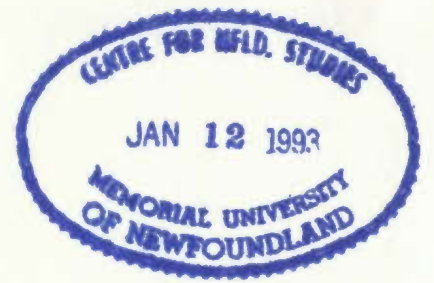


AXISYMMETRIC INERTIAL GRAVITY OSCILLATIONS
OF A COMPRESSIBLE STRATIFIED FLUID
IN A RIGID SPHERE: AN APPROXIMATION
TO THE EARTH'S OUTER CORE

SUSAN JANE WEBB, B.Sc.





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AXISYMMETRIC INERTIAL GRAVITY OSCILLATIONS OF A COMPRESSIBLE
STRATIFIED FLUID IN A RIGID SPHERE: AN APPROXIMATION TO THE
EARTH'S OUTER CORE

BY

© Susan Jane Webb, B.Sc.

A thesis submitted to the School of Graduate
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Department of Earth Sciences (Geophysics)
Memorial University of Newfoundland
February 1992

St. John's

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Abstract. This thesis investigates several theoretical methods for solving the sub-seismic wave equation (SSWE) in an incompressible or slightly stratified liquid bounded by a rigid fixed spherical boundary as an approximation to the Earth's liquid core. The solution of the SSWE yields eigenvalues for the inertia/gravity wave spectrum. The first method considered uses a variational principle based on a more general functional than has been previously used to evaluate the SSWE. The resulting eigenvalues are compared with previous work done by Aldridge and Toomre (1969). The work is then extended to include a density profile and a shift is noted in the eigenspectrum. A perturbation approach was used to include stably stratified regions characterised by negative values of the stability parameter β ; unfortunately this method only works for such small values of β as to be uninteresting. Finally a Galerkin approach was adopted that allows for the evaluation of negative β values. This results in a further change of the inertial eigenspectrum and the appearance of gravity modes.

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◊MACSYMA.

TABLE OF CONTENTS

CHAPTER 1	1
1.1 Introduction and Motivation	1
1.2 Review of Previous Work	5
1.2.1 State of the Liquid Core	5
1.2.2 Theory of Inertial Waves	7
1.2.3 Inertia/Gravity Wave Theory for the Earth's Core: the Subseismic Wave Equation (SSWE)	9
1.3 The Governing Equations of Core Dynamics	11
1.3.1 Density Gradient	14
1.3.2 Sub-seismic Approximation	16
1.3.3 Boundary Conditions	20
1.4 Solving the Subseismic Wave Equation: Prospectus	22
CHAPTER 2	23
2.1 Introduction	23
2.2 Variational Principle	23
2.3 Approximations to Produce the Poincaré Equation	26
2.4 General Form of the Trial Function	28
2.5 General Solution Method	29
2.6 Specific Form of χ and Results	31

CHAPTER 3	39
3.1 Introduction	39
3.2 Density Profile (PREM)	39
3.3 Variational Formulation of the Eigenvalue Problem (PREM Core)	40
3.4 Eigenvectors	46
CHAPTER 4	57
4.1 Introduction	57
4.2 Perturbation Theory for the Eigenvalue Problem (Slightly Stable Core)	57
4.3 Discussion	68
CHAPTER 5	72
5.1 Introduction	72
5.2 Galerkin Method	72
5.3 Application to the SSWE	74
5.4 Density Profile	79
5.4.1 Least Squares	79
5.4.2 Orthogonality Relationship	84
5.5 Expansion of Equation (5.16)	84
5.6 Results	89
5.6.1 $B = 0$, $\rho_c = \text{PREM}$, and $\rho_c = \text{NEUT}$	89
5.6.2 $B < 0$, $ B \leq 0.0001$, $\rho_c = \text{PREM}$	92
5.6.3 $B < 0$, $ B \geq 0.0001$, $\rho_c = \text{NEUT}$	94

CHAPTER 6	97
6.1 Introduction	97
6.2.1 Inner Core	97
6.2.2 Normal Modes Carrying Linear or Angular Momentum	98
6.2.3 More Realistic Boundary Conditions	101
6.2.4 Azimuthal Modes	101
6.2.5 Improved Representations of Density Profile	102
6.2.6 Viscosity	103
6.2.7 Density of the Eigenspectrum	103
6.2.8 Density of the Eigenspectrum	103
6.3 Summary of Research Results	103
 References	 105
 Appendix A: Three methods for computing an integral.	 108
 Appendix B: Program to calculate the eigenfrequencies of the axisymmetric normal modes of the Poincaré problem in a liquid sphere with rigid fixed boundaries.	 110
 Appendix C: Program to calculate the eigenfrequencies of the axisymmetric normal modes of the Poincaré problem in a liquid sphere including a neutral density profile from PREM.	 116

Appendix D:	This program finds the eigenvector coefficients for the eigenfrequencies previously calculated.	123
Appendix E:	This program evaluates equation (82) from Rochester (1989).	131
Appendix F:	This program evaluates negative betas using the Galerkin method.	139

LIST OF TABLES

Table 1.	Comparison of Dimensionless Eigenfrequencies Obtained by Aldridge and Toomre (1969) and from the Application of the Variational Principle to the Poincaré Equation	38
Table 2.	Comparison of Dimensionless Eigenfrequencies in an Homogeneous, Neutrally Stratified Sphere and in a Neutrally Stratified Sphere With a Density Profile of the Outer Core of PREM	45
Table 3.	Eigenvectors Associated With Eigenfrequencies for a Neutrally Stratified Sphere With a Density Profile of the Outer Core From PREM	51
Table 4.	Coefficients of a Seventh Degree Polynomial that Approximates the Value of $1/\alpha^2$	66
Table 5.	Comparison of Reference State Eigenvalues, σ_0 , and Eigenvalues, σ , Found From the Perturbation Method	69
Table 6.	Coefficients ρ_m of Neutral Density Profile Fitted by (a) Least Squares and (b) Orthogonality Relation	82

Table 7.	Comparison of Actual Values of δ for Neutral Density Profile Fitted by (a) Least Squares and (b) Orthogonality Relation	83
Table 8.	Comparison of Dimensionless Eigenfrequencies Calculated Using the Variational Principle (VP) and the Galerkin Method	91
Table 9.	Comparison of Dimensionless Eigenfrequencies Calculated Using the Perturbation and Galerkin Methods, With $\rho_0 =$ PREM	93
Table 10.	Tentative Comparison of Dimensionless Eigenfrequencies Calculated Using the Galerkin Method, with $\rho_0 =$ NEUT	96

x

LIST OF FIGURES

- Figure 1. Variation of beta with radius in the liquid core. Values are 41
from PREM

List of Symbols

Symbols are listed in the order in which they appear with similar symbols being grouped together.

All symbols are in italics with vectors and dyadics in boldface.

<u>Symbol</u>	<u>Significance</u>	<u>Equation Where Initially Used</u>
g_0	Gravity	1.1
V_0, V_1	Gravitational Potential	1.1, 1.5
ρ_0, ρ_1, ρ_0^1	Density	1.2, 1.5, 1.22
v	Velocity	1.2
t	Time	1.2
Ω	Angular Velocity	1.3
P_1, P, P	Pressure, Linear Momentum, Summation Limit	1.3, 6 2, 2.25
α	Compressional Wave Speed	1.4
G, G_m	Gravitational Constant, Matrix	1.5, 2.21
u	Displacement	1.6
ω	Frequency	1.7
r	Position, Spherical Coordinate, Summation Index	1.7, 2.25, 2.34

χ, χ_0	Scalar Field Variable	1.2, 4.6
β	Stability or buoyancy parameter	1.15
N	Brunt-Väisälä frequency, Summation Limit	1.16, 2.25
W_0	Gravitational Potential	1.23
$\Gamma, \Gamma_p, \Gamma_2$	Dyadics	1.26, 1.27, 2.12
$\mathbf{1}$	Unit Dyadic	1.27
\mathbf{e}_3, \mathbf{n}	Unit Vectors	1.27, 1.32
\mathbf{C}, \mathbf{C}_0	Vector Definitions	1.28, 4.7
\mathbf{B}, \mathbf{B}	Scalar, Matrix	1.29, 3.11
σ, σ_2	Dimensionless Frequency	1.30, 4.2
F	Functional, Definition	2.1, 5.9
\mathcal{L}	Linear Operator	2.1
Ψ	Arbitrary Function	2.1
dv, ds	Volume, Surface Integration	2.1
ϕ	Azimuthal Coordinate, Trial Function	2.16, 2.19
m	Azimuthal Number, Summation Index	2.16, 3.1
R	Cylindrical Radial Coordinate	2.17
c_p, c_{ij}, c	Constants, Function	2.19, 2.28, 2.32
p, q	Summation Index	2.19
r, θ, z	Spherical Coordinates	2.25
a	Radius of Core Mantle Boundary	2.25
Z	Cylindrical Height Coordinate	2.26
i, j, k, l	Summation Index	2.28, 2.32
f	Function, Definition	2.33, 5.29

d, d_{ij}	Function, Constants	3.5, 6.1
A, A	Square Matrix, Constant	3.10, 4.2
λ	Eigenvalue	3.10
x	Eigenvector	3.10
q	Summation Index	3.13
Q	Summation Limit	3.13
D_m, D	Matrix, Definition	3.16, 5.8
s	Summation Index	4.28
cu_m	Expressions	4.30
E	Definition	5.10
H_Ψ, H	Matrix, Total Angular Momentum	5.27, 6.4
γ	Definition	5.28
$I(i, j), I$	Definition, Inertia Tensor	5.30, 6.7
A_Ψ, B_Ψ, \dots	Matrices	5.31 - 5.44
t, v, w	Summation Index	5.36, 5.41, 5.36
h	Flow	6.9

CHAPTER 1

1.1 Introduction and Motivation

Unlike the Earth's surface, the core of the Earth is not directly accessible for study, consequently much is still unknown about its behaviour. Attempts have been made to characterize this region using theoretical and observational studies of ray seismology, short-period free oscillations, tides and wobble/nutation. These studies have generated a great deal of information about the core.

Specifically, studies of ray seismology, tides and wobble/nutation have established the fluidity of the core. Observation and analysis of seismic wave propagation have allowed determination of the velocity structure within the liquid core, and established the existence of the inner core. Studies of short-period free oscillations have helped to determine the solidity of the inner core. This work has been summarized in an historical sense by Brush (1980). Currently, seismic tomography is being used to map the topography of the core mantle boundary (Anderson & Dziewonski 1984).

These short-period free oscillations are what seismologists simply call free oscillations and have periods on the order of one hour. They are termed short-period here, as a way of distinguishing them from the much longer period free oscillations that will be discussed later in this work.

Unfortunately, several key properties of the core cannot be firmly established from these studies. One of the most important is the value and sign of β , a dimensionless parameter that represents the fractional departure of the density gradient from a strictly neutral, or adiabatic, value. This

parameter is defined in section 1.3.2, equation (1.15) and was originally developed by Pekeris & Accad (1972). The parameter β gives an indication of the ability of a region of the core to convect and shall be referred to as the stability parameter. If the value is positive then that region will be unstable, if it is negative then it will be stably stratified and if the value is zero then the region will be neutrally stratified. A study by Masters (1979) using ray seismology estimates an upper limit of the absolute value of β to be less than 0.03 - 0.05. The value of β is related to another parameter, N , called the Brünt-Väisälä frequency. This value is defined in terms of β in equation (1.16) in section 1.3.2. This frequency refers to the oscillations of a particle about its equilibrium position in a stably stratified fluid. Thus the oscillations will only occur if β is negative. If the value of β is of the size estimated above and the sign is negative, then the period of this free oscillation in the Earth's core would be greater than several hours. This is called a long-period free oscillation as the period is significantly longer than that for short-period free oscillations.

There are actually two distinct types of long-period free oscillations that could occur in the Earth's core. The type mentioned above are called gravity waves or core undertones and depend on negative buoyancy for a restoring force. The presence of negative buoyancy implies stable stratification at some location in the Earth's core. Thus, some part of the Earth's core would have to be stably stratified, indicated by a negative β value, in order for gravity waves to exist. Their periods are on the order of $2\pi/N_{\max}$ and longer, where N_{\max} is the limiting Brünt-Väisälä frequency. As noted earlier, this can be several hours depending on the magnitude of β .

The other type of long-period free oscillation that could exist in the core is termed an inertial wave. This type of oscillation depends upon the Coriolis effect as the restoring force. Thus

inertial waves require rotation for their very existence. Their periods are on the order of half a day and longer.

If the stratification present in the core is such that the periods of the gravity waves are comparable with the periods of the inertial waves, then the presence of negative buoyancy will modify the inertial waves, and the gravity waves will be altered by rotation. This leads to the terminology of "inertia/gravity waves" when referring to both types of long-period free oscillations when their spectra overlap. This will occur in a rotating stably stratified liquid for which the magnitude of the rotation period and $2\pi/N_{\text{max}}$ are comparable. This may well be the case in some regions in the Earth's core.

The range of frequency response in the core covered by the short-period free oscillation data is relatively small, having periods on the order of one hour. This means that only a very small portion of the Earth's free oscillation spectrum has actually been well studied. The dominant restoring force for short-period free oscillations is elasticity, as opposed to the Coriolis effect and negative buoyancy for the inertia/gravity wave spectrum.

Theoretical studies of short-period free oscillations have generally used spherical harmonic expansions as a normal mode can be described by a single spherical harmonic. Unfortunately this is not the case for inertia/gravity wave studies. The cylindrical symmetry that is imposed by rotation on the problem means that a normal mode can no longer be described by a single spherical harmonic. A whole chain of spherical harmonics must be used instead. In order to solve the problem this chain must, at some stage, be truncated. Unfortunately, this truncation makes the solutions inaccurate and other methods of solution need to be investigated.

In an effort to simplify the mathematics of the inertia/gravity wave problem, Smylie & Rochester (1981) introduced the subseismic approximation (SSA). It was hoped that this simplification would suggest an alternative method of solution that avoided or simplified the spherical harmonic expressions. The equation resulting from this simplification, known as the subseismic wave equation (SSWE), can be solved using a novel separation of variables under very specific conditions (Smylie & Rochester 1986a). However, solving this equation using spherical harmonics still results in a truncation problem. Other methods of solution of this equation have also been summarized by Rochester (1989).

Despite the progress made on the theoretical front, recognition of a signal from inertia/gravity waves will be difficult. The detection of a long period gravity signal interpretable as evidence of core waves could provide additional constraints on core properties, in particular β , as was first pointed out by Smylie (1974). However, such waves produce very small changes in density at great distances from the instrument and consequently a very small gravity signal is produced. The extreme sensitivity of superconducting gravimeters suggests they are the instrument of choice for observing these signals, and close to a dozen are now deployed worldwide with this as one of the principal objectives. While Melchior & Ducarme (1986) reported observations suggesting the detection of gravity waves, and Aldridge & Lumb (1987) argued that these same observations could be interpreted as due to inertial waves, both of these explanations have been contested by Zurn *et al.* (1987). At present all that can be said with certainty is that more data from globally distributed instruments are needed before unambiguous interpretations become possible.

Setting aside these observational difficulties, the mathematical description of the inertia/gravity wave spectrum of a rotating liquid-filled spherical shell still presents challenges which must be

met before the observations can be reliably interpreted. This thesis is concerned with one aspect of this theory, namely the use of a variational principle to describe the axisymmetric modes of a compressible neutral or stably stratified rotating liquid filling a rigid spherical container, as a first approximation to representing the Earth's liquid core.

1.2 Review of Previous Work

This section summarizes previous work on: (1) the state of the liquid core, (2) inertial wave theory, and (3) inertia/gravity wave theory applied to the Earth's core.

1.2.1 State of the Liquid Core

The most readily available source of Earth-penetrating energy is that supplied by earthquakes. This energy was used by Jeffreys (1926), who established the fluidity of the Earth's core. He showed that the presence of a large zero-rigidity core surrounded by a mantle with a rigidity determined from ray seismology led to an Earth with a mean rigidity corresponding to that inferred earlier from solid Earth tides and the Chandler wobble period. The idea that the fluid core is convecting has been argued on several different points, including considerations of heat, summarized by Stacey (1977). While the arguments for thermal convection given by Elsasser (1950) as an explanation of the geodynamo were convincing at the time, evidence now suggests that convection in the core may be compositional. This would be due to gravitational segregation of light and heavy fractions of the inhomogeneous core material. In fact it may be possible for

dynamo action to occur in stably stratified regions of the core (Singer & Olson 1984).

Other observations of Earth behaviour have provided information about the core. Benioff *et al.* (1954) devised a long period strain meter in an attempt to record natural periods of whole-Earth oscillations. These were predicted theoretically by Love (1911). While the Kamchatka earthquake of 1952 provided initial results that were considered to be short-period free oscillations, an unambiguous observation of this type of oscillation was made immediately following the 1960 Chilean earthquake.

These early observations were used by Alterman *et al.* (1959) to test several theoretical models (Bullen 1950, Bullard 1957) which were based on data from higher frequency ray seismology. Their inference that free oscillations correlated better with the Bullen B model, with an inner core, was really only confirmed by the data from the Chilean earthquake just mentioned.

An earthquake in Alaska in 1964 further confirmed the existence of whole Earth short-period free oscillations, and provided a wealth of new data. Dziewonski & Gilbert (1972) used this data to refine the current model of the Earth's interior and to establish the solidity of the inner core. While the density distribution is fairly well known from short-period free oscillation data, the details of its departure from a strictly neutral gradient are not well constrained by these data. Consequently, the sign and size of the stability parameter in the core remain unknown in spite of the information provided by short-period free oscillation data.

A compilation of seismic, free oscillation and nutation data enabled Dziewonski & Anderson (1981) to establish the preliminary reference Earth model or PREM. This model provides a basic

reference state for many Earth parameters, including density, for consistent studies of the Earth. In this widely accepted model of the Earth, the liquid core is mostly neutrally stratified. An argument for stable regions in the core has also been given by Gubbins *et al.* (1982), who inferred properties of the core's outer layers from the geomagnetic secular variation. Thus the state of the Earth's liquid core may allow the existence of long-period free oscillations, as inertia/gravity waves. While initial results have been interpreted as both core undertones (Melchior & Ducarme 1986) and as inertial waves (Aldridge & Lumb 1987) more data needs to be collected before these results can be fully understood.

1.2.2 Theory of Inertial Waves

The theory of inertial waves was established by Poincaré (1885), who developed the equation for oscillations in an homogeneous, incompressible, rotating, and inviscid fluid. This equation, (later called the Poincaré equation), describes pure inertial waves and can be solved analytically. A method of solution was developed by Bryan (1889), using a double transformation to obtain an "oblate spheroidal" coordinate system, which allowed separable solutions to the Poincaré problem and were represented as products of Legendre functions. Thus, the theoretical foundation for inertial waves in a rigid, spherical container was established.

Kudlick (1966) greatly expanded the theoretical work on this problem by investigating several effects. He was able to expand the linear theory to include the effects of viscosity by using a superposition of the natural oscillatory modes of the inviscid problem that had been corrected to first order for the effects of viscosity. He also extended the theory to include containers of

arbitrary shape and investigated the effects of an oscillatory body force applied to the container walls. Of particular interest to studies of the Earth, he found a first order inviscid solution for a precessing, fluid filled, rotating spheroid.

Greenspan (1968) provided an excellent summary of rotating fluid dynamic theory. This work considered rotating fluids exclusively, and included a comprehensive investigation of viscosity effects.

A simplified form of Bryan's (1889) solution to the Poincaré problem was found by Aldridge & Toomre (1969). The simplified solutions, which can only be obtained for the case of an incompressible liquid inside a rigid ellipsoidal boundary, were presented as polynomials in cylindrical coordinates for the spherical and spheroidal modes. They also conducted experiments on fluids contained in a rotating sphere for the case when viscosity is considered. The observed spectrum of oscillations obtained from these trials matched well with the values they predicted theoretically, where the analytical solution is possible only for the case of an incompressible liquid inside a rigid ellipsoidal boundary. This experimental set up can be considered as a crude model of the Earth, and the presence of these waves gave impetus to further investigation of more realistic Earth models.

The actual configuration of the Earth's core, a thick shell, presents mathematical difficulties in the theory of inertial waves, as the presence of an inner core renders the problem ill-posed (Stewartson & Rickard 1969). An attempt was made by Aldridge (1972) to circumvent the ill-posed nature of the inertial wave problem in a thick shell by using a variational formulation of the Poincaré equation. Although no analytical solution was found, his experimental results

indicated that inertial waves do exist in the thick shell configuration.

1.2.3 Inertia/Gravity Wave Theory for the Earth's Core: the Subseismic Wave Equation (SSWE)

A study of gravity waves by Pekeris & Accad (1972) in a nonrotating Earth model was done using asymptotic theory to the solution of the short-period free oscillation formulation. However, as the model considered was nonrotating, the Coriolis force was missing in the treatment of this problem; this was shown later to have a very large effect on the solutions (Smylie 1974, Johnson & Smylie 1977). By including the effects of rotation the mathematics of the problem were considerably complicated. When traditional vector spherical harmonics were used to solve this formulation of the problem, the rotational coupling of formerly independent displacement fields of different degree resulted in each normal mode displacement being represented by an infinite chain of spheroidal and toroidal fields. Numerical calculations of course require severe truncation of such chains, which can make the results obtained inconclusive.

Smylie & Rochester (1981) sought to reformulate the problem of inertia/gravity waves in the Earth's core by means of a scaling argument applied to the basic equations. They were able to make the 'subseismic' approximation (SSA) which neglects the effects of flow pressure on density at sub-acoustic frequencies. This in turn led to the subseismic wave equation (SSWE), which governs large scale rotating core dynamics in the sub-acoustic frequency range. It should be noted that the SSWE reduces to the familiar Poincaré equation when the core is treated as homogeneous and incompressible. They also considered the possible regions of stability in the core that can support these oscillations. Further work by Crossley (1984) investigated the

possible frequency regimes, excitation and damping mechanism for these waves. See also the summary by Smylie *et al.* (1984).

A qualitative investigation by Friedlander (1985) determined several regions in the Earth's core where inertia/gravity waves could exist. The structure and location of these regions is very sensitive to the stability structure within the core, confirming the suitability of these waves for use in studying the stability parameter. This work was extended in Friedlander (1987) where asymptotic solutions were extended from limited known solutions. Further investigations in Friedlander (1988) also considered very long period oscillations affected by a magnetic field.

A solution of the SSWE using the separation of variables technique was attempted by Smylie & Rochester (1986a). These solutions were restricted to the case where the Brünt-Väisälä frequency N , and the local compressibility number were uniform throughout the core. This work does demonstrate the relationship between the solutions and physical conditions in the core. A variational principle developed by Smylie & Rochester (1986b) offered an alternative and more powerful method of solution. Unfortunate mistakes in this formulation were discovered and later corrected by Rochester (1989) in a summary of work done on the SSWE. In this summary, several different solution techniques for the SSWE were explored, including separable solutions, asymptotic solutions and the variational principle. A discussion of the formulation of a variational principle for the SSWE alone (i.e. decoupled from the Poisson equation) is given by Wu & Rochester (1990), where they note that it can be developed for a stratified core with deformable boundaries, but only if the stratification is neutral. Recent work by Crossley & Rochester (1991) has shown that for long-period oscillations the subseismic approximation is valid in the interior of the liquid core but becomes unusable at the boundaries.

1.3 The Governing Equations of Core Dynamics

In order to study motion within the Earth's core, an equilibrium reference state must first be chosen. The departures from this state will then define the motions of interest. The reference state chosen here, as an approximation to real core conditions, is of an inviscid fluid in hydrostatic equilibrium in a uniformly rotating coordinate system. The equilibrium gravity, g_0 , is given by:

$$g_0 = -\nabla V_0 \quad (1.1)$$

where V_0 is the equilibrium gravitational potential. The Lorentz force is negligible for the time scale of the Earth's rotation and is ignored (Crossley & Smylie, 1975). Starting with these considerations the SSWE can be derived from the basic equations of fluid dynamics as found in Greenspan (1968, pp. 11-12). Notation has been changed here and the formulation includes the Poisson equation to account for the effects of self-gravitation.

The linearized equations of motion, that describe the Eulerian departure of the density (ρ_1), pressure (P_1), and gravitational potential (V_1) from their equilibrium values (ρ_0 , P_0 , V_0), are the conservation of mass, momentum, entropy and gravitational flux for a self-gravitating system:

$$\frac{\partial \rho_1}{\partial t} = -(\mathbf{v} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{v}) \quad (1.2)$$

$$\frac{\partial \mathbf{v}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{v} = \nabla V_1 - \frac{1}{\rho_0} \nabla P_1 - \frac{\rho_1}{\rho_0} \nabla V_0 \quad (1.3)$$

$$\frac{\partial P_1}{\partial t} = -\rho_0 (\alpha^2 \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla V_0) \quad (1.4)$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \quad (1.5)$$

where \mathbf{v} is velocity, t is time, α is the compressional wave speed in the core, G is the gravitational constant and $\boldsymbol{\Omega}$ is the angular velocity of the steadily rotating reference frame. As the liquid is treated as inviscid, dissipative effects are missing from equations (1.3) and (1.4). The system of equations, (1.2-1.5), must be solved to describe the motions in the core.

In order to solve these equations the Lagrangian displacement from an equilibrium location of a liquid mass element is defined as \mathbf{u} . As small departures from equilibrium are being considered, this displacement \mathbf{u} can be expressed as:

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \quad (1.6)$$

The frequency of these small oscillations will be given by ω . These oscillations can then be represented by:

$$u(r, t) = \text{Re} [u(r) e^{i\omega t}] \quad (1.7)$$

where $u(r)$ is generally a complex expression. This will replace equations involving real quantities with equations that are real parts of complex equations. Similar expressions can also be written for ρ_1 , V_1 , and P_1 . Substituting (1.6) and (1.7) into the governing equations (1.2 - 1.5) results in:

$$\rho_1 = -\nabla \cdot (\rho_0 u) \quad (1.8)$$

$$-\omega^2 u + 2i\omega \Omega \times u = \nabla V_1 - \frac{1}{\rho_0} \nabla P_1 - \frac{\rho_1}{\rho_0} \nabla V_0 \quad (1.9)$$

$$P_1 = -\rho_0 (\alpha^2 \nabla \cdot u - u \cdot \nabla V_0) \quad (1.10)$$

$$\nabla^2 V_1 = -4\pi G \rho_1 \quad (1.11)$$

where the equations now relate complex field quantities but involve only spatial differentiation.

For the sake of future simplification, the scalar field variable χ can be defined as:

$$\chi = \frac{P_1}{\rho_0} - V_1. \quad (1.12)$$

This can be substituted into equation (1.9) to give:

$$-\omega^2 \mathbf{u} + 2i\omega \boldsymbol{\Omega} \times \mathbf{u} = -\nabla \chi + P_1 \nabla \left(\frac{1}{\rho_0} \right) - \frac{P_1}{\rho_0} \nabla V_0. \quad (1.13)$$

for the momentum equation.

1.3.1 Density Gradient

In order to consider departures of the equilibrium density gradient ($\nabla \rho_0$) from the adiabatic stratification assumed by the Adams-Williamson equation (Melchior 1986), Pekeris & Accad (1972) introduced β , the stability parameter. This parameter changes the Adams-Williamson equation:

$$\nabla \rho_0 = \frac{\rho_0 g_0}{\alpha^2} \quad (1.14)$$

that describes an adiabatically or neutrally stratified fluid, to:

$$\nabla \rho_0 = (1 - \beta) \frac{\rho_0 g_0}{\alpha^2} \quad (1.15)$$

which can also describe a fluid that is stably or unstably stratified. If $\beta = 0$, the fluid is neutrally stratified and the equation reduces to the Adams-Williamson equation. Otherwise as $\beta < 0$ or $\beta > 0$, the fluid is locally stably or unstably stratified, respectively. In general β is expected to be radially dependent, but for simplicity core models with a constant β will be considered.

The stability parameter, β , is related to the Brünt-Väisälä frequency (Melchior 1986), N , which describes the frequency of oscillation of a particle about its equilibrium point in a stably stratified density profile. This parameter is more commonly used in oceanography and is related to the stability parameter by:

$$N^2 = -\frac{\beta g_0^2}{\alpha^2}. \quad (1.16)$$

The equations which govern dynamics in the core can now be written with the inclusion of the stability parameter as:

$$-\omega^2 u + 2i\omega \Omega \times u = -\nabla \chi + \beta g_0 \nabla \cdot u \quad (1.17)$$

$$\nabla^2 V_1 = 4\pi G \left[\rho_0 \nabla \cdot \mathbf{u} + \frac{(1 + \beta)}{\alpha^2} \rho_0 \mathbf{u} \cdot \mathbf{g}_0 \right] \quad (1.18)$$

where both the conservation of mass equation (1.8) and the entropy conservation equation (1.10) have been decoupled from the equations by substituting for ρ_1 and P_1 respectively. Even this reduced system of equations in general cannot be solved exactly without encountering the truncation problem already mentioned.

1.3.2 Sub-Seismic Approximation

In previous studies of long-period free oscillations of the core (e.g. Greenspan 1968), the solenoidal flow approximation,

$$\nabla \cdot \mathbf{u} = 0 \quad (1.19)$$

has been made in the entropy conservation equation. The incompressibility approximation is made in an attempt to simplify the mathematics of obtaining solutions in the presence of rotation. While this approximation is applicable in laboratory conditions, it should not be used in the core where compression is a factor. Therefore, a different approximation from the solenoidal flow approximation needs to be made in the governing equations. This approximation needs to simplify the mathematics yet preserve the physics of a compressible core.

This approximation was formulated by Smylie & Rochester (1981) who used a scaling argument to eliminate a term in the entropy conservation equation (1.10). This was done by calculating the magnitude of individual terms in the entropy conservation equation after choosing realistic values for particular Earth properties. Equation (1.10) can be rewritten as:

$$\nabla \cdot \mathbf{u} = - \frac{P_1}{\rho_0 \alpha^2} - \frac{\mathbf{u} \cdot \mathbf{g}_0}{\alpha^2}. \quad (1.20)$$

These terms relate the local compression of the LHS, to the effect of compression due to flow pressure, and compression due to transport through the equilibrium pressure gradient, the first and second terms of the RHS. By writing the governing equations in dimensionless form, using characteristic values for Earth properties, and considering a longer time scale than that used in conventional seismology, Smylie & Rochester (1981) demonstrated that the first term of the right hand side can be eliminated from the system of equations. This is because it is at least 3 orders of magnitude smaller than other terms, and contrasts to the situation in the acoustic frequency range where this term is substantial. This neglects the effect of compression due to flow pressure in the governing equations and is termed the subseismic approximation. It leads to a simplified expression for the entropy conservation equation:

$$\nabla \cdot \mathbf{u} = - \frac{\mathbf{u} \cdot \mathbf{g}_0}{\alpha^2}. \quad (1.21)$$

This expression can also be rewritten as:

$$\nabla \cdot (\rho_0^1 u) = 0 \quad (1.22)$$

where

$$\rho_0^1 = \rho_0 e^{-\int \frac{\beta}{\alpha^2} dW_0} \quad (1.23)$$

and where W_0 is the gravitational potential that includes rotation effects. The simplified form of the entropy conservation equation (1.22) will be used later.

Making the substitution of (1.21) into the governing equations (1.17 and 1.18) we arrive at:

$$-\omega^2 u + 2i\omega \Omega \times u = -\nabla \chi + \frac{\beta}{\alpha^2} g_0 u \cdot g_0 \quad (1.24)$$

$$\nabla^2 V_1 = -4\pi G \frac{\beta \rho_0}{\alpha^2} u \cdot g_0. \quad (1.25)$$

Equation (1.12) is used to obtain P_1 from χ and V_1 after these potentials have been obtained from equations (1.24) and (1.25).

At this point no further approximations are needed and the subseismic wave equation (SSWE) can be derived directly from equation (1.24). To do this, both the dot and cross products of (1.24)

with e_3 , the unit vector parallel to Ω , must be formed. These results are substituted back into (1.24). This expression is then solved for u by forming various other products of the original expression. When this is finished the result is a simple expression:

$$u = \Gamma \cdot \nabla \chi \quad (1.26)$$

where:

$$\Gamma = \frac{1}{\omega^2(\sigma^2 - 1)} \{ \sigma^2 \mathbf{1} - e_3 e_3 - \frac{C^* C}{B} + i\sigma e_3 \times \mathbf{1} \} \quad (1.27)$$

$$C = -\sigma^2 g_0 + (e_3 \cdot g_0) e_3 + i\sigma e_3 \times g_0 \quad (1.28)$$

$$B = \frac{\omega^2 \alpha^2 (\sigma^2 - 1)}{\beta} + \sigma^2 g_0^2 - (e_3 \cdot g_0)^2 \quad (1.29)$$

and σ is defined as:

$$\sigma = \frac{\omega}{2\Omega} \quad (1.30)$$

where C^* is the complex conjugate of C , and $\mathbf{1}$ is the unit dyadic. The SSWE can now be given, by combining (1.22) and (1.26) as:

$$\nabla \cdot (\rho_0^t \Gamma \cdot \nabla \chi) = 0. \quad (1.31)$$

The momentum equation (1.26) and the subseismic wave equation (1.31) were first given in this form by Rochester (1989).

1.3.3 Boundary Conditions

The final consideration is that of the boundary conditions. Three types of boundaries seem plausible for problems of the kind we are considering: rigid fixed, deformable, and free boundaries. The boundary condition explored here is a rigid fixed boundary, an approximation to real Earth conditions at the core mantle boundary (CMB). This means there is no mechanical energy exchange between the core and the rest of the Earth. The free boundary condition would be used in the study of stars and the deformable condition could be used in a more detailed study of the core than that presented here. The rigid fixed boundary condition is formed by noting that the normal component of the displacement vector must be zero at the boundary, or:

$$u \cdot n = 0 \quad (1.32)$$

where n is the unit normal vector. By noting that the normal vector can be written as:

$$-n = \frac{g_0}{g_0} \quad (1.33)$$

because the equilibrium gravity, g_0 , is everywhere normal to the core mantle boundary, an alternative expression for the boundary condition can be given as:

$$C \cdot \nabla \chi = 0. \quad (1.34)$$

When this is expanded as:

$$-\sigma^2 g_0 n \cdot \nabla \chi + (e_3 \cdot g_0)(e_3 \cdot \nabla \chi) + i\sigma(e_3 \times g_0) \cdot \nabla \chi \quad (1.35)$$

it is seen to be a very complicated boundary condition as it involves several different components of $\nabla \chi$ and also contains σ , the dimensionless frequency. Fortunately, in chapter 2, (1.34) will be shown to be a 'natural' boundary condition when solving (1.31) by means of a variational principle, and thus need not be invoked independently to constrain the solutions. The solution to the problem of long period free oscillations in the core is then found by solving (1.31) for χ and subsequently solving for u , ρ_1 , V_1 and P_1 in equations (1.26), (1.8), (1.25) and (1.12) respectively.

1.4 Solving the Subseismic Wave Equation: Prospectus

There are several methods that can be used to solve the subseismic wave equation. Many of these have been examined by Rochester (1989): separable solutions, asymptotic solutions, and the variational principle. In this work, the variational principle, asymptotic solutions and a Galerkin approach are explored.

This thesis investigates several different methods of solution of the subseismic wave equation in an incompressible or slightly stably stratified liquid bounded by a rigid fixed spherical surface, as a preliminary to an attack on the inertial/gravity wave spectrum for more realistic Earth models. In the next chapter, the variational principle is applied to the Poincaré equation (as a limiting form of the SSWE) to reproduce the results obtained by Aldridge & Toomre (1969). This will demonstrate the utility of the variational principle for the SSWE, which retains the ability to examine more of the problem than the formulation provided by Aldridge & Toomre (1969). Such is done in chapter 3 where the effect of compressibility is studied using a radial density profile from the Preliminary Reference Earth Model (PRFM) of Dziewonski & Anderson (1981). In the fourth chapter the perturbation approach is employed to establish the response of the system when the stability parameter, β , is nonzero. Finally, in chapter 5, the Galerkin approach is used with both a neutrally and stably stratified density profile. Several variations of this approach are considered to delineate changes in inertial frequencies and detect new gravity modes. The results are then compared with the previous methods. The final chapter summarizes the work done and briefly explores some of the possibilities for further research.

CHAPTER 2

2.1 Introduction

The purpose of this chapter is first to review the demonstration that a variational principle can be used as a method of solution for the SSWE. The resulting functional, which is more general than that derived by Aldridge (1972) for axisymmetric solutions of the Poincaré equation, is then used to reproduce the eigenvalue results of Aldridge and Toomre (1969), hereafter referred to as A&T.

2.2 Variational Principle

The variational principle will be applied to a functional where the SSA has been made throughout the fluid volume and the outer boundary is considered to be rigid and fixed. The functional chosen for this work is:

$$F = \int_V \chi^* \mathcal{L} \chi \, dv + \int_S \psi^* \mathbf{n} \cdot \mathbf{u} \, ds \quad (2.1)$$

where χ and ψ are arbitrary functions we are free to choose and \mathcal{L} is defined as the linear operator for the SSWE:

$$\mathcal{L}\chi = \nabla \cdot (\rho_0^i \Gamma \cdot \nabla \chi). \quad (2.2)$$

We need to ensure that this functional is suitable for the implementation of a variational principle with trial functions χ which do not satisfy the rigid fixed boundary condition:

$$n \cdot u = 0. \quad (2.3)$$

Implementing a variation in the functional, $\delta\chi$ and $\delta\psi$, we obtain:

$$\begin{aligned} \delta F = & \int_V (\delta\chi^* \mathcal{L}\chi + \delta\chi (\mathcal{L}\chi)^*) dv \\ & + \int_S (\delta\psi^* n \cdot u + \psi^* n \cdot \delta u) ds \\ & + \omega^2 (\sigma^2 - 1) \int_S (\rho_0^i \chi^* n \cdot \delta u - \rho_0^i \delta\chi n \cdot u^*) ds \end{aligned} \quad (2.4)$$

where $\delta\chi$ and $\delta\psi$ are arbitrary in the volume and on the boundary. Rearranging terms we can write:

$$\begin{aligned} \delta F = & \int_V (\delta\chi^* \mathcal{L}\chi + \delta\chi (\mathcal{L}\chi)^*) dv \\ & - \int_S \delta\chi^* n \cdot u ds + \omega^2 (\sigma^2 - 1) \int_S \rho_0^i \delta\chi n \cdot u^* ds \\ & = \int_S (\psi^* + \omega^2 (\sigma^2 - 1) \rho_0^i \chi^*) n \cdot \delta u ds \\ & = 0 \end{aligned} \quad (2.5)$$

for a choice of:

$$\psi = -\omega^2(\sigma^2 - 1)\rho_0^i \chi \quad (2.6)$$

at the boundary without restricting the choice of $\delta\chi$ or δu . Therefore with ψ chosen as above we have demonstrated that:

$$\delta F = 0 \quad \rightarrow \quad \begin{array}{ll} \mathcal{L}\chi = 0 & \text{in } \nu \\ \text{and} & \\ n \cdot u = 0 & \text{on the boundary.} \end{array} \quad (2.7)$$

since $\delta F = 0$ only if both of these statements are true in equation (2.4) above. Thus for the functional F so chosen, we have demonstrated that a variational principle can be implemented with trial functions χ that do not need to independently satisfy the boundary condition. This boundary condition is then called a natural boundary condition.

The functional F in equation (2.1) can be rewritten, first by substituting for ψ , and \mathcal{L} to get:

$$F = \int_{\nu} \chi^* \nabla \cdot (\rho_0^i \Gamma \cdot \nabla \chi) d\nu - \omega^2(\sigma^2 - 1) \int_{\nu} \rho_0^i \chi^* n \cdot u ds. \quad (2.8)$$

Next substituting for u and using a vector theorem yields:

$$F = \int_V [\nabla \cdot (\chi^* \rho_0^i \Gamma \cdot \nabla \chi) - \nabla \chi^* \rho_0^i \Gamma \cdot \nabla \chi] dv - \int_S \rho_0^i \chi^* \mathbf{n} \cdot \Gamma \cdot \nabla \chi ds. \quad (2.9)$$

Then using the divergence theorem this becomes:

$$F = \int_S \chi^* \rho_0^i \mathbf{n} \cdot \Gamma \cdot \nabla \chi ds - \int_V \nabla \chi^* \rho_0^i \Gamma \cdot \nabla \chi dv - \int_S \rho_0^i \chi^* \mathbf{n} \cdot \Gamma \cdot \nabla \chi ds. \quad (2.10)$$

And we are left with:

$$F = - \int_V \rho_0^i \nabla \chi^* \cdot \Gamma \cdot \nabla \chi dv. \quad (2.11)$$

This is only true for the rigid fixed boundary condition; if free or deformable boundaries were being considered, then other terms would need to be added to the functional in equation (2.1) to account for these conditions.

2.3 Approximations to Produce the Poincaré Equation

Attention can now be given to the specific problem that A&T considered, which consisted of axisymmetric, inertial oscillations in a homogeneous, incompressible fluid-filled sphere. Several approximations can now be made to the governing subseismic wave equation. The homogeneous condition means that density is independent of location and consequently ρ_0^i is a constant. The incompressible condition implies that the acoustic wave speed is infinite, with $\alpha = \infty$. In reality

α is observed to be 8-10 km/s. This assumption makes the value of $B = \infty$, thus eliminating the third term in (1.27), and reduces the expression for Γ to Γ_p :

$$\Gamma_p = \{\sigma^2 \mathbf{1} - \mathbf{e}_3 \mathbf{e}_3 + i\sigma \mathbf{e}_3 \times \mathbf{1}\} \quad (2.12)$$

and since ρ_0^1 is now a constant, the SSWE (1.31) reduces to:

$$\nabla \cdot (\Gamma_p \cdot \nabla \chi) = 0 \quad (2.13)$$

which when expanded yields:

$$\sigma^2 \nabla^2 \chi - (\mathbf{e}_3 \cdot \nabla)^2 \chi = 0 \quad (2.14)$$

the well known Poincaré equation. The rigid fixed boundary condition ($\mathbf{u} \cdot \mathbf{n} = 0$) is still natural in the reduced expression and the functional can simply be written as:

$$F(\chi) = \int_V \nabla \chi \cdot \Gamma_p \cdot \nabla \chi \, dv \quad (2.15)$$

and the variational principle can be applied.

2.4 General Form of the Trial Function

Up to now, no consideration has been given to the construction of the trial function χ . The azimuthal dependence ϕ , can be separated out by writing:

$$\chi(\psi_1, \psi_2, m, \sigma) e^{im\phi} \quad (2.16)$$

with ψ_1 and ψ_2 any independent curvilinear coordinates orthogonal to ϕ : cylindrical coordinates (R, Z) have been adopted in this thesis. This form for χ is now substituted in the simplified functional, (2.15), and can be expanded as:

$$F = \sigma^2 \int_V |\nabla \chi|^2 dv - \int_V |\mathbf{e}_3 \cdot \nabla \chi|^2 dv + \int_V |\chi|^2 \left\{ \frac{m\sigma(\mathbf{e}_3 \times \mathbf{n}) \cdot \boldsymbol{\phi}}{R} \right\} ds \quad (2.17)$$

where R is the radial coordinate of cylindrical coordinates. This formulation explicitly demonstrates the azimuthal dependence of the third term which results from the application of the divergence theorem. It is also apparent that the separation of azimuthal dependence from χ has left the integrands in (2.17) in a form which clearly indicates that $\chi(R, Z, m, \sigma)$ can be treated as entirely real as all terms are squared. However, in order to consider the axisymmetric case, as A&T did using a stream function, m , the azimuthal number must be set equal to zero. The option to explore nonaxisymmetric cases demonstrates one advantage of the use of χ , over that of a stream function. The nonaxisymmetric case would involve the introduction of new oscillation modes and result in the modification of previously discovered modes due to the

presence of σ in the final term of the functional. Considering the axisymmetric case, $m = 0$, the functional (2.17) reduces to:

$$F = \int_V \{ \sigma^2 |\nabla \chi|^2 - |e_3 \cdot \nabla \chi|^2 \} dV. \quad (2.18)$$

The application of the variational principle can now be considered in more detail.

2.5 General Solution Method

In order to proceed with the variational principle, the trial function χ is represented as a linear combination of basis functions as:

$$\chi = \sum_p c_p \phi_p \quad (2.19)$$

where the c_p 's are constants to be determined, and the ϕ_p 's are the basis functions that must be selected. By substituting this general form of χ in the axisymmetric functional (2.18), the result is:

$$F = \sum_p \sum_q c_p c_q \int_V \{ \sigma^2 (\nabla \phi_p) \cdot (\nabla \phi_q) - (e_3 \cdot \nabla \phi_p)(e_3 \cdot \nabla \phi_q) \} dV. \quad (2.20)$$

This can be simplified by defining the matrix G_{pq} in the functional as:

$$F = \sum_p \sum_q c_p c_q G_{pq} \quad (2.21)$$

Now the variational principle can be applied by taking the variation of the functional with respect to constant, c_p to give:

$$\frac{\partial F}{\partial c_p} = 0, \text{ then } \sum_p c_p G_{pq} + \sum_q c_q G_{qp} = 0 \quad (2.22)$$

where the variation has now been transferred to the c_j constants. The matrix G is obviously symmetric, since all the terms involve squares of quantities. The problem thus reduces to:

$$\sum_p c_p G_{pq} = 0 \quad \text{or} \quad Gc = 0 \quad (2.23)$$

an eigenvalue/eigenvector problem where:

$$\det G = 0 \quad (2.24)$$

gives the eigenvalues, σ , and the constants, c_p , give the associated eigenvectors.

2.6 Specific Formulation of χ and Results

At first it would seem that any linearly independent set of functions could be used to construct the trial functions. For example, in spherical coordinates, (r and θ), χ might be constructed from:

$$\chi = \sum_{p=0}^P \sum_{n=0}^N c_{p,n} \sin \frac{p\pi r}{a} P_n^m(\cos\theta) \quad \text{where } n \geq |m| \quad (2.25)$$

with a as the radius to the core-mantle boundary. The different values of p explore the radial dependence and the various values of n explore the angular dependence. However, the problem as a whole must be carefully considered to ensure that an appropriate trial function is chosen.

When constructing a suitable trial function, it is helpful to compare the procedure used by A&T with our work. Since they considered the axisymmetric case for an incompressible fluid ($\alpha = \infty$), they were able to formulate the problem in terms of a stream function. A comparison can be made between this stream function ψ of A&T and the current formulation for the trial function χ by noting that displacement u is proportional to the velocity v used in A&T. The two are related since the displacement, u leads to a velocity of: $v = i \omega u$, which when equated with the velocity components from the stream function of A&T results in:

$$\sigma^2 R \frac{\partial \chi}{\partial R} = \frac{\partial \psi}{\partial Z} \quad \text{and} \quad R(\sigma^2 - 1) \frac{\partial \chi}{\partial Z} = -\frac{\partial \psi}{\partial R} \quad (2.26)$$

when expressed in cylindrical coordinates (R,Z). Thus a possible construction for the trial function χ is suggested by the formulation of ψ used by Aldridge (1972, p. 34). In order to compare results with those obtained by A&T, attention is confined to those axisymmetric modes compatible with the excitation method used in the experiments performed by A&T, namely:

$$\frac{\partial \chi}{\partial R} = 0 \quad \text{on} \quad R = 0 \quad \text{and} \quad \frac{\partial \chi}{\partial Z} = 0 \quad \text{on} \quad Z = 0. \quad (2.27)$$

These restrictions mean that there is no flow across the rotation axis and no flow across the equatorial plane respectively. Thus χ should have the form (in cylindrical coordinates) of:

$$\chi = \sum_{i=0}^N \sum_{j=\delta_{00}}^{N-i} c_{ij} R^{2i} Z^{2j} \quad (2.28)$$

where δ_{00} is the Kronecker delta symbol and N is sufficiently large to ensure that all the eigenfrequencies of interest are found. That this should be a successful form for χ to take is also apparent from the similarity to the trial function used by Aldridge (1967), which provides an exact solution to the problem in the homogeneous, incompressible case, the same as that being considered here.

It is important to note that cylindrical coordinates have been chosen here. While this may seem like an odd choice when working with a spherical boundary, a comparison of the expanded functional in the cylindrical:

$$F = \int_v \left[\sigma^2 \left(\frac{\partial \chi}{\partial R} \right)^2 + (\sigma^2 - 1) \left(\frac{\partial \chi}{\partial Z} \right)^2 \right] dv \quad (2.29)$$

and spherical coordinate systems:

$$F = \sigma^2 \int_v \left\{ \left(\frac{\partial \chi}{\partial r} \right)^2 + \frac{(1 - z^2)}{r^2} \left(\frac{\partial \chi}{\partial z} \right)^2 - \left[z \frac{\partial \chi}{\partial r} + \frac{(1 - z^2)}{r} \frac{\partial \chi}{\partial z} \right] \right\} dv \quad (2.30)$$

suggests that the equations are much more straightforward in cylindrical coordinates. This is due to the cylindrical symmetry imposed on the problem by rotation, which is more important than the spherical shape of the boundary. The actual integration of resulting expressions can be carried out in cylindrical or spherical coordinates, depending on which is simpler.

Since in this formulation the boundary condition is natural, no attempt is made to force the trial function to satisfy it. The trial function (2.28) is then substituted into the functional (2.29), resulting in:

$$F = \sigma^2 c - (1 - \sigma^2) f \quad (2.31)$$

where

$$c = \sum_{i=1}^N \sum_{j=0}^{N-1} \sum_{k=1}^N \sum_{l=0}^{N-k} c_{ij} c_{kl} i k \int_0^1 \int_0^{\sqrt{1-z^2}} R^{2(i+k-1)} Z^{2(j+l)} R dZ dR \quad (2.32)$$

and

$$f = \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} c_U c_M j l \int_0^1 \int_0^{\sqrt{1-z^2}} R^{2(i+k)} Z^{2(j+l-1)} R dZ dR \quad (2.33)$$

and a constant factor of 32π has been dropped, the integration limits result from normalizing the radius of the spherical boundary to unity. These terms can then be integrated in cylindrical coordinates to yield:

$$c = \sum_{i=1}^N \sum_{j=0}^{N-i} \sum_{k=1}^N \sum_{l=0}^{N-k} c_U c_M \frac{ik}{i+k} \sum_{r=0}^{i+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[2(j+l+r)+1]} \quad (2.34)$$

and

$$f = \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} c_U c_M \frac{2jl}{i+k+1} \sum_{r=0}^{i+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[4(j+l+r)^2-1]}. \quad (2.35)$$

The integration over the Z coordinate was performed using binomial coefficients since:

$$\begin{aligned} \sum_p \sum_q \int_0^1 z^{2p} (1-z^2)^q dz &= \sum_p \sum_q \sum_{r=0}^q \frac{q!(-1)^r}{r!(q-r)!} \int_0^1 z^{2r+2p} dz \\ &= \sum_p \sum_q \sum_{r=0}^q \frac{q!(-1)^r}{r!(q-r)!} \frac{1}{2(r+p)+1}. \end{aligned} \quad (2.36)$$

This integration can be performed in other equivalent ways that are detailed in Appendix A. Taking the variation of (2.31) with respect to the c_{ij} 's results in the eigenvalue problem that needs to be solved for the dimensionless frequency σ . By noting that c and f are symmetric with respect to the pairs of indices (i,j) and (k,l) , the resulting eigenvalue/eigenvector equations can be written as:

$$\begin{aligned} & \sigma^2 \sum_{k=1}^N \sum_{l=0}^{N-k} \left\{ \frac{ik}{i+k} \sum_{r=0}^{i+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[2(j+l+r)+1]} \right\} c_{lk} \\ & - (1-\sigma^2) \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} \left\{ \frac{2jl}{i+k+1} \sum_{r=0}^{i+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[4(j+l+r)^2-1]} \right\} c_{lk} = 0. \end{aligned} \quad (2.37)$$

The summation limits have been changed to reflect only those contributions that are non-zero. The non-zero contributions given by i and j correspond exactly to these limits. The restructuring of the limits allows for simplification of the calculations because values of $\sigma^2 = 0$ and 1 can be immediately eliminated, as the eigenfrequencies for this boundary value problem lie in the range $0 < \sigma < 1$. This is done by noting three distinct contributions to the eigenvalue/eigenvector equations where now the range of i and j do not exactly correspond to the range of k and l :

1. For $i = 0; j = 1$ to N

the contribution is:

$$\sum_{k=0}^{N-1} \sum_{l=1}^{N-k} \left\{ \frac{2l}{k+1} \sum_{r=0}^k \frac{k!}{r!(k-r)!} \frac{(-1)^r}{[4(j+l+r)^2-1]} \right\} c_{kl} = 0. \quad (2.38)$$

2. For $i = 1$ to N ; $j = 0$

the contribution is:

$$\sum_{k=1}^N \sum_{l=0}^{N-k} \left\{ \frac{k}{l+k} \sum_{r=0}^{l+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[2(l+r)+1]} \right\} c_{kl} = 0. \quad (2.39)$$

3. For $i = 1$; $j = 1$ to $N-i$

the contribution is:

$$\begin{aligned} & \sigma^2 i \sum_{k=1}^N \sum_{l=0}^{N-k} \left\{ \frac{k}{l+k} \sum_{r=0}^{l+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[2(j+l+r)+1]} \right\} c_{kl} \\ & - (1 - \sigma^2) j \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} \left\{ \frac{2l}{i+k+1} \sum_{r=0}^{l+k} \frac{(i+k)!}{r!(i+k-r)!} \frac{(-1)^r}{[4(j+l+r)^2-1]} \right\} c_{kl} = 0. \end{aligned} \quad (2.40)$$

In order to express these in matrix form, as $Gc = 0$, it is necessary to transform from the double index (i, j) to a single index (say p). This is accomplished with:

$$\begin{aligned} p &= j + \frac{i(2N+3-i)}{2} \\ q &= l + \frac{k(2N+3-k)}{2} \end{aligned} \quad (2.41)$$

and the problem can be expressed as:

$$\sum_i G_{ni} c_i = 0. \quad (2.42)$$

The eigenvalues are found from:

$$\det G = 0. \quad (2.43)$$

This is now an algebraic problem that can be solved using a computer program. The details of the program sp5.f are found in Appendix B. The frequency range searched was from zero to one. The program uses the IMSL subroutine DZREAL to search for the eigenvalues from the determinant. The value used for N was initially $N = 3$, then $N = 4$ and finally $N = 5$. The value $N = 5$ reproduced the results of A & T. Larger values of N had no effect on these eigenfrequencies, but of course permitted new ones to be found.

The results obtained from the program exactly match those given by A&T (Table I). This demonstrates the validity of the application of the variational principle as a method of solution for the Poincaré equation, when implemented with trial functions which take advantage of the 'natural' property of the rigid fixed boundary condition.

The next chapter will introduce a density profile into the equations as a better approximation of a realistic Earth model.

Table 1. Comparison of Dimensionless Eigenfrequencies Obtained by Aldridge and Toomre (1969) and from the Application of the Variational Principle to the Poincaré Equation.

<u>Aldridge and Toomre (1969)</u>	<u>Program with N = 3</u>	<u>Program with N = 4</u>	<u>Program with N = 5</u>
0.6547	0.6547	0.6547	0.6547
0.4688	0.4688	0.4688	0.4688
0.8302	0.8302	0.8302	0.8302
0.3631		0.3631	0.3631
0.6772		0.6772	0.6772
0.8998		0.8998	0.8998
0.2958			0.2958
0.5652			0.5652
0.7845			0.7845
0.9340			0.9340

CHAPTER 3

3.1 Introduction

In this chapter a density profile is included in the model of a neutrally stratified, rotating, self gravitating liquid sphere enclosed by a rigid fixed boundary to determine its effect on the inertial wave eigenfrequencies. The eigenvectors for these eigenfrequencies are also obtained as they will be used in the next chapter.

3.2 Density Profile (PREM)

In order to make the working model approach conditions in the Earth's core, the density profile chosen is that given by Dziewonski and Anderson (1981) in their Preliminary Reference Earth Model (PREM), for the outer core. The density expression is a third degree polynomial that is extended to the centre of the working model since the inner core is currently not being considered. There are three problems with this that must be addressed. The first is that the chosen density profile is not actually valid at the centre of a sphere because the second term in the polynomial does not ensure that gravity is zero there. However, this profile is valid everywhere else in the model core so its use is justified as a first approximation to actual core conditions. The second problem is that the stability parameter, β , resulting from this density profile is nonzero near the inner core boundary and near the outer core boundary. However, it is neutral in a major portion of the liquid core, so using $\beta = 0$ in the formulation of the problem

is justified for this initial approximation. Figure 1 demonstrates the range of values of $\beta(r)$ throughout the liquid core of the PREM model. The nonzero portions of $\beta(r)$ are most likely due to the choice by Dziewonski and Anderson (1981), of third degree polynomials for representing Earth parameters. Later, in Chapter 5, a density profile will be developed which is essentially neutral throughout the core. The third difficulty is the ill-posed nature of the problem when an inner core boundary is included (Stewartson & Rickard 1969). This is avoided in the present work by using a fluid sphere.

3.3 Variational Formulation of the Eigenvalue Problem (PREM Core)

The actual density profile used is:

$$\rho_o(u) = \sum_{m=1}^4 \rho_m u^{m-1} \quad (3.1)$$

where the coefficients of the polynomial expression from PREM have been renormalized to an average outer core radius of 3480 km and are given by:

$$\rho_1 = 12.5815$$

$$\rho_2 = -0.6903191$$

$$\rho_3 = -1.0868125$$

$$\rho_4 = -0.900929$$

where the units are gms/cm³. The dimensionless radius u is defined as: $u = r/b$ where b is the radius of the sphere representing the outer core boundary and r is the radius in the outer core which has been extended to the centre. The symbol m has been used as a summation index here

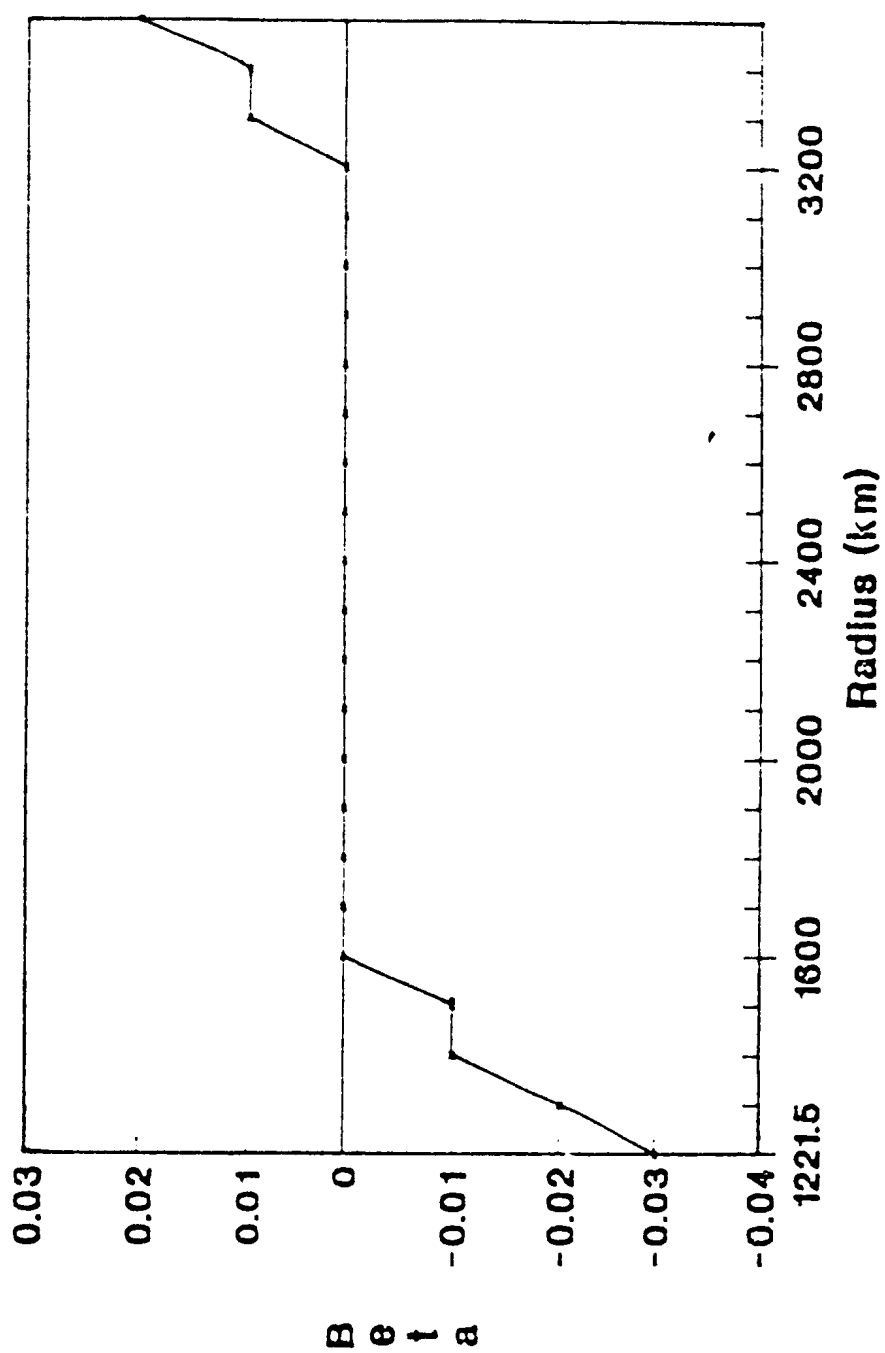


Figure 1. Variation of beta with radius in the liquid core. Values are from PREM.

and henceforth as it is no longer needed to stand for an azimuthal quantum number as our attention has been restricted to axisymmetric oscillations.

The addition of a density profile does not significantly change the mathematics of the problem.

The functional now becomes:

$$F = \sigma^2 \int_V \rho_0 |\nabla \chi|^2 dv - \int_V \rho_0 |e_3 \cdot \nabla \chi|^2 dv \quad (3.2)$$

where ρ_0 is the polynomial density expression given above and renormalized from PREM. This is analogous to equation (2.18) in Chapter 2. The boundary condition is still natural and does not need to be considered further. This formulation will give a quick check on the effect, upon the inertial wave eigenfrequencies calculated for a homogeneous liquid, of introducing compressibility via a neutral density profile.

The functional (3.2) above, can be expanded to:

$$F = \int_V (\sigma^2 \rho_0 (\frac{\partial \chi}{\partial R})^2 + (\sigma^2 - 1) \rho_0 (\frac{\partial \chi}{\partial Z})^2) dv \quad (3.3)$$

where the terms of ρ_0 are given in spherical coordinates and the derivatives are in cylindrical coordinates. By transforming the derivative expressions into spherical coordinates just for the integration of the expression, the mathematics of the problem are significantly simplified. Care must be taken to ensure that the density coefficients are properly normalized. The integration

results in:

$$\begin{aligned}
 F = & \sigma^2 \sum_{i=1}^N \sum_{j=0}^{N-i} \sum_{k=1}^N \sum_{l=0}^{N-k} d(i,j; k,l) c_{ij} c_{kl} \\
 & + (\sigma^2 - 1) \sum_{i=0}^{N-1} \sum_{j=1}^{N-i} \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} f(i,j; k,l) c_{ij} c_{kl}
 \end{aligned} \tag{3.4}$$

where

$$d(i,j; k,l) = ik \sum_{m=1}^4 \frac{\rho_m}{m+2(i+j+k+l)} \sum_{r=0}^{i+k-1} \frac{(i+k-1)! (-1)^r}{r! (i+k-1-r)! 2(j+l+r)+1} \tag{3.5}$$

and

$$f(i,j; k,l) = jl \sum_{m=1}^4 \frac{\rho_m}{m+2(i+j+k+l)} \sum_{r=0}^{i+k} \frac{(i+k)! (-1)^r}{r! (i+k-r)! 2(j+l+r)-1} \tag{3.6}$$

and ρ_m are the coefficients of the PREM polynomial for density in the outer core as given earlier.

The summation limits indicate the nonzero contributions. Again applying the variational principle results in an eigenvalue system similar to equation (2.37) in Chapter 2. The equations are now:

$$\sigma^2 d(i,j;k,l) c_{kl} - (1 - \sigma^2) f(i,j;k,l) c_{kl} = 0 \quad (3.7)$$

where d and f are defined above and the summations have been dropped for clarity. At no point in the summation do the denominators of these terms go to zero, so all terms can be calculated. These expressions can now be easily programmed by the addition of a loop in the computer program `sp5.f` to account for the density summation. The revised program `spden5.f` is found in Appendix C. The eigenfrequencies obtained from this program are shown in Table 2 and are compared with those obtained for the homogeneous model considered in Chapter 2. The frequency range that was searched by the program was between 0 and 1. The program uses the IMSL subroutine `DZREAL` to search for the eigenvalues from the determinate. The value of N used was $N = 5$, as the results from the previous chapter showed that all eigenvalues of interest were found using this value and we are presently interested in the modification of these eigenvalues.

Table 2. Comparison of Dimensionless Eigenfrequencies in an Homogeneous, Neutrally Stratified Sphere and in a Neutrally Stratified Sphere with a Density Profile of the Outer Core of PREM.

<u>Homogeneous</u>	<u>PREM</u>
.2958	.2976
.3631	.3653
.4688	.4714
.5652	.5660
.6547	.6572
.6772	.6779
.7845	.7847
.8302	.8306
.8998	.8998
.9340	.9340

These results indicate that the addition of a density profile actually increases the eigenfrequencies, more significantly at the lower end of the spectrum than at the higher end where there is little or no effect. This contradicts Friedlander (1987), who predicted a decrease in the values of the eigenspectrum with the addition of a density profile. This may be due to her choice of a reference state of an homogeneous, incompressible sphere which was perturbed to a non-neutral, compressible state. In fact one would expect that replacing a homogeneous incompressible liquid by a neutrally stratified compressible one would increase the eigenfrequencies, since the compressibility provides a small restoring force supplementing the dominant one due to the Coriolis effect. This thesis, in Chapter 5, will show that the addition of a non-neutral density profile is not a small perturbation and thus treating both compressibility and non-neutral stratification as perturbations is not an acceptable approach. The development followed here has as an initial model an homogeneous, incompressible sphere, as used in Chapter 2. This is then altered to the reference state, a neutrally stratified, compressible model as is developed in the present chapter. In the succeeding chapter a perturbation method will be applied to this reference state to determine the effect of non-neutral stratification.

3.4 Eigenvectors

At this point it is worthwhile to consider the eigenvectors associated with each eigenvalue for the case of a neutrally stratified, compressible model as these will be used in the next chapter. The trial function used:

$$x = \sum_{i=1}^N \sum_{j=\delta_0}^{N-1} c_{ij} R^{2i} Z^{2j} \quad (3.8)$$

when $N = 5$, results in 20 coefficients c_{ij} for the eigenvector. These are ordered using the double to single index given previously as equation (2.41). Thus,

$$c_p = c_{i, \frac{(2N-3-j)}{2}} \quad (3.9)$$

In a conventional eigensystem the eigenvalue/vector equations can be expressed as:

$$A x = \lambda x \quad (3.10)$$

where A is a square matrix, x is the eigenvector, and λ is the eigenvalue. While the formulation being considered here is not a conventional eigensystem it can be written in a form analogous to equation (3.10) by rewriting equation (3.7) as:

$$A c = \frac{1}{\sigma^2} B c \quad (3.11)$$

or in component form:

$$A_{pq} c_p = \frac{1}{\sigma^2} B_{pq} c_p \quad (3.12)$$

where A is formed from $d(l,j;k,l) + f(l,j;k,l)$, B from $f(l,j;k,l)$ and c is the eigenvector with p^{th} row c_p . Again equation (2.41) has been used to obtain the transformation between double and single indices. Unfortunately, the eigenvectors cannot be found using this formulation because the matrix A is very nearly singular. However, the eigenvectors can be successfully found by reformulating the problem.

The eigenequations can be written as:

$$\sum_{q=1}^Q G_{pq} c_q = 0 \quad (3.13)$$

where G_{pq} is obtained from equation (3.4) in analogy with equation (2.42). Note that because the eigenvalues have already been obtained there are only $Q-1$ parameters that can be still be obtained from the equations (3.13). Thus one of the coefficients of the eigenvectors must be found some other way. This is done by noting that these are free oscillations, i.e. of arbitrary amplitude, so there is a degree of freedom that must be constrained by choosing one of the coefficients of the eigenvectors. By making the first coefficient 1, the rest of the coefficients can then be found and they will be scaled to this initial choice. The eigenequations can then be rewritten as:

$$\sum_{q=2}^Q G_{pq} c_q = -G_{p1} \quad (3.14)$$

where $q = 2, 3, \dots, Q$ and letting $c_1 = 1$. This is now a linear system of equations that can be solved for the remaining eigenvector coefficients c_{q+1} after the appropriate eigenvalue has been substituted into the matrix G_{pq} . The system of equations is now one degree smaller than it previously was; the first row and column have been removed from the matrix. By letting:

$$p \rightarrow p + 1 \quad \text{and} \quad q \rightarrow q + 1 \quad (3.15)$$

where the arrows indicate that a value is replaced by another, then defining:

$$D_{pq} = G_{p+1, q+1} \quad \text{and} \quad d_q = c_{q+1} \quad (3.16)$$

the equations can be written as:

$$\sum_{q=1}^{Q-1} D_{pq} d_q = -G_{p+1, 1} \quad (3.17)$$

and the eigenvectors d_q solved for using a linear systems solution package. The program `spden5.f` was modified to include the IMSL subroutine `DLSASF` to solve the system and is given in Appendix D as `wspdlinsys5.f`. The resulting eigenvectors with their associated eigenfrequency

are shown in Table 3, where the first coefficient has been assigned a value of 1. As before, the eigenvalue range searched was between 0 and 1. Also the value of N is again $N = 5$. The program is similar to `spden5.f` described earlier, except that now the eigenvalues are used in the subroutine `DLSASF` to determine the eigenvectors. These eigenvectors will be used in the next chapter in the formulation of a perturbation solution.

Table 3. Eigenvectors Associated With Eigenfrequencies for a Neutrally Stratified Sphere With a Density Profile of the Outer Core From PREM.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.2976	1	1.0000
	2	-2.6500
	3	8.9165
	4	-10.7797
	5	4.397
	6	0.3537
	7	13.8265
	8	-83.6591
	9	98.7765
	10	-42.1415
	11	-0.1354
	12	-28.2156
	13	156.6337
	14	-81.6015
	15	-2.4620 x 10 ⁻²
	16	15.0030
	17	-86.7444
	18	-5.5746 x 10 ⁻³
	19	-0.1975
	20	-2.6719 x 10 ⁻²

Table 3. Continued.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.3653	1	1.0000
	2	-2.6002
	3	8.8842
	4	-10.7879
	5	4.4066
	6	0.3455
	7	18.0207
	8	-107.9571
	9	123.2600
	10	-49.5360
	11	-0.1268
	12	-42.0971
	13	236.6772
	14	-124.5127
	15	-1.9056 x 10 ⁻²
	16	25.1843
	17	-144.6092
	18	-3.5114 x 10 ⁻³
	19	-0.1846
	20	-3.2102 x 10 ⁻²
.4714	1	1.0000
	2	-2.1625
	3	7.3522
	4	-8.9393
	5	3.6524
	6	0.3198
	7	6.0140
	8	-40.7797
	9	61.7661
	10	-33.8769
	11	-0.1416
	12	4.1845
	13	-31.1128
	14	13.8618
	15	-1.7779 x 10 ⁻²
	16	-13.0839
	17	72.6477
	18	6.1467 x 10 ⁻³
	19	-0.2506
	20	-7.5419 x 10 ⁻³

Table 3. Continued.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.5660	1	1.0000
	2	-1.7370
	3	6.5811
	4	-8.2377
	5	3.3769
	6	0.1793
	7	20.3612
	8	-127.5279
	9	159.0353
	10	-67.2105
	11	-0.1031
	12	-31.7695
	13	175.4742
	14	-107.0989
	15	2.2310×10^{-2}
	16	5.5569
	17	-33.1810
	18	3.3977×10^{-2}
	19	-0.1821
	20	-4.7135×10^{-4}
.6572	1	1.0000
	2	-2.2776
	3	6.9005
	4	-8.0848
	5	3.3154
	6	0.7054
	7	-12.2665
	8	75.8173
	9	-85.8010
	10	22.2029
	11	-0.1848
	12	30.9130
	13	-185.3521
	14	129.8382
	15	-8.5156×10^{-2}
	16	-10.3549
	17	56.0833
	18	-5.0501×10^{-2}
	19	-0.4764
	20	-8.4625×10^{-2}

Table 3. Continued.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.6779	1	1.0000
	2	-2.0021
	3	6.3320
	4	-7.5182
	5	3.0836
	6	0.6069
	7	-5.6797
	8	35.0166
	9	-37.5676
	10	4.7975
	11	-0.1634
	12	17.7919
	13	-110.4150
	14	82.3259
	15	-5.9867 x 10 ⁻²
	16	-6.5822
	17	34.7164
	18	-3.1142 x 10 ⁻²
	19	-0.4394
	20	-7.4840 x 10 ⁻²
.7847	1	1.0000
	2	-0.9445
	3	3.2661
	4	-3.8223
	5	1.5024
	6	0.5064
	7	-0.8593
	8	10.5255
	9	-24.8698
	10	6.0642
	11	-0.1009
	12	8.6148
	13	-69.2478
	14	78.5027
	15	-6.3849 x 10 ⁻³
	16	-5.6108
	17	28.4398
	18	-7.6611 x 10 ⁻³
	19	-0.5220
	20	-0.1074

Table 3. Continued.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.8306	1	1.0000
	2	-0.4435
	3	2.1245
	4	-2.8513
	5	1.2077
	6	0.4549
	7	13.9078
	8	-102.6534
	9	167.3023
	10	-85.0624
	11	-5.8474 x 10 ²
	12	-25.8536
	13	165.1072
	14	-151.1843
	15	3.7774 x 10 ²
	16	10.6117
	17	-64.3893
	18	4.0071 x 10 ²
	19	-0.5644
	20	0.1422
.8998	1	1.0000
	2	2.4922
	3	-8.5904
	4	11.5842
	5	-5.3751
	6	0.3849
	7	7.8422
	8	-57.7494
	9	81.2868
	10	-39.1405
	11	0.3668
	12	-18.2271
	13	101.1813
	14	-83.3434
	15	0.2750
	16	3.0354
	17	-28.0091
	18	0.2005
	19	-1.6236
	20	-0.5825

Table 3. Continued.

Eigenvalue	Coefficient Number	Eigenvector Coefficients
.9340	1	1.0000
	2	-23.6395
	3	93.4954
	4	-130.7021
	5	61.2706
	6	2.1455
	7	24.2786
	8	-139.7413
	9	267.0663
	10	-183.7994
	11	-5.6188
	12	31.2635
	13	-161.8030
	14	230.8040
	15	-2.8322
	16	22.1556
	17	-50.5829
	18	-1.9131
	19	13.0792
	20	5.2476

CHAPTER 4

4.1 Introduction

A perturbation of the reference state developed in Chapter 3 to include non-neutral stratification is made here to determine the effect on the eigenfrequencies. A corrected version of equation (82) from Rochester (1989) is developed and used.

4.2 Perturbation Theory for the Eigenvalue Problem (Slightly Stable Core)

In Chapter 3 a solution to the problem:

$$\nabla \cdot (\rho_0 \Gamma \cdot \nabla \chi) = 0 \quad (4.1)$$

was found using the variational principle. This is the zeroth order problem given by Rochester (1989) where ρ_0 has been substituted for ρ_0^* . This problem can now be perturbed to determine the effect of a small non-zero stability parameter β . The value for the perturbed eigenfrequency will be found from:

$$\sigma = \sigma_0 + A \sigma_1 + A^2 \sigma_2 \quad (4.2)$$

where σ_0 is the eigenvalue found in Chapter 3, σ_1 and σ_2 are the eigenvalues from the first and second order perturbations, respectively. The value of A is defined by:

$$A = \frac{N_{\max}}{2\Omega}. \quad (4.3)$$

Since

$$\bar{N} = \frac{N}{N_{\max}} \quad (4.4)$$

then

$$\bar{N}^2 A^2 = \frac{-\beta g_0^2}{4\Omega^2 \alpha^2} \quad (4.5)$$

where here N is the Brunt-Väisälä frequency. The solution of the first order problem results in $\sigma_1 = 0$ as noted by both Friedlander (1987) and Rochester (1989). The second order perturbation is found directly from Rochester (1989) equation (81), with ρ_0^* replaced by ρ_0 :

$$\int_V \rho_0 \nabla \chi_0 \cdot \Gamma_2 \cdot \nabla \chi_0 \, dV = 0 \quad (4.6)$$

where

$$\Gamma_2 = 2\sigma_2\sigma_0 \mathbf{1} + \frac{\bar{N}^2}{\sigma_0^2(\sigma_0^2 - 1)} \mathbf{C}_0 \cdot \mathbf{C}_0 + i\sigma_2 \mathbf{e}_3 \times \mathbf{1}. \quad (4.7)$$

Substituting (4.7) into (4.6) results in 3 terms. The first of these is given by:

$$2\sigma_0\sigma_2 \int_V \rho_0 |\nabla \chi_0|^2 dv \quad (4.8)$$

and the second term gives

$$\frac{\bar{N}^2}{\sigma_0^2(\sigma_0^2 - 1)} \int_V \rho_0 |\mathbf{C}_0 \cdot \nabla \chi_0|^2 dv. \quad (4.9)$$

The third term initially gives

$$i\sigma_2 \int_V \rho_0 \nabla \chi_0 \cdot (\mathbf{e}_3 \times \nabla \chi_0) dv \quad (4.10)$$

which can be expanded as:

$$i\sigma_2 \int_V \rho_0 \nabla \cdot (\nabla \chi_0 \times \chi_0 \mathbf{e}_3) dv \quad (4.11)$$

since

$$\nabla \times \nabla \chi_0^* = 0. \quad (4.12)$$

This can be rewritten as:

$$i\sigma_2 \int_V (\nabla \cdot (\rho_0 \nabla \chi_0^* \times \chi_0 e_3) - \nabla \rho_0 \cdot (\nabla \chi_0^* \times \chi_0 e_3)) dv \quad (4.13)$$

and the divergence theorem is applied to yield:

$$i\sigma_2 \int_S \rho_0 \chi_0 n \cdot (\nabla \chi_0^* \times e_3) ds - i\sigma_2 \int_V \chi_0 \nabla \rho_0 \cdot (\nabla \chi_0^* \times e_3) dv \quad (4.14)$$

which, when rearranged is:

$$i\sigma_2 \int_S \rho_0 \chi_0 \nabla \chi_0^* \cdot (e_3 \times n) ds - i\sigma_2 \int_V \chi_0 \nabla \chi_0^* \cdot (e_3 \times \nabla \rho_0) dv. \quad (4.15)$$

The cross products in these two terms yield components of ϕ only. The dot product is then taken with the ϕ component of $\nabla \chi_0$ and results in:

$$\begin{aligned}
 & i \sigma_2 \int_s \rho_0 \chi_0 \left[\frac{-im \chi_0}{R} \right] \Phi \cdot (\epsilon_3 \times n) ds \\
 & - i \sigma_2 \int_v \chi_0 \left[\frac{-im \chi_0}{R} \right] \Phi \cdot \left(\epsilon_3 \times \frac{\rho_0 g_0}{a^2} \right) dv.
 \end{aligned} \tag{4.16}$$

In this instance however, $m = 0$, thus all the elements of the third term can be dropped from the formulation and the equation for the second order perturbation consisting solely of the first two terms is:

$$2 \sigma_0 \sigma_2 \int_v \rho_0 |\nabla \chi_0|^2 dv + \frac{1}{\sigma_0^2 (\sigma_0^2 - 1)} \int_v \bar{N}^2 \rho_0 |C_0 \cdot \nabla \chi_0|^2 dv = 0 \tag{4.17}$$

which when rearranged gives this expression for σ_2 :

$$\sigma_2 = \frac{-\frac{1}{\sigma_0^2 (\sigma_0^2 - 1)} \int_v \frac{\beta g_0^2}{4a^2 a^2} \rho_0 |C_0 \cdot \nabla \chi_0|^2 dv}{2 \sigma_0 \int_v \rho_0 |\nabla \chi_0|^2 dv} \tag{4.18}$$

where β is a constant. When expression (4.16) is included in the denominator, (4.18) becomes the corrected equation (82) in Rochester (1989).

In order to evaluate equation (4.18) the denominator is considered first as it is a simpler expression than the numerator. The denominator integral can be expanded as:

$$2 \sigma_0 \int_V \rho_0(r) \left[\left(\frac{\partial \chi}{\partial R} \right)^2 + \left(\frac{\partial \chi}{\partial Z} \right)^2 \right] dv \quad (4.19)$$

where χ is still given by equation (2.28) in Chapter 2 with $N = 5$. The appropriate expressions for derivatives of χ are substituted into the integral above and then the coordinates are changed to spherical coordinates to be compatible with the density expression. The density profile is that given by Dzierwonski and Anderson (1981) for the outer core, as used in Chapter 3 equation (3.1) and is already in spherical coordinates. The double to single index as given in the first element of equation (2.41) is still used.

With the whole denominator from equation (4.18) in spherical coordinates, the integral can be evaluated. Then equation (4.19) becomes:

$$4 \sigma_0 \sum_{l=0}^N \sum_{j=0}^{N-l} \sum_{k=0}^N \sum_{l=0}^{N-k} c_{lj} c_{kl} \left[\sum_{m=1}^4 \frac{\rho_m}{2(i+j+k+l)+m} \right] \cdot \left[\frac{ik \Gamma(j+l+\frac{1}{2}) \Gamma(i+k) + jl \Gamma(j+l-\frac{1}{2}) \Gamma(i+k+1)}{\Gamma(i+j+k+l+\frac{1}{2})} \right] \quad (4.20)$$

where $\Gamma(x)$ is the gamma function resulting from the evaluation of the z integrals. This expression for the denominator can now be easily programmed.

The more complicated numerator will now be considered in several segments. The first expression to be considered is:

$$|C_o \cdot \nabla \chi_o|^2 \quad (4.21)$$

where

$$C_o = \frac{[-\sigma_o^2 g_o + (e_3 \cdot g_o) e_3 + i \sigma_o e_3 \times g_o]}{g_o} \quad (4.22)$$

and χ is defined as above. Expanded in a mix of cylindrical and spherical coordinates, $C_o \cdot \nabla \chi_o$ gives:

$$C_o \cdot \nabla \chi_o = -\sigma_o^2 \sin \theta \frac{\partial \chi}{\partial R} + (1 - \sigma_o^2) \cos \theta \frac{\partial \chi}{\partial z}. \quad (4.23)$$

Substituting in for χ , converting to only spherical coordinates, squaring the expression, and evaluating the z integral yields:

$$\begin{aligned} \int_v |C_o \cdot \nabla \chi_o|^2 dz &= 4 \sum_{i=0}^N \sum_{j=0,10}^{N-i} \sum_{k=0}^N \sum_{l=0,10}^{N-k} c_{ij} c_{kl} \\ &\cdot (\sigma_o^4 i k - \sigma_o^2 (1 - \sigma_o^2) (il + jk) + (1 - \sigma_o^2)^2 jl) \\ &\cdot \{ u^{2(i+j+k+l-1)} \} \cdot \left\{ \frac{\Gamma(j+l+\frac{1}{2}) \Gamma(i+k+1)}{2 \Gamma(i+j+k+l+\frac{3}{2})} \right\} \end{aligned} \quad (4.24)$$

where the z integral has been evaluated as this is the only expression containing z terms. The remaining u integral for the numerator can now be written as:

$$\int \frac{\rho_0 g_0}{\alpha^2} u^{2(l+j+k+1-1)} u^2 du. \quad (4.25)$$

The expression for ρ_0 is that given earlier as equation (3.1). In the spirit of performing analytic integrations as far as possible, I shall not evaluate equation (4.25) numerically but instead develop polynomial expressions for g_0 and $1/\alpha^2$, and will rely on MACSYMA to evaluate complicated expressions. First an expression for g_0^2 will be developed using the definition for g_0 :

$$g_0 = \frac{4\pi G b}{u^2} \int_0^u \rho_0(u) u^2 du \quad (4.26)$$

where b is the core-mantle boundary radius making u dimensionless. Although (4.26) ignores the rotational contribution to gravity, the error involved is of the order 0.4%. Substituting the polynomial expression for ρ_0 in equation (4.26) and integrating yields:

$$g_0 = 4\pi G b \sum_{m=1}^4 \frac{\rho_m u^m}{m+2} \quad (4.27)$$

which can then be squared.

To develop a simple polynomial expression for $1/\alpha^2$, the expression from Dziewonski and Anderson (1981) for α^2 in the outer core is used to find values of $1/\alpha^2$, where the lower range of the radius has been extended to 0. These values are then used to develop the polynomial expression for $1/\alpha^2$ by using the IMSL subroutine DRCURV. This routine fits a polynomial curve to supplied data using the least squares method. A seventh degree polynomial was developed, which can be expressed as:

$$\frac{1}{\alpha^2} = \frac{1}{[\alpha(0)]^2} \sum_{j=1}^8 \alpha_j u^{j-1} \quad (4.28)$$

and that matches the original values to four significant figures of precision. The coefficients of this polynomial are given in Table 4.

Table 4. Coefficients of a Seventh Degree Polynomial that Approximates the Value of $1/a^2$.

Coefficient	Value
α_1	8.1914×10^{-3}
α_2	3.2968×10^{-3}
α_3	-1.6245×10^{-3}
α_4	5.5783×10^{-3}
α_5	-9.3306×10^{-3}
α_6	1.9870×10^{-2}
α_7	-1.8170×10^{-2}
α_8	7.5631×10^{-3}

The u integral, equation (4.25) is now a very extensive expression given by:

$$(4\pi Gb)^2 \int \left\{ \sum_{m=1}^4 \rho_m u^{m-1} \right\} \left\{ \sum_{r=1}^4 \frac{\rho_r u^r}{r+2} \right\} \left\{ \frac{1}{[\alpha(0)]^2} \sum_{s=1}^8 \alpha_s u^{s-1} \right\} \{ u^{2(i+j+k+l-1)} \} r^2 dr. \quad (4.29)$$

In order to reduce the risk of algebraic error, this expression was expanded and integrated using MACSYMA, an algebraic manipulation program. The resulting coefficients of u from the integration were exported from MACSYMA in FORTRAN format to be incorporated into the perturbation program.

The final expansion for the numerator is then:

$$\frac{8\beta\pi^2 G^2 b^2}{\Omega^2 \sigma_0^2 (\sigma_0^2 - 1)} \sum_{i=0}^N \sum_{j=\delta_{i0}}^{N-1} \sum_{k=0}^N \sum_{l=\delta_{k0}}^{N-k} c_{ij} c_{kl} (\sigma_0^4 ik - \sigma_0^2 (1 - \sigma_0^2) (il + jk) + (1 - \sigma_0^2)^2 jl) \\ \left\{ \frac{\Gamma(j+l+\frac{1}{2}) \Gamma(i+j+1)}{\Gamma(i+j+k+l+\frac{3}{2})} \right\} \left\{ \sum_{m=2}^{18} \frac{cu_m}{2(i+j+k+l) + m + 1} \right\} \quad (4.30)$$

where the cu_m 's are complicated expressions involving α_s and ρ_m resulting from the u integration that was evaluated using MACSYMA. These expressions can be found in Appendix E in the program Perturbation.f. By combining expressions (4.20) for the denominator and (4.30) for the numerator, a final expression for the perturbation, σ_2 , is obtained. These values are then added to the reference state eigenvalues to give the perturbed eigenfrequencies. This can now be easily

programmed using the eigenvalues, σ_n , and associated eigenvector elements, c_{ij} , from the reference state developed in Chapter 3. The program used, *Perturbation.f*, is given in Appendix E. The results from this program for various values of β are given in Table 5 below.

4.3 Discussion

The perturbation method clearly fails for the larger values of β . In fact, it is questionable even at low frequencies where $\beta = -0.0005$. For even smaller values of β , the spectrum varies only marginally from that considered previously, unfortunately values of β this small are uninteresting. For the perturbation treatment to be valid A , as given in equation (4.2), must be $\ll 1$. The larger values of β do not ensure this (A is then $\sim .15$, when $\beta = -0.0005$), so the perturbation method is inappropriate for them. However, the eigenvalues obtained for $\beta = -0.0001$ are valid, as they ensure $A \ll 1$. In order to determine how good the first order perturbation calculations could be, a second order perturbation analysis would be necessary. However, a comparison of these perturbation results for $\beta = -0.0001$ and results from Chapter 5 using a Galerkin method of solution demonstrate that the perturbation results are valid for this small value of the stability parameter.

Table 5. Comparison of Reference State Eigenvalues, σ_0 , and Eigenvalues, σ , Found From the Perturbation Method.

β	σ_0	σ
-0.0001	0.2976	0.2988
	0.3653	0.3653
	0.4714	0.4714
	0.5660	0.5661
	0.6572	0.6572
	0.6779	0.6779
	0.7847	0.7848
	0.8306	0.8307
	0.8998	0.9002
	0.9304	0.9346
-0.0001	0.2976	0.3090
	0.3653	0.3697
	0.4714	0.4738
	0.5660	0.5671
	0.6572	0.6580
	0.6779	0.6786
	0.7847	0.7856
	0.8306	0.8316
	0.8998	0.9043
	0.9340	0.9403

Table 5. Continued.

β	σ_0	σ
-0.0005	0.2976	0.3546
	0.3653	0.3871
	0.4714	0.4833
	0.5660	0.5716
	0.6572	0.6611
	0.6779	0.6817
	0.7847	0.7891
	0.8306	0.8359
	0.8998	0.9225
	0.9340	0.9655
-0.001	0.2976	0.4116
	0.3653	0.4088
	0.4714	0.4952
	0.5660	0.5772
	0.6572	0.6650
	0.6779	0.6855
	0.7847	0.7934
	0.8306	0.8413
	0.8998	0.9451
	0.9340	0.9970

Table 5. Continued.

β	σ_0	σ
-0.003	0.2976	0.6396
	0.3653	0.4958
	0.4714	0.5427
	0.5660	0.5997
	0.6572	0.6806
	0.6779	0.7008
	0.7847	0.8109
	0.8306	0.8627
	0.8998	1.0358
	0.9340	1.1230

CHAPTER 5

5.1 Introduction

In this chapter the Galerkin method is used to solve the SSWE. A density profile is developed that ensures a neutral stability parameter throughout the range of the fluid core. This density profile is utilized in the Galerkin solution to the SSWE. The Galerkin method is used to reproduce the results of Chapters 3 and 4. The method is then used to produce results for long-period axisymmetric oscillations when β is nonzero, i.e. small and negative.

5.2 Galerkin Method

The Galerkin method of solution is an approximate method for solving differential equations. We seek a solution of the partial differential equation:

$$\mathcal{L}\chi = 0 \tag{5.1}$$

where \mathcal{L} is a linear operator. The trial function χ can still be written as equation (2.28) found in Chapter 2 as was used for the variational principle. The ϕ_p 's must be members of a linearly independent set and have the orthogonality properties described below. Recognizing that the trial solution does not satisfy equation (5.1) exactly, the Galerkin procedure is to achieve this as nearly

as possible by requiring that:

$$\mathcal{L}(\sum_p c_p \phi_p) \quad (5.2)$$

be orthogonal to each ϕ_q over the domain of validity of equation (5.1). If the trial functions do not directly satisfy the boundary conditions they can be included using Lagrange multipliers; then the orthogonality requirement, in this case, can be written as:

$$\sum_p c_p \int_V \phi_q^* \mathcal{L} \phi_p dV + \sum_p c_p \int_S \psi_q^* \mathbf{n} \cdot \mathbf{u} ds = 0 \quad (5.3)$$

where the ψ_q 's are the Lagrange multipliers.

This leads to the following set of equations, for determining the corresponding c_q :

$$\sum_p G_{qp} c_p = 0 \quad (5.4)$$

where

$$G_{qp} = \int_V \phi_q^* \mathcal{L} \phi_p dV \quad (5.5)$$

The eigenvalues, which are dispersed throughout the operator \mathcal{L} , can be solved for if the determinant of G_A equals zero. Unfortunately, this matrix will not in general be symmetric. This is the theory for the Galerkin method which will be applied to the SSWE.

5.3 Application to the SSWE

When the functional for the SSWE is written in its full form with no approximations as

$$F = \int_V \rho_0^i \nabla \chi^* \cdot \Gamma \cdot \nabla \chi \, dV \quad (5.6)$$

where Γ and its components are defined by equations (1.27 - 1.30) in Chapter 1, there is a potential problem with solving the integral. When $B = 0$, the integral becomes improper and the SSWE,

$$\nabla \cdot (\rho_0^i \Gamma \cdot \nabla \chi) = 0 \quad (5.7)$$

becomes singular. As we are only concerned with negative B values this restriction applies to the following frequency ranges:

- i) in a weakly stably stratified region ($0 < N \leq 1$) when $0 < \sigma^2 \leq (N^2)_{\max}$
- ii) in a strongly stably stratified region ($N > 1$) when $0 < \sigma^2 < 1$ (Wu & Rochester 1990).

Previously this has not been a problem, as the term involving B has disappeared for one reason

or another. In Chapter 2, the density is a constant and the compressional wave speed is taken as infinite, thus eliminating the term containing B from consideration. This formulation produces solutions for pure inertial waves. The inclusion of a neutral radial density profile in Chapter 3 does not require the consideration of the B term, nor does the perturbation performed in Chapter 4. The inclusion of the neutral density profile did not introduce new frequencies, it only shifted those found in Chapter 2. The perturbation method similarly only caused a shift of the results of Chapter 3. However, because both B and C contain σ , the inclusion of this term will introduce new frequencies in addition to altering those already found.

In order to alleviate the difficulties caused by the possibility of the B term equalling zero, the SSWE is rewritten and the Galerkin method is used to solve it. The following definitions are made to facilitate rewriting the SSWE:

$$D = \frac{\beta B}{\alpha^2} = \omega^2(\sigma^2 - 1) + \frac{\beta}{\alpha^2} E \quad (5.8)$$

$$F = -\sigma^2 g_0 + (\epsilon_3 \cdot g_0) \epsilon_3 \quad (5.9)$$

and

$$E = \sigma^2 g_0^2 - (\epsilon_3 \cdot g_0)^2 \quad (5.10)$$

Making these substitutions and multiplying through by D^2 , the SSWE (5.7) can be written as:

$$\begin{aligned}
 & D^2 \nabla \cdot (\rho_0^1 \Gamma_p \cdot \nabla \chi) - \frac{\beta D}{\alpha^2} \nabla \cdot (\rho_0^1 C \cdot C \cdot \nabla \chi) \\
 & = \omega^2 (\sigma^2 - 1) \rho_0^1 C \cdot \nabla \chi \cdot F \cdot \nabla \left(\frac{\beta}{\alpha^2} \right) - \left(\frac{\beta}{\alpha^2} \right)^2 \rho_0^1 C \cdot \nabla \chi \cdot F \cdot \nabla E
 \end{aligned} \tag{5.11}$$

where Γ_p as defined in equation (2.12) has been used. As was done in Chapter 4, this formulation of the SSWE can be perturbed. However, in this case the value of B can equal zero and there is no constraint on the frequency range where the perturbation method is valid. The perturbation of order β/α^2 on the rewritten SSWE, equation (5.11) above, results in:

$$\sigma_2 = \frac{\frac{1}{4\alpha^2 \sigma_0^2 (\sigma_0^2 - 1)} \int_V \rho_0^1 g_0^2 \frac{\beta}{\alpha^2} |C \cdot \nabla \chi_0|^2 dv}{2 \sigma_0 \int_V \rho_0^1 |\nabla \chi_0|^2 dv + i \int_V \rho_0^1 \nabla \chi_0 \cdot (\epsilon_3 \cdot \nabla \chi_0) dv} \tag{5.12}$$

which is the same result as found in Chapter 4. Thus the values found in Chapter 4 do not need to be altered to consider the case when $B = 0$.

Attention can now be focused on the expression for the modified density profile, ρ_0^1 . This was introduced by Friedlander (1988) and expressed by Rochester (1989) as:

$$\begin{aligned}
 \rho_0^1 &= \rho_0 e^{-\int \frac{1}{\alpha^2} dW_0} \\
 W_0 &= V_0 - \frac{1}{2} |\Omega \times r|^2
 \end{aligned} \tag{5.13}$$

where W_0 is the gravity potential. This can be used to rewrite equation (5.11) since:

$$\nabla \rho_0^1 = \frac{g_0}{a^2} \rho_0^1 \quad (5.14)$$

Then the SSWE becomes:

$$\begin{aligned} D^2 \nabla \cdot (\Gamma_p \cdot \nabla \chi) + D^2 \frac{g_0}{a^2} \cdot \Gamma_p \cdot \nabla \chi - \frac{\beta D}{a^2} \nabla \cdot (C^* C \cdot \nabla \chi) \\ - \frac{\beta D}{a^2} \frac{g_0 \cdot C^*}{a^2} C \cdot \nabla \chi = D C \cdot \nabla \chi F \cdot \nabla \left(\frac{\beta}{a^2} \right) - \frac{\beta}{a^2} C \cdot \nabla \chi F \cdot \nabla \left(\frac{\beta E}{a^2} \right) \end{aligned} \quad (5.15)$$

The Galerkin method, using the orthogonality requirement expressed in equation (5.3) can now be applied to this formulation of the SSWE resulting in:

$$\begin{aligned} \sum_p c_p \int_V \rho_0 [D \nabla \phi_p \cdot (D \Gamma_p - \frac{\beta}{a^2} C^* C) \cdot \nabla \phi_p + 2 \frac{\beta}{a^2} \phi_p \cdot \nabla E \cdot (D \Gamma_p - \frac{\beta}{a^2} C^* C) \cdot \nabla \phi_p \\ + 2 \omega^2 (\sigma^2 - 1) \frac{\beta}{a^2} \nabla \cdot (\phi_p \cdot \frac{E}{g_0} C \cdot \nabla \phi_p g_0) + \omega^2 (\sigma^2 - 1) \frac{\beta}{a^2} (D + 2(1 - \beta) \frac{E}{a^2}) \phi_p \cdot C \cdot \nabla \phi_p] dv \\ - \omega^2 (\sigma^2 - 1) \int_S \{ \psi_p + [\omega^2 (\sigma^2 - 1) - \frac{\beta E}{a^2}] \phi_p \} \frac{C \cdot \nabla \phi_p}{g_0} ds = 0 \end{aligned} \quad (5.16)$$

where u in the boundary condition,

$$n \cdot u = 0 \quad \text{on } S \quad (5.17)$$

has been rewritten using the definitions in equations (5.8 - 5.9) and ψ_q are Lagrange multipliers used to include the boundary condition. The other terms in the surface integral come from the rearrangement of the expression using the divergence theorem. This surface integral can be eliminated by choosing the ψ_q such that:

$$\psi_q + [\omega^2(\sigma^2 - 1) - \frac{\beta E}{\alpha^2}] \phi_q = 0 \quad (5.18)$$

Thus, in a sense $C \cdot \nabla \chi = 0$ is a natural boundary condition for the Galerkin procedure, in the sense that the trial functions used to implement the latter need not themselves be made to satisfy the boundary condition a priori.

When $\beta = 0$, the Galerkin equations reduce to a slightly rearranged Poincaré problem as was considered earlier in Chapter 2:

$$\sum_p c_p \int_V \rho_0 [D \nabla \phi_p \cdot (D \Gamma_p \cdot \nabla \phi_p)] dv = 0 \quad (5.19)$$

where, when $\beta = 0$, $D = \omega^2 (\sigma^2 - 1)$ is just a constant. The additional terms in equation (5.16) that occur when β is slightly negative account for additional frequencies (gravity waves) and the modification of existing ones due to the presence of negative buoyancy ($\beta < 0$). To solve for the eigenvalues, equation (5.16) is written as:

$$\sum_p G_{pq} c_p = 0 \quad (5.20)$$

and the eigenvalues are found from:

$$\det G = 0. \quad (5.21)$$

Unfortunately, G is not a symmetric matrix. At this point the expressions in G_{ρ} can be expanded and programmed for solution by the computer. But first a closer look is needed at the density profile to be used.

5.4 Density Profile

As was discussed in Chapter 3, the density profile previously used from PREM does not lead to a core exactly neutrally stratified throughout. This discrepancy will now be greatly reduced by the development of a density profile that ensures that the stability parameter, β , is less than 1 part in 10^3 throughout the entire spherical model core range. Two ways of accomplishing this will be briefly discussed here: a least squares approach and an orthogonality relationship.

5.4.1 Least Squares

The least squares approach requires that the difference between β , as defined by the density gradient formula (1.15), and the prescribed value β_p , is minimized over the core in a least squares

sense, i.e. we minimize the expression:

$$F = \int_0^1 \left[1 - \beta_p + \left(\frac{u^2}{\rho_0 r_0 b} \right) \frac{d\rho_0}{du} \right]^2 u^2 du \quad (5.22)$$

where the normalized radius defined $u = r/b$ has been used.

The expression for α^2 from the PREM model can be used if it is assumed valid when extended to the centre of the model as it is a smooth function in the inner core range. The density is considered to be a polynomial that can be expanded as:

$$\rho_0 = \sum_{m=1}^M \rho_m u^{m-1} \quad (5.23)$$

and the value of ρ_1 will be constrained by the mass enclosed within the sphere being considered.

The fact that gravity equals zero at the centre of the sphere requires that the value of ρ_2 equal zero. In order for F to be a minimum,

$$\frac{\partial F}{\partial \rho_i} = 0 \quad \text{for } i = 3, N. \quad (5.24)$$

These equations then form the basis for an iterative process. The starting values for the polynomial of ρ_0 are taken from PREM with $\rho_2 = 0$. At each stage of the iteration the values of the coefficients ρ_n are recalculated using values of ρ_0 , g_0 obtained from the ρ_n of the previous stage. If β becomes small enough or values of the ρ_n 's no longer change, the iterative process is halted. The coefficient values for a prescribed value $\beta_p = 0$ using the least squares method are found in Table 6, and the values of β for various locations in the model sphere are in Table 7.

Table 6. Coefficients ρ_m of Neutral Density Profile Fitted by (a) Least Squares and (b) Orthogonality Relation.

Coefficient index m	(a)	(b)
1	12.365415	12.365415
2	0.0	0.0
3	-2.1355473	-2.1357751
4	-0.5724153	-0.5687766
5	0.4755506	0.4511777
6	-0.2017774	-0.1185908
7	0.0276528	-0.1303701
8	-0.2573055	-0.0878988
9	0.2394251	0.1435077
10	-0.0905760	-0.0682682

Table 7. Comparison of Actual Values of β for Neutral Density Profile Fitted by (a) Least Squares and (b) Orthogonality Relation.

Normalized Radius	(a)	(b)
0.0	0.10418×10^{-3}	-0.17078×10^{-5}
0.05	0.14402×10^{-4}	-0.94609×10^{-5}
0.1	-0.64605×10^{-5}	-0.72470×10^{-5}
0.15	-0.41660×10^{-5}	-0.48525×10^{-5}
0.2	0.10167×10^{-5}	-0.45842×10^{-5}
0.25	0.28317×10^{-5}	-0.55451×10^{-5}
0.3	0.15468×10^{-5}	-0.61821×10^{-5}
0.35	-0.59195×10^{-6}	-0.55616×10^{-5}
0.4	-0.17503×10^{-5}	-0.37353×10^{-5}
0.45	-0.13720×10^{-5}	-0.15112×10^{-5}
0.5	-0.53384×10^{-7}	0.85017×10^{-7}
0.55	0.11062×10^{-5}	0.39176×10^{-6}
0.6	0.12656×10^{-5}	-0.49905×10^{-6}
0.65	0.36728×10^{-6}	-0.17233×10^{-5}
0.7	-0.81458×10^{-6}	-0.20767×10^{-5}
0.75	-0.11879×10^{-5}	-0.81707×10^{-6}
0.8	-0.25509×10^{-6}	0.14805×10^{-5}
0.85	0.10813×10^{-5}	0.26582×10^{-5}
0.9	0.78620×10^{-6}	0.37067×10^{-6}
0.95	-0.14095×10^{-5}	-0.32568×10^{-5}
1.0	0.37869×10^{-5}	0.85257×10^{-5}

5.4.2 Orthogonality Relationship

The development of an orthogonality relationship is similar to the least squares method. Now, however, we require that the difference between β , as defined by the density gradient formula (1.15), and the prescribed value β_p be orthogonal to the lowest $M-2$ powers of the normalized radius, i.e. that

$$\int_0^1 \left[1 - \beta_p + \left(\frac{c^2}{\rho_0 g_0 b} \right) \frac{d\rho_0}{du} \right] u^{m-1} du = 0 \quad (5.25)$$

for $m = 1, \dots, M-2$. Again ρ_1 is known from the total mass constraint and $\rho_2 = 0$. The iterative process is again used and the results for this method are shown in Tables 6 and 7. The coefficient values for the orthogonality relationship give a density profile that is neutral to within 1 part in 10^5 , which is essentially neutral throughout the entire core range. Thus this is the density profile to be used in the Galerkin method when equation (5.16) is invoked, in the case of a neutral ρ_0 . This density profile shall be referred to as $\rho_0 = \text{NEUT}$.

5.5 Expansion of Equation (5.16)

Equation (5.16) can now be expanded and the elements of the matrix G_{ϕ} determined. At this point only those terms without azimuthal (ϕ) dependence are considered, as has been done in the previous Chapters. After a substantial amount of algebra G_{ϕ} can be rewritten as:

$$G_{qp} = 4\sigma^2 (\sigma^2 - 1) H_{qp} \quad (5.26)$$

where

$$\begin{aligned} H_{qp} = & \sigma^0 (A_{qp} + \gamma \beta f B_{qp}) + \sigma^4 [C_{qp} + \gamma \beta f D_{qp} + \beta f E_{qp} + \gamma \beta f^2 (2 - \beta) G_{qp}] \\ & + \sigma^2 [P_{qp} + \gamma \beta f Q_{qp} + \beta f R_{qp} + \gamma \beta f^2 (2 - \beta) T_{qp} + \beta^2 f^2 U_{qp}] \\ & + [\beta f V_{qp} + \gamma \beta f^2 (2 - \beta) X_{qp} + \beta^2 f^2 Y_{qp}] \end{aligned} \quad (5.27)$$

and where

$$\gamma = \frac{\Omega^2}{2\pi G \rho_0(0)} \quad (5.28)$$

and

$$f = \frac{4\pi^2 G^2 [\rho_0(0)]^2 b^2}{\Omega^2 [\alpha(0)]^2} \quad (5.29)$$

G is the gravitational constant, $1/\alpha^2$ is defined by equation (4.28), $\alpha(0)$ is the compressional p-wave speed evaluated at $u = 0$, similarly $\rho(0)$ is the density evaluated at $u = 0$. For brevity in defining the matrix elements $A_{qp} \dots Y_{qp}$ we introduce $I(i, j)$:

$$I(i, j) = \int_0^1 z^{2i} (1 - z^2)^j dz = \frac{\Gamma(i + \frac{1}{2}) \Gamma(j + 1)}{2 \Gamma(i + j + \frac{3}{2})} \quad (5.30)$$

where p, q are related to i, j, k , and l by (2.41). Then the individual matrices comprising H_Ψ are as follows:

$$A_{qp} = - \sum_{m=1}^M \frac{\rho_m}{2(i+j+k+l)+m} [ik I(i+k-1, j+l) + jl I(i+k, j+l-1)] \quad (5.31)$$

$$B_{qp} = - \sum_{m=1}^M \sum_{n=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \alpha_s (i+j)}{(n+2)[2(i+j+k+l)+m+n+s]} I(i+k, j+l) \quad (5.32)$$

$$C_{qp} = \sum_{m=1}^M \frac{\rho_m}{2(i+j+k+l)+m} [ik I(i+k-1, j+l) + 2jl I(i+k, j+l-1)] \quad (5.33)$$

$$D_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \alpha_s (i+2j)}{(n+2)[2(i+j+k+l)+m+n+s]} I(i+k, j+l) \quad (5.34)$$

$$\begin{aligned} E_{qp} = & \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s}{(n+2)(r+2)[2(i+j+k+l)+m+n+r+s-1]} \\ & \{-i(2i+3k-1) I(i+k-1, j+l+1) + i(2i+k-1) I(i+k-1, j+l) \\ & + [3l(i+j) + j(4i+2j+3k-1) + 2i+2j] I(i+k, j+l) - 2jl I(i+k, j+l-1)\} \end{aligned} \quad (5.35)$$

$$G_{qp} = - \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \sum_{t=1}^M \sum_{w=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s \rho_t \alpha_w}{(n+2)(r+2)(t+2)} \quad (5.36)$$

$$\left(\frac{(i+j)I(i+k, j+l)}{2(i+j+k+l)+m+n+r+s+t+w-1} \right)$$

$$P_{qp} = - \sum_{m=1}^M \frac{\rho_m}{2(i+j+k+l)+m} [jl I(i+k, j+l-1)] \quad (5.37)$$

$$Q_{qp} = - \sum_{m=1}^M \sum_{n=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \alpha_s j}{(n+2)[2(i+j+k+l)+m+n+s]} I(i+k, j+l) \quad (5.38)$$

$$R_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s}{(n+2)(r+2)[2(i+j+k+l)+m+n+r+s-1]} \quad (5.39)$$

$$\{ i(2i+2k-1) I(i+k-1, j+l+2) - i(2i-1) I(i+k-1, j+l-1) \\ - [j(2j+2k+2l-1) + 2i(2j+l) + 4(i+j)] I(i+k, j+l+1) \\ - [l(i+2j) + j(2i+2j+3k-1)] I(i+k, j+l) + 2jl I(i+k, j+l-1) \}$$

$$T_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \sum_{t=1}^M \sum_{w=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s \rho_t \alpha_w}{(n+2)(r+2)(t+2)} \quad (5.40)$$

$$\left(\frac{(i+j)I(i+k, j+l+1) + j I(i+k, j+l)}{2(i+j+k+l)+m+n+r+s+t+w-1} \right)$$

$$U_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \sum_{t=1}^M \sum_{v=1}^M \sum_{w=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s \rho_t \rho_v \alpha_w}{(n+2)(r+2)(t+2)(v+2)} \quad (5.41)$$

$$\left(\frac{-ik I(i+k-1, j+l+1) + (jk+il+jl) I(i+k, j+l) - jl I(i+k, j+l-1)}{2(i+j+k+l) + m+n+r+s+t+v+w-2} \right)$$

$$V_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s}{(n+2)(r+2)[2(i+j+k+l) + m+n+r+s-1]} \quad (5.42)$$

$$([j(2i+2j+2k+2l-1) + 4j] I(i+k, j+l+1) - (jl+2j) I(i+k, j+l))$$

$$X_{qp} = - \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \sum_{t=1}^M \sum_{v=1}^M \sum_{w=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s \rho_t \rho_v \alpha_w}{(n+2)(r+2)(t+2)} \quad (5.43)$$

$$\left(\frac{j I(i+k, j+l+1)}{2(i+j+k+l) + m+n+r+s+t+w-1} \right)$$

$$Y_{qp} = \sum_{m=1}^M \sum_{n=1}^M \sum_{r=1}^M \sum_{s=1}^8 \sum_{t=1}^M \sum_{v=1}^M \sum_{w=1}^8 \frac{\rho_m \rho_n \rho_r \alpha_s \rho_t \rho_v \alpha_w}{(n+2)(r+2)(t+2)(v+2)} \quad (5.44)$$

$$\left(\frac{ik I(i+k-1, j+l+2) - [jl+il+jk+2(i+j)] I(i+k, j+l+1) + (jl+2j) I(i+k, j+l)}{2(i+j+k+l) + m+n+r+s+t+v+w-2} \right)$$

The summation values deserve some attention. The sums over m, n, r, t, v all go from 1 to M , where M is either 4 or 10, depending on which density profile is selected, either $\rho_o = \text{PREM}$ with values on page 40, or $\rho_o = \text{NEUT}$ with values from Table 6. The sums over s and w all go from 1-8, as this is the polynomial approximation for $1/\alpha^2$, and is given by equation (4.28). The function $I(p, q)$ results from integration over the z terms, such as are detailed in Appendix

A. A value of $N=5$ has been used in the trial function (2.28) to produce the results presented here.

The expression for H_{ω} can now be programmed. To facilitate changing parameters, two programs were actually used. The first, biggerbetas.f, calculates the various elements of the matrices inside H_{ω} , and is found in Appendix F. The second, called readbb.f also in Appendix F, evaluates H_{ω} at an interval spacing of approximately .008 and determines where $\det H$ changes sign. These sign changes are then used to zero in on the eigenfrequencies. These particular parameters were chosen as the results of the previous chapters are accurately reproduced without taking exorbitant amounts (> 150 hours) of computer time. The program does take a substantial amount of time (> 6 hours) to run even with the use of a computer supporting the vector capabilities of FORTRAN, which is recommended (e.g. CONVEX C-1).

5.6 Results

Several different values of β were used, and the density polynomials, $\rho_s = \text{PREM}$ and $\rho_o = \text{NEUT}$ were used in programming the Galerkin solution. The various configurations are considered individually with their results.

5.6.1 $\beta = 0$, $\rho_s = \text{PREM}$, and $\rho_o = \text{NEUT}$

This comparison was made to see if the results using the Galerkin method would match the results

found in Chapter 3 for the different density profiles. The results are listed in Table 8. A comparison of values shows only one minor difference with previous results when $\rho_c = \text{NEUT}$. The values obtained when $\rho_c = \text{NEUT}$ are preferable because $\rho_c = \text{PREM}$ is not valid in a sphere, as it does not ensure that gravity is zero at the sphere's centre. However, since the values match so closely it is apparent that the approximation of using $\rho_c = \text{PREM}$ was acceptable to use. Thus as a first approximation, the variational principle can be used with $\rho_c = \text{PREM}$ to obtain the inertial eigenfrequencies in a fluid filled sphere. These results also demonstrate that the $\rho_c = \text{NEUT}$ density profile does not significantly alter the response of the model core.

Table 8. Comparison of Dimensionless Eigenfrequencies Calculated Using the Variational Principle (VP) and the Galerkin Method.

σ from Variational Principle $\beta = 0, \rho_0 = \text{PREM}$	σ from Galerkin Method $\beta = 0, \rho_0 = \text{PREM}$	σ from Galerkin Method $\beta = 0, \rho_0 = \text{NEUT}$
0.2976	0.2976	0.2976
0.3653	0.3653	0.3653
0.4714	0.4714	0.4714
0.5660	0.5660	0.5660
0.6572	0.6572	0.6572
0.6779	0.6779	0.6778
0.7847	0.7847	0.7847
0.8306	0.8306	0.8306
0.8998	0.8998	0.8998
0.9340	0.9340	0.9340

5.6.2 $\beta < 0$, $|\beta| \leq 0.0001$, $\rho_c = \text{PREM}$

This selection of parameters was made to compare the results of the Galerkin method with those of the perturbation method. Since the Galerkin method used here will give a direct calculation of the result, an exact match of the eigenvalues from the perturbation method is not expected. The results are shown in Table 9. It should be noted that due to the presence of gravity modes, new eigenfrequencies could appear in the Galerkin calculation since β is non-zero. Presumably the gravity modes associated with these values of β have periods so long as to lie outside the frequency range examined for zeros of $\det H$. However, as seen in Table 9, for $\rho_c = \text{PREM}$ this is not the case. The comparison has been made only between the smallest values of β as the perturbation method is known to fail for the larger values. It should be noted that for the extremely small values of $|\beta| = 0.00001$ the resulting eigenvalues are all very close to the results for $\beta = 0$. Thus an extremely small stability parameter will be indistinguishable from $\beta = 0$ for this model of the core. It must be noted here that the density profile $\rho_c = \text{PREM}$ is not actually valid for nonzero β , as it implies that $\beta = 0$ throughout most of the core. This density profile was used as a to provide a comparison with the previously calculated perturbation results.

Table 9. Comparison of Dimensionless Eigenfrequencies Calculated using the Perturbation and Galerkin Methods, with $\rho_s = \text{PREM}$.

$\beta = -0.00001$		$\beta = -0.0001$	
Perturbation	Galerkin	Perturbation	Galerkin
0.2988	0.2978	0.3090	0.2992
0.3658	0.3655	0.3697	0.3667
0.4717	0.4715	0.4738	0.4725
0.5661	0.5661	0.5671	0.5670
0.6573	0.6573	0.6580	0.6581
0.6779	0.6779	0.6786	0.6787
0.7848	0.7848	0.7856	0.7855
0.8307	0.8306	0.8316	0.8313
0.9002	0.8999	0.9043	0.9005
0.9346	0.9340	0.9403	0.9346

5.6.3 $\beta < 0$, $|\beta| \geq 0.0001$, $\rho_0 = \text{NEUT}$

The first thing to notice about the results in Table 10 is the appearance of new frequencies. Previous results have all yielded only 10 eigenfrequency values within the frequency range searched for zeroes of $\det H$. The first new frequency to appear occurs when $\beta = -0.0001$, however, the period of this extra mode is close to 12 hours indicating that it is not a newly appearing gravity mode. Most likely this is another inertial mode that was missed in the calculation of $\det H$. This is possible due to the high density of inertial modes in the frequency domain as the dimensionless frequency approaches the value 1 from below. Thus it is possible as the modes close to the dimensionless frequency 1 are shifted due to different β values they will start to appear as distinct frequencies. Another possibility is that $\rho_0 = \text{NEUT}$ is a better representation of density by virtue of its larger number of coefficients.

The periods of the expected newly appearing gravity waves can be quickly estimated by ignoring rotation and noting that the period will be given by:

$$\frac{2\pi}{N_{\max}} \quad (5.45)$$

where for uniform values of β

$$N_{\max} = (-\beta)^{\frac{1}{2}} \left(\frac{g_0}{\alpha} \right)_{\max} \quad (5.46)$$

thus for a $\beta = -0.0001$ the minimum period would be near 130 hours and would yield a σ value of .09231. But the actual minimum value should be lower than this due to the presence of rotation adding the Coriolis effect as an additional restoring force to negative buoyancy.

Another point to notice is that the negative buoyancy will add a restoring force to the inertial modes, their eigenfrequencies should increase as β becomes more negative. Thus, in Table 10 an attempt has been made to correlate these frequencies as they increase with increasing β . At the larger values of the eigenfrequencies this becomes difficult due to different eigenfrequencies appearing that may have been missed by previous calculation due to the density of the frequencies as we approach 1.

At some point as $-\beta$ increases the negative buoyancy and the Coriolis effect become comparable restoring forces and it becomes difficult to distinguish between the inertial and gravity modes.

It must be noted that this calculation is only a first approximation to stable core conditions as ideally a separate density profile should be developed for each β value being considered. While the density profile of the core is expected to be nearly neutral, as in PREM, the results in Table 10 indicate that small changes in the stratification lead to large changes in the eigenspectrum. This indicates how useful these calculations will be in determining the stability structure of the Earth's core.

Table 10. Tentative Comparison of Dimensionless Eigenfrequencies
Calculated Using the Galerkin Method, with $\rho_0 = \text{NEUT.}$

$\beta = -0.0001 \quad \beta = -0.001 \quad \beta = -0.002 \quad \beta = -0.003 \quad \beta = -0.004$				
		0.06376	0.07623	0.08578
	0.1734	0.1669	0.2026	0.2315
	0.2760	0.2496	0.3131	0.4480
0.2993	0.3020	0.3090	0.3470	
0.3668	0.3786		0.4227	
			0.4453	
0.4726	0.4824	0.4930	0.5019	
0.5670	0.5758	0.5843	0.5909	0.5952
0.6581	0.6654	0.6736	0.6816	0.6894
0.6787	0.6867	0.6952	0.7032	
		0.7918	0.7445	
0.7855	0.7926	0.8117	0.8165	
0.8313	0.8377	0.8449	0.8520	0.8583
			0.8782	0.8869
	0.8872	0.9126	0.9152	
0.9004	0.9061	0.9226	0.9292	
0.9346	0.9443			
	0.9611	0.9650	0.9756	0.9968
	0.9698			
0.9876	0.9832	0.9864	0.9982	0.9999
	0.9937			

CHAPTER 6

6.1 Introduction

In this thesis a variety of theoretical methods have been used to investigate the axisymmetric inertial wave spectrum, and its modification by small negative buoyancy, for a model approximately representing conditions in the Earth's liquid core. These methods, the variational principle, the perturbation, and the Galerkin methods have all proven useful in calculating results of an approximate Earth model progressing to a more complex and realistic one. However, several important aspects have been neglected; these will be detailed below, and could be the subject of further research.

6.2.1 Inner Core

The most obvious difference between the model presented here and the actual Earth is the lack of a solid inner core. This could be accounted for by using a trial function similar to that already used but including extra terms, such as:

$$\chi = \sum_{i=0}^N \sum_{j=0}^{N-1} c_{ij} R^{2i} Z^{2j} + \sum_{i=1}^M \sum_{j=1}^M d_{ij} \frac{Z^{2j}}{(R^2 + Z^2)^{i/2}} \quad (6.1)$$

The second term in this expression can only be used when an inner core is present, as it will degenerate in a sphere, as R and Z both approach zero. This second term substantially increases the amount of work to be done, as the d_y 's must be found independently of the c_y 's. It is best to use cylindrical coordinates, as the effect of rotation appears to impose its configuration on the problem more so than the physical spherical shape of the boundaries. The functionals for the variational principle and the Galerkin method will remain the same when an inner core is included, except for the integration limits. The inner core boundary condition can also be shown to be a natural boundary condition. The presence of the inner core will significantly affect the eigenfrequencies as has been shown in experimental work by Aldridge (1967) and Aldridge (1972). Unfortunately, the inclusion of the inner core makes the problem an ill-posed one, unless viscosity is introduced as in section 6.2.6.

6.2.2 Normal Modes Carrying Linear or Angular Momentum

Both linear (translational) and angular momentum were ignored in this model. Linear momentum, expressed as:

$$P = i\omega \int_V \rho_0 \mathbf{u} \, dV \quad (6.2)$$

needs to be considered in order to filter out those frequencies due to simple translation, however it should not be a problem with the simple model presented here as the rigid fixed boundary condition prevents any transfer of linear momentum to the fluid outside, or vice versa.

Consideration of linear momentum is especially important when the inner core is included due to the possibility of Slichter modes where the solid inner core "sloshes" within the fluid outer core. These modes can be filtered out using Lagrange multipliers associated with the various components of u , i.e. by adding to the functional

$$\sum_i \lambda_i \int_V \rho_0 u_i dv. \quad (6.2)$$

This filtering only needs to be done for the cases when $m = 0, \pm 1$, where the ϕ dependence has been expressed in the trial function as $e^{im\phi}$.

The angular momentum, which was also disregarded, is somewhat more complicated. The angular momentum can be changed in several ways: altering the flow, changing the relative angular momentum and changing the moment of inertia. This is of most concern when the boundaries are deformable and not spherical. These effects must be filtered out with Lagrange multipliers. The total angular momentum can be expressed as:

$$\Delta H = \int_m (r + u) \times \frac{d}{dt}(r + u) dm - \int_m r \times \frac{dr}{dt} dm. \quad (6.4)$$

This can be expressed as a complex expression with no time dependence as:

$$\Delta H = \int_m [u \times (\Omega \times r) + r \times (\Omega \times u) + i\omega r \times u] dm, \quad (6.5)$$

where the first two terms account for changes in the moment of inertia and the last term for changes in the flow. Alternatively, this can be written as:

$$\Delta H = h + \Omega \cdot \Delta I, \quad (6.6)$$

where the change in the inertia tensor I is given by:

$$\Delta I = \int_m [2 \mathbf{r} \cdot \mathbf{u} \mathbf{1} - (\mathbf{r}\mathbf{u} + \mathbf{u}\mathbf{r})] dm. \quad (6.7)$$

The changes in the inertia tensor can be expanded as:

$$\int_s \rho_0 \mathbf{n} \cdot \mathbf{u} [\mathbf{r}^2 \mathbf{1} - \mathbf{r}\mathbf{r}] ds - \int_v [u \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u}] [\mathbf{r}^2 \mathbf{1} - \mathbf{r}\mathbf{r}] dv, \quad (6.8)$$

where the first terms express displacement in the boundaries, the second term describes departures from homogeneity, and the final term depicts flow that departs from incompressible flow. The surface integral is over the inner core boundary and the core-mantle boundary. When the boundaries are rigid, and the fluid is homogeneous and incompressible, then no changes to the inertia tensor can occur and only the changes in the flow h need to be considered:

$$h = i\omega \int_m \mathbf{r} \times \mathbf{u} dm. \quad (6.9)$$

As with the linear momentum, only those cases where $m = 0, \pm 1$ need to be filtered out using Lagrange multipliers.

6.2.3 More Realistic Boundary Conditions

The boundary condition used here was a rigid spherical core-mantle boundary. For a more realistic model both the inner core and the core-mantle boundary must be deformable and nonspherical. A nonspherical boundary would allow a transfer of torque between the core and the mantle.

6.2.4 Azimuthal Modes

In this work the azimuthal modes were ignored by considering only the case when $m = 0$. If these modes are to be considered, both the trial function and the functionals used must be changed. For example, for $m \neq 0$, the functional for implementing the variational principle with rigid fixed boundaries but non-neutral stratification becomes:

$$\begin{aligned}
 F = & \sigma^2 \int_V \rho_0^i |\nabla \chi|^2 dv - \int_V \rho_0^i |e_3 \cdot \nabla \chi|^2 dv - \int_V \frac{\rho_0^i}{B} |C \cdot \nabla \chi|^2 dv \\
 & + m \sigma \int_V \frac{\rho_0^i}{a^2} |\chi|^2 \frac{(g_0 \times e_3) \cdot \phi}{R} dv + \rho_0^i (S) \int_S |\chi|^2 \frac{m \sigma (e_3 \times n) \cdot \phi}{R} ds
 \end{aligned} \tag{6.10}$$

which is essentially equation (86) from Rochester (1989).

Similarly, the Galerkin method will also include additional terms. The trial function will now be:

$$\chi = \sum_{p=1} c_p \phi_p (R, Z) e^{im\phi}. \quad (6.11)$$

Another point is to remove the restriction placed on the trial function forcing it to satisfy:

$$\frac{\partial \chi}{\partial R} = 0 \quad \text{on } R = 0 \quad \text{and} \quad \frac{\partial \chi}{\partial Z} = 0 \quad \text{on } Z = 0, \quad (6.12)$$

as was done in Chapter 2. Modes involving motion across the equatorial axis and plane are possible.

6.2.5 Improved Representations of Density Profile

The stability parameter used here was $\beta = 0$ or a small negative number. Positive values of β cannot be considered as gravity waves will not exist when the fluid is nowhere stably stratified. In any case, β is most likely a complicated function of radius. This aspect needs to be considered so that regions of stability can be dealt with separately from those that may be unstable.

As was noted in Chapter 5, the two density profiles used, $\rho_0 = \text{PREM}$ and $\rho_0 = \text{NEUT}$, lead to a neutrally stratified region. As β is made negative, the density profile must be changed to ensure that the model is consistent.

6.2.6 Viscosity

In this thesis the problem of viscosity has been ignored. This has been considered by Rieutord (1987, 1990). The presence of viscosity will selectively damp the smaller wavelength modes and reduce the severity of the truncation to small values of N necessary for computational purposes. Viscosity will also help in removing the degeneracy of the ill-posedness which seems to introduce discontinuous eigenfunctions in the spherical shell geometry.

6.2.7 Density of the Eigenspectrum

According to Valette (1989) the inertia/gravity eigenspectrum for an elastic, uniformly rotating, self-gravitating body with a fluid inclusion is a continuous one. However an unpublished proof by Rochester demonstrates that the eigenspectrum should be discrete (but dense) if the density profile of the liquid core is neutral or stable everywhere. This issue needs further investigation.

6.3 Summary of Research Results

Whereas the scope of this thesis is limited, it has yielded many useful results. First of all, the variational principle used is based on a more general functional than that used by Aldridge (1972). He used a stream function formulation, which while more desirable mathematically, is limiting in the amount of physics that it elucidates, as it cannot be used when the fluid is compressible. The variational principle presented here can be used in an inhomogeneous

compressible fluid. Aldridge (1972) also only considered those modes where $m = 0$. The variational principle used in this thesis for axisymmetric modes can also be used when m is nonzero. So far as I am aware, no published work has yet reported numerical results for axisymmetric oscillations based on this variational principle.

The second original contribution of this thesis is a numerical test of the perturbation treatment of the effect, on the eigenspectrum of a neutrally stratified fluid, of very small negative β . It appears that first-order perturbation theory is useful only for values of $-\beta$ which are so small as to be comparatively uninteresting.

Finally this thesis makes one further original contribution in the numerical application of a Galerkin procedure for studying the effects of increasingly negative β on the eigenspectrum, namely increase in the inertial mode frequencies and the appearance of gravity modes.

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APPENDIX A: Three methods for computing an integral.

The integral under consideration is:

$$\sum_p \sum_q \int_0^1 z^{2p} (1 - z^2)^q dz \quad (\text{A.1})$$

This integral can be evaluated in three different ways. The first uses the half integer gamma function, the second employs binomial coefficients and the third utilises factorials for a solution.

1) Half Integer Gamma

This solution can be expressed as:

$$\sum_{p=0}^P \sum_{q=0}^Q \int_0^1 z^{2p} (1 - z^2)^q dz = \sum_{p=0}^P \sum_{q=0}^Q \frac{\Gamma(p + \frac{1}{2}) \Gamma(q + 1)}{2 \Gamma(p + q + \frac{3}{2})} \quad (\text{A.2})$$

This solution can be programmed in FORTRAN using the IMSL subroutine DGAMMA.

2) Binomial Coefficients

This solution involves the inclusion of another summation:

$$\begin{aligned}
 \sum_{p=0}^P \sum_{q=0}^Q \int_0^1 z^{2p} (1-z^2)^q dz &= \sum_{p=0}^P \sum_{q=0}^Q \sum_{r=0}^q \frac{q!(-1)^r}{r!(q-r)!} \int_0^1 z^{(2r+2p)} dz \\
 &= \sum_{p=0}^P \sum_{q=0}^Q \sum_{r=0}^q \frac{q!(-1)^r}{r!(q-r)!} \frac{1}{2(r+p)+1}
 \end{aligned}
 \tag{A.3}$$

This can then be programmed using the IMSL routine DBINOM.

3) Factorial Solution

This solution is given by:

$$\sum_{p=0}^P \sum_{q=0}^Q \int_0^1 z^{2p} (1-z^2)^q dz = \sum_{p=0}^P \sum_{q=0}^Q \frac{(2p)! q! (p+q+1)! 2^{2q+1}}{p! (2(p+q+1))!}
 \tag{A.4}$$

This can also be programmed using the IMSL subroutine DFAC, however numbers may get very large. To alleviate this problem, logarithms can be used to make the numbers more tractable.

APPENDIX B: Program to calculate the eigenfrequencies of the axisymmetric normal modes of the Poincaré problem in a liquid sphere with rigid fixed boundaries.

C sp5.f

C Program to calculate eigenfrequencies of axisymmetric normal modes of the Poincare

C problem in a liquid sphere. Program calls on the subroutines for calculating the

C determinant of a matrix which has no zeroes on the main diagonal.

C The IMSL subroutine DZREAL, is used to locate the zeroes of the determinant,

C with subroutine DET and DETM to evaluate the determinant.

C Many write statements are included, but have been commented out. They were used to

C ensure that values were correct.

C Reminder: Change dimension and data specifications in the function DETM when

C changing NN, so that C, F, H, are MM by MM. NOTE: $MM = NN*(NN+3)/2$

C Both NN, MMM and steps may need to be changed for different runs.

implicit real*8(a-h, o-z)

parameter (NN = 5, MMM = $NN*(NN+3)/2$)

parameter (nroot = $NN*(NN-1)/2$)

integer info(nroot)

real*8 detm, x(nroot), xguess(nroot)

external wrrn, dzreal

external detm

common/a/ C(MMM,MMM), F(MMM,MMM), MM

MM = MMM

C NN is the size of the summation in the trial function,

- C MM the number of terms in the trial function.
- C FACTNN is a factor pulled out of the calculation.
- C FLOG is a factor multiplied in to make the numbers more manageable.

FLOG = 0.D0

DO 5, LL = 1, NN

DI = 2*LL + 1

FLOG = FLOG + DLOG10(DI)

- 5 CONTINUE

FACTNN = 10.D0**FLOG

- C Matrix elements

- C The matrix elements have two indices, JJ and LL, for row and column
- C respectively. (JJ depends on I and J, and LL depends on K and L.)
- C Since the row increases slowest, this loop starts first.
- C Starting the outer loop, to determine the row number.

do 10 I = 0, NN

kk=0

if (I.eq.0) kk=1

do 20 J = kk, NN-I

- C Starting the inner loop, to determine the column number.

do 30 K = 0, NN

kk=0

if (K.eq.0) kk=1

do 40 L = kk, NN-k

- C Start the calculation.


```

      csumm = 0
      fsumm = 0
do 50 M = 0, I + K
      BINO = BINOM(I + K, M)
      IM = (-1)**M
      CBOTM = 2*(J + L + M) + 1
      FBOTM = CBOTM*(2*(J + L + M) - 1)
      CSUMM = CSUMM + BINO * IM/CBOTM
      FSUMM = FSUMM + BINO * IM/FBOTM
50  CONTINUE
      IF (I + K .EQ. 0) THEN
          CC = 0
      ELSE
          CC = I*K* CSUMM/((I+K)*2)
      ENDIF
      FF = J * L * FSUMM/(I + K + 1)
C      Set up the indexing
      JJ = J + I * (2 * NN + 3 - I)/2
      LL = L + K * (2 * NN + 3 - K)/2
      C(JJ,LL) = FACTNN * CC
      F(JJ,LL) = FACTNN * FF
40  continue
30  continue
20  continue

```

```

10      continue

C      This will use the IMSL subroutine DZREAL to find the roots from DETM
data xguess/0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99/

EPS      = 1.0D-8

ERRABS    = 1.0D-11

ERRREL    = 1.0D-11

ETA      = 1.0D-7

ITMAX     = 300

      Call dzreal (detm, errabs, errrel, eps, eta, nroot, itmax, xguess,
&              x, info)

      Call dwrrm ('the zeroes are', 1, nroot, x, 1, 0)

      STOP

      END

C      CALCULATES DETERMINANT OF N BY N MATRIX WITH NO ZERO
C      ELEMENTS ON THE MAIN DIAGONAL. THE MATRIX IS AN N BY N ARRAY
C      VALUE OF DETERMINANT IS DETT TIMES (10 TO THE IE'TH POWER)
C      (THIS IS DONE BY TRIANGULARIZING THE MATRIX)

      SUBROUTINE DET(A,N,IE,DETT)

      IMPLICIT REAL*8(A-H,O-Z)

      DIMENSION A(N,N)

      NM1 = N - 1

      DETLOG = 0.D0

      SIGN = 1.D0

      DO 1 K = 1, NM1

```

```

      KP1 = K + 1
      R = 1.D0/A(K,K)
      IF (A(K,K).GT.0.D0) GO TO 3
      SIGN = -SIGN
3     DETLOG = DETLOG + DLOG10(DABS(A(K,K)))
      DO 2 J = KP1, N
2     A(K,J) = R*A(K,J)
      DO 1 I = KP1, N
      S = A(I,K)
      DO 1 L = KP1, N
1     A(I,L) = A(I,L) - S*A(K,L)
      IF (A(N,N).GT.0.D0) GO TO 4
      SIGN = -SIGN
4     DETLOG = DETLOG + DLOG10(DABS(A(N,N)))
      IE = IDINT(DETLOG)
      DETLOG = DETLOG - DFLOAT(IE)
      DETT = SIGN*(10.D0**DETLOG)
      RETURN
      END
      REAL*8 FUNCTION DETM(X)
      IMPLICIT REAL*8(A-H, O-Z)
      parameter (NN = 5, MMM = NN*(NN+3)/2)
      COMMON/A/ C(MMM,MMM), F(MMM,MMM), MM
      DIMENSION H(MMM,MMM)

```

```
      DO 1 J = 1, MM
      DO 2 L = 1, MM
      IF (J.LE.5) GO TO 3
      IF (J.EQ.6) GO TO 4
      IF (J.EQ.11) GO TO 4
      IF (J.EQ.15) GO TO 4
      IF (J.EQ.18) GO TO 4
      IF (J.EQ.20) GO TO 4
      GO TO 5
3      H(J,L) = F(J,L)
      GO TO 6
4      H(J,L) = C(J,L)
      GO TO 6
5      H(J,L) = X*X*(C(J,L) + F(J,L)) - F(J,L)
6      CONTINUE
2      CONTINUE
1      CONTINUE
      CALL DET(H,MM,IE,DETT)
      DETM = DETT * (10.DO**IE)
      RETURN
      END
```

APPENDIX C: Program to calculate the eigenfrequencies of the axisymmetric normal modes of the Poincaré problem in a liquid sphere including a neutral density profile from PREM.

```

C          spden5.f

C      Program to calculate eigenfrequencies of axisymmetric normal modes of the Poincare
C      problem in a liquid sphere INCLUDING DENSITY from PREM. Program calls on a
C      subroutine for calculating the determinant of a matrix which has no zeroes on the main
C      diagonal. The IMSL subroutine DZREAL, is used to locate the zeroes of the determinant,
C      using the external function DETM(X). NN is the size of the summation in the trial
C      function, MM the number of terms in the trial function. NOTE:  $MM = NN*(NN+3)/2$ 
C
C      NN will need to be changed for different runs.
C
C      NN must be changed in the parameter statements both on top
C
C      and in the external function
C
C      in DETM(X) must also change the goto statements for different
C
C      numbers of zeros.

implicit real*8(a-h, o-z)

parameter (NN = 5, MMM = NN*(NN+3)/2)

parameter (nroot = NN*(NN-1)/2)

integer    info(nroot)

real*8     detm, x(nroot), xguess(nroot), density(4)

external   dwrrrn, dzreal

external detm

common/a/ C(MMM,MMM), F(MMM,MMM), MM

open(10, file = 'spd5zeroes',status = 'unknown')

MM = MMM

```

C This calculates the values of the matrix needed in the program.

C This is a factor to make the values of the matrix tractable.

FLOG = 0.D0

DO 5, LL = 1, NN

DI = 2*LL + 1

FLOG = FLOG + DLOG10(DI)

5 CONTINUE

FACTNN = 10.D0**FLOG

C These are the coefficients for the density polynomial given by D&A (renormalized)

densty(1) = 12.5815

densty(2) = -1.2638 * .546225

densty(3) = -3.6426 * .546225 * .546225

densty(4) = -5.5281 * .546225 * .546225 * .546225

C Matrix elements

C The matrix elements have two indicies, JJ and LL, for row and column

C respectively. (JJ depends on I and J, and LL depends on K and L.)

C Since the row increases slowest, this loop starts first.

C Starting the outer loop, to determine the row number.

do 10 I = 0, NN

kk=0

if (I.eq.0) kk=1

do 20 J = kk, NN-I

C Starting the inner loop, to determine the column number.

do 30 K = 0, NN

```

      kk=0
      if (K.eq.0) kk=1
do 40 L = kk, NN-k
  csumm = 0
  fsumm = 0
do 50 M = 0, I + K
  FBINO = DBINOM(I + K, M)
  IM = (-1)**M
  FBOTM = 2*(J + L + M) - 1
  FSUMM = FSUMM + FBINO * IM/FBOTM
50  Continue
do 55 M = 0, I + K - 1
  CBINO = DBINOM((I + K - 1), M)
  IM = (-1)**M
  CBOTM = 2*(J + L + M) + 1
  CSUMM = CSUMM + CBINO * IM/CBOTM
55  Continue
C    This loop accounts for the "density effect"
SUMN = 0
  DO 60 N = 1,4
    DSUMN = N + 2*(I + J + K + L)
    SUMN = SUMN + DENSTY(N)/DSUMN
60  CONTINUE
C    Calculate the elements.

```

CC = I * K * CSUMM * SUMN

FF = J * L * FSUMM * SUMN

C Set up indexing....

JJ = J + I * (2 * NN + 3 - I)/2

LL = L + K * (2 * NN + 3 - K)/2

C(JJ,LL) = CC*500.0

F(JJ,LL) = FF*500.0

40 continue

30 continue

20 continue

10 continue

C This will use the IMSL subroutine dzreal to find the roots from the

C function detm. Guesses must be the correct number, but need not be close to the actual

C value. First set up the guesses.

do 4999 kkk=1,nroot

 xguess(kkk) = dble(kkk)/11.0

4999 continue

C EPS and ETA spread criteria for multiple zeroes.

C ERRABS absolute error (a stopping criterion)

C ERRREL relative error (second stopping criterion)

C ITMAX maximum number of iterations per zero

C INFO number of iterations actually used

EPS = 1.0D-11

ERRABS = 1.0D-12

ERRREL = 1.0D-12

ETA = 1.0D-10

ITMAX = 400

Call dzreal (detm, errabs, errrel, eps, eta, nroot, itmax, xguess,

& x, info)

Call dwrrrn ('the zeroes are', 1, nroot, x, 1, 0)

STOP

END

C CALCULATES DETERMINANT OF N BY N MATRIX WITH NO ZERO ELEMENTS
 C ON THE MAIN DIAGONAL. THE MATRIX IS STORED IN N BY N ARRAY.
 C THE VALUE OF THE DETERMINANT IS DETT TIMES (10 TO THE IE'TH
 C POWER). THIS IS DONE BY TRIANGULARIZING THE MATRIX.

SUBROUTINE DET(A,N,IE,DETT)

IMPLICIT REAL*8(A-H,O-Z)

DIMENSION A(N,N)

NM1 = N - 1

DETLOG = 0.D0

SIGN = 1.D0

DO 1 K = 1, NM1

KP1 = K + 1

R = 1.D0/A(K,K)

IF (A(K,K).GT.0.D0) GO TO 3

SIGN = -SIGN

3 DETLOG = DETLOG + DLOG10(DABS(A(K,K)))

```

      DO 2 J = KP1, N
2     A(K,J) = R*A(K,J)
      DO 1 I = KP1, N
        S = A(I,K)
      DO 1 L = KP1, N
1     A(I,L) = A(I,L) - S*A(K,L)
      IF (A(N,N).GT.0.D0) GO TO 4
      SIGN = -SIGN
4     DETLOG = DETLOG + DLOG10(DABS(A(N,N)))
      IE = IDINT(DETLOG)
      DETLOG = DETLOG - DFLOAT(IE)
      DETT = SIGN*(10.D0**DETLOG)
      RETURN
      END
      REAL*8 FUNCTION DETM(X)
      IMPLICIT REAL*8(A-H, O-Z)
      parameter (NN = 5, MMM = NN*(NN+3)/2)
      COMMON/A/ C(MMM,MMM), F(MMM,MMM), MM
      DIMENSION H(MMM,MMM)
C     DATA H/81*0.D0/
      DO 1 J = 1, MM
      DO 2 L = 1, MM
        IF (J.LE.5) GO TO 3
        IF (J.EQ.6) GO TO 4

```

```
      IF (J.EQ.11) GO TO 4
      IF (J.EQ.15) GO TO 4
      IF (J.EQ.18) GO TO 4
      IF (J.EQ.20) GO TO 4
      GO TO 5
3     H(J,L) = F(J,L)
      GO TO 6
4     H(J,L) = C(J,L)
      GO TO 6
5     H(J,L) = X*X*(C(J,L) + F(J,L)) - F(J,L)
6     CONTINUE
2     CONTINUE
1     CONTINUE
      CALL DET(H,MM,IE,DETT)
      DETM = DETT * (10.D0**IE)
      RETURN
      END
```

APPENDIX D: This program finds the eigenvector coefficients for the eigenfrequencies previously calculated.

```

C          wspdlinsys5.f
C
C   THIS PROGRAM USES A LINEAR SYSTEM ROUTINE TO FIND THE
C   EIGENVECTOR COEFFICIENTS FOR THE PREVIOUSLY DETERMINED
C   EIGENFREQUENCIES. FIRST PART OF THE PROGRAM IS EXACTLY AS IN
C   SPDEN5.F, NN is the size of the summation in the trial function,
C   MM the number of terms in the trial function. NOTE:  $MM = NN*(NN+3)/2$ 
C
C   NN will need to be changed for different runs.
C
C   NN must be changed in the parameter statements both on top
C
C   and in the external function
C
C   in DETM(X) must also change the goto statements for different runs.

implicit real*8(a-h, o-z)

parameter (NN = 5, MMM = NN*(NN+3)/2)

parameter (nroot = NN*(NN-1)/2)

integer    info(nroot)

real*8     detm, x(nroot), xguess(nroot), densty(4)

real*8     D(MMM-1, MMM-1), E(MMM-1), EIG(MMM-1), HH(MMM,MMM)

external   dwrrrn, dzreal

external   detm

common/a/ C(MMM,MMM), F(MMM,MMM), MM

open(1, file='eigenvalues.out',status='unknown')

      open(2, file='eigenvector.out',status='unknown')

MM = MMM

```

C This calculates the values of the matrix needed in the program.

C This is a factor to make the values of the matrix tractable.

FLOG = 0.D0

DO 5, LL = 1, NN

DI = 2*LL + 1

FLOG = FLOG + DLOG10(DI)

5 CONTINUE

FACTNN = 10.D0**FLOG

C These are the coefficients for the density polynomial given by D&A (renormalized)

densty(1) = 12.5815

densty(2) = -1.2638 * .546225

densty(3) = -3.6426 * .546225 * .546225

densty(4) = -5.5281 * .546225 * .546225 * .546225

C The matrix elements have two indicies, JJ and LL, for row and column

C respectively. (JJ depends on I and J, and LL depends on K and L.)

C Since the row increases slowest, this loop starts first.

C Starting the outer loop, to determine the row number.

do 10 I = 0, NN

kk=0

if (I.eq.0) kk=1

do 20 J = kk, NN-I

do 30 K = 0, NN

kk=0

if (K.eq.0) kk=1

```

do 40 L = kk, NN-k
    csumm = 0
    fsumm = 0
do 50 M = 0, I + K
    FBINO = DBINOM(I + K, M)
    IM = (-1)**M
    FBOTM = 2*(J + L + M) - 1
    FSUMM = FSUMM + FBINO * IM/FBOTM
50 CONTINUE
do 55 M = 0, I + K - 1
    CBINO = DBINOM((I + K - 1), M)
    IM = (-1)**M
    CBOTM = 2*(J + L + M) + 1
    CSUMM = CSUMM + CBINO * IM/CBOTM
55 Continue
    SUMN = 0
    DO 60 N = 1,4
        DSUMN = N + 2*(I + J + K + L)
        SUMN = SUMN + DENSTY(N)/DSUMN
60 CONTINUE
C   Calculate the elements.
    CC = I * K * CSUMM * SUMN
    FF = J * L * FSUMM * SUMN
C   Set up indexing.

```

$JJ = J + I * (2 * NN + 3 - I)/2$

$LL = L + K * (2 * NN + 3 - K)/2$

$C(JJ,LL) = CC*1000.0$

$F(JJ,LL) = FF*1000.0$

40 continue

30 continue

20 continue

10 continue

C This will use the IMSL subroutine dzreal to find the roots from the function detm.

C Guesses must be the correct number, but need not be close to the actual value.

EPS = 1.0D-9

ERRABS = 1.0D-12

ERRREL = 1.0D-12

ETA = 1.0D-8

ITMAX = 300

C first establish the initial guesses

do 4999 kkk=1,nroot

 xguess(kkk) = dble(kkk)/11.0

4999 continue

 Call dzreal (detm, errabs, errrel, eps, eta, nroot, itmax, xguess,

& x, info)

 Call dwrrrn ('THE EIGENVALUES ARE:', 1, nroot, x, 1, 0)

C THIS SECTION WILL FIND THE EIGENVECTORS USING A LINEAR SYSTEM

C SUBROUTINE FROM IMSL. NOTE THAT THE SYSTEM SOLVED IS OF ONE

C ORDER LESS THAN THE ORIGINAL SYSTEM. THIS IS BECAUSE THE FIRST
C CONSTANT HAS BEEN CHOSEN AS ONE. FIRST THE EIGENVALUES JUST
C FOUND ARE SUBBED INTO THE MATRIX HH.

Do 100 i = 1, nroot

Do 110 J = 1, MM

Do 120 L = 1, MM

IF (J.LE.5) GO TO 130

IF (J.EQ.6) GO TO 140

IF (J.EQ.11) GO TO 140

IF (J.EQ.15) GO TO 140

IF (J.EQ.18) GO TO 140

IF (J.EQ.20) GO TO 140

GO TO 150

130 HH(J,L) = F(J,L)

GO TO 160

140 HH(J,L) = C(J,L)

GO TO 160

150 HH(J,L) = X(i)*X(i)*(C(J,L) + F(J,L)) - F(J,L)

160 CONTINUE

120 CONTINUE

110 CONTINUE

Do 170 J = 1, MM - 1

Do 180 L = 1, MM - 1

D(J,L) = HH(J+1, L+1)

$E(J) = -HH(J+1, 1)$

180 Continue

170 Continue

Call DLSASF(MM-1, D, MM-1, E, EIG)

Call DWRRRN('THE EIGENVECTOR IS:', 1, MM-1, EIG, 1, 0)

Do 200 kkk = 1, MM-1

write(2,*) eig(kkk)

200 Continue

write(2,*)

100 Continue

STOP

END

C CALCULATES DETERMINANT OF N BY N MATRIX WITH NO ZERO
C ELEMENTS ON THE MAIN DIAGONAL. THE MATRIX IS STORED IN AN N BY
C N ARRAY. THE VALUE OF THE DETERMINANT IS DETT TIMES (10 TO THE
C IE'TH POWER). THIS IS DONE BY TRIANGULARIZING THE MATRIX.

SUBROUTINE DET(A,N,IE,DETT)

IMPLICIT REAL*8(A-H,O-Z)

DIMENSION A(N,N)

NM1 = N - 1

DETLOG = 0.D0

SIGN = 1.D0

DO 1 K = 1, NM1

KP1 = K + 1

```

      R = 1.D0/A(K,K)
      IF (A(K,K).GT.0.D0) GO TO 3
      SIGN = -SIGN
3     DETLOG = DETLOG + DLOG10(DABS(A(K,K)))
      DO 2 J = KP1, N
2     A(K,J) = R*A(K,J)
      DO 1 I = KP1, N
      S = A(I,K)
      DO 1 L = KP1, N
1     A(I,L) = A(I,L) - S*A(K,L)
      IF (A(N,N).GT.0.D0) GO TO 4
      SIGN = -SIGN
4     DETLOG = DETLOG + DLOG10(DABS(A(N,N)))
      IE = IDINT(DETLOG)
      DETLOG = DETLOG - DFLOAT(IE)
      DETT = SIGN*(10.D0**DETLOG)
      RETURN
      END
      REAL*8 FUNCTION DETM(X)
      IMPLICIT REAL*8(A-H, O-Z)
      parameter (NN = 5, MMM = NN*(NN+3)/2)
      COMMON/A/ C(MMM,MMM), F(MMM,MMM), MM
      DIMENSION H(MMM,MMM)
C     DATA H/81*0.D0/

```

```
DO 1 J = 1, MM
DO 2 L = 1, MM
    IF (J.LE.5) GO TO 3
    IF (J.EQ.6) GO TO 4
    IF (J.EQ.11) GO TO 4
    IF (J.EQ.15) GO TO 4
    IF (J.EQ.18) GO TO 4
    IF (J.EQ.20) GO TO 4
    GO TO 5
3    H(J,L) = F(J,L)
    GO TO 6
4    H(J,L) = C(J,L)
    GO TO 6
5    H(J,L) = X*X*(C(J,L) + F(J,L)) - F(J,L)
6    CONTINUE
2    CONTINUE
1    CONTINUE

    CALL DET(H,MM,IE,DETT)
    DETM = DETT * (10.D0**IE)
    RETURN
END
```

APPENDIX E: This program evaluates equation (82) from Rochester (1989).

C perturbation.f

C This is a program for eqn(82) in R's '89 paper. The eigenvalues and eigenvector

C coefficients are read in from a file. The first eigenvector value is always one

parameter (ivalue = 10, ivectr = 20)

implicit real*8(a-h, o-z)

real*8 eigval(ivalue), eigvec(ivalue,ivectr)

real*8 cr(20), rho(4), rintn

integer p, q, qq

open(1,file='eigenvalues.out',status='old')

open(2,file='eigenvector.out',status='old')

open(3,file='zeros.out', status='unknown')

beta = -0.000010d0

C Set up loops to read in eigenvalues/vector coefficients, the outer loop is for the

C eigenvalues, each eigenvalue is associated with an eigenvector comprising ivectr entries.

do 10 i = 1, ivalue

read(1,*) eigval(i)

eigvec(i,1) = 1.0d0

read(2,*) (eigvec(i,j),j=2,ivectr)

!0 continue

C The constants needed: alf* are coefficients of the polynomial representing $1/\alpha^2$, rho*

C are renormalized values of PREM density for CMB = $r = 1$, cr* are coefficients of the

C powers of r that have been integrated using MACSYMA and exported in FORTAN

C format. These are the coefficients for the 7th degree polynomial for $1/\alpha^2$ obtained

C using the IMSL routine called drcurv. (r has been renormalized). Units are $(\text{sec/km})^2$.

alf1 = .0081914306880072d0

alf2 = .0032968419226614d0

alf3 = -.0016245386362791d0

alf4 = .0055783009628760d0

alf5 = -.0093306230249849d0

alf6 = .019869926098212d0

alf7 = -.018169982295218d0

alf8 = .0075630663016279d0

rho1 = 12.5815d0

rho2 = -.6903192d0

rho3 = -1.0868115d0

rho4 = -.9009295d0

rho(1) = rho1

rho(2) = rho2

rho(3) = rho3

rho(4) = rho4

C Info. for the coefficient of g_0^2 . $(16\pi^2 G^2) b^2$, (units are $(\text{m}^3/(\text{kg}\cdot\text{sec}^2))^2 \cdot \text{km}^2$)

G = 6.6732d-11

b = 3.480d3

pi = 3.1415926d0

comega = 7.292115d-05

C Coefficients of powers of r integrated by MACSYMA and imported in FORTRAN format.

```
cr(18) = alf8*rho4**3/36.0d0
```

$$cr(17) = \rho_4^{**2} \cdot (5 \cdot \alpha_7 \cdot \rho_4 + 17 \cdot \alpha_8 \cdot \rho_3) / 180.0d0$$

$$\text{cr}(16) = \frac{\rho_4^4 (25 \alpha_6^2 \rho_4^2 + 85 \alpha_7 \rho_3 \rho_4 + 100 \alpha_8 \rho_2 \rho_4 + 96 \alpha_8^2 \rho_3^2)}{900.0d_0}$$

$$\text{cr}(15) = (25*\text{alf}5*\text{rho}4^{**3} + 85*\text{alf}6*\text{rho}3*\text{rho}4^{**2} + 100*\text{alf}7*\text{rho}2*\text{rho}4^{**2} + 125*\text{alf}8*\text{rho}1*\text{rho}4^{**2} + 96*\text{alf}7*\text{rho}3^{**2}*\text{rho}4 + 225*\text{alf}8*\text{rho}2*\text{rho}3*\text{rho}4 + 36*\text{alf}8*\text{rho}3^{**3})/900.0d0$$

$$\text{cr}(14) = (100*\text{af}4*\text{rho}4^{**3} + 340*\text{af}5*\text{rho}3*\text{rho}4^{**2} + 400*\text{af}6*\text{rho}2*\text{rho}4^{**2} + 500*\text{af}7*\text{rho}1*\text{rho}4^{**2} + 384*\text{af}6*\text{rho}3^{**2}*\text{rho}4 + 900*\text{af}7*\text{rho}2*\text{rho}3*\text{rho}4 + 1120*\text{af}8*\text{rho}1*\text{rho}3*\text{rho}4 + 525*\text{af}8*\text{rho}2^{**2}*\text{rho}4 + 144*\text{af}7*\text{rho}3^{**3} + 504*\text{af}8*\text{rho}2*\text{rho}3^{**2})/3600.0d0$$

$$\text{cr}(13) = (100*\text{alf}^3*\text{rho}^4^{**3} + 340*\text{alf}^4*\text{rho}^3*\text{rho}^4^{**2} + 400*\text{alf}^5*\text{rho}^2*\text{rho}^4^{**2} + 500*\text{alf}^6*\text{rho}^1*\text{rho}^4^{**2} + 384*\text{alf}^5*\text{rho}^3^{**2}*\text{rho}^4 + 900*\text{alf}^6*\text{rho}^2*\text{rho}^3*\text{rho}^4 + 1120*\text{alf}^7*\text{rho}^1*\text{rho}^3*\text{rho}^4 + 525*\text{alf}^7*\text{rho}^2^{**2}*\text{rho}^4 + 1300*\text{alf}^8*\text{rho}^1*\text{rho}^2*\text{rho}^4 + 144*\text{alf}^6*\text{rho}^3^{**3} + 504*\text{alf}^7*\text{rho}^2*\text{rho}^3^{**2} + 624*\text{alf}^8*\text{rho}^1*\text{rho}^3^{**2} + 585*\text{alf}^8*\text{rho}^2^{**2}*\text{rho}^3)/3600.0\text{d}0$$

$$\begin{aligned} r(12) = & (100*\alpha_2^2*\rho_4^3 + 340*\alpha_3*\rho_3*\rho_4^2 + 400*\alpha_4^2*\rho_2*\rho_4^2 \\ & + 500*\alpha_5*\rho_1*\rho_4^2 + 384*\alpha_4^2*\rho_3^2*\rho_4 + 900*\alpha_5^2*\rho_2^2*\rho_4 \\ & + 1120*\alpha_6*\rho_1*\rho_3*\rho_4 + 525*\alpha_6^2*\rho_2^2*\rho_4 + 1300*\alpha_7*\rho_1*\rho_2*\rho_4 \\ & + 800*\alpha_8*\rho_1^2*\rho_4 + 144*\alpha_5^3*\rho_3^3 + 504*\alpha_6^2*\rho_2^2*\rho_3^2 \\ & + 624*\alpha_7*\rho_1*\rho_3^2 + 585*\alpha_7^2*\rho_2^2*\rho_3 + 1440*\alpha_8*\rho_1*\rho_2*\rho_3 \\ & + 225*\alpha_8^2*\rho_2^2)/3600.0d0 \end{aligned}$$

$$(11) = (100*\alpha f_1*\rho_4^{**3} + 340*\alpha f_2*\rho_3*\rho_4^{**2} + 400*\alpha f_3*\rho_2*\rho_4^{**2} + 500*\alpha f_4*\rho_1*\rho_4^{**2} + 384*\alpha f_3*\rho_3^{**2}*\rho_4 + 900*\alpha f_4*\rho_2*$$

$$\begin{aligned}
& 2 \quad \rho_3 \rho_4 + 1120 \alpha_5 \rho_1 \rho_3 \rho_4 + 525 \alpha_5 \rho_2^2 \rho_4 + 1300 \alpha_6 \rho_1 \rho_2 \rho_4 + 800 \alpha_7 \rho_1^2 \rho_4 + 144 \alpha_4 \rho_3^3 + 504 \alpha_4 \rho_3 \rho_4^2 \\
& 4 \quad 15 \rho_2^2 \rho_3^2 + 624 \alpha_6 \rho_1 \rho_3^2 + 585 \alpha_6 \rho_2^2 \rho_3 + 1440 \alpha_6 \rho_1 \rho_2 \rho_3 + 880 \alpha_8 \rho_1^2 \rho_3 + 225 \alpha_7 \rho_2^3 + 825 \alpha_7 \rho_1 \rho_2^2 \\
& 6 \quad 25 \alpha_8 \rho_1 \rho_2^2) / 3600.0d0
\end{aligned}$$

$$\begin{aligned}
& \text{cr}(10) = (340 \alpha_1 \rho_3 \rho_4^2 + 400 \alpha_2 \rho_2 \rho_4^2 + 500 \alpha_3 \rho_1 \rho_4^2 + 384 \alpha_2 \rho_3^2 \rho_4 + 900 \alpha_3 \rho_2 \rho_3 \rho_4 + 1120 \alpha_3 \rho_1 \rho_3 \rho_4 + 525 \alpha_4 \rho_1 \rho_3 \rho_4 + 525 \alpha_4 \rho_2^2 \rho_4 + 1300 \alpha_5 \rho_1 \rho_2 \rho_4 + 800 \alpha_6 \rho_1^2 \rho_4 + 144 \alpha_3 \rho_3^3 + 504 \alpha_4 \rho_2 \rho_3^2 + 624 \alpha_5 \rho_1 \rho_3^2 + 585 \alpha_5 \rho_2^2 \rho_3 + 1440 \alpha_6 \rho_1 \rho_2 \rho_3 + 880 \alpha_7 \rho_1^2 \rho_3 + 225 \alpha_6 \rho_2^3 + 825 \alpha_7 \rho_1 \rho_2^2 + 1000 \alpha_8 \rho_1^2 \rho_2) / 3600.0d0
\end{aligned}$$

$$\begin{aligned}
& \text{cr}(9) = (400 \alpha_1 \rho_2 \rho_4^2 + 500 \alpha_2 \rho_1 \rho_4^2 + 384 \alpha_1 \rho_3^2 \rho_4 + 900 \alpha_2 \rho_2 \rho_3 \rho_4 + 1120 \alpha_3 \rho_1 \rho_3 \rho_4 + 525 \alpha_4 \rho_1 \rho_3 \rho_4^2 + 1300 \alpha_5 \rho_1 \rho_2 \rho_4 + 800 \alpha_6 \rho_1^2 \rho_4 + 144 \alpha_2 \rho_3^3 + 504 \alpha_3 \rho_2 \rho_3^2 + 624 \alpha_4 \rho_1 \rho_3^2 + 585 \alpha_5 \rho_2^2 \rho_3 + 1440 \alpha_6 \rho_1 \rho_2 \rho_3 + 880 \alpha_7 \rho_1^2 \rho_3 + 225 \alpha_6 \rho_2^3 + 825 \alpha_7 \rho_1 \rho_2^2 + 1000 \alpha_8 \rho_1^2 \rho_2) / 3600.0d0
\end{aligned}$$

$$\begin{aligned}
& \text{cr}(8) = (500 \alpha_1 \rho_1 \rho_4^2 + 900 \alpha_1 \rho_2 \rho_3 \rho_4 + 1120 \alpha_2 \rho_1 \rho_3 \rho_4 + 525 \alpha_2 \rho_2^2 \rho_4 + 1300 \alpha_3 \rho_1 \rho_2 \rho_4 + 800 \alpha_4 \rho_1^2 \rho_4 + 144 \alpha_1 \rho_3^3 + 504 \alpha_2 \rho_2 \rho_3^2 + 624 \alpha_3 \rho_1 \rho_3^2 + 585 \alpha_3 \rho_2^2 \rho_3 + 1440 \alpha_4 \rho_1 \rho_2 \rho_3 + 880 \alpha_5 \rho_1^2 \rho_3 + 225 \alpha_4 \rho_2^3 + 825 \alpha_5 \rho_1 \rho_2^2 + 1000 \alpha_6 \rho_1^2 \rho_2) / 3600.0d0
\end{aligned}$$

5 1000*alf6*rho1**2*rho2+400*alf7*rho1**3)/3600.0d0

cr(7) = (1120*alf1*rho1*rho3*rho4+525*alf1*rho2**2*rho4+1300*alf2*

1 rho1*rho2*rho4+800*alf3*rho1**2*rho4+504*alf1*rho2*rho3**2+624*

2 alf2*rho1*rho3**2+585*alf2*rho2**2*rho3+1440*alf3*rho1*rho2*rho

3 3+880*alf4*rho1**2*rho3+225*alf3*rho2**3+825*alf4*rho1*rho2**2+

4 1000*alf5*rho1**2*rho2+400*alf6*rho1**3)/3600.0d0

cr(6) = (1300*alf1*rho1*rho2*rho4+800*alf2*rho1**2*rho4+624*alf1*r

1 ho1*rho3**2+585*alf1*rho2**2*rho3+1440*alf2*rho1*rho2*rho3+880*

2 alf3*rho1**2*rho3+225*alf2*rho2**3+825*alf3*rho1*rho2**2+1000*a

3 lf4*rho1**2*rho2+400*alf5*rho1**3)/3600.0d0

cr(5) = (160*alf1*rho1**2*rho4+288*alf1*rho1*rho2*rho3+176*alf2*rh

1 o1**2*rho3+45*alf1*rho2**3+165*alf2*rho1*rho2**2+200*alf3*rho1*

2 *2*rho2+80*alf4*rho1**3)/720.0d0

cr(4) = rho1*(176*alf1*rho1*rho3+165*alf1*rho2**2+200*alf2*rho1*rh

1 o2+80*alf3*rho1**2)/720.0d0

cr(3) = rho1**2*(5*alf1*rho2+2*alf2*rho1)/18.0d0

cr(2) = alf1*rho1**3/9.0d0

C Start the loop for the eigenvalue usage.

do 80 isig = 1,ivalue

rinten = 0.0d0

rinted = 0.0d0

zinten = 0.0d0

zinted = 0.0d0

signum = 0.0d0


```

sigdem = 0.0d0
signew = 0.0d0
sigtwo = 0.0d0
do 100 l = 0,5
    nn=0
    if ( l .eq. 0 ) nn = 1
do 110 n = nn,5-1
do 120 p = 0,5
    qq=0
    if ( p .eq. 0 ) qq = 1
do 130 q = qq,5-p
C   Set up conversion from double index to single index
    jj = n + l*5 - (l-3)*1/2
    kk = q + p*5 - (p-3)*p/2
    rinten = 0.0d0
do 140 ii = 2,18
    rinten=rinten+cr(ii)/(2.0d0*dble(l+p+n+q)+dble(ii)+1.0d0)
140 continue
C   Calculate the z integral of the numerator (zinten)
    zntop=DGAMMA(dble(n+q)+0.5d0)*DGAMMA(dble(l+p)+1.0d0)
    znbot = DGAMMA(dble(n + q + l + p) + 1.5d0)
    zinten = zntop/znbot
C   Calculate the coefficient from C*gradX for the numerator
    ccgrad = (eigval(isig)**4.0*l*p-(eigval(isig)**2.0)*

```

```
& (1.0d0-eigval(isig)**2.0)*(l*q+n*p)
& + (1.0d0-eigval(isig)**2.0)**2.0*n*q)
```

C Choosing the correct eigenvector value to multiply in to the numerator.

```
cjj = eigvec(isig,jj)
```

```
ckk = eigvec(isig,kk)
```

```
signum = signum + (cjj*ckk*ccgrad*zinten*rinten)
```

C Calculating the denominator.

```
rinted = 0.0d0
```

```
do 150 ir = 1,4
```

```
rinted = rinted + rho(ir)/(2.0d0*dble(l+n+p+q)+dble(ir))
```

150 continue

```
if ((l+p).EQ.0)then
```

```
zdtop1 = 0.0d0
```

```
else
```

```
zdtop1 = DGAMMA(dble(n+q)+0.5d0)*DGAMMA(dble(l+p))
```

```
endif
```

```
zdbot1 = DGAMMA(dble(n+q+l+p)+0.5d0)
```

```
if ((n+q).EQ.0) then
```

```
zdtop2 = 0.0d0
```

```
else
```

```
zdtop2 = DGAMMA(dble(n+q)-0.5d0)*DGAMMA(dble(l+p)+1.0d0)
```

```
endif
```

```
zdbot2 = DGAMMA(dble(n+q+l+p)+0.5d0)
```

```

      zinted = l*p*(zdtop1/zdbot1) + n*q*(zdtop2/zdbot2)
      sigdem = sigdem + cjj*ckk*rinted*zinted
130  continue
120  continue
110  continue
100  continue

      coetop = 8.0d0*beta*((pi*G*b)**2.0d0)*1.0d6/
& ((comega**2.0d0)*eigval(isig)**2.0d0*(eigval(isig)**2.0d0-1.0d0))

      coebot = 4.0d0*eigval(isig)

      sigtwo = coetop*signum/(coebot*sigdem)

      signew = sigtwo + eigval(isig)

      write(*,900) sigtwo, signew, eigval(isig)

      write(3,920) signew

80  continue
900  format(f10.8, 5x, f10.8, 5x, f10.8)
920  format(d10.4)

      stop

      end

```

APPENDIX F: These programs evaluate negative betas using the Galerkin method.

```

C          Biggerbetas.f
C
C   THIS SETS UP THE MATRICIES TO BE READ BY readbb.f, MAKE SURE THAT
C   THE VALUE FOR BETA IS THE SAME IN BOTH PROGRAMS. This program is used
C   to find eigenvalues for the problem with nonzero negative betas. The equations are from
C   solving the SSWE when B can vanish in a sphere and allows for larger negative betas to
C   be used than the perturbation approach does. All integers are prefaced by "i's", except in
C   Functions, where they are prefaced by "n's", all arrays are prefaced by "A's" outside loops
C   range from 50 to 90 write loops range from 10 to 40 (for checking) outer index loops
C   range from 100 to 190, inner summation loops range from 200 up
C
implicit real*8(a-h, o-z)

parameter (NN = 5, LL = NN*(NN+3)/2, KK = LL)

real*8 AA(LL,KK), AB(LL,KK), AC(LL,KK), AD(LL,KK), AE(LL,KK)
real*8 AG(LL,KK), AP(LL,KK), AQ(LL,KK), AR(LL,KK), AT(LL,KK)
real*8 AU(LL,KK), AV(LL,KK), AX(LL,KK), AY(LL,KK)
real*8 ArhoP(4), ArhoR(10), Arho(10)
real*8 Aalpha(8)

open(11, file = 'AA.dat',status = 'unknown')
open(12, file = 'AB.dat',status = 'unknown')
open(13, file = 'AC.dat',status = 'unknown')
open(14, file = 'AD.dat',status = 'unknown')
open(15, file = 'AE.dat',status = 'unknown')
open(16, file = 'AG.dat',status = 'unknown')
open(17, file = 'AP.dat',status = 'unknown')

```

```

open(18, file = 'AQ.dat',status = 'unknown')
open(19, file = 'AR.dat',status = 'unknown')
open(20, file = 'AT.dat',status = 'unknown')
open(21, file = 'AU.dat',status = 'unknown')
open(22, file = 'AV.dat',status = 'unknown')
open(23, file = 'AX.dat',status = 'unknown')
open(24, file = 'AY.dat',status = 'unknown')
LLL = LL

```

C Choose value for Beta

```
beta = 0.000d0
```

C Choose values for irho (either 4 (PREM) or 10 (NEUT))

```
irho = 4
```

C irho = 10

```
crho20 = 12.5815d3
```

```
calph0 = .0081914306d0
```

```
Aalpha(1) = .0081914306880072d0/calph0
```

```
Aalpha(2) = .0032968419226614d0/calph0
```

```
Aalpha(3) = -.0016245386362791d0/calph0
```

```
Aalpha(4) = .0055783009628760d0/calph0
```

```
Aalpha(5) = -.0093306230249849d0/calph0
```

```
Aalpha(6) = .019869926098212d0/calph0
```

```
Aalpha(7) = -.018169982295218d0/calph0
```

```
Aalpha(8) = .0075630663016279d0/calph0
```

C Density from PREM

$$\text{ArhoP}(1) = 12.5815\text{d}3/\text{crhoP}0$$

$$\text{ArhoP}(2) = -.6903192\text{d}3/\text{crhoP}0$$

$$\text{ArhoP}(3) = -1.0868115\text{d}3/\text{crhoP}0$$

$$\text{ArhoP}(4) = -.9009295\text{d}3/\text{crhoP}0$$

C Density = NEUT

$$\text{ArhoR}(1) = .1134369\text{d}1$$

$$\text{ArhoR}(2) = 0.0\text{d}0$$

$$\text{ArhoR}(3) = -.1959092\text{d}0$$

$$\text{ArhoR}(4) = -.5251182\text{d}-1$$

$$\text{ArhoR}(5) = .4362577\text{d}-1$$

$$\text{ArhoR}(6) = -.1851055\text{d}-1$$

$$\text{ArhoR}(7) = .2536884\text{d}-2$$

$$\text{ArhoR}(8) = -.2360454\text{d}-1$$

$$\text{ArhoR}(9) = .2196424\text{d}-1$$

$$\text{ArhoR}(10) = -.8309286\text{d}-2$$

C Decide which density profile is to be used.

do 50 i = 1, irho

if (irho.eq.10) then

$$\text{Arho}(i) = \text{ArhoR}(i)$$

else

$$\text{Arho}(i) = \text{ArhoP}(i)$$

endif

50 continue

C Starting the loops for the calculations of the 14 matrices

```

do 100 ii = 0,NN
    iij = 0
    if(ii.eq.0) iij = 1
do 110 ij = iij,NN-ii
do 120 ip = 0,NN
    iqq = 0
    if(ip.eq.0) iqq = 1
do 130 iq = iqq,NN-ip
    il = ii*(2*NN+3-ii)/2 + ij
    ik = ip*(2*NN+3-ip)/2 + iq
C    Start filling in the 14 arrays needed
C    Array "AA"
    arhosum = 0.0d0
    azint = 0.0d0
    do 200 im = 1, irho
        arhosum = arhosum + Arho(im)/(2.0d0*(ii + ij + ip + iq) + im)
200    continue
    if ((ii + ip).eq.0) then
        aFIZ1 = 0.0d0
    else
        aFIZ1 = FIZ(ii + ip - 1, ij + iq)
    endif
    if((ij + iq).eq.0)then
        aFIZ2 = 0.0d0

```

```

else
    aFIZ2 = FIZ(ii + ip, ij + iq - 1)
endif

azint = ii*ip*aFIZ1 + ij*iq*aFIZ2

AA(il,ik) = -arhosum*azint

write(11,900) il, ik, AA(il,ik)

C   Array "AB"

brhosum = 0.0d0

bzint = 0.0d0

do 210 im = 1, irho
do 220 in = 1, irho
do 230 is = 1,8

bdenom = (in+2.0d0)*(2.0d0*(ii+ij+ip+iq)+im+in+is)

brhosum = brhosum + Arho(im)*Arho(in)*Aalpha(is)/bdenom

230 continue
220 continue
210 continue

bFIZ1 = FIZ(ii + ip, ij + iq)

bzint = (ip + iq)*bFIZ1

AB(il,ik) = -brhosum*bzint

write(12,900) il, ik, AB(il,ik)

C   Array "AC"

crhosum = 0.0d0

czint = 0.0d0

```



```

do 240 im = 1, irho
  crhosum = crhosum + Arho(im)/(2.0d0*(ii+ij+ip+iq)+im)
240 continue
  if ((ii + ip).eq.0) then
    cFIZ1 = 0.0d0
  else
    cFIZ1 = FIZ(ii + ip - 1, ij + iq)
  endif
  if((ij + iq).eq.0)then
    cFIZ2 = 0.0d0
  else
    cFIZ2 = FIZ(ii + ip, ij + iq - 1)
  endif
  czint = ii*ip*cFIZ1 + 2.0d0*ij*iq*cFIZ2
  AC(il,ik) = crhosum*czint
write(13,900) il, ik, AC(il,ik)
C   Array "AD"
drhosum = 0.0d0
dzint = 0.0d0
do 250 im = 1, irho
do 260 in = 1, irho
do 270 is = 1,8
ddenom = (in+2.0d0)*(2.0d0*(ii+ij+ip+iq)+im+in+is)
drhosum = drhosum + Arho(im)*Arho(in)*Alpha(is)/ddenom

```

```

270  continue
260  continue
250  continue

    dFIZ1 = FIZ(ii + ip, ij + iq)

        dzint = (ip + 2.0d0*iq)*dFIZ1

    AD(il,ik) = drhosum*dzint
    write(14,900) il, ik, AD(il,ik)

C    Array "AE"
    erhosum = 0.0d0
        ezint = 0.0d0
        do 280 im = 1, irho
            do 290 in = 1, irho
                do 300 ir = 1, irho
                    do 310 is = 1,8

                        edenom=(in+2.0d0)*(ir+2.0d0)

                        &  *(2.0d0*(ii+ij+ip+iq)+im+in+ir+is-1.0d0)

                        erhosum=erhosum + Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)/edenom

310  continue
300  continue
290  continue
280  continue

    if ((ii + ip).eq.0) then

        eFIZ1 = 0.0d0

    else

```

```

    eFIZ1 = FIZ(ii + ip - 1, ij + iq + 1)
endif
if ((ii + ip).eq.0) then
    eFIZ2 = 0.0d0
else
    eFIZ2 = FIZ(ii + ip - 1, ij + iq)
endif
eFIZ3 = FIZ(ii + ip, ij + iq)
    if((ij + iq).eq.0)then
        eFIZ4 = 0.0d0
    else
        eFIZ4 = FIZ(ii + ip, ij + iq - 1)
    endif
    ezint1 = -ip*(3.0d0*ii+2.0d0*ip-1.0d0)*eFIZ1
    ezint2 = ip*(ii+2.0d0*ip-1.0d0)*eFIZ2
    ezint3=(3.0d0*ij*(ip+iq)+iq*(3.0d0*ii+4.0d0*ip+2.0d0*iq-1.0d0)
& +2.0d0*(ip+iq))*eFIZ3
    ezint4 = -2.0d0*ij*iq*eFIZ4
    ezint = ezint1 + ezint2 + ezint3 + ezint4
    AE(il,ik) = erhosum*ezint
write(15,900) il, ik, AE(il,ik)
C    Array "AG"
grhosum = 0.0d0
gzint = 0.0d0

```

```

do 320 im = 1, irho
do 330 in = 1, irho
do 340 ir = 1, irho
do 350 is = 1,8
do 360 it = 1, irho
do 370 iw = 1,8

gnumer = Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)*Arho(it)*Aalpha(iw)

gdenom = (in+2.0d0)*(ir+2.0d0)*(it+2.0d0)*
& (2.0d0*(ii+ij+ip+iq)+im+in+ir+is+it+iw-1.0d0)

grhosum = grhosum + gnumer/gdenom
370 continue
360 continue
350 continue
340 continue
330 continue
320 continue

gFIZ1 = FIZ(ii + ip, ij + iq)
gzint = (ip + iq)*gFIZ1
AG(il,ik) = -grhosum*gzint
write(16,900) il, ik, AG(il,ik)

C   Array "AP"
prhosum = 0.0d0
pzint = 0.0d0
do 380 im = 1, irho

```

```

prhosum = prhosum + Arho(im)/(2.0d0*(ii + ij + ip + iq) + im)
380  continue

      if((ij + iq).eq.0)then
        pFIZ1 = 0.0d0
      else
        pFIZ1 = FIZ(ii + ip, ij + iq - 1)
      endif
      pzint = ij*iq*pFIZ1
      AP(il,ik) = -prhosum*pzint
      write(17,900) il, ik, AP(il,ik)
C    Array "AQ"
      qrhosum = 0.0d0
      qzint = 0.0d0
      do 390 im = 1, irho
        do 400 in = 1, irho
          do 410 is = 1,8
            qdenom = (in + 2.0d0)*(2.0d0*(ii + ij + ip + iq) + im + in + is)
            qrhosum = qrhosum + Arho(im)*Arho(in)*Aalpha(is)/qdenom
410    continue
400    continue
390    continue

      qFIZ1 = FIZ(ii + ip, ij + iq)
      qzint = iq*qFIZ1
      AQ(il,ik) = -qrhosum*qzint

```

```

write(18,900) il, ik, AQ(il,ik)

C   Array "AR"

rrhosum = 0.0d0

    rzint = 0.0d0

    do 420 im = 1, irho
    do 430 in = 1, irho
    do 440 ir = 1, irho
    do 450 is = 1,8

    rdenom=(in+2.0d0)*(ir+2.0d0)
&   *(2.0d0*(ii+ij+ip+iq)+im+in+ir+is-1.0d0)

    rrhosum=rrhosum + Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)/rdenom
450   continue
440   continue
430   continue
420   continue

    if ((ii + ip).eq.0) then
        rFIZ1 = 0.0d0
    else
        rFIZ1 = FIZ(ii + ip - 1, ij + iq + 2)
    endif

    if ((ii + ip).eq.0) then
        rFIZ2 = 0.0d0
    else
        rFIZ2 = FIZ(ii + ip - 1, ij + iq + 1)

```

```

endif

rFIZ3 = FIZ(ii + ip, ij + iq + 1)
rFIZ4 = FIZ(ii + ip, ij + iq)

  if((ij + iq).eq.0)then
    rFIZ5 = 0.0d0
  else
    rFIZ5 = FIZ(ii + ip, ij + iq - 1)
  endif

rzint1 = ip*(2.0d0*ii+2.0d0*ip-1.0d0)*rFIZ1
      rzint2 = -ip*(2.0d0*ip-1.0d0)*rFIZ2
      rzint3 = -(iq*(2.0d0*ii+2.0d0*ij+2.0d0*iq-1.0d0)
& +2.0d0*ip*(ij+2.0d0*iq)+4.0d0*(ip+iq))*rFIZ3
      rzint4 = -(ij*(ip+2.0d0*iq)
& +iq*(3.0d0*ii+2.0d0*ip+2.0d0*iq-1.0d0))*rFIZ4
      rzint5 = 2.0d0*ij*iq*rFIZ5
      rzint = rzint1 + rzint2 + rzint3 + rzint4 + rzint5
      AR(il,ik) = rrhosum*rzint

write(19,900) il, ik, AR(il,ik)

```

C Array "AT"

```

trhosum = 0.0d0

      tzint = 0.0d0

      do 460 im = 1, irho

      do 470 in = 1, irho

      do 480 ir = 1, irho

```

```

do 490 is = 1,8

do 500 it = 1, irho

do 510 iw = 1,8

tnumer=Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)*Arho(it)*Aalpha(iw)

tdenom = (in+2.0d0)*(ir+2.0d0)*(it+2.0d0)*
& (2.0d0*(ii+ij+ip+iq)+im+in+ir+is+it+iw-1.0d0)

trhosum = trhosum + tnumer/tdenom

510 continue

500 continue

490 continue

480 continue

470 continue

460 continue

tFIZ1 = FIZ(ii + ip, ij + iq + 1)

tFIZ2 = FIZ(ii + ip, ij + iq)

tzint = (ip + iq)*tFIZ1 + iq*tFIZ2

AT(il,ik) = trhosum*tzint

write(20,900) il, ik, AT(il,ik)

C   Array "AU"

urhosum = 0.0d0

uzint = 0.0d0

do 520 im = 1, irho

do 530 in = 1, irho

do 540 ir = 1, irho

```



```

do 550 is = 1,8
do 560 it = 1, irho
do 570 iv = 1, irho
do 580 iw = 1,8

unumer = Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)
& *Arho(it)*Arho(iv)*Aalpha(iw)

udenom = (in+2.0d0)*(ir+2.0d0)*(it+2.0d0)*(iv+2.0d0)*
& (2.0d0*(ii+ij+ip+iq)+im+in+ir+is+it+iv+iw-2.0d0)

urhosum = urhosum + unumer/udenom
580 continue
570 continue
560 continue
550 continue
540 continue
530 continue
520 continue

if((ii+ip).eq.0)then
    uFIZ1 = 0.0d0
else
    uFIZ1 = FIZ(ii + ip - 1, ij + iq + 1)
endif

uFIZ2 = FIZ(ii + ip, ij + iq)

if((ij+iq).eq.0)then
    uFIZ3 = 0.0d0

```

```

else
    uFIZ3 = FIZ(ii + ip, ij + iq - 1)
endif
uzint = -ii*ip*uFIZ1 + (ii*iq + ij*ip + ij*iq)*uFIZ2 - ij*iq*uFIZ3
AU(il,ik) = urhosum*uzint
write(21,900) il, ik, AU(il,ik)
C   Array "AV"
vrhosum = 0.0d0
vzint = 0.0d0
do 590 im = 1, irho
do 600 in = 1, irho
do 610 ir = 1, irho
do 620 is = 1,8
vnumber = Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)
vdenom = (in+2.0d0)*(ir+2.0d0)*
& (2.0d0*(ii+ij+ip+iq)+im+in+ir+is-1.0d0)
vrhosum = vrhosum + vnumber/vdenom
620 continue
610 continue
600 continue
590 continue
vFIZ1 = FIZ(ii + ip, ij + iq + 1)
vFIZ2 = FIZ(ii + ip, ij + iq)
vzint1 = (iq*(2.0d0*(ii+ij+ip+iq)-1.0d0)+4.0d0*iq)*vFIZ1

```

```

        vzint2 = -(ij*iq + 2.0d0*iq)*tFIZ2
vzint = vzint1 + vzint2

        AV(il,ik) = vrhosum*vzint
write(22,900) il, ik, AV(il,ik)
C      Array "AX"
xrhosum = 0.0d0

        xzint = 0.0d0

        do 630 im = 1, irho
        do 640 in = 1, irho
        do 650 ir = 1, irho
        do 660 is = 1,8
        do 670 it = 1, irho
        do 680 iw = 1,8

        xnumer = Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)*Arho(it)*Aalpha(iw)
        xdenom = (in+2.0d0)*(ir+2.0d0)*(it+2.0d0)*
& (2.0d0*(ii+ij+ip+iq)+im+in+ir+is+it+iw-1.0d0)

        xrhosum = xrhosum + xnumer/xdenom
680      continue
670      continue
660      continue
650      continue
640      continue
630      continue

        xFIZ1 = FIZ(ii + ip, ij + iq + 1)

```

```

      xzint = iq*xFLZ1
      AX(il,ik) = -xrhosum*xzint
      write(23,900) il, ik, AX(il,ik)
C   Array "AY"
      yrhosum = 0.0d0
      yzint = 0.0d0
      do 690 im = 1, irho
      do 700 in = 1, irho
      do 710 ir = 1, irho
      do 720 is = 1,8
      do 730 it = 1, irho
      do 740 iv = 1, irho
      do 750 iw = 1,8
      ynumer = Arho(im)*Arho(in)*Arho(ir)*Aalpha(is)
&   *Arho(it)*Arho(iv)*Aalpha(iw)
      ydenom = (in+2.0d0)*(ir+2.0d0)*(it+2.0d0)*(iv+2.0d0)*
&   (2.0d0*(ii+ij+ip+iq)+im+in+ir+is+it+iv+iw-2.0d0)
      yrhosum = yrhosum + ynumer/ydenom
750   continue
740   continue
730   continue
720   continue
710   continue
700   continue

```

```

690  continue

      if((ii + ip).eq.0)then
          yFIZ1 = 0.0d0
      else
          yFIZ1 = FIZ(ii + ip - 1, ij + iq + 2)
      endif

      yFIZ2 = FIZ(ii + ip, ij + iq + 1)
      yFIZ3 = FIZ(ii + ip, ij + iq)
      yzint1 = ii*ip*yFIZ1

      yzint2 = -(ij*iq + ij*ip + ii*iq + 2.0d0*(ip + iq))*yFIZ2
      yzint3 = (ij*iq + 2.0d0*iq)*yFIZ3
      yzint = yzint1 + yzint2 + yzint3

      AY(il,ik) = yrhosum*yzint
      write(24,900) il, ik, AY(il,ik)
130  continue
120  continue
110  continue
100  continue
900  format(i3, 3x, i3, 3x, d15.8)

      stop
      end

C    Function to calculate the z integrals using the gamma function.

      real*8 function FIZ(ni,nj)

      Implicit real*8 (a-h, o-z)

```

```

      rni = dble(ni)
      rnj = dble(nj)
      call factlog(2*nj, cone)
      call factlog(ni, ctwo)
      call factlog(ni + nj + 1, cthree)
      call factlog(nj, cfour)
      call factlog(2*(ni + nj + 1), cfive)
      csix = dlog10(2.0**(2.0*rni + 1))
      clogs = cone + ctwo + cthree - cfour - cfive + csix
      FIZ = 10.d0*clogs
      return
      end

```

C Calculates the base-10 logarithm of the factorial of an integer

```

      subroutine factlog(int, fctlg)
      implicit real*8(a-h, o-z)
      w = 0.d0
      do 100 i = 1, int
      di = dble(i)
      w = w + dlog10(di)
100   continue
      fctlg = w
      return
      end

```

C readbb.f

C Similar to above program, but reads matrices in and does evaluation.

```
implicit real*8(a-h, o-z)
```

```
parameter (NN = 5, LL = NN*(NN+3)/2, KK = LL)
```

```
character zer*15, iter*10, bet*10, den*35
```

C Values for IMSL routine DZBREN

```
real*8 ERRABS, ERRREL, A, B
```

```
integer maxfn
```

```
parameter (nroot = 120)
```

```
real*8 detm, xguess(nroot)
```

```
external detm, DZBREN
```

```
common/a/ LLL, AA(LL,KK), AB(LL,KK), AC(LL,KK), AD(LL,KK)
```

```
common/a/ AE(LL,KK), AG(LL,KK), AP(LL,KK), AQ(LL,KK)
```

```
common/a/ AR(LL,KK), AT(LL,KK), AU(LL,KK), AV(LL,KK)
```

```
common/a/ AX(LL,KK), AY(LL,KK)
```

```
common/a/ beta, gamma, f
```

```
open(11, file = 'AA.dat', status = 'old')
```

```
open(12, file = 'AB.dat', status = 'old')
```

```
open(13, file = 'AC.dat', status = 'old')
```

```
open(14, file = 'AD.dat', status = 'old')
```

```
open(15, file = 'AE.dat', status = 'old')
```

```
open(16, file = 'AG.dat', status = 'old')
```

```
open(17, file = 'AP.dat', status = 'old')
```

```
open(18, file = 'AQ.dat', status = 'old')
```

```

open(19, file = 'AR.dat',status = 'old')
open(20, file = 'AT.dat',status = 'old')
open(21, file = 'AU.dat',status = 'old')
open(22, file = 'AV.dat',status = 'old')
open(23, file = 'AX.dat',status = 'old')
open(24, file = 'AY.dat',status = 'old')
open(26, file = 'Zeros.out',status = 'unknown')
open(27, file = 'Hbefor.out',status = 'unknown')

```

```

LLL = LL

```

```

beta = 0.000d0

```

```

C   Select value for irho

```

```

    irho = 4

```

```

C   irho = 10

```

```

zer = 'Zeroes'

```

```

    iter = 'Iterations'

```

```

write(26, 1071) zer, iter

```

```

C   Values from PREM or NEUT, comeqa sec-1, cG m3/(kg sec2), crhoP0 kg/m3,

```

```

C   cb km, calph0 sec/km, cgatb km/sec-2)

```

```

comeqa = 7.292115d-5

```

```

cG = 6.6732d-11

```

```

cpi = 3.1415927d0

```

```

crhoP0 = 12.5815d3

```

```

cb = 3480.0d0

```

```

calph0 = .0081914306d0

```



```

cgatb = 1.06823d-2
if (irho.EQ.4) then
    gammat = comeqa**2.0d0
    gammab = 2.0d0*cpi*cG*crhoP0
    gamma = gammat/gammab
else
    gammat = 2.0d0*cb*comeqa**2.0d0
    gammab = 3.0d0*gatb
    gamma = gammat/gammab
endif
if (irho .EQ. 4) then
    ftop = 4.0d0*calph0*(cpi*cG*crhoP0*cd)**2.0d0
    fbot = comeqa**2.0d0
    f = ftop/fbot
else
    ftop = 9.0d0*calph0*gatb**2.0d0
    fbot = 4.0d0*comeqa
    f = ftop/fbot
endif

```

C Starting the loops for the reading of the 14 matrices

```

do 100 ii = 0,NN
    iij = 0
    if(ii.eq.0) iij = 1
    do 110 ij = iij,NN-ii

```

```
do 120 ip = 0,NN
    iqq = 0
    if(ip.eq.0) iqq = 1
    do 130 iq = iqq,NN-ip
        il = ii*(2*NN+3-ii)/2 + ij
        ik = ip*(2*NN+3-ip)/2 + iq
```

C Start reading in the 14 arrays needed

```
read(11,*) ia, ib, AA(il, ik)
read(12,*) ia, ib, AB(il, ik)
read(13,*) ia, ib, AC(il, ik)
read(14,*) ia, ib, AD(il, ik)
read(15,*) ia, ib, AE(il, ik)
read(16,*) ia, ib, AG(il, ik)
read(17,*) ia, ib, AP(il, ik)
read(18,*) ia, ib, AQ(il, ik)
read(19,*) ia, ib, AR(il, ik)
read(20,*) ia, ib, AT(il, ik)
read(21,*) ia, ib, AU(il, ik)
read(22,*) ia, ib, AV(il, ik)
read(23,*) ia, ib, AX(il, ik)
read(24,*) ia, ib, AY(il, ik)
```

130 continue

120 continue

110 continue

100 continue

C The value of the determinant of H is found from the subroutine detm. Where this value
 C changes from pos to neg is where the zeroes are searched for. To start with, nroot
 C guesses, are evaluated to see where the determinant changes sign. Set up the guesses.

do 4999 kkk = 1, nroot

 xguess(kkk) = dble(kkk)/(nroot + 1)

4999 continue

C Find the zeroes using the IMSL subroutine DZBREN

 ERRABS = 0.0d0

 ERRREL = 1.0d-6

 maxfn = 1.0d4

 A = 0.0d0

 B = 0.0d0

 do 5000 kkk = 1, nroot

 write(27, 1060) kkk, xguess(kkk), detm(xguess(kkk))

 A = xguess(kkk)

 if (kkk.EQ.nroot) then

 B = xguess(kkk)

 else

 B = xguess(kkk + 1)

 endif

 Aval = detm(A)

 Bval = detm(B)

 if (Aval*Bval.LT.0.0d0) then

```

        maxfn = 1000

        call DZBREN(detm, ERRABS, ERRREL, A, B, maxfn)

        write(26, 1070) B, maxfn

    else

        goto 5000

    endif

5000    continue

1060    format(i5, 3x, d15.8, 3x, d15.9)

1070    format(d15.4, 3x, i10)

1071    format(5x, a15, 3x, a10/)

        stop

        end

C      Real function Detm(x) used to find the determinant of H(l,k) using
C      the IMSL subroutines DLFTRG and DLFDRG. This uses the LU factorization
C      of the matrix. This seems to be ok, even though there are zeroes on
C      the main diagonal.

      REAL*8 FUNCTION DETM(X)

      IMPLICIT REAL*8(A-H, O-Z)

      parameter (NN = 5, LL = NN*(NN+3)/2, KK = LL)

      parameter (LDA = 20, LDFAC = 20, N = 20)

      integer IPVT(N)

      real*8 H(LDA, LDA), DET1, DET2, FAC(LDFAC, LDFAC)

      common/a/ LLL, AA(LL, KK), AB(LL, KK), AC(LL, KK), AD(LL, KK)

      common/a/ AE(LL, KK), AG(LL, KK), AP(LL, KK), AQ(LL, KK)

```

```

common/a/ AR(LL, KK), AT(LL, KK), AU(LL, KK), AV(LL, KK)
common/a/ AX(LL, KK), AY(LL, KK)
common/a/ beta, gamma, f
DO 1 L = 1, LLL
DO 2 K = 1, LLL
H(L, K) = 1.0d5 * (X**6.0d0 * (AA(L, K) + gamma*beta*f*AB(L, K))
& + X**4.0d0 * (AC(L, K) + gamma*beta*f*AD(L, K) + beta*f*AE(L, K)
& + gamma*beta*f*f*(2.0d0-beta)*AG(L, K))
& + X**2.0d0 * (AP(L, K) + gamma*beta*f*AQ(L, K) + beta*f*AR(L, K)
& + gamma*beta*f*f*(2.0d0-beta)*AT(L, K) + beta*beta*f*f*AU(L, K))
& + beta*f*AV(L, K) + gamma*beta*f*f*(2.0d0-beta)*AX(L, K)
& + beta*beta*f*f*AY(L, K))
2 CONTINUE
1 CONTINUE
call DLFTRG (N, H, LDA, FAC, LDFAC, IPVT)
call DLFDRG (N, FAC, LDFAC, IPVT, DET1, DET2)
detm = det1*10.0**dex2
RETURN
END

```