $\{i_j : X_j \hookrightarrow X\}_{j \in J}$.

Characterization 2.21 *Let $\{X_j\}_{j \in J}$ be an expanding sequence of subspaces, let $X = \bigcup_j X_j$ and let $\{i_j : X_j \hookrightarrow X\}_{j \in J}$ be the family of inclusions. The following conditions each determine precisely the same topology on X:*

(i) X carries the final topology relative to the inclusions $i_j : X_j \to X$, i.e. if $U \subset X$, then U is open in X if and only if $i_j^{-1}(U)$ is open in X_j for all j,

(ii) If $C \subset X$, then C is closed if and only if $i_j^{-1}(C)$ is closed in X_j for all j,

(iii) **Universal Property:** *Given any space Y and any function $f : X \to Y$, then f is continuous if and only if $f \circ i_j = f|X_j$ is continuous for all j,*

(iv) X has the finest topology such that i_j is continuous for all j.

Proof: Similar to 2.10. ■

In the case of a union of an expanding sequence of subspaces, notice the universal property takes a form similar to that for the topological sum of a denumerable family of spaces. The universal property in this case being illustrated by the familiar commutative diagram.

$$\begin{tikzcd}
 & Z \\
X_j \arrow[ur, "f \circ i_j"] \arrow[r, hook, "i_j"'] & X \arrow[u, "f"]
\end{tikzcd}$$

Chapter 3

FINAL TOPOLOGIES II

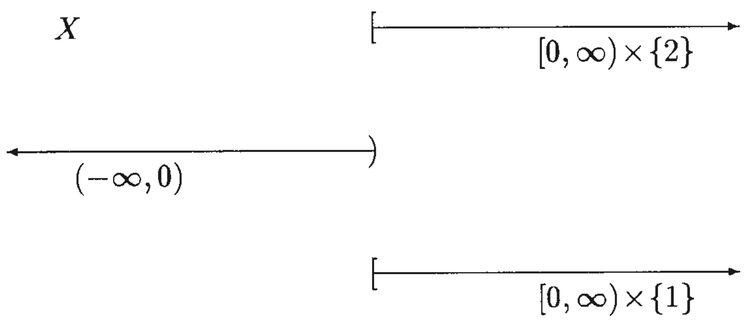
3.1 A useful local property

There is a desire for a nicer form of compactness and Hausdorffness. It is possible to consider every point in a space to be contained in a compact and Hausdorff neighbourhood.

Definition 3 *A topological space X is* **locally compact-Hausdorff** *if for any neighbourhood, N_x, of each $x \in X$ there is an open set $U \subset X$ such that $\bar{U}$ is both compact and Hausdorff where $x \in U \subset \bar{U} \subset N_x$.*

The reader should note that if X is a Hausdorff space, then the locally compact-Hausdorff and locally compact and Hausdorff concepts coincide. Every locally compact and Hausdorff space is locally compact-Hausdorff, but in general the converse is false. Consider the following nontrivial counterexample of a locally compact-Hausdorff space.

Example 3.1 *Let* $X = (-\infty, 0) \sqcup ([0,\infty)\times\{1\}) \sqcup ([0,\infty)\times\{2\})$.



A basis for the open sets of X *is given by sets of the following three types:*

(i) (α,β) *with* $\alpha < \beta \leq 0$,

(ii) $(\alpha,0)\cup([0,\beta)\times\{i\})$ *with* $\alpha < 0$, $\beta > 0$ *for* $i = 1$ *and* 2,

(iii) $(\alpha,\beta)\times\{i\}$ *with* $0 < \alpha < \beta$ *for* $i = 1$ *and* 2.

For any point $x \in X$ there is an open subset U containing x such that $\bar{U}$ is compact and Hausdorff. About the point $(0,1)$ take $U = (-\epsilon, 0)\cup([0,\epsilon)\times\{1\})$ for any $\epsilon > 0$. Then $\bar{U} = [-\epsilon, 0)\cup([0,\epsilon]\times\{1\})$. Thus X is locally compact-Hausdorff.

Clearly X is not compact, but it is locally compact. For any neighbourhood $V \subset X$, $\bar{V}$ is a compact subset of X. Since there are no disjoint open sets containing the points $(0,1)$ and $(0,2)$, it follows that X is not Hausdorff. Thereby confirming our earlier assertion of a false converse.

3.2 k-space topology

The older theory of *k-spaces* is mentioned briefly before studying *compactly generated spaces* in greater detail.

In this section there is deviation from denoting the topology of a space with a subscript. The original theory of k-spaces is based on a definition which is reminiscent of final topologies. Consider a standard definition [23].

Definition 4 *X is a k-space provided the following condition is satisfied: $A \subset X$ is closed if and only if $A \cap K$ is closed in K for each compact subset $K \subset X$.*

This definition is quite comparable to a characterization of a final topology on X.

Characterization 3.1 *Let X be a space, let $\{K_\alpha\}_{\alpha \in \Lambda}$ be the family of compact subspaces of X and let $\{i_\alpha : K_\alpha \hookrightarrow X\}$ be the family of inclusions of these subspaces into X. The following conditions each determine precisely the same topology kX on the set X:*

(i) *kX carries the final topology relative to the inclusions $i_\alpha : K_\alpha \hookrightarrow X$, i.e. if $U \subset X$, then U is open in kX if and only if $i_\alpha^{-1}(U)$ is open in K_α for all α,*

(ii) *If $C \subset X$, then C is closed in kX if and only if $i_\alpha^{-1}(C)$ is closed in K_α for all α,*

(iii) **Universal Property:** *Given any space Y and any function $f : kX \to Y$, then f is continuous if and only if $f \circ i_\alpha$ is continuous for all α,*

(iv) *kX has the finest topology such that i_α is continuous for all α.*

Proof: Similar to 2.10. ■

The unique topology satisfying conditions (i)-(iv) of this last theorem is called the k-*space topology.* The universal property for k-spaces is illustrated in the next section. A space X endowed with the above k-space topology is denoted by kX. In practice, X is a k-space if $X \cong kX$. Moreover, kX is always a k-space. This last fact is proven next.

Proposition 3.2 *In general,* $kX \cong kkX$.

Proof: There is the following diagram which commutes:

$$\begin{tikzcd} K_\alpha \arrow[r, "1_{K_\alpha}"'] \arrow[d, hook, "i_\alpha"'] & K_\alpha \subset kX \arrow[d, hook, "i_\alpha"'] \\ kX \arrow[r, "1"'] & kkX \end{tikzcd}$$

The result follows from Characterization 3.1(iii) and the fact that identity 1_{K_α} is a homeomorphism for all α. ■

This section concludes with a nice result similar to a theorem of Kelley [16]. A contrapositive method is employed to prove sufficiency for Characterization 3.1(ii).

Proposition 3.3 *If X is locally compact-Hausdorff then X is a k-space.*

Proof: Let C be closed in X where X is a locally compact-Hausdorff space. Then $i_\alpha^{-1}(C) = C \cap K_\alpha$ is closed in each K_α by the continuity of each i_α.

Next we assume that $C \subset X$ is not closed and will prove that $C \cap K_\alpha$ is not closed in X for some compact $K_\alpha \subset X$.

Assume that $C \subset X$ is not closed. Then there exists $x \notin C$ which is a limit point of C. By definition 3 there exists a compact Hausdorff $\bar{U} \subset X$ such that $x \in U \subset \bar{U}$ where U is open. Since $x \notin C \cap \bar{U}$ is a limit point of this set, it follows that $C \cap \bar{U}$ is not closed. Now $\bar{U} \in \mathcal{CH}$. Hence, C is not closed implies that $i_\alpha^{-1}(C) = C \cap K_\alpha$ is not closed in K_α. Consequently Characterization 3.1(ii) is satisfied and X is a k-space. ∎

3.3 Compactly generated topology

By using a convention similar to that employed in the k-space section, a suitable characterization of compactly generated spaces is realized. Instead of coinducing a final topology relative to inclusions of compact subspaces, a final topology is coinduced relative to all incoming maps of compact Hausdorff spaces.

Henceforth, let $\mathcal{CH}$ be the category of all compact Hausdorff spaces. Now coinduce a final topology on any space X by retopologizing X relative to the family of functions $\{k_\alpha : K \to X\}_{\alpha \in \Lambda}$ for all $K \in \mathcal{CH}$ and all maps $k_\alpha \in Map(K, X)$ for an arbitrary index Λ.

Characterization 3.4 *Let X be a space, let $\mathcal{CH}$ be the category of all compact Hausdorff spaces and let $k_\alpha \in Map(K, X)$. The following conditions each determine precisely the same topology $\mathcal{K}X$ on the set X:*

(i) $\mathcal{K}X$ carries the final topology relative to mappings $k_\alpha : K \to X$, i.e. if $U \subset X$, then U is open in $\mathcal{K}X$ if and only if $k_\alpha^{-1}(U)$ is open in K for all $K \in \mathcal{CH}$ and all α,

(ii) If $C \subset X$, then C is closed in $\mathcal{K}X$ if and only if $k_\alpha^{-1}(C)$ is closed in K for all $K \in \mathcal{CH}$ and all α,

(iii) **Universal Property:** *Given any space Y and any function $f : \mathcal{K}X \to Y$, then f is continuous if and only if $f \circ k_\alpha$ is continuous for all α,*

(iv) $\mathcal{K}X$ has the finest topology such that k_α is continuous on each $K \in \mathcal{CH}$ for all α.

Proof: Similar to 2.10. ■

The unique topology satisfying conditions (i)-(iv) described above is called the *compactly generated topology* on X or *cg-ification* of X, and is denoted by $\mathcal{K}X$. A space X will be said to be a *compactly generated space* or *cg-space* if $X \cong \mathcal{K}X$. The universal property commutative diagram in this instance closely resembles that of the past examples.

$$\begin{tikzcd} & Z \\ K \arrow[ur, "f \circ k_\alpha"] \arrow[r, "k_\alpha"'] & \mathcal{K}X \arrow[u, "f"'] \end{tikzcd}$$

The universal property for k-spaces is illustrated by the diagram above when $K \subset X$ is compact and $k_\alpha = i_\alpha$ for all α.

Proposition 3.5 *In general, $\mathcal{K}X \cong \mathcal{K}\mathcal{K}X$.*

Proof: The following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "1_K"'] \arrow[d, "k_\alpha"'] & K \subset \mathcal{K}X \arrow[d, "k_\alpha"'] \\ \mathcal{K}X \arrow[r, "1"'] & \mathcal{K}\mathcal{K}X \end{tikzcd}$$

The result follows from 3.4(iii) and the fact that identity 1_K is a homeomorphism for all $K \in \mathcal{CH}$. ∎

Proposition 3.6 *The identity function* $1 : \mathcal{K}X \to kX$ *is continuous.*

Proof: Let X be an arbitrary space. For any $K \in \mathcal{CH}$ and $k_\alpha \in Map(K, X)$, $k_\alpha(K)$ is a compact subspace in X. Then the diagram below commutes:

$$\begin{tikzcd} K \arrow[r, "k_\alpha"'] \arrow[d, "k_\alpha"'] & k_\alpha(K) \arrow[d, hook, "i_\alpha"'] \\ \mathcal{K}X \arrow[r, "1"'] & kX \end{tikzcd}$$

It follows from 3.1(i) that $i_\alpha : k_\alpha(K) \hookrightarrow kX$ is continuous for each $K \in \mathcal{CH}$ and each $k_\alpha \in Map(K, X)$. Thus each composite $i_\alpha \circ k_\alpha$ is continuous for all α; whence $1 \circ k_\alpha : K \to kX$ is continuous for all k_α. Hence 3.4(iii) ensures that $1 : \mathcal{K}X \to kX$ is continuous. ∎

Proposition 3.7 *Let X be Hausdorff. Then the identity function* $1 : kX \to \mathcal{K}X$ *is continuous.*

Proof: Assume that X is Hausdorff. Let A be a compact subspace of X, then the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, hook, "i_\alpha"'] & A \arrow[d, "k_\alpha"] \\ kX \arrow[r, "1"'] & \mathcal{K}X \end{tikzcd}$$

Now A is a compact Hausdorff space, by 3.4(i) k_α is continuous. Thus each composite $k_\alpha \circ 1_A$ is continuous. Hence $1 \circ i_\alpha : A \to \mathcal{K}X$ is continuous. It follows by 3.1(iii) that $1 : kX \to \mathcal{K}X$ is continuous. ■

The two previous propositions justify the following result.

Proposition 3.8 *Let X be Hausdorff then X is a k-space if and only if X is a compactly generated space.*

Proof: The result follows directly from 3.6 and 3.7. ■

Proposition 3.9 *In general, the identity function: $1_X : \mathcal{K}X \to X$ is continuous.*

Proof: Let X be an arbitrary space. Consider the identify function 1_X : $\mathcal{K}X \to X$. Then the following diagram commutes for all $K \in \mathcal{CH}$ and all $k_\alpha \in Map(K, X)$.

$$\begin{tikzcd} & X \\ K_\alpha \arrow[ur, "k_\alpha"] \arrow[r, "k_\alpha"'] & \mathcal{K}X \arrow[u, "1_X"'] \end{tikzcd}$$

With respect to this diagram, $1_X \circ k_\alpha = k_\alpha$ where each k_α is continuous by 3.4(i). Thus each composite $1_X \circ k_\alpha : K \to X$ is continuous for all $K \in \mathcal{CH}$ and all $k_\alpha \in Map(K, X)$. It follows by 3.4(iii) that $1_X : \mathcal{K}X \to X$ is continuous. ■

Proposition 3.10 *Let X be an arbitrary space and $K \in \mathcal{CH}$. Then a function $k : K \to \mathcal{K}X$ is continuous if and only if $k(= 1_X \circ k) : K \to X$ is continuous.*

Proof: Assume the hypothesis.

Let $k : K \to \mathcal{K}X$ be continuous. It follows from 3.9 that $1_X : \mathcal{K}X \to X$ is continuous. Hence $1_X \circ k = k : K \to X$ is continuous.

Let $k : K \to X$ be continuous. Then from 3.4(i) it follows $k : K \to \mathcal{K}X$ is continuous. ■

Let us consider one more result for this section, a special case of the Composition Rule for Final Topologies.

Proposition 3.11 *Let X have the final topology relative to $\{g_j : X_j \to X\}_{j \in J}$, where J is an arbitrary set. If each X_j is a cg-space, then X is a cg-space.*

Proof: Now 3.9 gives that $1_X : \mathcal{K}X \to X$ is continuous. To verify the proposition it is necessary to prove that $1_X : X \to \mathcal{K}X$ is continuous.

Each X_j has a final topology relative to $\{k_\alpha : K \to X_j\}_{\alpha\in\Lambda}$ where $K \in \mathcal{CH}$, $k_\alpha \in Map(K, X_j)$, and Λ is an arbitrary index set. Now $g_j \circ k_\alpha \in Map(K, X)$ for all $j \in J$ and all $\alpha \in \Lambda_j$, i.e. each composite is a coinducing map for the cg-topology on $\mathcal{K}X$. Hence the following diagram commutes.

$$\begin{tikzcd}
 & & \mathcal{K}X \\
K \arrow[rru, "g_j \circ k_\alpha"] \arrow[r, "k_\alpha"'] & X_j \arrow[r, "g_j"'] & X \arrow[u, "1_X"']
\end{tikzcd}$$

It follows from the diagram above that $1_X \circ g_j \circ k_\alpha$ is continuous for all $j \in J$ and all $\alpha \in \Lambda_j$. Then $1_X \circ g_j$ is continuous by 3.4(iii). It follows that 1_X is continuous by 2.5. Hence the result. ∎

3.4 χ-product topology

We shall define a specific final topology on the Cartesian product of two spaces and explore some of the more redeeming qualities of such a topology. The origin of this product is in [8]. However, the examination contained herein is in the more categorical flavour of [5].

Let X and Y be two arbitrary topological spaces. On the set $X \times Y$ induce a final topology by taking as coinducing maps the inclusions:

$$i_x : \{x\} \times Y \hookrightarrow X \times Y \qquad \forall x \in X,$$

and the maps:

$$1_X \times k : X \times K \longrightarrow X \times Y \qquad \forall K \in \mathcal{CH}, \forall k \in Map(K, Y).$$

We will denote the inclusion i_x simply by i when no confusion can occur. Before proceeding with some important results and properties concerning this final product topology, consider the usual characterizing theorem.

Characterization 3.12 *Let X and Y be arbitrary spaces and let i and $1_X \times k$ be as defined above. The following conditions each determine precisely the same topology on the set $X \times Y$:*

(i) *$X \times Y$ carries the final topology relative to $i : \{x\} \times Y \hookrightarrow X \times Y$ and $1_X \times k : X \times K \to X \times Y$, i.e. if $\mathcal{U} \subset X \times Y$, then $\mathcal{U}$ is open in $X \times Y$ if and only if $i^{-1}(\mathcal{U})$ is open in $\{x\} \times Y$ for all $x \in X$ and $(1_X \times k)^{-1}(\mathcal{U})$ is open in $X \times K$ for all $K \in \mathcal{CH}$ and all $k \in Map(K, Y)$,*

(ii) *If $\mathcal{C} \subset X \times Y$, then $\mathcal{C}$ is closed in $X \times Y$ if and only if $i^{-1}(\mathcal{C})$ is closed in $\{x\} \times Y$ for all $x \in X$ and $(1_X \times k)^{-1}(\mathcal{C})$ is closed in $X \times K$ for all $K \in \mathcal{CH}$ and all $k \in Map(K, Y)$,*

(iii) **Universal Property:** *Given any space Z and any function $f : X \times Y \to Z$, then f is continuous if and only if $f \circ i : \{x\} \times X \to Z$ is continuous for all $x \in X$ and $f \circ (1_X \times k) : X \times K \to Z$ is continuous for all $K \in \mathcal{CH}$ and for all $k \in Map(K, Y)$,*

(iv) *X has the finest topology such that each $i : \{x\} \times Y \hookrightarrow X \times Y$ and each $1_X \times k : X \times K \to X \times Y$ is continuous for all x and all k.*

Proof: Similar to 2.10. ■

The unique topology of this last theorem satisfying conditions (i)-(iv) will be named the *χ-product topology* for reasons which will be made clear in the following chapter. Henceforth denote the set $X \times Y$ with the χ-product topology by $X \times_\chi Y$. There is the usual commutative diagram illustrating the universal property.

$$\begin{tikzcd}
 & Z & \\
\{x\} \times Y \arrow[ur, "f \circ i"] \arrow[r, hook, "i"'] & X \times_\chi Y \arrow[u, "f"] & X \times K \arrow[l, "1_X \times k"] \arrow[ul, "f \circ (1_X \times k)"']
\end{tikzcd}$$

The reader should be made aware that the diagram above commutes for all $x \in X$ and $K \in Map(K, X)$. First, the χ-product is proven to be natural.

Proposition 3.13 *Let A, B, X and Y be arbitrary spaces and let $f : A \to X$ and $g : B \to Y$ be mappings. Then the function $f \times_\chi g : A \times_\chi B \to X \times_\chi Y$ is continuous. It is defined by*

$$(f \times_\chi g)(a, b) = (f(a), g(b)) \quad \forall a \in A,\ \forall b \in B.$$

Proof: $A \times_\chi B$ has the final topology relative to the mappings

$$i : \{a\} \times B \hookrightarrow A \times_\chi B \quad \forall a \in A,$$
$$1_A \times k : A \times K \longrightarrow A \times_\chi B \quad \forall K \in \mathcal{CH}, \forall k \in Map(K, B).$$

$X \times_\chi Y$ has the final topology relative to the mappings

$$j : \{x\} \times Y \hookrightarrow X \times_\chi Y \quad \forall x \in X,$$
$$1_X \times h : X \times K \longrightarrow X \times_\chi Y \quad \forall K \in \mathcal{CH}, \forall h \in Map(K, Y).$$

Then the following diagrams commute for all $a \in A$, all $K \in \mathcal{CH}$, and all $k \in Map(K, B)$.

$$\begin{tikzcd} \{a\}\times B \arrow[r, "f\times g"'] \arrow[d, hook, "i"'] & \{f(a)\}\times Y \arrow[d, hook, "j"'] \\ A\times_\chi B \arrow[r, "f\times_\chi g"'] & X\times_\chi Y \end{tikzcd} \qquad \begin{tikzcd} A\times K \arrow[r, "f\times 1_K"'] \arrow[d, "1_A\times k"'] & X\times K \arrow[d, "1_X\times(g\circ k)"'] \\ A\times_\chi B \arrow[r, "f\times_\chi g"'] & X\times_\chi Y \end{tikzcd}$$

So

$$(f \times_\chi g) \circ i = j \circ (f \times g), \text{ and}$$
$$(f \times_\chi g) \circ (1_A \times k) = [1_X \times (g \circ k)] \circ (f \times 1_K).$$

Now $f\times g : \{a\}\times B \to \{f(a)\}\times Y$ is continuous and the inclusion j is continuous by the 3.12(i). It follows that each composite $j \circ (f\times g) : \{a\}\times B \to X \times_\chi Y$ is continuous for all $a \in A$. In the second diagram each k is continuous by 3.12(i). Then the composite $[1_X \times (g \circ k)] \circ (f \times 1_K) : A\times K \to X \times_\chi Y$ is continuous for all $K \in \mathcal{CH}$ and all $k \in Map(K, B)$. Hence all mappings $(f \times_\chi g) \circ i$ and $(f \times_\chi g) \circ (1_A \times k)$ are continuous. It follows by 3.12(iii) that the function $f \times_\chi g : A \times_\chi B \to X \times_\chi Y$ is continuous. ■

Commutativity and associativity of the χ-product are examined next.

Proposition 3.14 *The natural projections $\pi_X : X \times_\chi Y \to X$ and $\pi_Y : X \times_\chi Y \to Y$ are continuous.*

Proof: Let $X\times Y$ have the χ-product topology. Then the following diagrams are commutative for all $x \in X$, all $K \in \mathcal{CH}$ and all $k \in Map(K, Y)$.

$$\begin{tikzcd}
 & X & \\
\{x\}\times Y \arrow[ur, "\pi_X"] \arrow[r, hook, "i"'] & X\times_\chi Y \arrow[u, "\pi_X"] & X\times K \arrow[ul, "\pi_X"'] \arrow[l, "1_X\times k"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y & \\
\{x\}\times Y \arrow[ur, "\pi_Y"] \arrow[r, hook, "i"'] & X\times_\chi Y \arrow[u, "\pi_Y"] & X\times K \arrow[ul, "k\circ\pi_K"'] \arrow[l, "1_X\times k"]
\end{tikzcd}$$

The four functions $\pi_X : \{x\}\times Y \to X$, $\pi_X : X\times K \to X$, $\pi_Y : \{x\}\times Y \to Y$, and $k\circ\pi_K : X\times K \to Y$ are continuous. The continuity of π_X and π_Y follows by 3.12(iii). ■

Lemma 3.15 *Let X be a non-empty space. The natural projection $\pi_Y : X\times_\chi Y \to Y$ is an identification.*

Proof: It can be shown that π_Y satisfies 2.11(iii). Let $f : Y \to Z$ be a function into an arbitrary space Z. Suppose f is continuous, then $f\circ\pi_Y : X\times_\chi Y \to Z$ is continuous. Now suppose that $f\circ\pi_Y$ is continuous. Then f is the composite $Y \cong \{x\}\times Y \hookrightarrow X\times_\chi Y \to Z$ which is continuous for each $x\in X$. It follows that π_Y is an identification. ■

Proposition 3.16 *If X is an arbitrary space then $X\times_\chi \{*\} \cong X$.*

Proof: Proposition 3.14 provides for the continuity of $\pi_X : X \times_\chi \{*\} \to X$ when $Y = \{*\}$. Clearly π_X is bijective in this situation. It is necessary to prove that π_X is an open mapping in order to show it is a homeomorphism. Let $\mathcal{V} = V \times \{*\} \subset X \times_\chi \{*\}$ be a non-empty open set for appropriate non-empty $V \subset X$. Then it follows from 3.12(i) with $K = \{*\}$ and $k = 1_{\{*\}}$ that $(1 \times k)^{-1}(\mathcal{V}) = V \times \{*\}$ is open in $X \times \{*\}$. Thus V is open in X by 1.10(i). Consequently π_X is an open mapping. Hence it is a homeomorphism. ∎

It will be shown shortly that the χ-product is not necessarily commutative. However, it is possible to prove a two sided identity.

Proposition 3.17 *If Y is an arbitrary space then $\{*\} \times_\chi Y \cong Y$.*

Proof: Let Y be an arbitrary space, from 3.14 with $X = \{*\}$ it follows that π_Y is continuous. The bijective nature of π_Y can be verified easily. Let $\mathcal{U} = \{*\} \times U \subset \{*\} \times_\chi Y$ be open for an appropriate $U \subset Y$. It follows that $\{*\} \times U$ is open in $\{*\} \times Y$ by 3.12(i) and that U is open in Y from 1.10(i). Therefore π_Y is an open mapping, thus it is a homeomorphism. Hence the result. ∎

Under certain conditions the χ-product coincides with the **S**-*product topology* of Brown including cases where the latter is known to be non-commutative.

Definition 5 *Let X and Y be Hausdorff spaces. The* **S**-*product topology is the final topology coinduced on the set $X \times Y$ by all mappings of the form*

$$i : \{x\} \times Y \hookrightarrow X \times Y \quad \forall x \in X,$$
$$1_X \times i_B : X \times B \hookrightarrow X \times Y \quad \forall B \subset Y \text{ with } B \text{ compact.}$$

For specific details concerning this product the reader is referred to [8].

Proposition 3.18 *The identity function* $1 : X \times_{\chi} Y \to X \times_{\mathbf{S}} Y$ *is continuous.*

Proof: The following diagrams commute for all $x \in X$, $K \in \mathcal{CH}$, and $k \in Map(K, k(K))$.

$$
\begin{tikzcd}
\{x\}\times Y \arrow[r, "\cong"'] \arrow[d, hook, "i"'] & \{x\}\times Y \arrow[d, hook, "i"'] \\
X \times_{\chi} Y \arrow[r, "1"'] & X \times_{\mathbf{S}} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times K \arrow[r, "1_X \times k"'] \arrow[d, "1_X \times k"'] & X \times k(K) \arrow[d, "1_X \times i_{k(K)}"'] \\
X \times_{\chi} Y \arrow[r, "1"'] & X \times_{\mathbf{S}} Y
\end{tikzcd}
$$

The remaining details of the proof are similar to that of 3.6 and follow from the commutative diagrams above and from 3.12(iii). ■

Proposition 3.19 *If* Y *is Hausdorff then the identity function* $1 : X \times_{\mathbf{S}} Y \to X \times_{\chi} Y$ *is continuous.*

Proof: The following diagrams commute for all $x \in X$ and for all compact $B \subseteq Y$.

$$
\begin{tikzcd}
\{x\}\times Y \arrow[r, "\cong"'] \arrow[d, hook, "i"'] & \{x\}\times Y \arrow[d, hook, "i"'] \\
X \times_{\mathbf{S}} Y \arrow[r, "1"'] & X \times_{\chi} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times B \arrow[r, "\cong"'] \arrow[d, "1_X \times i_B"'] & X \times B \arrow[d, "1_X \times i_B"'] \\
X \times_{\mathbf{S}} Y \arrow[r, "1"'] & X \times_{\chi} Y
\end{tikzcd}
$$

The remaining details of the proof are similar to that of 3.7 and follow from the commutative diagrams above and 2.5 for the **S**-product. ■

The following proposition can be justified quite easily.

Proposition 3.20 *Let Y be Hausdorff, then $X \times_\chi Y \cong X \times_{\mathbf{S}} Y$.*

Proof: The proof follows from 3.19 and 3.18. ■

Now suppose that X is a compact Hausdorff space, and J is an uncountable discrete space. Then corollary 6.5 of [8] shows that the natural map of $X \times_{\mathbf{S}} \mathbb{R}^J$ and $\mathbb{R}^J \times_{\mathbf{S}} X$ is not a homeomorphism. Hence it follows that in general the χ-product topology is not commutative.

There are still more useful propositions to be proved.

Proposition 3.21 *The identity function $1 : X \times_\chi Y \to X \times Y$ is continuous .*

Proof: The result follows by way of 3.12(iii) since $1 \circ i = i$ and $1 \circ (1_X \times k) = 1_X \times k$. Alternatively, the result follows immediately from 3.14 and 1.10(ii). ■

A nice result pertaining to the preservation of Hausdorffness under the χ-product topology can be proven.

Proposition 3.22 *If X and Y are Hausdorff spaces then $X \times_\chi Y$ is a Hausdorff space.*

Proof: Let $(x,y),(x',y') \in X \times_{\chi} Y$ be two arbitrary points. Since X and Y are Hausdorff there exists open subsets $x \in U \subset X$ and $x' \in U' \subset X$ such that $U \cap U' = \emptyset$; and $y \in V \subset Y$ and $y' \in V' \subset Y$ such that $V \cap V' = \emptyset$. Then $U \times V$ and $U' \times V'$ are two subsets of $X \times_{\chi} Y$ containing (x,y) and (x',y') respectively, such that $(U \times V) \cap (U' \times V') = \emptyset$. Proposition 3.21 ensures that these two sets are open in $X \times_{\chi} Y$. Hence $X \times_{\chi} Y$ is Hausdorff. ■

Proposition 3.23 *In general, the identity function $1_{X \times Y} : \mathcal{K}(X \times Y) \to X \times_{\chi} Y$ is continuous.*

Proof: All mappings $k : K \to \mathcal{K}(X \times Y)$ where $K \in \mathcal{CH}$ are of the form $k = (f,g)$, where $f : K \to X$ and $g : K \to Y$ are continuous by 3.10. Then there is the following commutative diagram.

$$\begin{tikzcd} K \arrow[r, "(f,1_K)"'] \arrow[d, "k=(f,g)"'] & X \times K \arrow[d, "1_X \times g"'] \\ \mathcal{K}(X \times Y) \arrow[r, "1"'] & X \times_{\chi} Y \end{tikzcd}$$

From the diagram above $1 \circ k = (1_X \times g) \circ (f, 1_K)$. It follows from 3.12(i) that $(1_X \times g)$ is continuous for all K. Both f and 1_K are continuous therefore $(f, 1_K) : K \to X \times K$ is continuous for all $K \in \mathcal{CH}$. Hence the composite $(1_X \times g) \circ (f, 1_K)$ is continuous. It follows $1 \circ k : K \to X \times_{\chi} Y$ is continuous, thus $1 : \mathcal{K}(X \times Y) \to X \times_{\chi} Y$ is continuous by 3.4(iii). ■

Proposition 3.24 *If $X \times Y$ is a cg-space, then $X \times Y \cong X \times_{\chi} Y$.*

Proof: Let X and Y be arbitrary spaces. It follows from 3.21 and 3.23 that there is a short chain of continuous identities.

$$\mathcal{K}(X \times Y) \xrightarrow{1} X \times_\chi Y \xrightarrow{1} X \times Y.$$

If $X \times Y$ is a cg-space, i.e. $\mathcal{K}(X \times Y) \cong X \times Y$. Thus $1 : X \times Y \to \mathcal{K}(X \times Y)$ is continuous. Hence $X \times Y \cong X \times_\chi Y$. ■

Thus the χ-product topology coincides with the standard product topology when spaces are compactly generated. However, it is possible to get the above result with weaker assumptions concerning $X \times Y$.

Proposition 3.25 *If Y is locally compact-Hausdorff, then $X \times Y \cong X \times_\chi Y$.*

Proof: Let Y be locally compact-Hausdorff and X be an arbitrary space. We show the identity function $1 : X \times Y \to X \times_\chi Y$ is continuous. Select a point $y \in Y$. By definition 3 there exists an open subset $V_y \subset Y$ about y such that $\bar{V}_y$ is a compact and Hausdorff subset of Y. Then the composite $j = (1 \times k) \circ i : X \times V_y \hookrightarrow X \times \bar{V}_y \to X \times_\chi Y$ is continuous, since the inclusion i is continuous. As well, $1 \times k$ is continuous by 3.12(i).

Let $\mathcal{U} \subset X \times_\chi Y$ be open, then $j^{-1}(\mathcal{U}) = \mathcal{U} \cap (X \times V_y) \subset X \times V_y$ is open. All subsets of the form $X \times V_y$ for each $y \in Y$ are an open covering of $X \times Y$. Moreover

$$\mathcal{U} = \bigcup_{y \in Y} \mathcal{U} \cap (X \times V_y),$$

where each $\mathcal{U} \cap (X \times V_y)$ is open in $X \times Y$. Since $\mathcal{U}$ is an arbitrary union of open sets in $X \times Y$, it is open in $X \times Y$. So any open subset in $X \times_\chi Y$ is open in $X \times Y$, thus $1 : X \times Y \to X \times_\chi Y$ is continuous.

Clearly it is bijective and its continuity in the reverse direction follows from 3.21. Consequently the identity function is a homeomorphism, that is, $X \times Y \cong X \times_\chi Y$. ■

By considering the previously mentioned chain it is possible to derive a result similar to 3.10.

Proposition 3.26 *Let $K \in \mathcal{CH}$ and $f : K \to \mathcal{K}(X \times Y)$ be a function, then the following conditions are equivalent:*

(i) $f : K \to X \times_\chi Y$ is continuous,

(ii) $f : K \to X \times Y$ is continuous,

(iii) $f : K \to \mathcal{K}(X \times Y)$ is continuous.

Proof: Assume (iii), then (i) follows by 3.23. Proposition 3.21 ensures (i) $\Rightarrow$ (ii) and 3.10 provides (ii) $\Rightarrow$ (iii). ■

The standard exponential law is employed to establish a useful lemma concerning the commutativity of the final topology and a specific product topology. We use this lemma to prove the associativity of the χ-product. Assume for the time being that any function space has the *compact-open topology* (see section 4.1 for further details concerning the exponential law and proper definitions). Let X and Y be Hausdorff, Z be arbitrary, and additionally suppose that Y is compact. Then there is a homeomorphism $Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$ given by the rule $f \rightsquigarrow f'$ where $f(x, y) = f'(x)(y)$ for all $x \in X$ and all $y \in Y$.

Lemma 3.27 *Let J be an indexing set and let X have the final topology relative to a family of functions $\{g_j : X_j \to X\}_{j \in J}$. Also, let K be a compact Hausdorff space and let $1_K : K \to K$ be the identity function on K. Then the product topology on $X \times K$ coincides with a final topology relative to the family of functions $\{g_j \times 1_K : X_j \times K \to X \times K\}_{j \in J}$.*

Proof: Let Z be an arbitrary space and $\phi : X \times K \to Z$ be a function. It will be proven that ϕ is continuous if and only if each composite $\phi \circ (g_j \times 1_K) : X_j \times K \to Z$ is continuous for each $j \in J$. Under the exponential law, if ϕ is continuous there is a corresponding map $\phi' : X \to Map(K, Z)$ such that

$$\phi(x, y) = \phi'(x)(y) \quad \forall x \in X,\ \forall y \in Y.$$

For a continuous map $\theta'_j : X_j \to Map(K, Z)$ there is a corresponding map $\theta_j : X_j \times K \to Z$ such that for each $j \in J$

$$\theta_j(x_j, y) = \theta'_j(x_j)(y) \quad \forall x_j \in X_j,\ \forall y \in Y.$$

Assume that ϕ is continuous. Recall that the hypothesis gives that 1_K and each g_j is continuous for all j. It follows that each composite $\phi \circ (g_j \times 1_K) : X_j \times K \to Z$ is continuous for $j \in J$. Set $\theta_j = \phi \circ (g_j \times 1_K)$. Next, let θ_j be continuous for each $j \in J$. Under the exponential law, there is a continuous map $\theta'_j : X_j \to Map(K, Z)$ defined appropriately. Now

$$\theta'_j(x_j)(k) = \theta(x_j, k) = \phi \circ (g_j \times 1_K)(x_j, k) = \phi(g_j(x_j), k) = \phi'(g_j(x_j))(k),$$

for all $x_j \in X_j$ and all $k \in K$. It follows by 2.5 that $\phi' : X \to Map(K, Z)$ is continuous.

Apply the exponential law to ϕ' and we see that $\phi : X \times K \to Z$ is continuous. Hence $X \times K$ has a final topology relative to the family $\{g_j \times 1_K : X_j \times K \to X \times K\}_{j \in J}$ by 2.6. ■

Proposition 3.28 *The χ-product topology is associative. Given arbitrary spaces X, Y, and Z then $(X \times_\chi Y) \times_\chi Z \cong X \times_\chi (Y \times_\chi Z)$.*

Proof: Lemma 3.27 and 2.7 permit a description of the final topologies on $(X \times_\chi Y) \times_\chi Z$ and $X \times_\chi (Y \times_\chi Z)$ in the following manner.

$(X \times_\chi Y) \times_\chi Z$ has the final topology relative to the mappings

$$\begin{aligned} i_x \times i_y \times 1_Z : \{x\} \times \{y\} \times Z &\hookrightarrow (X \times_\chi Y) \times_\chi Z, \\ i_x \times 1_Y \times k : \{x\} \times Y \times K &\longrightarrow (X \times_\chi Y) \times_\chi Z, \\ 1_X \times k' \times k : X \times K' \times K &\longrightarrow (X \times_\chi Y) \times_\chi Z, \end{aligned}$$

for all $x \in X$, all $y \in Y$, all $K, K' \in \mathcal{CH}$, all $k \in Map(K, Z)$ and all $k' \in Map(K', Y)$.

$X \times_\chi (Y \times_\chi Z)$ has the final topology relative to the mappings

$$\begin{aligned} i_x \times i_y \times 1_Z : \{x\} \times \{y\} \times Z &\hookrightarrow X \times_\chi (Y \times_\chi Z), \\ i_x \times 1_Y \times k : \{x\} \times Y \times K &\longrightarrow X \times_\chi (Y \times_\chi Z), \\ 1_X \times \bar{k} : X \times \bar{K} &\longrightarrow X \times_\chi (Y \times_\chi Z), \end{aligned}$$

for all $x \in X$, all $y \in Y$, all $K, \bar{K} \in \mathcal{CH}$, all $k \in Map(K, Z)$ and all $\bar{k} \in Map(\bar{K}, Y \times_\chi Z)$.

Let $1 : (X \times_\chi Y) \times_\chi Z \to X \times_\chi (Y \times_\chi Z)$ be the identity function and $1'$ be the identity function in the reverse direction. These functions will be shown to be continuous.

It is necessary to define some mappings for all $\bar{K} \in \mathcal{CH}$ and all continuous maps $\bar{k} : \bar{K} \to Y \times_X Z$. Define $\bar{k}' : \bar{K} \to Y$ by

$$\bar{k}'(w) = (\pi_Y \circ \bar{k})(w) \quad \forall w \in \bar{K}.$$

Define $\bar{k}'' : \bar{K} \to Z$ by

$$\bar{k}''(w) = (\pi_Z \circ \bar{k})(w) \quad \forall w \in \bar{K}.$$

Since $\bar{k}, \pi_y$, and π_z are everywhere continuous, it follows that all mappings $\bar{k}'$ and $\bar{k}''$ will be continuous as well. Lastly, let $\Delta : \bar{k} \to \bar{k} \times \bar{k}$ represent the continuous diagonal mapping. Then the following six diagrams commute:

$$\begin{tikzcd}
\{x\}\times\{y\}\times Z \arrow[r, "1_X\times 1_Y\times 1_Z"'] \arrow[d, hook, "i_X\times i_Y\times 1_Z"'] & \{x\}\times\{y\}\times Z \arrow[d, hook, "i_X\times i_Y\times 1_Z"'] \\
(X\times_X Y)\times_X Z \arrow[r, "1"'] & X\times_X (Y\times_X Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\{x\}\times Y\times K \arrow[r, "1_X\times 1_Y\times 1_K"'] \arrow[d, "i_X\times 1_Y\times k"'] & \{x\}\times Y\times K \arrow[d, "i_X\times 1_Y\times k"'] \\
(X\times_X Y)\times_X Z \arrow[r, "1"'] & X\times_X (Y\times_X Z)
\end{tikzcd}$$

$$\begin{tikzcd}
X\times K'\times K \arrow[r, "1_X\times 1_{K'}\times 1_K"'] \arrow[d, "1_X\times k'\times k"'] & X\times K'\times K \arrow[d, "1_X\times k'\times k"'] \\
(X\times_X Y)\times_X Z \arrow[r, "1"'] & X\times_X (Y\times_X Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\{x\}\times\{y\}\times Z \arrow[r, "1_X\times 1_Y\times 1_Z"'] \arrow[d, hook, "i_X\times i_Y\times 1_Z"'] & \{x\}\times\{y\}\times Z \arrow[d, hook, "i_X\times i_Y\times 1_Z"'] \\
X\times_X (Y\times_X Z) \arrow[r, "1'"'] & (X\times_X Y)\times_X Z
\end{tikzcd}$$

$$\begin{tikzcd}
\{x\}\times Y\times K \arrow[r, "1_X\times 1_Y\times 1_K"'] \arrow[d, "i_X\times 1_Y\times k"'] & \{x\}\times Y\times K \arrow[d, "i_X\times 1_Y\times k"'] \\
X\times_X (Y\times_X Z) \arrow[r, "1'"'] & (X\times_X Y)\times_X Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X\times\bar{K} \arrow[r, "1_X\times\Delta"'] \arrow[d, "1_X\times\bar{k}"'] & X\times\bar{K}\times\bar{K} \arrow[d, "1_X\times\bar{k}'\times\bar{k}''"'] \\
X\times_X (Y\times_X Z) \arrow[r, "1'"'] & (X\times_X Y)\times_X Z
\end{tikzcd}$$

Thus

$$\begin{aligned}
1 \circ (i_x \times i_y \times 1_Z) &= (i_x \times i_y \times 1_Z) \circ (1_X \times 1_Y \times 1_K), \\
1 \circ (i_x \times 1_Y \times k) &= (1_X \times 1_Y \times 1_K) \circ (i_x \times 1_Y \times k), \\
1 \circ (1_X \times k' \times k) &= (1_X \times k' \times k) \circ (1_X \times 1_{K'} \times 1_K), \\
1' \circ (i_x \times i_y \times 1_Z) &= (i_x \times i_y \times 1_Z) \circ (1_X \times 1_Y \times 1_K), \\
1' \circ (i_x \times 1_Y \times k) &= (1_X \times 1_Y \times 1_K) \circ (i_x \times 1_Y \times k), \\
1' \circ (1_X \times \bar{k}) &= (1_X \times \bar{k}' \times \bar{k}'') \circ (1_X \times \Delta),
\end{aligned}$$

for all $x \in X$, all $y \in Y$, all $K, K', \bar{K} \in \mathcal{CH}$, all $k \in Map(K, Z)$, all $k' \in Map(K', Y)$, all $\bar{k} \in Map(\bar{K}, Y \times_\chi Z)$, all $\bar{k}' \in Map(\bar{K}, Y)$ and all $\bar{k}'' \in Map(\bar{K}, Z)$.

For each of the first three equalities the functions on the left-hand side are continuous. It follows by 2.5 that $1 : (X \times_\chi Y) \times_\chi Z \to X \times_\chi (Y \times_\chi Z)$ is continuous. Similarly the final three left-hand side functions are continuous. Again, by 2.5, $1' : X \times_\chi (Y \times_\chi Z) \to (X \times_\chi Y) \times_\chi Z$ is continuous.

Clearly 1 is bijective. Hence it is a homeomorphism, i.e. $(X \times_\chi Y) \times_\chi Z \cong X \times_\chi (Y \times_\chi Z)$ for arbitrary spaces X, Y and Z. ∎

Chapter 4

DEALING IN INCONVENIENCE

The goal is a modified exponential law which utilizes less restrictions on the spaces involved. A law which can be safely applied in the usual category of topological spaces. The proof of 3.27 utilized an exponential law for mapping and product spaces. To apply the law it was necessary to utilize the compact-open topology and assume certain conditions on the spaces involved, i.e. compact and Hausdorff, or compactly generated.

We commence with a clear account of an exponential law as an introduction. Key notions are highlighted in this section. They reappear in the proof of a modified exponential law.

4.1 The exponential law: an introduction

Studies of function spaces in algebraic topology usually involve the *compact-open topology*, introduced by Fox [13].

Definition 6 *Let X and Y be arbitrary spaces. The* **compact-open topology** *on $Map(X,Y)$ has subbasis of the form*

$$M(K,U) = \{f : X \to Y \text{ such that } f(K) \subset U\},$$

where K ranges over the compact subsets of X and U ranges over the open subsets of Y.

Henceforth, the set $Map(X,Y)$ with the compact-open topology is denoted by $Map_{CO}(X,Y)$, with the exception of this section.

There are two conditions associated with the exponential correspondence between $Map(X \times Y, Z)$ and $Map(X, Map(Y,Z))$.

Definition 7 *The* **Admissible Condition** *states that if X, Y and Z are arbitrary spaces and there is a map $f' : X \to Map(Y,Z)$, then the rule $f'(x)(y) = f(x,y)$ determines a map $f : X \times Y \to Z$.*

Definition 8 *The* **Proper Condition** *states that if X, Y and Z are arbitrary spaces and there is a map $f : X \times Y \to Z$, then the rule $f(x,y) = f'(x)(y)$ determines a map $f' : X \to Map(Y,Z)$.*

The proof of both the admissible and proper conditions comes down to the proof of the continuity of the following functions.

Definition 9 *The* **evaluation function** $e : Map(X,Y) \times X \to Y$ *is defined by*

$$e(f,x) = f(x) \quad \forall x \in X, \ \forall f \in Map(X,Y).$$

Definition 10 *The* **coevaluation function** $e' : X \to Map(Y, X \times Y)$ *is defined by*

$$e'(x)(y) = (x,y) \quad \forall x \in X, \ \forall y \in Y.$$

Continuity of the evaluation function is necessary to prove the admissible condition. While continuity of the coevaluation function is key to validating the proper condition. Once these conditions are satisfied, and some mild assumptions concerning X and Y are made, we have the following well known results [19].

Theorem 4.1 : Exponential Correspondence

Let X, Y and Z be spaces, and additionally suppose that either (i) Y is locally compact Hausdorff or (ii) $X \times Y$ is compactly generated. Then a function $f : X \times Y \to Z$ is continuous if and only if the corresponding function $f' : X \to Map(Y,Z)$ is continuous. The functions are related by the rule $f(x,y) = f'(x)(y)$. The elements of $Map(X \times Y, Z)$ and $Map(X, Map(Y,Z))$ are in bijective correspondence via the rule $f \rightsquigarrow f'$.

If more assumptions concerning the spaces involved are made, then a much stronger result can be proven.

Theorem 4.2 : Exponential Law

Let X, Y, and Z be spaces that satisfy one of the conditions of the previous result. If X and Y are Hausdorff, then the rule $f \rightsquigarrow f'$ is a homeomorphism, i.e. $Map(X, Map(Y,Z)) \cong Map(X \times Y, Z)$.

4.2 A modified exponential law

Several results need deriving before a conditionless version of 4.1 and modified exponential law can be proven. An essential step is modification of the compact-open topology. It is necessary to consider a particular case of a set-open topology for $Map(X,Y)$ [5]. The focus is a specific $\mathcal{A}$-open topology where $\mathcal{A} = \mathcal{CH}$.

Definition 11 *Let X and Y be arbitrary topological spaces. The* **compact Hausdorff image-open topology** *on $Map(X,Y)$ has subbasis of the form*

$$M(g(K),U) = \{f : X \to Y \text{ such that } f \circ g(K) \subset U\},$$

where K ranges over $\mathcal{CH}$, g ranges over $Map(K,X)$, and U ranges over the open subsets of Y.

Henceforth we denote the set $Map(X,Y)$ with the compact Hausdorff image-open topology, or the χ-open topology, by $Map_\chi(X,Y)$.

Lemma 4.3 *Let X and Y be arbitrary spaces. If X is Hausdorff, then the χ-open topology and compact-open topology on $Map(X,Y)$ coincide.*

Proof: Let X be a Hausdorff space, $K \in \mathcal{CH}$, and $g \in Map(K,X)$. Then the compact Hausdorff images $g(K)$ are precisely the compact subsets of X. Hence in that case $Map_\chi(X,Y) \cong Map_{\mathcal{CO}}(X,Y)$. ■

4.2.1 Modified admissible condition

The continuity of the evaluation function is key to a positive admissible condition result.

Proposition 4.4 *The evaluation function* $e : Map_X(X,Y) \times_X X \to Y$ *is continuous.*

Proof: First, set $1 = 1_{Map_X(X,Y)}$. The reader will recall two facts. The function $e : Map_X(X,Y) \times_X X \to Y$ is defined by $e(f,x) = f(x)$ for all $f \in Map_X(X,Y)$ and all $x \in X$. The space $Map_X(X,Y) \times_X Y$ has the final topology relative to the mappings

$$\begin{array}{lr} i : \{f\} \times X \hookrightarrow Map_X(X,Y) \times_X X & \forall f \in Map_X(X,Y), \\ 1 \times k : Map_X(X,Y) \times K \to Map_X(X,Y) \times_X X & \forall K \in \mathcal{CH}, \forall k \in Map(K,X). \end{array}$$

It is necessary to show that the composites of e with these coinducing maps are continuous.

For each $f \in Map_X(X,Y)$ we notice that $e \circ i : \{f\} \times X \to Y$ is the map given by $(f,x) \rightsquigarrow f(x)$ where $x \in X$. This is the composite of the canonical homeomorphism $\{f\} \times X \to X$ and $f : X \to Y$. It follows that $e \circ i$ is continuous.

For each $K \in \mathcal{CH}$ define an evaluation function $e_k : Map_X(X,Y) \times K \to Y$ by

$$e_k = e \circ (1 \times k) \quad \forall k \in Map(K,X).$$

Let U be an open subset of Y and $(f,z) \in e_k^{-1}(U)$ where $z \in K$. So $f \circ k(z) \in U$ implies that $z \in (f \circ k)^{-1}(U)$. Since $K \in \mathcal{CH}$, i.e. K is compact Hausdorff, it follows that K is regular by the corollary to proposition 1 §9.2 [6]. Thus there exists an open subset $V \subset K$ such that $z \in V \subset \bar{V} \subset (f \circ k)^{-1}(U)$. $\bar{V}$ is compact and Hausdorff so $k|\bar{V}$ is a mapping of a compact and Hausdorff space into X, i.e. $k|\bar{V} \in Map(\bar{V},X)$ for $\bar{V} \in \mathcal{CH}$. Then $W = M(k(\bar{V}),U)$ is a subbasic open set of $Map_X(X,Y)$.

It follows that $W \times V$ is an open subset of $Map_{\chi}(X,Y) \times K$ such that $(f,z) \in W \times V \subset e_k^{-1}(U)$ for all $k \in Map(K,X)$. Hence $e_k^{-1}(U)$ is open; consequently e_k is continuous.

Now, $e \circ i : \{f\} \times X \to Y$ is continuous for all $f \in Map_{\chi}(X,Y)$ and $e \circ (1 \times k) : Map_{\chi}(X,Y) \times K \to Y$ is continuous for all $k \in Map(k,X)$. It follows by 3.12(iii) that $e : Map_{\chi}(X,Y) \times_{\chi} X \to Y$ is continuous. ■

The proof of the admissible condition in this setting is an easy consequence of the last section.

Proposition 4.5 : Modified Admissible Condition

Let X, Y, and Z be arbitrary spaces. If $f' : X \to Map_{\chi}(Y,Z)$ is continuous, then the function $f : X \times_{\chi} Y \to Z$ defined by

$$f(x,y) = f'(x)(y) \quad \forall x \in X,\ \forall y \in Y,$$

is continuous.

Proof: Let 1_Y be identity function on Y. It follows by 3.13 that the function $f' \times_{\chi} 1_Y : X \times_{\chi} Y \to Map_{\chi}(X,Y) \times_{\chi} Y$ is continuous. Then the composite $f = e \circ (f' \times_{\chi} 1_Y) : X \times_{\chi} Y \to Z$ is continuous by 4.4. Whence

$$f(x,y) = e \circ (f' \times_{\chi} 1_Y)(x,y) = e(f'(x),y) = f'(x)(y) \quad \forall x \in X,\ \forall y \in Y,$$

as required. ■

4.2.2 Modified proper condition

The proof of a modified Proper condition requires a familiar type of mapping space result.

Proposition 4.6 *Let X, Y, and Z be arbitrary spaces and $f : Y \to Z$ be a map. Then the induced function $f_* : Map_X(X,Y) \to Map_X(X,Z)$ given by*

$$f_*(g) = f \circ g \quad \forall g \in Map_X(X,Y),$$

is continuous.

Proof: Let $W = M(g(K),U)$ be a subbasic open set for $Map_X(X,Z)$ and $f_*(h) \in W$ for some $h \in Map_X(X,Y)$. Then $W' = M(g(K), f^{-1}(U))$ is an open subset in $Map_X(X,Y)$ such that $h \in W' \subset f_*^{-1}(W)$. Hence f_* is continuous. ■

A toned down result by way of theorem 5.12 [16] plays a role in establishing the continuity of the coevaluation in our setting.

Lemma 4.7 *Let X and Z be arbitrary spaces and let K be a compact space. Also, let $f : X \times K \to Z$ be a map with $x \in X$, and $f(\{x\} \times K) \subset U$, where U is an open subset of Z. Then there exists an open subset $V \subset X$ such that $x \in V$ and $f(V \times K) \subset U$.*

Proof: It follows by theorem 5.12 [16] that there exists an open subset $V \subset X$ such that $x \in V$ and $V \times K \subset f^{-1}(U)$. Hence $f(V \times K) \subset U$. ■

Proposition 4.8 *The coevaluation function $e' : X \to Map_X(Y, X \times_X Y)$ is continuous.*

Proof: The function $e'(x) : Y \to X \times_X Y$ is the composite of the canonical homeomorphism $Y \to \{x\} \times Y$ and the coinducing map $\{x\} \times Y \to X \times_X Y$.

Fix an $x \in X$. Now $e'(x) = i : \{x\} \times Y \to X \times_X Y$. Hence it is continuous for fixed $x \in X$. Thus for an arbitrary $x \in X$, $e' : X \to Map_X(Y, X \times_X Y)$ is well defined.

Let $W = M(g(K), \mathcal{U})$ be a subbasic open set of $Map_X(Y, X \times_X Y)$ where $\mathcal{U}$ is open in $X \times_X Y$, $g \in Map(K, Y)$ and $K \in \mathcal{CH}$. Select an arbitrary $x \in (e')^{-1}(W)$ then $e'(x) \in W$

$$\begin{aligned}
&\Rightarrow \; e'(x)(g(K)) \subseteq \mathcal{U} \\
&\Rightarrow \; \{x\} \times g(K) \subseteq \mathcal{U} \\
&\Rightarrow \; (1_X \times g)(\{x\} \times K) \subseteq \mathcal{U} \\
&\Rightarrow \; \{x\} \times K \subseteq (1_X \times g)^{-1}(\mathcal{U}).
\end{aligned}$$

The function $1_X \times g : X \times K \to X \times_X Y$ is a coinducing map for $X \times_X Y$ therefore it is continuous by 3.12(i). Hence $(1_X \times g)^{-1}(\mathcal{U})$ is open in $X \times K$. By 4.7 there exists an open subset $V \subset X$ such that $x \in V$ and $V \times K \subseteq (1_X \times g)^{-1}(\mathcal{U})$. Then $(1_X \times g)(V, K) \subseteq \mathcal{U}$

$$\begin{aligned}
&\Rightarrow \; V \times g(K) \subseteq \mathcal{U} \\
&\Rightarrow \; e'(V)(g(K)) \subseteq \mathcal{U} \\
&\Rightarrow \; e'(V) \subset W = M(g(K), \mathcal{U}).
\end{aligned}$$

Hence $e' : X \to Map_X(Y, X \times_X Y)$ is continuous. ■

Proposition 4.9 : Modified Proper Condition

Let X, Y, and Z be arbitrary spaces. If $f : X \times_X Y \to Z$ is continuous, then the function $f' : X \to Map_X(Y, Z)$ defined by

$$f'(x)(y) = f(x, y) \quad \forall x \in X, \; \forall y \in Y,$$

is well defined and continuous.

Proof: Since f is continuous, it follows by 4.6 that $f_* : Map_\chi(X,Y) \to Map_\chi(X,Z)$ is continuous. Continuity of $e' : X \to Map_\chi(Y, X \times_\chi Y)$ is ensured by 4.8. Thus the composite $f_* \circ e' : X \to Map_\chi(Y,Z)$ is continuous. Now for each $x \in X$ and $y \in Y$

$$\begin{aligned}(f_* \circ e')(x)(y) &= f_*(e'(x))(y) = (f \circ e'(x))(y)\\ &= f(e'(x)(y)) = f(x,y) = f'(x)(y).\end{aligned}$$

Hence $f' = f_* \circ e'$, therefore it is well defined and continuous. ■

4.2.3 χ-exponential law

Theorem 4.10 : χ-Exponential Correspondence

Let X, Y and Z be spaces. Then there is a bijective correspondence between maps $f : X \times_\chi Y \to Z$ and maps $f' : X \to Map_\chi(Y,Z)$ determined by the rule

$$f'(x)(y) = f(x,y) \quad \forall x \in X\ \forall y \in Y.$$

Proof: The result follows from 4.5 and 4.9. ■

There is one more result to consider before proving a modified exponential law. It can then be shown that the standard exponential law follows from the modified result.

Proposition 4.11 *Let X, Y and Z be arbitrary spaces. Then the composition function $(\circ) : Map_\chi(Y,Z) \times_\chi Map_\chi(X,Y) \to Map_\chi(X,Z)$ defined by*

$$(\circ)(g,f) = g \circ f \quad \forall f \in Map_\chi(X,Y),\ \forall g \in Map_\chi(Y,Z),$$

is continuous.

Proof: It follows from 3.28, 4.4, and 3.13 that the following composite is continuous.

$$(Map_\chi(Y,Z) \times_\chi Map_\chi(X,Y)) \times_\chi X \xrightarrow{1} Map_\chi(Y,Z) \times_\chi (Map_\chi(X,Y) \times_\chi X)$$
$$\xrightarrow{1_X \times_\chi e} Map_\chi(Y,Z) \times_\chi Y \xrightarrow{e} Z.$$

Then 4.9 ensures the continuity of

$$(\circ) : Map_\chi(Y,Z) \times_\chi Map_\chi(X,Y) \to Map_\chi(X,Z).$$

For all $x \in X$, all $f \in Map_\chi(X,Y)$, and all $g \in Map_\chi(Y,Z)$ it follows that

$$(\circ)(g,f)(x) = e \circ (1 \times_\chi e)(g,f,x) = e(g,f(x)) = g(f(x)) = (g \circ f)(x),$$

as required. ■

Theorem 4.12 : χ-Exponential Law

Let X, Y and Z be spaces. Then there is a homeomorphism

$$\phi : Map_\chi(X, Map_\chi(Y,Z)) \to Map_\chi(X \times_\chi Y, Z)$$

defined by

$$\phi(f') = f \textit{ where } f'(x)(y) = f(x,y), \quad \forall\, x \in X,\ \forall y \in Y.$$

Proof: First we prove that ϕ is continuous. It follows from 3.28, 3.13 and 4.4 that the following composite given by $(f',(x,y)) \rightsquigarrow f(x,y)$ is continuous.

$$Map_\chi(X, Map_\chi(Y,Z)) \times_\chi (X \times_\chi Y) \xrightarrow{1} (Map_\chi(X, Map_\chi(Y,Z)) \times_\chi X) \times_\chi Y$$
$$\xrightarrow{e \times_\chi 1_Y} Map_\chi(Y,Z) \times_\chi Y \xrightarrow{e} Z.$$

Then 4.9 ensures the continuity of $\phi : Map_\chi(X, Map_\chi(Y, Z)) \to Map_\chi(X \times_\chi Y, Z)$.

Next consider the function $\varphi : Map_\chi(X \times_\chi Y, Z) \to Map_\chi(X, Map_\chi(Y, Z))$ defined by

$$\varphi(f) = f' \text{ where } f(x, y) = f'(x)(y), \quad \forall x \in X,\ \forall y \in Y.$$

The following composite given by $(f, x) \rightsquigarrow (y \rightsquigarrow f(x, y))$ is continuous by 4.8, 3.13 and 4.11.

$$\begin{aligned} Map_\chi(X \times_\chi Y, Z) \times_\chi X &\xrightarrow{1\times_\chi e'} Map_\chi(X \times_\chi Y, Z) \times_\chi Map_\chi(Y, X \times_\chi Y) \\ &\xrightarrow{(\circ)} Map_\chi(Y, Z). \end{aligned}$$

Then 4.9 ensures the continuity of φ.

Now, $\phi \circ \varphi = 1$ and $\varphi \circ \phi = 1'$. Hence, ϕ is a homeomorphism, i.e. $Map_\chi(X, Map_\chi(Y, Z)) \overset{\phi}{\cong} Map_\chi(X \times_\chi Y, Z)$. ■

Corollary 4.12.1 *If X and Y are Hausdorff, then $Map_{\mathcal{CO}}(X \times_\chi Y, Z) \cong Map_{\mathcal{CO}}(X, Map_{\mathcal{CO}}(Y, Z))$.*

Proof: The result follows by 4.12, 3.22 and 4.3. ■

The following result shows that under the normal assumptions on the spaces involved the usual exponential law can be proven using a modified exponential law result.

Corollary 4.12.2 *If, in addition, either (i) Y is locally compact or (ii) $X \times Y$ is a cg-space, then $Map_{\mathcal{CO}}(X \times Y, Z) \cong Map_{\mathcal{CO}}(X, Map_{\mathcal{CO}}(Y, Z))$.*

Proof: Let Y be locally compact, then Y is locally compact and Hausdorff, it follows that Y is locally compact-Hausdorff. The result follows from 4.12.1 and 3.25.

Now suppose that $X \times Y$ is a cg-space. The result follows from 4.12.1 and 3.24. ■

Chapter 5

INITIAL AND FINAL COMMUTATIVITY

A generalized version of 3.27 is key to many of the results of this chapter

5.1 Product and final commutativity

Theorem 5.1 and theorem 5.2 are two nice general results that are to be used as legs for the final chapter.

Theorem 5.1 *Let J be an arbitrary set and X carry the final topology relative to the family of functions $\{g_j : X_j \to X\}_{j \in J}$. Further, suppose that for each $x \in X$ there exists an $x_j \in X_j$ such that $g_j(x_j) = x$ for some $j \in J$. Then for an arbitrary space W, $X \times_{\chi} W$ carries the final topology relative to the family of functions $\{g_j \times_{\chi} 1_W : X_j \times_{\chi} W \to X \times_{\chi} W\}_{j \in J}$.*

Proof: Let Z be an arbitrary space and $f : X \times_X W \to Z$ be a function.

Suppose that f is continuous and it follows from 3.13 that each function $g_j \times_X 1_W$ is continuous. Thus $f \circ (g_j \times_X 1_W) = \theta_j : X_j \times_X W \to Z$ is continuous for each $j \in J$.

Conversely drop the assumption that f is continuous, but suppose each θ_j is continuous. Theorem 4.12 ensures that for all j there is a continuous mapping $\theta'_j : X_j \to Map_X(W, Z)$ defined by

$$\theta'_j(x_j)(w) = f \circ (g_j \times_X 1_W)(x_j, w) = f(g_j(x_j), w) \quad \forall x_j \in X_j,\ \forall w \in W.$$

Define a function $f' : X \to Fn(W, Z)$ by

$$f'(x)(w) = f(x, w) \quad \forall x \in X,\ \forall w \in W.$$

The pseudo-surjectivity assumption implies that for each $x \in X$

$$f'(x)(w) = f'(g_j(x_j))(w) = f(g_j(x_j), w) = \theta_j(x_j, w) = \theta'_j(x_j)(w).$$

Whence it follows that $f'(X) \subset Map_X(W, Z)$. As well, for each $j \in J$

$$\theta'_j(x_j)(w) = \theta_j(x_j, w) = f(g_j(x_j), w) = f'(g_j(x_j))(w) = (f' \circ g_j)(x_j)(w).$$

Then the following triangle is commutative for all j.

$$\begin{tikzcd}
 & Map_X(W,Z) \\
X_j \arrow[ur, "\theta'_j"] \arrow[r, "g_j"'] & X \arrow[u, "f'"']
\end{tikzcd}$$

It follows from 2.5 and the continuity of θ_j' that f' is continuous. Thus f is continuous.

Hence $X \times_X W$ carries the final topology coinduced by $\{g_j \times_X 1_W : X_j \times_X W \to X \times_X W\}_{j \in J}$. ■

Corollary 5.1.1 *In addition to the hypothesis of 5.1, suppose that either (i) W is a locally compact-Hausdorff space or (ii) $X \times W$ is a cg-space. Then for an arbitrary space W, $X \times W$ carries the final topology relative to the family of functions $\{g_j \times 1_W : X_j \times W \to X \times W\}_{j \in J}$.*

Proof: The result using (i) follows from 5.1 and 3.25.

Assume condition (ii). It follows from 3.21 $X \times_X W \cong X \times W$ and from 3.24 it follows that $1 : X_j \times_X W \to X_j \times W$ is continuous for each $j \in J$. The diagram below commutes.

$$\begin{tikzcd} X_j \times_X W \arrow[r, "1"'] \arrow[d, "g_j \times_X 1_W"'] & X_j \times W \arrow[d, "g_j \times 1_W"'] \\ X \times_X W \arrow[r, "\cong"'] & X \times W \end{tikzcd}$$

Let $f : X \times W \to Z$ be a function for an arbitrary space Z.

Suppose f is continuous, then $f \circ (g_j \times 1_W) : X_j \times W \to Z$ is continuous.

Now suppose that $f \circ (g_j \times 1_W)$ is continuous. It follows from the equation

$$[(f \circ (g_j \times 1_W)] \circ 1 = f \circ [(g_j \times 1_W) \circ 1] = f \circ (g \times_X 1_W),$$

and 2.5 that f is continuous. The result follows from 2.6. ■

5.2 Sum and final commutativity

Theorem 5.2 *Let J be an arbitrary set and X carry the final topology relative to the family of functions $\{g_j : X_j \to X\}_{j \in J}$. Then for an arbitrary space W, $X \sqcup W$ carries the final topology relative to the family of functions $\{g_j \sqcup 1_W : X_j \sqcup W \to X \sqcup W\}_{j \in J}$.*

Proof: To prove this result we must show (i) that each composite $g_j \sqcup 1_W$ is continuous, and (ii) satisfy the universal property for final topologies.

(i) There is the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, hook, "i_X"'] & X \sqcup W & W \arrow[l, hook', "i_W"] \\
X_j \arrow[u, "g_j"] \arrow[r, hook, "i_{X_j}"'] & X_j \sqcup W \arrow[u, "g_j \sqcup 1_W"'] & W \arrow[l, hook', "i_W"] \arrow[u, "1_W"']
\end{tikzcd}$$

It follows for all $j \in J$ that

$$i_X \circ g_j = (g_j \sqcup 1_W) \circ i_{X_j},$$
$$i_W \circ 1_W = (g_j \sqcup 1_W) \circ i_W.$$

The functions i_X, i_W, 1_W, and g_j for each $j \in J$ are continuous. From 2.16(iii) it follows that each $g_j \sqcup 1_W$ is continuous for all j.

(ii) Now, let $f : X \sqcup W \to Z$ be a function for an arbitrary space Z. Suppose that f is continuous, it follows that each composite $f \circ (g_j \sqcup 1_W) : X_j \sqcup W \to Z$ is continuous for all $j \in J$.

Next suppose each $f \circ (g_j \sqcup 1_W)$ is continuous. The composites $(f \circ (g_j \sqcup 1_W)) \circ i_{X_j} : X_j \to Z$ and $(f \circ (g_j \sqcup 1_W)) \circ i_W : W \to Z$ are continuous by 2.16(iii). The continuity of $(f \circ i_X) \circ g_j$ for each $j \in J$ follows from

$$\begin{aligned}(f \circ (g_j \sqcup 1_W)) \circ i_{X_j} &= f \circ ((g_j \sqcup 1_W) \circ i_{X_j}) \\ &= f \circ (i_X \circ g_j) \\ &= (f \circ i_X) \circ g_j.\end{aligned}$$

Then 2.5 ensures $f \circ i_X$ is continuous. The continuity of $f \circ i_W$ for each $j \in J$ follows from

$$\begin{aligned}(f \circ (g_j \sqcup 1_W)) \circ i_W &= f \circ ((g_j \sqcup 1_W) \circ i_W) \\ &= f \circ (i_W \circ 1_W) \\ &= f \circ i_W.\end{aligned}$$

Now 2.16(iii) ensures the continuity of $f : X \sqcup W \to Z$. The result follows by 2.6. ■

Some significant specific results follow from these general results.

5.3 Sum and identification commutativity

Theorem 5.3 *If $p : X \to Y$ is an identification and Z an arbitrary space, then $p \sqcup 1_Z : X \sqcup Z \to Y \sqcup Z$ is an identification.*

Proof: Clearly, $p \sqcup 1_W : X \sqcup W \to Y \sqcup W$ is surjective. Then this is a special case of 5.2 with $|J| = 1$ where $g_1 = p : X \to Y$ is an identification. ■

Corollary 5.3.1 *If $p : X \to Y$ and $q : W \to Z$ are identifications, then $p \sqcup q : X \sqcup W \to Y \sqcup Z$ is an identification.*

Proof: Theorem 5.3 ensures that there are identifications $p \sqcup 1_W : X \sqcup W \to Y \sqcup W$ and $q \sqcup 1_Y : W \sqcup Y \to Z \sqcup Y$. Moreover, $1_Y \sqcup q : Y \sqcup W \to Y \sqcup Z$ is an identification by the commutativity of disjoint topological sums. Then $p \sqcup q = (p \sqcup 1_W) \circ (1_Y \sqcup q)$ is a composite of identifications and the result follows by 2.12. ■

5.4 Subspace and product commutativity

Theorem 5.4 *Let X and Y be arbitrary spaces and A be a closed subspace of X. Then the χ-product topology on $A \times Y$, denoted by $A \times_\chi Y$, coincides with the subspace topology on $A \times Y$ relative to the inclusion $j : A \times Y \hookrightarrow X \times_\chi Y$.*

Proof: It follows by 3.21 that $1 : A \times_\chi Y \to A \times Y$ is continuous. It will be shown that $1 : A \times Y \to A \times_\chi Y$ is continuous.

Select $K \in \mathcal{CH}$ and $k \in Map(K, Y)$. Now A is a closed subspace in X. It follows from $i_a^{-1}(A \times Y) = \{a\} \times Y$, where $a \in A$, and $(1 \times k)^{-1}(A \times Y) = A \times K$ that $A \times Y$ is closed in $X \times_\chi Y$ by 3.12(ii).

Let W be closed in $A \times_\chi Y$, $\bar{i}_a$ be the restriction of i to $i_a^{-1}(A \times_\chi Y)$ and $\overline{1 \times k}$ be the restriction of $1 \times k$ to $(1 \times k)^{-1}(A \times_\chi Y)$. Then $i_a^{-1}(W) = \bar{i}_a^{-1}(W) = \{a\} \times Y$ is closed in $\{x\} \times Y$ and $(1 \times k)^{-1}(W) = (\overline{1 \times k})^{-1}(W)$ is closed in $X \times K$. It follows from 3.12(ii) that W is closed in $X \times_\chi Y$, hence it is closed in $A \times Y$ as a subspace of $X \times_\chi Y$. Hence $1 : A \times Y \to A \times_\chi Y$ is continuous and the result follows. ■

5.5 Product and identification commutativity

Theorem 5.5 *If $p : X \to Y$ is an identification and W is an arbitrary space, then $p \times_{\chi} 1_W : X \times_{\chi} W \to Y \times_{\chi} W$ is an identification.*

Proof: This is a special case of 5.1. Where $|J| = 1$ and $g_1 = p$. ■

Corollary 5.5.1 *In addition to the hypothesis of 5.5, suppose that either (i) W is a locally compact-Hausdorff space or (ii) $Y \times W$ is a cg-space. Then $p \times 1_W : X \times W \to Y \times W$ is an identification.*

Proof: The proof of this result is similar to 5.1.1 and follows from 5.5. ■

It is a standard result that 5.5.1 holds when W is a locally compact and Hausdorff space. The result given here is an improvement on this known result.

Corollary 5.5.2 *Let W, X, Y and Z be spaces, and $p : X \to Y$ and $q : W \to Z$ be identifications. If either (i) W and Y are locally compact-Hausdorff, or (ii) X and Z are locally compact-Hausdorff, or (iii) $Y \times W$ and $Y \times Z$ are cg-spaces, or (iv) $X \times Z$ and $Y \times Z$ are cg-spaces. Then $p \times q : X \times W \to Y \times Z$ is an identification.*

Proof: The result is proven for (i) only. The proof using condition (iii) uses a similar argument and follows via 5.5.1(ii).

For (i), it follows by 5.5.1(i) that $p \times 1_W : X \times W \to Y \times W$ and $q \times 1_Y : W \times Y \to Z \times Y$ are identifications. Moreover, $1_Y \times q : Y \times W \to Y \times Z$ is an identification by the commutativity of the standard product topology.

The mapping $p \times q : X \times W \to Y \times Z$ can be factored into the above identifications, i.e. $p \times q = (p \times 1_W) \circ (1_Y \times q)$. Hence the result follows from 2.12.

For (ii) and (iv), it is necessary to factor $p \times q : X \times W \to Y \times Z$ as the composite of the identifications $1_X \times q$ and $p \times 1_Z$ instead, otherwise the arguments are similar. ∎

5.6 Adjunction and product commutativity

Theorem 5.6 *Let W and X be arbitrary spaces and $A \subset X$ be closed. Let $f : A \to Y$ be continuous and let $X \sqcup_f Y$ have the adjunction space topology coinduced by composites $\pi_f \circ i_Y$ and $\pi_f \circ i_X$. Then $(X \sqcup_f Y) \times_{\chi} W$ has the adjunction space topology coinduced by composites $(\pi_f \circ i_Y) \times_{\chi} 1_W$ and $(\pi_f \circ i_X) \times_{\chi} 1_W$.*

Proof: This is a special of 5.1 with $|J| = 2$, $X_1 = X$, $X_2 = Y$, $g_1 = \pi_f \circ i_X$, and $g_2 = \pi_f \circ i_Y$. ∎

Corollary 5.6.1 *In addition to the hypothesis of 5.6, suppose that either (i) W is a locally compact-Hausdorff space or (ii) $(X \sqcup_f Y) \times W$ is a cg-space. Then $(X \sqcup_f Y) \times W$ is the adjunction space coinduced by the maps $(\pi_f \circ i_Y) \times 1_W$ and $(\pi_f \circ i_X) \times 1_W$.*

Proof: The proof of this result is similar to 5.1.1, but follows from 5.6. ∎

If we use different coinducing maps and the same coinducing spaces, then we obtain a different adjunction space homeomorphic to the one realized in the previous theorem.

Define the relation $\sim$ on the set $(X \times_X W) \sqcup (Y \times_X W)$ by $(x, w) \sim (y, w)$ if and only if $x \in A$ and $f(x) = y$. Now $\sim$ generates the equivalence relation $\mathcal{R}'$. Define

$$(X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W) := (X \times_X W) \sqcup (Y \times_X W)/\mathcal{R}'.$$

In this situation

$$\begin{aligned}
\pi_f &:= \pi_{f \times_X 1_W} : (X \times_X W) \sqcup (Y \times_X W) \to (X \times_X W) \sqcup (Y \times_X W)/\mathcal{R}' \\
i_X &:= i_{X \times_X W} : X \times_X W \hookrightarrow (X \times_X W) \sqcup (Y \times_X W) \\
i_Y &:= i_{Y \times_X W} : Y \times_X W \hookrightarrow (X \times_X W) \sqcup (Y \times_X W)
\end{aligned}$$

Then the following diagram commutes.

$$\begin{tikzcd}
A \times_X W \arrow[r, "f \times_X 1_W"'] \arrow[d, hook, "i \times_X 1_W"'] & Y \times_X W \arrow[d, "\pi_{f \times_X 1_W} \circ i_{Y \times_X W}"'] \\
X \times_X W \arrow[r, "\pi_{f \times_X 1_W} \circ i_{X \times_X W}"'] & (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)
\end{tikzcd}$$

The elements of $(X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$ can be described explicitly. For each class $[(\alpha, w)]$ exclusively

$$\begin{aligned}
[(\alpha, w)] &= \{(x, w) \text{ such that } x \in X \backslash A,\ w \in W\}, \text{ or} \\
[(\alpha, w)] &= \{(f(a), w) \text{ such that } a \in A \subset X,\ w \in W\}, \text{ or} \\
[(\alpha, w)] &= \{(y, w) \text{ such that } y \in Y \backslash f(A),\ w \in W\}.
\end{aligned}$$

A similar description of the elements of $(X \sqcup_f Y) \times_X W$ is

$$\begin{aligned}
([\alpha], w) &= \{x \text{ such that } x \in X \backslash A, w \in W\}, \text{ or} \\
([\alpha], w) &= \{f(a) \text{ such that } a \in A \subset X, w \in W\}, \text{ or} \\
([\alpha], w) &= \{y \text{ such that } y \in Y \backslash f(A), w \in W\}.
\end{aligned}$$

It is now possible to generate a result which shows that two adjunction spaces we have discussed are homeomorphic.

Theorem 5.7 *The function* $\beta : (X \sqcup_f Y) \times_X W \to (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$ *defined by the rule* $([\alpha], w) \rightsquigarrow [(\alpha, w)]$ *is a homeomorphism. That is,* $(X \sqcup_f Y) \times_X W \cong (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$.

Proof: Let $(X \sqcup_f Y) \times_X W$ have the adjunction space topology coinduced by $(\pi_f \circ i_X) \times_X 1_W : X \times_X W \to (X \sqcup_f Y) \times_X W$ and $(\pi_f \circ i_Y) \times_X 1_W : Y \times_X W \to (X \sqcup_f Y) \times_X W$. Then the function β is such that for each $x \in X$, $y \in Y$ and $w \in W$

$$\beta \circ ((\pi_f \circ i_X) \times_X 1_W)(x, w) = \beta([\alpha], w) = [(\alpha, w)] = \pi_{f \times_X 1_W} \circ i_{X \times_X W}(x, w),$$
$$\beta \circ ((\pi_f \circ i_Y) \times_X 1_W)(y, w) = \beta([\alpha], w) = [(\alpha, w)] = \pi_{f \times_X 1_W} \circ i_{Y \times_X W}(y, w).$$

Where both composites $\pi_{f \times_X 1_W} \circ i_{X \times_X W} : (X \times_X W) \to (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$ and $\pi_{f \times_X 1_W} \circ i_{Y \times_X W} : (Y \times_X W) \to (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$ are continuous. It follows by 2.18(v) that β is continuous.

Consider the function β^{-1} defined by $[(\alpha, w)] \rightsquigarrow ([\alpha], w)$. For each $x \in X$, $y \in Y$ and $w \in W$

$$\begin{aligned} \beta^{-1} \circ (\pi_{f \times_X 1_W} \circ i_{X \times_X W})(x, w) &= \beta^{-1}([(\alpha, w)]) \\ &= ([\alpha], w) = (\pi_f \circ i_X) \times_X 1_W(x, w), \\ \beta^{-1} \circ (\pi_{f \times_X 1_W} \circ i_{Y \times_X W})(y, w) &= \beta^{-1}([(\alpha, w)]) \\ &= ([\alpha], w) = (\pi_f \circ i_Y) \times_X 1_W(y, w). \end{aligned}$$

Thus $\beta^{-1} : (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W) \to (X \sqcup_f Y) \times_X W$ is continuous. Clearly we have $\beta \circ \beta^{-1} = 1$ and $\beta^{-1} \circ \beta = 1$. Hence β is indeed a homeomorphism, i.e. $(X \sqcup_f Y) \times_X W \cong (X \times_X W) \sqcup_{f \times_X 1_W} (Y \times_X W)$. ∎

5.7 Expanding sequence of subspaces and product commutativity

Theorem 5.8 *Let W be an arbitrary space, let X be the union of an expanding sequence of subspaces, i.e. $X = \cup X_j$, and X_j is closed in X_{j+1} for all j. Then $X \times_X W$ is the union of an expanding of sequence of subspaces. We have $X \times_X W = \bigcup(X_j \times_X W)$ with $X_1 \times_X W \subset X_2 \times_X W \subset X_3 \times_X W \subset \ldots$, and $X_j \times_X W$ is closed in $X_{j+1} \times_X W$ for all j.*

Proof: It follows, via 5.4. that this is a special case of 5.1 with $g_j = i_j : X_j \hookrightarrow X$. ■

Corollary 5.8.1 *In addition to the hypothesis of 5.8, suppose that either (i) W is a locally compact-Hausdorff space or (ii) $X \times W$ is a cg-space. Then $X \times W$ is the union of an expanding sequence of subspaces, i.e. $X \times W = \bigcup(X_j \times W)$ where $X_1 \times W \subset X_2 \times W \subset X_3 \times W \subset \ldots$.*

Proof: The proof of this result is similar to 5.1.1, but follows from 5.8. ■

5.8 Identifications, pullbacks and subspaces I

To complete the final sections of this work it is necessary to introduce some notation from [2]. Let X and Y be arbitrary spaces, A be a subspace of Y, and $p : X \to Y$ be a map. Denote the subspace $p^{-1}(A)$ of X by $X|A$, and the restriction of p to $X|A$ by $p|A$.

Proposition 5.9 *If $p : X \to Y$ is an identification and A is a closed (or open) subspace of Y, then $p|A : X|A \to A$ is an identification.*

Proof: Suppose firstly that A is closed. The following square commutes.

$$\begin{tikzcd} X|A \arrow[r, hook] \arrow[d, "p|A"'] & X \arrow[d, "p"] \\ A \arrow[r, hook] & Y \end{tikzcd}$$

Now $p : X \to Y$ is continuous by 2.11(i). Then $p|A$ is the restriction of p to $X|A$, hence it is continuous. Let W be a subset of A, it follows from the continuity of $p|A$ that if W is closed in A then $(p|A)^{-1}(W)$ is closed in $X|A$.

Suppose that $(p|A)^{-1}(W)$ is closed in $X|A$. It follows from $(p|A)^{-1}(W) = p^{-1}(W)$ and 2.11(ii) that W is closed in Y. Hence W is closed in A as a closed subspace of Y and the result follows by 2.11(ii).

The proof of the result employing the open condition of the hypothesis follows in a similar manner. ∎

Recall that the usual topology on $X \sqcap Y$ is the subspace topology relative to the inclusion $i : X \sqcap Y \hookrightarrow X \times Y$.

Definition 12 *Let $X \sqcap_X Y$ denote the set $X \sqcap Y$ with the subspace topology relative to the inclusion $i : X \sqcap Y \hookrightarrow X \times_X Y$.*

Theorem 5.10 *Let X_1, X_2 and W be arbitrary spaces and B be a Hausdorff space. Also, let $p_1 : X_1 \to B$, $p_2 : X_2 \to B$, and $q : W \to B$ be maps;*

$p : X_1 \to X_2$ be an identification; and 1_W be the identity map on W. Then $p \sqcap_X 1_W : X_1 \sqcap_X W \to X_2 \sqcap_X W$ is an identification.

Proof: There is the following commutative diagram.

$$\begin{tikzcd}
X_1 \sqcap_X W \arrow[r, hook] \arrow[d, "p \sqcap_X 1_W"'] & X_1 \times_X W \arrow[d, "p \times_X 1_W"'] \\
X_2 \sqcap_X W \arrow[r, hook] \arrow[d, "p_2 \times_X q"'] & X_2 \times_X W \arrow[d, "p_2 \times_X q"'] \\
\Delta B \arrow[r, hook] & B \times_X B
\end{tikzcd}$$

B is Hausdorff, thus ΔB is closed in $B \times B$. It follows from 3.21 that ΔB is closed in $B \times_X B$. Now $p_2 \times_X q : X_2 \times_X W \to B \times_X B$ is continuous by 3.13, and it follows that $(p_2 \times_X q)^{-1}(\Delta B) = X_2 \sqcap_X W$ is closed in $X_2 \times_X W$. Theorem 5.5 ensures that $p \times_X 1_W : X_1 \times_X W \to X_2 \times_X W$ is an identification. Then the result follows by 5.9. ■

Corollary 5.10.1 *In addition to the hypothesis of 5.10, suppose that either W is a locally compact-Hausdorff space or (ii) $X_2 \times W$ is a cg-space. Then $p \sqcap 1_W : X_1 \sqcap W \to X_2 \sqcap W$ is an identification.*

Proof: The proof of this result is similar to 5.1.1 and follows from 5.10. ■

Corollary 5.10.2 *Let X_1, X_2, Y_1 and Y_2 be arbitrary spaces and B be a Hausdorff space. Also, let $p_1 : X_1 \to B$, $p_2 : X_2 \to B$, $q_1 : Y_1 \to B$ and $q_2 : Y_2 \to B$ be maps, and $p : X_1 \to X_2$ and $q : Y_1 \to Y_2$ identifications. If either (i) X_2 and Y_1 are locally compact-Hausdorff, or (ii) X_1 and Y_2 are locally compact-Hausdorff, or (iii) $X_2 \times Y_1$ and $X_2 \times Y_2$ are cg-spaces, or (iv) both $X_1 \times Y_2$ and $X_2 \times Y_2$ are cg-spaces. Then $p \sqcap q : X_1 \sqcap Y_1 \to X_2 \sqcap Y_2$ is an identification.*

Proof: The result is proven for (i) only. The proof using condition (iii) uses a similar argument and follows via 5.10.1(ii).

For (i), it follows by 5.10.1(i) that $p \sqcap 1_{Y_1} : X_1 \sqcap Y_1 \to X_2 \sqcap Y_1$ and $q \sqcap 1_{X_2} : Y_1 \sqcap X_2 \to Y_2 \sqcap X_2$ are identifications. Moreover, $1_{X_2} \sqcap q : X_2 \sqcap Y_1 \to X_2 \sqcap Y_2$ is an identification by the commutativity of the standard product topology. The mapping $p \sqcap q : X_1 \sqcap Y_1 \to X_2 \sqcap Y_2$ can be factored into the above identifications, i.e. $p \sqcap q = (p \sqcap 1_{Y_1}) \circ (1_{X_2} \sqcap q)$. Hence the result follows from 2.12.

For (ii) and (iv), it is necessary to factor $p \sqcap q : X_1 \sqcap Y_1 \to X_2 \sqcap Y_2$ as the composite of the identifications $1_{X_1} \sqcap q$ and $p \sqcap 1_{Y_2}$. ■

5.9 Identifications, pullbacks and subspaces II

Consider the following special case of the definition 12 where $B \sqcap Y$ is the pullback of maps $q : Y \to B$ and $1_B : B \to B$.

Lemma 5.11 *Let Y be a cg-space. Then $q^*(1_B) : B \sqcap_\chi Y \to Y$ is a homeomorphism that identifies $(1_B)^*(q) : B \sqcap_\chi Y \to B$ with q.*

Proof: It follows from 3.14 that $\pi_Y : B \times_X Y \to Y$ is continuous. Then $q^*(1_B)$ is the restriction of π_Y to $B \sqcap_X Y$, and hence it is continuous.

The following diagram commutes for all $K \in \mathcal{CH}$, $k \in Map(K, Y)$ and $y \in Y$.

$$\begin{tikzcd}
K \arrow[rr, "(q \circ k, 1_K)"'] \arrow[d, "k"'] & & B \times K \arrow[d, "1 \times k"] \\
Y \arrow[r, "(q, 1_Y)"'] & B \sqcap_X Y \arrow[r, hook, "i"'] & B \times_X Y
\end{tikzcd}$$

The composite $(1 \times k) \circ (q \circ k, 1_K) : K \to B \times_X Y$ is continuous. It follows that the composite $i \circ (q, 1_Y) \circ k : K \to B \times_X Y$ is continuous. Characterization 3.4(iii) ensures the continuity of the composite $i \circ (q, 1_Y) : Y \to B \times_X Y$. Hence 1.9(ii) ensures the continuity of $(q, 1_Y) : Y \to B \sqcap_X Y$.

Now $q^*(1_B) : B \sqcap_X Y \to Y$ is continuous and its inverse $(q, 1_Y)$ is continuous. So $q^*(1_B)$ is bijective, and hence is the desired homeomorphism. ∎

Definition 13 *Let $Y!_X Z$ denote the set $\bigcup_{b \in B} Map(Y|b, Z)$ with an initial topology relative to the functions*

$$\begin{array}{llll}
q!Z : Y!Z \longrightarrow B & \textit{defined by} & (q!Z)(h) = b & \textit{where } h : Y|b \to Z, \textit{ and} \\
i : Y!Z \longrightarrow Map_X(Y, Z^\omega) & \textit{defined by} & i(f) = f^\omega & \textit{where } f \in Map(Y|b, Z).
\end{array}$$

It is possible to generate a modified exponential law type result involving $X \sqcap_X Y$ and $Y!_X Z$.

Theorem 5.12 : Fibred χ-Exponential Correspondence

Let X, Y, and Z be arbitrary spaces and B be a Hausdorff space. Let $p : X \to B$ and $q : Y \to B$ be maps.

(i) If $f : X \sqcap_\chi Y \to Z$ is continuous, then the function $f' : X \to Y!_\chi Z$ defined by

$$f'(x)(y) = f(x,y) \quad \forall x \in X,\ \forall y \in Y,$$

is well defined and continuous.

(ii) If $f' : X \to Y!_\chi Z$ is continuous, then the function $f : X \sqcap_\chi Y \to Z$ defined by

$$f(x,y) = f'(x)(y) \quad \forall x \in X,\ \forall y \in Y,$$

is well defined and continuous.

Proof: (i) Let $f : X \sqcap_\chi Y \to Z$ be a map. Then f can be extended to a map $\tilde{f} : X \times_\chi Y \to Z^\omega$ given by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } p(x) = q(y), \\ \omega & \text{otherwise.} \end{cases}$$

It follows by 4.12 that the rule $\tilde{f}(x,y) \rightsquigarrow \tilde{f}'(x)(y)$ defines a map, $\tilde{f}' : X \to Map_\chi(Y, Z^\omega)$. Note that $\tilde{f}'(x)(y) = \omega$ whenever $p(x) \neq q(y)$.

Define a function $f' : X \to Y!_\chi Z$ by

$$f'(x)(y) = \begin{cases} \tilde{f}'(x)(y) & \text{if } p(x) = q(y), \\ \text{undefined} & \text{if } p(x) \neq q(y). \end{cases}$$

So $f'(x)(y) = \tilde{f}'(x)(y) = \tilde{f}(x,y) = f(x,y)$ if and only if $p(x) = q(y)$. If $p(x) = b$, then f' is defined for each $y \in Y|b$. Now $(q!_\chi Z) \circ f' = p$ and $i \circ f' = \tilde{f}'$. Now both p and $\tilde{f}'$ are continuous, so f' is continuous by 1.3.

(ii) Suppose on the other hand that p and $\tilde{f}'$ are continuous. We reverse the argument in (i), in the sense that a map $f' : X \to Y!_X Z$ can be extended to a map $\tilde{f}' : X \to Map_X(Y, Z^\omega)$, which determines an $\tilde{f}$ and hence the required f.

The result follows from (i) and (ii) above. ■

Theorem 5.13 *Let Y be an arbitrary space, X be a cg-space, and B be a Hausdorff space. Let $f : X \to B$ be a map and $q : Y \to B$ be an identification. Then $q^* f : Y \sqcap_X X \to X$ is an identification.*

Proof: It will be proven that $q^* f : Y \sqcap_X X \to X$ satisfies 2.11(iii). Let Z be an arbitrary space and $g : X \to Z$ be a function. Suppose that g is continuous, then $g \circ (q^* f)$ is continuous.

Now, suppose that $g \circ (q^* f)$ is continuous. It follows from 5.12 that there is a continuous map, $(g \circ (q^* f))' : Y \to X!_X Z$ defined by

$$(g \circ (q^* f))'(y)(x) = g \circ (q^* f)(y, x) = g(x) \quad \forall y \in Y,\ \forall x \in X.$$

Define a function $g_b : X|b \to Z$ by

$$g_b(x) = g \circ i_{X|b}(x) \quad \forall x \in X \text{ such that } f(x) = b.$$

It follows from 3.13, 3.14, and the continuity of $g \circ (q^* f)$ that the following diagram commutes.

$$\begin{tikzcd}
(Y|b) \times_X (X|b) \arrow[r, "i_{Y|b} \times_X i_{X|b}"'] \arrow[d, "\pi_{X|b}"'] & Y \sqcap_X X \arrow[d, "g \circ (q^* f)"'] \\
X|b \arrow[r, "g_b"'] & Z
\end{tikzcd}$$

The continuity of g_b follows from the commutativity of the diagram and 2.11(iii), since $\pi_{X|b} : (Y|b) \times_X (X|b) \to X|b$ is an identification by 3.15.

So the rule $b \rightsquigarrow g_b$ where $g_b \in X!_X Z$ is a well defined function. Let $g_\bullet : B \to X!_X Z$ represent this function. Then the following triangle commutes.

$$\begin{tikzcd} & X!_X Z \\ Y \arrow[ur, "(g \circ (q^* f))'"] \arrow[r, "q"'] & B \arrow[u, "g_\bullet"'] \end{tikzcd}$$

So $g_\bullet$ is continuous by 2.11(iii). Next 5.12 ensures there is a continuous map $g'_\bullet : B \sqcap_X X \to Z$ given by

$$g'_\bullet(b, x) = g_\bullet(b)(x) \quad \forall x \in X \text{ such that } f(x) = b.$$

Lemma 5.11 now ensures that the the map $(f, 1_X) : X \to B \sqcap_X X$ is a homeomorphism. Moreover, for every $f(x) = b$

$$g'_\bullet \circ (f, 1_X)(x) = g'_\bullet(b, x) = g_\bullet(b)(x) = g_b(x) = g \circ i_{X|b}(x) = g(x).$$

Hence $g : X \to Z$ is continuous and $q^* f : Y \sqcap_X X \to X$ is an identification. ■

Corollary 5.13.1 *The map $q^* f : Y \sqcap X \to X$ is an identification.*

Proof: The result follows from 5.13, and 2.13 by way of 3.21. ■

Corollary 5.13.2 *Let Y be a space, B a Hausdorff space, and A a compactly generated subspace of B. Let $q : Y \to B$ be an identification. Then $q|A : Y|A \to A$ is an identification.*

Proof: This is a special case of 5.13 with $X = A \subset B$ and $f = i : A \hookrightarrow B$.
$\blacksquare$

Chapter 6

Conclusion

6.1 Closing thoughts and questions

The theory presented in this work contains some improvements on known results, in particular the results concerning locally compact-Hausdorff spaces and some results concerning χ-open topologies. It is the intention of the author to reformulate some of these results for publication.

This study undertaken in this thesis is by no means complete. Indeed, one can explore further examples of commutativity of final and initial topologies. An immediate open question would be to determine sufficient conditions for a general result concerning Initial and Final Topology Commutativity. Another question arising is the possibility of replacing the χ versions of product and mapping space topologies with 'local' χ versions, i.e. consider incoming maps from the class of a locally compact-Hausdorff spaces. Would our theory work in this context?

6.2 Historical notes concerning the χ-product

The χ-product of two spaces, X and Y, as defined in section 3.4, is a particular case of the $\mathcal{A}$-product, Definition 1.1 of [5]. In our case, we take $\mathcal{A}$ to be the class $\mathcal{CH}$ of all compact Hausdorff spaces. The $\mathcal{A}$ product traces its origin to the **S**-product of [8].

Several of our χ-product results are particular cases of results in [5]. We list these coincidences here. Proposition 3.16 and 3.17 are particular cases of Proposition 1.2. Propositions 3.24 and 3.25 are particular cases of Proposition 3.3. Proposition 3.28 is a particular case of Proposition 1.4. Propositions 4.5 and 4.9 are particular cases of Propositions 2.3 and 2.1 respectively. And Theorems 4.10 and 4.12 are particular cases of Theorem 2.4.

Bibliography

[1] Richard F. Arens, A topology for spaces of transformations, Annals of Mathematics Vol. 47, No. 3 (1946) 480-495.

[2] Peter I. Booth, Cofibrations, cohomology, fibrations, identifications, and mapping spaces (to appear).

[3] Peter I. Booth, Mapping spaces for homotopy theory (to appear).

[4] Peter I. Booth and Ronald Brown, Spaces of partial maps, fibred mapping spaces and the compact-open topology, General Topology and its Applications 8 (1978) 181-195.

[5] Peter I. Booth and Joy Tillotson, Monoidal closed, cartesian closed and convenient categories of topological spaces, Pacific Journal of Mathematics Vol. 88, No. 1 (1980) 35-53.

[6] Nicolas Bourbaki, *General Topology, Part I* (Spring-Verlag, New York, 1989).

[7] Ronald Brown, *Topology: A Geometric Account of General Topology, Homotopy Types, and the Fundamental Groupoid* (Ellis Horwood Limited, West Sussex, 1988).

[8] Ronald Brown, Ten topologies for $X \times Y$, Quarterly Journal of Mathematics, (Oxford) (2), 14 (1963) 303-319.

[9] Ronald Brown, Function spaces and product topologies, Quarterly Journal of Mathematics, (Oxford) (2), 15 (1964) 238-250.

[10] Charles O. Christensen and William L. Voxman, *Aspects of Topology* (Marcel Dekker, Inc., New York, 1977).

[11] Brian J. Day and G. Max Kelly, On topological quotient maps preserved by pullbacks or products, Proc. Cambridge Phil. Soc. 67 (1970) 553-558.

[12] James Dugundji, *Topology* (Allyn and Bacon, Inc., Boston, 1966).

[13] Ralph H. Fox, On topologies for function spaces, Bulletin of the American Mathematical Society 51 (1945) 428-432.

[14] Sze-Tsen Hu, *Homotopy Theory* (Academic Press, Inc., New York, 1971).

[15] Donald W. Kahn, *Topology: An Introduction to the Point-Set and Algebraic Areas* (Dover Publications Inc., New York, 1995).

[16] John L. Kelley, *General Topology* (D. Van Nostrand Company, Inc., New Jersey, 1957).

[17] William S. Massey, *Algebraic Topology: An Introduction* (Harcourt-Brace-World, New York, 1967).

[18] M.C. Mccord, Classifying spaces and infinite symmetric products, Trans. Amer. Math. Soc. 146 (1969) 273-298.

[19] Edwin H. Spanier, *Algebraic Topology* (McGraw-Hill, New York, New York, 1966).

[20] Norman E. Steenrod, A convenient category of topological spaces, Michigan Mathematics Journal 14 (1967) 133-152.

[21] Joy Tillotson, The convenient category of sequential spaces, Masters thesis, Memorial University of Newfoundland, 1978.

[22] Rainer M. Vogt, Convenient categories of topological spaces for homotopy theory, Archiv der Mathematik 22 (1971) 545-555.

[23] Stephen Willard, *General Topology* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1970).

[24] Oswald Wyler, Convenient categories for topology, General Topology and its Applications 3 (1973) 225-242.