FINITE CW-COMPLEXES

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by

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Abstract

Topological spaces of the same homotopy type as CW-complexes are considered (CW-spaces). Constructive methods are used to give algebraic characterizations of CW-complexes, firstly of finite type, and secondly, of finite dimension. When both conditions are satisfied, there is an element of the projective class group of the integral group ring of the fundamental group of the space, a homotopy type invariant, whose vanishing is necessary and sufficient to guarantee that a CW-complex be finite.

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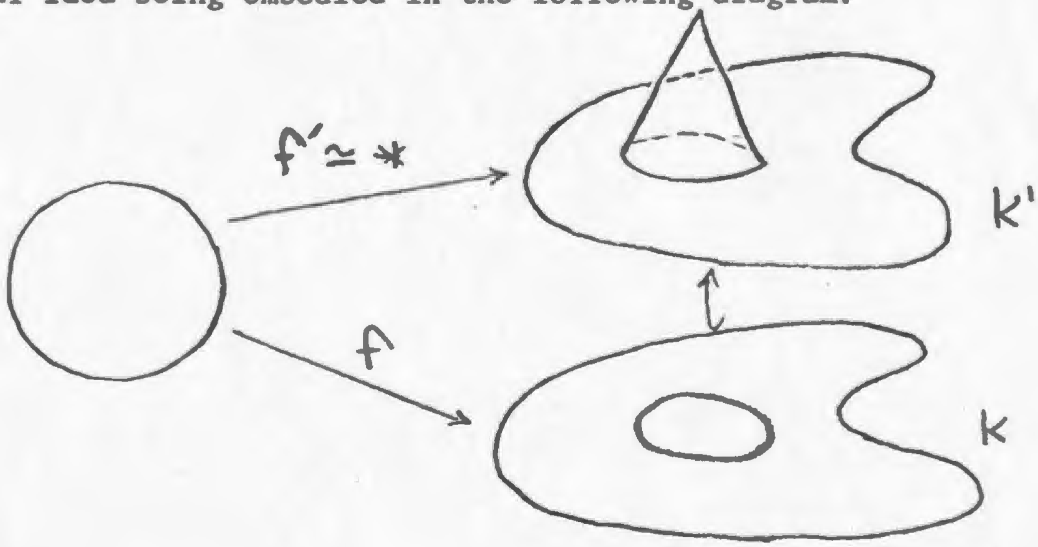
Introduction

CW-complexes, first introduced by J. H. C. Whitehead [19] almost thirty years ago, were shown by Whitehead himself to possess properties very useful in the study of algebraic topology. This thesis concerns itself with the homotopy type of such complexes. There are various ways of phrasing the right questions to be asked in this context; historically, because of its importance in differential topology, the approach took the following form:

Q. Suppose X is a space which is dominated by (a) a CW-complex, (b) a countable CW-complex, or (c) a finite CW-complex; then when is X of the same homotopy type as (a), (b) or (c)?

In fact, the two conditions "dominated by" and "of the same homotopy type" were seen to be equivalent in the case (a) [18] (1950), in the case (b) [9] (1957) and, for simply connected spaces X, in the case (c) [16] and [6] (1959). It was conjectured that the two conditions were equivalent in the case (c) for arbitrary spaces X. The question was now taken up by C. T. C. Wall [16] and C. B. de Lyra [7], who realized that a closer examination of the algebraic properties (by this, I mean homotopy and homology groups) of such spaces was required; in particular, a close examination of the action of the fundamental group on the higher homotopy groups. Of the two, Wall was the more successful, and he gave an almost complete characterization in the case (c), demonstrating the falsity of the conjecture mentioned above.

In Chapter One, we carry out the preliminary work necessary to establish the properties of CW-complexes which we will require. Of particular interest is the process of attaching cones to a space, K, in order to make a given map f homotopically trivial (see I.4.1); the essential idea being embodied in the following diagram.



In Chapter Two, we use this construction as in Wall's paper to answer the question above in the case (c). First, we give an algebraic characterization of CW-complexes of finite type and then of CW-complexes of finite dimension. If a space X is of the homotopy type of a CW-complex of finite type and of the homotopy type of a CW-complex of finite dimension, then there is an element of the (reduced) projective class group, $\mathbf{K}$, of the integral group ring R of the fundamental group G of X, which is a homotopy type invariant and X is of the homotopy type of a finite CW-complex if and only if this element vanishes (see Theorem II.3.4).

CHAPTER ONE

Preliminaries

§ I.1 CW-spaces

We show first how to construct a CW-complex; the references here are [10] and [11]. Consider the following diagram in Top, the category of topological spaces and continuous maps,

$$K^0 \xrightarrow{\bar{i}_0} K^1 \xrightarrow{\bar{i}_1} K^2 \xrightarrow{\bar{i}_2} \cdots \xrightarrow{\bar{i}_{n-2}} K^{n-1} \xrightarrow{\bar{i}_{n-1}} K^n \xrightarrow{\bar{i}_n} \cdots, \tag{1.1}$$

where K^0 is any discrete space and the maps $\bar{i}_n$ are one to one and closed. Assume that K^{n-1} has been constructed and let J_n be any given indexing set such that for each $j \in J_n$, we have an $(n-1)$-sphere denoted S_j^{n-1} and a map $f_j^{n-1} : S_j^{n-1} \to K^{n-1}$. By the universal property of coproducts, this gives rise to a unique map $f^{n-1} : \coprod_{j \in J_n} S_j^{n-1} \to K^{n-1}$. Now for each $j \in J_n$, S_j^{n-1} is a closed subspace of $CS_j^{n-1} \cong E_j^n$ and so $\coprod_j S^{n-1}$ is a closed subspace of $\coprod_j E^n$. We now define K^n to be the space obtained by adjunction of $\coprod_j E^n$ to K^{n-1} via the map f^{n-1}, as shown in the following pushout diagram.

$$\begin{tikzcd}
\coprod_j E^n \arrow[r, "\bar{f}^{n-1}"] & K^n \\
\coprod_j S^{n-1} \arrow[u, hook, "i"] \arrow[r, "f^{n-1}"] & K^{n-1} \arrow[u, hook, "\bar{i}_{n-1}"']
\end{tikzcd} \tag{1.2}$$

We call this process _attaching_ n-_cells_ _to_ K^{n-1}. Since $\coprod_j S^{n-1}$ is a closed subspace of $\coprod_j E^n$, it follows that the map $\bar{i}_{n-1}$ is one to one and closed. The image of $E^n_j \setminus S^n_j$, denoted e^n_j, in K^n is called an n-_cell_ of K^n and the restriction of $\bar{f}^{n-1}$ to this domain is called a _characteristic_ _map_; a _closed_ n-_cell_ being the image of E^n_j.

We now define a _CW-complex_ K to be the union of all the K^n, $n \geq 0$, with the weak topology with respect to the inclusions $K^n \subset K$. The spaces K^n, $n \geq 0$, are called the n-_skeletons_ of K. If there exists an integer $m \geq 0$ such that $\forall n \geq m$, we have $K^n = K^m$; we say that K is of _finite_ _dimension_ m; if all the sets J_n used in the construction are finite or countable, we say that K is of _finite_ _type_ or _countable_, respectively. If K is of finite type and is finite dimensional, then we say that K is a _finite_ CW-complex.

With respect to the above construction, we note that if we are working in _Top_, we may take the coproduct to be the disjoint union, but if we give our spaces base points and consider base point preserving maps, then we take the coproduct to be the wedge (one point union).

CW-complexes possess particularly nice topological properties; they satisfy the separation axioms T_0 through T_4 and are paracompact and locally contractible; for CW-complexes, the concepts of connectedness and path connectedness are equivalent. In what follows, we always take _CW_ to be the category of pointed connected CW-complexes and pointed continuous functions. Also, we always take _W_ to be category of pointed connected CW-*spaces* and pointed maps; a CW-space being a space X of $\underline{\mathrm{Top}}$ which is of the homotopy type of some (pointed, connected) CW-complex. In the case of $\underline{\mathrm{CW}}$, we may take K^0 to consist of just a single point. Any compact subset of K intersects only finitely many cells of K. Note that the skeletons of K are themselves CW-complexes and that the inclusions $K^n \hookrightarrow K$ and $K^n \hookrightarrow K^{n+1}$ are cofibrations.

We are concerned here with spaces and maps only up to homotopy and, hence, we adopt the common practice of replacing spaces up to homotopy type and maps up to homotopy at our convenience. For instance, given $f : X \to Y$ in $\underline{W}$, we may take X and Y to be in $\underline{\mathrm{CW}}$ and may replace f by a cellular map. Furthermore, taking cellular maps, $\underline{\mathrm{CW}}$ is closed under the formation of mapping cones, mapping cylinders and mapping tracks. Therefore, we can, at our leisure, take f as above as a fibration from a CW-complex into a CW-complex, or as an inclusion (cofibration) of a subcomplex into a CW-complex, and in the sequel, we shall do so, sometimes without comment. In particular, we usually replace maps by inclusions into mapping cylinders.

In what follows, we always consider a space $X \in \underline{W}$ with a *fundamental group* denoted G. We denote the *integral group ring* of such a group G by either $\mathbb{Z}G$ or R; R consists of all formal finite linear sums $n_i g_i$, $n_i \in \mathbb{Z}$, $g_i \in G$ with the obvious multiplication; note that this is just the free abelian group on the set G with the induced multiplication, that is $\oplus_G \mathbb{Z}$. So in the sequel, we often take X, G and R given as above without further comment.

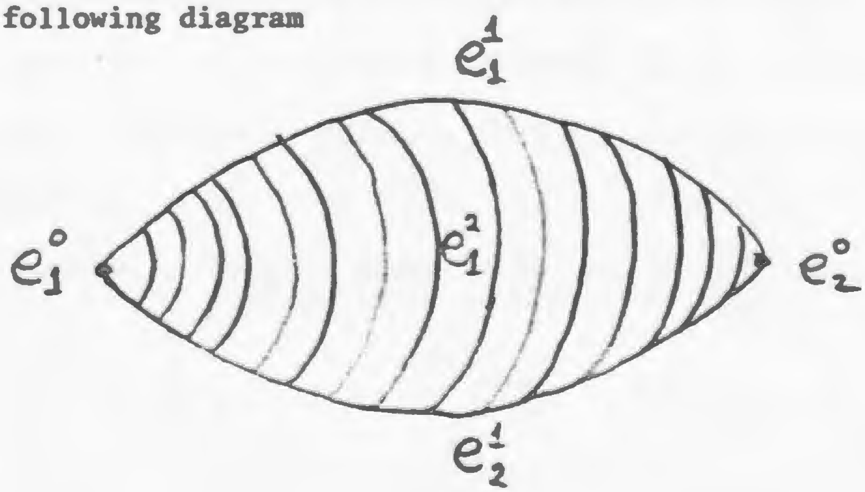
We now quote as a theorem a first result, which will be very useful later on. Recall that if F is the free group on a set of generators, $A = \{x_i : i \in I\}$ and R is the least normal subgroup of F containing any set $B = \{r_j \in F : j \in J\}$ then the sets A and B are said to give a group presentation of F/R which is said to be finite if both A and B are finite.

Theorem 1.3 - Given $K \in \underline{CW}$ such that K^n is finite, then

(a) if $n \geq 1$, $\pi_1(K)$ is finitely generated; and

(b) if $n \geq 2$, $\pi_1(K)$ is finitely presented.

Proof - The details of what follows are given in Massey [8, ch. 6, 7]. Intuitively, what happens is the following. The 1-skeleton K^1 is a (connected) collection of points and edges; from this, we remove a maximal collection of edges that contains no closed paths (a "tree"). Then the edges that remain freely generate $\pi_1(K^1)$. We see later (1.10) that the fundamental group of K depends only on its 2-skeleton. So we must consider $K^1 \hookrightarrow K^2$; it turns out that $\pi_1(K^2)$ is given by the generators above with one relation for every 2-cell; intuitively, we have the following diagram



in which $\pi_1(K^1)$ is the free group on two generators $\{e_1^1, e_2^1\}$ and $\pi_1(K^2)$ is this group subject to the relation $e_0^1 \cdot e_1^1 = 1$. //

We now proceed to a brief description of cellular homology on <u>CW</u>, and so on <u>W</u>. We use Switzer [15]. Let H denote the usual singular homology and define the <u>cellular chain complex</u> of K by setting $C_n(K) = H_n(K^n, K^{n-1})$ and taking the boundary operator to be the connecting homomorphism, Δ, of the triple (K^n, K^{n-1}, K^{n-2}),

$$\ldots \to H_n(K^n, K^{n-2}) \to H_n(K^n, K^{n-1}) \xrightarrow{\Delta} H_{n-1}(K^{n-1}, K^{n-2}) \to \ldots .$$

This chain complex is chain homotopic to the singular chain complex of K and hence the two have the same homology. But the inclusion $K^{n-1} \subset K^n$ is a cofibration and so it follows that $H_n(K^n, K^{n-1}) = H_n(K^n/K^{n-1}) = H_n(\vee_{j\in J_n} S_j^n) = \oplus_{J_n} H_n(S_j^n) = \oplus_{J_n} \mathbf{Z}$; so we may say that the n-chains of K are (freely) generated by the n-cells of K. It is now immediate that a CW-complex of finite type has finitely generated homology and that a CW-complex, K, of finite dimension m has homology $H_n(K) = 0$ for all $n > m$, and $H_m(K)$ is a free (abelian) group. Note that, just as for singular homology, the cellular homology of a pair (X, A) (by which we mean $A \hookrightarrow X$) is just the homology of the complex $C(X)/C(A)$.

To obtain singular or cellular homology with coefficients in an abelian group B, denoted by $H_n(K; B)$, we just take the homology of the chain complex $C(K)\otimes_{\mathbf{Z}} B$, where $C(K)$ is the singular or cellular chain complex of K. Similarly, the singular of cellular cohomology with coefficients in an abelian group B, denoted by $H^n(K; B)$ or in the case $B = \mathbb{Z}$ by $H^n(K)$, to be the homology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C(K); B)$, where we take $C(K)$ to be either the singular or the cellular chain complex of K. It is now immediate that if K is of finite dimension m, then $H^n(K) = 0$ for all $n > m$.

Now it often happens that $\pi_1(X) = G$ acts non-trivially on an abelian group B in which case B becomes a (non-trivial) module over $R = \mathbb{Z}G$ in a natural way. Then one wishes to study the sub(co)chain complex of $C(X) \otimes_{\mathbb{Z}} B$ or $\mathrm{Hom}_{\mathbb{Z}}(C(X), B)$ which consists of those chains or cochains which, roughly speaking, commute with the action of G on B. This is known as homology or cohomology _with local coefficients_; we return to this in more detail later on at the end of Section I.2.

We now give as theorems some of the basic facts relating homotopy and homology groups.

Theorem 1.4 - For a pair (X, A) in $\underline{W}$, if $n \geq 2$ or if $n = 1$ and $A = *$, with base points understood, there is a (Hurewicz) homomorphism $h : \pi_n(X, A) \to H_n(X, A)$ such that

(a) for $n \geq 2$, the following square commutes,

$$\begin{array}{ccc} \pi_n(X, A) & \xrightarrow{\sigma} & \pi_{n-1}(X, A) \\ \downarrow h & & \downarrow h \\ H_n(X, A) & \xrightarrow{\sigma} & H_{n-1}(X, A); \end{array} \qquad (1.5)$$

(b) given $f : (X, A) \to (Y, B)$, the following square commutes,

$$\begin{tikzcd} \pi_n(X, A) \arrow[r, "f_\#"] \arrow[d, "h"] & \pi_n(Y, B) \arrow[d, "h"] \\ H_n(X, A) \arrow[r, "f_*"] & H_n(Y, B) \end{tikzcd} \tag{1.6}$$

Proof - [14, 7.4.3]. //

Note that for $A = *$, $H_n(X, A) \cong H_n(X)$. The theorem below, known as the Hurewicz isomorphism theorem, gives conditions under which h above becomes an isomorphism. First of all, however, consider a map $f : X \to Y$ in $\underline{W}$; replacing this by an inclusion of a subcomplex into a CW-complex, we define $\pi_q(f)$ to be $\pi_q(Y, X)$ and $H_q(f)$ to be $H_q(Y, X)$. Such a map is said to be n-connected in case $\pi_q(f) = 0$ for all $q \leq n$; this is equivalent to saying that $f_\# : \pi_q(X) \to \pi_q(Y)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.

Theorem 1.7 - Take $f : X \to Y$ in $\underline{W}$ to be a map of simply connected spaces; then, for $n \geq 2$, f is n-connected if and only if $H_q(f) = 0$ for $q \leq n$; in either case, $h : \pi_{n+1}(f) \to H_{n+1}(f)$ is an isomorphism.

Proof - Spanier [14, 7.5.4]. //

If we take $X = *$ in this theorem, we have that the first non-vanishing homology group of Y is isomorphic to the first non-vanishing homotopy group of Y.

The following is known as the Whitehead Comparison Theorem.

Theorem 1.8 - Let $f : X \to Y$ be in $\underline{W}$, then f is n-connected implies that $H_q(f) = 0$ for $q \leq n$ and if X and Y are simply connected, the converse holds.

Proof - Spanier [14, 7.5.9]. //

A map which is n-connected for all n is called a weak homotopy equivalence; it induces isomorphisms in homotopy of all dimensions. The niceness of CW-complexes for homotopy theory is illustrated by the following.

Theorem 1.9 - A map $f : X \to Y$ in $\underline{W}$ is a homotopy equivalence if and only if it is a weak homotopy equivalence. Furthermore, a weak homotopy equivalence induces isomorphisms in homology of all dimensions. Conversely, if X and Y are simply connected, then a map inducing isomorphisms in homology is a weak homotopy equivalence and, hence, a homotopy equivalence.

Proof - Spanier [14, 7.6.24, 7.6.25]. //

Note that the existence of a map inducing the isomorphisms is critical if X and Y are to have the same homotopy type; there are spaces with isomorphic homotopy which are not of the same homotopy type; see appendix A. We also need the following.

Theorem 1.10 - The inclusions $K^n \hookrightarrow K^{n+1}$ and $K^n \hookrightarrow K$ are n-connected.

Proof - Consider $K^n \xhookrightarrow{i} K^{n+1} \xrightarrow{j} K^{n+1}/K^n$ in which we replace j by the mapping track of j, $E_j = K^{n+1}$, which gives us a fibration $p : E_j \to K^{n+1}/K^n$ which has a fibre of the same homotopy type as K^n and so just take the homotopy exact sequence of $K^n \hookrightarrow E_j \longrightarrow K^{n+1}/K^n = \vee_{J_n} S^{n+1}$. To see that the fibre really is of the homotopy type of K^n, we consider the pullback diagram,

$$\begin{tikzcd} E_j \arrow[r] \arrow[d] \arrow[dr, dashed, "j\pi_{e_0} = p" description] & P(K^{n+1}/K^n) \arrow[d, "e_0"] \\ K^{n+1} \arrow[r, "j"'] & K^{n+1}/K^n \end{tikzcd}$$

where $P(K^{n+1}/K^n)$ is the space of paths on K^{n+1}/K^n and e_0 is the evaluation map (at 0). Then the fibre $F = \{(x, \ell) \in E_j = K^{n+1}\,{}_j\pi_{e_0}\,P(K^{n+1}/K^n) : p(x, \ell) = *\} = \{(x, \ell) \in K^{n+1} \times P(K^{n+1}/K^n) : j(x) = *$ and $e_0(\ell) = \ell(0) = *\} = \{(x, \ell) \in K^{n+1} \times P(K^{n+1}/K^n) : x \in K^n$ and $\ell \in P^+(K^{n+1}/K^n)\} = K^n \times P^+(K^{n+1}/K^n)$. Now $P^+(K^{n+1}/K^n)$ is contractible and hence the result. //

This theorem says that up to dimension $n - 1$, the homotopy and homology of K depends only on its n-skeleton.

Now we sometimes wish to know when an n-connected map $(n \geq 2)$ $f : X \to Y$ in $\underline{W}$ possesses a homotopy right inverse $g : Y \to X$. For this, we need a few facts from obstruction theory, and we refer to Steenrod [13].

We replace f by an n-connected fibration $(n \geq 2)$ $f : X \to Y$ with fibre F (note that F is simply connected). The map g is defined successively over the skeletons of Y; we take $g_0 : Y^0 = * \to X$ to be the constant map. So we may suppose that $g_n : Y^n \to X$ is given such that $fg_n = i : Y^n \hookrightarrow Y$. The question to answer now is whether or not we can extend g_n over the $(n+1)$-cells of Y. So let X_e denote the inverse image in X, $f^{-1}(e^{n+1})$ of an $(n+1)$-cell e^{n+1} of Y. It follows from the fact that e^{n+1} is contractible that X_e is homotopy equivalent to F. To see this, take the pullback diagram

$$\begin{tikzcd}
F \arrow[r, hook, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & X_e \arrow[d, "f"] \\
* \arrow[r, hook, "\simeq"] & e^{n+1}
\end{tikzcd}$$

Now the boundary of e^{n+1} denoted $\dot{e}^{n+1}$ lies in Y^n and also in e^{n+1} and so $g_n|\dot{e}^{n+1} \subset X_e$. Altogether, we have a diagram as follows

$$\begin{tikzcd}
F \arrow[d] & X_e \arrow[l, "\simeq"'] \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] & & & \\
* & e^{n+1} \arrow[l, "\simeq"] \arrow[r, hook] & Y & Y^n \arrow[l, hook'] \arrow[ul, "g_n"'] & \dot{e}^{n+1} \arrow[l, hook'] \arrow[ulll, bend right, "g_n|\dot{e}^{n+1}"'] & S^n \arrow[l, "h"']
\end{tikzcd}$$

where we take $h : S^n \to Y$ to be the attaching map. So we claim that g_n determines an element denoted $\bar{g}_n$ of $C^{n+1}(Y; \pi_n(F)) = \mathrm{Hom}(C_{n+1}(Y); \pi_n(F))$ where $C_{n+1}(Y)$ is the free R-module on the $(n+1)$-cells of Y; so define $\bar{g}_n(e^{n+1}) = [g_n|\dot{e}^{n+1}] \in \pi_n(F)$. The element $\bar{g}_n$ is called the <u>obstruction cocycle</u> of g_n; its vanishing is necessary and sufficient to ensure that $g_{n+1} : Y^{n+1} \to X$ exists with the right properties (of course, one has to show that $\bar{g}_n$ is in fact a cocycle and then one uses the fact that e^{n+1}

is the cone over $\dot{e}^{n+1}$). However, the theorem of interest to us is the following.

<u>Theorem 1.11</u> - Given an n-connected $(n \geq 2)$ fibration $f : X \to Y$ with fibre F, f possesses a right inverse (sometimes called a section or a cross-section) $g : Y \to X$ when, in particular, all the groups $H^r(Y; \pi_{r-1}(F))$ vanish. //

The situation in which a space X dominates another space Y gives rise to the situation where the fundamental group H of Y is a <u>retract</u> of the fundamental group G of X, that is, there are maps $H \underset{r}{\overset{j}{\rightleftarrows}} G$ such that $r_j = 1_H$.

<u>Lemma 1.12</u> - Let G be a finitely presented group and let H be a retract of G. Then H is finitely presented.

<u>Proof</u> - Let $(g_i : r_j)$ be the finite presentation for G and let F be the free group on the g_i with $p : F \to G$ the corresponding surjection with kernel R; then $R \overset{i}{\rightarrowtail} F \overset{p}{\twoheadrightarrow} G$ is exact, that is, $G \cong F/R$ and R may be taken as the smallest normal subgroup of F containing all the r_j. Consider the composition $F \xrightarrow{p} G \xrightarrow{r} H \xrightarrow{j} G$ and take $jrp(g_i)$; since p is surjective, we take w_i in F such that $p(w_i) = jrp(g_i)$. We claim that $(g_i : r_j, g_i^{-1}w_i)$ is a presentation for H. Let $(g_i : r_i, g_i^{-1}w_i)$ define L and let $u : G \to L$ be the obvious projection. Note that $rp(\bar{r}_j) = r(1) = 1$ (where $\bar{r}_j = i(r_j)$) and that $rp(g_i^{-1}w_i) = rp(g_i^{-1})rp(w_i) = rp(g_i^{-1})rjrp(g_i) = rp(g_i^{-1}g_i) = 1$, so the relations hold in H and also

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[dr, "r"] & L \arrow[d, "v"] \\ & H \arrow[ul, bend left, dashed, "j"] \end{tikzcd} \tag{1.13}$$

commutes. Now $vuj = r_j = 1$, and uj is onto since $ujrp(g_i) = up(w_i) = up(g_i)$ give the generators of L. Hence, u and u_j are inverse isomorphisms. //

It is worth noting that in the previous proof, H is just G with some extra relations and we know precisely what the kernel of $r : G \to H$ is (examine the proof).

§ I.2 Universal Covering Spaces

For a given space $X \in \underline{W}$, a covering space for X is a CW-space $\tilde{X}$ together with an onto map $p : \tilde{X} \to X$ such that each point $x \in X$ has a neighbourhood $U(x)$ such that the subspace $p^{-1}(U(x))$ is the disjoint union of open sets $U(\tilde{x})$, one for each $\tilde{x} \in p^{-1}(x)$, each of which is homeomorphic to $U(x)$ under p. A universal cover for X is a simply connected covering space for X. Hereafter, $\tilde{X}$ denotes a universal cover.

We need to know quite a few things about universal coverings; we start with a brief discussion of the action of the fundamental group. First of all, consider a fibration $p : E \to B$ with fibre $F \equiv p^{-1}(*)$, $* \in B$, and a map $w : I \to B$ with $[w] \in \pi_1(B, *)$, that is, $w(0) = w(1) = *$. Since we have a fibration, given $x \in p^{-1}(*)$, we may lift w to a map $\tilde{w} : I \to E$ such that $p\tilde{w} = w$, and so, $\tilde{w}(0)$ and $\tilde{w}(1) \in p^{-1}(*)$. So w induces a continuous map, still called $w : F \to F$. Notice that this is an action of $\pi_1(B)$ on F and, hence, induces an action of $\pi_1(B)$ on the chain complex of F.

Recall also that the fundamental group G acts on the higher homotopy groups of a space X; intuitively, the loops of G pull the elements of $\pi_n(X)$, which we think of as images of n-spheres in X, around themselves to a new element of $\pi_n(X)$ (see [8], [14], [15]).

Now recall the integral group ring of G, $R \quad \mathbb{Z}(G)$, which we took to consist of all formal finite linear combinations $g_1^{n_1} + \ldots + g_k^{n_k}$,

$g_i \in G$, $n_i \in \mathbb{Z}$; for example, taking $G = \mathbb{Z}$, $R = \oplus_{\mathbb{Z}} \mathbb{Z}$ (when G is abelian, we write the linear combinations as $n_1 g_1 + \ldots + n_k g_k$). When the group G acts on another group A, then it is easy to see that A becomes an $R \equiv \mathbb{Z}(G)$-module.

Now there is a particularly nice way to see how the fundamental group of $X \in \underline{W}$ acts on the homology and homotopy of its universal cover $\tilde{X}$; unfortunately, this relies on a construction for killing homotopy groups not given till Section I.4. [1] However, we proceed as follows. We may attach cells of dimension ≥ 2 to X to yield a space K which has fundamental group G and vanishing higher homotopy, and there is an inclusion $X \to K$. Deforming this map to a fibration (by replacing, if necessary, X by a mapping track, which is homotopy equivalent to X, still called X) and taking the fibre F, a simple examination of the homotopy exact sequence yields the fact that F is simply connected and has higher homotopy isomorphic to that of X; hence, the fundamental group G of X acts on both the homotopy and the chain complex of F. Replacing F up to homotopy, one obtains the space $\tilde{X}$. So the homology and homotopy of $\tilde{X}$ are all R-modules and, further, since the G-action is free, the chain complex of $\tilde{X}$ consists of free R-modules; we examine this further below.

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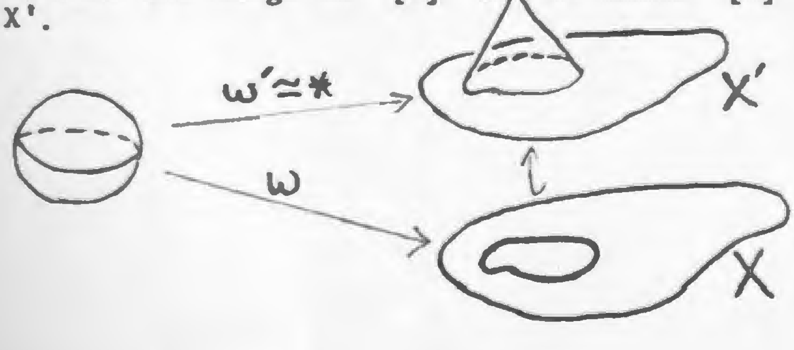
[1] An intuitive picture is as follows: take $[w] \in \pi_n(X)$, then attaching a cell over the image of $[w]$ in X makes $[w]$ homotopically trivial in X'.



The cellular structure of $\tilde{X}$ is completely determined by the cellular structure of X and the covering map p. One may say either that the characteristic maps of $\tilde{X}$ consist of lifts of the characteristic maps of X, or that the cellular structure of $\tilde{X}$ is given by the sets $p^{-1}(e_j^n)$, $n \geq 0$, $j \in J_n$. It turns out that, if we fix e_j^n in X, then the set $p^{-1}(e_j^n) = \{\tilde{e}_{j_i}^n, \ i \in I\}$ is determined by $g \cdot \tilde{e}_{j_r}^n$, say, as g runs over $\pi_1(X) \equiv G$, for a fixed $r \in I$. So if $X \in \underline{CW}$ is of finite dimension m, it is immediate that $\tilde{X}$ is also of finite dimension m. Now suppose X is of finite type so that the sets J_n are finite and consider an element of $C_n(\tilde{X})$ say x, and for convenience, let us write the n-cells of $\tilde{X}$ as $g \cdot e_j$, $j \in J_n$, $g \in G$, which we can do by the above; since J_n is finite, we may write these as $g \cdot e_j$, with $j = i, \ldots, s$. Now $C_n(\tilde{X})$ is the free abelian group on J_n or, if we like, the e_i, so that we can write x as $\Sigma_{j=1}^s n_1^j g_1^j e_j + \ldots + n_{k_j}^j e_j = \Sigma_{j=1}^s (\Sigma_{i=1}^{k_s} n_i g_i) e_j = r_1 e_1 + \ldots + r_s e_s$, $r_i \in R$; that is $C_n(\tilde{X})$ is a free R-module of <u>finite rank</u>.

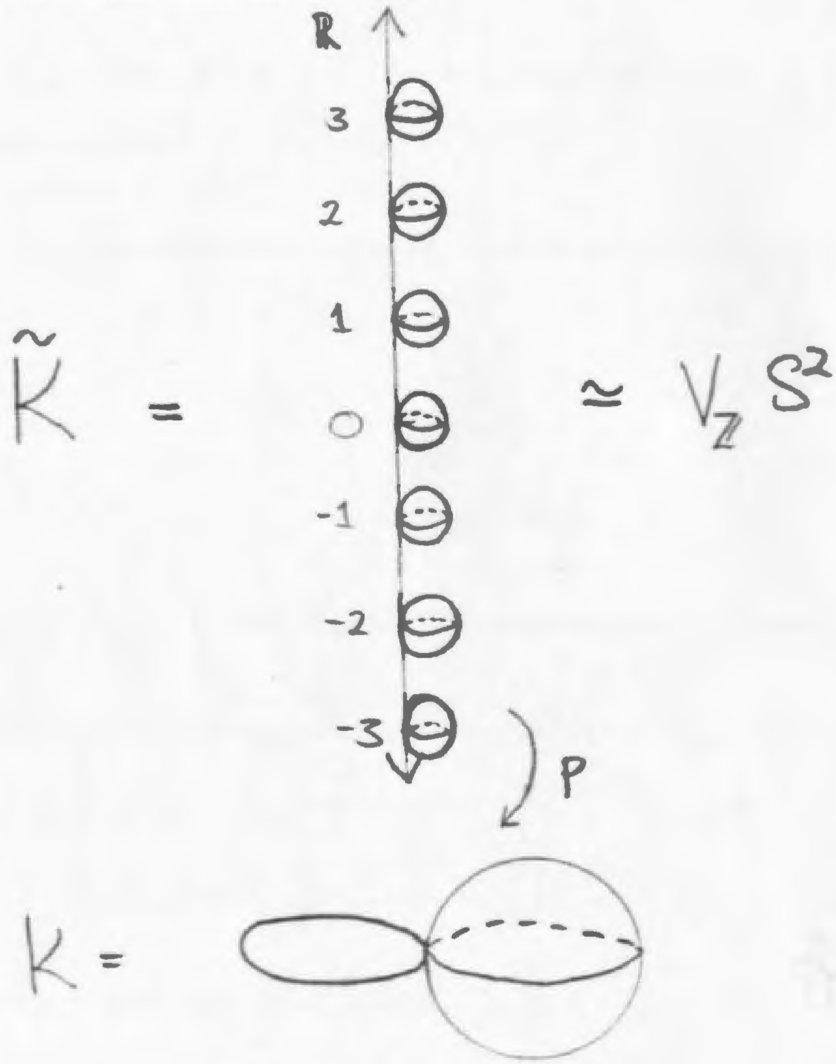
An example is overdue and we give the following extremely simple finite CW-complex K which possesses the following properties:

(1) $\pi_2(K)$ is not finitely generated as a Z-module but is finitely generated as an $R \equiv \mathbb{Z}(G) \equiv \mathbb{Z}(\pi_1(K))$-module;

(2) $\tilde{K}$ is finite dimensional but not even of the homotopy type of a finite CW-complex. Take $K = S^1 \vee S^2$, the universal cover of which is

seen to be the real line $\mathbb{R}$ with a 2-sphere at every integer point; so $\tilde{K}$ is of the homotopy type of a bouquet of 2-spheres, one for each integer. We have the following.

(2.1)



So $H_0(\tilde{K}) = 0$, $\pi_0(\tilde{K}) = 0$ $\qquad$ $H_0(K) = 0$, $\pi_0(K) = 0$

$H_1(\tilde{K}) = 0$, $\pi_1(\tilde{K}) = 0$ $\qquad$ $H_1(K) = \mathbb{Z}$, $\pi_1(K) = \mathbb{Z}$

$H_2(\tilde{K}) \cong \pi_2(\tilde{K}) = \oplus_{\mathbb{Z}} \mathbb{Z} = R$, $\qquad$ $H_2(K) = \mathbb{Z}$, $\pi_2(K) = R$.

Before going on to give a brief account of cohomology with local coefficients, we recall a few standard results relating maps of base spaces to maps of covering spaces.

Theorem 2.2 - If $f : X \to Y$ is n-connected, then so is the induced or lifted map denoted $\tilde{f} : \tilde{X} \to \tilde{Y}$.

Proof - This follows from the following commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which gives rise to the following commutative diagram

$$\begin{tikzcd} \pi_{r+1}(\tilde{X}) \arrow[r] \arrow[d, "p_*"', "\cong"] & \pi_{r+1}(\tilde{Y}) \arrow[r] \arrow[d, "q_*"', "\cong"] & \pi_{r+1}(\tilde{f}) \arrow[r] \arrow[d] & \pi_r(\tilde{X}) \arrow[r] \arrow[d, "p_*"', "\cong"] & \pi_r(\tilde{Y}) \arrow[d, "q_*"', "\cong"] \\ \pi_{r+1}(X) \arrow[r] & \pi_{r+1}(Y) \arrow[r] & \pi_{r+1}(f) \arrow[r] & \pi_r(X) \arrow[r] & \pi_r(Y) \end{tikzcd}$$

to which we apply the five lemma $(r \leq n - 1)$. //

Theorem 2.3 - If Y dominates X, that is, we have maps $j : X \to Y$ and $r : Y \to X$ such that $rj \simeq 1_X$, then $\tilde{r}\tilde{j} \simeq 1_{\tilde{X}}$, that is, $\tilde{Y}$ dominates $\tilde{X}$ and, furthermore, $H_{i+1}(\tilde{r}) \cong H_i(\tilde{j})$, for all $i \geq 1$.

Proof - If $h_t : X \to X$, $t \in I = [0, 1]$, is a homotopy between 1_X and rj, then $\tilde{h}_t : \tilde{X} \to \tilde{X}$, $t \in I$, is a homotopy between $1_{\tilde{X}}$ and $\widetilde{rj}$; it is easy to show that $\widetilde{rj} = \tilde{r}\tilde{j}$. To see the last claim, we replace first $\tilde{j}$ and then $\tilde{r}$ by inclusions into mapping cylinders to obtain a triple which we still call $\tilde{X} \overset{\tilde{j}}{\hookrightarrow} \tilde{Y} \overset{\tilde{r}}{\hookrightarrow} \tilde{X}$; the result follows from the exact homology sequence of this triple

$$\ldots\ 0 \longrightarrow H_{i+1}(\tilde{r}) \longrightarrow H_i(\tilde{j}) \longrightarrow H_i(1_{\tilde{X}}) = 0 \longrightarrow H_i(\tilde{r}) \longrightarrow \ldots . \quad //$$

As well, a simple application of the Hurewicz theorem (1.7) yields the following.

Theorem 2.4 - If $f : X \to Y$ is n-connected, then $\pi_{n+1}(f) \cong H_{n+1}(\tilde{f})$. //

We deal here briefly with cohomology with local coefficients, the relevant facts are found in Steenrod [12], [13], Eilenberg [3], and Hilton-Stammbach [4]. Given $X \in \underline{W}$ with fundamental group G, suppose that there is a G-action on an abelian group K; the question is what happens to cohomology with coefficients in K when we take into account the action of G. Of course, when this action is trivial, we get back our usual cohomology.

So let X be in $\underline{W}$ with base point $*$ and let $f : I \to X$ be a path in X beginning at $f(0) = x$ and ending at $f(1) = y$, say. The class of paths from x to y homotopic to f (rel end points) is denoted by f_{xy}. The class of the inverse path is denoted by f^{-1}_{xy} or

f_{yx}, and elements of G by f_*, g_*, etc. With these notations, we say that we have a <u>system of local coefficients</u> if: (1) For each point x of X, there is a given group say K_x; (2) For each class of paths f_{xy} there is an isomorphism $K_x \to K_y$; (3) The isomorphism given by the composition of the isomorphisms corresponding to f_{xy} and f_{yz} is the isomorphism corresponding to the composition of f_{xy} and f_{yz}.

Now it is relatively easy to see that any R-module K gives rise to a local system $\mathbb{K}$ (in which, obviously, each K_x is isomorphic to K) and further, each local system $\mathbb{K}$ corresponds to an R-module K, in an obvious way (choose $K = K_*$). One can then go on to define the homology and cohomology of X with coefficients in a local system, but this is not of interest here.

In fact, what we do need is the following. First of all, the cohomology group $H^n(X; \mathbb{K})$ is isomorphic to $H_n(\mathrm{Hom}_R(\tilde{X}; K))$ and secondly, under conditions that hold, for instance, when the universal cover $\tilde{X}$ is contractible, $H^2(X; \mathbb{K})$ is isomorphic to $H^2(G; K)$. Furthermore, this last term is isomorphic to the set of extensions of K by G which correspond to the given G-action.

§ I.3 Finitely Generated Projective R-modules

In the sequel, we give conditions under which a map $f : K \to X$ in $\underline{W}$ has the property that $\pi_n(f)$ is finitely generated and projective over R. So we must be able to manipulate such objects. Let R be any ring and consider functions f which assign to each finitely generated projective R-module X a value $f(X)$ in some abelian group subject to the conditions:

(i) if $X \cong Y$ then $f(X) = f(Y)$;

(ii) $f(X \oplus Y) = f(X) + f(Y)$;

(iii) $f(R) = 0$.

Among all such functions f there is one which is universal (its universality is not of interest here and is omitted); the group in which it takes its values is called the <u>reduced projective class group of</u> R, and we write it as $\underline{\mathbf{K}}$; if the f's are required to satisfy only the first two axioms, the universal group is just the projective class group written [1] $\mathbf{K}$.

There is an extensive theory of such groups and taken in much greater generality, but here we require only a few simple facts. First of all, we build the groups $\mathbf{K}$ and $\underline{\mathbf{K}}$. So let $\underline{A}$ be the class of all finitely generated projective R-modules and $\underline{B}$ be the subclass of all finitely generated free R-modules; denote also their sets of isomorphism classes by $[\underline{A}]$ and $[\underline{B}]$, respectively.

---

(1) The usual notations are $\mathbf{K} \equiv K_0(R)$ and $\underline{\mathbf{K}} \equiv \tilde{K}_0(R)$; see, for example, [1] and [2].

Now let $\mathbf{F}$ be the free abelian group on $[\underline{A}]$ and $\mathbf{G}$ the free abelian group on $[\underline{B}]$; note that $\mathbf{G} \subset \mathbf{F}$. Let $\mathbf{R}$ be the subgroup of $\mathbf{F}$ generated by all elements of the form $1[X \oplus Y] - 1[X] - 1[Y]$. We define $\mathbf{K}$ as $\mathbf{F}/\mathbf{R}$; consider the subgroup $\mathbf{G} + \mathbf{R}$ of $\mathbf{K}$, and define $\underline{\mathbf{K}}$ as $\mathbf{K}/\mathbf{G} + \mathbf{R}$; if $X \in \underline{A}$, we denote the class of $\{X\}$ in $\mathbf{K}$ by $X = [X] + \mathbf{R}$ and its class in $\underline{\mathbf{K}}$ by $\underline{X} = \{X\} + \mathbf{G}$.

<u>Theorem 3.1</u> - $\forall\, P, Q \in \underline{A}$, $\underline{P} = \underline{Q}$ if and only if there exist $F, G \in \underline{B}$ such that $P \oplus F \cong Q \oplus G$.

<u>Proof</u> - Suppose $P \oplus F \cong Q \oplus G$ then $[P \oplus F] = [Q \oplus G] \Rightarrow \{P \oplus F\} = \{Q \oplus G\} \Rightarrow \{P\} + \{F\} = \{Q\} + \{G\} \Rightarrow \underline{P} = \underline{Q}$, so suppose that $\underline{P} = \underline{Q}$. Then $\underline{P} - \underline{Q} = 0$, that is, $\{P\} - \{Q\} \in \mathbf{G}$ so that $\{P\} - \{Q\} = \Sigma_i\{F_i\} - \Sigma_j\{F_j\}$ where the F_i's and F_j's are in $\underline{B}$ (we allow repetitions). But $\Sigma_i\{F_i\} = \{\oplus_i F_i\} \equiv \{L\}$ and similarly, $\Sigma_j\{F_j\} \equiv \{K\}$, both L and K in $\underline{B}$. So $\{P\} - \{Q\} = \{L\} - \{K\} \Rightarrow \{P\} + \{K\} = \{L\} + \{Q\} \Rightarrow \{P \oplus K\} = \{L \oplus Q\} \Rightarrow [P \oplus K] - [L \oplus Q] \in \mathbf{R} \Rightarrow [P \oplus K] - [L \oplus Q] = \Sigma_r([X_r \oplus Y_r] - [X_r] - [Y_r]) - \Sigma_s([X_s \oplus Y_s] - [X_s] - [Y_x])$ where X_r, X_s, Y_r, $Y_s \in \underline{A} \Rightarrow [P \oplus K] + \Sigma_s[X_s \oplus Y_x] + \Sigma_r[X_r] + {}_r[Y_r] = [L \oplus Q] + \Sigma_s[X_s] + \Sigma_s[Y_s] + \Sigma_r[X_r \oplus Y_r]$. But note that these are all basis elements of $\mathbf{F}$, where $\mathbf{F}$ is free abelian; hence, it must be that $P \oplus K \oplus E \cong L \oplus Q \oplus E$, where $E \equiv (\oplus_r[X_r \oplus Y_r]) \oplus (\oplus_s[X_s \oplus Y_s])$. That is, E is in $\underline{A}$ and, hence, there exists an M in $\underline{A}$ such that $E \oplus M$ is

in $\underline{B}$ and so $K \oplus E \oplus M \equiv F$ is in B and $L \oplus E \oplus M \equiv G$ is in $\underline{B}$, so that $P \oplus F \equiv Q \oplus G$, as required. //

There is only one other small thing to note; that if $P \oplus Q \equiv F \in B$ with $P, Q \in \underline{A}$, then $\underline{P} = -\underline{Q}$, for $[P \oplus Q] - [P] - [Q] \in \mathbf{R} \Rightarrow \{P \oplus Q\} - \{P\} - \{Q\} = 0 \in K$ or $\{F\} - \{P\} - \{Q\} = 0 \in K \Rightarrow -\underline{P} - \underline{Q} = 0 \in \underline{K}$.

§ I.4 Constructions

In this section, our aim is to give constructions which will enable us to approximate, if you like, a given CW-space X by building a CW-complex K of the same homotopy type by methods which allow us to count the number of cells of K. Note that the proofs as well as the statements of theorems will be important. We use our construction in various situations which appear to be artificial at first glance, but which arise in a natural way in the next chapter.

Theorem 4.1 - Given an $(n-1)$-connected map $f : K \to X$, where $K \in \underline{CW}$ and $X \in \underline{W}$, $n \geq 2$, we can attach n-cells from $\pi_n(f)$ (this phrase is explained below) to K to form a CW-complex L and an n-connected map $g : L \to X$.

Proof - Consider the exact homotopy sequence of f,

$$\cdots \longrightarrow \pi_n(f) \xrightarrow{\delta} \pi_{n-1}(K) \xrightarrow{f_\#} \pi_{n-1}(X) \longrightarrow \pi_{n-1}(f) \longrightarrow \cdots .$$

If $n \geq 3$, then f induces an isomorphism of fundamental groups and so the above sequence may be taken as a sequence of R-modules, where R is $\mathbb{Z}(G)$. Here G denotes the fundamental group of either X or K (they are isomorphic). In this case, choose R-generators of $\pi_n(f)$, say $\{[h_i], i \in I\}$. In the case $n = 2$, we consider the sequence as a sequence of (abelian) groups, and choose the $[h_i]$ as $\mathbb{Z}$-generators of the abelian group $\pi_2(f)$. In either case, the $\delta[h_i]$ belong to $\pi_{n-1}(K)$ (and, hence, by exactness, to the kernel of $f_\#$) and choosing representatives h_i of the $\delta[h_i]$, we have $h_i : S^{n-1} \to K$ and these define a map $h : V_I S^{n-1} \to K$; take the pushout,

$$
\begin{tikzcd}
V_I E^n \arrow[r, "\overline{h}"] & K_{\pi_n}(V_I E^n) \equiv L \\
V_I S^{n-1} \arrow[r, "h"] \arrow[u, hook] & K \arrow[u, hook]
\end{tikzcd}
\tag{4.2}
$$

Now the $\delta[h_i] \in \ker f_\#$, so that $f_\#[h_i] = [fh_i] = [*]$, that is $fh \simeq *$ and thus extends to a map $\overline{f} : V_I E^n \to X$ so that the outer square commutes in (4.3) below,

$$
\begin{tikzcd}
 & & X \\
V_I E^n \arrow[r, "\overline{h}"] \arrow[urr, bend left, "\overline{f}"] & L \arrow[ur, dashed, "g"] & \\
V_I S^{n-1} \arrow[r, "h"] \arrow[u, hook] & K \arrow[u, hook] \arrow[uur, bend right, "f"'] &
\end{tikzcd}
\tag{4.3}
$$

Hence, there exists a $g : L \to X$ such that $gi = f$ and $g\overline{h} = f$. Since K^{n-1} is the (n-1)-skeleton of L by construction, it follows from (1.10) that $i : K \to L$ is (n-1)-connected. Then since f is also (n-1)-connected, it follows immediately from the homotopy sequence of the triple (X, L, K) that g is (n-1)-connected; to show that g is also n-connected, consider the same sequence,

$$\ldots \longrightarrow \pi_n(i) \xrightarrow{r} \pi_n(f) \longrightarrow \pi_n(g) \longrightarrow \pi_{n-1}(i) = 0 \longrightarrow \ldots,$$

in which it suffices to show that the map r is onto. But as in Section 1.2, by construction $\pi_n(i) \cong H_n(\tilde{i}) \cong C_n(\tilde{L})$, where now $C_n(\tilde{L})$ is the free R-module on the n-cells of $\tilde{L}$ of which there is one for each generator (R or Z) of $\pi_n(f)$, so that r is onto. Hence $\pi_n(g) = 0$, as required. //

<u>Theorem 4.4</u> - Let $K \in \underline{CW}$ and $X \in \underline{W}$ and let $f : K \to X$ be (n-1)-connected. Then we can attach cells of dimension $\geq n$ to K to form a CW-complex L and a homotopy equivalence $g : L \to X$.

<u>Proof</u> - We use (4.1) to obtain an n-connected map g^n from a CW-complex L^n to X. We now apply (4.1) repeatedly to obtain (n+r)-connected maps $g^{n+r} : L^{n+r} \to X$ and finally let $L = \coprod_{r \geq 0} L^{n+r}$ with the weak topology, so we obtain $g : L \to X$ which is m-connected, $\forall m$, and hence (by 1.9) a homotopy equivalence. //

We now give the first use of our construction which might appear artificial.

<u>Theorem 4.5</u> - Given an (n-1)-dimensional CW-complex K and an (n-1)-connected map $f : K \to X$, $X \in \underline{W}$, such that $\pi_n(f)$ is free as a module over R and with $H_r(\tilde{X}) = 0$, $\forall r > n$, then we can attach n-cells to K to get an n-dimensional CW-complex L and a map $g : L \to X$ which is a homotopy equivalence.

Proof - Perform the construction of (4.1) to get L and $g : h \to X$ using free R (or $\mathbf{Z}$, if $n = 2$) generators of $\pi_n(f)$; L is certainly n-dimensional and g is n-connected. By (4.4), we may attach cells of higher dimension to get a CW-complex homotopy equivalent to X which has L as its n-skeleton; we call this new CW-complex X also. We want to show that the inclusion of L in X is a homotopy equivalence, and for this, we use universal covers; it is enough to show that all the $H_r(\tilde{g})$ vanish, where we have $K \overset{j}{\hookrightarrow} L \overset{g}{\hookrightarrow} X$ with $gj = f$, for then $\tilde{L}$ and $\tilde{X}$ have isomorphic homology and, hence, by (1.9) isomorphic homotopy, so that L and X have isomorphic homotopy for $r > 1$ and also for $r = 1$ since g is n-connected and so are homotopy equivalent through g.

That $H_r(\tilde{g})$ vanishes for all r is easy to check. First of all, by construction, $H_r(\tilde{j})$ vanishes for $r \neq n$, and $H_r(\tilde{f})$ vanishes for $r < n$.

Note now that $H_r(\tilde{f})$ vanishes also for $r > n$ since K is $(n-1)$-dimensional and $H_r(\tilde{X}) = 0$ for $r > n$ by hypothesis. Hence all that remains to be shown is that $H_n(\tilde{j}) \cong H_n(\tilde{f})$ (this by exact homology sequence of the triple $(\tilde{X}, \tilde{L}, \tilde{K})$) but $H_n(\tilde{j}) = C_n(\tilde{L})$ which is the free R-module on the generators of $\pi_n(f) \cong H_n(\tilde{f})$ which is itself this same free R-module by hypothesis, and hence the result. //

Next we show various results that can be obtained when attaching trivial cells, that is, cells that are attached via trivial maps (explained further below).

<u>Theorem 4.6</u> - If L is a CW-complex obtained from $K \in \underline{CW_*}$ by means of attaching trivial cells, then L dominates K.

<u>Proof</u> - Let maps $f_i : S^{n-1} \to K$, all constant to the base point $*$ of K, define $f : V_I S^{n-1} \to K$ and form the pushout,

$$\begin{tikzcd} V_I E^n \arrow[r, "\bar{f}"] & L \\ V_I S^{n-1} \arrow[u, hook] \arrow[r, "f"] & K \arrow[u, hook, "j"'] \end{tikzcd} \tag{4.7}$$

This is what is meant by "attaching trivial cells".

Now define a function $r : L \to K$ by taking r on $K \subset L$ to be the identity and r on the n-spheres of L, that is, the images under $f : VE^n \to L$ (these E^n all have S^{n-1} collapsed so their image in L is an n-sphere) to be the constant map; clearly, r is continuous and $rj = 1$. //

In the sequel, given an (n-1)-connected map $f : K \to X$, we are sometimes interested in what happens when $\pi_n(f) \equiv P$ turns out to be a projective finitely generated R-module, where $R \equiv \mathbb{Z}(G)$, and G is the fundamental group (of both K and X). Then certainly there is a free R-module F equipped with a submodule Q such that $F \cong P \oplus Q$. There are two constructions of interest: one where we build an n-dimensional CW-complex L and an n-connected map $g : L \to X$ with the property that

$\pi_{n+1}(g) \cong Q$; and another where we construct an n-dimensional CW-complex Y and a map $h : Y \to X$ such that $\pi_{n+1}(h)$ is free.

<u>Theorem 4.8</u> - Given an (n-1)-connected map $f : K \to X$ where $K \in \underline{CW}$ is finite of dimension n and $X \in \underline{W}$ has $H_r(\tilde{X}) = 0$, $\forall r \geq n+1$, such that $\pi_n(f) \cong P$ is a finitely generated R-module, then if F is free with $F \cong P \oplus Q$, there is an $L \in \underline{CW}$ and an n-connected map $g : L \to X$ with the property that $\pi_{n+1}(g) \cong Q$.

<u>Proof</u> - Since P is finitely generated, we can take F to have finite rank and, as well, take these generators as coming from either P or Q. Attach n-cells from P to K as in (4.1) and attach trivial n-cells from Q to this, to form the complex L and map $g : L \to X$; g is n-connected as in (4.1) since the cells from Q are all trivial and since $\pi_{n+1}(f) \cong H_{n+1}(\tilde{f}) = H(\tilde{X}, \tilde{K}) = 0$ and $\pi_{n+1}(i) \cong H_{n+1}(\tilde{i}) = H_{n+1}(\tilde{L}, \tilde{K}) = F$, the homology exact sequence of the triple $\tilde{K} \xhookrightarrow{i} \tilde{L} \xhookrightarrow{g} \tilde{X}$ becomes

$$0 \longrightarrow \pi_{n+2}(g) \longrightarrow \pi_{n+1}(i) = F \longrightarrow \pi_{n+1}(f) = P \longrightarrow 0.$$

This splits as required because P is projective. //

<u>Theorem 4.9</u> - Let K be an (n-1)-dimensional CW-complex and let $f : K \to X$, $X \in \underline{W}$, be (n-1)-connected with $\pi_n(f)$ being a projective R-module. Then we can attach trivial n-cells to K to give a CW-complex L and an (n-1)-connected map $g : L \to X$ such that $\pi_n(g)$ is free.

<u>Proof</u> - Writing $\pi_n(f)$ as B there is a free R-module F equipped with a submodule A such that $F \cong A \oplus B$, since B is projective. Now consider the module

$$P = B \oplus A \oplus B \oplus A \oplus \ldots;$$

this is isomorphic to the free module $H = F \oplus F + \ldots,$ bracketing after even terms and to the module $B \oplus H$, bracketing after the odd terms. We attach trivial n-cells to K one for each generator of H to obtain a CW-complex Y which dominates K and a map $g : Y \to X$ which is still n-connected, all of this by the preceding results of this section. Since Y dominates K, the exact sequence of the triple $\tilde{K} \xhookrightarrow{\tilde{j}} \tilde{Y} \xhookrightarrow{\tilde{g}} \tilde{X}$ (with $\tilde{g}\tilde{j} = \tilde{f}$) splits so that $H_n(\tilde{g}) = H_n(\tilde{f}) \oplus H_n(\tilde{j})$; but $H_n(\tilde{j}) = C_n(\tilde{Y}) = H$, by construction, and then by (2.4), this becomes $\pi_n(g) \cong \pi_n(f) \oplus H \cong B \oplus H = H$, which is free as required. //

Now something interesting happens to the cell attached to the universal cover $\tilde{X}$ when we attach a trivial cell to X.

<u>Theorem 4.10</u> - Suppose we have an $X \in \underline{W}$ such that $\pi_n(X) \cong P \oplus B$, $n \geq 2$, where B and P are finitely generated projective R-modules such that $A \oplus B = F$ is free (and finitely generated), where A is some finitely generated projective R-module. If we now attach to X some (n+1)-cells, one for each free R-generator of F by means of the projection $F \twoheadrightarrow B \subset \pi_n(X)$, this has the effect of attaching to $\tilde{X}$, (n+1)-cells by the images of the free $\mathbb{Z}$-generators of F.

Proof - Since F is finitely generated over R, we have that $F \cong R \oplus \ldots \oplus R$, say n times. Now $R = \mathbb{Z}(G)$, where G is the fundamental group of X; we can take R to be the free abelian group on the set G (together with a multiplicative structure which we ignore here); that is, $R = \oplus_G \mathbb{Z}$. Altogether, we can say that if x_k, $k = 1, \ldots, n$ generate F over R then gx_k, $g \in G$, $k = 1, \ldots, n$ generate F over $\mathbb{Z}$.

Now consider the following commutative diagram,

$$\begin{tikzcd} & p_*^{-1}(B) \arrow[r, hook, "i'"] \arrow[d, "p_*"] \arrow[d, phantom, "G\circlearrowleft\ \cong"'] & \pi_n(\tilde{X}) \arrow[d, "p_*"] \arrow[d, phantom, "G\circlearrowleft\ \cong"'] \\ F \arrow[ur, dashed, "\tilde{j}"] \arrow[r, two heads, "j"] & B \arrow[r, hook, "i"] & \pi_n(X) \end{tikzcd} \qquad (n \geq 2); \qquad \begin{tikzcd} & i'\tilde{j}(x_k) \arrow[d, "p_*"] \\ x_k \arrow[ur, mapsto] \arrow[r, mapsto] & ij(x_k) \end{tikzcd}$$

So attaching an $(n+1)$-cell to X via $ij(x_k)$ (which is a map $S^n \to X$) attaches (many) $(n+1)$-cells to $\tilde{X}$ via the $g \cdot i'\tilde{j}(x_k)$, $g \in G$ (this due to the cellular structure of $\tilde{X}$ described in Section 1.2), but this is just the statement of the theorem. //

We also require a result of Whitehead's.

Theorem 4.11 - Given a $(n-1)$-connected map $f : K \to X$, with K finite of dimension $(n-1)$ and X finite of dimension n, then there is a finite $(n+1)$-dimensional CW-complex Y homotopy equivalent to X, which has K as its n-skeleton.

Proof - This is a special case of [17, Thm. 15]. //

Now, just for fun, we examine the case of a simply connected $X \in \underline{W}$ which is dominated by a finite CW-complex. First of all, we prove a theorem which allows us, in this special situation, to kill homology groups (this phrase is explained below).

<u>Theorem 4.12</u> - Let X and K be simply connected with $X \in \underline{W}$ and $K \in \underline{CW}$, finite of dimension n, $H_{n+1}(X) = 0$. and let $f : K \to X$ be n-connected. Then we can attach (n+1)-cells to K from $H_n(f)$ (this phrase is explained below) to form a CW-complex L such that $H_{n+1}(L) = 0$ and a map $g : L \to X$ which is (n+1)-connected.

<u>Proof</u> - We have that $\pi_{n+1}(f) \cong H_{n+1}(\tilde{f})$ by (2.4) and that $H_n(K)$ is a finitely generated free abelian group. Taking the m_i, $i = 1, 2, 3$ below to be the Hurewicz homomorphism, we obtain the following commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(f) \arrow[r, "\delta^*"] & H_n(K) \arrow[r, "f_*"] & H_n(X) \arrow[r] & \cdots & \text{exact} \\
\cdots \arrow[r] & \pi_{n+1}(f) \arrow[r, "\delta\#"] \arrow[u, "m_1", "\cong"'] \arrow[ur, phantom, "\circlearrowleft"] & \pi_n(K) \arrow[r, "f_\#"] \arrow[u, "m_2"] \arrow[ur, phantom, "\circlearrowleft"] & \pi_n(X) \arrow[r] \arrow[u, "m_3"] & \cdots & \text{exact.}
\end{tikzcd} \tag{4.13}$$

Take $\{\tilde{k}_1, \ldots, \tilde{k}_p\}$ to be a set of free generators for the kernel of $f_* : H_n(K) \to H_n(X)$; by exactness, we have $\{\tilde{d}_1, \ldots, \tilde{d}_p\} \in H_{n+1}(f)$ such that $\delta_*(\tilde{d}_i) = \tilde{k}_i$. Now let $[d_i] = m_1^{-1}(\tilde{d}_i)$ and then let $[k_i] = \delta_\#([d_i])$, $i = 1, \ldots, p$. By commutativity $m_2([k_i]) = \tilde{k}_i$, for $i = 1, \ldots, p$, and also by exactness the composition $f_\#[k_i]$ is trivial; that is if

$k_i \in [k_i]$ are representatives $fk_i \simeq * \cdot S^n \to X$. Now attaching $(n+1)$-cells to K by means of $k = Vk_i : VS^n \to K$ and extending f over the VE^{n+1}, we obtain the following diagram,

$$\begin{tikzcd}
 & & X \\
VE^{n+1} \arrow[r, "\bar{k}"] \arrow[urr, bend left, "\bar{f}"] & L \arrow[ur, "g"] & \\
VS^n \arrow[u] \arrow[r, "\bar{k}"] & K \arrow[u, "j"] \arrow[uur, bend right, "f"'] &
\end{tikzcd} \tag{4.14}$$

As usual, g is n-connected, since f and j are n-connected. We now use the commutative diagram (4.15). Consider the triangle labelled I, since $H_n(j) \simeq \pi_n(j) = 0$, the map j_* is epic; f_* is epic by hypothesis and, hence, g_* is epic because the triangle commutes. In order to show that g_* is monic we need only to show that $\ker j_* = \ker f_*$, for if $g_*(x) = 0$, then $j_*(y) = x$ for some y (j_* is epic), hence, $j_*g_*(y) = f_*(y) = 0$ and so $y \in \ker f_* = \ker j_* \Rightarrow x = 0$, as required. Now $\ker j_* \subset \ker f_*$ is immediate by commutativity, so consider a generator of $\ker f_*$, $\bar{k}_i$ and recall that we have $[k_i] \in \pi_n(K)$ such that $m_2([k_i]) = \bar{k}_i$. Furthermore, $k_i \in [k_i]$ induced a map $\bar{k}_i : (E^{n+1}, S^*) \to (L, K)$ such that $\delta_\#([\bar{k}_i]) = [k_i]$ and, hence, by commutativity of the appropriate square, we have $\delta_* n_1([\bar{k}_i]) = \tilde{k}_i$ and so $i_*(\tilde{k}_i) = i_*\delta_*(n_1]\bar{k}_i]) = 0$, by exactness, as required.

Now L is just the mapping cone of $K : V_1S^n \to K$ so consider the homology exact sequence,

(4.15)

$$
\begin{tikzcd}[column sep=large]
\cdots \arrow[r] & \pi_{n+1}(j) \arrow[r, "\delta_\#"] \arrow[d, "n_1", "\wr"'] & \pi_n(K) \arrow[r, "j_\#"] \arrow[dd, "n_2=m_2"] & \pi_n(L) \arrow[r, "p_\#"] \arrow[d, "n_3"] & \pi_n(j) = 0 \arrow[r] \arrow[d, "\wr"'] & \cdots \ \text{exact} \\
\cdots \arrow[r] & H_{n+1}(j) \arrow[dr] \arrow[dd, "g^*"] & & H_n(L) \arrow[r, "p_*"] \arrow[dd, "I"'] \arrow[dd, bend left, dashed, "g_*"] & H_n(j) = 0 \arrow[r] \arrow[dd, "g^*"] & \cdots \ \text{exact} \\
& & H_n(K) \arrow[ur, "j_*"] \arrow[dr, "f_*"] & & & \\
\cdots \arrow[r] & H_{n+1}(f) \arrow[ur] & & H_n(X) \arrow[r, "\bar{p}_*"] & H_n(f) = 0 \arrow[r] & \cdots \ \text{exact} \\
\cdots \arrow[r] & \pi_{n+1}(f) \arrow[r, "\delta_\#"] \arrow[u, dash, "m_1", "\wr"'] & \pi_n(K) \arrow[r] \arrow[uu, "m_2=n_2"] & \pi_n(X) \arrow[r, "\bar{p}_\#"] \arrow[u, "m_3"] & \pi_n(f) = 0 \arrow[r] \arrow[u, "\wr"'] & \cdots \ \text{exact}
\end{tikzcd}
$$

$$\cdots \longrightarrow H_{n+1}(K) \xrightarrow{j_*} H_{n+1}(L) \xrightarrow{\delta} H_n(V_1^p S^n) \xrightarrow{k_*} H_n(K) \longrightarrow \cdots,$$

in which $H_{n+1}(K) = 0$ since K is n-dimensional and so δ is monic. Note that all the terms above are finitely generated free abelian groups. By exactness, to show $H_{n+1}(L) = 0$, we need only show that k_* is monic.

So consider the following commutative diagram,

$$\begin{array}{ccccccc} H_n(S^n) & \xrightarrow{(k_i)_*} & H_n(K) & & 1 & \longmapsto & k_i \\ \uparrow \cong & & \uparrow n & & \uparrow & & \uparrow \\ \pi_n(S^n) & \xrightarrow{(k_i)_\#} & \pi_n(K) & & [1] & \longmapsto & [k_i]. \end{array} \qquad (4.16)$$

The map $(k_i)_\#$ takes the generator of $\pi_n(S^n)$, $[1_{S^n}]$ to the corresponding $[k_i] \in \pi_n(K)$, by definition of $(k_i)_\# : (S^n \xrightarrow{1_{S^n}} S_i^n \xrightarrow{k_i} K)$; and it follows that $(k_i)_*$ takes the generator of $H_n(S^n)$ to $\tilde{k}_i \in H_n(K)$ since the square commutes. Hence, $(k_i)_*$ is monic and hence, also, is k_*, as required. //

Theorem 4.17 - Let $X \in \underline{W}$ be simply connected and be dominated by a finite CW-complex L of dimension n. Then X is of the homotopy type of a finite n-dimensional CW-complex.

Proof - The homology groups of X are just retracts of those of L in this situation, and so they are all finitely generated since the homology of L is finitely generated and this tells us also $H_m(X) = 0$, $\forall m > n$.

We use inductively the construction of (4.1) to build a finite n-dimensional CW-complex K_n and a map $f_n : K_n \to X$ which is n-connected and then simply apply (4.11).

So take $K_0 = K_1 = *$ and $f_1 : K_1 \to X$ the constant map and inductively assume $f_m : K_m \to X$ is m-connected and K_m is finite of dimension m. Using (4.1), we have only to show that $\pi_{m+1}(f_m)$ is finitely generated. There are two ways to do this; by (1.7) or by (1.8). In either case, the homology or homotopy sequence of f_m consists of finitely generated groups as required. //

CHAPTER TWO

We are now in a position to examine the algebraic conditions which characterize finite CW-complexes. This is accomplished in three steps. First of all, we study conditions which ensure that a CW-complex has only finitely many cells in each dimension and then we study conditions which ensure that a CW-complex has finite dimension. In the last section, we take all these conditions together.

§ II.1 CW-spaces of Finite Type

We give various conditions on $X \in \underline{W}$ (with fundamental group G and group ring R).

(F1): The group G is finitely generated;

(F2): The group G is finitely presented and for any finite 2-dimensional $K \in \underline{CW}$ and map $f : K \to X$ inducing an isomorphism of fundamental groups, we have that $\pi_2(f)$ is a finitely generated R-module;

(Fn): Condition $F(n - 1)$ holds, and for any finite $(n-1)$-dimensional CW-complex K and $(n-1)$-connected map $f : K \to X$, $\pi_n(f)$ is a finitely generated R-module, $(n \geq 3)$;

(F): All the (Fn) hold.

Notes:

(1) Since, for $n \geq 2$, we have that $\pi_1(K) \cong \pi_1(X) = G$, it follows that $\pi_r(K)$ and $\pi_r(f)$ are always R-modules for $r \geq 1$;

(2) Suppose we wish to check whether or not a given space satisfies (Fn), assuming $F(n - 1)$ holds. At first glance, it is not apparent whether any admissible maps f exist at all, and supposing

they do, say $f_i : K_i \to X$, apparently we must check that each $\pi_n(f_i)$ is finitely generated. However, we will show later that, if $n \geq 3$, then $F(n - 1)$ holding implies that admissible maps f_i exist and further, that if any one $\pi_n(f_i)$ is finitely generated for $n \geq 2$, then so are all the others.

Theorem 1.1 - The following conditions on X are equivalent:

(i) X is homotopy equivalent to a CW-complex with finite n-skeleton (respectively of finite type);

(ii) X is dominated by a CW-complex with finite n-skeleton (respectively of finite type);

(iii) X satisfies (Fn) (respectively F).

Proof - We prove (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) and start with (i) $\Rightarrow$ (iii). If $n \geq 1$, then (F1) holds by (I.1.3), and if $n \geq 2$, G is finitely presented, also by (I.1.3). We proceed by letting $n \geq 3$, and taking an (n-1)-connected map $f : K \to X$ where $K \in \underline{CW}$ is finite of dimension $(n - 1)$. Note that we need not demonstrate the existence of such a map, only that, if it exists, $\pi_n(f)$ is finitely generated.

Note that we may take X to be a CW-complex with finite n-skeleton and that we may take f to be the composition $K \overset{g}{\hookrightarrow} X^n \overset{j}{\hookrightarrow} X$ (replacing f by a cellular map and applying (I.1.10)). Further, using (I.4.11), we may replace X^n by a finite (n+1)-dimensional CW-complex Y with K as (n-1)-skeleton. Now consider the exact homology sequence of the triple $\tilde{K} \overset{\tilde{g}}{\hookrightarrow} \tilde{X}^n \overset{\tilde{j}}{\hookrightarrow} \tilde{X}$, $\tilde{j}\tilde{g} = \tilde{f}$,

$$\dots \longrightarrow H_{n+1}(\tilde{j}) \longrightarrow H_n(\tilde{g}) \xrightarrow{i_\#} H_n(\tilde{f}) \longrightarrow H_n(\tilde{j}) \longrightarrow \dots .$$

Then $H_n(\tilde{j}) = 0$ since $j : X^n \hookrightarrow X$ is n-connected, so the map $i_\#$ is an epimorphism and thus $H_n(\tilde{f}) \cong H_n(\tilde{g})/\ker i_\#$, that is, $H_n(\tilde{f})$ is a quotient of $H_n(\tilde{g})$. But a simple examination of the exact homology sequences of the pairs $\tilde{K} \overset{\tilde{g}}{\hookrightarrow} \tilde{X}^n$ and $\tilde{K} \overset{\tilde{h}}{\hookrightarrow} \tilde{Y}$ shows that $H_n(\tilde{g}) \cong H_n(\tilde{h})$.

So we consider the triple $\tilde{K} \overset{\tilde{m}}{\hookrightarrow} \tilde{Y}^n \overset{\tilde{k}}{\hookrightarrow} \tilde{Y}$, $\tilde{k}\tilde{m} = \tilde{h}$, and the homology sequence,

$$\dots \longrightarrow H_n(\tilde{m}) \longrightarrow H_n(\tilde{h}) \longrightarrow H_n(\tilde{k}) \longrightarrow \dots .$$

As before $H_n(\tilde{k}) = 0$, since $\tilde{Y}^n$ is the n-skeleton of $\tilde{Y}$, and so $H_n(\tilde{h})$ is a quotient of $H_n(\tilde{m})$ which is a free R-module of finite rank, by section (I.2), since K is the (n-1)-skeleton of Y. Thus $\pi_n(f)$ is finitely generated.

To complete this implication, we take the case $n = 2$; let X^2 be finite and $f : K \to X$ be such that $f_* : \pi_1(K) \xrightarrow{\cong} G$ and K is finite of dimension 2. By (I.4.11), we replace X^2 by a 3-dimensional finite CW-complex Y which is obtained from K by attaching 2 and 3-cells.

Then, by the above argument, we find that $\pi_2(f)$ is a quotient of $H_2(\tilde{Y}, \tilde{K})$, and this is again finitely generated as above.

To show (iii) $\Rightarrow$ (i) is easy. Suppose X satisfies (F1) so G is finitely generated by say $\{f_i : i = 0, \ldots, n\}$, and let $K = V_i S_i^1$ and let $f = V_i f_i : V_i S_i^1 \to X$; then f is surjective and so 1-connected and apply (I.4.4) to obtain the result. If X satisfies (F2), we can construct a finite CW-complex L' and a map $g' : L' \to X$ which induces an isomorphism of fundamental groups by attaching 2-cells to the K constructed above. By (F2), $\pi_2(g')$ is finitely generated so we apply (I.4.1) to attach finitely many 2-cells to L' to form L and a 2-connected map $g : L \to X$ and apply (I.4.4). The same argument applies when X satisfies (Fn).

It is clear that (i) $\Rightarrow$ (ii) and so take maps $X \xrightarrow{j} Y \xrightarrow{r} X$ be given such that $rj \simeq 1_X$ and firstly supppose that Y^1 is finite; then by (I.1.3), $\pi_1(Y)$ is finitely generated and then since $r_* : \pi_1(Y) \to G$ is epic, G is a quotient of $\pi_1(Y)$ and so X satisfies (F1). Now suppose Y^n is finite; inductively, we claim the map $r : Y \to X$ is n-connected, for $n \geq 2$. Supposing this is true then, using (I.4.4), we attach cells of dimension $\geq n + 1$ to make r a homotopy equivalence, and we will be through.

Now given that Y^2 is finite, we have that, by (I.1.3), $\pi_1(Y)$ is finitely presented and then, by (I.1.12), that G is also finitely presented. In fact, (I.1.12) yields that $p(g_i^{-1} w_i)$ generate the kernel of $r_* : \pi_1(Y) \to G$; hence by attaching a wedge of 2-cells to Y by means of representatives of these elements, we obtain a space Y_1 and a map $r_1 : Y_1 \to X$ which induces a fundamental group isomorphism; furthermore, Y_1 dominates X $(X \xrightarrow{j} Y \subset Y_1 \xrightarrow{r_1 | Y = r} X)$; hence, $(r_1)_* : \pi_2(Y) \to G$ is an epimorphism and so r_1 is 2-connected.

Now suppose, for $n \geq 3$, that Y^n is finite. By the induction hypothesis, since Y^{n-1} is certainly finite, we may suppose that the retraction $r : Y \to X$ is (n-1)-connected, perhaps replacing Y by some Y_{n-1}, obtained from Y by attaching cells of dimension $\leq n - 1$. We then apply (I.4.4) and so we may suppose that X is obtained from Y by attaching cells of dimension $\geq n$; in other words, that Y and X have the same skeletons up to and including dimension $n - 1$. Now, by (I.2.3), we have $\pi_n(r) \cong H_n(\tilde{r}) \cong H_{n-1}(\tilde{j})$. Take $k = j | X^{n-2}$, then $H_{n-1}(\tilde{j})$ is a quotient of $H_{n-1}(\tilde{k})$, as seen in (i) $\Rightarrow$ (iii);

$$(\tilde{Y}, \tilde{X}, \tilde{X}^{n-2}) : \ldots \to H_{n-1}(\tilde{k}) \to H_{n-1}(\tilde{j}) \to \underset{\substack{\| \\ 0}}{H_{n-2}(\tilde{X}, \tilde{X}^{n-2})} \to \ldots .$$

But r is just the inclusion, and $rj \simeq 1$ in X; since the cells of $X - Y$ have dimension $\geq n$, this homotopy restricted to X^{n-2} can be taken to lie in Y, so we can assume k is just the inclusion of $X^{n-2} = Y^{n-2}$ in Y. So consider $(\tilde{Y}, \tilde{Y}^{n-1}, \tilde{Y}^{n-2})$,

$$\ldots \to H_{n-1}(\tilde{Y}^{n-1}, \tilde{Y}^{n-2}) \to H_{n-1}(\tilde{Y}, \tilde{Y}^{n-2}) \to H_{n-1}(\tilde{Y}, \tilde{Y}^{n-1}),$$

where the last term is zero. So $H_{n-1}(\tilde{Y}, \tilde{Y}^{n-2}) = H_{n-1}(\tilde{k})$ is a quotient of $H_{n-1}(\tilde{Y}^{n-1}, \tilde{Y}^{n-2})$ a finitely generated R-module by section (1.2).

Now we are through, for consider the following sequence,

$$\ldots \to \pi_n(r) \to \pi_{n-1}(Y) \xrightarrow{r_*} \pi_{n-1}(X) \to \ldots ;$$

to make r n-connected, we attach cells by means of representatives of the generators of $\ker r_*$ which is finitely generated as a quotient of $\pi_n(r)$. //

<u>Corollary 1.2</u> - For $n = 2$, if G is finitely presented then there exist finite 2-dimensional CW-complexes K and maps $f : K \to X$ which induce isomorphisms of fundamental groups. For $n \geq 3$, if $F(n - 1)$ holds, then there exist finite (n-1)-dimensional CW-complexes K and (n-1)-connected maps $f : K \to X$ such that $\forall n \geq 2$ if one such $\pi_n(f)$ is finitely generated over R, so are all others.

<u>Proof</u> - If G is finitely presented, we have seen that such a K and $f : K \to X$ exist. If X now satisfies $F(n - 1)$, then X is equivalent to a complex with finite (n-1)-skeleton. If one such $\pi_n(f)$ is finitely generated, we add a finite number of n-cells to make f n-connected and then cells of dimension $\geq n + 1$ to make f a homotopy equivalence. Then X satisfies (i) and so also (iii). //

The conditions Fn become simpler when the group ring R is noetherian; we examine this situation more closely in Appendix B.

§ II.2 CW-Spaces of Finite Dimension

Once again, we give various conditions on X.

Dn : $H_i(\tilde{X}) = 0$, for $i > n$ and $H^{n+1}(X; \mathbf{B}) = 0$ for all systems of local coefficient $\mathbf{B}$ (see section (I.2)).

Note that we do not require that $D(n - 1)$ holds. The case $n = 1$ is entirely different from the rest, and we treat it first.

Theorem 2.1 - If X satisfies $D1$, it has the homotopy type of a bouquet of circles.

Proof - All the homology groups of $\tilde{X}$ vanish and thus, so do all of its homotopy groups so X is contractible. In any case, X has only one non-vanishing homotopy group $\pi_1(X) = G$. So if $\mathbf{B}$ has fibre F, we can identify $H^2(X; \mathbf{B})$ with the set of extensions of F by G which corresponds to the action of G on F. So let

$$F \rightarrowtail K \overset{\alpha}{\twoheadrightarrow} G,$$

be an exact sequence where K is a free group. By the remarks above, this extension splits and so G is isomorphic to a subgroup of the free group K and so is also free. The result is now immediate, for spaces with only one non-vanishing homotopy group are unique up to homotopy type. //

Note that if X now also satisfies $F1$ then X is a finite one-dimensional CW-space, and further if Y dominates X and satisfies $D1$, then X satisfies D1 also, because the homology of $\tilde{X}$, respectively, cohomology of X, are retracts of the homology of $\tilde{Y}$, respectively, cohomology of Y.

Now, given X and $n \geq 3$, our constructive methods allow us to build an $(n-1)$-dimensional CW-complex K (which need not, however, be finite) and an $(n-1)$-connected map $f : K \to X$. Furthermore, we can attach to K cells of dimension $\geq n$ to make f a homotopy equivalence, and so can take X to be a CW-complex with K as $(n-1)$-skeleton (and f as the inclusion).

<u>Lemma 2.2</u> - Given $f : K \hookrightarrow X$ as above with X satisfying Dn $(n \geq 3)$, then $\pi_n(f)$ is a projective R-module.

<u>Proof</u> - Results which are familiar by now show that $\pi_n(f) \cong H_n(\tilde{f}) = H_n(\tilde{X}, \tilde{K}) = C_n(\tilde{X})/B_n(\tilde{X}) = C_n/B_n$.

We have only to prove that the exact sequence,

$$B_n \overset{j}{\rightarrowtail} C_n \twoheadrightarrow \pi_n(f)$$

splits; then since C_n is a free R-module, $\pi_n(f)$ is R-projective as a summand of a free R-module.

Now as an R-module, by section (I.2), B_n defines a coefficient bundle $\mathbf{B}$ over X with $H^{n+1}(X; \mathbf{B}) = 0$ since X satisfies Dn. But we have $H^p(X; \mathbf{B}) = H^p(\mathrm{Hom}_R(C_*, B_n)$ where C_* is the R-free chain complex of X. So consider

$$C_{n+2} \xrightarrow{d} C_{n+1} \overset{c}{\twoheadrightarrow} B_n \overset{j}{\rightarrowtail} C_n .$$

Now the R-homomorphism c (c is the restriction of d_{n+1} to its image) is certainly an $(n+1)$-cochain of $(X, \mathbf{B})$ and it is also a cocycle since $d_{n-1}C = d_{n-1}d_n = 0$ and hence, since $H^{n+1}(X, \mathbf{B}) = 0$, a coboundary, and so there exists an $s \in \mathrm{Hom}(C_n, B_n)$ such that $c = sd_n = sjc$; but c is onto and so $sj = 1$, that is, s splits the sequence as required. //

In the case X satisfies D2, we take a 2-connected map f from a 2-dimensional CW-complex K to X. Then we have again $\pi_3(f) \cong H_3(\tilde{f}) = H_3(\tilde{X}, \tilde{K}^2) = C_3(\tilde{X})/B_3(\tilde{X})$, since $\tilde{K}^2$ may be taken as the 2-skeleton of $\tilde{X}$; but $H_3(\tilde{X}) = 0$ is given, so $B_3(\tilde{X}) = Z_3(\tilde{X})$ and hence $\pi_3(f) \cong C_3(\tilde{X})/Z_3(\tilde{X}) \cong B_2(\tilde{X})$. Now the above proof applies to show that

$$B_2(\tilde{X}) \rightarrowtail C_2(\tilde{X}) \twoheadrightarrow C_2(\tilde{X})/B_2(\tilde{X})$$

splits, so that $\pi_3(f)$ is projective over R as required.

<u>Theorem 2.3</u> - X satisfies Dn, $n \geq 3$, if and only if it is homotopy equivalent to an n-dimensional CW-complex K. Moreover, if X also satisfies Fr, $0 \leq r < n$, we may take K to have finite r-skeleton.

<u>Proof</u> - If X is homotopy equivalent to an n-dimensional CW-complex K, the result is clear. Hence, suppose X satisfies Fr and Dn, $0 \leq r < n$, $n \geq 3$, and apply the results of section (I.4) to obtain in turn an r-connected map $f : K \to X$ with K a finite r-dimensional CW-complex and

an (n-1)-connected map $g : L \to X$ with L an (n-1)- dimensional CW-complex with K as r-skeleton. By the previous lemma (2.2), $\pi_n(g)$ is a projective R-module, and we find ourselves precisely in the situation of (I.4.9) so we can attach trivial n-cells to L and obtain a CW-complex M and an (n-1)-connected map $h : M \to X$ such that $\pi_n(h)$ is free over R. Now we just apply (I.4.5) which gives us the result. //

If X now satisfies D2, it is not hard to see that X is homotopy equivalent to a 3-dimensional CW-complex. For then, constructing a 2-connected map $f : K \to X$ where K is a 2-dimensional CW-complex, we have that $\pi_3(f)$ is R-projective. Proceeding as in the theorem, we attach 2-spheres to K and obtain $g : L \to X$ with $\pi_3(g)$ free and so may attach 3-cells to L to obtain a homotopy equivalence.

<u>Corollary 2.4</u> - For $r \leq s$, Dr implies Ds.

<u>Proof</u> - This is clear for $r = s$ and for $r < s$, an r-dimensional complex certainly satisfies Ds. //

§ II.3 The Obstruction to Finiteness

We are, in this section, primarily concerned with CW-spaces X satisfying both Fn and Dn. It turns out that such a space is only dominated by a finite n-dimensional complex, this domination becoming equivalence only under the conditions described below. But certainly, we can find an (n-1)-connected map $f : K \to X$ with K being finite of dimension $n - 1$, and with $\pi_n(f)$ being finitely generated and projective over R. Hence, $\pi_n(f)$ determines an element $w \in \underline{K}$; we wish to show that w depends only on X. Write P for $\pi_n(f)$ and let F be R-free of finite rank with $F \cong P \oplus Q$ for some (finitely generated projective) R-module Q. As in (I.4.8), for each generator of F attach an n-sphere to K by taking an (n-1)-sphere to the base point of K (by a constant map) to form an n-connected map $g : L \to X$ with L finite of dimension n, and $\pi_{n+1}(g) \cong Q$. Observe that the class of Q in $\underline{K}$ is minus that of P.

Lemma 3.1 - Let X satisfy Dn and $g : L \to X$ be n-connected. Then g has a homotopy right inverse, so L dominates X.

Proof - Replace g by an equivalent fibre map, still called g and note that $\pi_{r-1}(F) \cong \pi_r(g)$, where F is the fibre of g. According to (I.1.11), the obstructions to finding a cross-section lie in the groups $H^r(X; \pi_r(g))$. But for $i \leq n$, $\pi_i(g)$ vanishes, and for $r > n$, the cohomology group vanishes by $D(r - 1)$ which holds by Dn and (I.2.4). Thus, there are no obstructions and a section exists. //

Lemma 3.2 - Let X satisfy Fn and Dn and let K and L be finite n-dimensional complexes, and $f : K \to X$ and $g : L \to X$ n-connected maps, and let $P = \pi_n(f)$ and $Q = \pi_n(g)$. Then P and Q have the same class in $\underline{R}$.

Proof - It is immediate that $H_j(\tilde{f}) \cong H_j(\tilde{g}) \cong H_j(\tilde{X})$, for $j < n$; and for $j > n$, $H_j(\tilde{f}) = H_j(\tilde{g}) = 0$; and for $j = n$, $H_n(\tilde{K}) = H_n(\tilde{X}) \oplus P$, $H_n(\tilde{L}) = H_n(\tilde{X}) \oplus Q$ by (3.1). The induced maps $\tilde{f}_*$ and $\tilde{g}_*$ project on to the first summands and are split by maps $\tilde{r}_*$ and $\tilde{s}_*$ where r and s are right inverses of f and g respectively. Now the composite $sf : K \to L$ certainly induces an isomorphism of fundamental groups, and by the above, isomorphisms of the homology of the universal cover, up to and including dimension $n - 1$ and so is $(n-1)$-connected.

Hence, L is homotopy equivalent, by (I.4.11), to a finite $(n+1)$-dimensional CW-complex still called L with K as a subcomplex and cells outside of K only in dimensions n and $n + 1$. So we may take the chain complex of the pair $(\tilde{L}, \tilde{K})$ to be

$$0 \longrightarrow C_{n+1} \xrightarrow{d} C_n \longrightarrow 0,$$

and we claim that we may take the homology of this complex to consist of P and Q. For consider the exact homology sequence of the pair $(\tilde{L}, \tilde{k})$ to obtain

$$0 \longrightarrow H_{n+1}(\tilde{L}, \tilde{K}) \longrightarrow H_n(\tilde{X}) \oplus P \xrightarrow{(sf)_*} H_n(X) \oplus Q \longrightarrow H_n(\tilde{L}, \tilde{K}) \longrightarrow 0,$$

the map $(sf)_*$ is seen to be $(sf)_*(x, y) = (x, 0)$ and the claim above now follows easily by the exactness.

So now we have the following two exact sequence,

$$P \to C_{n+1} \to B_n \qquad \text{and} \qquad B_n \to C_n \to Q.$$

The second splits since Q is projective, hence, $C_n \cong B_n \oplus Q$ and, therefore, B_n is projective and so the first splits, hence, $C_{n+1} \cong B_n \oplus P$. Altogether then $P \oplus C_n \cong P \oplus Q \oplus B_n \cong Q \oplus C_{n+1}$ and since C_n and C_{n+1} are R-free, it follows that P and Q have the same class in $\underline{\mathbf{K}}$ (see section (I.3)). //

<u>Theorem 3.4</u> - X is dominated by a finite CW-complex of dimension n if and only if X satisfies Fn and Dn. When this holds for $n \geq 2$, there is an obstruction $w(X)$ in $\underline{\mathbf{K}}$ depending only on the homotopy type of X which vanishes if X is of the homotopy type of a finite CW-complex and whose vanishing is sufficient for X to be homotopy equivalent to a finite CW-complex of dimension $\max(3, n)$. Furthermore, any $(n-1)$-dimensional $(n \geq 2)$ homotopy type contains CW-complexes X satisfying Fn and Dn with $w(X)$ any arbitrary element of $\underline{\mathbf{K}}$.

<u>Proof</u> - If X satisfies Fn and Dn, we use our construction and (3.1) and if X is dominated by a finite n-dimensional CW-complex then X satisfies Fn and Dn by (1.1) and (2.3). So we define the obstruction $w(X)$ as $(-1)^n w$ where w is the class of Q in $\underline{\mathbf{K}}$, Q as in (3.2), $w(X)$ being independent of the CW-complex K and independent of n by

the remarks immediately preceding (2.1). That is, we have an n-connected map $f : K \to X$ where K is finite of dimension n and $\pi_n(f) \equiv Q$; if X is now finite of dimension n then f is a homotopy equivalence and so $\pi_n(f) = 0$, hence, $w(X) = 0$. Now let $w(X) = 0$ in $\underline{K}$, that is, let $f : K \to X$ be an (n-1)-connected map from a finite (n-1)-dimensional CW-complex K with $\pi_n(f) = P$ and let the class of P, w, be zero. For the case $n = 2$, the argument below proceeds taking $n - 1 = 2$ and $n = 3$. If w is zero in $\underline{K}$ then there is a finitely generated free R-module F such that $F \oplus P$ is free. A familiar situation indeed; for each generator of F we attach n-spheres to K and extend f by collapsing (n-1)-spheres onto the base point of K. As usual, if we call the result $g : L \to X$, we have that $\pi_n(g)$ is free of finite rank and hence may be extended over a finite number of n-cells to a homotopy equivalence.

We now have only to construct the examples, so let K be any finite n-dimensional CW-complex with fundamental group G. Let A and B be any finitely generated projective R-modules with $A \oplus B = F$ a free R-module. So we attach a trivial n-cell (an n-sphere) to K, one for each R-generator of F, by mapping (n-1)-spheres onto the base point of K; then if we call the resulting CW-complex $X(n)$, $X(n)$ dominates K by (I.4.6), and so $\pi_n(X(n)) \cong \pi_n(K) \oplus \pi_n(X(n), K)$ where $\pi_n(X_n, K) \cong H_n(\tilde{X}_n, \tilde{K}) \cong F$. Recall now (I.4.10) to see that $\tilde{X}_n$ has the homotopy type of $\tilde{K}$ wedged to one n-sphere for each $\mathbb{Z}$-generator of F. Also, we can take the $\mathbb{Z}$-generators of F to be the union of $\mathbb{Z}$-bases for A and B

and so we write $\tilde{X}(n) \simeq K \vee (V_A S^n) \vee (V_B S^n) = Y \vee (V_B S^n)$, where $Y = K \vee (V_A S^n)$. So suppose inductively that we have constructed $X(m)$, $m = n + 2k$ such that $\tilde{X}(m) \simeq Y \vee (V_B S^m)$, that is $\tilde{X}(m)$ is of the homotopy type of a Y with m-spheres attached corresponding to a decomposition $\pi_m(X(m)) \cong \pi_m(Y) \oplus B$. Now we form $X(m + 1)$ as follows: consider the composition $p : F \twoheadrightarrow B \rightarrowtail \pi_m(X(m))$ and let a_i be an R-generator of F, then $p(a_i) \in \pi_m(X(m))$ and so we attach (m+1)-cells to $\tilde{X}(m)$ by means of these. As in (I.4.10), this attaches an (m+1)-cell to $X(m)$ by the images of the free $\mathbf{Z}$-generators of F. Now $F \cong A \oplus B$ and so the generators of A yield trivially attached cells and in the resulting space $X(m + 1)$, the cells attached corresponding to the generators of B will have killed (made homotopically trivial) all the spheres in $V_B S^m$, this by construction and so $\tilde{X}(m + 1) \simeq Y \vee (V_A S^{m+1})$, and $\pi_{m+1}(X(m+1)) \cong \pi_{m+1}(Y) \oplus A$. So we attach (m+2)-cells to $X(m + 1)$ via the composition $F \twoheadrightarrow A \rightarrowtail \pi_{m+1}(X(m + 1))$ and the resulting space $X(m + 2)$ now satisfies the induction hypothesis. Thus we attach cells of all dimensions and eventually determine a space X in which $V_B S^n$ has been destroyed so that $\tilde{X} \simeq \tilde{Y}$. The conditions Fn and Dn still hold for X since $X(n)$ dominates X just as in (3.1); the inclusion $i : K \hookrightarrow X$ is (n-1)-connected and has $\pi_n(i) \cong A$ so that the obstruction $w(X)$ is $(-1)^n$ times the class of A in $\underline{\mathbf{K}}$ and, hence, $w(X)$ is arbitrary. //

BIBLIOGRAPHY

[1] Adams, J. F., Algebraic Topology in the Last Decade, Proceedings of Symposia in Pure Math. (edited by A. Liulevicius), VXXII, AMS (1971).

[2] Bass, H., Algebraic K-Theory, W. A. Benjamin, Inc., New York.

[3] Eilenberg, S., Homology of Spaces with Operators I, Trans. Amer. Math. Soc. (1947), 175.

[4] Hilton, P. J., Staminbach, V., Homological Algebra, Springer-Verlag, New York (1971).

[5] Hu, S. T., Homotopy Theory, Academic Press, New York (1959).

[6] Lyra, C. B. de, On Spaces of the Same Homotopy Type as Polyhedra, Bol. Soc. Mat. de S. Paulo, V. 12 (1960), 43-62.

[7] Lyra, C. B. de, On a Conjecture in Homotopy Theory, An. da Acad. Brasil. de Ciências (1965), 499.

[8] Massey, W. S., Algebraic Topology: An Introduction, Harcourt, Brace and World, Inc., New York (1967).

[9] Milnor, J., On Spaces Having the Homotopy Type of a CW-Complex, Ann. of Math., V. 65 (1957), 357-362.

[10] Morgan, C. C. G., On Spaces of the Same Homotopy Type as a CW-Complex, (mimeographed) Memorial University (1977).

[11] Piccinini, R. A., CW-Complexes, Homology Theory, Queen's University Press, Kingston (1973).

[12] Steenrod, N. E., Homology with Local Coefficients, Ann. of Math., V. 44 (1943), 610-627.

[13] Steenrod, N. E., The Topology of Fibre Bundles, Princeton University Press, Princeton (1956).

[14] Spanier, E. H., Algebraic Topology, McGraw-Hill, New York (1966).

[15] Switzer, R., Algebraic Topology, Springer-Verlag, New York (1975).

[16] Wall, C. T. C., Finiteness Conditions for CW-Complexes, Ann. of Math., V. 81 (1965), 56-69.

[17] Whitehead, J. H. C., Simple Homotopy Types, Amer. J. Math., V. 72 (1950), 1-57.

[18] Whitehead, J. H. C., A Certain Exact Sequence, Ann. of Math., V. 52 (1) (1950), 51-110.

[19] Whitehead, J. H. C., Combinatorial Homotopy I, Bull. Amer. Math. Soc., Vol. 55 (1949), 213-245.

Appendix A

We give an example of two CW-complexes with the same homotopy groups but which are not of the same homotopy type; we take this example from Switzer [15, p. 90]. They are the 3-dimensional lens spaces $L(p, q)$ where p and q are positive integers with greatest common divisor 1. We may take $S^3 \subset Q^2$ to be the subspace $\{(Z_0, Z_1) \in Q^2 : |Z_0|^2 + |Z_1|^2 = 1\}$. Then we define $g : S^3 \to S^3$ by setting $g(Z_0, Z_1) = (Z_0 \exp(2\pi i/p), Z_1 \exp(2\pi qi/p))$; if g^K denotes g composed with itself K times, then $g^P = 1_{S^3}$. Hence, g defines an action of $\mathbb{Z}_p$ on S^3 by $\overline{K} \quad X = g^K(x)$ for $\overline{K} \in \mathbb{Z}_p$, $x \in S^3$ and since this action is without fixed points, it follows that the projection $S^3 \xrightarrow{q} S^3/\mathbb{Z}_p$ is a covering.

Now we define $L(p, q) = S^3/\mathbb{Z}_p$; provided we divide S^3 into cells such that $g : S^3 \to S^3$ is cellular, $L(p, q)$ inherits a natural CW-structure through the covering q, with one n-cell in each dimension $n = 0, 1, 2, 3$. Since $\pi_1(S^3) = 0$ and q is a covering, one can show that $\pi_1(L(p, q)) \cong \mathbb{Z}_p$; we already know that $q_* : \pi_n(L(p, q)) \to \pi_n(S^3)$ is an isomorphism, and what is more, q_* is also a morphism of $\mathbb{Z}(\mathbb{Z}_p)$-modules. But it is easy to see that $g : S^3 \to S^3$ is (freely) homotopic to 1_{S^3} through the homotopy $h_t : S^3 \to S^3$ defined by $h_t(Z_0, Z_1) = (Z_0 \exp(2\pi it/p), Z_1 \exp(2\pi itq/p))$ and this means that $\mathbb{Z}_p$ acts trivially on $\pi_n(S^3)$ and hence on $\pi_n(L(p, q))$, $n \geq 1$. That is, the homotopy groups of $L(p, q)$ are independent of q as $\mathbb{Z}(\mathbb{Z}_p)$-modules.

Under these circumstances, one asks if $L(p, q) \simeq L(p, q')$ for $p \neq q$. It turns out, however, that the cohomology groups of $L(p, q)$ and $L(p, q')$ are isomorphic as groups but not as rings unless $qq' \equiv m^2 (\mathrm{mod}\ p)$ for some m [see Hilton, P. J. and Wylie, S.; Homology Theory; Cambridge University Press, New York (1960), pp. 223-225]. Thus, although $L(5, 1)$ and $L(5, 2)$ have the same homotopy groups, they are not of the same homotopy type.

Appendix B

We examine here the conditions Fn when the group ring R is noetherian (see section II.1).

Definition B.1 - A ring R with unit is said to be noetherian if every ideal of R is finitely generated [see Northcott; Ideal Theory; Cambridge Tracts, ed. W. Hodge, Cambridge University Press, London (1953), p. 19].

This is equivalent to the condition that every ascending chain sequence $R_1 \subset R_2 \subset R_3 \quad \ldots$ of ideals be such that $R_n = R_m$, $\forall n \geq m$, some m [see Northcott, p. 20]. If R is noetherian, we have that every submodule of a finitely generated module over R is finitely generated.

We simplify the conditions Fn as follows:

(NF2): G is finitely presented and $H_2(\tilde{X})$ is finitely generated over R;

(NFn): NF(n - 1) holds and $H_n(\tilde{X})$ is finitely generated over R.

Lemma B.2 - If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence of modules over a noetherian ring, and if A and C are finitely generated, then so is B.

Proof - If C is finitely generated then so is $\mathrm{Im}(\beta)$, so take $\mathrm{Im}(\beta)$ to have generators $\beta(f_j)$ and take A to have generators e_i. Then $\alpha(e_i)$ and f_j generate B. For take $x \in B$ and write $\beta(X) = \sum\lambda_j\beta(f_j) = \beta(\sum\lambda_j f_j) \Rightarrow \beta(X - \sum\lambda_j f_j) = 0 \Rightarrow X - \sum\lambda_j f_j = \sum\lambda_i\alpha(e_i) \Rightarrow X = \sum\lambda_j f_j + \sum\lambda_i\alpha(e_i)$ as required. //

<u>Theorem B.3</u> - Assume R is noetherian. Then G is finitely generated and for $n \geq 2$, (Fn) is equivalent to NF(n).

<u>Proof</u> - The proof we give here that G is finitely generated may be found in [Zalesskiy and Mihalev; <u>Group Rings</u> (in Russian); Modern Problems in Mathematics, Viniti, Moscow (1973), p. 12, 65]. Assume an infinite number of generators $\{x_1, x_2, \ldots\}$ for G, chosen so that $x_{j+1} \notin \langle x_1, \ldots, x_j \rangle$, $j = 2, 3, \ldots$, where $\langle x_1, \ldots, x_j \rangle$ denotes the subgroup generated by the $\{x_1, \ldots, x_j\}$. Now consider the sequence of ideals $(1 - x_1)$, $(1 - x_1, 1 - x_2)$, $\ldots$, $(1 - x_1, 1 - x_2, \ldots, 1 - x_j)$, $\ldots$, where $(1 - x_i)$ denotes the ideal generated by $\{1 - x_i\}$ in the augmentation ideal J of R, that is, the kernel of the natural map $R \to \mathbb{Z}$. Being ideals in J, they are all ideals of R and, furthermore, it follows from the choice of the $\{x_i\}$ that this is a strictly increasing chain, contradicting the hypothesis; hence, G is finitely generated. So let K be a finite CW-complex with $\pi_1(K) = G$, then $C_i(\tilde{K})$ is free of finite rank, hence, $Z_i(\tilde{K})$ and so also $H_i(\tilde{K})$ are finitely generated. Now if $f : K \to X$ is an (n-1)-connected map, $\pi_n(f) \cong H_n(\tilde{f})$ by (I.2.4), and we have the exact sequence

$$H_n(\tilde{K}) \to H_n(\tilde{X}) \to H_n(\tilde{f}) \to H_{n-1}(\tilde{K})$$

with the extreme terms finitely generated. Hence by (B.2), $H_n(\tilde{X})$ is finitely generated if and only if $\pi_n(f)$ is finitely generated. //