

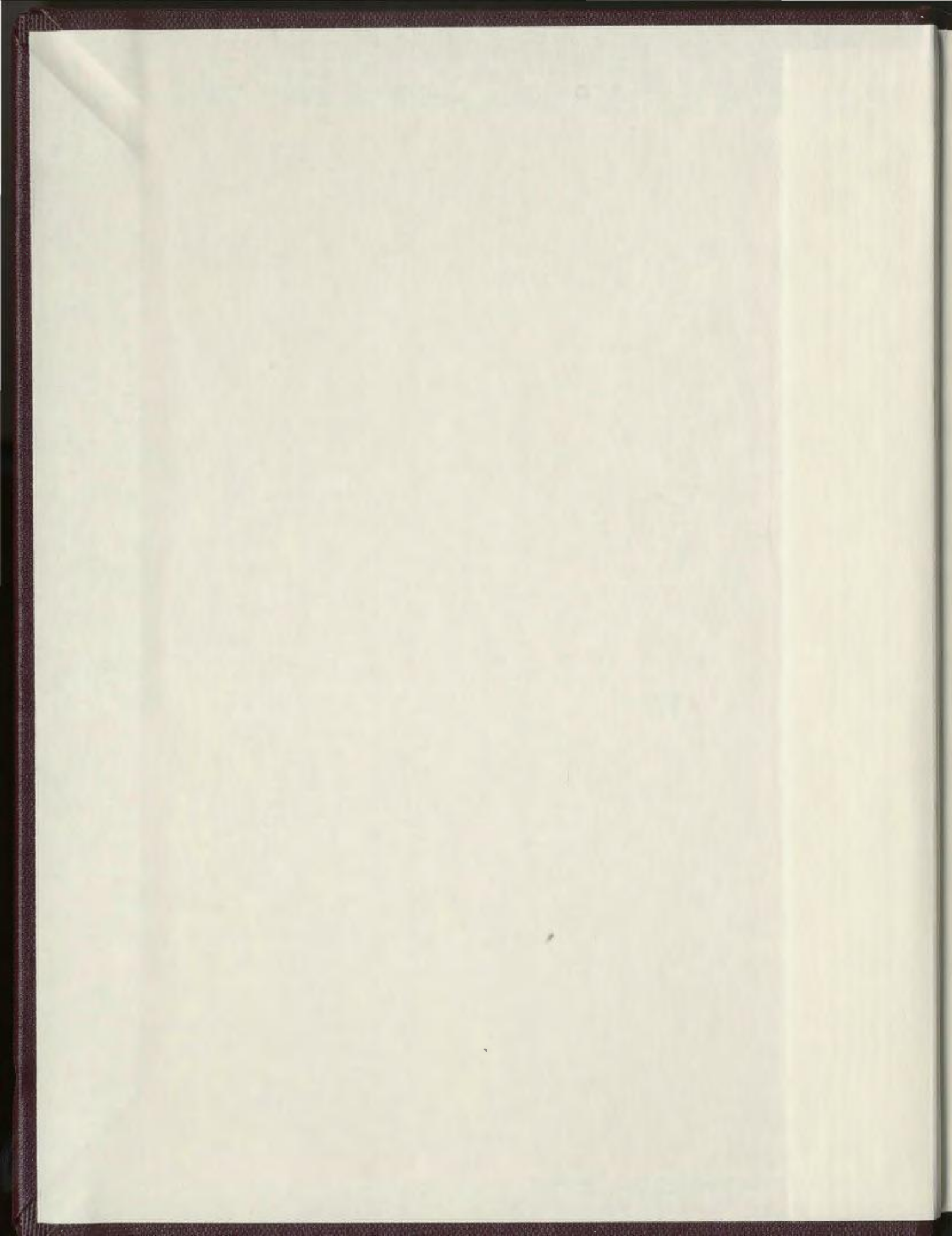
WOBBLE/NUTATION OF A ROTATING ELLIPSOIDAL
EARTH WITH LIQUID OUTER CORE:
IMPLEMENTATION OF A NEW SET OF EQUATIONS
DESCRIBING DYNAMICS OF ROTATING FLUIDS

CENTRE FOR NEWFOUNDLAND STUDIES

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WOBBLE/NUTATION OF A ROTATING ELLIPSOIDAL EARTH
WITH LIQUID OUTER CORE: Implementation of a
New Set of Equations Describing Dynamics
of Rotating Fluids

by

©Behnam Seyed-Mahmoud

A thesis submitted to the
School of Graduate Studies
in partial fulfilment of the
requirements for the degree of
Master of science

Department of Earth Sciences (Geophysics)
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ABSTRACT

In this thesis I implement a new set of scalar equations describing the free oscillations of rotating self-gravitating compressible fluids, to solve for the wobble and inertial modes of a rotating Earth with rigid mantle and liquid core. A Galerkin method is used to integrate these equations in the liquid core. It is shown that by using the divergence theorem it is possible to make use of the '*natural*' boundary conditions to reduce the order of the derivatives from second to first in the Galerkin formulation of the governing equations. As a partial test of the reliability of our formulation, the eigenperiods of the Earth's Chandler wobble (CW), nearly diurnal free wobble (NDFW) and some of the other inertial modes are computed for the case of a compressible, but neutrally stratified, core and compared to those for the homogeneous incompressible core model.

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LIST OF SYMBOLS

symbol	description	page appeared first
A_E	moment of inertia of the Earth with respect to an equatorial axis	60
A_m	moment of inertia of the mantle with respect to an equatorial axis	36
a_n	constant coefficient	11, 43
B	scalar notation	25
B_1	scalar notation	28
b	mean radius of CMB	14
C	vector notation	24
C_1	vector notation	28
C_m	moment of inertia of the Earth/mantle with respect to a polar axis	36
CMB	core mantle boundary	12
CW	Chandler wobble	i
\hat{E}_i	principal axis fixed in the mantle	31
e_E	dynamical ellipticity of the Earth	60
e_m	$(C_m - A_m)/A_m$	36
\hat{e}_i	unit vector in i direction	4
f_i	trial function	44
G	gravitational constant	5
g_0	gravitational acceleration	5
H_E	total angular momentum of the Earth	35
H_{lc}	angular momentum of the liquid core	35
H_m	angular momentum of the mantle	35
\mathbf{i}_m	mantle's inertia tensor	35
ICB	inner core boundary	11
L	integer denoting the degree of truncation	44
$M(r)$	total mass of a body enclosed by a shell of radius r	14
N	Brunt-Väisälä frequency	5
N	integer denoting the degree of truncation	13, 44
NDFW	nearly diurnal free wobble	i
PDE	partial differential equation	3
P_n^m	Legendre polynomials of degree n and azimuthal order m	8
PREM	preliminary reference Earth model	11
P_0	pressure before deformation	5
p_1	Eulerian pressure disturbance	6
R	mean radius of the surface of the Earth	11

r	distance from the geocentre	8
r_0	mean radius of equipotential surface	16
SSA	subseismic approximation	3
SSWE	subseismic wave equation	3
S_n^m	spheroidal displacement field	8
THPD	three potential description	4
TPD	two potential description	4
T_n^m	toroidal displacement field	8
\mathbf{u}	Lagrangian displacement	6
V_0	gravitational potential	34
V_1	Eulerian perturbation in gravitational potential	6
\mathbf{v}	velocity	6
W_0	gravity potential	5
x	r/R	11
Y_n^m	spherical harmonics of degree n and azimuthal order m	8
α	compressional wave speed	5
β	stability parameter	1
Γ_p	dyadic	24
γ	amplitude of the wobble	32
ε	ellipticity	16
ε_{ijk}	three dimensional Levi-Civita symbol	33
ζ	$\alpha^2 \nabla \cdot \mathbf{u}$ (scalar potential)	23
ζ_n^m	scalar function	29
θ	colatitude	8
λ	Lamé parameter	8
μ	Lamé parameter	8
ρ_0	density before deformation	5
ρ_1	Eulerian perturbation in density	6
$\langle \rho \rangle$	average density	15
σ	$2\Omega/\omega$	24
σ_n	constant coefficient	33
$\tilde{\tau}$	stress tensor	7
ϕ	longitude	8
Φ_n^m	scalar function	8
χ	scalar potential	23
χ_n^m	scalar function	29
Ω	rate of rotation of the Earth	4
ω	oscillation frequency	6
ω_m	angular velocity of the mantle	31
ω_r	angular velocity of the reference frame	4
\mathbf{i}	unit dyadic	35

CHAPTER 1: INTRODUCTION

The innermost region of the earth consists of a liquid outer core and a solid inner core. The radius of the inner core is estimated to be 1221.5 km and that of the outer core 3480 km. The fluidity of the outer core and the solidity of the inner core have been established through studies of ray seismology, tides and the Earth's normal modes.

The core of the Earth plays a key role in many geophysical studies. An example of this is the role of the liquid core on the periods of the Earth's Chandler wobble and the nearly diurnal free wobble. A perfectly rigid body with the same mass distribution as the Earth is expected to have a free Eulerian wobble with period of about 306 days. The presence of a liquid core the size of the Earth's changes this period to about 270 days and gives rise to an additional retrograde wobble of nearly diurnal period, corresponding to a nutation of about 350 days. Elasticity and oceans lengthen the Eulerian wobble period to 435 days (the Chandler wobble) and the free core nutation period to 460 days, assuming that the steadily-rotating configuration is one of hydrostatic equilibrium.

Since the core is not directly accessible, much is still unknown about its properties. The distributions of material properties such as the density ρ and Lamé parameter λ in the core are established through theoretical and observational studies using ray seismology and free oscillations. However, the stability parameter β (Pekeris and Accad 1972) cannot so far be inferred from the seismological data and is therefore poorly controlled, uncertain and varies from one model to another.

The spectrum of free oscillations possible in the liquid core can be conveniently divided into:

(a) short period free oscillations, with periods of the order of an hour or less, for which elasticity is the primary restoring force. For these oscillations the effects of rotation and ellipticity can be treated as small perturbations on the solution of the governing equations for a non-rotating spherical Earth;

(b) long period free core oscillations, with periods of the order of half a day and longer. These oscillations are considerably affected by the rotation and, in some cases, the ellipticity of the Earth.

Long period free core oscillations are of three types:

(a) Slichter (inner core translational) modes, with periods of several hours, for which the primary restoring force is gravitational (due to the density difference across the inner core boundary);

(b) gravity waves (or core undertones) whose essential restoring force is negative buoyancy. Since negative buoyancy implies stable stratification, some parts of the outer core must be stably stratified ($\beta < 0$) for gravity waves to exist;

(c) inertial waves, which depend on the Coriolis effect as their dominant restoring force. Therefore, the rotation of the Earth is necessary to the existence of these modes.

Among the first who studied the inertial modes of the Earth were Hough (1895) and Poincaré (1910). The Earth model they considered consists of an inviscid incompressible homogeneous liquid filling the region bounded by an ellipsoidal container

which is rigid but can change its orientation. Analytical solutions were found for the Earth's Chandler wobble and nearly diurnal free wobble and other inertial modes using this simple Earth model.

During the last four decades the theory of Hough and Poincaré has been greatly extended to treat more realistic Earth models. Jeffreys & Vicente (1957a, 1957b) considered Earth models with radially stratified elastic mantle and a homogeneous incompressible liquid core. Molodensky (1961) included core compressibility. Shen and Mansinha (1976) used the theory of Molodensky (1961) but added non-neutral stratification in the liquid core. Smith (1974) derived the elastic-gravitational normal modes theory in which an infinite set of coupled differential equations describe the dynamics of a rotating, slightly elliptical Earth. This theory was utilized by Smith (1977) to compute the periods of the Earth's Chandler wobble and the nearly diurnal free wobble, with inner core. Also Wahr (1981) used Smith's formulation to study the effects of Earth's rotation and ellipticity on the body tides. Moon (1982) derived the linearized equation of motion for the slightly elliptical rotating earth in order to study Earth's free oscillations, free wobbles and core modes.

In the hope of finding an alternative, more simplified, solution to the governing equations of core dynamics Smylie & Rochester (1981) made the subseismic approximation (SSA) to derive the subseismic wave equation (SSWE). The SSWE is a scalar second order partial differential equation (PDE) involving only one scalar potential, and (to the extent the SSA is valid) once the SSWE is solved, all other dynamical

variables can be readily obtained. As shortcomings in the subseismic description became apparent (Crossley and Rochester 1992, Rochester and Peng 1993), Wu & Rochester (1990) derived the exact two potential description of core dynamics (TPD). They showed that the dynamics of the inviscid liquid core is described without approximation by two scalar second order PDEs involving two scalar potentials. Although the TPD is mathematically elegant, it also has some shortcomings which we discuss in chapter 3.

Rochester (unpublished) has derived an alternative set of three scalar PDEs involving three scalar potentials which exactly describe the dynamics of the inviscid liquid core (hereafter we refer to this description as THPD). The motivation for this new description and its advantages will be discussed later in this thesis. In this study, we implement the THPD to solve for some of the normal modes of a rotating ellipsoidal liquid core.

1.1 The Governing Equations and Boundary Conditions

The liquid core is taken as inviscid, with the reference state being one of hydrostatic equilibrium in a coordinate system rotating steadily with angular velocity

$$\boldsymbol{\omega}_r = \Omega \hat{\boldsymbol{e}}_3 \quad (1.1)$$

where Ω is the rate of rotation of the Earth. The rate of change of direction of a unit vector, $\hat{\boldsymbol{e}}_j$, is therefore

$$\frac{d\mathbf{e}_j}{dt} = \Omega \mathbf{e}_3 \times \mathbf{e}_j \quad (1.2)$$

In the reference frame stated above the equilibrium density ρ_0 , pressure p_0 and gravitational acceleration \mathbf{g}_0 are related by

$$\nabla P_0 = \rho_0 \mathbf{g}_0 \quad (1.3)$$

$$\mathbf{g}_0 = \nabla W_0 \quad (1.4)$$

$$\nabla^2 W_0 = -4\pi G \rho_0 + 2\Omega^2 \quad (1.5)$$

$$\nabla \rho_0 = (1 - \beta) \rho_0 \frac{\mathbf{g}_0}{\alpha^2} \quad (1.6)$$

where G , α , β are respectively the gravitational constant, the local compressional wave speed and the stability parameter.

The parameter β measures the extent and sign of any departure of density gradient from purely adiabatic stratification; it is proportional to the square of the local Brunt-Väisälä frequency N :

$$\beta = -\frac{\alpha^2 N^2}{g_0^2} \quad (1.7)$$

A region is stably stratified if $\beta < 0$, unstably stratified if $\beta > 0$ and neutral if $\beta = 0$ (i.e. the Adams Williamson condition is satisfied). Although an exact value for β has not been established, Masters (1979) shows that current seismological data sets a limit $|\beta| < 0.03$ -

0.05.

The equations governing the isentropic small oscillations of an inviscid liquid core are given (Rochester 1989) by the conservation laws for mass, momentum, gravitational flux and entropy. These equations are as follows:

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}) \quad (1.8)$$

$$\frac{\partial \mathbf{v}}{\partial t} + 2\Omega \boldsymbol{\epsilon}_3 \times \mathbf{v} = -\frac{1}{\rho_0} \nabla p_1 + \nabla V_1 + \frac{\rho_1}{\rho_0} \mathbf{g}_0 \quad (1.9)$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \quad (1.10)$$

$$\frac{\partial p_1}{\partial t} = \alpha^2 \frac{\partial \rho_1}{\partial t} - \beta \rho_0 \mathbf{v} \cdot \mathbf{g}_0 \quad (1.11)$$

In the above equations \mathbf{v} , ρ_1 , p_1 and V_1 (all regarded as first order departures from the equilibrium reference state) stand for velocity, the Eulerian perturbation in density, Eulerian pressure disturbance and the Eulerian perturbation in the gravitational potential respectively, with

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \quad (1.12)$$

where \mathbf{u} is the Lagrangean displacement from equilibrium.

In dealing with free oscillations it is convenient to assume that all the disturbance variables have time dependence $e^{i\omega t}$. After operating with the time derivative on the

disturbance variables in equations (1.8)-(1.11) and cancelling the common factors we get

$$\rho_1 = -\nabla \cdot (\rho_0 \mathbf{u}) \quad (1.13)$$

$$-\omega^2 \mathbf{u} + 2i\omega \Omega \mathbf{e}_3 \times \mathbf{u} = -\frac{1}{\rho_0} \nabla p_1 + \nabla V_1 + \frac{\rho_1}{\rho_0} \mathbf{g}_0 \quad (1.14)$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \quad (1.15)$$

$$p_1 = -\mathbf{u} \cdot \nabla p_0 + \alpha^2 (\rho_1 + \mathbf{u} \cdot \nabla \rho_0) \quad (1.16)$$

We use (1.13) to write equations (1.14)-(1.16) as

$$(\omega^2 \mathbf{u} - 2i\omega \Omega \mathbf{e}_3 \times \mathbf{u}) = \frac{1}{\rho_0} \nabla p_1 - \nabla V_1 + \frac{\mathbf{g}_0}{\rho_0} \nabla \cdot (\rho_0 \mathbf{u}) \quad (1.17)$$

$$\nabla^2 V_1 = -4\pi G \nabla \cdot (\rho_0 \mathbf{u}). \quad (1.18)$$

$$\frac{p_1}{\rho_0} = -(\alpha^2 \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{g}_0) \quad (1.19)$$

The five differential equations above [three components of the vector equation (1.17), equations (1.18) and (1.19)] are linear in five variables: three components of \mathbf{u} , p_1 and V_1 . These are the equations governing the dynamics of the liquid core.

To these equations we add the boundary conditions, which require continuity of $\hat{\mathbf{n}} \cdot \mathbf{u}$, V_1 , $\hat{\mathbf{n}} \cdot (\nabla V_1 - 4\pi G \rho_0 \mathbf{u})$ and $\hat{\mathbf{n}} \cdot \bar{\boldsymbol{\tau}}$ across core boundaries. Here $\hat{\mathbf{n}}$ is the unit vector normal to the boundary surface and $\bar{\boldsymbol{\tau}}$ is the stress tensor. In the outer core the stress tensor can be written as

$$\bar{\tau} = -(\rho_1 + \mathbf{u} \cdot \nabla \rho_0) \bar{\mathbf{I}} \quad (1.20)$$

and in the solid mantle it takes the form

$$\bar{\tau} = (\lambda \nabla \cdot \mathbf{u}) \bar{\mathbf{I}} + 2\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \quad (1.21)$$

where μ and λ are the Lamé parameters and $\bar{\mathbf{I}}$ is the unit dyadic.

The traditional approach to solving these equations is to represent the variation fields in spherical polar coordinate by spherical harmonics

$$\mathbf{u} = \sum_{m=-n}^n \sum_{n=|m|}^{\infty} S_n^m + T_n^m \quad (1.22)$$

$$V_1, P_1 = \sum_{m=-n}^n \sum_{n=|m|}^{\infty} [\phi_n^m, \psi_n^m] Y_n^m \quad (1.23)$$

with

$$S_n^m = [u_n^m \hat{r} + r v_n^m \nabla] Y_n^m \quad (1.24)$$

$$T_n^m = -t_n^m \hat{r} \times \nabla Y_n^m$$

$$Y_n^m = P_n^m(\cos \theta) e^{im\phi} \quad (1.25)$$

where u_n^m , v_n^m , t_n^m , ϕ_n^m and ψ_n^m are functions of r only and P_n^m is the associated Legendre function of degree n and azimuthal order m .

For the acoustic modes (short period free oscillations) the effects of rotation and ellipticity are small and equations with different degree n are independent when these effects are neglected (Alterman et al. 1959). Therefore, the traditional approach is an effective tool in solving the governing equations for the acoustic modes.

However, for the long period free oscillations, where neglecting rotation and, in some cases, ellipticity is no longer valid, equations with different order m are decoupled but those of different degree are not. Instead to each m there correspond solutions in the form of two coupling chains

$$S_{|m|}^m + T_{|m|+1}^m + S_{|m|+2}^m + T_{|m|+3}^m + \dots \quad (1.26)$$

$$T_{|m|}^m + S_{|m|+1}^m + T_{|m|+2}^m + S_{|m|+3}^m + \dots \quad (1.27)$$

(Smith 1974). Numerical estimates of the eigenperiods and eigenfunctions then necessarily rest on heavy truncations of these coupling chains.

Shen & Mansinha (1976) used a three term truncation, e.g.

$$T_1^1 + S_2^1 + T_3^1 \quad (1.28)$$

to compute the periods of the Earth's nearly diurnal free wobble and some of the liquid core undertones of a rotating ellipsoidal earth. Smith (1977) seriously questions the validity of the numerical results for undertone periods based on such heavy truncations, and concludes that the latter are inadequate for the free wobble of the inner core and other internal core modes. He was convinced, however, that (1.28) was sufficient to yield

reliable periods for the Chandler and nearly diurnal free wobble. This belief seems not yet to have been seriously tested by using less severe truncation than (1.28).

The problems of truncation presented by the traditional approach motivated a search for alternative solutions to the governing equations [Smylie & Rochester (1981), Wu & Rochester (1990)]. The three-potential description of core dynamics is the result of one such attempt. We will give the derivation of the THPD in this thesis (chapter 3). To test the reliability of our description, we use the well-known analytical solutions to compute the periods of the Earth's Chandler wobble, nearly diurnal free wobble and some of the other inertial modes using a simplified Earth model consisting of a rigid mantle and a homogeneous incompressible liquid core. These results will then be compared to the ones computed using the THPD.

CHAPTER 2: THE EARTH MODEL

We adopt PREM [the Preliminary Reference Earth Model (Dziewonski & Anderson 1981)], as the base for our Earth model. The mantle is assumed to be rigid and its properties are taken directly from PREM. The presence of the solid inner core is ignored for simplicity. The liquid core is assumed to be neutrally stratified and therefore we must make sure that $\beta=0$ everywhere in its interior. Also, since we ignore the presence of the inner core in computational work, we will make sure that the density profile ρ and the compressional wave speed profiles α in the liquid core are extended to the centre of the Earth as smooth functions of the radius. The method to compute the ellipticity of the equipotential surfaces will also be described in this chapter.

2.1 Modification of α in the Liquid Core

In the liquid core of PREM α is expressed as a third order polynomial

$$\alpha = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \quad (2.1)$$

where $c_1=11.0487$, $c_2=-4.0362$, $c_3=4.8023$, $c_4=-13.5732$, $x=r/R$ and R is the mean radius of the surface of the Earth. Since the first derivative of this profile with respect to r does not vanish at $r=0$, this profile can not be used when the inner core is ignored.

In the inner core of the PREM α is a smooth function of the radius and has the form

$$\alpha_{ic} = a_1 + a_2 x^2 \quad (2.2)$$

with $a_1=11.2622$ and $a_2=-6.3640$. Since p-waves travel faster in solid media than in fluids, we will have a maximum error of about 17% (at the CMB) if we choose the inner core profile for α as an approximation to that in the liquid core. Compromising to reduce this discrepancy, we modify α so that in our model its first derivative vanishes at $r=0$ and its numerical values are close to those of PREM in the range ICB-CMB.

The function

$$\alpha = 10.6752 - 8.7490 x^2 \quad (2.3)$$

satisfies the requirement of vanishing first derivative at $x=0$, gives the PREM values at the ICB and CMB, and involves a maximum error of about 0.7% (at $r=2900$ km). Table 1 shows the α distribution in the liquid core using: (a) equation (2.3) and (b) expression from PREM.

2.2 Modification of the Density Profile in the Liquid Core

As mentioned in chapter 1, the density gradient in the liquid core has the form

$$\nabla \rho_0 = (1-\beta) \frac{\rho_0 g_0}{\alpha^2} \quad (2.4)$$

For a spherically stratified Earth (2.4) becomes

$$\frac{d\rho_0}{dr} = -(1-\beta) \frac{\rho_0 g_0}{\alpha^2} \quad (2.5)$$

which can be written as

$$\frac{\alpha^2}{\rho_0 g_0} \frac{d\rho_0}{dr} + (1-\beta) = 0 \quad (2.6)$$

As suggested by PREM, we choose a density profile which has the form

$$\rho_0 = \sum_{j=1}^N d_j x^{j-1} \quad (2.7)$$

where x is defined as in previous section, N is an integer and d_j are constant. Substituting (2.7) into (2.5) [and setting $d_2=0$ to satisfy (2.5) at $x=0$] we get

$$\sum_{j=3}^N (j-1) \frac{\alpha^2 R}{\rho_0 g_0} d_j x^{j-2} + (1-\beta) = 0 \quad (2.8)$$

where

$$\frac{d\rho_0}{dr} = \frac{d\rho_0}{dx} \frac{dx}{dr} = R \sum_{j=3}^N (j-1) d_j x^{j-2} \quad (2.9)$$

To solve (2.8) for a best fitting density function, ρ_0 , we use a Galerkin method with weight functions x^{i-1} ($i=1, \dots, N-2$):

$$\sum_{j=3}^N (j-1)d_j \int_0^{\frac{b}{R}} \frac{R\alpha^2}{\rho_0 g_0} x^{i+j-3} dx + (1-\beta) \int_0^{\frac{b}{R}} x^{i-1} dx = 0 \quad (2.10)$$

for each $i, i=1, \dots, N-2$, b is the mean radius of CMB. We write (2.10) as

$$\sum_{j=3}^N A_{ij} d_j = F_i \quad (2.11)$$

where

$$A_{ij} = (j-1) \int_0^{\frac{b}{R}} \frac{R\alpha^2}{\rho_0 g_0} x^{i+j-3} dx \quad (2.12)$$

$$F_i = -\frac{1-\beta}{i} \left(\frac{b}{R}\right)^i \quad (2.13)$$

In (2.13) β is set to a desired value, β_d , and PREM density profile is used as a starting value to find ρ_0 and g_0 in (2.12). The starting values for the coefficients of the density profile, therefore, are: $d_1=12.5815$, $d_2=0$, $d_3=-3.6426$, $d_4=-5.5281$, all other $d_i=0$. In PREM $N=4$, but our computational experience has shown that a higher value of N results in a much faster convergence of β to β_d , and here we set $N=12$.

The gravitational acceleration, g_0 , in the liquid core is computed as

$$g_0(r) = \frac{GM(r)}{r^2} \quad (2.14)$$

where $M(r)$ is the total mass of the body enclosed by the shell of radius r and is given

as

$$M(r) = 4\pi \int_0^r \rho_0 r'^2 dr' \quad (2.15)$$

Using (2.7), (2.15) becomes

$$M(r) = 4\pi R^3 \sum_{i=1}^3 d_i \int_0^x x'^{i+1} dx' = 4\pi R^3 \sum_{i=1}^3 \frac{d_i}{i+2} x^{i+2} \quad (2.16)$$

Substituting (2.16) into (2.14) we get

$$g_0(r) = 4\pi G R \sum_{i=1}^N \frac{d_i x^i}{i+2} \quad (2.17)$$

IMSL subroutine dslsarg is then called to solve (2.12) for the coefficients d_3, \dots, d_N .

To solve for d_1 we use the mass conservation of the liquid core as a constraint and proceed as follows:

$$\int_{LC} \rho_0 dV = \frac{4}{3} \pi \langle \rho \rangle b^3 \quad (2.18)$$

where $\langle \rho \rangle$ is the average density of the outer core. Using (2.8) we can write (2.18) as

$$4\pi R^3 \sum_{i=1}^N d_i \int_0^{\frac{b}{R}} x^{i+1} dx = \frac{4\pi \langle \rho \rangle}{3} b^3 \quad (2.19)$$

$$\therefore d_1 = \langle \rho \rangle - 3\left(\frac{R}{b}\right)^3 \sum_{i=3}^N \frac{d_i}{i+2} \left(\frac{b}{R}\right)^{i+2} \quad (2.20)$$

Once d_i are known, we use them to solve for β in

$$\beta = 1 + \frac{\alpha^2}{\rho_0 g_0} \frac{d\rho_0}{dr} \quad (2.21)$$

If $|\beta - \beta_d| > \kappa$, where κ is the desired accuracy, we use the new d_i as the starting values. This process is repeated until $|\beta - \beta_d| \leq \kappa$. In this problem we set $\beta = 0$ and $\kappa = 10^{-5}$. The coefficients of the density profile in the modified liquid core are given in table 2. Tables 3 and 4 show the density and β distributions in the outer core before and after modification respectively.

2.3 Ellipticity

Since the departure of the shape of the Earth from sphericity is small, we use a first order theory to include the effect of ellipticity in the governing equations. We follow Chandrasekhar and Roberts (1963) in assigning the equipotential surfaces of mean radius r_0 (over which the equilibrium properties are constant) the polar equation

$$r = r_0 \left(1 - \frac{2}{3} \epsilon(r_0) P_2(\cos\theta)\right) \quad (2.22)$$

For a hydrostatically prestressed Earth Clairaut's equation

$$\frac{d^2e}{dr^2} + \frac{6}{r^2} \left[\frac{\rho_0}{\langle \rho_0 \rangle} - 1 \right] e + \frac{6}{r} \frac{\rho_0}{\langle \rho_0 \rangle} \frac{de}{dr} = 0 \quad (2.23)$$

(Jeffreys 1970) describes the ellipticity of the equipotential surfaces. In writing (2.23) we use the fact that $\epsilon(r_0) = \epsilon(r)$ to first order.

We use a Runge-Kutta integration method to solve (2.23) for ϵ . Since equation (2.23) is singular at $r=0$, it cannot be used directly to start the integration from the geocentre. To get around this problem we proceed as follows. Using Taylor expansion we can write ϵ and ρ near the geocentre as

$$e = e(0) + \frac{1}{2} \left(\frac{d^2e}{dr^2} \right)_0 r^2 + \dots \quad (2.24)$$

$$\rho_0 = d_0 + d_2 r^2 + d_3 r^3 + \dots \quad (2.25)$$

The average density $\langle \rho \rangle$ is given as

$$\langle \rho \rangle = \frac{4\pi \int_0^r \rho x^2 dx}{\frac{4}{3}\pi r^3} = \frac{3}{r^3} \int_0^r [d_0 + d_2 x^2 + \dots] x^2 dx = d_0 + \frac{3}{5} d_2 r^2 + \dots \quad (2.26)$$

Substituting (2.24)-(2.26) into (2.23) we get

$$\left(\frac{d^2e}{dr^2} \right)_0 = -\frac{12}{35} \frac{d_2}{d_0} e(0) \quad (2.27)$$

We use (2.27) and an arbitrary value for $\epsilon(0)$ to start the integration at the geocentre, and then scale $\epsilon(0)$ so that $\epsilon(R)$ is the observed hydrostatic value of Earth's surface ellipticity.

Table 5 shows the distribution of the ellipticity in the interior of the Earth.

Table 1. Distribution of α in the Liquid Core

a: Modified Model

b: PREM Model

	(a)	(b)
radius km	α km/s	α km/s
0.00000D+00	0.106776D+02	
0.81433D+02	0.106762D+02	
0.16287D+03	0.106719D+02	
0.24430D+03	0.106647D+02	
0.40717D+03	0.106418D+02	
0.48860D+03	0.106261D+02	
0.57003D+03	0.106075D+02	
0.65147D+03	0.105860D+02	
0.73290D+03	0.105617D+02	
0.81433D+03	0.105345D+02	
0.89577D+03	0.105045D+02	
0.97720D+03	0.104716D+02	
0.10586D+04	0.104358D+02	
0.11401D+04	0.103972D+02	
0.12215D+04	0.103557D+02	0.103557D+02
0.13404D+04	0.102900D+02	0.102857D+02
0.14592D+04	0.102182D+02	0.102131D+02
0.15781D+04	0.101403D+02	0.101373D+02
0.16970D+04	0.100563D+02	0.100578D+02
0.19347D+04	0.987003D+01	0.988576D+01
0.20536D+04	0.976774D+01	0.979209D+01
0.21724D+04	0.965936D+01	0.969263D+01
0.22913D+04	0.954489D+01	0.958684D+01
0.24102D+04	0.942431D+01	0.947420D+01
0.25291D+04	0.929764D+01	0.935417D+01
0.26479D+04	0.916487D+01	0.922624D+01
0.27668D+04	0.902601D+01	0.908987D+01
0.28857D+04	0.888105D+01	0.894453D+01
0.30045D+04	0.872999D+01	0.878969D+01
0.31234D+04	0.857283D+01	0.862483D+01
0.32423D+04	0.840958D+01	0.844941D+01
0.33611D+04	0.824023D+01	0.826290D+01
0.34800D+04	0.806479D+01	0.806479D+01

**Table 2. Coefficients of the Density Profile
in the Modified Liquid Core**

kg/m³

$$\begin{aligned}d_1 &= 12.4780 \times 10^3 \\d_2 &= 0.0 \\d_3 &= -7.7460 \times 10^3 \\d_4 &= 1.1811 \times 10^{-1} \\d_5 &= -2.5083 \times 10^3 \\d_6 &= 2.7304 \times 10 \\d_7 &= -1.0422 \times 10^3 \\d_8 &= 7.6678 \times 10^{-1} \\d_9 &= -2.3423 \times 10^3 \\d_{10} &= 3.4341 \times 10^3 \\d_{11} &= -3.3502 \times 10^3 \\d_{12} &= 1.3934 \times 10^3\end{aligned}$$

**Table 3. Distribution of Density and Stability Parameter
in the Liquid Core of PREM**

radius km	ρ kg/m ³	β
0.12215D+04	0.121663D+05	-0.278235D-01
0.12968D+04	0.121267D+05	-0.209509D-01
0.13721D+04	0.120852D+05	-0.150552D-01
0.14474D+04	0.120416D+05	-0.101290D-01
0.15226D+04	0.119959D+05	-0.612425D-02
0.15979D+04	0.119482D+05	-0.297092D-02
0.16732D+04	0.118982D+05	-0.588442D-03
0.17485D+04	0.118460D+05	0.110760D-02
0.18238D+04	0.117916D+05	0.220173D-02
0.18991D+04	0.117347D+05	0.277648D-02
0.19743D+04	0.116755D+05	0.291137D-02
0.20496D+04	0.116139D+05	0.268248D-02
0.21249D+04	0.115497D+05	0.216239D-02
0.22002D+04	0.114829D+05	0.142035D-02
0.22755D+04	0.114136D+05	0.522422D-03
0.23508D+04	0.113416D+05	-0.468165D-03
0.24260D+04	0.112668D+05	-0.149081D-02
0.25013D+04	0.111893D+05	-0.248731D-02
0.25766D+04	0.111089D+05	-0.340165D-02
0.26519D+04	0.110257D+05	-0.417991D-02
0.27272D+04	0.109395D+05	-0.477018D-02
0.28025D+04	0.108503D+05	-0.512257D-02
0.28777D+04	0.107580D+05	-0.518931D-02
0.29530D+04	0.106626D+05	-0.492491D-02
0.30283D+04	0.105641D+05	-0.428642D-02
0.31036D+04	0.104624D+05	-0.323377D-02
0.31789D+04	0.103573D+05	-0.173026D-02
0.32542D+04	0.102490D+05	0.256870D-03
0.33294D+04	0.101373D+05	0.275574D-02
0.34047D+04	0.100221D+05	0.578897D-02
0.34800D+04	0.990344D+04	0.937268D-02

Table 4. Distribution of Density and Stability Parameter
in the Liquid Core of the Modified Earth model

radius km	ρ kg/m ³	β
0.0000D+00	0.124775D+05	-0.116976D-05
0.1200D+03	0.124748D+05	-0.908067D-06
0.2400D+03	0.124665D+05	-0.835353D-06
0.3600D+03	0.124528D+05	-0.818009D-06
0.4800D+03	0.124335D+05	-0.800050D-06
0.6000D+03	0.124086D+05	-0.765664D-06
0.7200D+03	0.123782D+05	-0.716400D-06
0.8400D+03	0.123421D+05	-0.658616D-06
0.9600D+03	0.123004D+05	-0.597716D-06
0.1080D+04	0.122529D+05	-0.536469D-06
0.1200D+04	0.121995D+05	-0.475462D-06
0.1320D+04	0.121403D+05	-0.414311D-06
0.1440D+04	0.120752D+05	-0.352816D-06
0.1560D+04	0.120039D+05	-0.291631D-06
0.1680D+04	0.119265D+05	-0.232358D-06
0.1800D+04	0.118478D+05	-0.177163D-06
0.1920D+04	0.117527D+05	-0.128157D-06
0.2040D+04	0.116560D+05	-0.867970D-07
0.2160D+04	0.115527D+05	-0.535222D-07
0.2280D+04	0.114425D+05	-0.277709D-07
0.2400D+04	0.113253D+05	-0.835182D-08
0.2520D+04	0.112008D+05	0.596607D-08
0.2640D+04	0.110690D+05	0.159118D-07
0.2760D+04	0.109295D+05	0.215309D-07
0.2880D+04	0.107822D+05	0.226098D-07
0.3000D+04	0.106267D+05	0.196996D-07
0.3120D+04	0.104629D+05	0.152089D-07
0.3240D+04	0.102904D+05	0.131259D-07
0.3360D+04	0.101089D+05	0.144800D-07
0.3480D+04	0.991810D+04	0.344820D-08

Table 5. Distribution of Ellipticity in the Modified Earth Model

radius km	ellipticity
0.00000D+00	0.2461D-02
0.18316D+03	0.2461D-02
0.36632D+03	0.2462D-02
0.54947D+03	0.2463D-02
0.73263D+03	0.2464D-02
0.91579D+03	0.2466D-02
0.10989D+04	0.2469D-02
0.12821D+04	0.2472D-02
0.14653D+04	0.2475D-02
0.16484D+04	0.2479D-02
0.18316D+04	0.2483D-02
0.20147D+04	0.2488D-02
0.21979D+04	0.2494D-02
0.23811D+04	0.2500D-02
0.25642D+04	0.2507D-02
0.27474D+04	0.2514D-02
0.29305D+04	0.2522D-02
0.31137D+04	0.2531D-02
0.32968D+04	0.2541D-02
0.34800D+04	0.2551D-02
0.34800D+04	0.2551D-02
0.36865D+04	0.2574D-02
0.38930D+04	0.2614D-02
0.40995D+04	0.2665D-02
0.45125D+04	0.2783D-02
0.47190D+04	0.2846D-02
0.49255D+04	0.2909D-02
0.51320D+04	0.2972D-02
0.53385D+04	0.3034D-02
0.57515D+04	0.3153D-02
0.59580D+04	0.3212D-02
0.61645D+04	0.3273D-02
0.63710D+04	0.3335D-02

CHAPTER 3: THE THREE POTENTIAL DESCRIPTION OF LIQUID CORE DYNAMICS

In section 3.1 we give the derivation of the THPD following unpublished notes by Rochester, and discuss its advantages. In section 3.2 we include the effect of ellipticity in the equations.

3.1 The Three Potential Description

First we define two scalar variables

$$\zeta = \alpha^2 \nabla \cdot \mathbf{u} \quad (3.1)$$

$$\chi = \frac{p_1}{\rho_0} - V_1 \quad (3.2)$$

Using these variables and equation (1.6) we can rewrite equations (1.17)-(1.19) as

$$\omega^2 \mathbf{u} - 2i\omega \Omega \hat{\mathbf{e}}_3 \times \mathbf{u} = \nabla \chi + \frac{\beta \zeta}{\alpha^2} \mathbf{g}_0 \quad (3.3)$$

$$\frac{\alpha^2}{4\pi G \rho_0} \nabla^2 V_1 = \beta \zeta - (1 - \beta)(\chi + V_1) \quad (3.4)$$

$$\mathbf{u} \cdot \mathbf{g}_0 = -(\zeta + \chi + V_1) \quad (3.5)$$

We cross multiply $\hat{\mathbf{e}}_3$ with equation (3.3) to get

$$\omega^2 \hat{e}_3 \times u - 2i\omega \Omega [\hat{e}_3 \hat{e}_3 \cdot u - u] = \hat{e}_3 \times \nabla \chi + \frac{\beta \zeta}{\alpha^2} \hat{e}_3 \times \mathbf{g}_0 \quad (3.6)$$

To find $\hat{e}_3 \cdot u$ we take the dot product of \hat{e}_3 with equation (3.3) to get

$$\hat{e}_3 \cdot u = \frac{1}{\omega^2} [\hat{e}_3 \cdot \nabla \chi + \frac{\beta \zeta}{\alpha^2} \hat{e}_3 \cdot \mathbf{g}_0] \quad (3.7)$$

Using (3.6) and (3.7), the momentum equation can be written as

$$\omega^2 (1 - \sigma^2) u = \Gamma_p \cdot \nabla \chi - \frac{\beta C \cdot \zeta}{\alpha^2} \quad (3.8)$$

with $\sigma = 2\Omega/\omega$ and

$$\Gamma_p = 1 - \sigma^2 \hat{e}_3 \hat{e}_3 + i\sigma \hat{e}_3 \times \mathbf{1} \quad (3.9)$$

$$C = -\mathbf{g}_0 + \sigma^2 \hat{e}_3 \mathbf{g}_0 \hat{e}_3 + i\sigma \hat{e}_3 \times \mathbf{g}_0 \quad (3.10)$$

It should be emphasized that our σ is the inverse of the definition used by Smylie and Rochester (1981), Wu and Rochester (1990). Taking divergence of (3.8) and using (3.2) we get

$$\omega^2 (1 - \sigma^2) \frac{\zeta}{\alpha^2} = \nabla \cdot (\Gamma_p \cdot \nabla \chi) - \nabla \cdot \left(\frac{\beta C \cdot \zeta}{\alpha^2} \right) \quad (3.11)$$

Taking the dot product of (3.8) with \mathbf{g}_0 we get

$$\omega^2 (1 - \sigma^2) u \cdot \mathbf{g}_0 = -C \cdot \nabla \chi + \frac{\beta}{\alpha^2} [g_0^2 - \sigma^2 (\hat{e}_3 \cdot \mathbf{g}_0)^2] \zeta \quad (3.12)$$

Substituting for $u \cdot \mathbf{g}_0$ from (3.5) into (3.12) we get

$$\omega^2(1-\sigma^2)(\chi+V_1)+\frac{\beta B}{\alpha^2}\zeta=C\cdot\nabla\chi \quad (3.13)$$

where

$$B=\frac{\omega^2 \alpha^2(1-\sigma^2)}{\beta}+\mathbf{g}_0\cdot\mathbf{g}_0-\sigma^2(\mathbf{g}_0\cdot\mathbf{g}_0)^2 \quad (3.14)$$

Substituting (3.1) and (3.2) into Poisson's equation we get

$$\frac{\alpha^2}{4\pi G\rho_0}\nabla^2 V_1=\beta\zeta-(1-\beta)(\chi+V_1) \quad (3.15)$$

Rewrite equations (3.13), (3.11) and (3.15) as

$$C\cdot\nabla\chi=\omega^2(1-\sigma^2)(\chi+V_1)+\frac{\beta B}{\alpha^2}\zeta \quad (3.16)$$

$$\nabla(\Gamma_p\cdot\nabla\chi)=\nabla\left(\frac{\beta C\cdot\zeta}{\alpha^2}\right)+\omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2} \quad (3.17)$$

$$\nabla^2 V_1=\frac{4\pi G\rho_0}{\alpha^2}[\beta\zeta-(1-\beta)(\chi+V_1)] \quad (3.18)$$

The three *scalar* differential equations above, two of which are of second order and one of first, constitute the THPD. They describe without approximation the dynamics of the inviscid, non-conducting liquid core through three *scalar* potentials χ , ζ and V_1 . We note again that these equations are linearized in the field variables and therefore the Earth's configuration at all time is assumed to be close to its equilibrium configuration.

If the liquid core is assumed to be incompressible and homogeneous, $\zeta/\alpha^2=0$ and

equation (3.17) reduces to the Poincaré equation

$$\nabla(\Gamma_p \cdot \nabla \chi) = \nabla^2 \chi - \sigma^2 (\mathbf{e}_3 \cdot \nabla)^2 \chi = 0 \quad (3.19)$$

and (3.18) reduces to the Poisson equation.

If we solve for ζ in (3.16)

$$\zeta = \frac{\alpha^2}{\beta B} [C \cdot \nabla \chi - \omega^2 (1 - \sigma^2) (\chi + V_1)] \quad (3.20)$$

and substitute into (3.17) and (3.18) we get

$$\begin{aligned} \nabla[\Gamma_p \cdot \nabla \chi - \frac{C \cdot \nabla \chi - \omega^2 (1 - \sigma^2) (\chi + V_1)}{B}] \\ - \frac{\omega^2 (1 - \sigma^2)}{\beta B} [C \cdot \nabla \chi - \omega^2 (1 - \sigma^2) (\chi + V_1)] = 0 \end{aligned} \quad (3.21)$$

$$\frac{\nabla^2 V_1}{4\pi G \rho_0} - \frac{C \cdot \nabla \chi - \omega^2 (1 - \sigma^2) (\chi + V_1)}{B} - \frac{(1 - \beta) (\chi + V_1)}{\alpha^2} = 0 \quad (3.22)$$

Equations (3.21) and (3.22) constitute the two potential description (Wu and Rochester 1990). The subseismic approximation (SSA) is equivalent to dropping the term in $\chi + V_1$. Then (3.21) reduces to the subseismic wave equation (SSWE) [Smylie & Rochester 1981, Rochester 1989].

The advantages of THPD are that:

- (a) all dependent field variables are scalar, so there is no need to introduce the spheroidal/toroidal representation of vector fields;

(b) in a non-neutrally stratified liquid core it is possible that $B=0$ for some frequency ranges i.e. when

$$\frac{\beta}{\alpha^2} [g_0 z_0 - \sigma^2 (g_3 z_0)^2] = \omega^2 (1 - \sigma^2) \quad (3.23)$$

Since B appears in denominators in both equations of TPD (Wu and Rochester 1990), this description is not appropriate at and near these frequency ranges and the equations must be replaced by a more complicated form of TPD. In THPD this disadvantage never occurs;

(c) in THPD B appears only in the combination βB which remains finite as $\beta \rightarrow 0$;

(d) the description involves no derivatives higher than second. In chapter 5 we show that when a Galerkin method is used to solve the governing equations, the divergence theorem can be used to replace the volume integrals involving second derivatives by surface integrals involving first derivatives in a form for which the boundary conditions are 'natural';

(e) THPD can be solved using a variational principle as shown by Rochester.

3.2 Adjustments for Ellipticity

There are different methods of including the effect of ellipticity in the equations of core dynamics. Wu (1993) adopts a coordinate system in which all variables are functions of r_0 , the mean radius of the equipotential on which the field point is positioned.

We find it more appropriate to adopt a procedure similar to, but slightly improving on, that suggested by Smith (1974). This procedure is as follows.

In the interior of the outer core we let all variables be functions of r, θ, ϕ . The governing equations are then integrated over the volume of radius r given by (2.22). In the interior of the outer core

$$\rho(r) \equiv \rho_0(r_0) = \rho_0(r + \frac{2}{3}reP_2) = \rho_0(r) + \frac{2}{3}eP_2 \frac{d\rho_0}{dr} \quad (3.24)$$

$$\alpha(r) \equiv \alpha(r_0) = \alpha(r + \frac{2}{3}reP_2) = \alpha(r) + \frac{2}{3}eP_2 \frac{d\alpha}{dr} \quad (3.25)$$

using Taylor expansion and ignoring terms higher than first order in ellipticity. The expression for the gravitational acceleration, g_0 , is given as

$$g_0(r) = -f[g_0(r) - \frac{2}{3}\Omega^2 r + \frac{2}{3} \frac{d}{dr}(reg_0(r)) P_2] - \frac{2}{3}eg_0(r)P_2^1 \quad (3.26)$$

(see derivation in appendix 1) where $g_0(r)$ in right hand side of (3.26) is the gravitational acceleration at point r in a spherical non-rotating body of radius r .

Substituting (3.24)-(3.26) into (3.16)-(3.18) we get

$$C \cdot \nabla \chi = \omega^2(1 - \sigma^2)(\chi + V_1) + \beta B_1 \zeta \quad (3.27)$$

$$\nabla(\Gamma_p \cdot \nabla \chi) = \nabla(\beta C_1 \zeta) - \omega^2(1 - \sigma^2) \left[1 - \frac{4}{3} \frac{reP_2}{\alpha} \frac{d\alpha}{dr} \right] \frac{\zeta}{\alpha^2} \quad (3.28)$$

$$\nabla^2 V_1 = \frac{4\pi G \rho_0}{\alpha^2} \left[1 - \frac{2}{3} r e P_2 \left(\frac{2}{\alpha} \frac{d\alpha}{dr} - \frac{1}{\rho_0} \frac{d\rho}{dr} \right) \right] [\beta \zeta - (1-\beta)(\chi + V_1)] \quad (3.29)$$

Equations (3.27)-(3.29) are the three scalar differential equations governing the dynamics of the liquid core in the three potential description. Here B_1 and C^*_1 are defined as

$$C^*_1 = \frac{1}{\alpha^2} \left[C^* - \frac{4}{3} \frac{r e P_2}{\alpha} \frac{d\alpha}{dr} \left(g_0 - \frac{2}{3} \Omega^2 r \right) [r^2 - \sigma^2 \cos^2 \theta \delta_3 + i \sigma \delta_3 \times r] \right] \quad (3.30)$$

$$B_1 = \frac{\omega^2(1-\sigma^2)}{\beta} + \frac{1}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[(1-\sigma^2 \cos^2 \theta) \left(g_0 + \frac{4}{3} \frac{d}{dr} (r e g_0) P_2 \right) - \frac{4}{9} \sigma^2 e g_0 (P_2^1)^2 \right] \quad (3.31)$$

$$- \frac{4}{3\alpha} \frac{d\alpha}{dr} g_0 \left(g_0 - \frac{2}{3} \Omega^2 r \right) (1-\sigma^2 \cos^2 \theta) - \frac{2}{3} (1-\sigma^2 \cos^2 \theta) g_0 \Omega^2 r \left[1 - \frac{4}{3\alpha} r e P_2 \frac{d\alpha}{dr} \right]$$

using Taylor expansion and ignoring terms higher than first order in ellipticity. Spherical harmonics are then used to represent the field variables in (2.27)-(2.29) as

$$(\chi, \zeta, V_1) = \sum_{n=|m|}^{\infty} (\chi_n^m, \zeta_n^m, \Phi_n^m) Y_n^m \quad (3.32)$$

In this thesis we consider constant values of β . In more realistic Earth models we could substitute

$$\beta(r) = \beta(r) + \frac{2}{3} r e P_2 \frac{d\beta}{dr} \quad (3.33)$$

in the equations and keep terms to first order in ellipticity, although even this may be an unnecessary refinement since β is so poorly known.

CHAPTER 4: BOUNDARY CONDITIONS ON AN ELLIPSOIDALLY-STRATIFIED EARTH

The boundary conditions require continuity in a number of functions $f(r, \theta, \phi)$ [normal displacement, gravitational potential, gravitational flux, normal stress] across the surfaces (2.22), i.e. continuity in

$$\left[f(r, \theta, \phi) \right]_{r=RHS (2.22)} = f(r_0, \theta, \phi) - \frac{2}{3} r_0 e(r_0) P_2 \left(\frac{\partial f}{\partial r} \right)_{r_0} \quad (4.1)$$

using Taylor expansion to first order in ellipticity. Therefore at the core mantle boundary (CMB) we require continuity in

$$\hat{n} \left[u(r_0) - \frac{2}{3} r_0 e P_2 \frac{\partial u}{\partial r} \right] \quad (4.2)$$

$$V_1(r_0) - \frac{2}{3} r_0 e P_2 \frac{\partial V_1}{\partial r} \quad (4.3)$$

$$\hat{n} \left[\nabla V_1(r_0) - 4\pi G \rho_0 u(r_0) - \frac{2}{3} r_0 e P_2 \left(\frac{\partial}{\partial r} \nabla V_1 - 4\pi G \rho_0 \frac{\partial u}{\partial r} \right) \right] \quad (4.4)$$

$$\hat{n} \left[\bar{\tau}(r_0) - \frac{2}{3} r_0 e P_2 \frac{\partial \bar{\tau}}{\partial r} \right] \quad (4.5)$$

where for brevity we write

$$\left(\frac{\partial f}{\partial r} \right)_{r=r_0} \equiv \frac{\partial f}{\partial r} \quad (4.6)$$

We use conservation of the Earth's total angular momentum in place of (4.5). To satisfy (4.2)-(4.4) we need information from the dynamics of the liquid core as well as

those of the mantle. We have already dealt with the dynamics of the outer core in chapters 1 and 3. The dynamics of the mantle are dealt with in this chapter. In section 4.1 the expressions for \mathbf{u} and V_1 in the mantle are derived; in section 4.2 we derive the expression for the conservation of the Earth's total angular momentum and in section 4.3 we expand the boundary conditions (4.2)-(4.4). For lack of time we implement these boundary conditions only for the Earth model described in chapter 2. However, these boundary conditions can easily be adjusted to include the inner core boundary (ICB) as well.

4.1 Dynamics of a Rigid Wobbling Mantle

In section 1.1 we chose a reference frame which rotates steadily with angular velocity $\Omega \hat{\mathbf{e}}_3$. We let a set of principal axes $\hat{\mathbf{E}}_i$ fixed in the (supposedly rigid) mantle rotate with angular velocity

$$\boldsymbol{\omega}_m = \Omega(\hat{\mathbf{e}}_3 + \mathbf{m}) \quad (4.7)$$

where \mathbf{m} is purely oscillatory. Therefore,

$$\frac{d\hat{\mathbf{E}}}{dt} = \boldsymbol{\omega}_m \times \hat{\mathbf{E}} \quad (4.8)$$

In this thesis we are concerned with pure wobble, with no change in the length of the day, therefore we set $\hat{\mathbf{E}}_3 \cdot \boldsymbol{\omega}_m = \Omega$, i.e. $m_3 = 0$. Therefore

$$m = \gamma(\hat{e}_1 - i\hat{e}_2)e^{i\omega t} \quad (4.9)$$

where γ is real and positive, provided we adopt the convention that $\omega > 0$ corresponds to a prograde wobble and $\omega < 0$ to a retrograde one. A particle dm at \mathbf{r} in the wobbling mantle will have a velocity relative to the reference frame given by

$$\mathbf{v} = (\omega_{\mathbf{m}} - \omega_{\mathbf{p}}) \times \mathbf{r} = \Omega \mathbf{m} \times \mathbf{r} \quad (4.10)$$

Now

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} = i\omega \mathbf{u} \quad (4.11)$$

Therefore we can write the displacement \mathbf{u} in the disturbed mantle as

$$\mathbf{u} = \frac{\Omega \mathbf{m} \times \mathbf{r}}{i\omega} \quad (4.12)$$

In terms of \hat{e}_j , $\hat{\mathbf{E}}_i$ can be written as

$$\hat{\mathbf{E}}_i = \sum_j a_{ij} \hat{e}_j \quad (4.13)$$

Since $\hat{\mathbf{E}}_i$ remain in close alignment with \hat{e}_i , we can write

$$a_{ij} = \delta_{ij} + \beta_{ij} \quad (4.14)$$

where β_{ij} are of order γ . Substituting (4.14) and (4.13) into (4.8) we get

$$\frac{d\beta_{ij}}{dt} = \Omega \sum_k \epsilon_{ijk} m_k \quad (4.15)$$

to first order in small quantities. ϵ_{ijk} in (4.15) is the standard permutation symbol. From (4.10) and (4.15) we conclude that

$$\begin{aligned} \beta_{11} &= \beta_{22} = \beta_{33} = 0, \\ \beta_{12} &= 0 \quad (\text{since } m_3 = 0), \\ (\beta_{31}, \beta_{32}) &= -\frac{\Omega}{\omega} \gamma(1, -i) e^{i\omega t} \end{aligned} \quad (4.16)$$

At a point r in the disturbed mantle the Eulerian perturbation in the gravitational potential V_1 is

$$V_1 = V_1(m) + V_1(lc) \quad (4.17)$$

where $V_1(m)$ and $V_1(lc)$ are the contributions to V_1 from rearrangement of mass of the mantle and liquid core respectively. Outside the liquid core $V_1(lc)$ must be a solution of Laplace's equation and must vanish as r tends to infinity, so we write

$$V_1(lc) = \sum_{n=1}^N \sigma_n \left(\frac{b}{r}\right)^{n+1} P_n^{-m}(\cos\theta) e^{-im\phi} \quad (4.18)$$

where b is the mean radius of the CMB and is used as a scale factor so that σ_n have the dimensions of potential. The factor $e^{i\omega t}$ is dropped from (4.18) for convenience.

To evaluate $V_1(m)$ we proceed as follows. In the equilibrium configuration the gravitational potential due to the mantle, at a point $\mathbf{r}=(r,\theta,\phi)$ in the mantle, is

$$\begin{aligned}
 V_0(r) = 4\pi G \left(\int_b^r \rho'_0 \left[r'_0 - \frac{2}{15} \frac{d}{dr'_0} (r'_0{}^5 e(r'_0))}{r^3} P_2(\cos\theta) \right] dr'_0 \right. \\
 \left. + \int_r^R \rho'_0(r'_0) \left[r'_0 - \frac{2}{15} \frac{de(r'_0)}{dr'_0} r^2 P_2(\cos\theta) \right] dr'_0 \right)
 \end{aligned} \tag{4.19}$$

using D.10 but replacing the lower limit 0 in the first integral by b , the mean radius of the CMB. Here r_0 is the mean radius of the spheroid of constant density through r . In the wobbling configuration the mass element at r is displaced to $r+u$, where u is given by (4.12). The gravitational potential there, due to the mantle, is

$$V(r+u) = V_0(r) \tag{4.20}$$

since the mantle is taken as rigid. The Eulerian perturbation in gravitational potential at r is

$$V_1(m) \equiv V(r) - V_0(r) \tag{4.21}$$

$$= V_0(r-u) - V_0(r) \tag{4.22}$$

using (4.20)

$$= -u \cdot \nabla V_0 \tag{4.23}$$

to first order in u . Using (4.19) and (4.12) we reduce (4.21) to

$$V_1(m) = -\frac{\Omega}{i\omega} m \phi \frac{\partial V_0}{\partial \theta} \tag{4.24}$$

$$= \frac{16\pi G\Omega}{5\omega r^3} \gamma \left[\int_b^r \rho_0(\xi) \frac{d}{d\xi} (\xi^5 e(\xi)) d\xi + r^5 \int_r^R \rho_0(\xi) \frac{de(\xi)}{d\xi} d\xi \right] P_2^{-1}(\cos\theta) e^{-i\omega t} \quad (4.25)$$

dropping the factor $e^{i\omega t}$ and using (4.19), (4.9) and the result

$$\sin\theta \cos\theta = 2 P_2^{-1} \quad (4.26)$$

4.1 Angular Momentum Conservation of the Earth

The equation for the conservation of Earth's total angular momentum H_E (for our Earth model) is

$$\frac{dH_E}{dt} = \frac{dH_m}{dt} + \frac{dH_c}{dt} = L \quad (4.27)$$

where H_m and H_c are respectively the angular momentum of the mantle and liquid core and L is the external torque exerted on the Earth; $L=0$ for free wobble.

The angular momentum of a wobbling rigid mantle has the form

$$H_m = \int_{\text{mantle}} \xi \times \frac{d\xi}{dt} dm = \int_{\text{mantle}} \xi \times (\omega_m \times \xi) dm = \tilde{I}_m \cdot \omega_m \quad (4.28)$$

where ξ is the position vector of dm in the wobbling mantle and \tilde{I}_m is the mantle's inertia tensor,

$$\tilde{I}_m = \int_{\text{mantle}} (\xi \cdot \xi \tilde{1} - \xi \xi) dm = A_m \tilde{1} + (C_m - A_m) \hat{E}_3 \hat{E}_3 \quad (4.29)$$

since \hat{E}_i are principal axes for the wobbling mantle. Substituting for \hat{E}_i from (4.13) we get

$$\tilde{I}_m = A_m \tilde{I} + (C_m - A_m) [\hat{e}_3 \hat{e}_3 + \sum_k \beta_{3k} (\hat{e}_3 \hat{e}_k + \hat{e}_k \hat{e}_3)] \quad (4.30)$$

where \tilde{I} is the unit dyadic. Substituting (4.30) and (4.16) into (4.28) we find

$$H_m = \Omega (C_m \hat{e}_3 + A_m [1 - \frac{\Omega}{\omega} e_m] m) \quad (4.31)$$

where A_m and C_m are the moments of inertia of the mantle with respect to an equatorial and a polar axis respectively and e_m is defined as

$$e_m = \frac{C_m - A_m}{A_m} \quad (4.32)$$

$$\begin{aligned} \therefore \frac{dH_m}{dt} &= A_m \Omega (1 - \frac{\Omega}{\omega} e_m) \frac{dm}{dt} = A_m \Omega (1 - \frac{\Omega}{\omega} e_m) (\sum_k \frac{dm_k}{dt} \hat{e}_k + \Omega \hat{e}_3 \times m) \\ &= A_m \Omega (1 - \frac{\Omega}{\omega} e_m) (\omega + \Omega) \hat{e}_3 \times m \end{aligned} \quad (4.33)$$

The angular momentum of the liquid core is

$$H_k = C_k \Omega \hat{e}_3 + \Omega \int_k [2u \cdot r \hat{e}_3 - (u \cdot r + r \cdot u) \hat{e}_3] dm + \int_k r \times v dm \quad (4.34)$$

(Wu 1993). Substituting for u from (4.12) into above equation we get

$$H_k = C_k \Omega \hat{e}_3 + \Omega \int_k [2u \cdot r \hat{e}_3 - (u \cdot r + r \cdot u) \hat{e}_3] dm + i\omega \int_k r \times u dm \quad (4.35)$$

Therefore

$$\frac{dH_k}{dt} = \sum_i \frac{dH_{k,i}}{dt} \hat{e}_i + \Omega \hat{e}_3 \times H_k \quad (4.36)$$

Substituting (4.35) into (4.36) we get

$$\begin{aligned} \frac{dH_k}{dt} = & i\omega \Omega \int_k [2 u \tau \hat{e}_3 - (ur+ru) \hat{e}_3] dm - \omega^2 \int_k r \times u dm \\ & - \Omega^2 \int_k \hat{e}_3 \times [(ur+ru) \hat{e}_3] dm + i\omega \Omega \int_k \hat{e}_3 \times (r \times u) dm \end{aligned} \quad (4.37)$$

In a slightly elliptical liquid core the displacement u takes the form

$$\omega^2(1-\sigma^2)u = \Gamma_p \cdot \nabla \chi - \beta C_1^* \zeta \quad (4.38)$$

using (3.8).

We substitute (4.33), (4.37) into (4.27) and use (4.38) and the following two identities

$$P_2 P_n^m = A_n^m P_{n-2}^m + B_n^m P_n^m + C_n^m P_{n+2}^m \quad (4.39)$$

$$P_2^1 \frac{dP_n^m}{d\theta} = 2(n+1)A_n^m P_{n-2}^m + 3B_n^m P_n^m - 2nC_n^m P_{n+2}^m \quad (4.40)$$

where

$$A_n^m = \frac{3(n+m)(n+m-1)}{2(2n+1)(2n-1)}, \quad B_n^m = \frac{n(n+1)-3m^2}{(2n+3)(2n-1)}, \quad (4.41)$$

$$C_n^m = \frac{3(n+2-m)(n+1-m)}{2(2n+3)(2n+1)},$$

and define

$$Sg = g_0 - \frac{2}{3}\Omega^2 r - \frac{2}{3}eg_0$$

(4.42)

$$Se = \frac{2}{3} \left[r \frac{d}{dr} (eg_0) - eg_0 + re \left(g_0 - \frac{2}{3}\Omega^2 r \right) \left(\frac{1}{\rho} \frac{d\rho}{dr} - \frac{2}{\alpha} \frac{d\alpha}{dr} \right) \right]$$

to obtain the final form of (4.27):

$$\begin{aligned} & 2 A_m \Omega \gamma \left(1 - \frac{\Omega}{\omega} e_m \right) (\omega + \Omega) - \frac{\pi \sigma^2 (2 + \sigma)}{15(1 + \sigma)} \int_0^b [\rho_0 \left(\frac{d\chi_2^{-1}}{dr} + \frac{3\chi_2^{-1}}{r} \right) \\ & + \frac{2}{3} re \frac{d\rho_0}{dr} \left(A_4^{-1} \left(\frac{d\chi_4^{-1}}{dr} + 5 \frac{\chi_4^{-1}}{r} \right) + (5B_2^{-1} - 2) \frac{\chi_2^{-1}}{r} + B_2^{-1} \frac{d\chi_2^{-1}}{dr} \right. \\ & + \left. \frac{3}{2} \frac{\omega}{\Omega} \left(\frac{\omega}{\Omega} + 1 \right) \frac{\chi_2^{-1}}{r} \right] r^3 dr - \frac{2}{3} b^4 e \rho_0(b) \left(A_4^{-1} \left(\frac{d\chi_4^{-1}}{dr} + 5 \frac{\chi_4^{-1}}{r} \right) \right. \\ & + \left. (5B_2^{-1} - 2) \frac{\chi_2^{-1}}{r} + B_2^{-1} \frac{d\chi_2^{-1}}{dr} + \frac{3}{2} \frac{\omega}{\Omega} \left(\frac{\omega}{\Omega} + 1 \right) \frac{\chi_2^{-1}}{r} \right) + \int_0^b \frac{\beta \rho_0}{\alpha^2} (Sg \zeta_2^{-1} \\ & + Se (A_4^{-1} \zeta_4^{-1} + B_2^{-1} \zeta_2^{-1}) - \left(\frac{\omega}{2\Omega} + 1 \right) \left(\frac{\omega}{\Omega} - 1 \right) eg_0 \zeta_2^{-1}) r^3 dr \\ & - \frac{2}{3} b^4 e(b) \left(\frac{\beta \rho_0}{\alpha^2} \left(g_0 - \frac{2}{3}\Omega^2 r \right) (A_4^{-1} \zeta_4^{-1} + B_2^{-1} \zeta_2^{-1}) \right) = 0 \end{aligned} \quad (4.43)$$

Here we have made use of the orthogonality relations among associated Legendre functions. Expressions for A_m and e_m are given in appendix 1. Expression (4.43) indicates that only quantities of azimuthal order number $m=-1$ contribute to angular momentum in our representation.

4.3 Expansion of the Boundary Conditions

We now turn to the boundary conditions (4.2)-(4.4). We treat \hat{n} as pointing opposite to \mathbf{g}_0 on the core-mantle boundary. The first boundary condition requires

$$\left[\hat{n} \left(u(r_0) - \frac{2}{3} r e P_2 \frac{\partial u}{\partial r} \right) \right]_{(b^-, \theta, \phi)} = \left[\hat{n} \left(u(r_0) - \frac{2}{3} r e P_2 \frac{\partial u}{\partial r} \right) \right]_{(b^+, \theta, \phi)} \quad (4.44)$$

where b^- and b^+ denote the fluid and solid side of the CMB respectively. Substituting for u in the mantle from (4.12) and using

$$\hat{n} = \hat{r} - 4\epsilon(r) P_2^{-1} \hat{\theta} \quad (4.45)$$

(Smith 1974) and keeping terms up to first order in ellipticity we get

$$\left[\hat{n} \left(u(r_0) - \frac{2}{3} r e P_2 \frac{\partial u}{\partial r} \right) \right]_{b^-, \theta, \phi} = \frac{4\Omega\gamma}{\omega} \epsilon(b) P_2^{-1} e^{-\psi} \quad (4.46)$$

Since we are making use of the 'natural' character of the boundary conditions in the next chapter we need not expand (4.46) any farther.

At the CMB, equation (4.3) implies

$$\left[V_1(r_0) - \frac{2}{3} r e(r) P_2 \frac{\partial V_1}{\partial r} \right]_{(b^-, \theta, \phi)} = \left[V_1(r_0) - \frac{2}{3} r e(r) P_2 \frac{\partial V_1}{\partial r} \right]_{(b^+, \theta, \phi)} \quad (4.47)$$

Using (4.18), (4.25), (3.29) and keeping terms up to first order in ellipticity, (4.47) reduces to

$$\begin{aligned}
& \frac{6G\Omega}{\omega} \gamma \left[\frac{8\pi b^2}{15} \int_b^R \rho_0 \frac{de}{dr} dr \right] \delta_{n2} + \sigma_n \\
& + \frac{2}{3} \epsilon(b) [(n+3)A_{n+2}^{-1} \sigma_{n+2} + (n+1)B_n^{-1} \sigma_n + (n-1)C_{n-2}^{-1} \sigma_{n-2}] \\
& = \Phi_n^{-1} - \frac{2}{3} b \epsilon(b) \left[A_{n+2}^{-1} \left(\frac{d\Phi_{n+2}^{-1}}{dr} \right)_b + B_n^{-1} \left(\frac{d\Phi_n^{-1}}{dr} \right)_b + C_{n-2}^{-1} \left(\frac{d\Phi_{n-2}^{-1}}{dr} \right)_b \right]
\end{aligned} \tag{4.48}$$

using the linear independence of the associated Legendre functions. In (4.48) $\delta_{n2}=1$ if $n=2$ and 0 otherwise. Also at the CMB the boundary condition (4.4) requires

$$\begin{aligned}
& \left[\frac{\partial V_1}{\partial r} - \frac{2}{3} r \epsilon(r) P_2 \frac{\partial^2 V_1}{\partial r^2} + \frac{2\epsilon(r)}{3r} P_2 \frac{\partial V_1}{\partial \theta} \right]_{(b^+, \theta)} \\
& = \frac{16\pi G\Omega}{\omega} \gamma \epsilon(b) b [\rho_0(b^+) - \rho_0(b^-)] P_2^{-1} e^{-\theta}
\end{aligned} \tag{4.49}$$

where we have made use of the continuity of $\hat{n} \cdot \mathbf{u}(r)$ across CMB. Using (4.18), (4.25), (4.39) and (4.40), (4.49) can be written

$$\begin{aligned}
& \frac{16\pi G\Omega b}{5\omega} \left[2 \int_b^R \rho_0 de + 5\rho_0(b^-) \epsilon(b^-) \right] \gamma + \frac{\sigma_n}{b} \left[-(n+1) - \frac{2}{3} \epsilon P_2(n+1)(n+2) \right] P_n^{-1} \\
& + \frac{2}{3} \epsilon P_2^{-1} \frac{dP_n^{-1}}{d\theta} \Big|_b - \left[\left(\frac{d\Phi_n^{-1}}{dr} - \frac{2}{3} r \epsilon P_2 \frac{d^2 \Phi_n^{-1}}{dr^2} \right) P_n^{-1} + \frac{2}{3} \frac{\epsilon}{r} \Phi_n^{-1} P_2 \frac{dP_n^{-1}}{d\theta} \right] \Big|_b = 0
\end{aligned} \tag{4.50}$$

In the next chapter it will become clear why we have not removed P_n^{-1} from equation (4.50).

CHAPTER 5: THE GALERKIN METHOD AND THE NUMERICAL INTEGRATION OF THE GOVERNING EQUATIONS

In this chapter we use the Galerkin method to obtain approximate solutions of the governing equations of core dynamics. In section 5.1 we give a summary of the Galerkin method applicable to a system of simultaneous PDEs subject to boundary conditions. In section 5.2 this method is applied to the governing equations and in section 5.3 we describe a method which we use to integrate the Galerkin equations with respect to θ .

5.1 The Galerkin Method

Suppose we have a set of functions

$$\chi = (\chi_1, \chi_2, \dots, \chi_N) \quad (5.1)$$

which satisfies, in a region V , the set of simultaneous PDEs

$$\sum_{j=1}^N L_{ij} \chi_j = 0 \quad (5.2)$$

for every i ($i=1, \dots, N$) where L_{ij} are linear partial differential operators. Suppose also that there are a number of associated boundary conditions which have to be satisfied on the boundary S of V

$$\sum_{j=1}^N B_{ij} \chi_j = 0 \quad (5.3)$$

for every i ($i=1, \dots, N$) where B_{ij} are linear (possibly partial differential) operators.

Using a basis set f_k , $k=1, \dots, L$, we introduce trial functions

$$\chi_j = \sum_{k=1}^L C_{jk} f_k \quad (j=1, \dots, N) \quad (5.4)$$

which need not a priori satisfy the boundary conditions. The Galerkin method tries to make $\sum_j L_{ij} \chi_j$ as nearly null as possible by requiring

$$\sum_{j=1}^N \sum_{k=1}^L \int_V f_l^* L_{ij} C_{jk} f_k dV = 0 \quad (5.5)$$

for $l=1, \dots, L$, where * denotes the complex conjugate. The resulting set of linear homogeneous Galerkin equations have the form

$$\sum_{j=1}^N \sum_{k=1}^L H_{ljk} C_{jk} = 0 \quad (5.6)$$

with

$$H_{ljk} = \int_V f_l^* L_{ij} f_k dV \quad (5.7)$$

Since in general the trial functions do not a priori satisfy the boundary conditions, we choose a set of basis functions ψ_i equal in number to those used in constructing the trial functions χ_j and extend (5.5) to require that

$$\sum_{j=1}^N \sum_{k=1}^L \left[\int_V f_l^* L_{ij} (C_{jk} f_k) dV + \int_S \psi_i^* B_{ij} (C_{jk} f_k) ds \right] = 0 \quad (5.8)$$

The Galerkin equations then take the form

$$\sum_{k=1}^L \sum_{j=1}^N F_{ijk} C_{jk} = 0 \quad (5.9)$$

where

$$F_{ijk} = H_{ijk} + \int_S \psi_l^* B_{ij} f_k dS \quad (5.10)$$

By proper choice of weight functions, and by use of the divergence theorem, it may be possible to remove the surface integral from the boundary conditions (i.e. by choosing ψ_l so that the surface integral arising from the boundary conditions is cancelled by the surface integral produced by using the divergence theorem.) When this is possible, the boundary conditions are called *natural*.

We can write (5.7) in matrix form by setting up one-to-one correspondence $m=N(l-1)+i$, $n=N(k-1)+j$, so $C_{jk}=a_n$, $H_{ijk}=G_{mn}$. Then (5.7) becomes

$$\sum_n G_{mn} a_n = 0, \quad m=1, \dots, LN. \quad (5.11)$$

\tilde{G} is then a $(LN) \times (LN)$ matrix which hereafter we refer to as the coefficient matrix.

5.2 Application of the Galerkin Method to THPD

Since only spherical harmonics of order -1 are needed to represent the field variables associated with wobble/nutation, we represent the functions defined in chapter 3 as:

$$(\chi, \zeta, \psi) = \sum_{n=1}^N (\chi_n^{-1}, \zeta_n^{-1}, \Phi_n^{-1}) Y_n^{-1} \quad (5.12)$$

where N denotes the degree of truncation. We construct the trial functions as follows:

$$\zeta_n^{-1} = \sum_{l=1}^L a_{[L(n-1)+l]} f_l(r) \quad (5.13)$$

$$\chi_n^{-1} = \sum_{l=1}^L a_{[LN+n-1+l]} f_l(r) \quad (5.14)$$

$$\Phi_n^{-1} = \sum_{l=1}^L a_{[L(2N+n-1)+l]} f_l(r) \quad (5.15)$$

where a_l are constant and f_l are functions of r only. Before proceeding further we should note how we integrate the equations over ellipsoidal volumes. Suppose we have an integral over the liquid core volume,

$$I = \int_k f(r) g(\theta) dv \quad (5.16)$$

To first order in ellipticity we can write this as

$$I = 2\pi \int_0^{\pi} \int_0^{b(1-\frac{2}{3}\epsilon(b)P_2)} f(r) g(\theta) r^2 \sin\theta d\theta dr \quad (5.17)$$

after integrating over ϕ . To first order in ellipticity we can write (5.17) as

$$I = 2\pi \int_0^\pi g(\theta) \left[\int_0^b f(r) r^2 dr - \frac{2}{3} b^3 \epsilon(b) f(b) P_2(\cos\theta) \right] \sin\theta d\theta \quad (5.18)$$

The volume integral in (5.18) is over a sphere of radius b and the surface integral corrects this volume integral for an ellipsoidal volume of radius $b[1 - 2/3 \epsilon(b) P_2(\cos \theta)]$.

5.2.1 Galerkin Formulation of the Entropy Equation

Rewrite equation (3.27) as

$$C \cdot \nabla \chi - \beta B_1 \zeta - \omega^2 (1 - \sigma^2) (\chi + V_1) = 0 \quad (5.19)$$

The Galerkin representation of (5.19) is

$$\sum_{n=1}^N \sum_{l=1}^L \int f_k P_j^{-1} e^{i\phi} a_{L(N+n-1),l} [C \cdot \nabla (f_l P_n^{-1} e^{-i\phi}) - \omega^2 (1 - \sigma^2) f_l P_n^{-1} e^{-i\phi}] - [\beta B_1 a_{L(n-1),i} + \omega^2 (1 - \sigma^2) a_{L(2N+n-1),i}] (f_l P_n^{-1} e^{-i\phi}) dv = 0 \quad (5.20)$$

for each pair of indices (k, j) , $k=1, \dots, L$, $j=1, \dots, N$. We have not added any of the boundary conditions to equation (5.20). Substituting for C and B_1 from (3.10), (3.26) and integrating over ϕ equation (5.20) becomes

$$\begin{aligned}
& 2\pi \left[\sum_{l=1}^L \sum_{n=1}^N \int_0^{b_n} \int_0^{\pi} f_k P_j^{-1} [a_{L(N+n-1)} \cdot (s_1 P_n^{-1} + s_2 P_2 P_n^{-1} + s_3 P_2^{-1} \frac{dP_n^{-1}}{d\theta} \right. \\
& \quad + s_4 (P_2)^2 P_n^{-1} + s_5 P_2 P_2^{-1} \frac{dP_n^{-1}}{d\theta}) - a_{L(n-1)} \cdot (t_1 P_n^{-1} + t_2 P_2 P_n^{-1} \\
& \quad + t_4 (P_2)^2 P_n^{-1}) - \omega^2 (1 - \sigma^2) a_{L(2N+n-1)} \cdot f_l P_n^{-1}] r^2 \sin\theta d\theta dr \\
& \quad \left. - \frac{2}{3} r e(b) \int_0^{\pi} P_2 \langle f_k P_j^{-1} [a_{L(N+n-1)} \cdot (s_6 P_n^{-1} + s_7 P_2 P_n^{-1} \right. \\
& \quad \left. + s_8 P_2^{-1} \frac{dP_n^{-1}}{d\theta}) - a_{L(N+n-1)} \cdot (t_6 P_n^{-1} + t_7 P_2 P_n^{-1})] \rangle b^2 \sin\theta d\theta \right] = 0
\end{aligned} \tag{5.21}$$

for each (k,j). In the next section we describe a method which we use to integrate with respect to θ functions of the type (5.21). Different terms in (5.21) are defined as follows:

$$s_1 = (g_0 - \frac{2}{3} \Omega^2 r) [(1 - \sigma^2) \frac{df_l}{dr} - \frac{\sigma}{r} f_l] + \frac{2}{3} e g_0 (-\frac{\sigma^2}{3} \frac{df_l}{dr} + \frac{\sigma}{r} f_l) \tag{5.22}$$

$$\begin{aligned}
s_2 = & -\frac{2\sigma^2}{3} (g_0 - \frac{2}{3} \Omega^2 r) \frac{df_l}{dr} + \frac{2}{3} \frac{d}{dr} (r e g_0) [(1 - \frac{1}{3} \sigma^2) \frac{df_l}{dr} - \frac{\sigma}{r} f_l] \\
& + \frac{2}{3} e g_0 (-\frac{\sigma^2}{3} \frac{df_l}{dr} + \frac{2\sigma}{r} f_l)
\end{aligned} \tag{5.23}$$

$$s_3 = [-\frac{\sigma^2}{3r} (g_0 - \frac{2}{3} \Omega^2 r) + \frac{2}{3r} e g_0 (1 - \frac{2}{3} \sigma^2)] f_l \tag{5.24}$$

$$s_4 = -\frac{4\sigma^2}{9} r \left(g_0 \frac{de}{dr} + e \frac{dg_0}{dr} \right) \frac{df_i}{dr} \quad (5.25)$$

$$s_5 = \frac{2\sigma^2}{9} \left[e g_0 - r \left(g_0 \frac{de}{dr} + e \frac{dg_0}{dr} \right) \right] \frac{f_i}{r} \quad (5.26)$$

$$s_6 = \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[(1 - \sigma^2) \frac{df_i}{dr} - \frac{\sigma}{r} f_i \right] \quad (5.27)$$

$$s_7 = -\frac{2\sigma^2}{3} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \frac{df_i}{dr} \quad (5.28)$$

$$s_8 = -\frac{\sigma^2}{3r} \left(g_0 - \frac{2}{3} \Omega^2 r \right) f_i \quad (5.29)$$

$$t_1 = \left(\omega^2 (1 - \sigma^2) + \frac{\beta}{\alpha^2} \left[\left(1 - \frac{\sigma^2}{3} \right) \left(g_0^2 - \frac{4}{3} g_0 \Omega^2 r \right) - \frac{8}{9} e g_0 \left(g_0 - 2 \Omega^2 r \right) \sigma^2 \right] \right) f_i \quad (5.30)$$

$$t_2 = \frac{\beta}{\alpha^2} \left[-\frac{2}{3} \left(\sigma^2 + \frac{2re}{\alpha} \frac{d\alpha}{dr} \left(1 - \frac{\sigma^2}{3} \right) \right) \left(g_0 - \frac{4}{3} \Omega^2 r \right) g_0 \right. \\ \left. + \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left(\frac{4}{3} \frac{d}{dr} (re g_0) \left(1 - \frac{\sigma^2}{3} \right) - \frac{8\sigma^2}{9} e g_0 \right) \right] f_i \quad (5.31)$$

$$t_4 = \frac{\beta}{\alpha^2} \frac{8\sigma^2}{9} \left[\left(g_0 - \frac{4}{3} \Omega^2 r \right) g_0 \frac{re}{\alpha} \frac{d\alpha}{dr} - \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left(\frac{d}{dr} (re g_0) - 2e g_0 \right) \right] f_i \quad (5.32)$$

$$t_6 = [\omega^2(1-\sigma^2) + \frac{\beta}{\alpha^2}(1-\frac{\sigma^2}{2})(g_0 - \frac{4}{3}\Omega^2 r)g_0]f_i \quad (5.33)$$

$$t_7 = -\frac{2}{3}\frac{\beta}{\alpha^2}\sigma^2 g_0(g_0 - \frac{4}{3}\Omega^2 r)f_i \quad (5.34)$$

with

$$\cos^2\theta = \frac{1}{3}(2P_2 + 1), \quad (P_2^1)^2 = 2[P_2 - 2(P_2)^2 + 1] \quad (5.35)$$

5.2.2 Galerkin Formulation of the Momentum Equation

Equation (3.28) can be written as

$$\nabla \cdot (\Gamma_p \cdot \nabla \chi - \beta C_1^* \zeta) - \omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2} = 0 \quad (5.36)$$

The Galerkin representation of (5.36) is

$$\int_k f_k P_j^{-1} e^{i\omega t} [\nabla \cdot (\Gamma_p \cdot \nabla \chi - \beta C_1^* \zeta) - \omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2}] dv \quad (5.37)$$

$$+ \int_{S_k} \psi_k^* [(\hat{n} \cdot \mathbf{u}) - (\hat{n} \cdot \mathbf{u})] ds = 0$$

where we have added boundary condition (4.2) to the above formulation. + and - in (5.37) denote the fluid and solid sides of the CMB. Applying the divergence theorem to (5.37) and using (4.37) we get

$$\begin{aligned} & \omega^2(1-\sigma^2) \int_{S_k} f_k P_j^{-1} e^{*}(\hat{n} \cdot u) ds - \int_k \nabla(f_k P_j^{-1} e^{*}) \cdot [\Gamma_p \cdot \nabla \chi - \beta C_1^* \zeta] dv \\ & - \omega^2(1-\sigma^2) \int_k f_i P_n^{-1} \frac{\zeta}{\alpha^2} dv + \int_{S_k} \psi_{kj}^* [(\hat{n} \cdot u) - (\hat{n} \cdot u)] ds = 0 \end{aligned} \quad (5.38)$$

Now, by choosing

$$\psi_{kj}^* = -\omega^2(1-\sigma^2) f_k P_j^{-1} e^{*} \quad (5.39)$$

we can make use of the *natural* property of the boundary condition (4.46) and cancel surface integral produced from using the divergence theorem and the one arising from the boundary condition and write (5.38) as

$$\begin{aligned} & \sum_{l=1}^L \sum_{n=1}^N [a_{LN \cdot n-1, l} \left(\int_k \nabla(f_k P_j^{-1} e^{*}) \cdot \Gamma_p \cdot \nabla(f_i P_n^{-1} e^{-*}) dv \right. \\ & \left. - \left[\frac{4\Omega\gamma}{\omega} b\epsilon(b) \right] \omega^2(1-\sigma^2) \int_{S_k} f_k^{-1} P_j^{-1} e^{*} (P_2^{-1} e^{-*}) ds \right) \\ & - a_{L(n-1), l} \int_k [\nabla(f_k P_j^{-1} e^{*}) \cdot \beta C_1^* + \frac{\omega^2}{\alpha^2} (1-\sigma^2) f_k P_j^{-1} e^{*} f_i P_n^{-1} e^{-*} dv] = 0 \end{aligned} \quad (5.40)$$

Substituting for Γ_p and C_1 into (5.40) we get

$$\begin{aligned}
& \sum_{l=1}^L \sum_{n=1}^N [a_{LN+n-1}] \int_k \left(\frac{df_k}{dr} P_j^{-1} [(1-\sigma^2 \cos^2 \theta) \frac{df_l}{dr} P_n^{-1} - \frac{f_l}{3r} P_2^1 \frac{dP_n^{-1}}{d\theta} - \frac{\sigma}{r} f_l P_n^{-1}] \right. \\
& \quad + \frac{f_k}{r} [-\sigma^2 \frac{df_l}{3dr} P_n^{-1} P_2^1 \frac{dP_j^{-1}}{d\theta} - \frac{\sigma^2 \sin^2 \theta}{r} f_l \frac{dP_n^{-1}}{d\theta} \frac{dP_j^{-1}}{d\theta} - \sigma \frac{df_l}{dr} P_n^{-1} P_j^{-1} \\
& \quad - \frac{\sigma}{r} f_l \frac{\cos \theta}{\sin \theta} (P_n^{-1} \frac{dP_j^{-1}}{d\theta} + P_j^{-1} \frac{dP_n^{-1}}{d\theta}) + \frac{f_l}{r} (\frac{dP_n^{-1}}{d\theta} \frac{dP_j^{-1}}{d\theta} + \frac{P_n^{-1} P_j^{-1}}{\sin^2 \theta}) \Big] dv \\
& - a_{LN+n-1} \int_k \left(\frac{\beta}{\alpha^2} [(g_0 - \frac{2}{3} \Omega^2 r + \frac{2}{3} \frac{d}{dr} (r e g_0) P_2^1) [(1-\sigma^2 \cos^2 \theta) \frac{df_k}{dr} P_j^{-1} \right. \\
& \quad - \frac{\sigma^2 P_2^1}{3r} f_k \frac{dP_j^{-1}}{d\theta} + \frac{\sigma}{r} f_k P_j^{-1}] + \frac{2}{3} e g_0 [-\sigma^2 (P_2^1)^2 \frac{df_k}{dr} P_j^{-1} + (1-\sigma^2 \sin^2 \theta) \\
& \quad \times \frac{df_k}{dr} P_2^1 \frac{dP_j^{-1}}{d\theta} - \frac{\sigma}{3r} \cos^2 \theta f_k P_j^{-1}] - \frac{4}{3} \frac{r e}{\alpha} \frac{d\alpha}{dr} P_2^1 (g_0 - \frac{2}{3} \Omega^2 r) \\
& \quad \left. [(1-\sigma^2 \cos^2 \theta) \frac{df_k}{dr} P_j^{-1} - \frac{\sigma^2 P_2^1}{3r} f_k \frac{dP_j^{-1}}{d\theta} + \frac{\sigma}{r} f_k P_j^{-1}] + \omega^2 (1-\sigma^2) \frac{f_k}{\alpha^2} P_j^{-1} \right. \\
& \left. [1 - \frac{4}{3\alpha} r e \frac{d\alpha}{dr} P_2^1] f_l P_n \right) dv - 4\Omega \omega (1-\sigma^2) b e a_{LN+n-1} \int_{s_k} f_k P_j^{-1} P_2^{-1} ds = 0
\end{aligned} \tag{5.41}$$

Here we have labelled $\gamma = a_{3LN+N+1}$. Using (5.34) and some other identity relations among associated Legendre functions (given in appendix 2) we write equation (5.41) as

$$\begin{aligned}
& 2\pi \left[\sum_{l=1}^L \sum_{n=1}^N \int_0^{b\pi} \int_0^\pi f_k P_j^{-1} [a_{L(N+n-1) \cdot l} (w_1 P_n^{-1} + w_2 P_2 P_n^{-1} \right. \\
& + w_3 P_2^{-1} \frac{dP_n^{-1}}{d\theta}) - a_{L(n-1) \cdot l} (z_1 P_n^{-1} + z_2 P_2 P_n^{-1} + z_4 (P_2)^2 P_n^{-1}) \\
& \left. - \omega^2 (1 - \sigma^2) a_{L(N+n-1) \cdot l} f_l P_n^{-1} \right] r^2 \sin\theta d\theta dr + \frac{2}{3} r e(b) \\
& \times \int_0^\pi P_2 \langle f_k P_j^{-1} [a_{L(N+n-1) \cdot l} [w_6 P_2 P_n^{-1} + w_7 P_2 P_n^{-1} + w_8 P_2^{-1} \frac{dP_n^{-1}}{d\theta} \\
& + w_9 (P_2)^2 P_n^{-1} + w_{10} P_2 P_2^{-1} \frac{dP_n^{-1}}{d\theta}] - a_{L(N+n-1) \cdot l} [z_6 P_n^{-1} + z_7 P_2 P_n^{-1} \\
& + z_8 P_2^{-1} \frac{dP_n^{-1}}{d\theta} + z_9 (P_2)^2 P_n^{-1} + z_{10} P_2 P_2^{-1} \frac{dP_n^{-1}}{d\theta}] \rangle b^2 \sin\theta d\theta \\
& - 8\pi \omega (1 - \sigma^2) a_{N(3L+1) \cdot 1} \Omega b e(b) \int_{S_k} f_k P_j^{-1} P_2^{-1} b^2 \sin\theta d\theta = 0
\end{aligned} \tag{5.42}$$

after integrating over ϕ . Different terms in (5.42) are

$$w_1 = \frac{df_k}{dr} \left[\left(1 - \frac{\sigma^2}{3}\right) \frac{df_l}{dr} - \frac{\sigma}{r} f_l \right] + \frac{f_k}{r} \left\langle -\frac{f_l}{r} \left[\frac{2}{3} n(n+1) - 1 \right] \sigma^2 - \sigma \frac{df_l}{dr} - \frac{\sigma f_l}{r} + \frac{f_l}{r} n(n+1) \right\rangle \tag{5.43}$$

$$w_2 = \sigma^2 \left\langle -\frac{2}{3} \frac{df_k}{dr} \frac{df_l}{dr} + \frac{f_k}{r} \left[-2 \frac{df_l}{dr} + \frac{2}{3} n(n+1) \frac{f_l}{r} \right] \right\rangle \tag{5.44}$$

$$w_3 = \sigma^2 \left\langle -\frac{1}{3} \frac{df_k f_l}{dr r} + \frac{f_k}{r} \left[\frac{1}{3} \frac{df_l}{dr} - \frac{2f_l}{3r} \right] \right\rangle \quad (5.45)$$

$$w_6 = -\frac{f_k}{r} \left[\frac{2\sigma^2}{3} \frac{df_l}{dr} - \frac{\sigma}{r} f_l \right] \quad (5.46)$$

$$w_7 = \frac{df_k}{dr} \left[\left(1 - \frac{\sigma^2}{3}\right) \frac{df_l}{dr} \right] + \frac{f_k}{r} \left(-\sigma \frac{df_l}{dr} \right. \\ \left. + \frac{2\sigma^2}{3} \frac{df_l}{dr} + \frac{f_l}{r} \left[-\left(\frac{2}{3}n(n+1) - 1\right)\sigma^2 - 3\sigma + n(n+1) \right] \right) \quad (5.47)$$

$$w_8 = \frac{f_k f_l}{r^2} \left(\frac{2}{3}\sigma^2 - 1 \right) \quad (5.48)$$

$$w_9 = \sigma^2 \left\langle -\frac{2}{3} \frac{df_k}{dr} \frac{df_l}{dr} + \frac{f_k}{r} \left[-\frac{10}{3} \frac{df_l}{dr} - \frac{2}{3} (n(n+1) - 1) \frac{df_l}{dr} \right] \right\rangle \quad (5.49)$$

$$w_{10} = \sigma^2 \left\langle -\frac{df_k f_l}{dr 3r} + \frac{f_k}{r} \left[\frac{1}{3} \frac{df_l}{dr} - \frac{4f_l}{3r} \right] \right\rangle \quad (5.50)$$

$$z_1 = \frac{\beta}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[\left(1 - \frac{\sigma^2}{3}\right) \frac{df_k}{dr} + \frac{\sigma}{r} f_k \right] + \frac{4\sigma^2}{9} \frac{d}{dr} (r e g_0) \frac{f_k}{r} \\ + \frac{2}{3} e g_0 \left[\sigma^2 \left(-\frac{2}{3} \frac{df_k}{dr} + \frac{4f_k}{3r} \right) - \frac{\sigma}{r} f_k - \frac{8\sigma^2}{9\alpha} \frac{d\alpha}{dr} f_k \right] \gamma_l - \frac{\omega^2 (1 - \sigma^2)}{\alpha^2} f_k f_l \quad (5.51)$$

$$\begin{aligned}
z_2 = & \frac{\beta}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[-\frac{2}{3} \frac{df_k}{dr} - \frac{f_k}{r} \right] \sigma^2 + \frac{2}{3} \frac{d}{dr} (re g_0) \left[\left(1 - \frac{\sigma^2}{3} \right) \frac{df_k}{dr} \right. \\
& + \frac{2}{3} \frac{f_k}{r} \frac{\sigma^2}{r} + \frac{\sigma}{r} f_k \left. \right] - \frac{2}{3} e g_0 \left[\frac{2\sigma^2}{3} \frac{df_k}{dr} - \left(1 - \frac{2\sigma^2}{3} \right) \frac{6f_k}{r} - \frac{4\sigma^2}{3} \frac{f_k}{r} + \frac{2\sigma}{r} f_k \right] \\
& - \frac{4}{3} \frac{re}{\alpha} \frac{d\alpha}{dr} \left[\left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[\left(1 - \frac{\sigma^2}{3} \right) \frac{df_k}{dr} + \frac{2\sigma^2}{3} \frac{f_k}{r} + \frac{\sigma}{r} f_k \right] \right] f_i \\
& + \frac{4}{3} \frac{re}{\alpha} \frac{d\alpha}{ar} \frac{\omega^2 (1 - \sigma^2)}{\alpha^2} f_k f_i
\end{aligned} \tag{5.52}$$

$$z_3 = \frac{\beta}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \frac{\sigma^2}{3r} \frac{f_k}{r} - \frac{2}{3} e g_0 \left(1 - \frac{2\sigma^2}{3} \right) \frac{f_k}{r} \Big) f_i \tag{5.53}$$

$$z_5 = \frac{\beta \sigma^2}{\alpha^2} \left(\frac{2}{3} \frac{d}{dr} (re g_0) \frac{f_k}{3r} + \frac{4}{9} e g_0 \frac{f_k}{r} - \frac{4}{3} \frac{re}{\alpha} \frac{d\alpha}{dr} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \frac{f_k}{3r} \right) f_i \tag{5.54}$$

$$z_6 = \frac{2f_k}{3r} \sigma^2 f_i \tag{5.55}$$

$$z_7 = \frac{\beta}{\alpha^2} \left(g_0 - \Omega^2 r \right) \left[\left(1 - \frac{\sigma^2}{3} \right) \frac{df_k}{dr} + \frac{2\sigma^2}{3} \frac{f_k}{r} - \frac{\sigma f_k}{r} \right] f_i + \frac{\omega^2 (1 - \sigma^2)}{\alpha^2} f_k f_i \tag{5.56}$$

$$z_8 = 0 \tag{5.57}$$

$$z_9 = \frac{\beta \sigma^2}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \left[-\frac{2}{3} \frac{df_k}{dr} - \frac{10}{3} \frac{f_k}{r} \right] f_i \tag{5.58}$$

$$z_{10} = \frac{\beta \sigma^2}{\alpha^2} \left(g_0 - \frac{2}{3} \Omega^2 r \right) \frac{f_k f_l}{3r} \quad (5.59)$$

5.2.3 Galerkin Formulation of Poisson's Equation

The Galerkin formulation of equation (3.29) and the corresponding boundary condition (4.50) is

$$\int_k f_k P_j^{-1} \nabla^2 V_1 dv - 4\pi G \int_k \frac{\rho_0}{\alpha^2} f_k P_j^{-1} \left[1 + \frac{2}{3} r e P_2 \left(\frac{1}{\rho_0} \frac{d\rho_0}{dr} - \frac{2}{\alpha} \frac{d\alpha}{dr} \right) \right] [\beta \zeta - (1-\beta)(\chi + V_1)] dv + \int_{s_k} f_k P_j^{-1} [\text{left hand side of (4.50)}] ds = 0 \quad (5.60)$$

Using the definition of the Laplacian operator and some of the identity relations (from appendix 2) among associated Legendre functions and defining

$$gt(r) = \frac{2}{r} \frac{df_l^{-1}}{dr} + \frac{d^2 f_l^{-1}}{dr^2} - \frac{n(n+1)}{r^2} f_l^{-1} \quad (5.61)$$

we rewrite (5.60) to first order in ellipticity as

$$\begin{aligned}
& 2\pi \sum_{n=1}^N \sum_{l=1}^L \int_0^b \int_0^\pi f_k P_j^{-1} \langle a_{LN+n-1, l} g(r) P_n^{-1} + \frac{4\pi G \rho_0 f_l}{\alpha^2} [P_n^{-1} \\
& + \frac{2}{3} re \left(\frac{1}{\rho_0} \frac{d\rho_0}{dr} - \frac{2}{\alpha} \frac{d\alpha}{dr} \right) P_2 P_n^{-1}] \{ (1-\beta)(a_{LN+n-1, l} + a_{L(2N+n-1), l} \\
& - \beta a_{L(n-1), l} \rangle r^2 dr \sin\theta d\theta - \frac{2}{3} be(b) \int_0^\pi f_k P_j^{-1} \langle a_{LN+n-1, l} g(r) \\
& + \frac{4\pi G \rho_0 f_l}{\alpha^2} \rangle \{ (1-\beta)(a_{LN+n-1, l} + a_{L(2N+n-1), l} \\
& - \beta a_{L(n-1), l} \rangle P_2 P_n^{-1} b^2 \sin\theta d\theta - a_{L(2N+n-1), l} \int_0^\pi P_j^{-1} f_k \frac{df_l}{dr} P_n^{-1} \\
& - \frac{2}{3} re \left[f_k \frac{d^2 f_l}{dr^2} + \frac{df_l}{dr} \left(\frac{df_k}{dr} + 2 \frac{f_k}{r} \right) \right] P_2 P_n^{-1} \\
& + \frac{2}{3} \frac{e}{r} f_l P_2^{-1} \frac{d\rho_n^{-1}}{d\theta} \Big] b^2 \sin\theta d\theta \Big] + a_{3LN+n} \int_0^\pi P_j^{-1} [\langle -(n+1) f_k \\
& - \frac{2}{3} re(b) \{ (n+2)(n+1) f_k - (r \frac{df_k}{dr} + 2 f_k)(n+1) \} P_2 \rangle P_n^{-1} \\
& + \frac{2e(b)}{3r} f_k P_2^{-1} \frac{dP_n^{-1}}{d\theta} \Big] b \sin\theta d\theta \Big] + \frac{16\pi G b \Omega}{5\omega} \left[\int_b^R \rho_0 \frac{de}{dr} \right. \\
& \left. + 5\rho_0(b^-) e(b) \right] a_{N(3L+1), 1} f_k(b^-) \int_0^\pi P_j^{-1} P_2^{-1} b^2 \sin^2\theta d\theta = 0
\end{aligned} \tag{5.62}$$

The contribution of $V_1(\omega)$ in the mantle, and the wobble of the mantle, introduce

$N+1$ additional unknown coefficients to the Galerkin formulation of the governing equations. We have labelled these coefficients a_{3LN+n} , $n=1,\dots,N+1$; i.e. every α_n in the boundary conditions corresponds to a_{3LN+n} and γ corresponds to $a_{3LN+N+1}$. To solve for these coefficients we need the two boundary conditions not yet incorporated into the Galerkin equations, namely (4.43) and (4.48). The coefficient matrix of our Galerkin formulation then has $(3LN+N+1) \times (3LN+N+1)$ entries.

5.3 Integration with Respect to θ

Suppose we have a functional of the form

$$I = E_n \int_{r_1}^{r_2} \int_0^\pi \left[F_{1,n}(r) P_n^m + F_{2,n}(r) P_2 P_n^m + F_{3,n}(r) P_2^1 \frac{dP_n^m}{d\theta} \right. \\ \left. + F_{4,n}(r) (P_2)^2 P_n^m + F_{5,n}(r) P_2 P_2^1 \frac{dP_n^m}{d\theta} \right] P_j^m \sin\theta \, d\theta \, dr \quad (5.63)$$

where E_n are constant. We could use a standard double integration technique to evaluate but this method takes too much CPU time, so we look for an alternative technique.

In this thesis we choose to use the orthogonality relations among associated Legendre functions to remove the θ dependence from the equations and then integrate with respect to r . In what follows we need the two identities (4.39) and (4.40), and in addition

$$\begin{aligned}
(P_2)^2 P_n^m &= A_n^m A_{n-2}^m P_{n-4}^m + A_n^m [B_{n-2}^m + B_n^m] P_{n-2}^m + [A_n^m C_{n-2}^m \\
&+ (B_n^m)^2 + C_n^m A_{n-2}^m] P_n^m + C_n^m [B_n^m + B_{n+2}^m] P_{n+2}^m + C_n^m C_{n+2}^m P_{n+4}^m
\end{aligned} \tag{5.64}$$

$$\begin{aligned}
P_2 P_2^1 \frac{dP_n^m}{d\theta} &= 2(n+1)A_n^m A_{n-2}^m P_{n-4}^m + A_n^m [2(n+1)B_{n-2}^m + 3B_n^m] P_{n-2}^m \\
&+ [2(n+1)A_n^m C_{n-2}^m + 3(B_n^m)^2 - 2nC_n^m A_{n+2}^m] P_n^m + C_n^m [3B_n^m - 2nB_{n+2}^m] P_{n+2}^m \\
&- 2nC_n^m C_{n+2}^m P_{n+4}^m
\end{aligned} \tag{5.65}$$

Using these identities (5.63) takes the form

$$I = E_n \int_0^\pi P_n^m P_q^m \sin\theta \, d\theta \int_{r_1}^{r_2} F_n(r) \, dr \tag{5.66}$$

The orthogonality relation can now be applied:

$$\int_0^\pi P_n^{-1} P_q^{-1} \sin\theta \, d\theta = \frac{2}{2q+1} \frac{(q+m)!}{(q-m)!} \delta_{q,n} \tag{5.67}$$

To apply the above technique to the Galerkin formulation of the governing equations we use a procedure which employs a five-step do loop to integrate the equations with respect to θ . The algorithm for this method is given in appendix 3. After integrating the Galerkin equations with respect to θ we let

$$f(r) = \left(\frac{r}{R}\right)^{l-1} \quad (5.68)$$

and the IMSL subroutine dqdag is called to integrate the equations with respect to r .

CHAPTER 6 : DISCUSSION, NUMERICAL RESULTS AND CONCLUSION

The aim of this thesis is to utilize the THPD by computing the periods of the Earth's Chandler wobble, nearly diurnal free wobble and some of the Earth's other inertial modes. In chapter 3 we showed the derivation of the equations constituting THPD and, in the same chapter, we included the effect of ellipticity in the equations. In chapter 4 we expanded the boundary conditions which must be invoked on a wobbling Earth. In chapter 5 we applied the Galerkin method to approximate the solution of the equations.

In this chapter we first apply the Galerkin method to the Poincaré Earth model, for which analytical solutions exist for the modes mentioned above. These results will then be compared to the ones computed using the Earth model described in chapter 2.

6.1 The Poincaré Earth model

The Poincaré Earth model consists of a rigid stratified mantle and an incompressible homogeneous liquid core. The analytical solution (Wu 1993, Rochester lecture notes) yields a period of about 270 days for the Earth's Chandler wobble (CW) and a nutation period of about 350 days for the Earth's nearly diurnal free wobble (NDFW).

The other inertial modes of the above Earth model have solutions of the form

$$(1-x^2) \frac{dP_n^m(x)}{dx} = m \left(\frac{1+f x^2}{1+f} \right)^{\frac{1}{2}} P_n^m(x) \quad (6.1)$$

(Aldridge and Lumb 1987) where

$$x = (\sigma^2 - f)^{-\frac{1}{2}} \quad (6.2)$$

and $f = 1/(1-\epsilon)^2 - 1$, ϵ is the ellipticity of the CMB and has a value of 0.002551 (table 5). We compute some of the Earth's inertial modes using a: equation (6.1) and b: applying the Galerkin method to Poincaré's equation to make sure that the same results are obtained. To compute periods of the Earth's CW and NDFW in the Poincaré model we use the well known analytical results

$$\frac{\Omega}{\omega} = \left[\frac{A_m}{A_E e_E}, -\left(1 + \frac{A_E e}{A_m}\right)^{-1} \right] \quad (6.3)$$

where e_E and A_E are respectively the Earth's dynamical ellipticity and moment of inertia with respect to an equatorial axis. The frequency of the free core nutation (FCN) is then given as

$$\frac{\omega_{FCN}}{\Omega} = 1 + \frac{\omega_{NDFW}}{\Omega} \quad (6.4)$$

As we showed in chapter 3, when the liquid core is assumed to be homogeneous and incompressible, the momentum equation reduces to the Poincaré equation

$$\nabla^2 \chi - \sigma^2 (\mathbf{e}_3 \cdot \nabla)^2 \chi = 0 \quad (6.5)$$

Since χ is the only dependent variable in (6.5) the entropy and Poisson equations are redundant and are removed from the formulation. Also the boundary conditions involving V_1 are redundant [V_1 does not appear explicitly in (6.5)] and hence removed. Therefore we are left with one Galerkin equation (5.43) which includes the boundary condition (4.46) and the boundary condition equivalent to the total angular momentum conservation of the Earth, which we treat as an independent equation to solve for γ (of course in these equations β and ζ/α^2 are both 0). The coefficient matrix of this formulation therefore has $(LN+1) \times (LN+1)$ entries. The roots of the determinant of the coefficient matrix, regarded as a function of frequency ω , correspond to the frequencies of the Earth's oscillations.

Table 6 shows the periods of the normal modes computed using the two methods described above. Except for the periods of the Chandler wobble and the nearly diurnal free wobble which differ slightly from those computed using (6.3) and (6.4), the inertial modes computed using the Galerkin method are identical, to 4 significant figures, with those computed using (6.1).

Table 7 shows the periods of the same modes as in table 6 using the formulation given in chapter 5. Except for the CW all the modes whose periods are reported here are retrograde modes.

6.2 Convergence of the Solutions

For the Poincaré case the convergence of the Galerkin formulation is very fast. For almost all the modes whose periods are reported, the solutions converge when we set $N=2$ and $L=4$. We tested our program for higher values of N and L to make sure of the convergence.

For the Earth model described in chapter 2, the solutions converge for the period of the Chandler wobble when $N=2$ and $L=5$, but for the case of nearly diurnal free wobble convergence required $N=2$ and $L=13$. We tested our program for the cases $N=3$ and 4. The results are almost identical to the case $N=2$. For some of the inertial modes whose periods are listed in table 6.2 we needed higher values of L i.e. $L>12$ for the solutions to converge. For some frequency ranges the determinant of the coefficient matrix becomes unstable when $L>16$ and many false roots appear in the eigenperiod spectrum. For this reason we were not able to achieve convergence for the modes whose periods are not listed in table 7.

6.3 Conclusion

The close agreement between the periods of the modes listed in tables 6 and 7 show that THPD may well be an effective tool in solving the governing equations of core dynamics for the long period free oscillations. Although we considered only the case of

a neutrally stratified liquid core in our computations, the equations and the Galerkin formulations were derived so that they include any value of β .

Future work on this subject should consider a non-neutrally stratified liquid core, include the presence of the solid inner core and take into account the elasticity of the mantle and the solid inner core. This would permit study of the effects of (a) non-neutral stratification of the liquid core on the inertial modes, including those involving wobble /nutation, and (b) truncation at low values of N .

Table 6 Periods of the Chandler Wobble (CW), Nearly Diurnal Free Wobble (NDFW), Free Core Nutation, and some of the other inertial modes of the Poincaré Earth Model

a: Using eq. (6.1), (6.3) and (6.4)

o: Using The Galerkin Method

(periods are in days)

Mode	a	b
CW	272.6	271.7
FCN	347.4	348.6
NDFW	0.99713	0.99714
P_2^{-1}	0.9980	0.9980
P_4^{-1}	0.5851	0.5851
	1.6313	1.6313
P_6^{-1}	0.5371	0.5371
	0.7645	0.7645
	2.2676	2.2676

Table 7 Periods of the same modes as in table 6 using the Earth model described in chapter 2

Mode	Period (in days)
CW	273 .2
FCN	347.4
NDFW	0.99713
P_2^{-1}	0.9973
P_4^{-1}	0.5849
P_6^{-1}	0.5371

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Appendix A

A.1 Derivation of the Expression for A_m

The equation for A_m is given as

$$\begin{aligned} A_m &= \int_{\text{mantle}} r^2 [\cos^2\theta + \sin^2\theta \sin^2\phi] dm \\ &= \int_V r^2 \rho_0(r) [\cos^2\theta + \sin^2\theta \sin^2\phi] dV \end{aligned} \quad (\text{A.1})$$

where θ is the colatitude and ϕ is the longitude. Integrating (A.1) with respect to ϕ we get

$$A_m = 2\pi \int_{\xi_1}^{\xi_2} \int_0^{\pi} r^4 \rho_0(r) [\cos^2\theta + \frac{1}{2} \sin^2\theta] \sin\theta \, d\theta dr \quad (\text{A.2})$$

where ξ_1 and ξ_2 are the radii of the CMB and the Earth's outer surface of the Earth respectively and

$$\int_0^{2\pi} \sin^2\phi \, d\phi = \pi \quad (\text{A.3})$$

Substituting for $\cos^2\theta$ and $\sin^2\theta$ from (5.34) and for r in terms of r_0 from (3.24) into (A.2) we get

$$(A.4) \quad A_m = \frac{2\pi}{3} \int_0^R \int_b^{\pi} \rho_0(r_0) (P_2+2) r_0^4 \left[1 - \frac{2}{3} P_2 \left(5e + r_0 \frac{de}{dr}\right)\right] \sin\theta \, d\theta \, dr$$

Integrating (A.4) with respect to θ we get

$$A_m = \frac{8\pi}{3} \int_b^R \rho_0(r_0) r_0^4 \left[1 - \frac{1}{15} \left(5e + r_0 \frac{de}{dr}\right)\right] dr_0 \quad (A.5)$$

A.2 Derivation of the Expression for $A_m e_m$

Since e_m appears in the equations in the combination $A_m e_m$ only we show the derivation of $A_m e_m$. To derive the expression for $A_m e_m$ we proceed as follows

$$\begin{aligned} A_m e_m &= C_m - A_m = \int_{\text{mantle}} [(r \sin\theta \cos\phi)^2 - (r \cos\theta)^2] dm \\ &= \int_{V_m} \rho_0(r) [\sin^2\theta \cos^2\phi - \cos^2\theta] r^2 \, dV \end{aligned} \quad (A.6)$$

where C_m is the Earth's moment of inertia with respect to a polar axis. Substituting for $\sin^2\theta$ and $\cos^2\theta$ in terms of Legendre polynomials and for r in terms of r_0 into (A.6) we get

$$A_m e_m = -2\pi \int_0^R \int_b^{\pi} \rho_0(r_0) \left[1 - \frac{2}{3} \left(5e + r_0 \frac{de}{dr}\right) P_2\right] P_2 r_0^4 \sin\theta \, d\theta \, dr_0 \quad (A.7)$$

Integrating (A.7) with respect to θ using orthogonality relation for associated Legendre

functions,

$$A_m e_m = \frac{8\pi}{15} \int_b^R \rho_0 \left[5e + r \frac{de}{dr} \right] r^4 dr \quad (\text{A.8})$$

Expressions (A.5) and (A.8) are then integrated numerically for A_m and $A_m e_m$ respectively.

A.3 Derivation of the Expression for g_0

In the interior of the Earth the gravity potential is given as

$$W_0(r) = W_0(r_0) = V_0(r) + \frac{1}{3} r^2 \Omega^2 (1 - P_2) \quad (\text{A.9})$$

where $V_0(r)$ is the gravitational potential at point r on a surface of mean radius r_0 . (A.9) can be written as

$$W_0(r_0) - \frac{1}{3} \Omega^2 r^2 = V_0(r) - \frac{P_2}{3} \Omega^2 r^2 = U_0(r_0) \quad (\text{A.10})$$

To first order in ellipticity, LHS of (A.10) is constant over the surface of mean radius r_0 and therefore so must be the RHS. We can interpret $U_0(r)$ as the gravitational potential at r due to the density distribution $\rho_0(r)$ in the non-rotating spherical Earth model. Therefore we can write (A.10) as

$$W_0(r) = U_0(r) + \frac{2}{3} r e \frac{dU_0}{dr} P_2 + \frac{1}{3} \Omega^2 r \quad (\text{A.11})$$

to first order in ellipticity. Then

$$\begin{aligned} g_0(r) &= -\nabla W_0 = -\nabla \left[g_0(r) - \frac{2}{3} \Omega^2 r \right] - \frac{2}{3} \nabla (r e g_0 P_2) \\ &= -\hat{r} \left[g_0(r) - \frac{2}{3} r \Omega^2 + \frac{2}{3} \frac{d}{dr} (r e g_0 P_2) \right] - \hat{\theta} \frac{2}{3} e g_0 P_2' \end{aligned} \quad (\text{A.12})$$

where $g_0(r)$ on the RHS is gravity in the non-rotating spherical Earth model. The correctness of (A.12) can be verified by showing that Clairaut's equation (2.23) results from requiring that (A.12) satisfy Poisson's equation (1.5).

Appendix B

Some Useful Identity Relations Among Spherical Harmonics

In what follows we have used integration by parts to derive the final expression.

The integrations are taken from 0 to π .

$$\int P_2^1 \frac{dP_n^{-1}}{d\theta} P_q^{-1} \sin\theta \, d\theta = \int P_n^{-1} [6P_2 P_q^{-1} - P_2^{-1} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta \quad (\text{B.1})$$

$$\begin{aligned} & \int [\sin^2\theta \frac{dP_n^{-1}}{d\theta} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta \\ & = \int P_n^{-1} [\frac{2}{3} P_2^1 \frac{dP_q^{-1}}{d\theta} + (\frac{2}{3} q(q+1)(1-P_2) - 1) P_q^{-1}] \sin\theta \, d\theta \end{aligned} \quad (\text{B.2})$$

$$\int \frac{\cos\theta}{\sin\theta} [P_q^{-1} \frac{dP_n^{-1}}{d\theta} + P_n^{-1} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta = \int P_n^{-1} P_q^{-1} \sin\theta \, d\theta \quad (\text{B.3})$$

$$\int [\frac{dP_n^{-1}}{d\theta} \frac{dP_q^{-1}}{d\theta} + \frac{P_n^{-1} P_q^{-1}}{\sin^2\theta}] \sin\theta \, d\theta = q(q+1) \int P_n^{-1} P_q^{-1} \sin\theta \, d\theta \quad (\text{B.4})$$

$$\int P_2 P_2^1 \frac{dP_n^{-1}}{d\theta} P_q^{-1} \sin\theta \, d\theta = -\int P_n^{-1} [2P_2 P_q^{-1} - 10(P_2)^2 P_q^{-1} + 2P_q^{-1} + P_2 P_2^1 \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta \quad (\text{B.5})$$

$$\int \sin^2\theta [P_2 \frac{dP_q^{-1}}{d\theta} \frac{dP_n^{-1}}{d\theta}] \sin\theta \, d\theta = -\int P_n^{-1} [\frac{2}{3}(1-2P_2)P_2^1 \frac{dP_q^{-1}}{d\theta} - P_2^1 (\frac{2}{3}q(q+1)(1-P_2)-1)P_q^{-1}] \sin\theta \, d\theta \quad (\text{B.6})$$

$$\int P_2 \frac{\cos\theta}{\sin\theta} [P_q^{-1} \frac{dP_n^{-1}}{d\theta} + P_n^{-1} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta = \int (3P_2+1)P_n^{-1} P_q^{-1} \sin\theta \, d\theta \quad (\text{B.7})$$

$$\int P_2 [\frac{dP_n^{-1}}{d\theta} \frac{dP_q^{-1}}{d\theta} - \frac{P_n^{-1} P_q^{-1}}{\sin^2\theta}] \sin\theta \, d\theta = \int P_n^{-1} [q(q+1)P_2 P_q^{-1} - P_2^1 \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta \quad (\text{B.8})$$

APPENDIX C

Integration with Respect to θ

Here we show the algorithm which we use to integrate the Galerkin formulation of the governing equations of liquid core dynamics with respect to θ . We rewrite equation (5.62) as

$$I = \int_0^\pi P_n^{-1} P_q^{-1} \sin\theta \, d\theta \left[E_{n+4} \int_{r_1}^{r_2} \xi_{1,n+4}(r) dr + E_{n+2} \int_{r_1}^{r_2} \xi_{2,n+2}(r) dr + E_n \int_{r_1}^{r_2} \xi_{3,n}(r) dr \right. \\ \left. + E_{n+2} \int_{r_1}^{r_2} \xi_{4,n-2}(r) dr + E_{n+4} \int_{r_1}^{r_2} \xi_{5,n+4}(r) dr \right] \quad (\text{C.1})$$

after operating on the associated Legendre functions with the identities (4.39), (4.40), (5.63), (5.64) and collecting terms of the same degree.

The algorithm we use to evaluate $\xi_{i,j}$ in (5.66) is as follows:

```

m=n+4
do 1 i=1,5
if(m.lt.2.or.m.gt.N) goto 2 {N is the degree of truncation}
if (i.eq.1) then
s1=0
s2=0
s3=0
s4=An+4-1An+2-1

```

$$s5=2(n+5)A_{n+4}^{-1}A_{n+2}^{-1}$$

$$k=n+4$$

else if (i.eq.2) then

$$s1=0$$

$$s2=A_{n+2}^{-1}$$

$$s3=2(n+3) A_{n+2}^{-1}$$

$$s4=A_{n+2}^{-1} [B_n^{-1} + B_{n+2}^{-1}]$$

$$s5=A_{n+2}^{-1} [2(n+3) B_n^{-1} + 3 B_{n+2}^{-1}]$$

$$k=n+2$$

else if(i.eq.3) then

$$s1=1$$

$$s2=B_n^{-1}$$

$$s3=3 B_n^{-1}$$

$$s4=A_n^{-1} C_{n-2}^{-1} + (B_n^{-1})^2 + A_{n+2}^{-1} C_n^{-1}$$

$$s5=2(n+1)A_n^{-1} C_{n-2}^{-1} + 3 (B_n^{-1})^2 - 2 n A_{n+2}^{-1} C_n^{-1}$$

else if(i.eq.4) then

$$s1=0$$

$$s2=C_{n-2}^{-1}$$

$$s3=-2(n-2)C_{n-2}^{-1}$$

$$s4=C_{n-2}^{-1} [B_{n-2}^{-1} + B_n^{-1}]$$

$$s5=C_{n-2}^{-1} [3 B_{n-2}^{-1} - 2(n-2) B_n^{-1}]$$

k=n-2

else if (i.eq.5) then

s1=0

s2=0

s3=0

s4= $C_{n-4}^{-1} C_{n-2}^{-1}$

s5= $-2(n-4) C_{n-4}^{-1} C_{n-2}^{-1}$

k=n-4

endif

$\xi_{i,k} = s1 F_{1,k} + s2 F_{2,k} + s3 F_{3,k} + s4 F_{4,k} + s5 F_{5,k}$ [F_{jk} are as defined in (5.62)]

2 m=m-2

1 continue

APPENDIX D

Gravitational Potential in an Ellipsoidally Stratified Body

The exterior gravitational potential, at a point (r, θ, ϕ) , of a homogeneous body of density ρ bounded by the surface

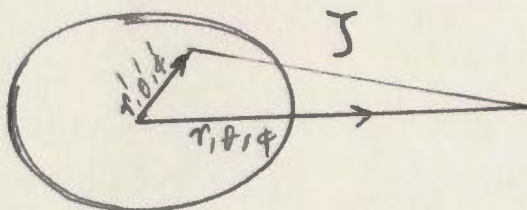
$$\xi = R \left[1 - \frac{2}{3} \epsilon P_2(\cos \theta') \right] \quad (\text{D.1})$$

is

$$V_e = \int_0^{2\pi} \int_0^\pi \int_0^\xi \frac{\rho}{\zeta} r' \sin \theta' d\theta' d\phi' \quad (\text{D.2})$$

where

$$\frac{1}{\zeta} = |\mathbf{r} - \mathbf{r}'|^{-1} = [r^2 - 2rr' \cos \alpha + r'^2]^{-\frac{1}{2}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos \alpha) \quad (\text{D.3})$$



Integrating (D.2) over r' using (D.3) we get

$$V_e = \frac{G\rho}{r} \sum_{n=0}^{\infty} \frac{R^{n+3}}{(n+3)r^n} \int_0^{2\pi} \int_0^{\pi} P_n(\cos\alpha) \left[1 - \frac{2}{3} e(R) P_2(\cos\theta') \right]^{n+3} \sin\theta' d\theta' d\phi' \quad (\text{D.4})$$

To evaluate (D.4) neglecting all quantities smaller than the first order in ellipticity we use the binomial theorem in the integral, then substitute for

$$P_2(\cos\theta')^2 = \frac{1}{5} + \frac{2}{3} P_2(\cos\theta') + \frac{18}{35} P_4(\cos\theta') \quad (\text{D.5})$$

and finally use the addition theorem for spherical harmonics, which gives

$$\int_0^{2\pi} \int_0^{\pi} P_n(\cos\alpha) P_j(\cos\theta') \sin\theta' d\theta' d\phi' = \frac{4\pi}{2n+1} P_n(\cos\theta) \delta_{nj} \quad (\text{D.6})$$

The result is that

$$V_e = 4\pi G\rho R^2 \left[\frac{R}{3r} - \frac{2}{15} e \left(\frac{R}{r} \right)^3 P_2(\cos\theta) \right] \quad (\text{D.7})$$

The internal potential of such a body must satisfy Poisson's equation and be continuous with (D.7) across the boundary (D.1). The function

$$V_i = 4\pi G\rho R^2 \left[\frac{1}{2} - \frac{r^2}{6R^2} - \frac{2}{15} e P_2(\cos\theta) \left(\frac{r}{R} \right)^2 \right] \quad (\text{D.8})$$

satisfies these requirements.

Now we shall suppose that the heterogeneous body of mass M is built up of a set

of concentric spheroidal surfaces

$$r=r_0[1-\frac{2}{3}\epsilon(r_0)P_2(\cos\theta)] \quad (\text{D.9})$$

on each of which the density $\rho=\rho(r_0)$ is constant, according to the hypothesis of hydrostatic equilibrium. Thus a particular equipotential is specified by its mean radius and ϵ is a function of r_0 .

For the heterogeneous mass the standard technique is to regard the contribution to the potential V at an interior point r from a shell of density ρ'_0 bounded by the spheroids r'_0 , $r'_0+dr'_0$ as just the difference between (a) the potential at r due to a homogeneous mass of density ρ'_0 bounded by the spheroid $r'_0+dr'_0$ and (b) the potential at r due to a homogeneous mass of the same density bounded by the spheroid r'_0 . Taking into account that the contributions from spheroids containing/not containing r will be given by (D.7)/(D.8) we have the gravitational potential at a point $r=(r,\theta,\phi)$ interior to the mass:

$$V=4\pi G \left(\int_0^{r_0} \rho'_0 \left[\frac{r_0^2}{r} - \frac{2}{15} \frac{d[e(r'_0)r'_0]}{r^3 dr'_0} P_2(\cos\theta) \right] dr'_0 \right. \\ \left. + \int_{r_0}^R \rho'_0 \left[r'_0 - \frac{2}{15} \frac{de(r'_0)}{dr'_0} r'^2 P_2(\cos\theta) \right] dr'_0 \right) \quad (\text{D.10})$$

