SUBSEISMIC DESCRIPTION OF THE SLICHTER MODES IN A ROTATING EARTH

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BY

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Abstract

This thesis reports a study of the triplet of (purely translational) Slichter modes of inner core oscillation for a simplified uniformly rotating Earth model with spherical elastic inner core, spherical rigid fixed mantle and neutrally stratified, compressible liquid core. A variational principle is used to solve the subseismic wave equation, which is used to model the liquid core dynamics. The investigation shows the utility of the subseismic wave equation for describing a long-period oscillation, with the effects of higher order harmonics in the displacement field taken into account. For the first time, a numerical estimate of error involved in making the subseismic approximation is given for a particular mode. The eigenperiod of the Slichter mode is found to be around 5 hours for neutrally stratified liquid core with total mass constrained by PREM (1981) data. The effect of the Earth's rotation is to split the mode into a triplet with eigenperiods 12% shorter, 2% shorter, and 10% longer. The effects of compressibility of the liquid core and elasticity of the inner core are to increase the eigenperiod by about 0.6% and 9% respectively. The study can be regarded as a preliminary numerical attempt to describe gravitational/inertial oscillations of the Earth by the subseismic wave equation.

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(in order of appearance)

ω : oscillation frequency (p. 2)

$\mathbf{S}_n^m$: spheroidal displacement field (p. 3)

$\mathbf{T}_n^m$: toroidal displacement field (p. 3)

T : oscillation period (p. 3)

ρ_{IC} : density of the inner core (p. 4)

ρ_o : density of the liquid core (p. 4)

N : Brunt-Väisälä frequency (p. 4)

α : compressional velocity (p. 4)

β : { shear velocity on page 4, 25-27 and Appendix A (p. 5)
stability parameter of the liquid core on other pages (p. 22) }

G : gravitational constant (p. 5)

CMB : core-mantle boundary (p. 6)

$\delta\hat{\mathbf{e}}_3$: inner core displacement in the simple model (p. 7)

ICB : inner core boundary (p. 7)

$\mathbf{F}_g$: gravitational restoring force on the inner core (p. 8)

M_{IC} : mass of the inner core (p. 9)

M_{LC} : mass of the liquid core (p. 9)

M_m : mass of the mantle (p. 9)

M : mass of the Earth (p. 9)

ϵ : correction factor of the Earth's centre of mass relative to geocentre (p. 9)

$\mathbf{u}$: displacement (p. 10)

$\vec{\tau}$: additional stress due to deformation or/and displacement (p. 10)

Ω : steady rotation speed of the Earth (p. 10)

V_1 : gravitational potential perturbation (p. 10)

$\mathbf{g}_0$: gravity in the undeformed Earth (p. 10)

F_1 : additional pressure due to deformation (p. 11)

$\mathbf{F}_p$: restoring force on the *ICB* by liquid core flow (p. 13)

χ : intermediary potential (p. 13)

$\mathbf{P}_{IC}$: linear momentum of the inner core (p. 17)

$\mathbf{P}_{LC}$: linear momentum of the liquid core (p. 17)

$\rho_0^{\dagger}$: modified density (p. 22)

λ : Lamé parameter (p. 25)

μ : rigidity (p. 25)

$h_1^{m,\prime}$: load Love number (p. 29)

e : elastic correction factor in the load Love number (p. 30)

u : radius as a fraction of that of *CMB* (p. 37)

1. INTRODUCTION

In order to explain an unidentified spectral peak (with period about 86 minutes) appearing at the lower frequency end of the gravity spectrum recorded during the large Chilean earthquake of May 22, 1960, Slichter (1961) proposed a new kind of mode. He suggested that a gravitational perturbation of the approximate observed magnitude and period could reasonably be produced by translational oscillations of the inner core as a rigid body in the surrounding liquid core. Because the period of this Slichter mode would be an extremely sensitive indicator of the density contrast across the inner core boundary, a series of efforts has been made to detect this mode, but so far unsuccessful. For example, after carefully analysing the earth tides data from South Pole, Jackson and Slichter (1974) were unable to detect evidence of inner core free oscillation in a record of nearly one year duration (October 1970 to September 1971) there. Theoretical studies of various possible effects on the mode (rotation, compressibility and viscosity of the liquid core, elasticity of the inner core and mantle, ellipticity of the Earth and geomagnetic field) have also been carried out since then.

The model that Slichter (1961) considered consists of a rigid inner core, an inviscid spherical liquid outer core and a rigid spherical mantle. Such a system behaves as a simple linear vibrator, and the circular frequency of the system is split by the Earth's rotation into a polar mode and equatorial modes. Slichter assumed that gravitation was the dominant restoring force, but also considered the effect of an additional restoring force due to rigidity of the outer core, small enough ($\sim 10^9$ N/m^2) to be consistent with failure to detect seismic shear waves there.

The effects of Coriolis force due to Earth's rotation on different polarizations of a Slichter oscillation were first discussed clearly by Busse (1974). He examined theoretically the effects of rotation and the finite radius of the outer core on calculations of the Slichter eigenperiod, and suggested that the action of the Coriolis force on the motion of the fluid in the outer core and its finite radius may change the period by as much as 50% from that predicted without taking these effects into account. He used cylindrical system of coordinates to formulate a mathematical approach, but restricted quantitative discussion to the polar mode (oscillation in the direction of the axis of rotation) and left the equatorial modes for a qualitative perturbation analysis. An asymmetric splitting of the frequency happens in such an oscillation (at least for a small ratio of rotation frequency to the polar frequency of the vibration), namely $\frac{1}{2}(\omega_+ + \omega_-) < \omega_p$, where ω_p is the polar frequency, ω_+ , ω_- are frequencies of the eastward and westward-travelling waves respectively due to vibrations in the equatorial plane viewed in the rotating frame. But note that Busse's equation (21) has misprints, which we take this opportunity to correct:

$$\omega_\pm = \mp \frac{\Omega(1 + \frac{\rho_o}{\rho_i}\alpha_1)}{(1 + \frac{\rho_o}{\rho_i}\alpha_o)}$$

$$\pm \frac{\{[\alpha_1(2 + \frac{\rho_o}{\rho_i}\alpha_1) - \alpha_o]\frac{\rho_o}{\rho_i}\Omega^2 + \frac{4\pi}{3}G\rho_o(1 - \frac{\rho_o}{\rho_i})(1 + \frac{\rho_o}{\rho_i}\alpha_o)\}^{1/2}}{(1 + \frac{\rho_o}{\rho_i}\alpha_o)}.$$

Busse assumed the liquid core as incompressible and homogeneous, but pointed out that compressibility and stratification would play an important role on the mode under discussion. The influence of the inner core elasticity on the oscillation is neglected in Busse's analysis.

The spherical harmonic representation of displacement fields leads to normal modes of spheroidal and toroidal type, which can exist independently in a non-rotating spherically-stratified Earth, but are coupled by rotation. Crossley (1975) was the first one to point out that the Slichter mode should be regarded as the first undertone (period > 1 hour) of the $n = 1$ set of spheroidal modes, corresponding to a displacement field $\mathbf{S}_1^m$. Rotation of the Earth and compressibility of the liquid core were taken into account. Following Crossley, we reserve the term 'Slichter-type oscillation' for the undertones of longer period which are possible when the liquid core is stably stratified. In the liquid core, Crossley took into account both the modification of $\mathbf{S}_1^m$ due to the Coriolis force ('self-coupling') and the toroidal field $\mathbf{T}_2^m$ which the Coriolis force induces from $\mathbf{S}_1^m$. All further spheroidal-toroidal couplings were neglected. In the solid inner core and mantle, only self-coupling was taken into account.

Smith (1976) also used a 2-term expansion to represent the displacement field, but in both solid inner core and mantle, as well as in the liquid core. By comparison with Busse's (1974) results, Smith found that stratification of the liquid core and elasticity of the inner core may have a significant influence on the Slichter mode: about 20% increment in eigenperiod of the polar mode was obtained in Smith's calculation for an elastic inner core and neutrally stratified liquid core model ($T =$ 7.653 hr), comparing to that of Busse ($T \approx 6.397$ hr), which assumed a homogeneous incompressible liquid core and a rigid inner core. Note that both models use a same fractional density jump across inner core boundary ($\frac{\rho_{IC} - \rho_o}{\rho_o} \approx 0.0328$), and $N^2 = 0$, where $N(r)$ is the Brunt Väisälä frequency. Ellipticity of the Earth was also taken into account by Smith.

In our study, we will try a different approach to the problem by applying the subseismic wave equation (Smylie and Rochester, 1981) to describe the Slichter mode for a simplified uniformly rotating Earth model with a spherical homogeneous elastic inner core, a spherical rigid fixed mantle and a neutrally stratified self-gravitating, compressible liquid core. With rotation, elasticity of the inner core, compressibility and stratification of the liquid core taken into account, we hope to investigate the Slichter mode more economically and completely. It will also be a test of the utility of the subseismic wave equation in long-period theoretical geodynamics. We will begin with a restricted Earth model to establish the basic characteristics of the Slichter mode. We then relax the restrictions step-by-step to reach a model which allows for (i) the elastic response of a homogeneous, self-gravitating inner core to flow pressure of the neutrally-stratified compressible liquid outer core, (ii) rotation, with self-coupling in the inner core and higher coupling in the liquid core. The governing equations are solved by applying a variational principle.

To fix ideas, we take Earth structure data from the PREM model (Dziewonski and Anderson, 1981): radius of the Earth 6371 km, radius of core-mantle boundary 3480 km, radius of the inner core 1221.5 km; total mass of the Earth 5.974×10^{24} kg; compressional velocity function (km s^{-1}) in the outer and inner core $\alpha(r)_{LC} = 11.0483 - 4.0362(r/c) + 4.8023(r/c)^2 - 13.5732(r/c)^3$,

$\alpha(r)_{IC}$ = 11.2622 - 6.3640$(r/c)^2$, shear velocity function (km s^{-1}) in the inner core $\beta(r)$ = 3.6678 - 4.4475$(r/c)^2$ (where c = 6371 km); average density of the inner core $\rho_{IC} = 12.8936 \times 10^3$ kg/m^3, which is the mean density of PREM; gravitational constant $G = 6.6732 \times 10^{-11}$ m^3 kg^{-1} s^{-2}. A density profile for a neutrally-stratified, compressible liquid core is obtained later, but as a comparison we will also use that of PREM in eigenperiod computation.

2. STARTING MODEL FOR THE SLICHTER MODE

2.1. Simple Pendulum Model

The simplest model we assume contains a spherical rigid inner core, a spherical rigid fixed mantle, and a homogeneous, inviscid, incompressible liquid outer core. The Slichter mode is an oscillation in which the geometric centre of the inner core suffers a purely translational displacement from the Earth's centre of mass (which we take as the geocentre). Conservation of linear momentum requires that the latter point remains fixed in inertial space, so the geometric centre of the mantle will have to move in the opposite direction to that of the inner core centre, but with a much smaller amplitude because of the enormous mass of mantle (68% of whole Earth). Because of this, and also for mathematical simplicity, we will assume here the core-mantle boundary is rigid-fixed. The principal restoring force on the displaced inner core is the gravitational force, produced by the attraction of that portion of the outer core fluid (with constant density ρ_o throughout) in the shaded 'cavity', on the sphere which has the radius of inner core a and the differential density $\rho_{IC} - \rho_o$, where ρ_{IC} is the average density of the inner core (Fig. 1).

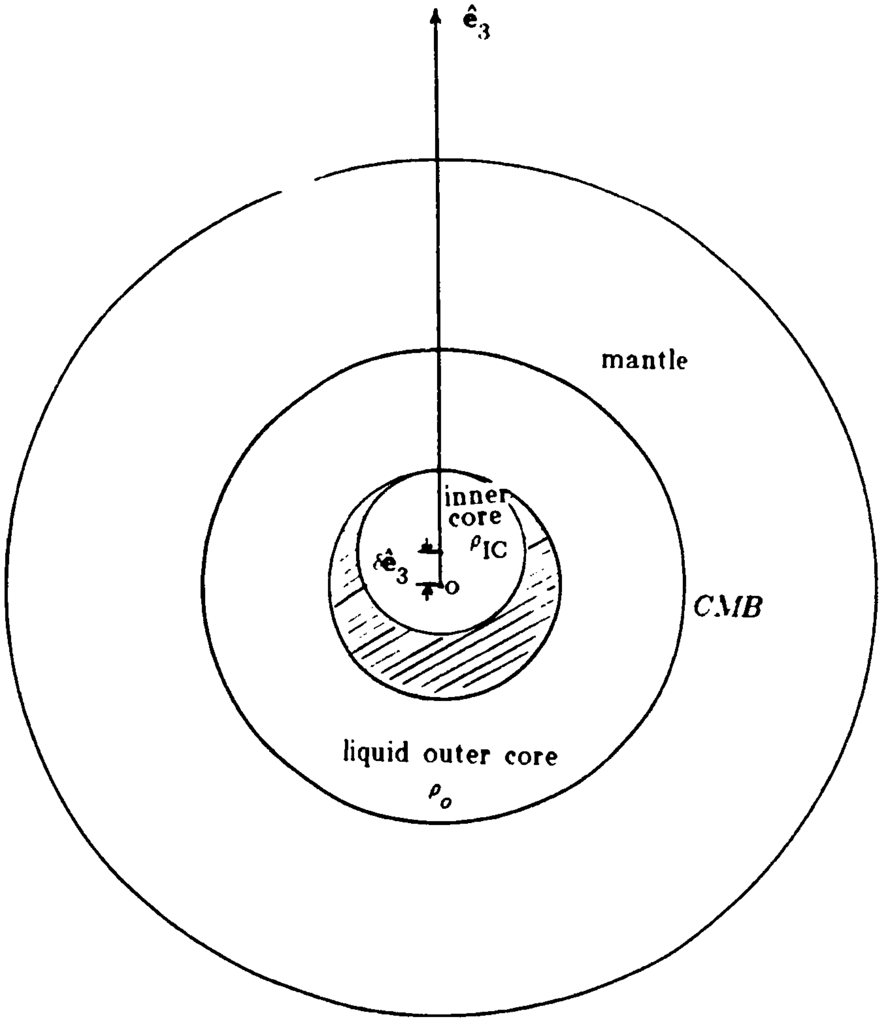

Figure 1: Displaced Inner-Outer Core System.
o: geocentre; *CMB*: core-mantle boundary

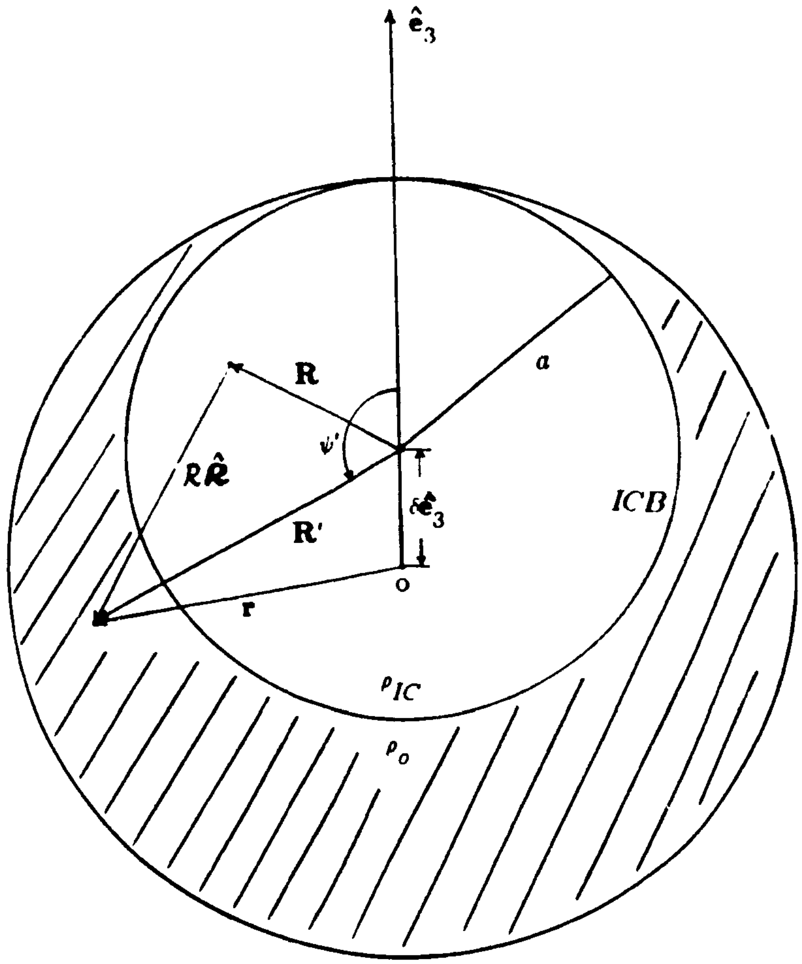

Figure 2: Vector Relationships in Displaced Inner-Outer Core System. ρ_{IC}: density of the innercore; ρ_o: density of the liquid core; a: radius of inner core boundary (ICB); $\delta\hat{\mathbf{e}}_3$: inner core displacement; o: geocentre; **R**: field vector; **R**': source vector

The gravitational restoring force on the inner core is

$$\mathbf{F}_g = G\rho_o \int_{cavity} d^3R' \int_{sphere} (\rho_{IC} - \rho_o) d^3R \frac{\hat{\mathcal{R}}}{\mathcal{R}^2} \tag{1}$$

where $\mathcal{R}\hat{\mathcal{R}} = \mathbf{R}' - \mathbf{R}$ (Fig. 2)

Since the inner core is spherically stratified, the second integral in expression (1) can be written as

$$\int_{sphere} (\rho_{IC} - \rho_o) d^3R \frac{\hat{\mathcal{R}}}{\mathcal{R}^2}$$

$$= \frac{4\pi}{3}(\rho_{IC} - \rho_o) a^3 \frac{\hat{\mathbf{R}}'}{R'^2}.$$

Thus we have

$$\mathbf{F}_g = \frac{4\pi}{3} G\rho_o (\rho_{IC} - \rho_o) a^3 \int_{cavity} \hat{\mathbf{R}}' \frac{d^3R'}{R'^2}. \tag{2}$$

Now suppose $\delta\hat{\mathbf{e}}_3$ is the displacement of the inner core from geocentre (we are free to choose it in the $\hat{\mathbf{e}}_3$ direction for convenience), then $\hat{\mathbf{R}}' \cdot \hat{\mathbf{e}}_3 = \cos\psi'$. Because of symmetry (Fig. 2), the integral in (2) is

$$\int_{cavity} \hat{\mathbf{R}}' \frac{d^3R'}{R'^2} = \hat{\mathbf{e}}_3 \int_{cavity} \cos\psi' \frac{d^3R'}{R'^2}$$

$$= \hat{\mathbf{e}}_3 \int_0^{2\pi} d\phi' \int_0^{\pi} \cos\psi' \sin\psi' \, d\psi' \int_{a(ICB)}^{outer\ boundary\ of\ cavity} dR'. \tag{3}$$

As in Fig. 2, on the outer boundary of the cavity, R′ can be found from relation

$$(a + \delta)^2 = R'^2 + \delta^2 + 2R'\delta\cos\psi',$$

i.e. $R' = -\delta\cos^2\psi' + [\delta^2\cos^2\psi' + (a+\delta)^2 - \delta^2]^{1/2}$

Substitute R′ and (3) into (2), we have

$$\mathbf{F}_g = -\omega_s^2 M_{IC}\,\delta\hat{\mathbf{e}}_3. \tag{4}$$

where M_{IC} is mass of the inner core, and

$$\omega_s^2 = \frac{4\pi}{3}G\rho_o(1 - \frac{\rho_o}{\rho_{IC}}), \tag{5}$$

which is identical to the expression found by Slichter (1961) and Busse (1974).

The whole-Earth centre of mass now is located between the geocentre and the centre of the displaced inner core, at distance $\epsilon\delta$ from the former, where ϵ is a correction factor. The mass $\frac{4\pi}{3}(\rho_{IC}-\rho_o)a^3$ is distributed spherically about the centre of the displaced inner core, and the mass $M_m + M_{LC} + 4\pi\rho_o a^3/3$ is distributed spherically about geocentre, where M_m and M_{LC} are mass of the mantle and liquid core respectively. The definition of the centre of mass gives

$$\frac{4\pi}{3}(\rho_{IC} - \rho_o)a^3(\delta - \epsilon\delta) = (M_m + M_{LC} + \frac{4\pi}{3}\rho_o a^3)\epsilon\delta, \tag{6}$$

by which the correction factor ϵ is obtained:

$$\epsilon = \frac{\rho_{IC} - \rho_o}{M}V_{IC}, \tag{7}$$

where $V_{IC} = 4\pi a^3/3$, M is the mass of whole Earth. For ϵ small enough we can reasonably take the geocentre as the centre of whole Earth mass, i.e. assume that the origin of the inertial reference frame coincides with the geocentre. Referring to

PREM, $\rho_{IC} - \rho_o = 0.7273 \times 10^3$ kg/m^3, ϵ is found to be about 0.001, which is adequate to support our assumption.

2.2. Effect of Flow Pressure in Outer Core

Beside the dominant gravitational restoring force acting on entire inner core, there is an additional force developed by liquid core flow acting on the inner core boundary. Recall the equation of motion in a deformable rotating solid subject to hydrostatic prestress

$$-\rho\omega^2\mathbf{u} + 2i\omega\rho\Omega\hat{\mathbf{k}}\times\mathbf{u} = \nabla\cdot\tilde{\tau} + \rho\nabla V_1 + \nabla(\rho\mathbf{u}\cdot\mathbf{g}_o) - \mathbf{g}_o\nabla\cdot(\rho\mathbf{u}), \quad (8)$$

(Smylie & Mansinha, 1971, and lecture notes of Rochester, 1986), where $\tilde{\tau}$ and V_1 are respectively the additional stress and the Eulerian disturbance in gravitational potential due to deformation or/and displacement, and Ω is the constant rotation speed of the Earth reference frame. For a homogeneous non-rotating rigid inner core, $\rho = \rho_{IC}$, $\mathbf{u} = \delta\hat{\mathbf{e}}_3$, $\Omega = 0$, this equation reduces to

$$-\rho_{IC}\,\omega^2\delta\hat{\mathbf{e}}_3 = \nabla\cdot\tilde{\tau} + \rho_{IC}\nabla V_1 + \rho_{IC}\nabla(\delta\hat{\mathbf{e}}_3\cdot\mathbf{g}_o). \quad (9)$$

Integrating (9) over the volume of the deformed inner core,

$$-\omega^2 M_{IC}\,\delta\hat{\mathbf{e}}_3 = \int_{IC}\nabla\cdot\tilde{\tau}\,dV + \rho_{IC}\int_{IC}\nabla(V_1 + \delta\hat{\mathbf{e}}_3\cdot\mathbf{g}_o)dV$$

$$= \int_{ICB}\hat{\mathbf{r}}\cdot\tilde{\tau}\,dS + \rho_{IC}\int_{ICB}\hat{\mathbf{r}}(V_1 + \delta\hat{\mathbf{e}}_3\cdot\mathbf{g}_o)dS, \quad (10)$$

using Gauss' theorem and the spherical shape of the inner core boundary. Invoking the continuity of the normal stress and V_1 across the inner core boundary, we have

$$-\omega^2 M_{IC}\, s\hat{\mathbf{e}}_3 = -\int_{ICB}\hat{\mathbf{r}}(P_1 + \rho_o \mathbf{u}\cdot\mathbf{g}_o)dS$$

$$+\rho_{IC}\int_{ICB}\hat{\mathbf{r}}(V_1 + s\hat{\mathbf{e}}_3\cdot\mathbf{g}_o)dS, \tag{11}$$

where all quantities in the integrals are evaluated on the liquid side of ICB, and P_1, the perturbation pressure, is defined by

$$\tilde{\tau} = -(P_1(\mathbf{r}) + \rho_o \mathbf{u}\cdot\mathbf{g}_o)\,\tilde{\mathbf{I}}$$

in the liquid core.

We have assumed a spherical homogeneous rigid inner core and a spherical homogeneous incompressible liquid outer core, hence $\mathbf{g}_o(\mathbf{r}) = -g_o(r)\hat{\mathbf{r}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ on ICB and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on CMB.

Substituting (9) and (11) into (8) and using Gauss' theorem give

$$-M_{IC}\,\omega^2 s\hat{\mathbf{e}}_3 = \int_{ICB}(P_1+\rho_o\mathbf{u}\cdot\mathbf{g}_o)_{LC}\,\hat{\mathbf{n}}dS - \rho_{IC}\int_{ICB}\hat{\mathbf{n}}V_1 dS$$

$$-\rho_{IC}\,g_o(a)\,s\hat{\mathbf{e}}_3\cdot\int_{ICB}\hat{\mathbf{r}}\hat{\mathbf{r}}dS. \tag{12}$$

Using continuity $[\mathbf{u}\cdot\mathbf{g}_o]_{ICB} = [s\hat{\mathbf{e}}_3\cdot\mathbf{g}_o]_{ICB}$, (12) can be written

$$-M_{IC}\,\omega^2 s\hat{\mathbf{e}}_3 = -\int_{ICB}P_1\hat{\mathbf{r}}dS - g_o(a)(\rho_{IC}-\rho_o)\,s\hat{\mathbf{e}}_3\cdot\int_{ICB}\hat{\mathbf{r}}\hat{\mathbf{r}}dS$$

$$+\rho_{IC}\int_{ICB}V_1\hat{\mathbf{r}}dS. \tag{13}$$

At point $\mathbf{r}$ in outer core, $V_1(\mathbf{r})$ can be expressed as

$$V_1(\mathbf{r}) = G\int_{LC+IC}\mathbf{u}'\cdot\nabla'(\frac{1}{R})dm'$$

$$= G\rho_o\int_{LC}\mathbf{u}'\cdot\nabla'(\frac{1}{R})dV' + G\rho_{IC}\,\delta\hat{\mathbf{e}}_3\cdot\int_{IC}\nabla'(\frac{1}{R})dV'. \tag{14}$$

First term on the RHS can be written

$$G\rho_o\int_{LC}[\nabla'\cdot(\mathbf{u}'\frac{1}{R}) - \frac{1}{R}\nabla'\cdot\mathbf{u}']dV',$$

but $\nabla'\cdot\mathbf{u}' = 0$ by incompressible assumption in the liquid core, then

$$V_1(\mathbf{r}) = G\rho_o\int_{LC}\nabla'\cdot(\mathbf{u}'\frac{1}{R})dV' + G\rho_{IC}\,\delta\hat{\mathbf{e}}_3\cdot\int_{IC}\nabla'(\frac{1}{R})dV'$$

$$= G\rho_o\int_{ICB+CMB}\mathbf{u}'\frac{1}{R}\cdot\hat{\mathbf{n}}'\,dS' + G\rho_{IC}\,\delta\hat{\mathbf{e}}_3\cdot\int_{ICB}\frac{1}{R}\hat{\mathbf{N}}'\,dS', \tag{15}$$

where $\hat{\mathbf{n}}'$ points out of liquid core and $\hat{\mathbf{N}}'$ is the surface normal of the inner core. By noting $\hat{\mathbf{n}}'\cdot\mathbf{u}' = 0$ on *CMB*, and $\hat{\mathbf{n}}'\cdot\mathbf{u}' = \hat{\mathbf{n}}'\cdot\delta\hat{\mathbf{e}}_3$ on *ICB*, we finally have

$$V_1(\mathbf{r}) = G(\rho_o-\rho_{IC})\delta\hat{\mathbf{e}}_3\cdot\int_{ICB}\frac{\hat{\mathbf{n}}'\,dS'}{R}. \tag{16}$$

R in (16) is the distance from source point $\mathbf{r}'$ to field point $\mathbf{r}$ which can be expanded in Legendre polynomials. For $r' < r$,

$$\frac{1}{R} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(\cos\psi'), \tag{17}$$

where $\cos\psi' = \hat{\mathbf{r}}'\cdot\hat{\mathbf{r}}$. Upon using (17), and setting $\hat{\mathbf{n}}' = -\hat{\mathbf{r}}'$ on the *ICB*, the surface integral in (16) for $\mathbf{r}$ in the outer core or on the *ICB* is found to be $-\frac{4\pi}{3}\frac{a^3}{r^3}\mathbf{r}$,

so

$$V_1(a) = \frac{4\pi}{3} G(\rho_{IC} - \rho_o) a \delta \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{r}}. \tag{18}$$

Substitute V_1(a) and $P_1 = \rho_o(\chi + V_1)$ into (13), we find that

$$-M_{IC} \omega^2 \delta \hat{\mathbf{e}}_3 = \mathbf{F}_g + \mathbf{F}_p \tag{19}$$

where

$$\begin{aligned}
\mathbf{F}_g &= (\rho_{IC} - \rho_o) \int_{ICB} \hat{\mathbf{r}}(V_1 + \delta \hat{\mathbf{e}}_3 \cdot \mathbf{g}_o) dS \\
&= -\int_{ICB} \rho_o V_1(r) \hat{\mathbf{r}} dS \\
&= -\frac{4\pi}{3} G(\rho_{IC} - \rho_o)\rho_o a \, \delta \hat{\mathbf{e}}_3 \cdot \int_{ICB} \hat{\mathbf{r}}\hat{\mathbf{r}} dS \\
&= -\omega_s^2 M_{IC} \, \delta \hat{\mathbf{e}}_3,
\end{aligned} \tag{20}$$

which is identical to equation (1) (considering non-homogeneity of the inner core gives the same result), and

$$\mathbf{F}_p = -\rho_o \int_{ICB} \chi \, \hat{\mathbf{r}} dS. \tag{21}$$

where for mathematical convenience we define

$$\chi = \frac{P_1(\mathbf{r})}{\rho_o} - V_1(\mathbf{r}), \tag{22}$$

(one may call it 'intermediary potential' for the role it plays between P_1/ρ_o and V_1).

In order to evaluate $\mathbf{F}_p$, we look at momentum conservation equation in the liquid outer core (Smylie & Rochester, 1981)

$$-\omega^2\mathbf{u} + 2i\omega\Omega\hat{\mathbf{k}}\times\mathbf{u} = -\nabla\chi - \beta\mathbf{g}_o\nabla\cdot\mathbf{u}. \tag{23}$$

In our case $\Omega = 0$ (non-rotating), $\beta = 0$ and $\nabla\cdot\mathbf{u} = 0$ (homogeneous incompressible), the flow motion is then governed by

$$\omega^2\mathbf{u} = \nabla\chi. \tag{24}$$

i.e. flow is irrotational, and χ also satisfies Laplace equation $\nabla^2\chi = 0$ in the liquid core. The latter suggests a solution with a form of

$$\chi = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}})P_n(\cos\theta). \tag{25}$$

There is no ϕ dependence here because of the absence of rotation. A_n and B_n are of order δ.

Boundary conditions at inner core boundary and core-mantle boundary are

$$\hat{\mathbf{n}}\cdot\mathbf{u} = 0, \quad \text{on } CMB \tag{26.a}$$
$$\hat{\mathbf{n}}\cdot\mathbf{u} = \hat{\mathbf{n}}\cdot\delta\hat{\mathbf{e}}_3 \quad \text{on } ICB \tag{26.b}$$

or by equation (24)

$$\hat{\mathbf{n}}\cdot\nabla\chi = 0, \quad \text{on } CMB \tag{27.a}$$
$$\hat{\mathbf{n}}\cdot\nabla\chi = \omega^2\hat{\mathbf{n}}\cdot\delta\hat{\mathbf{e}}_3, \quad \text{on } ICB \tag{27.b}$$

i.e.

$$\frac{\partial\chi}{\partial r} = 0 \qquad \text{on } CMB(r = b) \tag{28.a}$$
$$\frac{\partial\chi}{\partial r} = \delta\omega^2\cos\theta \qquad \text{on } ICB(r = a) \tag{28.b}$$

Substituting (25) into (28), on the core-mantle boundary we have

$$nA_n - (n+1)\frac{B_n}{b^{2n+1}} = 0, \qquad n = 0,1,2,\cdots \tag{29.a}$$

and on the inner core boundary we have

$$A_1 - \frac{2B_1}{a^3} = \delta\omega^2, \qquad n = 1 \tag{29.b}$$

$$nA_n - (n+1)\frac{B_n}{a^{2n+1}} = 0, \qquad n = 0,2,3,\cdots \tag{29.c}$$

From (29), the following results are obtained

$$A_1 = -\delta\omega^2\frac{a^3}{b^3-a^3}, \tag{30.a}$$

$$B_1 = -\frac{1}{2}\frac{\delta\omega^2 a^3 b^3}{b^3-a^3}, \tag{30.b}$$

$$A_2 = A_3 = \cdots = B_o = B_2 = B_3 = \cdots = 0, \tag{30.c}$$

and without loss of generality we take $A_o = 0$ since it merely adds a constant to χ. Thus we arrive at an analytic solution (Lamb, 1932)

$$\chi(r,\theta) = -\frac{\delta\omega^2 a^3}{b^3-a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta. \tag{31}$$

Placing this expression in (21), we have

$$\mathbf{F}_p = \frac{\delta\omega^2 a}{b^3-a^3}\left(\frac{b^3+2a^3}{2}\right)\rho_o\int_{ICB}\hat{\mathbf{r}}\cos\theta dS$$

$$= \omega^2 M_{IC} f\delta\hat{\mathbf{e}}_3, \tag{32}$$

where

$$f=\frac{\rho_o}{2\rho_{IC}}\left(\frac{b^3+2a^3}{b^3-a^3}\right). \tag{33}$$

The force $\mathbf{F}_p$ is seen to reduce the total restoring force in the system, it is equivalent to an increase of the inertia of the inner core. For PREM (Dziewonski and Anderson, 1981), $f\approx 0.536$, this increase is about 50%.

Substituting $\mathbf{F}_g$ and $\mathbf{F}_p$ back to (19), we find the oscillation frequency

$$\omega=\frac{\omega_s}{(1+f)^{1/2}}, \tag{34}$$

where ω_s is as in (5). Thus the simple model we presented here predicts the period of the Slichter mode for a non-rotating Earth with rigid inner core and mantle, homogeneous incompressible liquid outer core to be

$$T=\frac{2\pi}{\omega}=\frac{2\pi(1+f)^{1/2}}{\omega_s}. \tag{35}$$

T is very sensitive to the Earth model adopted. For PREM (1981), ρ_{IC} = 12.8936×10^3 kg/m^3, ρ_o = 12.1663×10^3 kg/m^3, we have T = 4.94 hr. (This result is reproduced in the limiting-case of our model in Section 7, see upper left-hand portion of Table 3.) But Busse (1974) used ρ_{IC} = 12.3×10^3 kg/m^3, ρ_o = 12.0×10^3 kg/m^3, which leads him to $T=7.59$ hr.

2.3. Effect of Linear Momentum of the Mantle

Finally we need to have a look at the linear momentum of the Earth in the model:

$$\mathbf{P}_{IC} + \mathbf{P}_{LC} \text{ (momentum of liquid core and inner core)}$$

$$= M_{IC}\,(i\omega\epsilon\hat{\mathbf{e}}_3) + \rho_o \int_{LC} i\omega\mathbf{u}\,dV$$

$$= i\omega M_{IC}\,\epsilon\hat{\mathbf{e}}_3 + \frac{i\rho_o}{\omega}\int_{LC} \nabla\chi\,dV$$

$$= i\omega[\,-\frac{\mathbf{F}_p + \mathbf{F}_g}{\omega^2} + \frac{\rho_o}{\omega^2}\int_{CMB+ICB} \hat{\mathbf{n}}\,\chi\,dS\,]$$

$$= \frac{i}{\omega}[\,-\mathbf{F}_g + \rho_o\int_{CMB} \hat{\mathbf{n}}\chi\,dS\,]$$

$$= \frac{i}{\omega}[\,-\mathbf{F}_g + \int_{CMB} \hat{\mathbf{n}}(P_1 - \rho_o V_1)dS\,]$$

$$= \frac{i}{\omega}[\int_{CMB} \hat{\mathbf{n}}P_1 dS - (\mathbf{F}_g + \rho_o\int_{CMB} \hat{\mathbf{n}}V_1 dS\,)], \tag{36}$$

using equation (19) and (21). For $\mathbf{r}$ in the outer core,

$$V_1(r) = \frac{4\pi}{3}G(\rho_{IC} - \rho_o)\frac{a^3}{r^3}\,\epsilon\hat{\mathbf{e}}_3\cdot\mathbf{r}, \tag{37}$$

and at core-mantle boundary

$$V_1(b) = \frac{4\pi}{3}G(\rho_{IC} - \rho_o)\frac{a^3}{b^2}\,\epsilon\hat{\mathbf{e}}_3\cdot\hat{\mathbf{r}}. \tag{38}$$

Then we have in equation (36)

$$\mathbf{F}_g + \rho_o\int_{CMB} \hat{\mathbf{n}}V_1 dS = 0. \tag{39}$$

Since $\mathbf{u}\cdot\mathbf{g}_0 = 0$ at core-mantle boundary, we can add a zero integral into (36), i.e.

$$\mathbf{P}_{IC} + \mathbf{P}_{LC} = \frac{i}{\omega}\int_{CMB} \hat{\mathbf{n}}(P - \rho_0 \mathbf{u}\cdot\mathbf{g}_o)dS$$

$$= -\frac{i}{\omega}\int_{CMB} \hat{\mathbf{n}}\cdot\tilde{\tau}\, dS$$

$$= \frac{i}{\omega}\int_{CMB} \hat{\mathbf{N}}\cdot\tilde{\tau}\, dS$$

$$= \frac{i}{\omega}\ \text{(rate of change of momentum of mantle)}, \qquad (40)$$

where $\hat{\mathbf{N}} = -\hat{\mathbf{n}}$ at core-mantle boundary, points out of mantle. Because of spherical symmetry and homogeneity assumption, neither inner core nor outer core exert a gravitational attraction on mantle, then

$$\mathbf{P}_{IC} + \mathbf{P}_{LC} = \frac{i}{\omega}\left[i\omega\ \text{(momentum of mantle)}\right]$$

$$= -\ \text{momentum of mantle}$$

$$= -i\omega M_m(-\epsilon\xi). \qquad (41)$$

So by neglecting ϵ we fail by a small amount to conserve momentum over the whole Earth.

From equation (30) we see that the only non-zero coefficients of the solution χ are A_1 and B_1. It can readily be proved by spheroidal and toroidal representation of displacement fields in a spherically-stratified Earth that the only possible contribution to change in location of centre of mass of the inner core (or the whole Earth) is from $n = 1$ spheroidal terms. This is the basis of Crossley's (1975) description of the Slichter mode as the first undertone of the $n = 1$ spheroidal modes.

Our simple model is still very far from representing the Earth realistically. To model the Slichter mode more rigorously, we are going to relax those three restrictions (assumptions of a non-rotating reference frame, homogeneous and incompressible outer core and rigid inner core) imposed on the starting model and try to find the effects of each restriction on the Slichter mode. Thus we arrive at a relatively more realistic Earth model (i.e. rotating, compressible, neutrally-stratified outer core and homogeneous elastic inner core). By further generalizing to non-neutral stratification of the liquid core, we allow negative buoyancy force to exist there, so all corresponding undertones (of longer period than the first) will appear. But we will not explore those undertones in the present study.

3. THE SUBSEISMIC WAVE EQUATION IN THE LIQUID CORE

3.1. The Subseismic Approximation

Several researchers (e.g. Busse (1974), Crossley (1975), Smith (1976)) have shown that taking off any (or all) restrictions on the simple model discussed in previous section will greatly increase the complexity and computational difficulty of the mathematical approach to the problem. While removing some restrictions can be said only to cause minor difficulties in analysis and computation, the effect of taking rotation into account is to introduce difficulties which are very grave. From previous review we see that the classical approach to free oscillation theory is to represent the perturbation fields as a superposition of spherical harmonics. The weakness of the spherical harmonic representation for such long-period oscillation (gravity/inertial) modes is that the presence of Coriolis force due to Earth's rotation couples the resulting sets of ordinary differential equations in the liquid core into an infinitely long chain with coupling parameter of order 1 ($\sim 2\Omega/\omega$), truncation of which is almost certain to raise serious questions about convergence. So far all attempts to include more terms in the spherical harmonic expansion have been severely limited because of the resulting computational burden. For example, Crossley (1975) and Smith (1976) carried the coupling chain of spherical harmonic representation only as far as $n = 2$ in the liquid core.

In the hope of overcoming this particular difficulty, Smylie and Rochester (1981) suggested filtering out acoustic modes by neglecting the effect of flow pressure on compression, compared to the effect of displacement through the hydrostatic pressure field. In the liquid core, the perturbation pressure P_1 is governed by equation of state

$$\frac{P_1}{\rho_o} = -\alpha^2 \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{g}_o, \tag{42}$$

which can be written as

$$-\nabla \cdot \mathbf{u} = \frac{P_1}{\rho_o \alpha^2} + \frac{\mathbf{u} \cdot \mathbf{g}_o}{\alpha^2}, \tag{43}$$

where α is the compressional wave speed. By applying subseismic approximation (Smylie and Rochester, 1981), (43) reduces to

$$\nabla \cdot \mathbf{u} = -\frac{\mathbf{u} \cdot \mathbf{g}_o}{\alpha^2}, \tag{44}$$

which is valid only as long as

$$\left|\frac{P_1}{\rho_o}\right| \ll |\mathbf{u}\cdot\mathbf{g}_o|,\ \alpha^2|\nabla\cdot\mathbf{u}|, \tag{45}$$

i.e.

$$|\chi + V_1| \ll |\mathbf{u}\cdot\mathbf{g}_o|,\ \alpha^2|\nabla\cdot\mathbf{u}|, \tag{46}$$

using $P_1 = \rho_o(\chi + V_1)$.

We assume that the simple model considered in section 2 is a good first approximation for evaluating eigenperiod of the Slichter mode. This suggests that results obtained there can be used to estimate $\chi + V_1$ and $\mathbf{u}\cdot\mathbf{g}_o$ in (46), with which to test the validity of the subseismic approximation as we approach a more realistic liquid core model. Because the simple model assumes incompressibility in the liquid core ($\alpha \to \infty$, $\nabla\cdot\mathbf{u} \to 0$), the second term in (46) is not directly involved.

Taking $\mathbf{u}$, χ and V_1 in the outer core from (24), (31) and (37):

$$\mathbf{u}\cdot\mathbf{g}_o = -g_o\hat{\mathbf{r}}\cdot\left(\frac{1}{\omega^2}\nabla\chi\right) = \frac{g_o a^3\delta}{b^3-a^3}\left(1-\frac{b^3}{r^3}\right)\cos\theta, \tag{47}$$

$$\chi+V_1 = \frac{\omega_s^2 a^3\delta}{(1+f)r^2}\left[\frac{\rho_{IC}}{\rho_o} - \left(\frac{r^3-a^3}{b^3-a^3}\right)\right]\cos\theta. \tag{48}$$

Then for subseismic approximation to be valid the following relation must hold:

$$\omega^2\frac{r}{g_o}\left[\frac{\rho_{IC}}{\rho_o} - \left(\frac{r^3-a^3}{b^3-a^3}\right)\right]\left(1-\frac{a^3}{b^3}\right) \ll \left(1-\frac{r^3}{b^3}\right). \tag{49}$$

Clearly the difficulties will be greatest as $r \to b$. For $T = 4.94$ hr, $r = 0.9b$, and

$g_o(r) = 9.8$ m s^{-2} , we have $\omega \approx 3.53\times10^{-4}$ s^{-1}, $\frac{r}{g_o} \approx 3.2\times10^5$ s^2, $\frac{\rho_{IC}}{\rho_o} \approx 1.06$, $\frac{r^3-a^3}{b^3-a^3} \approx 0.72$, and $(1-\frac{a^3}{b^3}) \approx 0.96$. Then LHS of (36) $\approx 1.3\times10^{-2}$, RHS of (36) ≈ 0.271, $\frac{LHS}{RHS} \approx 4.8\times10^{-2}$, i.e. the subseismic approximation in liquid core dynamics involves an error of less than 5% (even fairly close to the core-mantle boundary).

The error estimated here is rather large, compared to the expected error of order 0.1% from the scaling argument of Smylie & Rochester (1981). A more reliable calculation of the error involved in the subseismic approximation would require using the χ, V_1, $\mathbf{u}\cdot\mathbf{g}_o$ for the neutrally-stratified compressible outer core obtained by invoking that approximation (rather than the more stringent one of incompressibility). This would probably show that subseismic approximation is considerably better than our simple estimate suggests.

3.2. The Subseismic Wave Equation

Now the equation of state takes the form of (44), which can be written

$$\nabla\cdot(\rho_o^{+}\mathbf{u}) = 0, \tag{50}$$

where ρ_o^{+} is a modified density function (Friedlander, 1988):

$$\rho_o^{+}(r) = \rho_o(r)\exp\left(\int_a^r \frac{1}{\alpha^2}\beta(r)dW_o\right)$$

$$= \rho_o(a)\exp\left(-\int_a^r \frac{1}{\alpha^2}g_o(r)dr\right) \qquad \text{for spherical stratification ,} \tag{51}$$

where $\beta(r)$ is the stability parameter of the liquid core, which measures the size and sign of any departure of the density gradient from purely adiabatic stratification (Pekeris and Accad, 1972).

On substituting (44), the momentum conservation equation (23) becomes

$$-\omega^2\mathbf{u} + 2i\omega\Omega\hat{\mathbf{k}}\times\mathbf{u} = -\nabla\chi + \frac{\beta}{\alpha^2}\mathbf{g}_o\mathbf{u}\cdot\mathbf{g}_o \tag{52}$$

where $\Omega\hat{\mathbf{k}}$ is the angular velocity of the uniformly-rotating reference frame with respect to which displacements $\mathbf{u}$ are measured. Equation (52) can be abbreviated as

$$\mathbf{u} = \frac{1}{\omega^2(\sigma^2-1)}\tilde{\Gamma}\cdot\nabla\chi, \tag{53}$$

where

$$\sigma = \frac{\omega}{2\Omega} \tag{54.a}$$

$$\tilde{\Gamma} = \sigma^2\tilde{\mathbf{I}} - \hat{\mathbf{k}}\hat{\mathbf{k}} - \frac{\mathbf{C}^*\mathbf{C}}{B} + i\sigma\hat{\mathbf{k}}\times\tilde{\mathbf{I}} \tag{54.b}$$

$$\mathbf{C} = \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{g}_o) - \sigma^2\mathbf{g}_o + i\sigma\hat{\mathbf{k}}\times\mathbf{g}_o \tag{54.c}$$

$$B = \frac{\alpha^2\omega^2}{\beta}(\sigma^2-1) + \sigma^2 g_o{}^2 + (\hat{\mathbf{k}}\cdot\mathbf{g}_o)^2 \tag{54.d}$$

(Rochester 1989, Smylie & Rochester 1981).

By combining equations (50) and (53), we obtain the subseismic wave equation

$$\nabla\cdot(\rho_o^{+}\tilde{\Gamma}\cdot\nabla\chi) = 0 \tag{55}$$

(Rochester, 1939), which is subject to the boundary conditions appropriate to the model adopted.

V_1 satisfies Poisson's equation in the liquid core

$$\nabla^2 V_1 = 4\pi G \nabla\cdot(\rho_o \mathbf{u})$$
$$= 4\pi G(\nabla\rho_o\cdot\mathbf{u} + \rho_o\nabla\cdot\mathbf{u}). \tag{56}$$

But the density of the liquid core is subject to the following equation

$$\nabla\rho_o = (1-\beta)\frac{\rho_o}{\alpha^2}\mathbf{g}_o, \tag{57}$$

where α is the compressional wave speed in the liquid core. Then by using equation (57) and equation of state (44), the Poisson's equation becomes

$$\nabla^2 V_1 = -\,4\pi G\rho_o\beta\frac{\mathbf{u}\cdot\mathbf{g}_o}{\alpha^2}. \tag{58}$$

3.3. Neutral Stratification

For a neutrally stratified outer core, $\tilde{\Gamma}$ reduces to the Poincaré dyadic

$$\tilde{\Gamma}_p = \sigma^2\tilde{\mathbf{I}} - \hat{\mathbf{k}}\hat{\mathbf{k}} + i\sigma\hat{\mathbf{k}}\times\tilde{\mathbf{I}} \tag{59}$$

and $\rho_o^\dagger(r) = \rho_o(r)$. Thus (55) reduces to

$$\nabla\cdot(\rho_o\tilde{\Gamma}_p\cdot\nabla\chi) = 0 \tag{60}$$

or $\mathcal{L}_p\chi = 0$, where $\mathcal{L}_p$ is a linear operator. Now as $\beta \to 0$, $V_1(r)$ satisfies Laplace's equation in the neutrally stratified liquid core, i.e. $\nabla^2 V_1 = 0$. This result will be seen, in next section, to have such advantages that we will henceforth regard the liquid core as neutrally stratified.

4. ELASTODYNAMICS OF THE INNER CORE

The assumption of a rigid inner core has greatly simplified our analysis of the problem, but at the same time led us away from reality, especially for the Slichter modes which are generated by inner core translational oscillation. Elastic deformation of the inner core reduces the effective gravitational restoring force in the oscillation, thus increasing its vibration period.

4.1. AJP Description of Elastic Deformation of a Rotating, Homogeneous Inner Core

For simplicity, we consider a homogeneous, spherical elastic inner core in which Lamé parameter λ, rigidity μ and density ρ_{IC} are constant throughout. The gravity acceleration $g_0(r)$ in this homogeneous sphere is

$$g_0(r) = \Gamma r, \tag{61}$$

where

$$\Gamma = \frac{4\pi}{3} G \rho_{IC} \tag{62}$$

In a uniformly rotating reference frame, the inner core undergoes an elastic deformation responding to the force $\mathbf{F}_p$, i.e. to the intermediary potential χ. The equation of motion of the inner core in the deformed state is expressed by equation (8), with $\rho = \rho_{IC}$, $\mathbf{u} = \mathbf{u}_{IC}$,

$$
\begin{aligned}
&-\rho_{IC}\,\omega^2\mathbf{u}_{IC} + 2i\omega\rho_{IC}\,\Omega\hat{\mathbf{k}}\times\mathbf{u}_{IC} \\
&= \nabla\cdot\tilde{\tau} + \rho_{IC}\nabla V_1 + \nabla(\rho_{IC}\mathbf{u}_{IC}\cdot\mathbf{g}_o) - \mathbf{g}_o\nabla\cdot(\rho_{IC}\mathbf{u}_{IC})
\end{aligned}
\tag{63}
$$

But $\nabla\rho_{IC} = 0$ here since we have assumed a homogeneous inner core, so

$$
\begin{aligned}
&-\rho_{IC}\,\omega^2\mathbf{u}_{IC} + 2i\omega\rho_{IC}\,\Omega\hat{\mathbf{k}}\times\mathbf{u}_{IC} \\
&= \nabla\cdot\tilde{\tau} + \rho_{IC}\nabla V_1 + \rho_{IC}\nabla(\mathbf{u}_{IC}\cdot\mathbf{g}_o) - \rho_{IC}\,\mathbf{g}_o\nabla\cdot\mathbf{u}_{IC}
\end{aligned}
\tag{64}
$$

or

$$
\begin{aligned}
\nabla\cdot\tilde{\tau} = &-\rho_{IC}\,\omega^2\mathbf{u}_{IC} + 2i\omega\rho_{IC}\,\Omega\hat{\mathbf{k}}\times\mathbf{u}_{IC} + \rho_{IC}\,\mathbf{g}_o\nabla\cdot\mathbf{u}_{IC} \\
&-\rho_{IC}\nabla(\mathbf{u}_{IC}\cdot\mathbf{g}_o) - \rho_{IC}\nabla V_1,
\end{aligned}
\tag{65}
$$

which can be written as

$$
\begin{aligned}
&(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u}_{IC} - \mu\nabla\times(\nabla\times\mathbf{u}_{IC}) \\
&= -\rho_{IC}\,\omega^2\mathbf{u}_{IC} + 2i\omega\rho_{IC}\,\Omega\hat{\mathbf{k}}\times\mathbf{u}_{IC} + \rho_{IC}\,\mathbf{g}_o\nabla\cdot\mathbf{u}_{IC} \\
&\quad -\rho_{IC}\nabla(\mathbf{u}_{IC}\cdot\mathbf{g}_o) - \rho_{IC}\nabla V_1.
\end{aligned}
\tag{66}
$$

The displacement field $\mathbf{u}_{IC}$ must satisfy equation (66) and Poisson's equation

$$
\nabla^2 V_1 = 4\pi G\rho_{IC}\nabla\cdot\mathbf{u}_{IC}.
\tag{67}
$$

Following Alterman, Jarosch and Pekeris (1959), we represent $\mathbf{u}_{IC}$ and V_1 in spherical harmonics as follows:

$$
\mathbf{u}_{IC} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(u_n^m\hat{\mathbf{r}} + r v_n^m\nabla - t_n^m\mathbf{r}\times\nabla\right)Y_n^m,
\tag{68}
$$

$$
V_1 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\phi_n^m(r)Y_n^m,
\tag{69}
$$

and use their y_i notations to denote u_n^m, v_n^m and ϕ_n^m, each of which is a function of r:

$$y_1 = u_n^m,$$
$$y_3 = v_n^m,$$
$$y_5 = \phi_n^m, \tag{70}$$
$$y_6 = \frac{dy_5}{dr} - 4\pi G\rho_{IC}\, y_1,$$
$$y_7 = it_n^m,$$

then

$$\nabla\cdot\mathbf{u}_{IC} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{dy_1}{dr} + \frac{2y_1 - n(n+1)y_3}{r}\right]Y_n^m, \tag{71}$$

$$\nabla\times\mathbf{u}_{IC} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\{-\nabla\times[\mathbf{r}(\frac{dy_3}{dr} + \frac{y_3 - y_1}{r})Y_n^m]$$
$$- i[\frac{n(n+1)}{r^2}\mathbf{r}y_7 + \frac{d}{dr}(ry_7)\nabla]Y_n^m\}. \tag{72}$$

Define y_2, y_4 and y_8 as

$$y_2 = \lambda(\frac{dy_1}{dr} + \frac{2y_1 - n(n+1)y_3}{r}) + 2\mu\frac{dy_1}{dr}, \tag{73.a}$$
$$y_4 = \mu(\frac{dy_3}{dr} - \frac{y_3 - y_1}{r}), \tag{73.b}$$
$$y_8 = \mu(\frac{dy_7}{dr} - \frac{y_7}{r}). \tag{73.c}$$

We now substitute (68)-(73) into equations (66) and (67). Crossley and Rochester (1980) showed that the latter can be written, for $n = 1$,

$$\alpha^2\frac{d\Delta}{dr}+2\frac{\beta^2}{r}H+\omega^2 y_1-2\omega\Omega[my_3-\frac{3}{5}(m+2)y_7^{(2)}]+\Gamma(y_1-2y_3)+\frac{dy_5}{dr}=0, \qquad (74)$$

$$\frac{\alpha^2}{r}\Delta+\frac{\beta^2}{r}\frac{d}{dr}(rH)+\omega^2 y_3-\omega\Omega[m(y_1+y_3)+\frac{3}{5}(m+2)y_7^{(2)}]-\Gamma y_1+\frac{y_5}{r}=0, \qquad (75)$$

$$\beta^2[\frac{1}{r}\frac{d^2}{dr^2}(ry_7^{(2)})-\frac{6}{r^2}y_7^{(2)}]+\omega^2 y_7^{(2)}-\frac{1}{3}\omega\Omega[my_7^{(2)}-(2-m)(y_1-y_3)]=0, \qquad (76)$$

$$\frac{1}{r}\frac{d^2}{dr^2}(ry_5)-\frac{2}{r^2}y_5=3\Gamma\Delta, \qquad (77)$$

where

$$\Delta=\frac{dy_1}{dr}+2(y_1-y_3)/r, \qquad (78)$$

$$H=\frac{dy_3}{dr}-(y_1-y_3)/r, \qquad (79)$$

$$\alpha^2=\frac{\lambda+2\mu}{\rho_{IC}}, \qquad (80)$$

$$\beta^2=\frac{\mu}{\rho_{IC}}. \qquad (81)$$

α and β are respectively compressional and shear wave speeds in solid inner core. Note that in (76) we take into account only self-coupling by dropping terms in y_1 and y_3 of degree 3. Superscript (2), which will be dropped later, indicates degree 2.

4.2. Boundary Conditions at *ICB* and a Load Love Number

If we represent χ as

$$\chi(r,\vartheta,\phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \chi_n^m(r) P_n^m(\cos\vartheta) e^{im\phi}, \tag{82}$$

the boundary conditions prescribed at inner core boundary are

$$y_4(a) = 0, \tag{83}$$

$$\Gamma a y_1(a) - y_5(a) - \frac{y_2(a)}{\rho_o(a)} = \chi_1^m(a), \tag{84}$$

$$y_6(a) = -\frac{2}{a} y_5(a) - \frac{3\Gamma\rho_o(a)}{\rho_{IC}} y_1(a), \tag{85}$$

which are based respectively on continuity of the shear stress, of the radial component of the normal stress, and of the gravitational potential and gravitational flux across *ICB*. Here we emphasize that by assuming the liquid core to be neutrally stratified, the subseismic approximation permits us to take V_1 as harmonic there.

Now by analogy with the Love number concept from Earth tides analysis we represent the degree-1 radial displacement u_1^m at inner core boundary as

$$u_1^m(a) = y_1(a) = -h_1^{m\prime} \frac{\chi_1^m(a)}{g_o(a)}, \tag{86}$$

i.e.

$$[\hat{\mathbf{r}}\cdot\mathbf{u}]_{ICB} = - h_1^{m\,\prime} \cdot \frac{\chi_1^m(a)}{g_o(a)} P_1^m(\cos\theta) e^{im\phi}. \tag{87}$$

where $h_1^{m\,\prime}$ is a load Love number of degree 1 which describes the displacement of the inner core in response to the 'intermediary potential' χ. Load Love numbers n > 1 could be similarly defined by the response $\mathbf{u}_n^m(a)$ to the load $\chi_n^m(a)$, but for simplicity, we here neglect all fields of degree n > 1 in the inner core.

Upon substitution of (86) and use of boundary conditions (83)-(85), we can solve equations (74), (75), (76) and (77) to find the load Love number $h_1^{m\,\prime}$. For details see appendix A. Here we write the solution

$$\frac{1}{h_1^{m\,\prime}} = 1 - \frac{\rho_o(a)}{\rho_{IC}} - \frac{\omega(\omega - 2m\Omega)\rho_{IC}}{\Gamma\rho_o(a)}(1+e), \tag{88}$$

where e is the elastic correction factor

$$e \approx \frac{\{\frac{\Gamma a\rho_o(a)}{\omega\alpha\rho_{IC}}[1 - \frac{\omega^2\rho_{IC}}{\Gamma\rho_o(a)}(1 - \frac{2m\Omega}{\omega})]\}^2}{5(1 - \frac{2m\Omega}{\omega})(1 - \frac{4\beta^2}{3\alpha^2})}. \tag{89}$$

In the case of rigid inner core, $\alpha \rightarrow \infty, e \rightarrow 0$, (88) reduces to

$$\frac{1}{h_1^{m\,\prime}} = 1 - \frac{\rho_o(a)}{\rho_{IC}} - \frac{\omega(\omega - 2m\Omega)\rho_{IC}}{\Gamma\rho_o(a)}, \tag{90}$$

If further let $\Omega \rightarrow 0$ in (90), we get h_1' for the case of non-rotating, homogeneous rigid inner core limit

$$\frac{1}{h_1'} = 1 - \frac{\rho_o(a)}{\rho_{IC}} - \frac{\omega^2 \rho_{IC}}{\Gamma \rho_o(a)}, \tag{91}$$

(with m dropped because it is superfluous when rotation is ignored.) Using (5), (31), (34), (61) and (62) it can be shown that (91) is consistent with the results of section 2.

5. VARIATIONAL PRINCIPLE FOR THE SUBSEISMIC WAVE EQUATION IN THE LIQUID CORE

5.1. Establishing a Variational Principle

We appeal to a variational principle to solve the subseismic wave equation (55) in the liquid core which is assumed neutrally-stratified and compressible. We define the functional

$$F = \int_{LC} \chi^* \mathcal{L}_p \chi dV, \tag{92}$$

integrated over the liquid core, with $\mathcal{L}_p$ given by (60). The variation

$$\delta F = \int_{LC} \delta\chi^* \mathcal{L}_p \chi dV + \int_{LC} \chi^* \mathcal{L}_p \delta\chi dV. \tag{93}$$

We wish to show that if $\delta F = 0$ (i.e. F is stationary) for arbitrary $\delta\chi$, then χ must satisfy the subseismic wave equation, i.e.

$$\mathcal{L}_p \chi = 0, \quad \text{and} \quad (\mathcal{L}_p \chi)^* = 0. \tag{94}$$

From (93),

$$\delta F - \int_{LC} [\delta\chi^* \mathcal{L}_p \chi + \delta\chi (\mathcal{L}_p \chi)^*] dV$$

$$= \int_{LC} [\chi^* \mathcal{L}_p \delta\chi - \delta\chi (\mathcal{L}_p \chi)^*] dV$$

$$= \int_{LC} [\chi^* \nabla \cdot (\rho_o \tilde{\Gamma}_p \cdot \nabla \delta\chi) - \delta\chi \nabla \cdot (\rho_o \tilde{\Gamma}_p^* \nabla \chi^*)] dV$$

$$= \int_{LC} [\nabla \cdot (\chi^* \rho_o \tilde{\Gamma}_p \cdot \nabla \delta\chi) - \rho_o \nabla \chi^* \cdot \tilde{\Gamma}_p \cdot \nabla \delta\chi$$

$$+ \rho_o \nabla \delta\chi \cdot \tilde{\Gamma}_p^* \cdot \nabla \chi^* - \nabla \cdot (\delta\chi \rho_o \tilde{\Gamma}_p^* \cdot \nabla \chi^*)] dV. \tag{95}$$

From (59) $\tilde{\Gamma}_p$ is Hermitean, so this last expression reduces to

$$\int_{LC} \nabla \cdot (\chi^* \rho_o \tilde{\Gamma}_p \cdot \nabla \delta\chi - \delta\chi \rho_o \tilde{\Gamma}_p^* \cdot \nabla \chi^*) dV$$

$$= \int_{ICB+CMB} \rho_o [\chi^* \tilde{\Gamma}_p \cdot \nabla \delta\chi - \delta\chi \tilde{\Gamma}_p^* \cdot \nabla \chi^*] \hat{\mathbf{n}} dS$$

$$= - \int_{ICB} \rho_o [\chi^* \tilde{\Gamma}_p \cdot \nabla \delta\chi - \delta\chi \tilde{\Gamma}_p^* \cdot \nabla \chi^*] \hat{\mathbf{r}} dS$$

$$+ \omega^2(\sigma^2 - 1) \int_{CMB} \rho_o [\chi^* \delta(\mathbf{u} \cdot \hat{\mathbf{r}}) - \delta\chi \mathbf{u} \cdot \hat{\mathbf{r}}] dS,$$

$$= - \omega^2(\sigma^2 - 1) \int_{ICB} \rho_o [\chi^* \hat{\mathbf{r}} \cdot \delta(\delta \hat{\mathbf{e}}_3) - \delta\chi \hat{\mathbf{r}} \cdot (\delta^* \hat{\mathbf{e}}_3)] dS, \tag{96}$$

using rigid-fixed boundary condition on CMB. Substituting (86) and (87) into above result, we finally have

$$\delta F - \int_{LC} [\delta\chi^* \mathcal{L}_p \chi + \delta\chi (\mathcal{L}_p \chi)^*] dV$$

$$= \omega^2(\sigma^2 - 1) \frac{h_1^{m\,\prime} \rho_o(a)}{g_o(a)} \int_{ICB} [\chi^* \delta\chi_1^m(a) P_1^m e^{im\phi} - (\delta\chi) \chi_1^{m*}(a) P_1^m e^{-im\phi}] dS$$

$$= \omega^2(\sigma^2 - 1) \frac{h_1^{m\,\prime} \rho_o(a)}{g_o(a)} \int_{ICB} [\chi_1^{m*}(a) \delta\chi_1^m(a) - \delta\chi_1^m(a) \chi_1^{m*}(a)] (P_1^m)^2 dS$$

$$= 0, \tag{97}$$

using the orthogonality properties of the spherical harmonics. This establishes the validity of the variational principle based on the functional F. Now

$$
\begin{aligned}
F &= \int_{LC} \chi^* L_p \chi \, dV \\
&= \int_{LC} \chi^* \nabla \cdot (\rho_o \tilde{\Gamma}_p \cdot \nabla \chi) dV \\
&= \int_{LC} \nabla \cdot (\chi^* \rho_o \tilde{\Gamma}_p \cdot \nabla \chi) dV - \int_{LC} \rho_o \nabla \chi^* \cdot \tilde{\Gamma}_p \cdot \nabla \chi \, dV \\
&= \int_{ICB+CMB} \chi^* \rho_o \hat{\mathbf{n}} \cdot \tilde{\Gamma}_p \cdot \nabla \chi \, dS - \int_{LC} \rho_o \nabla \chi^* \cdot \tilde{\Gamma}_p \cdot \nabla \chi \, dV \\
&= -\omega^2(\sigma^2-1)\rho_o(a) \int_{ICB} \chi^* \hat{\mathbf{r}} \cdot \mathbf{u} \, dS - \int_{LC} \rho_o \nabla \chi^* \cdot \tilde{\Gamma}_p \cdot \nabla \chi \, dV. \qquad (98)
\end{aligned}
$$

The surface integral in the last expression,

$$
\begin{aligned}
& -\omega^2(\sigma^2-1)\rho_o(a) \int_{ICB} \chi^* \hat{\mathbf{r}} \cdot \mathbf{u} \, dS \\
&= \omega^2(\sigma^2-1) \frac{h_1^{m\,\prime} \rho_o(a) a^2}{g_o(a)} \int_{ICB} \sum_{q=0}^{\infty} \sum_{p=-q}^{q} \chi_q^{p*} P_q^p(x) e^{-ip\phi} \chi_1^m P_1^m(x) e^{im\phi} dx \, d\phi \\
&= \omega^2(\sigma^2-1) \frac{h_1^{m\,\prime} \rho_o(a) a^2}{g_o(a)} \sum_{q=|m|}^{\infty} \chi_1^m(a) \chi_q^{m*}(a) (2\pi) \int_{-1}^{1} P_q^m(x) P_1^m(x) dx \\
&= \frac{(1+m)!}{(1-m)!} \frac{\omega^2(\sigma^2-1)\rho_o(a) a}{G\rho_{IC}} h_1^{m\,\prime} |\chi_1^m(a)|^2, \qquad (99)
\end{aligned}
$$

where $x = \cos\theta$. The final form of the functional is therefore

$$
F = \frac{(1+m)!}{(1-m)!} \frac{\omega^2(\sigma^2-1)\rho_o(a) a}{G\rho_{IC}} h_1^{m\,\prime} |\chi_1^m(a)|^2 - \int_{LC} \rho_o \nabla \chi^* \cdot \tilde{\Gamma}_p \cdot \nabla \chi \, dV. \qquad (100)
$$

As mentioned earlier, when we let $e \rightarrow 0$, $\Omega \rightarrow 0$ and $\rho_o(r) = \rho_o(a)$, we reduce to the simple model of a non-rotating, homogeneous incompressible liquid core and rigid inner core. It must be interesting to test what result we will get by putting those limits into the functional (100). Using (5) and (91) we have

$$h_1' = \frac{1}{1 - \dfrac{\rho_o(a)}{\rho_{IC}} - \dfrac{\omega^2 \rho_{IC}}{\Gamma \rho_o(a)}} = \frac{4\pi G \rho_o(a)}{3(\omega_s{}^2 - \omega^2)}. \tag{101}$$

In order to avoid zeros in the denominator we build up a new functional by modifying (100) to

$$J = 4\Omega^2(\omega_s{}^2 - \omega^2)F. \tag{102}$$

When $e \to 0, \Omega \to 0$ and $\rho_o(r) = \rho_o(a)$, this becomes

$$J = 4\pi\omega^2 \rho_o(a) \left[\frac{\omega^2 \rho_o(a) a |\chi_1^m(a)|^2}{3\rho_{IC}} - \frac{(\omega_s{}^2 - \omega^2)}{4\pi} \int_{LC} \nabla\chi^* \cdot \nabla\chi dV \right]. \tag{103}$$

5.2. Trial Functions and Boundary Conditions

The trial function χ used in F need not satisfy the prescribed boundary conditions since the latter have been used when developing the functional (100). An explicit demonstration of this follows for the case $e \to 0, \Omega \to 0$ and $\rho_o(r) = \rho_o(a)$. By choosing χ as same as in section 2 (expression (25)), we have

$$\chi_1(a) = A_1 a + \frac{B_1}{a^2}, \tag{104}$$

and

$$\int_{LC} \nabla\chi^* \cdot \nabla\chi dV$$

$$=\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\int_{LC}\{[nA_n r^{n-1}-(n+1)\frac{B_n}{r^{n+2}}][qA_q r^{q-1}-(q+1)\frac{B_q}{r^{q+2}}]P_n(x)P_q(x)$$

$$+(A_n r^n+\frac{B_n}{r^{n+1}})(A_q r^q+\frac{B_q}{r^{q+1}})\frac{(1-x^2)}{r^2}\frac{dP_n(x)}{dx}\frac{dP_q(x)}{dx}\}r^2 dr dx d\phi$$

$$=4\pi\sum_{n=0}^{\infty}\frac{b^{2n+1}-a^{2n+1}}{2n+1}[nA_n^{\ 2}+(n+1)\frac{B_n^{\ 2}}{(ab)^{2n+1}}]. \tag{105}$$

so,

$$J=4\pi\omega^2\rho_o(a)[\frac{\omega^2\rho_o(a)a}{3\rho_{IC}}(A_1 a+\frac{B_1}{a^2})^2$$

$$-(\omega_s^{\ 2}-\omega^2)\sum_{n=0}^{\infty}\frac{b^{2n+1}-a^{2n+1}}{2n+1}(nA_n^{\ 2}+(n+1)\frac{B_n^{\ 2}}{(ab)^{2n+1}})]. \tag{106}$$

For J to be stationary,

$$\frac{\partial J}{\partial A_n}=0,\qquad \frac{\partial J}{\partial B_n}=0,\qquad n=0,1,2,\cdots \tag{107}$$

From (107) we find that the only non-zero quantities are A_1 and B_1, which satisfy

$$A_1+\frac{B_1}{a^3}-\frac{(\omega_s^{\ 2}-\omega^2)}{\omega^2}\frac{\rho_{IC}}{\rho_o(a)}(\frac{b^3}{a^3}-1)A_1=0, \tag{108.a}$$

$$A_1+\frac{B_1}{a^3}-\frac{(\omega_s^{\ 2}-\omega^2)}{\omega^2}\frac{\rho_{IC}}{\rho_o(a)}(\frac{b^3}{a^3}-1)\frac{2B_1}{b^3}=0, \tag{108.b}$$

A non-trivial solution requires the determinant of the coefficients of A_1, B_1 to vanish,

so, $\omega=\omega_s$, (.09)

$$\text{or } \omega = \frac{\omega_s}{(1+f)^{1/2}}. \tag{110}$$

Root (110) is identical to (34) in section 2, but root (109) was not predicted by simple model analysis. However it is clear from (101) that this root is just a mathematical by-product, since $\omega = \omega_s$ makes $h_1' \to \infty$, which is physically inadmissible. Substituting (110) into (108) we find $A_1 = 2B_1/b^3$ in agreement with (30).

The fact that we have obtained the correct expression for the Slichter eigenfrequency and eigenfunction by invoking the variational principle based on (75), using a trial function which does not satisfy the boundary conditions in advance, is evidence that we can do the same when we use (100).

6. DENSITY PROFILE IN THE LIQUID CORE

6.1. PREM Density Profile in the Liquid Core

For density distribution in the inner core and mantle, we have adopted the PREM (Dziewonski and Anderson, 1981). As written in section 3, the density gradient in the liquid outer core is given by

$$\nabla \rho_o = (1-\beta)\rho_o \mathbf{g}_o/\alpha^2. \tag{111}$$

For a spherically-stratified configuration, this becomes

$$\frac{d\rho_o}{dr} = -(1-\beta)\rho_o g_o/\alpha^2, \tag{112}$$

For a neutrally stratified liquid core, $\beta = 0$, i.e. the Adams-Williamson condition is satisfied. So far we haven't known exactly what β looks like in the liquid core, but many authors have taken constant value of it through the liquid core for convenience (Pekeris and Accad 1972, Crossley 1975, etc.). It has been pointed out that the liquid core density profile might be more complicated than this. A more realistic model might be stable in some region and neutral or unstable in other regions. For example, Kennedy & Higgins (1973) and Gubbins (1982) suggested that it is unstable and convective ($\beta > 0$) near the inner core boundary, and neutral or stable ($\beta \leq 0$) in the rest of the outer core. PREM (1981) is so constructed that it has an almost neutrally-stratified outer core, tending slightly towards stable stratification near the *ICB* and slightly towards unstable stratification near the *CMB*. Table 1 lists the density and β distribution in the liquid core for PREM (1981). Note that the density coefficients here are modified to use a radial variable $u = r/b$.

6.2. Modifying the Density Profile to a Prescribed β

In Section 3, we have assumed the liquid core is neutrally stratified. Consistent with that assumption we wish to modify the PREM density profile so that it is exactly neutral. We will do this as a special case, with the method we now develop, for fitting any prescribed β.

From (112), β is expressed as

$$\beta = 1 + \frac{\alpha^2}{\rho_o g_o} \frac{d\rho_o}{dr}, \tag{113}$$

then β, computed by (113) using our density profile, should be reasonably close to its prescribed value everywhere in the liquid core.

In order to reduce numerical computation error and make our analysis easier, we first scale the density function in the outer core in such a way that average density of outer core is unity, i.e.

$$\frac{4\pi b^3 \int_{a/b}^{1} \rho^* u^2 du}{[4\pi(b^3-a^3)/3]} = 1, \tag{114}$$

that is, the total mass of the outer core in this non-dimensional system is numerically equal to its volume

$$4\pi b^3 \int_{a/b}^{1} \rho^* u^2 du = \frac{4\pi}{3}(b^3-a^3), \tag{115}$$

where $u = r/b$, a and b are inner and outer core radius respectively, ρ^* is dimensionless density after scaling, which satisfies the relation

$$\rho_o = \bar{\rho}\rho^*. \tag{116}$$

Here scaling factor $\bar{\rho}$ is the average density of the outer core before scaling

$$\bar{\rho} = \frac{M_{LC}}{4\pi(b^3-a^3)/3}. \tag{117}$$

Similarly, α and g_o in (113) can also be scaled into dimensionless quantities

$$\alpha = \alpha_o \alpha^*, \tag{118}$$

$$g_o = g_b g^*, \tag{119}$$

where $\alpha_o = \alpha(0)$, the compressional wave speed in outer core for $r = 0$, g_b is proportional to gravity at core-mantle boundary $g_o(b)$. With new variable u,

$$\frac{d\rho_o}{dr} = \frac{1}{b}\frac{d\rho_o}{du}, \tag{120}$$

then (83) can be rewritten

$$\frac{\alpha^{*2}}{\rho^* g^*}\frac{d\rho^*}{du} + u(1-\beta) = 0, \tag{121}$$

where

$$w = \frac{bg_b}{\alpha_o^{\ 2}}. \tag{122}$$

For determining a best-fitting function ρ^*, we use a Galerkin method with weighting functions u^{i-1} to build up following statement

$$\int_{a/b}^{1} u^{i-1}\left[\frac{\alpha^{*2}}{\rho^* g^*}\frac{d\rho^*}{du} + u(1-\beta)\right]du = 0. \qquad \text{for each } i \tag{123}$$

We choose a representation ρ^* as

$$\rho^* = \sum_{j=1}^{N} d_j^* u^{j-1}, \tag{124}$$

then

$$\rho^{*\prime} = \sum_{j=2}^{N} (j-1) d_j^* u^{j-2}, \tag{125}$$

substituting (125) into (123), we form a set of non-linear equations

$$\sum_{j=2}^{N} d_j^* A_{ij} = F_i, \qquad i=1,2,\cdots,N-1 \tag{126}$$

where

$$A_{ij} = (j-1)\int_{a/b}^{1} \frac{a^{*2}}{\rho^* g^*}\, u^{i+j-3} du, \tag{127}$$

$$F_i = -\, w \int_{a/b}^{1} (1-\beta) u^{i-1} du. \tag{128}$$

Applying an iteration procedure, (126) can be solved to obtain coefficients d^*_2, d^*_3, ..., d^*_N. Here we use mass conservation as only constraint to the problem, leaving the total moment of inertia of outer core unconstrained (but this constraint could easily be added). From mass conservation (115), we have

$$\sum_{j=1}^{N} d_j^* \int_{a/b}^{1} u^{i+1} du = \frac{1}{3}(1 - \frac{a^3}{b^3}). \tag{129}$$

Thus d_1^* is found to be

$$d_1^* = 1 - 3\,\gamma \sum_{j=2}^{N} \frac{d_j^*}{j+2}[1 - (\frac{a}{b})^{j+2}], \tag{130}$$

where

$$\gamma = (1 - \frac{a^3}{b^3})^{-1}. \tag{131}$$

Note that in the iteration process for solving the non-linear system (123), we use previous results of d_j^*'s to compute ρ^* and g^*, so A_{ij} can be integrated numerically. In the outer core, gravity at r is

$$g_o(r) = \frac{GM_{IC}}{b^2u^2} + \frac{3GM_{LC}}{b^2u^2}\gamma \sum_{k=1}^{N} \frac{d_k^*}{k+2}[u^{k+2} - (a/b)^{k+2}], \tag{132}$$

we write this as

$$g_o(r) = \frac{3G(M_{IC} + M_{LC})}{b^2} g^* = 3g_o(b)g^*. \qquad (133)$$

where $g_o(b)$ is the gravity at core-mantle boundary. Then scaling factor

$$g_b = 3g_o(b), \qquad (134)$$

and

$$g^* = (1-\xi)\,\gamma\sum_{k=1}^{N} \frac{d_k^*}{k+2} u^k + \frac{1}{u^2}[\frac{\xi}{3} - (1-\xi)\,\gamma\sum_{k=1}^{N} \frac{d_k^*}{k+2} (\frac{a}{b})^{k+2}], \qquad (135)$$

where

$$\xi = \frac{M_{IC}}{M_{IC}+M_{LC}}. \qquad (136)$$

We also adopted $\alpha(r)$ in the outer core from PREM (as written in section 1)

$$\alpha(r) = \sum_{l=1}^{4} c_l(\frac{r}{R})^{l-1}. \qquad (137)$$

By definition (118) we have

$$\alpha^* = \sum_{l=1}^{4} c_l^* u^{l-1}. \qquad (138)$$

where

$$c_l^* = \frac{c_l}{\alpha_o}(\frac{b}{R})^{l-1}. \qquad (139)$$

Starting value of iteration process can be based on PREM's density profile

$$\rho_o = \sum_{j=1}^{4} d_j (\frac{r}{R})^{j-1}, \tag{140}$$

where $d_1 = 12.5818$, $d_2 = -1.2638$, $d_3 = -3.6426$ and $d_4 = -5.5281$. By definition (116)

$$\rho^* = \sum_{j=1}^{N} d_j^* u^{j-1}. \tag{141}$$

where

$$d_j^* = \frac{d_j}{\bar{\rho}} (\frac{b}{R})^{j-1}. \tag{142}$$

For $N = 4$, we can choose starting value as $d_1^* = 1.154193$, $d_2^* = -0.063328$, $d_3^* = -0.0997012$ and $d_4^* = -0.0826488$. But from computational experiments we have learned that for a good β profile (10^{-5} can be considered as a good approximation to prescribed zero value of β in neutrally stratified case), N must be expanded to at least 10. The starting values of d_5^*, d_6^*, ..., d_{10}^* can be chosen to be zero. Iteration is stopped by examining the difference of computed and prescribed β value at chosen points in the liquid core, say $|\beta - \beta_{prescribed}|_i < x$, where x is a pre-chosen error limit. In our case, $\beta_{prescribed} = 0$, we choose $x = 0.5 \times 10^{-5}$. Table 2 contains computed density coefficients d_j^*'s, the density distribution and the β distribution (which is evaluated by using computed density profile) for a neutrally stratified liquid core. For non-neutrally stratified cases, one only needs to prescribe $\beta(r)$ as a function (e.g. linear or non-linear), then corresponding density coefficients will be produced by the same procedure as in neutrally stratified case.

Table 1: Density and ϑ Distribution in the Liquid Core for PREM(1981)

---Density Coefficients---

$d_1 = 12.5815$
$d_2 = -0.6903$
$d_3 = -1.0868$
$d_4 = -0.9009$

---u--- ($=r/b$)	---Density--- ($\times 10^3$ kg/m^3)	---$\vartheta(u)$---
0.35101	12.166332	-0.02822
0.38346	12.106194	-0.01828
0.41591	12.041584	-0.01052
0.44835	11.972318	-0.00484
0.48080	11.898212	-0.00098
0.51325	11.819080	0.00134
0.54570	11.734738	0.00240
0.57815	11.645001	0.00247
0.61060	11.549684	0.00182
0.64305	11.448603	0.00067
0.67550	11.341572	-0.00073
0.70795	11.228408	-0.00218
0.74040	11.108926	-0.00346
0.77285	10.982941	-0.00438
0.80530	10.850267	-0.00473
0.83775	10.710721	-0.00431
0.87020	10.564118	-0.00291
0.90265	10.410273	-0.00035
0.93510	10.249001	0.00361
0.96755	10.080118	0.00917
1.00000	9.903438	0.01655

Table 2: Density and β Distribution for neutrally stratified Liquid Core

---Density Coefficients---

$d_1^* = 1.1547830$
$d_2^* = -0.1180724$
$d_3^* = 0.3291199$
$d_4^* = -1.5822028$
$d_5^* = 3.0599420$
$d_6^* = -4.0944424$
$d_7^* = 3.7293918$
$d_8^* = -2.2307839$
$d_9^* = 0.7866311$
$d_{10}^* = -0.1260219$

---u--- ($=r/b$)	---Density--- ($\times 10^3$ kg/m^3)	--- $\beta(u) \times 10^5$---
0.35101	12.168237	-0.102484
0.38346	12.109420	-0.177723
0.41591	12.045681	0.014823
0.44835	11.976896	-0.031802
0.48080	11.902957	-0.120506
0.51325	11.823764	-0.130272
0.54570	11.739219	-0.062772
0.57815	11.649221	0.001734
0.61060	11.553662	0.025610
0.64305	11.452420	-0.001624
0.67550	11.345361	-0.030754
0.70795	11.232331	-0.047418
0.74040	11.113152	-0.020070
0.77285	10.987622	0.017656
0.80530	10.855506	0.048758
0.83775	10.716534	0.028150
0.87020	10.570393	-0.006532
0.90265	10.416724	-0.042587
0.93510	10.255109	-0.019283
0.96755	10.085067	0.031823
1.00000	9.906041	-0.102484

7. SLICHTER MODE EIGENFREQUENCIES: NUMERICAL RESULTS

When using the variational principle based on (100), we have F as a functional of χ

$$F = F(\chi). \tag{143}$$

Upon choosing a trial function χ, we can find eigenfrequency for the Slichter oscillation by requiring functional F to take an extremal value. Initially we chose a trial function as

$$\chi^m = \sum_{l=1}^{L} \sum_{n=1}^{N} \frac{b}{r} [\cos \frac{l\pi(r-a)}{b-a} + i\sin \frac{l\pi(r-a)}{b-a}] P_n^m(\cos\theta) e^{im\phi}, \tag{144}$$

but some extra (false) roots were obtained besides the true one. The situation became worse when we increased N, L. Thus it is apparently important to choose a trial function which has a form more directly suggested by the analytic solution to the simple case (non-rotating, homogeneous incompressible liquid outer core, rigid inner core), as shown by expression (25).

Based on this idea, we now choose the trial function to have a form

$$\chi(r,\theta,\phi) = \sum_{j=1}^{N-1} \sum_{l=1}^{L} d_{j+(2N-2)(l-1)} r^{j-(N+1)} P_l^m(\cos\theta) e^{im\phi}$$

$$+ \sum_{j=N}^{2N-2} \sum_{l=1}^{L} d_{j+(2N-2)(l-1)} r^{j-(N-1)} P_l^m(\cos\theta) e^{im\phi}, \tag{145}$$

or

$$\chi(r,\theta,\phi) = \sum_{k=1}^{(2N-2)L} d_k \Psi_k^m e^{im\phi}, \tag{146}$$

where

$$\Psi_k^m = r^{\upsilon} P_l^m, \quad \begin{cases} \upsilon = j-(N+1) & for\ 1 \le j \le N-1, \\ \upsilon = j-(N-1) & for\ N \le j \le 2N-2. \end{cases} \tag{147}$$
$$k = j+(2N-2)(l-1)$$

N and L are pre-assigned integers specifying the truncation level for the powers of radius and degree of Legendre function respectively. Then the functional has the form

$$F = \sum_{k=1}^{(2N-2)L} \sum_{j=1}^{(2N-2)L} d_k d_j H_{kj}. \tag{148}$$

For F to be extremal, we require that

$$\frac{\partial F}{\partial d_p} = 0, \qquad p = 1,2,\cdots,(2N-2)L \tag{149}$$

that is

$$\sum_{j=1}^{(2N-2)L} d_j H_{pj} = 0, \qquad p=1,2,\cdots,(2N-2)L \tag{150}$$

or

$$\underset{\sim}{H}\, \underline{D} = \underline{0}. \tag{151}$$

For non-trivial $\underline{D}$,

$$Det(\underset{\sim}{H}) = 0. \tag{152}$$

Frequencies which satisfy equation (152) will be the eigenfrequencies of the Slichter mode. Table 3 contains results for the various cases discussed here, using (i) PREM density function, and (ii) density function calculated in the present study to ensure $\beta = 0$.

8. DISCUSSION

The Earth model we have considered in this study is simplified, but has most of the essential aspects to satisfy our requirement for numerical investigation, hence the results presented here can be regarded as a complement to or as extension of previous research on the Slichter mode, and in particular a good demonstration of the utility of the subseismic approximation in a long-period calculation. Our achievements can be summarised as follows:

(i) One of the principal achievements of this study is to demonstrate the utility of the variational principle in long-period free oscillation calculations based on the subseismic approximation. The main advantage of adopting a VP is reduction of computational effort, especially since integrals involving second derivatives of χ can be eliminated from the functional by using the 'natural' boundary conditions (26) and (87). For a neutrally stratified, compressible liquid core, the remaining integrals can be evaluated analytically. Even for a non-neutrally stratified liquid core, only one integral needs to be solved by numerical integration. Of course in order to ensure that F provides a variational principle, the linear operator $\mathcal{L}_p$ in our partial differential equation must be Hermitean, and the boundary conditions (26) and (87) ensure this.

(ii) A second major achievement of this study is that the effects of higher har-

Table 3: Eigenperiod of Slichter Mode (in hr)

I. Using density function of the liquid core from PREM

N,L	non-rotating	rotating $m=+1$	rotating $m=0$	rotating $m=-1$

(1) Rigid inner core, homogeneous incompressible liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	4.938	4.384	4.854	5.422
$N=3,L=4$	4.938	4.386	4.857	5.424

(2) Rigid inner core, neutrally stratified liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	4.968	4.412	4.880	5.448
$N=3,L=4$	4.968	4.415	4.884	5.450

(3) Elastic inner core, neutrally stratified liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	5.411	4.755	5.296	5.966
$N=3,L=3$	5.412	4.759	5.301	5.969
$N=4,L=4$	5.412	4.759	5.301	5.969
$N=5,L=5$	5.412	4.759	5.301	5.969
$N=6,L=6$	5.412	4.759	5.301	5.969

II. Using density function for a strictly neutral liquid core

N,L	non-rotating	rotating $m=+1$	rotating $m=0$	rotating $m=-1$

(1) Rigid inner core, homogeneous incompressible liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	4.945	4.389	4.860	5.430
$N=3,L=4$	4.945	4.391	4.863	5.432

(2) Rigid inner core, neutrally stratified liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	4.974	4.417	4.886	5.455
$N=3,L=4$	4.974	4.419	4.889	5.475

(3) Elastic inner core, neutrally stratified liquid core

N,L	non-rotating	$m=+1$	$m=0$	$m=-1$
$N=2,L=2$	5.420	4.761	5.304	5.975
$N=3,L=3$	5.420	4.765	5.309	5.979
$N=4,L=4$	5.420	4.765	5.309	5.979
$N=5,L=5$	5.420	4.765	5.309	5.979
$N=6,L=6$	5.420	4.765	5.309	5.979

monics in the displacement field of the liquid core are readily taken into account. In our numerical calculation, we have effectively extended the coupling chain beyond Crossley (1975) and Smith (1976), and have clearly obtained convergence to 0.02% by going to $L = 3$. By choosing simple models of inner core and mantle, we have been able to focus our attention and computing effort on the representation of the field χ in the liquid core.

(iii) It is significant that carrying terms in the trial function beyond $L = 3$ yields no change in the eigenperiod (to within 0.02%). This would appear to confirm a result of which Dahlen and Sailor (1979) were uncertain, namely that second-order perturbation theory is accurate enough to give the effects of rotation on the eigenperiod of the Slichter mode (which these authors describe as ${}_1S_1$). This follows from the fact that $L = 3$ represents taking the second stage from $L = 1$ in the Coriolis coupling chain.

(iv) For the first time we have estimated the numerical error committed in adopting the subseismic approximation for a particular long-period mode. A better (a posteriori) estimate of error would use the computed eigenfunctions for χ (and from them the appropriate V_1 and $\mathbf{u} \cdot \mathbf{g}_0$), but we reserve this for later study. Meanwhile the computational advantages of the subseismic approximation impel us to go ahead with calculations based on it.

(v) This study shows that choosing a proper trial function is quite critical. As we discussed in Section 7, one should be guided, in choosing a trial function, by the form of solution found for a simplified model. If a very different trial function is used, its own unnecessary complexities may (as in our experience) introduce destructive inputs.

(vi) It was also demonstrated that it is easy to separate the effects on the eigenperiod of rotation, of elasticity of the inner core and of compressibility of the outer core. In programming, this is controlled by appropriate parameters. The effect on the Slichter eigenperiod of liquid core compressibility is much smaller (0.6%) than that of elasticity (9%) or rotation (splitting by 12%). Our results reproduce one noted by Smith (1976)

$$\omega(m=+1) + \omega(m=-1) < 2\omega(m=0).$$

The 5-hr-eigenperiod Slichter mode found in the study appears to be the only undertone which exists in a strictly neutrally-stratified liquid core. Higher undertones can be obtained by extending the search range in conjunction with a properly modified stability parameter β. By 'properly' here we mean that it will ensure the (negative) value of β does not make the integrand, in one of the terms of the functional, singular at any point in the liquid core (otherwise the variational principle must be replaced by a Galerkin procedure applied to the subseismic wave equation).

As mentioned earlier, our model is still far from reality in representing the Earth. For further understanding of the translational inner core oscillations, we should try to include more natural properties of the Earth, roughly in order of probable importance: mantle elasticity, non-neutral stratification of the liquid core, higher-degree displacement fields in the solid inner core due to Coriolis coupling , and ellipticity of the Earth strata. It would also be interesting to compare mappings of eigenfunctions obtained by applying the subseismic approximation, with those of Crossley (1975) and Smith (1976). Finally, the subseismic approximation itself could be replaced by the exact two-potential description of outer core dynamics suggested by Wu and Rochester (1990).

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Appendix A

LOAD LOVE NUMBER $h_1^{m\prime}$ OF THE INNER CORE

The dynamics of a spherical, homogeneous, rotating, elastic inner core is governed by equations (74)-(77), and boundary conditions (83), (84) and (85). The procedure we follow is an extension of that outlined by Lapwood & Usami (1981, pp. 87 - 88, 219 - 220). First we do transforms

$$\{\frac{d}{dr}[r(75)] - (74)\}/r = 0,$$

$$[\frac{d}{dr} + \frac{2}{r}](74) - \frac{2}{r}(75) = 0,$$

we get

$$\beta^2 \mathcal{L}_1 H + \omega(\omega - m\omega\Omega)H = (\Gamma + m\omega\Omega)\Delta + \frac{3}{5}\omega\Omega(m+2)(\frac{d}{dr} + \frac{3}{r})y_7, \tag{a.1}$$

$$\alpha^2 \mathcal{L}_1 \Delta + (\omega^2 + 4\Gamma)\Delta = 2(\Gamma + m\omega\Omega)H - \frac{6}{5}\omega\Omega(m+2)(\frac{d}{dr} + \frac{3}{r})y_7, \tag{a.2}$$

Equation (76) and (77) can be written as

$$\mathcal{L}_2 y_7 + s^2 y_7 = t^2(y_1 - y_3), \tag{a.3}$$

$$\mathcal{L}_1 y_5 = 3\Gamma\Delta, \tag{a.4}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear operators $\mathcal{L}_n$ for $n = 1, 2$

$$\mathcal{L}_n = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{n(n+1)}{r^2},$$

and

$$s^2 = \frac{\omega(\omega - m\Omega/3)}{\beta^2},$$

$$t^2 = \frac{\omega\Omega(m-2)}{3\beta^2},$$

Eliminating H from (a.1) and (a.2), we have a single fourth-order ordinary differential equation in Δ:

$$[\mathcal{L}_1 + \frac{\omega(\omega - m\Omega)}{\beta^2}][\mathcal{L}_1 + \frac{\omega^2 + 4\Gamma}{\alpha^2}]\Delta - \frac{2(\Gamma + m\omega\Omega)^2}{\alpha^2\beta^2}\Delta$$
$$+ \frac{6\omega\Omega(m+2)}{5\alpha^2}[\mathcal{L}_1 + \frac{\omega(\omega - 3m\Omega) - 2\Gamma}{\beta^2}](\frac{d}{dr} + \frac{3}{r})y_7 = 0, \quad (a.5)$$

which suggests a combination of spherical Bessel functions as a solution. By reference to Pekeris & Jarosch's (1958) analytical solutions for a uniform non-rotating sphere, we assume here

$$\Delta = \sum_{i=1}^{2} A_i j_1(p_i r) + B j_1(sr) \quad (a.6.1)$$

$$H = \sum_{i=1}^{2} L_i j_1(p_i r) + M j_1(sr) \quad (a.6.2)$$

$$y_7 = \sum_{i=1}^{2} Q_i j_2(p_i r) + R j_2(sr) \quad (a.6.3)$$

$$y_1 - y_3 = \sum_{i=1}^{2} K_i j_2(p_i r) \tag{a.6.4}$$

Substituting (a.6) into (a.1) and using the recurrence relations

$$\frac{dj_1(p_i r)}{dr} = \frac{1}{r} j_1(p_i r) - p_i j_2(p_i r)$$

$$\frac{dj_2(p_i r)}{dr} = -\frac{3}{r} j_2(p_i r) + p_i j_1(p_i r)$$

we have

$$[\omega(\omega - m\Omega) - \beta^2 p_i^2] L_i = (\Gamma + m\omega\Omega) A_i + \frac{3}{5}\omega\Omega(m+2) p_i Q_i, \tag{a.7}$$

$$[\omega(\omega - m\Omega) - \beta^2 s^2] M = (\Gamma + m\omega\Omega) B + \frac{3}{5}\omega\Omega(m+2) s R \tag{a.8}$$

Similarly, from (a.2) we have

$$[\omega^2 + 4\Gamma - \alpha^2 p_i^2] A_i = 2(\Gamma + m\omega\Omega) L_i - \frac{6}{5}\omega\Omega(m+2) p_i Q_i, \tag{a.9}$$

$$[\omega^2 + 4\Gamma - \alpha^2 s^2] B = 2(\Gamma + m\omega\Omega) M - \frac{6}{5}\omega\Omega(m+2) s R \tag{a.10}$$

and from (a.3)

$$(s^2 - p_i^2) Q_i = t^2 K_i, \tag{a.11}$$

i.e.

$$Q_i = \frac{t^2 K_i}{s^2 - p_i^2}. \tag{a.12}$$

From the definition of H, we can write

$$\frac{d}{dr}(ry_3) = y_1 + rH. \tag{a.13}$$

Differentiate (a.13) and use (a.6),

$$L_1(ry_3) = \sum_{i=1}^{2} (A_i + 4L_i)\, j_1(p_i r) - r\sum_{i=1}^{2} p_i L_i\, j_2(p_i r)$$

$$+ (B + 4M) j_1(sr) - rsMj_2(sr). \tag{a.14}$$

By observation we know that the solution of (a.14) has the following form

$$ry_3 = f_1 j_1(p_i r) + f_2 r j_2(p_i r) + f_3 j_1(sr) + f_4 r j_2(sr). \tag{a.15.1}$$

Then

$$L_1(ry_3) = f_1 L_1 j_1(p_i r) + f_2 L_1[r j_2(p_i r)]$$

$$+ f_3 L_1 j_1(sr) + f_4 L_1[r j_2(sr)], \tag{a.15.2}$$

where

$$L_1 j_1(p_i r) = -p^2_i j_1(p_i r)$$
$$L_1 j_1(sr) = -s^2 j_1(sr)$$
$$L_1[r j_2(p_i r)] = 2p_i j_1(p_i r) - p_i^2 r j_2(p_i r)$$
$$L_1[r j_2(sr)] = 2s j_1(sr) - s^2 r j_2(sr)$$

By substituting (a.15) into (a.14), the f_i's are found to be

$$f_1 = -\sum_{i=1}^{2} \frac{A_i + 2L_i}{p_i^2} \tag{a.16.1}$$

$$f_2 = \sum_{i=1}^{2} \frac{L_i}{p_i} \tag{a.16.2}$$

$$f_3 = -\frac{B+2M}{s^2} \tag{a.16.3}$$

$$f_4 = \frac{M}{s} \tag{a.16.4}$$

For homogeneous equation $L_1(ry_3)=0$, there is a general solution like

$$ry_3 = Er + \frac{F}{r^2}$$

so we finally have a solution of y_3

$$y_3 = E + \frac{F}{r^3} - \frac{1}{r}\sum_{i=1}^{2}\frac{A_i+2L_i}{p_i^2}j_1(p_ir) + \sum_{i=1}^{2}\frac{L_i}{p_i}j_2(p_ir) - \frac{B+2M}{rs^2}j_1(sr) + \frac{M}{s}j_2(sr) \tag{a.17}$$

Then y_1 can be obtained from (a.13)

$$y_1 = \frac{d}{dr}(ry) - rH$$

$$= E - \frac{2F}{r^3} - \frac{1}{r}\sum_{i=1}^{2}\frac{A_i+2L_i}{p_i^2}j_1(p_ir) + \sum_{i=1}^{2}\frac{A_i}{p_i}j_2(p_ir) - \frac{B+2M}{rs^2}j_1(sr) + \frac{B}{s}j_2(sr) \tag{a.18}$$

Substituting y_1 and y_3 into (a.6.4), we have

$$-\frac{3F}{r^3} + \sum_{i=1}^{2}\frac{A_i-L_i}{p_i}j_2(p_ir) + \frac{B-M}{s}j_2(sr) = \sum_{i=1}^{2}K_ij_2(p_ir) \tag{a.19}$$

So we conclude that the assumption of (a.6.4) is verified, with

$$K_i = \frac{A_i - L_i}{p_i}, \tag{a.20.1}$$

$$B - M = 0, \tag{a.20.2}$$

$$F = 0. \tag{a.20.3}$$

By putting (a.20) into (a.12), the latter yields

$$Q_i = \frac{t^2}{p_i} \frac{(A_i - L_i)}{(s^2 - p_i^2)}. \tag{a.21}$$

Then from (a.7) and (a.?), we obtain two expressions for L_i/A_i

$$\frac{L_i}{A_i} = \frac{\Gamma + m\omega\Omega + \dfrac{(\omega\Omega)^2(m^2-4)}{5\beta^2(s^2-p_i^2)}}{\omega(\omega-\omega\Omega) - \beta^2 p_i^2 + \dfrac{(\omega\Omega)^2(m^2-4)}{5\beta^2(s^2-p_i^2)}}, \tag{a.22}$$

$$\frac{L_i}{A_i} = \frac{\omega^2 + 4\Gamma - \alpha^2 p_i^2 + \dfrac{2(\omega\Omega)^2(m^2-4)}{5\beta^2(s^2-p_i^2)}}{2[\Gamma + m\omega\Omega + \dfrac{(\omega\Omega)^2(m^2-4)}{5\beta^2(s^2-p_i^2)}]}. \tag{a.23}$$

These can be greatly simplified by examining

$$\left| \frac{(\omega\Omega)^2(m^2-4)}{5\beta^2(s^2-p_i^2)} \right| \le \frac{4}{5} \frac{(\omega\Omega)^2}{\beta^2} \frac{1}{|s^2-p_i^2|}. \tag{a.24}$$

If we take $\omega \approx 3.5 \times 10^{-4}$ s^{-1}, (i.e. $T = 5$ hr), $\Omega = 7.3 \times 10^{-5}$ s^{-1}, $\beta = 3.5 \times 10^3$ m s^{-1}, $\alpha \approx 10.0 \times 10^3$ m s^{-1}, then $(\frac{\omega\Omega}{\beta})^2 \approx 5.3 \times 10^{-23}$ m^{-2}. p_i^2 has a size of 2×10^{-13} m^{-2}, as suggested in Pekeris & Jarosch's non-rotating solid sphere dynamics

(Pekeris & Jarosch, 1958), then $|s^2-p_i^2|$ is of order 2×10^{-13} m^{-2}, and RHS of (a.21) is of order 2×10^{-10} s^{-2}, whereas $\Gamma+m\omega\Omega$ and $\omega^2+4\Gamma-\alpha^2p_i^2$ are of order 10^{-5} s^{-2}. Thus (a.22) and (a.23) can be approximated as

$$\frac{L_i}{A_i}=\frac{\Gamma+m\omega\Omega}{\omega(\omega-m\Omega)-\beta^2p_i^2} \tag{a.25}$$

$$\frac{L_i}{A_i}=\frac{\omega^2+4\Gamma-\alpha^2p_i^2}{2(\Gamma+m\omega\Omega)} \tag{a.26}$$

These give a quadratic equation in p_i^2

$$p_i^4-[\frac{\omega^2-m\omega\Omega}{\beta^2}+\frac{\omega^2+4\Gamma}{\alpha^2}]p_i^2$$

$$+\frac{1}{\alpha^2\beta^2}[(\omega^2+4\Gamma)(\omega^2-m\omega\Omega)-2(\Gamma+m\omega\Omega)^2]=0. \tag{a.27}$$

with roots

$$\begin{Bmatrix}p_1^2\\p_2^2\end{Bmatrix}=\frac{1}{2}\{\frac{\omega^2-m\omega\Omega}{\beta^2}+\frac{\omega^2+4\Gamma}{\alpha^2}\pm[(\frac{\omega^2-m\omega\Omega}{\beta^2}+\frac{\omega^2+4\Gamma}{\alpha^2})^2$$

$$-\frac{4(\omega^2+4\Gamma)(\omega^2-m\omega\Omega)-8(\Gamma+m\omega\Omega)^2}{\alpha^2\beta^2}]^{1/2}\} \tag{a.28}$$

Placing $B=M$ into (a.8) and (a.10), two more related expressions are obtained

$$\frac{6}{5}\omega\Omega(m+2)sR=[-\omega^2-4\Gamma+\alpha^2s^2+2(\Gamma+m\omega\Omega)]M \tag{a.29}$$

$$\frac{6}{5}\omega\Omega(m+2)sR=2[\omega^2-m\omega\Omega-\beta^2s^2-(\Gamma+m\omega\Omega)]M \tag{a.30}$$

i.e.

$$(3\omega^2-\alpha^2s^2-2\beta^2s^2-4m\omega\Omega+\Gamma)M=0. \tag{a.31}$$

This will hold only if $M = 0$, then $B = 0$, $R = 0$ follow, and only two terms are left in Δ, H and y_7 of (a.6).

Equation (a.4) indicates a solution with form of

$$y_5 = Nj_1(p_i r). \tag{a.32}$$

Substituting y_5 into (a.4), N is found as

$$N = -3r\sum_{i=1}^{2}\frac{A_i}{p_i^2}.$$

Adding the general solution of homogeneous equation $L_1 y_5 = 0$, i.e. $y_5 = Gr + \frac{J}{r^2}$, gives

$$y_5 = Gr + \frac{J}{r^2} - 3r\sum_{i=1}^{2}\frac{A_i}{p_i^2}j_1(p_i r). \tag{a.33}$$

Here we are seeking solutions of y_i's which must be regular at $r = 0$ in a solid sphere, this requirement sets $J = 0$ ($F = 0$ already). Hence

$$y_1 = E - \frac{1}{r}\sum_{i=1}^{2}\frac{A_i+2L_i}{p_i^2}j_1(p_i r) + \sum_{i=1}^{2}\frac{A_i}{p_i}j_2(p_i r), \tag{a.34}$$

$$y_3 = E - \frac{1}{r}\sum_{i=1}^{2}\frac{A_i+2L_i}{p_i^2}j_1(p_i r) + \sum_{i=1}^{2}\frac{L_i}{p_i}j_2(p_i r), \tag{a.35}$$

$$y_5 = Gr - 3r\sum_{i=1}^{2}\frac{A_i}{p_i^2}j_1(p_i r), \tag{a.36}$$

$$y_7 = t^2\sum_{i=1}^{2}\frac{A_i+L_i}{p_i(s^2-p_i^2)}j_2(p_i r). \tag{a.37}$$

Substituting Δ, H, y_1, y_3, y_5 and y_7 into equation (74), we obtain

$$G = (\Gamma - \omega^2 + 2m\omega\Omega)E,$$

so

$$y_5 = (\Gamma - \omega^2 + 2m\omega\Omega)rE - 3\Gamma \sum_{i=1}^{2} \frac{A_i}{p_i^2} j_1(p_i r). \tag{a.38}$$

So finally we are left with constants A_1, A_2 and E to be determined in terms of $\chi_1^m(a)$ by the boundary conditions (83), (84) and (85) (we have dropped an additional boundary condition $y_8 = 0$ here since we ignored the Coriolis induced-toroidal field in the inner core). As defined in (73) and (70),

$$y_2 = \lambda\Delta + 2\mu\frac{dy_1}{dr}$$

$$=(\alpha^2-2\beta^2)\rho_{IC}\Delta + 2\beta^2\rho_{IC}\frac{dy_1}{dr}$$

$$=\alpha^2\rho_{IC}[\sum_{i=1}^{2} A_i j_1(p_i r) - \frac{4}{r}\frac{\beta^2}{\alpha^2}\sum_{i=1}^{2}\frac{A_i-L_i}{p_i}j_2(p_i r)] \tag{a.39}$$

$$y_4 = \mu\,(\frac{dy_3}{dr} - \frac{y_3-y_1}{r})$$

$$=\mu\,[H + \frac{2}{r}(y_1-y_3)]$$

$$=\beta^2\rho_{IC}[\sum_{i=1}^{2} L_i j_1(p_i r) + \frac{2}{r}\sum_{i=1}^{2}\frac{A_i-L_i}{p_i}j_2(p_i r)] \tag{a.40}$$

$$y_6 = \frac{dy_5}{dr} - 3\Gamma y_1$$

$$= G - 3\Gamma E + \frac{6\Gamma}{r}\sum_{i=1}^{2}\frac{L_i}{p_i^2}j_1(p_i r) \qquad (a.41)$$

Since $\chi_1^m(a)$ is arbitrary, we set $\chi_1^m(a) = -\Gamma a^2$ to obtain $y_1(a) = h_1^{m\prime} a$. The boundary condition (84) becomes

$$\sum_{i=1}^{2} A_i\{\frac{1}{p_i a}j_2(p_i a) + \frac{2}{p_i^2 a^2}(1 - \frac{L_i}{A_i})j_1(p_i a)$$

$$- \frac{\alpha^2}{\Gamma a^2}\frac{\rho_{IC}}{\rho_o(a)}[j_1(p_i a) - \frac{4\vartheta^2}{\alpha^2}\frac{1}{p_i a}(1 - \frac{L_i}{A_i})j_2(p_i a)]\}$$

$$+ \frac{\omega(\omega-2m\Omega)E}{\Gamma}\frac{E}{a} = -1. \qquad (a.42)$$

Whereas boundary condition (85) is

$$\sum_{i=1}^{2} A_i\{\frac{2}{p_i^2 a^2}(1 - \frac{L_i}{A_i})j_1(p_i a) - \frac{\rho_o(a)}{\rho_{IC}}[\frac{1}{p_i a}j_2(p_i a) - \frac{1}{p_i^2 a^2}(1 + \frac{2L_i}{A_i})j_1(p_i a)]\}$$

$$+ [\frac{\omega(\omega-2m\Omega)}{\Gamma} - \frac{\rho_o(a)}{\rho_{IC}}]\frac{E}{a} = 0. \qquad (a.43)$$

Using (86), we can get an expression of $h_1^{m\prime}$ from definition (a.34)

$$h_1^{m\prime} = \frac{E}{a} + \sum_{i=1}^{2} A_i[\frac{1}{p_i a}j_2(p_i a) - \frac{1}{p_i^2 a^2}(1 + \frac{2L_i}{A_i})j_1(p_i a)] \qquad (a.44)$$

Thus (a.43) will be reduced to

$$2\sum_{i=1}^{2}\frac{A_i}{p_i^2a^2}(1-\frac{L_i}{A_i})j_1(p_ia)-\frac{\omega(\omega-2m\Omega)}{\Gamma}\frac{E}{a}=\frac{\rho_o(a)}{\rho_{IC}}h_1^{m\,\prime}. \qquad (a.45)$$

(a.45) can be changed by substituting $\frac{E}{a}$ from (a.44) into

$$[1-\frac{\Gamma\rho_o(a)}{\omega(\omega-2m\Omega)\rho_{IC}}]h_1^{m\,\prime}=\sum_{i=1}^{2}A_i\{\frac{1}{p_ia}j_2(p_ia)$$

$$-\frac{1}{p_i^2a^2}[1+\frac{2\Gamma}{\omega(\omega-2m\Omega)}+2(1-\frac{\Gamma}{\omega(\omega-2m\Omega)})\frac{L_i}{A_i}]j_1(p_ia)\}. \qquad (a.46)$$

Form equation (a.25) and (a.26) we will have

$$1+\frac{2\Gamma}{\omega(\omega-2m\Omega)}+2(1-\frac{\Gamma}{\omega(\omega-2m\Omega)})\frac{L_i}{A_i}=\frac{\alpha^2p_i^2}{\omega(\omega-2m\Omega)}(1+\frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i}), \qquad (a.47)$$

and this relation will further change (a.46) into

$$[1-\frac{\Gamma\rho_o(a)}{\omega(\omega-2m\Omega)\rho_{IC}}]h_1^{m\,\prime}=\sum_{i=1}^{2}A_i[\frac{1}{p_ia}j_2(p_ia)$$

$$-\frac{1}{a^2}\frac{\alpha^2}{\omega(\omega-2m\Omega)}(1+\frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i})j_1(p_ia)]. \qquad (a.48)$$

Now using (a.45) to reduce (a.42) as

$$\sum_{i=1}^{2}A_i\{\frac{1}{p_ia}j_2(p_ia)-\frac{\alpha^2}{\Gamma a^2}\frac{\rho_{IC}}{\rho_o(a)}[j_1(p_ia)$$

$$-\frac{4\beta^2}{\alpha^2}\frac{1}{p_ia}(1-\frac{L_i}{A_i})j_2(p_ia)]\}+1=-\frac{\rho_o(a)}{\rho_{IC}}h_1^{m\,\prime}. \qquad (a.49)$$

and using boundary condition $y_4(a)=0$ to further reduce (a.49) as

$$\sum_{i=1}^{2} A_i\{\frac{1}{p_i a} j_2(p_i a) - \frac{\alpha^2}{\Gamma a^2}\frac{\rho_{IC}}{\rho_o(a)}[1 + \frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i}] j_1(p_i a)\}$$

$$+ 1 = -\frac{\rho_o(a)}{\rho_{IC}} h_1^{m\,\prime}. \qquad (a.50)$$

Boundary condition (83) and equation (a.48) and (a.50) are basic equations from which $h_1^{m\,\prime}$ will be obtained. Those equations can be rewritten into the form

$$\sum_{i=1}^{2} A_i j_1(p_i a)\{\frac{L_i}{A_i} + 2(1 - \frac{L_i}{A_i})\frac{j_2(p_i a)}{p_i a j_1(p_i a)}\} = 0, \qquad (a.51)$$

$$\sum_{i=1}^{2} A_i j_1(p_i a)\{\frac{j_2(p_i a)}{p_i a j_1(p_i a)} - \frac{\alpha^2}{a^2\omega(\omega - 2m\Omega)}[1 + \frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i}]\}$$

$$= [1 - \frac{\Gamma\rho_o(a)}{\omega(\omega-2m\Omega)\rho_{IC}}] h_1^{m\,\prime} \qquad (a.52)$$

$$\sum_{i=1}^{2} A_i j_1(p_i a)\{\frac{j_2(p_i a)}{p_i a j_1(p_i a)} - \frac{\alpha^2}{\Gamma a^2}\frac{\rho_{IC}}{\rho_o(a)}[1 + \frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i}]\}$$

$$= -[1 + \frac{\rho_o(a)}{\rho_{IC}} h_1^{m\,\prime}]. \qquad (a.53)$$

By expanding $j_2(p_i a)/p_i a j_1(p_i a)$ in power series, we can make following approximation

$$\frac{j_2(p_i a)}{p_i a j_1(p_i a)} \simeq \frac{1}{5}(1 + x_i), \qquad (a.54)$$

where

$$x_i = \frac{(p_i a)^2}{35} \simeq 0.014,$$

and other higher order quantities in (a.54) were omitted.

Defining $A_i j_1(p_i a) = c_i$, then from (a.51) we get

$$c_2 = -\frac{[1 + \frac{3}{2}\frac{L_1}{A_1} + x_1(1 - \frac{L_1}{A_1})]}{[1 + \frac{3}{2}\frac{L_2}{A_2} + x_2(1 - \frac{L_2}{A_2})]} c_1. \tag{a.55}$$

Substituting (a.54) and (a.55) into (a.52)

$$\begin{aligned} &c_1[B_L(1)X_L(2) - B_L(2)X_L(1)] \\ &\quad = \frac{\rho_o(a)}{\rho_{IC}} \frac{\Gamma a^2}{\alpha^2}[1 - \frac{\rho_{IC}\omega(\omega-2m\Omega)}{\Gamma\rho_o(a)}]h_1^{m\prime} X_L(2). \end{aligned} \tag{a.56}$$

Similarly (a.53) turns to be

$$\begin{aligned} &c_1[E_L(1)X_L(2) - E_L(2)X_L(1)] \\ &\quad = \frac{\rho_o(a)}{\rho_{IC}} \frac{\Gamma a^2}{\alpha^2}[1 + \frac{\rho_o(a)}{\rho_{IC}} h_1^{m\prime}]X_L(2), \end{aligned} \tag{a.57}$$

where

$$B_L(i) = 1 + \frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i} - Y(1+x_i),$$

$$E_L(i) = 1 + \frac{2\beta^2}{\alpha^2}\frac{L_i}{A_i} - Z(1+x_i),$$

$$X_L(i) = 1 + \frac{3}{2}\frac{L_i}{A_i} + x_i(1 - \frac{L_i}{A_i}),$$

$$Y = \omega(\omega-2m\Omega)\frac{a^2}{5\alpha^2}.$$

$$Z = \frac{a^2}{5\alpha^2}\frac{\Gamma\rho_o(a)}{\rho_{IC}},$$

Divide (a.57) by (a.56)

$$\left(\frac{1}{h_1^{m\,\prime}} + \frac{\rho_o(a)}{\rho_{IC}}\right)(1 - U)^{-1} = 1 + \frac{(Y - Z)\,W}{B_L(1)X_L(2) - B_L(2)X_L(1)}, \qquad (a.58)$$

where

$$U = \frac{\rho_{IC}\omega(\omega - 2m\Omega)}{\Gamma\rho_o(a)},$$

$$W = (1 + x_1)\left(\frac{3}{2} - x_2\right)\frac{L_2}{A_2} - (1 + x_2)\left(\frac{3}{2} - x_1\right)\frac{L_1}{A_1}.$$

Now we make another approximation $x_i \simeq 0$, (a.58) becomes

$$\frac{1}{h_1^{m\,\prime}} - \left[1 - \frac{\rho_o(a)}{\rho_{IC}} - \frac{\rho_{IC}\omega(\omega - 2m\Omega)}{\Gamma\rho_o(a)}\right]$$

$$\simeq \left[1 - \frac{\rho_{IC}\omega(\omega - 2m\Omega)}{\Gamma\rho_o(a)}\right]\left[1 - \frac{\Gamma\rho_o(a)}{\rho_{IC}\omega(\omega - 2m\Omega)}\right]\frac{Y}{1 - \frac{4\beta^2}{3\alpha^2} - Y}. \qquad (a.59)$$

By quantities of ω, Ω and α adopted earlier, $Y \simeq 1/5000$, but $\frac{4\beta^2}{3\alpha^2} \simeq \frac{1}{6}$, which allow Y to be omitted in the denominator of the RHS of (a.59). Finally we obtain the load Love number $h_1^{m\,\prime}$

$$\frac{1}{h_1^{m\,\prime}} = 1 - \frac{\rho_o(a)}{\rho_{IC}} - \frac{\rho_{IC}\omega(\omega - 2m\Omega)}{\Gamma\rho_o(a)}(1 + e). \qquad (a.60)$$

where the elastic correction factor is

$$e \simeq \frac{\left[\frac{\Gamma a \rho_o(a)}{\omega \alpha \rho_{IC}}\left(1 - \frac{\omega^2 \rho_{IC}}{\Gamma \rho_o(a)}\left(1 - \frac{2m\Omega}{\omega}\right)\right)\right]^2}{5\left(1 - \frac{2m\Omega}{\omega}\right)\left(1 - \frac{4\beta^2}{3\alpha^2}\right)} \tag{a.61}$$

Appendix B

PROGRAMME CODE FOR EIGENPERIOD COMPUTATION

* The following routines are written in FORTRAN-77, and executed on VAX/VMS at Computing Service of Memorial University of Newfoundland. Executing time ranges from less than 1 second to 5 minutes, depending on the mode computed and on the degree of truncation.

(I) Main programme EIGENVALUE: This programme is used to search the eigenperiod of the Slichter mode by computing the value of the determinant det(H) at every minute in the range of p_b - p_e (using subroutine VALUE). Once the value of det(H) changes sign, it calls a subroutine FINDROOT to locate the eigenperiod more accurately

The input consists of end-points of search range (p_b-p_e)

1. m:=0

2. for p_i:=p_b,p_e,step 1

2.1 if m=0 then

2.1.1 p_o:=p_i,v_o=0

2.1.2 call VALUE(p_o,vv)

```
        2.1.3 v_o:=vv,m:=1
        else
        2.1.4 v_1:=0,p_1:=p_i
        2.1.5 call VALUE(p_1,vv)
        2.1.6 v_i=vv
        2.1.7 if v_1*v_o < 0  then
              2.1.7.1 x_o:=p_o,x_1:=p_1,f_o:=v_o,f_1:=v
              2.1.7.2 call FINDROOT(x_o,x_1,f_o,f_1)
              else
              2.1.7.3 p_o:=p_1,v_o:=v_1
    2.2 continue
3. halt
```

(II) Subroutine FINDROOT(p_o, p_1, v_o, v_1): This programme is used to locate the eigenperiod exactly

The input consists of two points p_o and p_1, for which $v_o(p_o)^*v_1(p_1) < 0$. Parameter E is used to stopping calculation

```
1. y_1:=p_o, y_o:=y_1+0.2, y_2:=y_o
2. while |p_1-y_1| > E
   2.1 if y_1 ≠ y_2
       then
       2.1.1 d:=[(p_o-p_1)(v_1/v_o)]/(1-v_1/v_o)
       2.1.2 if sign(d,d) ≠ sign(p_1-y_1,p_1-y_1)
                  or      |d| > |p_1-y_1|
```

```
            then

            2.1.2.1 d:=(p1-y1)/2

        else

        2.1.3 d:=(p1-y1)/2

    2.2 po:=p1, vo:=v1, p1:=p1-d

    2.3 call VALUE(p1,vd)

    2.4 v1:=vd, y2:=yo, yo:=y1

    2.5 if v1*vo < 0

        then

        2.5.1 y1:=po

    2.6 continue

3. if |vo| < |v1|

   then

   3.1 output eigenperiod {po}

   else

   3.2 output eigenperiod {p1}

4. halt
```

(III) Subroutine VALUE(p,vd): Computing the elements of the determinant at period p_i. Subroutine PIVOT is called to compute the value of the determinant

The input consists of period p_i, density ρ, order N and degree L of the trial function, control parameters 'homi', 'rot', 'elas' and mode parameter m(-1, 0, +1)

homi=0: homogeneous, incompressible liquid core, $\rho_o=\rho_o(a)$

homi=1: compressible, neutrally stratified liquid core, $\rho_o=\rho_o(r)$

rot=0: non-rotating

rot=1: rotating

elas=0: rigid inner core

elas=1: elastic inner core

1. $\omega:=2\pi/p$, $\Omega:=rot*\Omega$, $e:=0$
2. **if** elas $\neq$ 0 **and** homi $\neq$ 0

   then

   2.1 e:=equation (68)
3. $\omega_s{}^2:=\frac{4\pi}{3}G\rho_o(a)\,(1-\frac{\rho_o(a)}{\rho_{IC}})$

   $Q_\omega:=\omega_s{}^2-(\omega^2-2m\Omega\omega)\,(1+e)$

4. qrac$:=\frac{4\pi}{3}\omega^2(\omega^2-4\Omega^2)\frac{\rho^2(a)a(1+m)!}{\rho_{IC}\,(1-m)!}$
5. N2:=2N-2

```
6. for il:=1, L
   6.1 for ip:=1, N2
       6.1.1 for ik:=1, L
             6.1.1.1 for iq:=1, N2
                     i:=ip+N2(il-1)
                     j:=iq+N2(ik-1)
                     H_ij:=0
7. for i:=1, N2
   7.1 for j:=1, N2
       7.1.1 if i ≤ N-1
             then
             7.1.1.1 iv:=N+1
             else
             7.1.1.2 iv:=N-1
          7.1.2 if j ≤ N-1
             then
             7.1.2.1 jv:=N+1
             else
             7.1.2.2 jv:=N-1
       7.1.3 H_ij:=qrac*r_a^(i+j-iv-jv)
8. for il:=1, L
   8.1 for ip:=1, N2
```

```
8.1.1 for ik:=1, L
    8.1.1.1 for iq:=1, N2
        8.1.1.1.1 i:=(il-1)*N2+ip
        8.1.1.1.2 j:=(ik-1)*N2+iq
        8.1.1.1.3 if ip ≤ N-1
              then
              8.1.1.1.3.1 iv:=N+1
              else
              8.1.1.1.3.2 iv:=N-1
        8.1.1.1.4 if iq ≤ N-1
              then
              8.1.1.1.4.1 jv:=N+1
              else
              8.1.1.1.4.2 jv:=N-1
        8.1.1.1.5 PR1:=0, PR2:=0, PR3:=0, PR4:=0
        8.1.1.1.6 FP:=0, FM:=0
        8.1.1.1.7 if il = ik, then
              FP:=Fact(il+1,m), FM:=Fact(il-1,m)
                 if m=0 or (m ≠ 0 and il ≥ 2)
                 then
                  PR1:=PR1+2(il+m)^2*FM
                        /(2*il-1)(2*il+1)^2
                  PR2:=PR2+2(il+m)^2(il+1)*FM
                        /(2*il-1)(2*il+1)^2
```

```
PR3:=PR2
PR4:=PR4+2(il+m)^2(il+1)^2*FM
     /(2*il-1)(2*il+1)^2
else
PR1:=PR1+2(il-m+1)^2*FP
     /(2*il+3)(2*il+1)^2
PR2:=PR2-2*il(il-m+1)^2*FP
     /(2*il+3)(2*il+1)^2
PR3:=PR2
PR4:=PR4+2*il^2(il-m+1)^2*FP
     /(2*il+3)(2*il+1)^2
```

8.1.1.1.8 else if ik = il+2, then

```
PR1:=PR1+2(il-m+1)(ik+m)*FP
     /(2*il+1)(2*ik+1)(2*il+3)
PR2:=PR2-2*il(il-m+1)(ik+m)*FP
     /(2*il+1)(2*ik+1)(2*il+3)
PR3:=PR3+2(il-m+1)(ik+1)(ik+m)*FP
     /(2*il+1)(2*ik+1)(2*il+3)
PR4:=PR4-2*il(il-m+1)(ik+1)(ik+m)
     *FP/(2*il+1)(2*ik+1)(2*il+3)
```

8.1.1.1.9 else if ik = il-2, then

```
PR1:=PR1+2(il+m)(ik-m+1)*FM
     /(2*il+1)(2*ik+1)(2*il-1)
PR2:=PR2+2(ik-m+1)(il+1)(il+m)*FM
```

```
        /(2*il+1)(2*ik+1)(2*il-1)
    PR3:=PR3-2*ik(ik-m+1)(il+m)*FM
        /(2*il+1)(2*ik+1)(2*il-1)
    PR4:=PR4-2*ik(il+1)(ik-m+1)(il+m)
        *FM/(2*il+1)(2*ik+1)(2*il-1)
8.1.1.1.10 WA:=0, WM:=0, RPA:=0
8.1.1.1.11 nn:=ip+iq-iv-jv+1
8.1.1.1.12 if homi ≠ 0
           then
           8.1.1.1.12.1 for ii:=0, iρ
             (iρ:power of density function)
                  n:=nn+ii
                  if n = 0, then
                    RPA:=RPA+ρ_ii
                         *(dlog(b)-dlog(a))
                  else
                    RPA:=RPA+ρ_ii(b^n
                                  -a^n)/n
           else
           8.1.1.1.12.2 if nn = 0, then
                      RPA:=RPA+ρ_o(a)
                         (dlog(b)-dlog(a))
```

```
                        else
                           RPA:=RPA+ρ_o(a)(b^nn
                                 -a^nn)/nn
            8.1.1.1.13 if il = ik then
                  FT:=Fact(il,m)
                  WA:=-8mπωΩQ_ω(nn*RPA)*FT
                     /(2*il+1)-4πω^2
                     Q_ω[(ip-iv)(iq-jv)+il(il+1)]
                     *RPA*FT/(2*il+1)
            8.1.1.1.14 WM:=8πΩ^2Q_ω*RPA
                     *[(ip-iv)(iq-jv)*PR1
                     +(iq-jv)*PR2+(ip-iv)*PR3+PR4]
            8.1.1.1.15 H_ij:=H_ij+WA+WM
9.  js:=(2N-2)
10. call PIVOT(js,H,VD)
11. output {VD}
12. halt
```

(IV) Subroutine PIVOT(N,H,det): computing the value of a determinant by the Gaussian Elimination

Input N×N matrix H

```
1. D:=1.0, TINY:=1.0*10^-20
2. for i:=1, N
   2.1 MAX:=0
   2.2 for j:=1, N
       if |H_ij > MAX
       then
       2.2.1 MAX:=|H_ij|
   2.3 if MAX = 0
       stop
   2.4 V:=1/MAX
3. for j:=1, N
   3.1 if j > 1
       then
       3.1.1 for i:=1, j-1
             3.1.1.1 SUN:=H_ij
             3.1.1.2 if i > 1
                     then
                     3.1.1.2.1 for k:=1, i-1
                               SUM:=SUM-H_ik*H_kj
```

3.2 MAX:=0

3.3 for i:=j, N

  3.3.1 SUM:= H_{ij}

  3.3.2 if j > 1

    then

    3.3.2.1 for k:=1, j-1

      SUM:=SUM- H_{ik} * H_{kj}

    3.3.2.2 H_{ij} :=SUM

  3.3.3 DUM:= V_i * |SUM|

  3.3.4 if DUM $\geq$ MAX

    then

    3.3.4.1 l:=1

    3.3.4.2 MAX:=DUM

3.4 if j $\neq$ ix

  then

  3.4.1 for k:=1, N

    3.4.1.1 DUM:= H_{lk}

    3.4.1.2 H_{lk} := H_{jk}

    3.4.1.3 H_{jk} :=DUM

  3.4.2 D:=-D

  3.4.3 V_l := V_j

3.5 INDIX:=l

```
   3.6 if j ≠ N
       then
       3.6.1 if H_jj=0
             3.6.1.1 H_jj:=TINY
       3.6.2 DUM:=1/H_jj
       3.6.3 for i:=j+1, N
                 H_ij:=H_ij*DUM
4. if H_nn = 0
   4.1 H_nn:=TINY
5. DETLOG:=0, SIGN:=1
3. for i:=1, N
   6.1 DETLOG:=DETLOG+DLOG10(|H_ii|)
   6.2 sign:=sign*H_ii/|H_ii|
7. iE:=idint(DETLOG)
8. DETLOG:=DETLOG-DFLOAT(iE)
9. Det:=D*sign*(10*DETLOG)
10. output {Det}
11. halt
```

(V) Function FACT(k,m): computing ratio of factorials $\frac{(k+m)!}{(k-m)!}$

Input integer k, m

```
1. j1:=k+m, j2:=k-m, QL:=1.0, QJ:=1.0
2. while j1 > 1
   2.1 l:=1, N1:=0
   2.2 N1:=l+1
   2.3 QL:=QL*N1
   2.4 if N1 < j1
       then
       2.4.1 l:=N1, N1:=0
       2.4.2 go to 2.2
3. while j2 >  1
   3.1 j:=1, N2:=0
   3.2 N2:=j+1
   3.3 QJ:=QJ*N2
   3.4 if N2 < j2
       then
       3.4.1 j:=1, N2:=0
       3.4.2 go to 3.2
4. Fact:=QL/QJ
5. output {Fact}
6. halt
```