# WOBBLE/NUTATION OF A ROTATING ELLIPSOIDAL EARTH WITH LIQUID OUTER CORE: IMPLEMENTATION OF A NEW SET OF EQUATIONS DESCRIBING DYNAMICS OF ROTATING FLUIDS



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# WOBBLE/NUTATION OF A ROTATING ELLIPSOIDAL EARTH WITH LIQUID OUTER CORE: Implementation of a New Set of Equations Describing Dynamics of Rotating Fluids

by

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# ABSTRACT

In this thesis I implement a new set of scalar equations describing the free oscillations of rotating self-gravitating compressible fluids, to solve for the wobble and inertial modes of a rotating Earth with rigid mantle and liquid core. A Galerkin method is used to integrate these equations in the liquid core. It is shown that by using the divergence theorem it is possible to make use of the '*natural*' boundary conditions to reduce the order of the derivatives from second to first in the Galerkin formulation of the governing equations. As a partial test of the reliability of our formulation, the eigenperiods of the Earth's Chandler wobble (CW), nearly diurnal free wobble (NDFW) and some of the other inertial modes are computed for the case of a compressible, but neutrally stratified, core and compared to those for the homogeneous incompressible core model.

## ACKNOWLEDGEMENTS

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## LIST OF SYMBOLS

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|---|---|---|
| $A_E$ | moment of inertia of the Earth with respect to an equatorial axis | 60 |
| $A_m$ | moment of inertia of the mantle with respect to an equatorial axis | 36 |
| $a_n$ | constant coefficient | 11, 43 |
| B | scalar notation | 25 |
| $B_1$ | scalar notation | 28 |
| b | mean radius of CMB | 14 |
| **C** | vector notation | 24 |
| $\mathbf{C}_1$ | vector notation | 28 |
| $C_m$ | moment of inertia of the Earth/mantle with respect to a polar axis | 36 |
| CMB | core mantie boundary | 12 |
| CW | Chandler wobble | i |
| $\hat{\mathbf{E}}_i$ | principal axis fixed in the mantle | 31 |
| $e_E$ | dynamical ellipticity of the Earth | 60 |
| $e_m$ | $(C_m\text{-}A_m)/A_m$ | 36 |
| $\hat{\mathbf{e}}_i$ | unit vector in i direction | 4 |
| $f_l$ | trial function | 44 |
| G | gravitational constant | 5 |
| $g_0$ | gravitational acceleration | 5 |
| $\mathbf{H}_E$ | total angular momentum of the Earth | 35 |
| $\mathbf{H}_{lc}$ | angular momentum of the liquid core | 35 |
| $\mathbf{H}_m$ | angular momentum of the mantle | 35 |
| $\bar{\mathbf{I}}_m$ | mantle's inertia tensor | 35 |
| ICB | inner core boundary | 11 |
| L | integer denoting the degree of truncation | 44 |
| M(r) | total mass of a body enclosed by a shell of radius r | 14 |
| N | Brunt-Väisälä frequency | 5 |
| N | integer denoting the degree of truncation | 13, 44 |
| NDFW | nearly diurnal free wobble | i |
| PDE | partial differential equation | 3 |
| $P_n^m$ | Legendre polynomials of degree n and azimuthal order m | 8 |
| PREM | preliminary reference Earth model | 11 |
| $P_0$ | pressure before deformation | 5 |
| $p_1$ | Eulerian pressure disturbance | 6 |
| R | mean radius of the surface of the Earth | 11 |

| | | |
|---|---|---|
| $r$ | distance from the geocentre | 8 |
| $r_0$ | mean radius of equipotential surface | 16 |
| SSA | subseismic approximation | 3 |
| SSWE | subseismic wave equation | 3 |
| $S_n^m$ | spheroidal displacement field | 8 |
| THPD | three potential description | 4 |
| TPD | two potential description | 4 |
| $T_n^m$ | toroidal displacement field | 8 |
| $\mathbf{u}$ | Lagrangean displacement | 6 |
| $V_0$ | gravitational potential | 34 |
| $V_1$ | Eulerian perturbation in gravitational potential | 6 |
| $\mathbf{v}$ | velocity | 6 |
| $W_0$ | gravity potential | 5 |
| $x$ | r/R | 11 |
| $Y_n^m$ | spherical harmonics of degree n and azimuthal order m | 8 |
| $\alpha$ | compressional wave speed | 5 |
| $\beta$ | stability parameter | 1 |
| $\Gamma_p$ | dyadic | 24 |
| $\gamma$ | amplitude of the wobble | 32 |
| $\varepsilon$ | ellipticity | 16 |
| $\varepsilon_{ijk}$ | three dimensional Levi-Civita symbol | 33 |
| $\zeta$ | $\alpha^2\nabla\cdot\mathbf{u}$ (scalar potential) | 23 |
| $\zeta_n^m$ | scalar function | 29 |
| $\theta$ | colatitude | 8 |
| $\lambda$ | Lamé parameter | 8 |
| $\mu$ | Lamé parameter | 8 |
| $\rho_0$ | density before deformation | 5 |
| $\rho_1$ | Eulerian perturbation in density | 6 |
| $\langle\rho\rangle$ | average density | 15 |
| $\sigma$ | $2\Omega/\omega$ | 24 |
| $\sigma_n$ | constant coefficient | 33 |
| $\tilde{\tau}$ | stress tensor | 7 |
| $\phi$ | longitude | 8 |
| $\Phi_n^m$ | scalar function | 8 |
| $\chi$ | scalar potential | 23 |
| $\chi_n^m$ | scalar function | 29 |
| $\Omega$ | rate of rotation of the Earth | 4 |
| $\omega$ | oscillation frequency | 6 |
| $\omega_m$ | angular velocity of the mantle | 31 |
| $\omega_r$ | angular velocity of the reference frame | 4 |
| $\tilde{\mathbf{I}}$ | unit dyadic | 35 |

# CHAPTER 1: INTRODUCTION

The innermost region of the earth consists of a liquid outer core and a solid inner core. The radius of the inner core is estimated to be 1221.5 km and that of the outer core 3480 km. The fluidity of the outer core and the solidity of the inner core have been established through studies of ray seismology, tides and the Earth's normal modes.

The core of the Earth plays a key role in many geophysical studies. An example of this is the role of the liquid core on the periods of the Earth's Chandler wobble and the nearly diurnal free wobble. A perfectly rigid body with the same mass distribution as the Earth is expected to have a free Eulerian wobble with period of about 306 days. The presence of a liquid core the size of the Earth's changes this period to about 270 days and gives rise to an additional retrograde wobble of nearly diurnal period, corresponding to a nutation of about 350 days. Elasticity and oceans lengthen the Eulerian wobble period to 435 days (the Chandler wobble) and the free core nutation period to 460 days, assuming that the steadily-rotating configuration is one of hydrostatic equilibrium.

Since the core is not directly accessible, much is still unknown about its properties. The distributions of material properties such as the density $\rho$ and Lamé parameter $\lambda$ in the core are established through theoretical and observational studies using ray seismology and free oscillations. However, the stability parameter $\beta$ (Pekeris and Accad 1972) cannot so far be inferred from the seismological data and is therefore poorly controlled, uncertain and varies from one model to another.

The spectrum of free oscillations possible in the liquid core can be conveniently divided into:

(a) short period free oscillations, with periods of the order of an hour or less, for which elasticity is the primary restoring force. For these oscillations the effects of rotation and ellipticity can be treated as small perturbations on the solution of the governing equations for a non-rotating spherical Earth;

(b) long period free core oscillations, with periods of the order of half a day and longer. These oscillations are considerably affected by the rotation and, in some cases, the ellipticity of the Earth.

Long period free core oscillations are of three types:

(a) Slichter (inner core translational) modes, with periods of several hours, for which the primary restoring force is gravitational (due to the density difference across the inner core boundary);

(b) gravity waves (or core undertones) whose essential restoring force is negative buoyancy. Since negative buoyancy implies stable stratification, some parts of the outer core must be stably stratified ($\beta < 0$) for gravity waves to exist;

(c) inertial waves, which depend on the Coriolis effect as their dominant restoring force. Therefore, the rotation of the Earth is necessary to the existence of these modes.

Among the first who studied the inertial modes of the Earth were Hough (1895) and Poincaré (1910). The Earth model they considered consists of an inviscid incompressible homogeneous liquid filling the region bounded by an ellipsoidal container which is rigid but can change its orientation. Analytical solutions were found for the Earth's Chandler wobble and nearly diurnal free wobble and other inertial modes using this simple Earth model.

During the last four decades the theory of Hough and Poincaré has been greatly extended to treat more realistic Earth models. Jeffreys & Vicente (1957a, 1957b) considered Earth models with radially stratified elastic mantle and a homogeneous incompressible liquid core. Molodensky (1961) included core compressibility. Shen and Mansinha (1976) used the theory of Molodensky (1961) but added non-neutral stratification in the liquid core. Smith (1974) derived the elastic-gravitational normal modes theory in which an infinite set of coupled differential equations describe the dynamics of a rotating, slightly elliptical Earth. This theory was utilized by Smith (1977) to compute the periods of the Earth's Chandler wobble and the nearly diurnal free wobble, with inner core. Also Wahr (1981) used Smith's formulation to study the effects of Earth's rotation and ellipticity on the body tides. Moon (1982) derived the linearized equation of motion for the slightly elliptical rotating earth in order to study Earth's free oscillations, free wobbles and core modes.

In the hope of finding an alternative, more simplified, solution to the governing equations of core dynamics Smylie & Rochester (1981) made the subseismic approximation (SSA) to derive the subseismic wave equation (SSWE). The SSWE is a scalar second order partial differential equation (PDE) involving only one scalar potential, and (to the extent the SSA is valid) once the SSWE is solved, all other dynamical variables can be readily obtained. As shortcomings in the subseismic description became apparent (Crossley and Rochester 1992, Rochester and Peng 1993), Wu & Rochester (1990) derived the exact two potential description of core dynamics (TPD). They showed that the dynamics of the inviscid liquid core is described without approximation by two scalar second order PDEs involving two scalar potentials. Although the TPD is mathematically elegant, it also has some shortcomings which we discuss in chapter 3.

Rochester (unpublished) has derived an alternative set of three scalar PDEs involving three scalar potentials which exactly describe the dynamics of the inviscid liquid core (hereafter we refer to this description as THPD). The motivation for this new description and its advantages will be discussed later in this thesis. In this study, we implement the THPD to solve for some of the normal modes of a rotating ellipsoidal liquid core.

## 1.1 The Governing Equations and Boundary Conditions

The liquid core is taken as inviscid, with the reference state being one of hydrostatic equilibrium in a coordinate system rotating steadily with angular velocity

$$\boldsymbol{\omega}_r = \Omega \hat{\mathbf{e}}_3 \qquad (1.1)$$

where $\Omega$ is the rate of rotation of the Earth. The rate of change of direction of a unit vector, $\hat{\mathbf{e}}_j$, is therefore

$$\frac{d\hat{e}_j}{dt}=\Omega\hat{e}_3\times\hat{e}_j \tag{1.2}$$

In the reference frame stated above the equilibrium density $\rho_0$, pressure $p_0$ and gravitational acceleration $g_0$ are related by

$$\nabla P_0=\rho_0 g_0 \tag{1.3}$$

$$g_0=\nabla W_0 \tag{1.4}$$

$$\nabla^2 W_0=-4\pi G\rho_0+2\Omega^2 \tag{1.5}$$

$$\nabla\rho_0=(1-\beta)\rho_0\frac{g_0}{\alpha^2} \tag{1.6}$$

where G, $\alpha$, $\beta$ are respectively the gravitational constant, the local compressional wave speed and the stability parameter.

The parameter $\beta$ measures the extent and sign of any departure of density gradient from purely adiabatic stratification; it is proportional to the square of the local Brunt-Väisälä frequency N:

$$\beta=-\frac{\alpha^2 N^2}{g_0^2} \tag{1.7}$$

A region is stably stratified if $\beta<0$, unstably stratified if $\beta>0$ and neutral if $\beta=0$ (i.e. the Adams Williamson condition is satisfied). Although an exact value for $\beta$ has not been established, Masters (1979) shows that current seismological data sets a limit $|\beta|<0.03$-

0.05.

The equations governing the isentropic small oscillations of an inviscid liquid core are given (Rochester 1989) by the conservation laws for mass, momentum, gravitational flux and entropy. These equations are as follows:

$$\frac{\partial \rho_1}{\partial t} = -\nabla\cdot(\rho_0 \mathbf{v}) \qquad (1.8)$$

$$\frac{\partial \mathbf{v}}{\partial t} + 2\Omega \hat{\mathbf{e}}_3 \times \mathbf{v} = -\frac{1}{\rho_0}\nabla p_1 + \nabla V_1 + \frac{\rho_1}{\rho_0}\mathbf{g}_0 \qquad (1.9)$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \qquad (1.10)$$

$$\frac{\partial p_1}{\partial t} = \alpha^2 \frac{\partial \rho_1}{\partial t} - \beta \rho_0 \mathbf{v}\cdot\mathbf{g}_0 \qquad (1.11)$$

In the above equations v, $\rho_1$, $p_1$ and $V_1$ (all regarded as first order departures from the equilibrium reference state) stand for velocity, the Eulerian perturbation in density, Eulerian pressure disturbance and the Eulerian perturbation in the gravitational potential respectively, with

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \qquad (1.12)$$

where **u** is the Lagrangean displacement from equilibrium.

In dealing with free oscillations it is convenient to assume that all the disturbance variables have time dependence $e^{i\omega t}$. After operating with the time derivative on the disturbance variables in equations (1.8)-(1.11) and cancelling the common factors we get

$$\rho_1 = -\nabla\cdot(\rho_0 \boldsymbol{u}) \tag{1.13}$$

$$-\omega^2 \boldsymbol{u} + 2i\omega\Omega\hat{\boldsymbol{e}}_3 \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p_1 + \nabla V_1 + \frac{\rho_1}{\rho_0}\boldsymbol{g}_0 \tag{1.14}$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \tag{1.15}$$

$$p_1 = -\boldsymbol{u}\cdot\nabla p_0 + \alpha^2(\rho_1 + \boldsymbol{u}\cdot\nabla\rho_0) \tag{1.16}$$

We use (1.13) to write equations (1.14)-(1.16) as

$$(\omega^2 \boldsymbol{u} - 2i\omega\Omega\ \hat{\boldsymbol{e}}_3 \times \boldsymbol{u}) = \frac{1}{\rho_0}\nabla p_1 - \nabla V_1 + \frac{\boldsymbol{g}_0}{\rho_0}\nabla\cdot(\rho_0 \boldsymbol{u}) \tag{1.17}$$

$$\nabla^2 V_1 = -4\pi G \nabla\cdot(\rho_0 \boldsymbol{u}). \tag{1.18}$$

$$\frac{p_1}{\rho_0} = -(\alpha^2 \nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{g}_0) \tag{1.19}$$

The five differential equations above [three components of the vector equation (1.17), equations (1.18) and (1.19)] are linear in five variables: three components of **u**, $p_1$ and $V_1$. These are the equations governing the dynamics of the liquid core.

To these equations we add the boundary conditions, which require continuity of $\hat{\mathbf{n}}\cdot\mathbf{u}$, $V_1$, $\hat{\mathbf{n}}\cdot(\nabla V_1 - 4\pi G\rho_0\mathbf{u})$ and $\hat{\mathbf{n}}.\bar{\tau}$ across core boundaries. Here $\hat{\mathbf{n}}$ is the unit vector normal to the boundary surface and $\bar{\tau}$ is the stress tensor. In the outer core the stress tensor can be written as

$$\bar{\tau} = -(p_1 + u \cdot \nabla p_0)\bar{\mathbf{I}} \tag{1.20}$$

and in the solid mantle it takes the form

$$\bar{\tau} = (\lambda \nabla \cdot u)\bar{\mathbf{I}} + 2\mu[\nabla u + (\nabla u)^T] \tag{1.21}$$

where $\mu$ and $\lambda$ are the Lamé parameters and $\bar{\mathbf{I}}$ is the unit dyadic.

The traditional approach to solving these equations is to represent the variation fields in spherical polar coordinate by spherical harmonics

$$u = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} S_n^m + T_n^m \tag{1.22}$$

$$V_1, p_1 = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} [\phi_n^m, \psi_n^m] \; Y_n^m \tag{1.23}$$

with

$$\begin{aligned} S_n^m &= [u_n^m \, \hat{r} + r v_n^m \, \nabla] \; Y_n^m \\ T_n^m &= -t_n^m \, \hat{r} \times \nabla Y_n^m \end{aligned} \tag{1.24}$$

$$Y_n^m = P_n^m(\cos\theta) e^{im\phi} \tag{1.25}$$

where $u_n^m$, $v_n^m$, $t_n^m$, $\phi_n^m$ and $\psi_n^m$ are functions of r only and $P_n^m$ is the associated Legendre function of degree n and azimuthal order m.

For the acoustic modes (short period free oscillations) the effects of rotation and ellipticity are small and equations with different degree n are independent when these effects are neglected (Alterman et al. 1959). Therefore, the traditional approach is an effective tool in solving the governing equations for the acoustic modes.

However, for the long period free oscillations, where neglecting rotation and, in some cases, ellipticity is no longer valid, equations with different order m are decoupled but those of different degree are not. Instead to each m there correspond solutions in the form of two coupling chains

$$S^{m}_{|m|} + T^{m}_{|m|+1} + S^{m}_{|m|+2} + T^{m}_{|m|+3} + \dots \qquad (1.26)$$

$$T^{m}_{|m|} + S^{m}_{|m|+1} + T^{m}_{|m|+2} + S^{m}_{|m|+3} + \dots \qquad (1.27)$$

(Smith 1974). Numerical estimates of the eigenperiods and eigenfunctions then necessarily rest on heavy truncations of these coupling chains.

Shen & Mansinha (1976) used a three term truncation, e.g.

$$T^{1}_{1} + S^{1}_{2} + T^{1}_{3} \qquad (1.28)$$

to compute the periods of the Earth's nearly diurnal free wobble and some of the liquid core undertones of a rotating ellipsoidal earth. Smith (1977) seriously questions the validity of the numerical results for undertone periods based on such heavy truncations, and concludes that the latter are inadequate for the free wobble of the inner core and other internal core modes. He was convinced, however, that (1.28) was sufficient to yield reliable periods for the Chandler and nearly diurnal free wobble. This belief seems not yet to have been seriously tested by using less severe truncation than (1.28).

The problems of truncation presented by the traditional approach motivated a search for alternative solutions to the governing equations [Smylie & Rochester (1981), Wu & Rochester (1990)]. The three-potential description of core dynamics is the result of one such attempt. We will give the derivation of the THPD in this thesis (chapter 3). To test the reliability of our description, we use the well-known analytical solutions to compute the periods of the Earth's Chandler wobble, nearly diurnal free wobble and some of the other inertial modes using a simplified Earth model consisting of a rigid mantle and a homogeneous incompressible liquid core. These results will then be compared to the ones computed using the THPD.

# CHAPTER 2: THE EARTH MODEL

We adopt PREM [the Preliminary Reference Earth Model (Dziewonski & Anderson 1981)], as the base for our Earth model. The mantle is assumed to be rigid and its properties are taken directly from PREM. The presence of the solid inner core is ignored for simplicity. The liquid core is assumed to be neutrally stratified and therefore we must make sure that $\beta=0$ everywhere in its interior. Also, since we ignore the presence of the inner core in computational work, we will make sure that the density profile $\rho$ and the compressional wave speed profiles $\alpha$ in the liquid core are extended to the centre of the Earth as smooth functions of the radius. The method to compute the ellipticity of the equipotential surfaces will also be described in this chapter.

## 2.1 Modification of $\alpha$ in the Liquid Core

In the liquid core of PREM $\alpha$ is expressed as a third order polynomial

$$\alpha = c_1 + c_2\, x + c_3\, x^2 + c_4\, x^3 \qquad (2.1)$$

where $c_1$=11.0487, $c_2$=-4.0362, $c_3$=4.8023, $c_4$=-13.5732, x=r/R and R is the mean radius of the surface of the Earth. Since the first derivative of this profile with respect to r does not vanish at r=0, this profile can not be used when the inner core is ignored.

In the inner core of the PREM $\alpha$ is a smooth function of the radius and has the form

$$\alpha_{ic}=a_1+a_2\ x^2 \tag{2.2}$$

with $a_1$=11.2622 and $a_2$=-6.3640. Since p-waves travel faster in solid media than in fluids, we will have a maximum error of about 17% (at the CMB) if we choose the inner core profile for α as an approximation to that in the liquid core. Compromising to reduce this discrepancy, we modify α so that in our model its first derivative vanishes at r=0 and its numerical values are close to those of PREM in the range ICB-CMB.

The function

$$\alpha = 10.6752-8.7490\ x^2 \tag{2.3}$$

satisfies the requirement of vanishing first derivative at $x$=0, gives the PREM values at the ICB and CMB, and involves a maximum error of about 0.7% (at r≈2900 km). Table 1 shows the α distribution in the liquid core using: (a) equation (2.3) and (b) expression from PREM.

## 2.2 Modification of the Density Profile in the Liquid Core

As mentioned in chapter 1, the density gradient in the liquid core has the form

$$\nabla\rho_0 = (1-\beta)\frac{\rho_0 \mathbf{g}_0}{\alpha^2} \tag{2.4}$$

For a spherically stratified Earth (2.4) becomes

$$\frac{d\rho_0}{dr} = -(1-\beta)\frac{\rho_0 g_0}{\alpha^2} \tag{2.5}$$

which can be written as

$$\frac{\alpha^2}{\rho_0 g_0}\frac{d\rho_0}{dr}+(1-\beta)=0 \tag{2.6}$$

As suggested by PREM, we choose a density profile which has the form

$$\rho_o=\sum_{j=1}^{N} d_j x^{j-1} \tag{2.7}$$

where $x$ is defined as in previous section, N is an integer and $d_j$ are constant. Substituting (2.7) into (2.5) [and setting $d_2$=0 to satisfy (2.5) at $x$=0] we get

$$\sum_{j=3}^{N}(j-1)\frac{\alpha^2 R}{\rho_0 g_0}d_j x^{j-2}+(1-\beta)=0 \tag{2.8}$$

where

$$\frac{d\rho_0}{dr}=\frac{d\rho_0}{dx}\frac{dx}{dr}=R\sum_{j=3}^{N}(j-1)d_j\, x^{j-2} \tag{2.9}$$

To solve (2.8) for a best fitting density function, $\rho_0$, we use a Galerkin method with weight functions $x^{i-1}$ (i=1,...N-2):

$$\sum_{j=3}^{N}(j-1)d_j\int_0^{\frac{b}{R}}\frac{R\alpha^2}{\rho_0 g_0}x^{i+j-3}dx+(1-\beta)\int_0^{\frac{b}{R}}x^{i-1}dx=0 \tag{2.10}$$

for each i, i=1,...,N-2, b is the mean radius of CMB. We write (2.10) as

$$\sum_{j=3}^{N}A_{ij}\,d_j=F_i \tag{2.11}$$

where

$$A_{ij}=(j-1)\int_0^{\frac{b}{R}}\frac{R\alpha^2}{\rho_0 g_0}x^{i+j-3}dx \tag{2.12}$$

$$F_i=-\frac{1-\beta}{i}\left(\frac{b}{R}\right)^i \tag{2.13}$$

In (2.13) $\beta$ is set to a desired value, $\beta_d$, and PREM density profile is used as a starting value to find $\rho_0$ and $g_0$ in (2.12). The starting values for the coefficients of the density profile, therefore, are: $d_1$=12.5815, $d_2$=0, $d_3$=-3.6426, $d_4$=-5.5281, all other $d_i$=0. In PREM N=4, but our computational experience has shown that a higher value of N results in a much faster convergence of $\beta$ to $\beta_d$, and here we set N=12.

The gravitational acceleration, $g_0$, in the liquid core is computed as

$$g_0(r)=\frac{GM(r)}{r^2} \tag{2.14}$$

where M(r) is the total mass of the body enclosed by the shell of radius r and is given as

$$M(r) = 4\pi \int_0^r \rho_0 r^2 \, dr \tag{2.15}$$

Using (2.7), (2.15) becomes

$$M(r) = 4\pi R^3 \sum_{i=1}^{3} d_i \int_0^x x'^{\,i+1} \, dx' = 4\pi R^3 \sum_{i=1}^{3} \frac{d_i}{i+2} x^{i+2} \tag{2.16}$$

Substituting (2.16) into (2.14) we get

$$g_0(r) = 4\pi G R \sum_{i=1}^{N} \frac{d_i \, x^i}{i+2} \tag{2.17}$$

IMSL subroutine dslsarg is then called to solve (2.12) for the coefficients $d_3,\ldots,d_N$. To solve for $d_1$ we use the mass conservation of the liquid core as a constraint and proceed as follows:

$$\int_{LC} \rho_0 dv = \frac{4}{3}\pi \langle\rho\rangle b^3 \tag{2.18}$$

where $\langle\rho\rangle$ is the average density of the outer core. Using (2.8) we can write (2.18) as

$$4\pi R^3 \sum_{i=1}^{N} d_i \int_0^{\frac{b}{R}} x^{i+1} dx = \frac{4\pi \langle\rho\rangle}{3} b^3 \tag{2.19}$$

$$\therefore \quad d_1 = \langle\rho\rangle - 3\left(\frac{R}{b}\right)^3 \sum_{i=3}^{N} \frac{d_i}{i+2} \left(\frac{b}{R}\right)^{i+2} \tag{2.20}$$

Once $d_i$ are known, we use them to solve for $\beta$ in

$$\beta = 1 + \frac{\alpha^2}{\rho_0 g_0} \frac{d\rho_0}{dr} \tag{2.21}$$

If $|\beta-\beta_d|>\varkappa$, where $\varkappa$ is the desired accuracy, we use the new $d_i$ as the starting values. This process is repeated until $|\beta-\beta_d|\leq\varkappa$. In this problem we set $\beta$=0 and $\varkappa=10^{-5}$. The coefficients of the density profile in the modified liquid core are given in table 2. Tables 3 and 4 show the density and $\beta$ distributions in the outer core before and after modification respectively.

## 2.3 Ellipticity

Since the departure of the shape of the Earth from sphericity is small, we use a first order theory to include the effect of ellipticity in the governing equations. We follow Chandrasekhar and Roberts (1963) in assigning the equipotential surfaces of mean radius $r_0$ (over which the equilibrium properties are constant) the polar equation

$$r = r_0\left(1 - \frac{2}{3}\epsilon(r_0)\, P_2(\cos\theta)\right) \tag{2.22}$$

For a hydrostatically prestressed Earth Clairaut's equation

$$\frac{d^2\epsilon}{dr^2}+\frac{6}{r^2}[\frac{\rho_0}{<\rho_o>}-1]\epsilon+\frac{6}{r}\frac{\rho_0}{<\rho_0>}\frac{d\epsilon}{dr}=0 \tag{2.23}$$

(Jeffreys 1970) describes the ellipticity of the equipotential surfaces. In writing (2.23) we use the fact that $\varepsilon(r_0)=\varepsilon(r)$ to first order.

We use a Runge-Kutta integration method to solve (2.23) for ε. Since equation (2.23) is singular at r=0, it cannot be used directly to start the integration from the geocentre. To get around this problem we proceed as follows. Using Taylor expansion we can write ε and ρ near the geocentre as

$$\epsilon=\epsilon(0)+\frac{1}{2}(\frac{d^2\epsilon}{dr^2})_0\ r^2+\ldots \tag{2.24}$$

$$\rho_0=d_0+d_2r^2+d_3r^3+\ldots \tag{2.25}$$

The average density <ρ> is given as

$$<\rho>=[4\pi\int_0^r\rho x^2dx]\ /\ [\frac{4}{3}\pi r^3]=\frac{3}{r^3}\int_0^r[d_0+d_2x^2+\ldots]x^2dx\ =\ d_0+\frac{3}{5}d_2r^2+\ldots \tag{2.26}$$

Substituting (2.24)-(2.26) into (2.23) we get

$$(\frac{d^2\epsilon}{dr^2})_0=-\frac{12}{35}\frac{d_2}{d_0}\epsilon(0) \tag{2.27}$$

We use (2.27) and an arbitrary value for ε(0) to start the integration at the geocentre, and then scale ε(0) so that ε(R) is the observed hydrostatic value of Earth's surface ellipticity. Table 5 shows the distribution of the ellipticity in the interior of the Earth.

**Table 1.** Distribution of α in the Liquid Core
a: Modified Model
b: PREM Model

| radius<br>km | (a)<br>α<br>km/s | (b)<br>α<br>km/s |
|---|---|---|
| 0.00000D+00 | 0.106776D+02 | |
| 0.81433D+02 | 0.106762D+02 | |
| 0.16287D+03 | 0.106719D+02 | |
| 0.24430D+03 | 0.106647D+02 | |
| 0.40717D+03 | 0.106418D+02 | |
| 0.48860D+03 | 0.106261D+02 | |
| 0.57003D+03 | 0.106075D+02 | |
| 0.65147D+03 | 0.105860D+02 | |
| 0.73290D+03 | 0.105617D+02 | |
| 0.81433D+03 | 0.105345D+02 | |
| 0.89577D+03 | 0.105045D+02 | |
| 0.97720D+03 | 0.104716D+02 | |
| 0.10586D+04 | 0.104358D+02 | |
| 0.11401D+04 | 0.103972D+02 | |
| 0.12215D+04 | 0.103557D+02 | 0.103557D+02 |
| 0.13404D+04 | 0.102900D+02 | 0.102857D+02 |
| 0.14592D+04 | 0.102182D+02 | 0.102131D+02 |
| 0.15781D+04 | 0.101403D+02 | 0.101373D+02 |
| 0.16970D+04 | 0.100563D+02 | 0.100578D+02 |
| 0.19347D+04 | 0.987003D+01 | 0.988576D+01 |
| 0.20536D+04 | 0.976774D+01 | 0.979209D+01 |
| 0.21724D+04 | 0.965936D+01 | 0.969263D+01 |
| 0.22913D+04 | 0.954489D+01 | 0.958684D+01 |
| 0.24102D+04 | 0.942431D+01 | 0.947420D+01 |
| 0.25291D+04 | 0.929764D+01 | 0.935417D+01 |
| 0.26479D+04 | 0.916487D+01 | 0.922624D+01 |
| 0.27668D+04 | 0.902601D+01 | 0.908987D+01 |
| 0.28857D+04 | 0.888105D+01 | 0.894453D+01 |
| 0.30045D+04 | 0.872999D+01 | 0.878969D+01 |
| 0.31234D+04 | 0.857283D+01 | 0.862483D+01 |
| 0.32423D+04 | 0.840958D+01 | 0.844941D+01 |
| 0.33611D+04 | 0.824023D+01 | 0.826290D+01 |
| 0.34800D+04 | 0.806479D+01 | 0.806479D+01 |

**Table 2.** Coefficients of the Density Profile in the Modified Liquid Core

| kg/m$^3$ |
|---|
| $d_1 = 12.4780 \times 10^3$ |
| $d_2 = 0.0$ |
| $d_3 = -7.7460 \times 10^3$ |
| $d_4 = 1.1811 \times 10^{-1}$ |
| $d_5 = -2.5083 \times 10^3$ |
| $d_6 = 2.7304 \times 10$ |
| $d_7 = -1.0422 \times 10^3$ |
| $d_8 = 7.6678 \times 10^{-1}$ |
| $d_9 = -2.3423 \times 10^3$ |
| $d_{10} = 3.4341 \times 10^3$ |
| $d_{11} = -3.3502 \times 10^3$ |
| $d_{12} = 1.3934 \times 10^3$ |

**Table 3.** Distribution of Density and Stability Parameter in the Liquid Core of PREM

| radius<br>km | ρ<br>$kg/m^3$ | β |
|---|---|---|
| 0.12215D+04 | 0.121663D+05 | -0.278235D-01 |
| 0.12968D+04 | 0.121267D+05 | -0.209509D-01 |
| 0.13721D+04 | 0.120852D+05 | -0.150552D-01 |
| 0.14474D+04 | 0.120416D+05 | -0.101290D-01 |
| 0.15226D+04 | 0.119959D+05 | -0.612425D-02 |
| 0.15979D+04 | 0.119482D+05 | -0.297092D-02 |
| 0.16732D+04 | 0.118982D+05 | -0.588442D-03 |
| 0.17485D+04 | 0.118460D+05 | 0.110760D-02 |
| 0.18238D+04 | 0.117916D+05 | 0.220173D-02 |
| 0.18991D+04 | 0.117347D+05 | 0.277648D-02 |
| 0.19743D+04 | 0.116755D+05 | 0.291137D-02 |
| 0.20496D+04 | 0.116139D+05 | 0.268248D-02 |
| 0.21249D+04 | 0.115497D+05 | 0.216239D-02 |
| 0.22002D+04 | 0.114829D+05 | 0.142035D-02 |
| 0.22755D+04 | 0.114136D+05 | 0.522422D-03 |
| 0.23508D+04 | 0.113416D+05 | -0.468165D-03 |
| 0.24260D+04 | 0.112668D+05 | -0.149081D-02 |
| 0.25013D+04 | 0.111893D+05 | -0.248731D-02 |
| 0.25766D+04 | 0.111089D+05 | -0.340165D-02 |
| 0.26519D+04 | 0.110257D+05 | -0.417991D-02 |
| 0.27272D+04 | 0.109395D+05 | -0.477018D-02 |
| 0.28025D+04 | 0.108503D+05 | -0.512257D-02 |
| 0.28777D+04 | 0.107580D+05 | -0.518931D-02 |
| 0.29530D+04 | 0.106626D+05 | -0.492491D-02 |
| 0.30283D+04 | 0.105641D+05 | -0.428642D-02 |
| 0.31036D+04 | 0.104624D+05 | -0.323377D-02 |
| 0.31789D+04 | 0.103573D+05 | -0.173026D-02 |
| 0.32542D+04 | 0.102490D+05 | 0.256870D-03 |
| 0.33294D+04 | 0.101373D+05 | 0.275574D-02 |
| 0.34047D+04 | 0.100221D+05 | 0.578897D-02 |
| 0.34800D+04 | 0.990344D+04 | 0.937268D-02 |

**Table 4.** Distribution of Density and Stability Parameter in the Liquid Core of the Modified Earth model

| radius km | $\rho$ $kg/m^3$ | $\beta$ |
|---|---|---|
| 0.00000D+00 | 0.124775D+05 | -0.116976D-05 |
| 0.12000D+03 | 0.124748D+05 | -0.908067D-06 |
| 0.24000D+03 | 0.124665D+05 | -0.835353D-06 |
| 0.36000D+03 | 0.124528D+05 | -0.818009D-06 |
| 0.48000D+03 | 0.124335D+05 | -0.800050D-06 |
| 0.60000D+03 | 0.124086D+05 | -0.765664D-06 |
| 0.72000D+03 | 0.123782D+05 | -0.716400D-06 |
| 0.84000D+03 | 0.123421D+05 | -0.658616D-06 |
| 0.96000D+03 | 0.123004D+05 | -0.597716D-06 |
| 0.10800D+04 | 0.122529D+05 | -0.536469D-06 |
| 0.12000D+04 | 0.121995D+05 | -0.475462D-06 |
| 0.13200D+04 | 0.121403D+05 | -0.414311D-06 |
| 0.14400D+04 | 0.120752D+05 | -0.352816D-06 |
| 0.15600D+04 | 0.120039D+05 | -0.291631D-06 |
| 0.16800D+04 | 0.119265D+05 | -0.232358D-06 |
| 0.18000D+04 | 0.1184[illegible]8D+05 | -0.177163D-06 |
| 0.19200D+04 | 0.117527D+05 | -0.128157D-06 |
| 0.20400D+04 | 0.116560D+05 | -0.867970D-07 |
| 0.21600D+04 | 0.115527D+05 | -0.535222D-07 |
| 0.22800D+04 | 0.114425D+05 | -0.277709D-07 |
| 0.24000D+04 | 0.113253D+05 | -0.835182D-08 |
| 0.25200D+04 | 0.112008D+05 | 0.596607D-08 |
| 0.26400D+04 | 0.110690D+05 | 0.159118D-07 |
| 0.27600D+04 | 0.109295D+05 | 0.215309D-07 |
| 0.28800D+04 | 0.107822D+05 | 0.226098D-07 |
| 0.30000D+04 | 0.106267D+05 | 0.196996D-07 |
| 0.31200D+04 | 0.104629D+05 | 0.152089D-07 |
| 0.32400D+04 | 0.102904D+05 | 0.131259D-07 |
| 0.33600D+04 | 0.101089D+05 | 0.144800D-07 |
| 0.34800D+04 | 0.991810D+04 | 0.344820D-08 |

**Table 5.** Distribution of Ellipticity in the Modified Earth Model

| radius km | ellipticity |
|---|---|
| 0.00000D+00 | 0.2461D-02 |
| 0.18316D+03 | 0.2461D-02 |
| 0.36632D+03 | 0.2462D-02 |
| 0.54947D+03 | 0.2463D-02 |
| 0.73263D+03 | 0.2464D-02 |
| 0.91579D+03 | 0.2466D-02 |
| 0.10989D+04 | 0.2469D-02 |
| 0.12821D+04 | 0.2472D-02 |
| 0.14653D+04 | 0.2475D-02 |
| 0.16484D+04 | 0.2479D-02 |
| 0.18316D+04 | 0.2483D-02 |
| 0.20147D+04 | 0.2488D-02 |
| 0.21979D+04 | 0.2494D-02 |
| 0.23811D+04 | 0.2500D-02 |
| 0.25642D+04 | 0.2507D-02 |
| 0.27474D+04 | 0.2514D-02 |
| 0.29305D+04 | 0.2522D-02 |
| 0.31137D+04 | 0.2531D-02 |
| 0.32968D+04 | 0.2541D-02 |
| 0.34800D+04 | 0.2551D-02 |
| 0.34800D+04 | 0.2551D-02 |
| 0.36865D+04 | 0.2574D-02 |
| 0.38930D+04 | 0.2614D-02 |
| 0.40995D+04 | 0.2665D-02 |
| 0.45125D+04 | 0.2783D-02 |
| 0.47190D+04 | 0.2846D-02 |
| 0.49255D+04 | 0.2909D-02 |
| 0.51320D+04 | 0.2972D-02 |
| 0.53385D+04 | 0.3034D-02 |
| 0.57515D+04 | 0.3153D-02 |
| 0.59580D+04 | 0.3212D-02 |
| 0.61645D+04 | 0.3273D-02 |
| 0.63710D+04 | 0.3335D-02 |

# CHAPTER 3: THE THREE POTENTIAL DESCRIPTION OF LIQUID CORE DYNAMICS

In section 3.1 we give the derivation of the THPD following unpublished notes by Rochester, and discuss its advantages. In section 3.2 we include the effect of ellipticity in the equations.

## 3.1 The Three Potential Description

First we define two scalar variables

$$\zeta \equiv \alpha^2 \nabla \cdot \boldsymbol{u} \tag{3.1}$$

$$\chi \equiv \frac{p_1}{\rho_0} - V_1 \tag{3.2}$$

Using these variables and equation (1.6) we can rewrite equations (1.17)-(1.19) as

$$\omega^2 \boldsymbol{u} - 2i\omega\Omega\hat{\boldsymbol{e}}_3 \times \boldsymbol{u} = \nabla\chi + \frac{\beta\zeta}{\alpha^2}\boldsymbol{g}_0 \tag{3.3}$$

$$\frac{\alpha^2}{4\pi G\rho_0}\nabla^2 V_1 = \beta\zeta - (1-\beta)(\chi + V_1) \tag{3.4}$$

$$\boldsymbol{u}\cdot\boldsymbol{g}_0 = -(\zeta + \chi + V_1) \tag{3.5}$$

We cross multiply $\hat{e}_3$ with equation (3.3) to get

$$\omega^2\hat{e}_3\times u - 2i\omega\Omega[\hat{e}_3\hat{e}_3\cdot u - u] = \hat{e}_3\times\nabla\chi + \frac{\beta\zeta}{\alpha^2}\hat{e}_3\times g_0 \tag{3.6}$$

To find $\hat{e}_3\cdot\mathbf{u}$ we take the dot product of $\hat{e}_3$ with equation (3.3) to get

$$\hat{e}_3\cdot u = \frac{1}{\omega^2}[\hat{e}_3\cdot\nabla\chi + \frac{\beta\zeta}{\alpha^2}\hat{e}_3\cdot g_0] \tag{3.7}$$

Using (3.6) and (3.7), the momentum equation can be written as

$$\omega^2(1-\sigma^2)u = \Gamma_p\cdot\nabla\chi - \frac{\beta C^*\zeta}{\alpha^2} \tag{3.8}$$

with $\sigma=2\Omega/\omega$ and

$$\Gamma_p \equiv 1 - \sigma^2\hat{e}_3\hat{e}_3 + i\sigma\hat{e}_3\times 1 \tag{3.9}$$

$$C \equiv -g_0 + \sigma^2\hat{e}_3\cdot g_0\hat{e}_3 + i\sigma\hat{e}_3\times g_0 \tag{3.10}$$

It should be emphasized that our $\sigma$ is the inverse of the definition used by Smylie and Rochester (1981), Wu and Rochester (1990). Taking divergence of (3.8) and using (3.2) we get

$$\omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2} = \nabla\cdot(\Gamma_p\cdot\nabla\chi) - \nabla\cdot(\frac{\beta C^*\zeta}{\alpha^2}) \tag{3.11}$$

Taking the dot product of (3.8) with $g_0$ we get

$$\omega^2(1-\sigma^2)u\cdot g_0 = -C\cdot\nabla\chi + \frac{\beta}{\alpha^2}[g_0^2 - \sigma^2(\hat{e}_3\cdot g_0)^2]\zeta \tag{3.12}$$

Substituting for $\mathbf{u}.\mathbf{g}_0$ from (3.5) into (3.12) we get

$$\omega^2(1-\sigma^2)(\chi+V_1)+\frac{\beta B}{\alpha^2}\zeta=C\cdot\nabla\chi \tag{3.13}$$

where

$$B=\frac{\omega^2\ \alpha^2(1-\sigma^2)}{\beta}+\mathbf{g}_0\cdot\mathbf{g}_0-\sigma^2(\hat{e}_3\cdot\mathbf{g}_0)^2 \tag{3.14}$$

Substituting (3.1) and (3.2) into Poisson's equation we get

$$\frac{\alpha^2}{4\pi G\rho_0}\nabla^2V_1=\beta\zeta-(1-\beta)(\chi+V_1) \tag{3.15}$$

Rewrite equations (3.13), (3.11) and (3.15) as

$$C\cdot\nabla\chi=\omega^2(1-\sigma^2)(\chi+V_1)+\frac{\beta B}{\alpha^2}\zeta \tag{3.16}$$

$$\nabla\cdot(\Gamma_p\cdot\nabla\chi)=\nabla\cdot(\frac{\beta C^*\zeta}{\alpha^2})+\omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2} \tag{3.17}$$

$$\nabla^2V_1=\frac{4\pi G\rho_0}{\alpha^2}\ [\beta\zeta-\ (1-\beta)(\chi+V_1)] \tag{3.18}$$

The three *scalar* differential equations above, two of which are of second order and one of first, constitute the THPD. They describe without approximation the dynamics of the inviscid, non-conducting liquid core through three *scalar* potentials $\chi$, $\zeta$ and $V_1$. We note again that these equations are linearized in the field variables and therefore the Earth's configuration at all time is assumed to be close to its equilibrium configuration.

If the liquid core is assumed to be incompressible and homogeneous, $\zeta/\alpha^2=0$ and equation (3.17) reduces to the Poincaré equation

$$\nabla\cdot(\Gamma_p\cdot\nabla\chi)=\nabla^2\chi-\sigma^2(\hat{\ell}_3\cdot\nabla)^2\chi=0 \tag{3.19}$$

and (3.18) reduces to the Poisson equation.

If w. solve for $\zeta$ in (3.16)

$$\zeta=\frac{\alpha^2}{\beta B}[C\cdot\nabla\chi-\omega^2(1-\sigma^2)(\chi+V_1)] \tag{3.20}$$

and substitute into (3.17) and (3.18) we get

$$\begin{aligned}&\nabla\cdot[\Gamma_p\cdot\nabla\chi-\frac{C^*}{B}[C\cdot\nabla\chi-\omega^2(1-\sigma^2)(\chi+V_1)]]\\&-\frac{\omega^2(1-\sigma^2)}{\beta B}[C\cdot\nabla\chi-\omega^2(1-\sigma^2)(\chi+V_1)]=0\end{aligned} \tag{3.21}$$

$$\frac{\nabla^2 V_1}{4\pi G\rho_0}-\frac{C\cdot\nabla\chi-\omega^2(1-\sigma^2)(\chi+V_1)}{B}-\frac{(1-\beta)(\chi+V_1)}{\alpha^2}=0 \tag{3.22}$$

Equations (3.21) and (3.22) constitute the two potential description (Wu and Rochester 1990). The subseismic approximation (SSA) is equivalent to dropping the term in $\chi+V_1$. Then (3.21) reduces to the subseismic wave equation (SSWE) [Smylie & Rochester 1981, Rochester 1989].

The advantages of THPD are that:

(a) all dependent field variables are scalar, so there is no need to introduce the spheroidal/toroidal representation of vector fields;

(b) in a non-neutrally stratified liquid core it is possible that B=0 for some frequency ranges i.e. when

$$\frac{\beta}{\alpha^2}[\mathbf{g}_0\cdot\mathbf{g}_0-\sigma^2(\mathbf{e}_3\cdot\mathbf{g}_0)^2]=\omega^2(1-\sigma^2) \qquad (3.23)$$

Since B appears in denominators in both equations of TPD (Wu and Rochester 1990), this description is not appropriate at and near these frequency ranges and the equations must be replaced by a more complicated form of TPD. In THPD this disadvantage never occurs;

(c) in THPD B appears only in the combination $\beta B$ which remains finite as $\beta \to 0$;

(d) the description involves no derivatives higher than second. In chapter 5 we show that when a Galerkin method is used to solve the governing equations, the divergence theorem can be used to replace the volume integrals involving second derivatives by surface integrals involving first derivatives in a form for which the boundary conditions are 'natural';

(e) THPD can be solved using a variational principle as shown by Rochester.

### 3.2 Adjustments for Ellipticity

There are different methods of including the effect of ellipticity in the equations of core dynamics. Wu (1993) adopts a coordinate system in which all variables are functions of $r_0$, the mean radius of the equipotential on which the field point is positioned.

We find it more appropriate to adopt a procedure similar to, but slightly improving on, that suggested by Smith (1974). This procedure is as follows.

In the interior of the outer core we let all variables be functions of $r,\theta,\phi$. The governing equations are then integrated over the volume of radius r given by (2.22). In the interior of the outer core

$$\rho(r)=\rho_0(r_0)=\rho_0(r+\frac{2}{3}reP_2)=\rho_0(r)+\frac{2}{3}eP_2\frac{d\rho_0}{dr} \tag{3.24}$$

$$\alpha(r)=\alpha(r_0)=\alpha(r+\frac{2}{3}reP_2)=\alpha(r)+\frac{2}{3}eP_2\frac{d\alpha}{dr} \tag{3.25}$$

using Taylor expansion and ignoring terms higher than first order in ellipticity. The expression for the gravitational acceleration, $g_0$, is given as

$$g_0(r)=-\hat{r}[g_o(r)-\frac{2}{3}\Omega^2 r+\frac{2}{3}\frac{d}{dr}(reg_0(r))\ P_2]-\hat{\theta}\frac{2}{3}eg_0(r)P^1{}_2 \tag{3.26}$$

(see derivation in appendix 1) where $g_0$(r) in right hand side of (3.26) is the gravitational acceleration at point r in a spherical non-rotating body of radius r.

Substituting (3.24)-(3.26) into (3.16)-(3.18) we get

$$C\cdot\nabla\chi=\omega^2(1-\sigma^2)(\chi+V_1)+\beta B_1\zeta \tag{3.27}$$

$$\nabla\cdot(\Gamma_p\cdot\nabla\chi)=\ \nabla\cdot(\beta C^*{}_1\zeta)-\omega^2(1-\sigma^2)[1-\frac{4}{3}\frac{reP_2}{\alpha}\frac{d\alpha}{dr}]\frac{\zeta}{\alpha^2} \tag{3.28}$$

$$\nabla^2 V_1 = \frac{4\pi G\rho_0}{\alpha^2}[1-\frac{2}{3}reP_2(\frac{2}{\alpha}\frac{d\alpha}{dr}-\frac{1}{\rho_0}\frac{d\rho}{dr})][\beta\zeta-(1-\beta)(\chi+V_1)] \tag{3.29}$$

Equations (3.27)-(3.29) are the three scalar differential equations governing the dynamics of the liquid core in the three potential description. Here $B_1$ and $C^*_1$ are defined as

$$C^*_1=\frac{1}{\alpha^2}[C^*-\frac{4}{3}\frac{reP_2}{\alpha}\frac{d\alpha}{dr}(g_0-\frac{2}{3}\Omega^2 r)[\hat{r}-\sigma^2\cos\theta\ \hat{e}_3+i\sigma\hat{e}_3\times\hat{r}]] \tag{3.30}$$

$$B_1=\frac{\omega^2(1-\sigma^2)}{\beta}+\frac{1}{\alpha^2}\langle(g_0-\frac{2}{3}\Omega^2 r)[(1-\sigma^2\cos^2\theta)(g_0+\frac{4}{3}\frac{d}{dr}(reg_0)P_2)-\frac{4}{9}\sigma^2\epsilon g_0(P_2^1)^2$$
$$-\frac{4}{3\alpha}\frac{d\alpha}{dr}g_0(g_0-\frac{2}{3}\Omega^2 r)(1-\sigma^2\cos^2\theta)]-\frac{2}{3}(1-\sigma^2\cos^2\theta)g_0\Omega^2 r\ [1-\frac{4}{3\alpha}reP_2\frac{d\alpha}{dr}]\rangle \tag{3.31}$$

using Taylor expansion and ignoring terms higher than first order in ellipticity. Spherical harmonics are then used to represent the field variables in (2.27)-(2.29) as

$$(\chi,\ \zeta,\ V_1)=\sum_{n=|m|}^{\infty}(\chi_n^m,\ \zeta_n^m,\ \Phi_n^m)\ Y_n^m \tag{3.32}$$

In this thesis we consider constant values of $\beta$. In more realistic Earth models we could substitute

$$\beta(r)=\beta(r)+\frac{2}{3}reP_2\frac{d\beta}{dr} \tag{3.33}$$

in the equations and keep terms to first order in ellipticity, although even this may be an unnecessary refinement since $\beta$ is so poorly known.

# CHAPTER 4: BOUNDARY CONDITIONS ON AN ELLIPSOIDALLY-STRATIFIED EARTH

The boundary conditions require continuity in a number of functions $f(r,\theta,\phi)$ [normal displacement, gravitational potential, gravitational flux, normal stress] across the surfaces (2.22), i.e. continuity in

$$[f(r,\theta,\phi)\quad]_{r=RHS\,(2.22)} = f(r_0,\theta,\phi) - \frac{2}{3} r_0\ e(r_0)\ P_2\left(\ \frac{\partial f}{\partial r}\ \right)_{r_0} \tag{4.1}$$

using Taylor expansion to first order in ellipticity. Therefore at the core mantle boundary (CMB) we require continuity in

$$\hat{n}\cdot[u(r_0) - \frac{2}{3} re\ P_2 \frac{\partial u}{\partial r}] \tag{4.2}$$

$$V_1(r_0) - \frac{2}{3} re\ P_2 \frac{\partial V_1}{\partial r} \tag{4.3}$$

$$\hat{n}\cdot[\nabla V_1(r_0) - 4\pi G\rho_0\ u(r_0) - \frac{2}{3} re\ P_2(\frac{\partial}{\partial r}\nabla V_1 - 4\pi G\rho_0 \frac{\partial u}{\partial r})] \tag{4.4}$$

$$\hat{n}\cdot[\tilde{\tau}(r_0) - \frac{2}{3} re\ P_2 \frac{\partial \tilde{\tau}}{\partial r}] \tag{4.5}$$

where for brevity we write

$$(\frac{\partial f}{\partial r}\ )_{r=r_0} \equiv \frac{\partial f}{\partial r}. \tag{4.6}$$

We use conservation of the Earth's total angular momentum in place of (4.5). To satisfy (4.2)-(4.4) we need information from the dynamics of the liquid core as well as those of the mantle. We have already dealt with the dynamics of the outer core in chapters 1 and 3. The dynamics of the mantle are dealt with in this chapter. In section 4.1 the expressions for **u** and $V_1$ in the mantle are derived; in section 4.2 we derive the expression for the conservation of the Earth's total angular momentum and in section 4.3 we expand the boundary conditions (4.2)-(4.4). For lack of time we implement these boundary conditions only for the Earth model described in chapter 2. However, these boundary conditions can easily be adjusted to include the inner core boundary (ICB) as well.

### 4.1 Dynamics of a Rigid Wobbling Mantle

In section 1.1 we chose a reference frame which rotates steadily with angular velocity $\Omega\hat{e}_3$. We let a set of principal axes $\hat{E}_i$ fixed in the (supposedly rigid) mantle rotate with angular velocity

$$\omega_m = \Omega(\hat{e}_3 + m) \tag{4.7}$$

where **m** is purely oscillatory. Therefore,

$$\frac{d\hat{E}}{dt} = \omega_m \times \hat{E} \tag{4.8}$$

In this thesis we are concerned with pure wobble, with no change in the length of the day, therefore we set $\hat{E}_3 \cdot \omega_m = \Omega$, i.e. $m_3 = 0$. Therefore

$$\boldsymbol{m}=\gamma(\hat{\boldsymbol{e}}_1-i\hat{\boldsymbol{e}}_2)e^{i\omega t} \tag{4.9}$$

where $\gamma$ is real and positive, provided we adopt the convention that $\omega>0$ corresponds to a prograde wobble and $\omega<0$ to a retrograde one. A particle dm at **r** in the wobbling mantle will have a velocity relative to the reference frame given by

$$\boldsymbol{v}= (\omega_m-\omega_r)\times\boldsymbol{r}= \Omega\ \boldsymbol{m}\times\boldsymbol{r} \tag{4.10}$$

Now

$$\boldsymbol{v}= \frac{\partial \boldsymbol{u}}{\partial t}= i\omega\boldsymbol{u} \tag{4.11}$$

Therefore we can write the displacement **u** in the disturbed mantle as

$$\boldsymbol{u}=\frac{\Omega\boldsymbol{m}\times\boldsymbol{r}}{i\omega} \tag{4.12}$$

In terms of $\hat{e}_i$, $\hat{E}_i$ can be written as

$$\hat{E}_i=\sum_j a_{ij}\hat{e}_j \tag{4.13}$$

Since $\hat{E}_i$ remain in close alignment with $\hat{e}_i$, we can write

$$a_{ij}=\delta_{ij}+\beta_{ij} \tag{4.14}$$

where $\beta_{ij}$ are of order $\gamma$. Substituting (4.14) and (4.13) into (4.8) we get

$$\frac{d\beta_{ij}}{dt}=\Omega\sum_{k} e_{ijk}m_k \tag{4.15}$$

to first order in small quantities. $\epsilon_{ijk}$ in (4.15) is the standard permutation symbol. From (4.10) and (4.15) we conclude that

$$\begin{aligned} &\beta_{11}=\ \beta_{22}=\ \beta_{33}=0, \\ &\beta_{12}=0 \quad (since \quad m_3=0), \\ &(\beta_{31},\ \beta_{32})=\ -\frac{\Omega}{\omega}\gamma(1,-i)\ e^{i\omega t} \end{aligned} \tag{4.16}$$

At a point **r** in the disturbed mantle the Eulerian perturbation in the gravitational potential $V_1$ is

$$V_1=\ V_1(m)+V_1(lc) \tag{4.17}$$

where $V_1(m)$ and $V_1(lc)$ are the contributions to $V_1$ from rearrangement of mass of the mantle and liquid core respectively. Outside the liquid core $V_1(lc)$ must be a solution of Laplace's equation and must vanish as r tends to infinity, so we write

$$V_1(lc)=\ \sum_{n=1}^{N}\sigma_n\left(\frac{b}{r}\right)^{n+1}P_n^{-m}(\cos\theta)e^{-im\phi} \tag{4.18}$$

were b is the mean radius of the CMB and is used as a scale factor so that $\sigma_n$ have the dimensions of potential. The factor $e^{i\omega t}$ is dropped from (4.18) for convenience.

To evaluate $V_1(m)$ we proceed as follows. In the equilibrium configuration the gravitational potential due to the mantle, at a point **r**$=(r,\theta,\phi)$ in the mantle, is

$$V_0(r) = 4\pi G \left( \int_b^r \rho'_0 \left[ \frac{r'^2_0}{r} - \frac{2}{15} \frac{\frac{d}{dr'_0}(r'^5_0 e(r'_0))}{r^3} P_2(\cos\theta) \right] dr'_0 \right. \tag{4.19}$$

$$\left. + \int_r^R \rho'_0(r'_0) \left[ r'_0 - \frac{2}{15} \frac{de(r'_0)}{dr'_0} r^2 P_2(\cos\theta) \right] dr'_0 \right)$$

using D.10 but replacing the lower limit 0 in the first integral by b, the mean radius of the CMB. Here $r_0$ is the mean radius of the spheroid of constant density through **r**. In the wobbling configuration the mass element at **r** is displaced to **r+u**, where **u** is given by (4.12). The gravitational potential there, due to the mantle, is

$$V(\mathbf{r}+\mathbf{u}) = V_0(\mathbf{r}) \tag{4.20}$$

since the mantle is taken as rigid. The Eulerian perturbation in gravitational potential at **r** is

$$V_1(m) \equiv V(\mathbf{r}) - V_0(\mathbf{r}) \tag{4.21}$$

$$= V_0(\mathbf{r}-\mathbf{u}) - V_0(\mathbf{r}) \tag{4.22}$$

using (4.20)

$$= -\mathbf{u}\cdot\nabla V_0 \tag{4.23}$$

to first order in **u**. Using (4.19) and (4.12) we reduce (4.21) to

$$V_1(m) = -\frac{\Omega}{i\omega}\, \mathbf{m}\cdot\hat{\boldsymbol{\phi}}\, \frac{\partial V_0}{\partial\theta} \tag{4.24}$$

$$= \frac{16\pi G\Omega}{5\omega r^3}\gamma\left[\int_b^r \rho_0(\xi)\frac{d}{d\xi}(\xi^5 e(\xi))d\xi + r^5\int_r^R \rho_0(\xi)\frac{de(\xi)}{d\xi}d\xi\right]P_2^{-1}(\cos\theta)e^{-i\phi} \quad (4.25)$$

dropping the factor $e^{i\omega t}$ and using (4.19), (4.9) and the result

$$\sin\theta\ \cos\theta = 2\ P_2^{-1} \quad (4.26)$$

## 4.1 Angular Momentum Conservation of the Earth

The equation for the conservation of Earth's total angular momentum $H_E$ (for our Earth model) is

$$\frac{dH_E}{dt} = \frac{dH_m}{dt} + \frac{dH_{lc}}{dt} = L \quad (4.27)$$

where $H_m$ and $H_{lc}$ are respectively the angular momentum of the mantle and liquid core and L is the external torque exerted on the Earth; L=0 for free wobble.

The angular momentum of a wobbling rigid mantle has the form

$$H_m = \int_{mantle} \xi \times \frac{d\xi}{dt} dm = \int_{mantle} \xi \times (\omega_m \times \xi) dm = \tilde{I}_m \cdot \omega_m \quad (4.28)$$

where $\xi$ is the position vector of dm in the wobbling mantle and $\tilde{I}_m$ is the mantle's inertia tensor,

$$\tilde{I}_m = \int_{mantle} (\xi \cdot \xi\ \tilde{1} - \xi\xi) dm = A_m \tilde{1} + (C_m - A_m)\hat{E}_3\hat{E}_3 \quad (4.29)$$

since $\hat{\mathbf{E}}_i$ are principal axes for the wobbling mantle. Substituting for $\hat{\mathbf{E}}_i$ from (4.13) we get

$$\bar{I}_m = A_m \,\bar{1} + (C_m - A_m)[\hat{e}_3\hat{e}_3 + \sum_k \beta_{3k}(\hat{e}_3\hat{e}_k + \hat{e}_k\hat{e}_3)] \qquad (4.30)$$

where $\bar{1}$ is the unit dyadic. Substituting (4.30) and (4.16) into (4.28) we find

$$H_m = \Omega\langle C_m\hat{e}_3 + A_m[1 - \frac{\Omega}{\omega}e_m]m\rangle \qquad (4.31)$$

where $A_m$ and $C_m$ are the moments of inertia of the mantle with respect to an equatorial and a polar axis respectively and $e_m$ is defined as

$$e_m = \frac{C_m - A_m}{A_m} \qquad (4.32)$$

$$\therefore \quad \frac{dH_m}{dt} = A_m\,\Omega(1 - \frac{\Omega}{\omega}e_m)\,\frac{dm}{dt} = A_m\,\Omega(1 - \frac{\Omega}{\omega}e_m)(\sum_k \frac{dm_k}{dt}\hat{e}_k + \Omega\hat{e}_3 \times m)$$
$$= A_m\,\Omega(1 - \frac{\Omega}{\omega}e_m)(\omega + \Omega)\hat{e}_3 \times m \qquad (4.33)$$

The angular momentum of the liquid core is

$$H_{lc} = C_{lc}\,\Omega\hat{e}_3 + \Omega\int_{lc}[2u\cdot r\,\hat{e}_3 - (u\,r + r\,u)\cdot\hat{e}_3]dm + \int_{lc} r\times v\,dm \qquad (4.34)$$

(Wu 1993). Substituting for u from (4.12) into above equation we get

$$H_{lc} = C_{lc}\,\Omega\hat{e}_3 + \Omega\int_{lc}[2u\cdot r\,\hat{e}_3 - (u\,r + r\,u)\cdot\hat{e}_3]dm + i\omega\int_{lc} r\times u\,dm \qquad (4.35)$$

Therefore

$$\frac{dH_{lc}}{dt}=\sum_i \frac{dH_{lc,i}}{dt}\hat{e}_i+\Omega\ \hat{e}_3\times H_{lc} \tag{4.36}$$

Substituting (4.35) into (4.36) we get

$$\begin{aligned}\frac{dH_{lc}}{dt}=&i\omega\Omega\int_{lc}[2\ u\cdot r\ \hat{e}_3-(ur+ru)\cdot\hat{e}_3]\ dm-\omega^2\int_{lc}r\times u\ dm\\ &-\Omega^2\int_{lc}\hat{e}_3\times[(ur+ru)\cdot\hat{e}_3]\ dm+i\omega\Omega\int_{lc}\hat{e}_3\times(r\times u)\ dm\end{aligned} \tag{4.37}$$

In a slightly elliptical liquid core the displacement **u** takes the form

$$\omega^2(1-\sigma^2)u=\Gamma_p\cdot\nabla\chi-\beta C_1^*\zeta \tag{4.38}$$

using (3.8).

We substitute (4.33), (4.37) into (4.27) and use (4.38) and the following two identities

$$P_2\ P_n^m=A_n^m\ P_{n-2}^m+B_n^m\ P_n^m+C_n^m\ P_{n+2}^m \tag{4.39}$$

$$P_2^1\ \frac{dP_n^m}{d\theta}=2(n+1)A_n^m\ P_{n-2}^m+3B_n^m\ P_n^m-2nC_n^m\ P_{n+2}^m \tag{4.40}$$

where

$$A_m^n=\frac{3}{2}\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)},\quad B_n^m=\frac{n(n+1)-3m^2}{(2n+3)(2n-1)},\quad C_n^m=\frac{3}{2}\frac{(n+2-m)(n+1-m)}{(2n+3)(2n+1)}, \tag{4.41}$$

and define

$$Sg = g_0 - \frac{2}{3}\Omega^2 r - \frac{2}{3} e g_0,$$

$$Se = \frac{2}{3}[r\frac{d}{dr}(eg_0) - eg_0 + re(g_0 - \frac{2}{3}\Omega^2 r)(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{\alpha}\frac{d\alpha}{dr})] \tag{4.42}$$

to obtain the final form of (4.27):

$$\begin{aligned}
&2\, A_m \Omega\gamma\ (1 - \frac{\Omega}{\omega}\ e_m)(\omega + \Omega) - \frac{\pi\sigma^2(2+\sigma)}{15(1+\sigma)}[\int_0^b [\rho_0(\frac{d\chi_2^{-1}}{dr} + \frac{3\chi_2^{-1}}{r}) \\
&+ \frac{2}{3} re \frac{d\rho_0}{dr}\ \langle A_4^{-1}(\frac{d\chi_4^{-1}}{dr} + 5\frac{\chi_4^{-1}}{r}) + (5B_2^{-1} - 2)\frac{\chi_2^{-1}}{r} + B_2^{-1}\frac{d\chi_2^{-1}}{dr} \\
&+ \frac{3}{2}\frac{\omega}{\Omega}(\frac{\omega}{\Omega} + 1)\frac{\chi_2^{-1}}{r}\rangle] r^3\ dr - \frac{2}{3} b^4 e \rho_0(b^-)\langle\ A_4^{-1}(\frac{d\chi_4^{-1}}{dr} + 5\frac{\chi_4^{-1}}{r}) \\
&+ (5B_2^{-1} - 2)\frac{\chi_2^{-1}}{r} + B_2^{-1}\ \frac{d\chi_2^{-1}}{dr} + \frac{3}{2}\frac{\omega}{\Omega}(\frac{\omega}{\Omega} + 1)\frac{\chi_2^{-1}}{r}\rangle_b + \int_0^b \frac{\beta\rho_0}{\alpha^2}\langle Sg\ \zeta_2^{-1} \\
&+ Se\ (A_4^{-1}\ \zeta_4^{-1} + B_2^{-1}\ \zeta_2^{-1}) - (\frac{\omega}{2\Omega} + 1)(\frac{\omega}{\Omega} - 1) e g_0 \zeta_2^{-1}\rangle\ r^3 dr \\
&- \frac{2}{3} b^4 e(b)\langle\frac{\beta\rho_0}{\alpha^2}(g_0 - \frac{2}{3}\Omega^2 r)(A_4^{-1}\ \zeta_4^{-1} + B_2^{-1}\ \zeta_2^{-1})\rangle_b] = 0
\end{aligned} \tag{4.43}$$

Here we have made use of the orthogonality relations among associated Legendre functions. Expressions for $A_m$ and $e_m$ are given in appendix 1. Expression (4.43) indicates that only quantities of azimuthal order number m=-1 contribute to angular momentum in our representation.

## 4.3 Expansion of the Boundary Conditions

We now turn to the boundary conditions (4.2)-(4.4). We treat $\hat{\mathbf{n}}$ as pointing opposite to $\mathbf{g}_0$ on the core-mantle boundary. The first boundary condition requires

$$[\hat{n}\cdot(u(r_0)-\frac{2}{3}reP_2\frac{\partial u}{\partial r})\ ]_{(b^-,\theta,\phi)}=[\hat{n}\cdot(u(r_0)-\frac{2}{3}reP_2\frac{\partial u}{\partial r})\ ]_{(b^+,\theta,\phi)} \tag{4.44}$$

where $b^-$ and $b^+$ denote the fluid and solid side of the CMB respectively. Substituting for **u** in the mantle from (4.12) and using

$$\hat{n}=\hat{r}-4e(r)P_2^{-1}\ \hat{\theta} \tag{4.45}$$

(Smith 1974) and keeping terms up to first order in ellipticity we get

$$[\ \hat{n}\cdot(u(r_0)-\frac{2}{3}reP_2\frac{\partial u}{\partial r})\ ]_{b^-,\theta,\phi}=\frac{4\Omega\gamma}{\omega}e(b)P_2^{-1}\ e^{-i\phi} \tag{4.46}$$

Since we are making use of the '*natural*' character of the boundary conditions in the next chapter we need not expand (4.46) any farther.

At the CMB, equation (4.3) implies

$$[\ V_1(r_0)-\frac{2}{3}re(r)P_2\frac{\partial V_1}{\partial r}\ ]_{(b^-,\theta,\phi)}=[V_1(r_0)-\frac{2}{3}re(r)P_2\frac{\partial V_1}{\partial r}\ ]_{(b^+,\theta,\phi)} \tag{4.47}$$

Using (4.18), (4.25), (3.29) and keeping terms up to first order in ellipticity, (4.47) reduces to

$$\frac{6G\Omega}{\omega}\gamma[\frac{8\pi b^2}{15}\int_b^R \rho_0 \frac{de}{dr}dr]\delta_{n2}+\sigma_n$$

$$+\frac{2}{3}e(b)[(n+3)A_{n+2}^{-1}\sigma_{n+2}+(n+1)B_n^{-1}\sigma_n+(n-1)C_{n-2}^{-1}\sigma_{n-2}] \tag{4.48}$$

$$=\Phi_n^{-1}-\frac{2}{3}be(b)[A_{n+2}^{-1}\left(\frac{d\Phi_{n+2}^{-1}}{dr}\right)_{b^-}+B_n^{-1}\left(\frac{d\Phi_n^{-1}}{dr}\right)_{b^-}+C_{n-2}^{-1}\left(\frac{d\Phi_{n-2}^{-1}}{dr}\right)_{b^-}]$$

using the linear independence of the associated Legendre functions. In (4.48) $\delta_{n2}=1$ if n=2 and 0 otherwise. Also at the CMB the boundary condition (4.4) requires

$$[\frac{\partial V_1}{\partial r}-\frac{2}{3}re(r)P_2\frac{\partial^2 V_1}{\partial r^2}+\frac{2e(r)}{3r}P_2^1\frac{\partial V_1}{\partial \theta}]_{(b^-,\theta,\phi)}^{(b^+,\theta,\phi)}$$

$$=\frac{16\pi G\Omega}{\omega}\gamma e(b)b[\rho_0(b^+)-\rho_0(b^-)]P_2^{-1}e^{-i\phi} \tag{4.49}$$

where we have made use of the continuity of $\hat{n}\cdot\mathbf{u}(r)$ across CMB. Using (4.18), (4.25), (4.39) and (4.40), (4.49) can be written

$$\frac{16\pi G\Omega b}{5\omega}[2\int_b^R \rho_0 de+5\rho_0(b^-)e(b^-)]\gamma+\frac{\sigma_n}{b}\langle[-(n+1)-\frac{2}{3}eP_2(n+1)(n+2)]P_n^{-1}$$

$$+\frac{2}{3}eP_2^{-1}\frac{dP_n^{-1}}{d\theta}\rangle_b-\langle[\frac{d\Phi_n^{-1}}{dr}-\frac{2}{3}r\,eP_2\frac{d^2\Phi_n^{-1}}{dr^2}]P_n^{-1}+\frac{2}{3}\frac{e}{r}\Phi_n^{-1}P_2^1\frac{dP_n^{-1}}{d\theta}\rangle_{b^-}=0 \tag{4.50}$$

In the next chapter it will become clear why we have not removed $P^{-1}_n$ from equation (4.50).

# CHAPTER 5: THE GALERKIN METHOD AND THE NUMERICAL INTEGRATION OF THE GOVERNING EQUATIONS

In this chapter we use the Galerkin method to obtain approximate solutions of the governing equations of core dynamics. In section 5.1 we give a summary of the Galerkin method applicable to a system of simultaneous PDEs subject to boundary conditions. In section 5.2 this method is applied to the governing equations and in section 5.3 we describe a method which we use to integrate the Galerkin equations with respect to $\theta$.

## 5.1 The Galerkin Method

Suppose we have a set of functions

$$\chi=(\chi_1,\ \chi_2,\ \ldots,\ \chi_N) \tag{5.1}$$

which satisfies, in a region V, the set of simultaneous PDEs

$$\sum_{j=1}^{N} L_{ij}\ \chi_j = 0 \tag{5.2}$$

for every i (i=1,...,N) where $L_{ij}$ are linear partial differential operators. Suppose also that there are a number of associated boundary conditions which have to be satisfied on the boundary S of V

$$\sum_{j=1}^{M} B_{ij}\ \chi_j=0 \tag{5.3}$$

for every i (i=1,...,N) where $B_{ij}$ are linear (possibly partial differential) operators.

Using a basis set $f_k$, k=1,...,L, we introduce trial functions

$$\chi_j = \sum_{k=1}^{L} C_{jk} f_k \qquad (j=1, \ldots, N) \tag{5.4}$$

which need not a priori satisfy the boundary conditions. The Galerkin method tries to make $\Sigma_j L_{ij} \chi_j$ as nearly null as possible by requiring

$$\sum_{j=1}^{N} \sum_{k=1}^{L} \int_V f_l^* \, L_{ij} \, C_{jk} f_k \, dV = 0 \tag{5.5}$$

for $l$=1,...,L, where * denotes the complex conjugate. The resulting set of linear homogeneous Galerkin equations have the form

$$\sum_{j=1}^{N} \sum_{k=1}^{L} H_{lijk} \, C_{jk} = 0 \tag{5.6}$$

with

$$H_{lijk} = \int_V f_l^* \, L_{ij} \, f_k \, dV \tag{5.7}$$

Since in general the trial functions do not a priori satisfy the boundary conditions, we choose a set of basis functions $\psi_l$ equal in number to those used in constructing the trial functions $\chi_j$ and extend (5.5) to require that

$$\sum_{j=1}^{N} \sum_{k=1}^{L} \left[ \int_V f_l^* \, L_{ij}(C_{jk} f_k) \, dV + \int_S \psi_l^* \, B_{ij}(C_{jk} f_k) \, ds \right] = 0 \tag{5.8}$$

The Galerkin equations then take the form

$$\sum_{k=1}^{L}\sum_{j=1}^{N} F_{lijk}\, C_{jk} = 0 \qquad (5.9)$$

where

$$F_{lijk} = H_{lijk} + \int_S \psi_l^* \, B_{ij}\, f_k \, dS \qquad (5.10)$$

By proper choice of weight functions, and by use of the divergence theorem, it may be possible to remove the surface integral from the boundary conditions (i.e. by choosing $\psi_l$ so that the surface integral arising from the boundary conditions is cancelled by the surface integral produced by using the divergence theorem.) When this is possible, the boundary conditions are called *natural.*

We can write (5.7) in matrix form by setting up one-to-one correspondence $m=N(l-1)+i$, $n=N(k-1)+j$, so $C_{jk}=a_n$, $H_{lijk}=G_{mn}$. Then (5.7) becomes

$$\sum_n G_{mn} a_n = 0, \quad m=1,\ldots,LN. \qquad (5.11)$$

$\bar{G}$ is then a (LN) x (LN) matrix which hereafter we refer to as the coefficient matrix.

## 5.2 Application of the Galerkin Method to THPD

Since only spherical harmonics of order -1 are needed to represent the field variables associated with wobble/nutation, we represent the functions defined in chapter 3 as:

$$(\chi,\ \zeta,\ V_1)=\sum_{n=1}^{N}\ (\chi_n^{-1},\ \zeta_n^{-1},\ \Phi_n^{-1})\ Y_n^{-1} \tag{5.12}$$

where N denotes the degree of truncation. We construct the trial functions as follows:

$$\zeta_n^{-1}=\sum_{l=1}^{L} a_{[L(n-1)+l]}\ f_l(r) \tag{5.13}$$

$$\chi_n^{-1}=\sum_{l=1}^{L} a_{[L(N+n-1)+l]}\ f_l(r) \tag{5.14}$$

$$\Phi_n^{-1}=\sum_{l=1}^{L} a_{[L(2N+n-1)+l]}\ f_l(r) \tag{5.15}$$

where $a_i$ are constant and $f_l$ are functions of $r$ only. Before proceeding further we should note how we integrate the equations over ellipsoidal volumes. Suppose we have an integral over the liquid core volume,

$$I=\int_{lc} f(r)\ g(\theta)\ dv \tag{5.16}$$

To first order in ellipticity we can write this as

$$I=2\pi\int_0^{\pi}\int_0^{b(1-\frac{2}{3}e(b)P_2)} f(r)\ g(\theta) r^2\ \sin\theta d\theta\ dr \tag{5.17}$$

after integrating over $\phi$. To first order in ellipticity we can write (5.17) as

$$I=2\pi\int_0^{\pi} g(\theta)\left[\int_0^{b} f(r)\ r^2 dr-\frac{2}{3}b^3\epsilon(b)\ f(b)\ P_2(\cos\theta)\right]\ \sin\theta\ d\theta \tag{5.18}$$

The volume integral in (5.18) is over a sphere of radius b and the surface integral corrects this volume integral for an ellipsoidal volume of radius b[1-2/3 $\epsilon$(b) $P_2$(cos $\theta$)].

### 5.2.1 Galerkin Formulation of the Entropy Equation

Rewrite equation (3.27) as

$$C\cdot\nabla\chi-\beta B_1\zeta-\omega^2(1-\sigma^2)(\chi+V_1)= 0 \tag{5.19}$$

The Galerkin representation of (5.19) is

$$\sum_{n=1}^{N}\sum_{l=1}^{L}\int_{lc} f_k\ P_j^{-1}e^{i\phi}(a_{L(N+n-1)+l}\ [C\cdot\nabla(f_l\ P_n^{-1}e^{-i\phi})-\omega^2(1-\sigma^2)f_l\ P_n^{-1}e^{-i\phi}]$$
$$-[\beta B_1\ a_{L(n-1)+l}+\omega^2(1-\sigma^2)a_{L(2N+n-1)+l}](f_l\ P_n^{-1}e^{-i\phi}))dv=0 \tag{5.20}$$

for each pair of indices (k,j), k=1,...,L, j=1,...,N. We have not added any of the boundary conditions to equation (5.20). Substituting for C and $B_1$ from (3.10), (3.26) and integrating over $\phi$ equation (5.20) becomes

$$2\pi[\sum_{l=1}^{L}\sum_{n=1}^{N}\int_0^b\int_0^\pi f_k\, P_j^{-1}\ [a_{L(N+n-1)+l}\langle s_1\ P_n^{-1}+s_2\ P_2\ P_n^{-1}+s_3\ P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}$$

$$+s_4\ (P_2)^2\ P_n^{-1}+s_5\ P_2\ P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}\rangle-a_{L(n-1)+l}\langle t_1\ P_n^{-1}+t_2\ P_2\ P_n^{-1}$$

$$+t_4\ (P_2)^2\ P_n^{-1}\rangle-\omega^2(1-\sigma^2)a_{L(2N+n-1)+l}f_l\ P_n^{-1}]\ r^2\ \sin\theta d\theta\ dr$$

$$\frac{2}{3}r\epsilon(b)\int_0^\pi P_2\ \langle\ f_k\ P_j^{-1}[a_{L(N+n-1)+l}[s_6\ P_n^{-1}+s_7\ P_2\ P_n^{-1} \tag{5.21}$$

$$+s_8 P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}]-a_{L(N+n-1)+l}[t_6\ P_n^{-1}+t_7\ P_2\ P_n^{-1}]]\rangle b^2\sin\theta\ d\theta]=0$$

for each (k,j). In the next section we describe a method which we use to integrate with respect to $\theta$ functions of the type (5.21). Different terms in (5.21) are defined as follows:

$$s_1=(g_0-\frac{2}{3}\Omega^2 r)[(1-\sigma^2)\frac{df_l}{dr}-\frac{\sigma}{r}f_l]+\frac{2}{3}\epsilon g_0(-\frac{\sigma^2}{3}\frac{df_l}{dr}+\frac{\sigma}{r}f_l) \tag{5.22}$$

$$s_2=-\frac{2\sigma^2}{3}(g_0-\frac{2}{3}\Omega^2 r)\frac{df_l}{dr}+\frac{2}{3}\frac{d}{dr}(r\epsilon g_0)[(1-\frac{1}{3}\sigma^2)\frac{df_l}{dr}-\frac{\sigma}{r}f_1]$$

$$+\frac{2}{3}\epsilon g_0(-\frac{\sigma^2}{3}\frac{df_l}{dr}+\frac{2\sigma}{r}f_l) \tag{5.23}$$

$$s_3=[-\frac{\sigma^2}{3r}(g_0-\frac{2}{3}\Omega^2 r)+\frac{2}{3r}\epsilon g_0(1-\frac{2}{3}\sigma^2)]f_l \tag{5.24}$$

$$s_4=-\frac{4\sigma^2}{9}r(g_0\frac{d\epsilon}{dr}+\epsilon\frac{dg_0}{dr})\frac{df_l}{dr} \tag{5.25}$$

$$s_5=\frac{2\sigma^2}{9}[\epsilon g_0-r(g_0\frac{d\epsilon}{dr}+\epsilon\frac{dg_0}{dr})]\ \frac{f_l}{r} \tag{5.26}$$

$$s_6=(g_0-\frac{2}{3}\Omega^2 r)[(1-\sigma^2)\frac{df_l}{dr}-\frac{\sigma}{r}f_l] \tag{5.27}$$

$$s_7=-\frac{2\sigma^2}{3}(g_0-\frac{2}{3}\Omega^2 r)\frac{df_l}{dr} \tag{5.28}$$

$$s_8=-\frac{\sigma^2}{3r}(g_0-\frac{2}{3}\Omega^2 r)f_l \tag{5.29}$$

$$t_1=(\omega^2(1-\sigma^2)+\frac{\beta}{\alpha^2}[(1-\frac{\sigma^2}{3})(g_0^2-\frac{4}{3}g_0\Omega^2 r)-\frac{8}{9}\epsilon g_0(g_0-2\Omega^2 r)\sigma^2])f_l \tag{5.30}$$

$$\begin{aligned} t_2=\frac{\beta}{\alpha^2}[-\frac{2}{3}(\sigma^2+\frac{2r\epsilon}{\alpha}\frac{d\alpha}{dr}(1-\frac{\sigma^2}{3}))(g_0-\frac{4}{3}\Omega^2 r)g_0 \\ +(g_0-\frac{2}{3}\Omega^2 r)(\frac{4}{3}\frac{d}{dr}(r\epsilon g_0)(1-\frac{\sigma^2}{3})-\frac{8\sigma^2}{9}\epsilon g_0)]\ f_l \end{aligned} \tag{5.31}$$

$$t_4=\frac{\beta}{\alpha^2}\frac{8\sigma^2}{9}\ [(g_0-\frac{4}{3}\Omega^2 r)g_0\frac{r\epsilon}{\alpha}\ \frac{d\alpha}{dr}-(g_0-\frac{2}{3}\Omega^2 r)(\frac{d}{dr}(r\epsilon g_0)-2\epsilon g_0)]\ f_l \tag{5.32}$$

$$t_6=[\omega^2(1-\sigma^2)+\frac{\beta}{\alpha^2}(1-\frac{\sigma^2}{3})(g_0-\frac{4}{3}\Omega^2 r)g_0]f_l \tag{5.33}$$

$$t_7=-\frac{2}{3}\frac{\beta}{\alpha^2}\sigma^2\; g_0(g_0-\frac{4}{3}\Omega^2)f_l \tag{5.34}$$

with

$$\cos^2\theta=\frac{1}{3}(2\;P_2+1),\quad (P_2^1)^2=2[P_2-2(P_2)^2+1] \tag{5.35}$$

### 5.2.2 Galerkin Formulation of the Momentum Equation

Equation (3.28 ) can be written as

$$\nabla\cdot(\Gamma_p\cdot\nabla\chi-\beta C^{*1}\zeta)-\omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2}=0 \tag{5.36}$$

The Galerkin representation of (5.36) is

$$\begin{aligned}&\int_{l_k} f_k\, P_j^{-1} e^{i\phi}[\nabla\cdot(\Gamma_p\cdot\nabla\chi-\beta C_1^*\zeta)-\omega^2(1-\sigma^2)\frac{\zeta}{\alpha^2}]\;dv\\&+\int_{S_k}\Psi_{kj}^*\;[(\hat{n}\cdot u)_- -(\hat{n}\cdot u)_+]ds=0\end{aligned} \tag{5.37}$$

where we have added boundary condition (4.2) to the above formulation. + and - in (5.37) denote the fluid and solid sides of the CMB. Applying the divergence theorem to (5.37) and using (4.37) we get

$$\omega^2(1-\sigma^2)\int_{S_k} f_k\, P_j^{-1}\, e^{i\phi}(\hat{n}\cdot \underset{-}{u})\; ds - \int_{k} \nabla(f_k\, P_j^{-1}\, e^{i\phi})\cdot[\Gamma_p\cdot\nabla\chi - \beta C_1^{*}\zeta]dv$$
$$-\omega^2(1-\sigma^2)\int_{k} f_l\, P_n^{-1}\frac{\zeta}{\alpha^2}dv + \int_{S_k}\psi_{kj}^{*}[(\hat{n}\cdot \underset{-}{u}) - (\hat{n}\cdot \underset{+}{u})]ds = 0 \tag{5.38}$$

Now, by choosing

$$\psi_{kj}^{*} = -\omega^2(1-\sigma^2) f_k\, P_j^{-1}\, e^{i\phi} \tag{5.39}$$

we can make use of the *natural* property of the boundary condition (4.46) and cancel surface integral produced from using the divergence theorem and the one arising from the boundary condition and write (5.38) as

$$\sum_{l=1}^{L}\sum_{n=1}^{N}[a_{L(N+n-1)+l}\left(\int_{k}\nabla(f_k\, P_j^{-1}\, e^{i\phi})\cdot\Gamma_p\cdot\nabla(f_l\, P_n^{-1}\, e^{-i\phi})\; dv\right.$$
$$\left.-[\frac{4\Omega\gamma}{\omega}b\epsilon(b)]\omega^2(1-\sigma^2)\int_{S_k} f_k^{-1}\, P_j^{-1}e^{i\phi}(P_2^{-1}e^{-i\phi})ds\right) \tag{5.40}$$
$$-a_{L(n-1)+l}\int_{k}[\nabla(f_k\, P_j^{-1}\, e^{i\phi})\cdot\beta C_1^{*} + \frac{\omega^2}{\alpha^2}(1-\sigma^2) f_k\, P_j^{-1}\, e^{i\phi}] f_l P_n^{-1}e^{-i\phi}dv] = 0$$

Substituting for $\Gamma_p$ and $C_1$ into (5.40) we get

$$
\begin{aligned}
&\sum_{l=1}^{L}\sum_{n=1}^{N}[a_{L(N+n-1)+l}\int_{lc}\langle\frac{df_k}{dr}P_j^{-1}[(1-\sigma^2\cos^2\theta)\frac{df_l}{dr}P_n^{-1}-\frac{f_l}{3r}P_2^1\frac{dP_n^{-1}}{d\theta}-\frac{\sigma}{r}f_lP_n^{-1}] \\
&+\frac{f_k}{r}[-\sigma^2\frac{df_l}{3dr}P_n^{-1}P_2^1\frac{dP_j^{-1}}{d\theta}-\frac{\sigma^2\sin^2\theta}{r}f_l\frac{dP_n^{-1}}{d\theta}\frac{dP_j^{-1}}{d\theta}-\sigma\frac{df_l}{dr}P_n^{-1}P_j^{-1} \\
&-\frac{\sigma}{r}f_l\frac{\cos\theta}{\sin\theta}(P_n^{-1}\frac{dP_j^{-1}}{d\theta}+P_j^{-1}\frac{dP_n^{-1}}{d\theta})+\frac{f_l}{r}(\frac{dP_n^{-1}}{d\theta}\frac{dP_j^{-1}}{d\theta}+\frac{P_n^{-1}P_j^{-1}}{\sin^2\theta})]\rangle dv \\
&-a_{L(2N+n-1)+l}\int_{lc}\langle\frac{\beta}{\alpha^2}[(g_0-\frac{2}{3}\Omega^2r+\frac{2}{3}\frac{d}{dr}(reg_0)P_2)[(1-\sigma^2\cos^2\theta)\frac{df_k}{dr}P_j^{-1} \\
&-\frac{\sigma^2P_2^1}{3r}f_k\frac{dP_j^{-1}}{d\theta}+\frac{\sigma}{r}f_kP_j^{-1}]+\frac{2}{3}eg_0[-\sigma^2(P_2^1)^2\frac{df_k}{dr}P_j^{-1}+(1-\sigma^2\sin^2\theta) \\
&\times\frac{df_k}{dr}P_2^1\frac{dP_j^{-1}}{d\theta}-\frac{\sigma}{3r}\cos^2\theta\, f_kP_j^{-1}]-\frac{4}{3}\frac{re}{\alpha}\frac{d\alpha}{dr}P_2\langle(g_0-\frac{2}{3}\Omega^2r) \\
&[(1-\sigma^2\cos^2\theta)\frac{df_k}{dr}P_j^{-1}-\frac{\sigma^2P_2^1}{3r}f_k\frac{dP_j^{-1}}{d\theta}+\frac{\sigma}{r}f_kP_j^{-1}]\rangle+\omega^2(1-\sigma^2)\frac{f_k}{\alpha^2}P_j^{-1} \\
&[1-\frac{4}{3\alpha}re\frac{d\alpha}{dr}P_2]]f_lP_n\rangle\,dv]-4\Omega\;\omega(1-\sigma^2)bea_{3LN+N+1}\int_{S_k}f_kP_j^{-1}\,P_2^{-1}ds=0
\end{aligned}
\tag{5.41}
$$

Here we have labelled $\gamma=a_{3LN+N+1}$. Using (5.34) and some other identity relations among associated Legendre functions (given in appendix 2) we write equation (5.41) as

$$2\pi[\sum_{l=1}^{L}\sum_{n=1}^{N}\int_0^b\int_0^\pi f_k P_j^{-1}\ [a_{L(N+n-1)+l}\langle w_1\ P_n^{-1}+w_2\ P_2\ P_n^{-1}$$

$$+w_3\ P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}\rangle-a_{L(n-1)+l}\langle z_1\ P_n^{-1}+z_2\ P_2\ P_n^{-1}+z_4\ (P_2)^2\ P_n^{-1}\rangle$$

$$-\omega^2(1-\sigma^2)a_{L(2N+n-1)+l}\ f_l\ P_n^{-1}]\ r^2\ \sin\theta d\theta\ dr+\frac{2}{3}re(b)$$

$$\times\int_0^\pi P_2\ \langle\ f_k\ P_j^{-1}[a_{L(N+n-1)+l}[w_6\ P_2\ P_n^{-1}+w_7\ P_2\ P_n^{-1}+w_8 P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}$$

$$+w_9(P_2)^2\ P_n^{-1}+w_{10}P_2\ P_2^1\ \frac{dP_n^{-1}}{d\theta}]-a_{L(N+n-1)+l}[z_6\ P_n^{-1}+z_7\ P_2\ P_n^{-1} \tag{5.42}$$

$$+z_8 P_2^{-1}\frac{dP_n^{-1}}{d\theta}+z_9(P_2)^2\ P_n^{-1}+z_{10}P_2\ P_2^{-1}\ \frac{dP_n^{-1}}{d\theta}]]\rangle b^2\sin\theta\ d\theta]$$

$$-8\pi\ \ \omega(1-\sigma^2)\ a_{N(3L+1)+1}\ \ \Omega b\ \ e(b)\int_{S_k} f_k\ P_j^{-1}P_2^{-1}\ b^2\sin\theta\ d\theta=0$$

after integrating over $\phi$. Different terms in (5.42) are

$$w_1=\frac{df_k}{dr}[(1-\frac{\sigma^2}{3})\frac{df_l}{dr}-\frac{\sigma}{r}f_l]+\frac{f_k}{r}\langle-\frac{f_l}{r}[\frac{2}{3}n(n+1)-1]\sigma^2-\sigma\frac{df_l}{dr}-\frac{\sigma f_l}{r}+\frac{f_l}{r}n(n+1)\rangle \tag{5.43}$$

$$w_2=\sigma^2\langle-\frac{2}{3}\frac{df_k}{dr}\frac{df_l}{dr}+\frac{f_k}{r}[-2\frac{df_l}{dr}+\frac{2}{3}n(n+1)\ \frac{f_l}{r}]\rangle \tag{5.44}$$

$$w_3=\sigma^2\langle-\frac{1}{3}\frac{df_k}{dr}\frac{f_l}{r}+\frac{f_k}{r}[\frac{1}{3}\frac{df_l}{dr}-\frac{2}{3}\frac{f_l}{r}]\rangle \tag{5.45}$$

$$w_6=\frac{f_k}{r}[\frac{2\sigma^2}{3}\frac{df_l}{dr}-\frac{\sigma}{r}f_l] \tag{5.46}$$

$$w_7=\frac{df_k}{dr}[(1-\frac{\sigma^2}{3})\frac{df_l}{dr}]+\frac{f_k}{r}\langle-\sigma\frac{df_l}{dr}$$
$$+\frac{2\sigma^2}{3}\frac{df_l}{dr}+\frac{f_l}{r}[-(\frac{2}{3}n(n+1)-1)\sigma^2-3\sigma+n(n+1)]\rangle \tag{5.47}$$

$$w_8=\frac{f_k f_l}{r^2}(\frac{2}{3}\sigma^2-1) \tag{5.48}$$

$$w_9=\sigma^2\langle-\frac{2}{3}\frac{df_k}{dr}\frac{df_l}{dr}+\frac{f_k}{r}[-\frac{10}{3}\frac{df_l}{dr}-\frac{2}{3}(\ n(n+1)-1)\frac{df_l}{dr}]\rangle \tag{5.49}$$

$$w_{10}=\sigma^2\langle-\frac{df_k}{dr}\frac{f_l}{3r}+\frac{f_k}{r}[\frac{1}{3}\frac{df_l}{dr}-\frac{4}{3}\frac{f_l}{r}]\rangle \tag{5.50}$$

$$z_1=\frac{\beta}{\alpha^2}\langle(g_0-\frac{2}{3}\Omega^2 r)[(1-\frac{\sigma^2}{3})\frac{df_k}{dr}+\frac{\sigma}{r}f_k]+\frac{4\sigma^2}{9}\frac{d}{dr}(r\epsilon g_0)\frac{f_k}{r}$$
$$+\frac{2}{3}\epsilon g_0[\sigma^2(-\frac{2}{3}\frac{df_k}{dr}+\frac{4}{3}\frac{f_k}{r})-\frac{\sigma}{r}f_k-\frac{8\sigma^2}{9\alpha}\frac{\epsilon\, d\alpha}{dr}f_k]\rangle f_l-\frac{\omega^2(1-\sigma^2)}{\alpha^2}f_k f_l \tag{5.51}$$

$$
\begin{aligned}
z_2 = \frac{\beta}{\alpha^2} \langle (g_0 - \frac{2}{3}\Omega^2 r)[-\frac{2}{3}\frac{df_k}{dr} - 2\frac{f_k}{r}]\sigma^2 + \frac{2}{3}\frac{d}{dr}(reg_0)[(1-\frac{\sigma^2}{3})\frac{df_k}{dr} \\
+\frac{2}{3}\frac{f_l\ \sigma^2}{r} + \frac{\sigma}{r}f_k] - \frac{2}{3}eg_0[\frac{2\sigma^2}{3}\frac{df_k}{dr} - (1-\frac{2\sigma^2}{3})\frac{6f_k}{r} - \frac{4\sigma^2}{3}\frac{f_k}{r} + \frac{2\sigma\ f_k}{r}] \\
-\frac{4}{3}\frac{re}{\alpha}\frac{d\alpha}{dr}[(g_0 - \frac{2}{3}\Omega^2 r)[(1-\frac{\sigma^2}{3})\frac{df_k}{dr} + \frac{2\sigma^2}{3}\frac{f_k}{r} + \frac{\sigma}{r}f_k]]\rangle\ f_l \\
+\frac{4}{3}\frac{re}{\alpha}\frac{d\alpha}{ar}\frac{\omega^2(1-\sigma^2)}{\alpha^2}f_k\,f_l
\end{aligned}
\qquad (5.52)
$$

$$
z_3 = \frac{\beta}{\alpha^2} \langle (g_0 - \frac{2}{3}\Omega^2 r)\frac{\sigma^2 f_k}{3r} - \frac{2}{3}eg_0(1-\frac{2\sigma^2}{3})\frac{f_k}{r}\ \rangle\ f_l \qquad (5.53)
$$

$$
z_5 = \frac{\beta\sigma^2}{\alpha^2} \langle \frac{2}{3}\frac{d}{dr}(reg_0)\frac{f_k}{3r} + \frac{4}{9}eg_0\frac{f_k}{r} - \frac{4}{3}\frac{re}{\alpha}\frac{d\alpha}{dr}(g_0 - \frac{2}{3}\Omega^2 r)\frac{f_k}{3r}\rangle\ f_l \qquad (5.54)
$$

$$
z_6 = \frac{2f_k}{3r}\ \sigma^2\ f_l \qquad (5.55)
$$

$$
z_7 = \frac{\beta}{\alpha^2}\ (g_0 - \Omega^2 r)[(1-\frac{\sigma^2}{3})\frac{df_k}{dr} + \frac{2\sigma^2}{3}\frac{f_k}{r} - \frac{\sigma f_k}{r}]f_l + \frac{\omega^2(1-\sigma^2)}{\alpha^2}f_k\,f_l \qquad (5.56)
$$

$$
z_8 = 0 \qquad (5.57)
$$

$$
z_9 = \frac{\beta\sigma^2}{\alpha^2}(g_0 - \frac{2}{3}\Omega^2 r)[-\frac{2}{3}\frac{df_k}{dr} - \frac{10}{3}\frac{f_k}{r}]\ f_l \qquad (5.58)
$$

$$z_{10}=\frac{\beta\sigma^2}{\alpha^2}(g_0-\frac{2}{3}\Omega^2 r)\frac{f_k f_l}{3r} \tag{5.59}$$

### 5.2.3 Galerkin Formulation of Poisson's Equation

The Galerkin formulation of equation (3.29) and the corresponding boundary condition (4.50) is

$$\int_k f_k P_j^{-1}\nabla^2 V_1 dv - 4\pi G\int_k \frac{\rho_0}{\alpha^2} f_k P_j^{-1}[1+\frac{2}{3}re\ P_2(\frac{1}{\rho_0}\frac{d\rho_0}{dr}-\frac{2}{\alpha}\frac{d\alpha}{dr})][\beta\zeta$$
$$-(1-\beta)(\chi+V_1)]dv+\int_{S_k} f_k\ P_j^{-1}[\textit{left hand side of } (4.50)]ds=0 \tag{5.60}$$

Using the definition of the Laplacian operator and some of the identity relations (from appendix 2) among associated Legendre functions and defining

$$gt(r)=\frac{2}{r}\frac{df_l^{-1}}{dr}+\frac{d^2 f_l^{-1}}{dr^2}-\frac{n(n+1)}{r^2}f_l^{-1} \tag{5.61}$$

we rewrite (5.60) to first order in ellipticity as

$$
\begin{aligned}
&2\pi\sum_{n=1}^{N}\sum_{l=1}^{L}[\int_0^b\int_0^\pi f_k\, P_j^{-1}\,\langle\, a_{L(N+n-1)+l}\; gt(r)\; P_n^{-1}+\frac{4\pi G\rho_0 f_l}{\alpha^2}[\;P_n^{-1}\\
&+\frac{2}{3}re(\frac{1}{\rho_0}\frac{d\rho_0}{dr}-\frac{2}{\alpha}\frac{d\alpha}{dr})\;P_2\;P_n^{-1}][(1-\beta)(a_{L(N+n-1)+l}+a_{L(2N+n-1)+l})\\
&-\beta a_{L(n-1)+l}]\;\rangle\; r^2\; dr\;\sin\theta\; d\theta-\frac{2}{3}b\epsilon(b)\int_0^\pi\langle\, f_k\; P_j^{-1}\;\langle a_{L(N+n-1)+l}\; gt(r)\\
&+\frac{4\;\pi\;G\;\rho_0\,f_l}{\alpha^2}\rangle[(1-\beta)(a_{L(N+n-1)+l}+a_{L(2N+n-1)+l})\\
&-\beta a_{L(n-1)+l}]\rangle_b P_2 P_n^{-1} b^2\;\sin\theta d\theta-a_{L(2N+n-1)+l}\int_0^\pi\; P_j^{-1}[f_k\;\frac{df_l}{dr}P_n^{-1}\\
&-\frac{2}{3}r\epsilon\;[f_k\;\frac{d^2 f_l}{dr^2}+\frac{df_l}{dr}(\frac{df_k}{dr}+2\frac{f_k}{r})]\;P_2\;P_n^{-1}\\
&+\frac{2}{3}\frac{\epsilon}{r}f_l f P_2^{-1}\frac{dp_n^{-1}}{d\theta}]_b\; b^2\;\sin\theta\; d\theta\;]+a_{3LN+n}\int_0^\pi P_j^{-1}\;[\langle-(n+1)f_k \qquad (5.62)\\
&-\frac{2}{3}r\epsilon(b)[(n+2)(n+1)f_k-(r\frac{df_k}{dr}+2f_k)(n+1)]\;P_2\rangle\;P_n^{-1}\\
&+\frac{2\epsilon(b)}{3r}f_k P_2^1\;\frac{dP_n^{-1}}{d\theta}]_b\; b\sin\theta\; d\theta]+\frac{16\pi G\; b\Omega}{5\omega}[\int_b^R\rho_0\frac{d\epsilon}{dr}\\
&+5\rho_0(b^-)\epsilon(b)]a_{N(3L+1)+1}\; f_k(b^-)\int_0^\pi P_j^{-1}P_2^{-1}b^2\sin^2\theta d\theta=0
\end{aligned}
$$

The contribution of $V_1(oc)$ in the mantle, and the wobble of the mantle, introduce

N+1 additional unknown coefficients to the Galerkin formulation of the governing equations. We have labelled these coefficients $a_{3LN+n}$, n=1,...,N+1; i.e. every $\sigma_n$ in the boundary conditions corresponds to $a_{3LN+n}$ and $\gamma$ corresponds to $a_{3LN+N+1}$. To solve for these coefficients we need the two boundary conditions not yet incorporated into the Galerkin equations, namely (4.43) and (4.48). The coefficient matrix of our Galerkin formulation then has (3LN+N+1) x (3LN+N+1) entries.

### 5.3 Integration with Respect to θ

Suppose we have a functional of the form

$$I=E_n\int_{r_1}^{r_2}\int_0^{\pi}[F_{1,n}(r)\ P_n^m+F_{2,n}(r)\ P_2\ P_n^m+F_{3,n}(r)\ P_2^1\ \frac{dP_n^m}{d\theta}$$
$$+F_{4,n}(r)(P_2)^2\ P_n^m+F_{5,n}(r)\ P_2\ P_2^1\frac{dP_n^m}{d\theta}]P_j^m\ \sin\theta\ d\theta\ dr \qquad (5.63)$$

where $E_n$ are constant. We could use a standard double integration technique to evaluate but this method takes too much CPU time, so we look for an alternative technique.

In this thesis we choose to use the orthogonality relations among associated Legendre functions to remove the θ dependence from the equations and then integrate with respect to r. In what follows we need the two identities (4.39) and (4.40), and in addition

$$(P_2)^2 P_m^n = A_n^m A_{n-2}^m P_{n-4} + A_n^m [B_{n-2}^m + B_n^m] P_{n-2}^m + [A_n^m C_{n-2}^m + (B_n^m)^2 + C_n^m A_{n+2}^m] P_n^m + C_n^m [B_n^m + B_{n+2}^m] P_{n+2}^m + C_n^m C_{n+2}^m P_{n+4} \tag{5.64}$$

$$P_2 P_2^1 \frac{dP_m^n}{d\theta} = 2(n+1) A_n^m A_{n-2}^m P_{n-4} + A_n^m [2(n+1) B_{n-2}^m + 3B_n^m] P_{n-2}^m + [2(n+1) A_n^m C_{n-2}^m + 3(B_n^m)^2 - 2nC_n^m A_{n+2}^m] P_n^m + C_n^m [3B_n^m - 2nB_{n+2}^m] P_{n+2}^m - 2nC_n^m C_{n+2}^m P_{n+4} \tag{5.65}$$

Using these identities (5.63) takes the form

$$I = E_n \int_0^\pi P_n^m P_q^m \sin\theta \, d\theta \int_{r_1}^{r_2} F_n(r) \, dr \tag{5.66}$$

The orthogonality relation can now be applied:

$$\int_0^\pi P_n^{-1} P_q^{-1} \sin\theta \, d\theta = \frac{2}{2q+1} \frac{(q+m)!}{(q-m)!} \delta_{q,n} \tag{5.67}$$

To apply the above technique to the Galerkin formulation of the governing equations we use a procedure which employs a five-step do loop to integrate the equations with respect to $\theta$. The algorithm for this method is given in appendix 3. After integrating the Galerkin equations with respect to $\theta$ we let

$$f_l(r)=\left(\frac{r}{R}\right)^{l-1} \tag{5.68}$$

and the IMSL subroutine dqdag is called to integrate the equations with respect to r.

# CHAPTER 6 : DISCUSSION, NUMERICAL RESULTS AND CONCLUSION

The aim of this thesis is to utilize the THPD by computing the periods of the Earth's Chandler wobble, nearly diurnal free wobble and some of the Earth's other inertial modes. In chapter 3 we showed the derivation of the equations constituting THPD and, in the same chapter, we included the effect of ellipticity in the equations.In chapter 4 we expanded the boundary conditions which must be invoked on a wobbling Earth.In chapter 5 we applied the Galerkin method to approximate the solution of the equations.

In this chapter we first apply the Galerkin method to the Poincaré Earth model, for which analytical solutions exist for the modes mentioned above. These results will then be compared to the ones computed using the Earth model described in chapter 2.

## 6.1 The Poincaré Earth model

The Poincaré Earth model consists of a rigid stratified mantle and an incompressible homogeneous liquid core. The analytical solution (Wu 1993, Rochester lecture notes) yields a period of about 270 days for the Earth's Chandler wobble (CW) and a nutation period of about 350 days for the Earth's nearly diurnal free wobble (NDFW).

The other inertial modes of the above Earth model have solutions of the form

$$(1-x^2)\,\frac{dP_n^m(x)}{dx}=m\,\left(\frac{1+f\,x^2}{1+f}\right)^{\frac{1}{2}}\,P_n^m(x) \tag{6.1}$$

(Aldridge and Lumb 1987) where

$$x=(\sigma^2-f)^{-\frac{1}{2}} \tag{6.2}$$

and $f=1/(1-\varepsilon)^2-1$, $\varepsilon$ is the ellipticity of the CMB and has a value of 0.002551 (table 5). We compute some of the Earth's inertial modes using a: equation (6.1) and b: applying the Galerkin method to Poincaré's equation to make sure that the same results are obtained. To compute periods of the Earth's CW and NDFW in the Poincaré model we use the well known analytical results

$$\frac{\Omega}{\omega}=\left[\frac{A_m}{A_E e_E},\quad -\left(1+\frac{A_E e}{A_m}\right)^{-1}\right] \tag{6.3}$$

where $e_E$ and $A_E$ are respectively the Earth's dynamical ellipticity and moment of inertia with respect to an equatorial axis. The frequency of the free core nutation (FCN) is then given as

$$\frac{\omega_{FCN}}{\Omega}=1+\frac{\omega_{NDFW}}{\Omega} \tag{6.4}$$

As we showed in chapter 3, when the liquid core is assumed to be homogeneous and incompressible, the momentum equation reduces to the Poincaré equation

$$\nabla^2\chi - \sigma^2(\hat{e}_3\cdot\nabla)^2\chi = 0 \tag{6.5}$$

Since $\chi$ is the only dependent variable in (6.5) the entropy and Poisson equations are redundant and are removed from the formulation. Also the boundary conditions involving $V_1$ are redundant [$V_1$ does not appear explicitly in (6.5)] and hence removed. Therefore we are left with one Galerkin equation (5.43) which includes the boundary condition (4.46) and the boundary condition equivalent to the total angular momentum conservation of the Earth, which we treat as an independent equation to solve for $\gamma$ (of course in these equations $\beta$ and $\zeta/\alpha^2$ are both 0). The coefficient matrix of this formulation therefore has (LN+1)x(LN+1) entries. The roots of the determinant of the coefficient matrix, regarded as a function of frequency $\omega$, correspond to the frequencies of the Earth's oscillations.

Table 6 shows the periods of the normal modes computed using the two methods described above. Except for the periods of the Chandler wobble and the nearly diurnal free wobble which differ slightly from those computed using (6.3) and (6.4), the inertial modes computed using the Galerkin method are identical, to 4 significant figures, with those computed using (6.1).

Table 7 shows the periods of the same modes as in table 6 using the formulation given in chapter 5. Except for the CW all the modes whose periods are reported here are retrograde modes.

## 6.2 Convergence of the Solutions

For the Poincaré case the convergence of the Galerkin formulation is very fast. For almost all the modes whose periods are reported, the solutions converge when we set N=2 and L=4. We tested our program for higher values of N and L to make sure of the convergence.

For the Earth model described in chapter 2, the solutions converge for the period of the Chandler wobble when N=2 and L=5, but for the case of nearly diurnal free wobble convergence required N=2 and L=13. We tested our program for the cases N=3 and 4. The results are almost identical to the case N=2. For some of the inertial modes whose periods are listed in table 6.2 we needed higher values of L i.e. L>12 for the solutions to converge. For some frequency ranges the determinant of the coefficient matrix becomes unstable when L>16 and many false roots appear in the eigenperiod spectrum. For this reason we were not able to achieve convergence for the modes whose periods are not listed in table 7.

## 6.3 Conclusion

The close agreement between the periods of the modes listed in tables 6 and 7 show that THPD may well be an effective tool in solving the governing equations of core dynamics for the long period free oscillations. Although we considered only the case of a neutrally stratified liquid core in our computations, the equations and the Galerkin formulations were derived so that they include any value of $\beta$.

Future work on this subject should consider a non-neutrally stratified liquid core, include the presence of the solid inner core and take into account the elasticity of the mantle and the solid inner core. This would permit study of the effects of (a) non-neutral stratification of the liquid core on the inertial modes, including those involving wobble /nutation, and (b) truncation at low values of N.

**Table 6** Periods of the Chandler Wobble (CW), Nearly Diurnal Free Wobble (NDFW), Free Core Nutation, and some of the other inertial modes of the Poincaré Earth Model

a: Using eq. (6.1), (6.3) and (6.4)
b: Using The Galerkin Method

(periods are in days)

| Mode | a | b |
|---|---|---|
| CW | 272.6 | 271.7 |
| FCN | 347.4 | 348.6 |
| NDFW | 0.99713 | 0.99714 |
| $p_2^{-1}$ | 0.9980 | 0.9980 |
| $P_4^{-1}$ | 0.5851 | 0.5851 |
| | 1.6313 | 1.6313 |
| $P_6^{-1}$ | 0.5371 | 0.5371 |
| | 0.7645 | 0.7645 |
| | 2.2676 | 2.2676 |

**Table 7** Periods of the same modes as in table 6 using the Earth model described in chapter 2

| Mode | Period (in days) |
|---|---|
| CW | 273 .2 |
| FCN | 347.4 |
| NDFW | 0.99713 |
| $p_2^{-1}$ | 0.9973 |
| $p_4^{-1}$ | 0.5849 |
| $p_6^{-1}$ | 0.5371 |

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# Appendix A

## A.1 Derivation of the Expression for $A_m$

The equation for $A_m$ is given as

$$A_m = \int_{mantle} r^2[\cos^2\theta + \sin^2\theta \ \sin^2\phi]dm$$
$$= \int_V r^2 \rho_0(r)[\cos^2\theta + \sin^2\theta \ \sin^2\phi] \ dV \tag{A.1}$$

where $\theta$ is the colatitude and $\phi$ is the longitude. Integrating (A.1) with respect to $\phi$ we get

$$A_m = 2\pi \int_0^{\pi}\int_{\xi_1}^{\xi_2} r^4 \rho_0(r)[\cos^2\theta + \frac{1}{2}\sin^2\theta]\sin\theta \ d\theta dr \tag{A.2}$$

where $\xi_1$ and $\xi_2$ are the radii of the CMB and the Earth's outer surface of the Earth respectively and

$$\int_0^{2\pi} \sin^2\phi \ d\phi = \pi \tag{A.3}$$

Substituting for $\cos^2\theta$ and $\sin^2\theta$ from (5.34) and for r in terms of $r_0$ from (3.24) into (A.2) we get

~~(A.4)~~ $$A_m=\frac{2\pi}{3}\int_0^{\pi}\int_b^{R}\rho_0(r_o)\ (P_2+2)\ r_0^4[1-\frac{2}{3}P_2(5\epsilon+r_0\frac{d\epsilon}{dr})]\sin\theta\ d\theta dr$$

Integrating (A.4) with respect to $\theta$ we get

$$A_m=\frac{8\pi}{3}\int_b^{R}\rho_0(r_0)r_0^4[1-\frac{1}{15}(5\epsilon+r_0\frac{d\epsilon}{dr})]dr_0 \tag{A.5}$$

### A.2 Derivation of the Expression for $A_m e_m$

Since $e_m$ appears in the equations in the combination $A_m e_m$ only we show the derivation of $A_m e_m$. To derive the expression for $A_m e_m$ we proceed as follows

$$\begin{aligned} A_m e_m = C_m - A_m &= \int_{mantle}[(r\sin\theta\ \cos\phi)^2-(r\ \cos\theta)^2]dm \\ &= \int_{V_m}\rho_0(r)[\sin^2\theta\ \cos^2\phi-\cos^2\theta]r^2\ dV \end{aligned} \tag{A.6}$$

where $C_m$ is the Earth's moment of inertia with respect to a polar axis. Substituting for $\sin^2\theta$ and $\cos^2\theta$ in terms of Legendre polynomials and for r in terms of $r_0$ into (A.6) we get

$$A_m e_m=-2\pi\int_0^{\pi}\int_b^{R}\ \rho_0(r_0)[1-\frac{2}{3}(5\epsilon+r_0\frac{d\epsilon}{dr})P_2]\ P_2\ r_0^4\ \sin\theta\ d\theta\ dr_0 \tag{A.7}$$

Integrating (A.7) with respect to $\theta$ using orthogonality relation for associated Legendre functions,

$$A_m e_m = \frac{8\pi}{15}\int_b^R \rho_0[5e + r\frac{de}{dr}]\ r^4 dr \qquad \text{(A.8)}$$

Expressions (A.5) and (A.8) are then integrated numerically for $A_m$ and $A_m e_m$ respectively.

## A.3 Derivation of the Expression for $g_0$

In the interior of the Earth the gravity potential is given as

$$W_0(r) \equiv W_0(r_0) = V_0(r) + \frac{1}{3}r^2\Omega^2(1 - P_2) \qquad \text{(A.9)}$$

where $V_0(r)$ is the gravitational potential at point r on a surface of mean radius $r_0$. (A.9) can be written as

$$W_0(r_0) - \frac{1}{3}\Omega^2 r^2 = V_0(r) - \frac{P_2}{3}\Omega^2 r^2 \equiv U_0(r_0) \qquad \text{(A.10)}$$

To first order in ellipticity, LHS of (A.10) is constant over the surface of mean radius $r_0$ and therefore so must be the RHS. We can interpret $U_0(r)$ as the gravitational potential at r due to the density distribution $\rho_0(r)$ in the non-rotating spherical Earth model. Therefore we can write (A.10) as

$$W_0(r) = U_0(r) + \frac{2}{3} re \frac{dU_o}{dr} P_2 + \frac{1}{3}\Omega^2 r \tag{A.11}$$

to first order in ellipticity. Then

$$\begin{aligned} g_0(r) &= -\nabla W_0 = -\hat{r}[g_0(r) - \frac{2}{3}\Omega^2 r] - \frac{2}{3}\nabla(reg_0P_2) \\ &= -\hat{r}\ [g_0(r) - \frac{2}{3} r\Omega^2 + \frac{2}{3}\frac{d}{dr}(reg_0)P_2] - \hat{\theta}\ \frac{2}{3} eg_0P_2^1 \end{aligned} \tag{A.12}$$

where $g_0(r)$ on the RHS is gravity in the non-rotating spherical Earth model. The correctness of (A.12) can be verified by showing that Clairaut's equation (2.23) results from requiring that (A.12) satisfy Poisson's equation (1.5).

# Appendix B
# Some Useful Identity Relations Among Spherical Harmonics

In what follows we have used integration by parts to derive the final expression. The integrations are taken from 0 to $\pi$.

$$\int P_2^1 \frac{dP_n^{-1}}{d\theta} P_q^{-1} \sin\theta \, d\theta = \int P_n^{-1} [6P_2 \, P_q^{-1} - P_2^{-1} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta \tag{B.1}$$

$$\int [\sin^2\theta \frac{dP_n^{-1}}{d\theta} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta$$

$$= \int P_n^{-1} [\frac{2}{3} P_2^1 \frac{dP_q^{-1}}{d\theta} + (\frac{2}{3} q(q+1)(1-P_2) - 1) P_q^{-1}] \sin\theta \, d\theta \tag{B.2}$$

$$\int \frac{\cos\theta}{\sin\theta} [P_q^{-1} \frac{dP_n^{-1}}{d\theta} + P_n^{-1} \frac{dP_q^{-1}}{d\theta}] \sin\theta \, d\theta = \int P_n^{-1} P_q^{-1} \sin\theta \, d\theta \tag{B.3}$$

$$\int [\frac{dP_n^{-1}}{d\theta} \frac{dP_q^{-1}}{d\theta} + \frac{P_n^{-1} P_q^{-1}}{\sin^2\theta}] \sin\theta \, d\theta = q(q+1) \int P_n^{-1} P_q^{-1} \sin\theta \, d\theta \tag{B.4}$$

$$\int P_2\ P_2^1\ \frac{dP_n^{-1}}{d\theta}\ P_q^{-1}\ \sin\theta\ d\theta = -\int P_n^{-1}[2P_2\ P_q^{-1} - 10(P_2)^2 P_q^{-1}$$

$$+2P_q^{-1} + P_2\ P_2^1\ \frac{dP_q^{-1}}{d\theta}]\sin\theta\ d\theta \tag{B.5}$$

$$\int \sin^2\theta\ [P_2\ \frac{dP_q^{-1}}{d\theta}\ \frac{dP_n^{-1}}{d\theta}]\sin\theta\ d\theta = -\int P_n^{-1}[\frac{2}{3}(1-2P_2)P_2^1 \frac{dP_q^{-1}}{d\theta}$$

$$-P_2(\frac{2}{3}q(q+1)(1-P_2)-1)P_q^{-1}]\sin\theta\ d\theta \tag{B.6}$$

$$\int P_2 \frac{\cos\theta}{\sin\theta}[P_q^{-1}\ \frac{dP_n^{-1}}{d\theta} + P_n^{-1}\frac{dP_q^{-1}}{d\theta}]\sin\theta\ d\theta = \int (3P_2+1)P_n^{-1}\ P_q^{-1}\ \sin\theta\ d\theta \tag{B.7}$$

$$\int P_2[\frac{dP_n^{-1}}{d\theta}\frac{dP_q^{-1}}{d\theta} - \frac{P_n^{-1}\ P_q^{-1}}{\sin^2\theta}]\sin\theta\ d\theta$$

$$= \int P_n^{-1}[q(q+1)P_2\ P_q^{-1} - P_2^1\ \frac{dP_q^{-1}}{d\theta}]\sin\theta\ d\theta \tag{B.8}$$

# APPENDIX C
## Integration with Respect to θ

Here we show the algorithm which we use to integrate the Galerkin formulation of the governing equations of liquid core dynamics with respect to $\theta$. We rewrite equation (5.62) as

$$I=\int_0^\pi P_n^{-1}\ P_q^{-1}\sin\theta\ d\theta[E_{n+4}\int_{r_1}^{r_2}\xi_{1,n+4}(r)dr+E_{n+2}\int_{r_1}^{r_2}\xi_{2,n+2}(r)dr+E_n\int_{r_1}^{r_2}\xi_{3,n}(r)dr$$
$$+E_{n-2}\int_{r_1}^{r_2}\xi_{4,n-2}(r)dr+E_{n+4}\int_{r_1}^{r_2}\xi_{5,n-4}(r)dr] \quad \text{(C.1)}$$

after operating on the associated Legendre functions with the identities (4.39), (4.40), (5.63), (5.64) and collecting terms of the same degree.

The algorithm we use to evaluate $\xi_{i,j}$ in (5.66) is as follows:

m=n+4

do 1 i=1,5

if(m.lt.2.or.m.gt.N) goto 2 {N is the degree of truncation}

if (i.eq.1) then

s1=0

s2=0

s3=0

s4=$A_{n+4}^{-1}A_{n+2}^{-1}$

$s5=2(n+5)A_{n+4}^{-1}A_{n+2}^{-1}$

k=n+4

else if (i.eq.2) then

s1=0

$s2=A_{n+2}^{-1}$

$s3=2(n+3)\ A_{n+2}^{-1}$

$s4=A_{n+2}^{-1}\ [\ B_n^{-1}+\ B_{n+2}^{-1}]$

$s5=A_{n+2}^{-1}\ [2(n+3)\ B_n^{-1}+3\ B_{n+2}^{-1}\ ]$

k=n+2

else if(i.eq.3) then

s1=1

$s2=B_n^{-1}$

$s3=3\ B_n^{-1}$

$s4=A_n^{-1}\ C_{n-2}^{-1} + (B_n^{-1})^2 + A_{n+2}^{-1}\ C_n^{-1}$

$s5=2(n+1)A_n^{-1}\ C_{n-2}^{-1} +3\ (B_n^{-1})^2 -2\ n\ A_{n+2}^{-1}\ C_n^{-1}$

else if(i.eq.4) then

s1=0

$s2=C_{n-2}^{-1}$

$s3=-2(n-2)C_{n-2}^{-1}$

$s4=C_{n-2}^{-1}\ [\ B_{n-2}^{-1} + B_n^{-1}\ ]$

$s5=C_{n-2}^{-1}\ [3\ B_{n-2}^{-1} - 2(n-2)\ B_n^{-1}\ ]$

```
k=n-2
else if (i.eq.5) then
s1=0
s2=0
s3=0
```

$s4=C_{n-4}^{-1}\ C_{n-2}^{-1}$

$s5=-2(n-4)\ C_{n-4}^{-1}\ C_{n-2}^{-1}$

```
k=n-4
endif
```

$\xi_{i,k}= s1\ F_{1,k} + s2\ F_{2,k} + s3\ F_{3,k} + s4\ F_{4,k} + s5\ F_{5,k}$ [$F_{jk}$ are as defined in (5.62)]

```
2    m=m-2
1    continue
```

# APPENDIX D

## Gravitational Potential in an Ellipsoidally Stratified Body

The exterior gravitational potential, at a point (r,θ,ϕ), of a homogeneous body of density ρ bounded by the surface

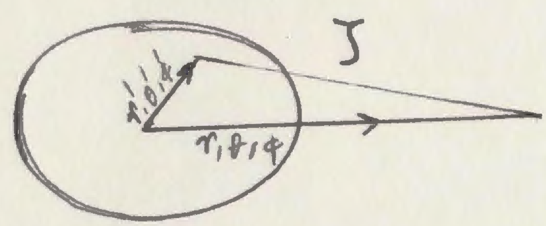
$$\xi = R[1 - \frac{2}{3}\epsilon P_2(\cos\theta')] \tag{D.1}$$

is

$$V_e = \int_0^{2\pi}\int_0^{\pi}\int_0^{\xi} \frac{\rho}{\zeta} r' \sin\theta' d\theta' d\phi' \tag{D.2}$$

where

$$\frac{1}{\zeta} = |r - r'|^{-1} = [r^2 - 2rr'\cos\alpha + r'^2]^{-\frac{1}{2}} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_2(\cos\alpha) \tag{D.3}$$



Integrating (D.2) over r′ using (D.3) we get

$$V_e=\frac{G\rho}{r}\sum_{n=0}^{\infty}\frac{R^{n+3}}{(n+3)r^n}\int_0^{2\pi}\int_0^{\pi}P_n(\cos\alpha)[1-\frac{2}{3}\varepsilon(R)P_2(\cos\theta')]^{n+3}\sin\theta' d\theta' d\phi' \quad \text{(D.4)}$$

To evaluate (D.4) neglecting all quantities smaller than the first order in ellipticity we use the binomial theorem in the integral, then substitute for

$$P_2(\cos\theta')^2=\frac{1}{5}+\frac{2}{3}P_2(\cos\theta')+\frac{18}{35}P_4(\cos\theta') \quad \text{(D.5)}$$

and finally use the addition theorem for spherical harmonics, which gives

$$\int_0^{2\pi}\int_0^{\pi}P_n(\cos\alpha)P_j(\cos\theta')\sin\theta' d\theta' d\phi'=\frac{4\pi}{2n+1}P_n(\cos\theta)\delta_{nj} \quad \text{(D.6)}$$

The result is that

$$V_e=4\pi G\rho R^2[\frac{R}{3r}-\frac{2}{15}\varepsilon(\frac{R}{r})^3P_2(\cos\theta)] \quad \text{(D.7)}$$

The internal potential of such a body must satisfy Poisson's equation and be continuous with (D.7) across the boundary (D.1). The function

$$V_i=4\pi G\rho R^2[\frac{1}{2}-\frac{r^2}{6R^2}-\frac{2}{15}\varepsilon P_2(\cos\theta)(\frac{r}{R})^2] \quad \text{(D.8)}$$

satisfies these requirements.

Now we shall suppose that the heterogeneous body of mass M is built up of a set of concentric spheroidal surfaces

$$r=r_0[1-\frac{2}{3}\epsilon(r_0)P_2(\cos\theta)] \tag{D.9}$$

on each of which the density $\rho=\rho(r_0)$ is constant, according to the hypothesis of hydrostatic equilibrium. Thus a particular equipotential is specified by its mean radius and $\epsilon$ is a function of $r_0$.

For the heterogeneous mass the standard technique is to regard the contribution to the potential V at an interior point r from a shell of density $\rho_0'$ bounded by the spheroids $r_0'$, $r_0'+dr_0'$ as just the difference between (a) the potential at **r** due to a homogeneous mass of density $\rho_0'$ bounded by the spheroid $r_0'+dr_0'$ and (b) the potential at **r** due to a homogeneous mass of the same density bounded by the spheroid $r_0'$. Taking into account that the contributions from spheroids containing/not containing **r** will be given by (D.7)/(D.8) we have the gravitational potential at a point **r**=(r,θ,ϕ) interior to the mass:

$$V=4\pi G\,\Big\langle\int_0^{r_0}\rho'_0[\frac{r_0'^2}{r}-\frac{2}{15}\frac{d[\epsilon(r'_0)r'_0]}{r^3dr'_0}P_2(\cos\theta)]dr'_0 + \int_{r_0}^{R}\rho'_0[r'_0-\frac{2}{15}\frac{d\epsilon(r'_0)}{dr_0}r^2P_2(\cos\theta)]dr'_0\Big\rangle \tag{D.10}$$