FREE WOBBLE/NUTATION OF THE EARTH: A NEW APPROACH FOR HYDROSTATIC EARTH MODELS

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ABSTRACT

One of the two conventional approaches to formulate a theory to describe the Earth's wobble/nutations is to regard these as among the set of free oscillations of a rotating oblate Earth model. This theory, now standard, predicts the eigenperiods of the Chandler wobble and free core nutation (FCN). Very-long-baseline interferometry and superconducting gravimetry data indicate a significant discrepancy between the inferred and theoretically-predicted values of FCN period. The widely-accepted explanation for this discrepancy is that the core-mantle boundary's ellipticity departs from its hydrostatic equilibrium value. However the standard theory for a hydrostatic Earth model has two mathematical shortcomings, which should be removed before abandoning hydrostatic equilibrium as the reference state. These shortcomings are: (1) the treatment of the governing equations in interior regions is not consistent with the treatment of the boundary conditions at surfaces of discontinuity in material properties; (2) formulation of the boundary conditions does not treat material properties properly. To remove these shortcomings, in this thesis spherical polar coordinates are replaced by a non-orthogonal coordinate system (r_0, θ, ϕ), named after Clairaut. A set of new variables in Clairaut coordinates is introduced, generalizing the conventional notation for a spherical-layered Earth model. The governing equations and boundary conditions, written in these new variables, give a consistent description of free wobble/nutation accurate to first order in ellipticity. A program to compute wobble/nutation eigenperiods has been written, and preliminary numerical results obtained, but either the program still contains errors or truncation is too severe.

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I would like to take this opportunity to thank the other members of my supervisory committee, Dr. L. Lines and Dr. J. Wright for their help. Many thanks are owed to my family and my friends for their constant understanding and encouragement. I also thank Memorial University of Newfoundland for the financial support of a Graduate Fellowship.

The original ideas for this research were contributed by Dr. Rochester. In earlier unpublished notes (1993, 1995) he carried out the transformations in Section 2.2, and first derived the new independent field variables from the boundary conditions as described in section 3.1-3.2. The derivation of the gravity field in an oblate rotating hydrostatic Earth in Section 4.1 and ellipticity calculation in section 5.3 were also done earlier by Dr. Rochester. I checked all this work independently at the start of my research program. Getting started understanding the issues and doing this checking took about 15% of my total research time.

The derivation of the governing equations using the new dependent variables in Section 4.2, 4.3, 4.4 and the starting solutions at the geocentre in Section 5.2 was done entirely independently by me and Dr. Rochester at the same time, so that by comparing our results at the end we could find errors in the other's work to ensure

that these very complicated expressions were as free of mistakes as possible. This work took about 45% of my research time, and (with sections 3.3 and 5.1) represents the primary contribution of this thesis.

The remaining 40% of my research time was spent doing independent work not yet checked by Dr. Rochester, and reported in the following sections:

3.3 done after Dr. Rochester (unpublished notes, 1997) suggested to me that Smith must not have correctly transformed the normal stress boundary conditions;

5.1 which gives the coefficients of the governing ordinary differential equations (ODEs) using scaled variables;

5.4 which outlines the FORTRAN 77 program I created for integrating the set of 10 coupled ODEs in 10 dependent variables across the inner core and mantle (or 4 ODEs and 3 algebraic equations across the outer core), and for determining eigenperiods using the surface boundary conditions. This program modified and greatly extended a program written ealier by Dr. Rochester for computing eigenperiods of spheroidal oscillations of a non-rotating spherical Earth model, i.e. involving only 6 ODEs and 6 dependent variables in the solid parts of the Earth, with far simpler coefficients;

5.5 summarizing and discussing the numerical results so far.

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Chapter 1

Introduction

The Earth's rotation is affected by the Earth's interactions with other astronomical bodies and by its internal movements. Irregularities of the Earth's rotation have been observed as changes both in the length of the day (LOD) and in the orientation of the Earth's rotation axis. Changes in the orientation of the Earth's rotation axis in the celestial reference frame (i.e. in space) include precession and nutation. Steady precession, caused by the attraction of the Moon and Sun on the Earth's equatorial bulge, is the slow periodic motion of the rotation axis in space, in which that axis describes a cone with apex angle 47° about the pole of the ecliptic in 26,000 years. The rotation axis also carries out shorter-period oscillatory motions in space, about its mean position (located on the cone of steady precession). During these small (at most 9.3 seconds of arc) amplitude oscillations the rotation axis moves toward and away from the ecliptic pole. Viewed from space this is a "nodding" motion of the rotation axis, and is called nutation. If observed relative to the terrestrial frame, changes in the orientation of the Earth's rotation axis are changes in its geographical location, and are called polar motion. Periodic polar motion is called wobble — in wobble the rotation axis traces out a cone in the Earth around the figure axis (i.e. the axis of symmetry) in periodic fashion. The law of angular momentum conservation requires that each wobble is accompanied by a nutation, and vice versa. However, the amplitudes corresponding to the two features are quite different. Whether the motion of the rotation axis should be called 'wobble' or 'nutation' depends on which of the two features has the larger amplitude (and is therefore in principle more easily detectable).

The terms free and forced wobble/nutation are introduced to describe the motions of the Earth's rotation axis driven by different factors. Forced nutation/wobble is caused by the periodic fluctuations in the lunar and solar gravitational torques on the Earth's equatorial bulge, whereas free wobble/nutation does not need any torque from external bodies, and results if the rotation axis of an oblate Earth is disturbed from alignment with the axis of figure. The first study of the forced nutation goes back to Bradley's (1726) discovery of the principal forced nutation, with 18.6 year period. There are also smaller forced nutations with shorter periods, e.g. 9.3 years, a year, half a year etc. Euler (1765) first predicted that a rigid Earth with a symmetric equatorial bulge would exhibit a prograde free wobble with a period of about 10 months. However this wobble was not detected until 1891, when its period was found to be 435 days rather than the 306 days predicted by Euler. It is named the Chandler wobble after its discoverer. Newcomb (1892) correctly attributed the lengthening of the Chandler wobble period from 306 days to 435 days to the real Earth's departure from rigidity, which causes a gyroscopic restoring torque weaker than that for a rigid

Earth. In an attempt to explain the lengthening of the Chandler period, Hough (1895) extended Hopkins' (1839) work and found that an inviscid homogeneous incompressible liquid contained inside a rigid ellipsoidal shell would shorten the Euler period rather than lengthen it. Of course the existence and size of the Earth's liquid core were first unambiguously demonstrated by Jeffreys (1926). Other studies (Larmor 1909, Love 1909, Dahlen 1976) showed that the oceans lengthen the period by about 33 days, but the bulk of the lengthening is due to elasticity of the mantle (Jeffreys & Vicente 1957, Molodenskii 1961).

Hough's work also showed (but he did not realize) that the liquid core would give rise to another free wobble mode, which is retrograde with period slightly shorter than a sidereal day, by a small amount proportional to the ellipticity of the core-mantle boundary (CMB). Hough's Earth model (with a rigid mantle, homogeneous incompressible liquid core) predicted that the period of this nearly diurnal free wobble (NDFW) is $\frac{1}{350}$ shorter than a day. So, the accompanying nutation of this NDFW has a period of 350 days in space. Jeffreys & Vicente (1957), Molodenskii (1961) and Smith (1977) found that this period in space is lengthened to about 460 days by mantle elasticity. Based on Poinsot's theorem which implies that the amplitudes of wobble and its accompanying nutation are in the same ratio as their periods, Toomre (1974) and Rochester et al. (1974) recognized that the nutation accompanying the NDFW has 460 times bigger amplitude than the wobble itself, so Toomre (1974) argued that free core nutation (FCN) is a better name for this NDFW.

One approach to describe the eigenperiods of the free wobble/nutation spectrum is to use the integral equations of angular momentum conservation for a deformable Earth model. The governing equations involving angular momentum of one part of the Earth and torques exerted on one part of the Earth by the others are represented by integrals over the volume occupied by the mantle, inner core and liquid outer core. The eigenperiods are then given by the roots of a determinant derived from the set of integral equations. Following Hough (1895), many researchers have made efforts along this line and improved the Earth model to make it much more realistic. Jeffreys & Vincente (1957b) considered an Earth model with radially-stratified elastic mantle, a homogeneous incompressible liquid outer core and a point inner core. Molodenskii (1961) included core compressibility in his Earth model but ignored the inner core. Shen & Mansinha (1976) extended the theory of Molodenskii to include the possibility of non-adiabatic stratification in the outer core. Sasao et al. (1980) reconstructed the theory of Molodenskii on a much simpler basis and included dissipative core-mantle coupling. Mathews et al. (1991), de Vries & Wahr (1991) and Dehant et al. (1993) separated the solid inner core as an independent dynamic system when solving the dynamics of the liquid outer core, and found eigenperiods for the free inner core nutation (FICN) and inner core wobble (ICW). However, it is still a challenge to adequately represent the integrated effects of gravitational and pressure torques on any part of the Earth, especially for a realistic Earth model (Xu & Szeto 1996).

Another approach, first developed by Smith (1974), is to treat the wobble/nutation modes of a realistic Earth model as a subset of the free oscillations of the Earth because wobble/nutation involves only small periodic departures from its reference state of steady rotation about the figure axis. Smith derived the elastic-gravitational normal mode theory for a rotating, slightly oblate, hydrostatically prestressed and oceanless Earth model in the form of an infinite set of coupled ordinary differential equations (ODEs) with dependent variables representing displacement, stress and gravitational potential and flux, with radius as independent variable. The eigenperiod spectrum will be given by numerically integrating these ODEs across interfaces of discontinuity in material properties and satisfying the boundary conditions at the surface of the Earth. This approach is suitable for realistic Earth models because of the high accuracy of its numerical calculation. Smith (1977) used this theory to explore numerically a portion of the elastic-gravitational normal mode spectrum. Smith's work (1974, 1977) has come to be regarded as the definitive analysis by most subsequent workers. Wahr's (1981a, b) calculations based on this theory were adopted by the International Astronomical Union (IAU) as the reference model for the Earth's forced and free nutations.

Each of these two approaches has advantages and disadvantages. The first (i.e. semi-analytical) approach, which is based on the torque-angular momentum equations, still requires certain approximations to be made regarding the flow in the outer core and the elastic deformation of the inner core and mantle, so as to model the gravitational and pressure torques acting on the mantle. These approximations are unnecessary when following Smith's approach. On the other hand, Smith's approach does not easily produce good approximations for such a long-period mode as the ICW, whose eigenfunction probably cannot be well represented by a heavily truncated series such as has been necessary (up to now anyway) for computing eigenperiods.

In the standard theory of free oscillations in a non-rotating, spherically-layered Earth, the displacement field $\mathbf{u}$ and the gravitational perturbation V_1 can be represented by a single spherical harmonic

$$\mathbf{u} \equiv \mathbf{S}_n^m \quad or \quad \mathbf{T}_n^m \tag{1.1}$$

$$V_1 \equiv \Phi_n^m Y_n^m \tag{1.2}$$

with

$$\mathbf{S}_n^m = (u_n^m \hat{\mathbf{r}} + r v_n^m \nabla) Y_n^m \tag{1.3}$$

$$\mathbf{T}_n^m = -t_n^m \hat{\mathbf{r}} \times \nabla Y_n^m \tag{1.4}$$

$$Y_n^m = P_n^m e^{im\phi} \tag{1.5}$$

where u_n^m, v_n^m, t_n^m and Φ_n^m are functions of r only, and P_n^m is the associated Legendre function (see equations (3.4)-(3.5)). $\mathbf{S}_n^m$ and $\mathbf{T}_n^m$ represent purely spheroidal and toroidal displacement field respectively. For the free oscillations most studied by seismologists (periods less than one hour), the effects of the Earth's rotation can be considered as small perturbations to the solutions for a non-rotating Earth model (e.g. Dahlen & Sailor 1979). However for modes with periods longer than one hour (core undertones, Slichter modes and wobble/nutation) the effects of the Coriolis and centrifugal forces can no longer be treated as small perturbations. Rotation and ellipticity make the displacement vector become an infinitely long coupled chain: either

$$\mathbf{u} = \mathbf{S}_{|m|}^m + \mathbf{T}_{|m|+1}^m + \mathbf{S}_{|m|+2}^m + \mathbf{T}_{|m|+3}^m + \cdots \tag{1.6}$$

or

$$\mathbf{u} = \mathbf{T}^m_{|m|} + \mathbf{S}^m_{|m|+1} + \mathbf{T}^m_{|m|+2} + \mathbf{S}^m_{|m|+3} + \cdots \tag{1.7}$$

These infinite chains must be truncated for numerical calculations. Smith (1977) used a three term truncated series.

$$\mathbf{T}^m_1 + \mathbf{S}^m_2 + \mathbf{T}^m_3 \tag{1.8}$$

to represent the wobble/nutation displacement field ($|m| = 1$), and claimed that this truncation was sufficient for the Chandler wobble and FCN. However he did not extend his computation to a less severe truncation than in (1.8), so his conclusion needs to be tested.

Smith (1974) took into account the ellipticity of the Earth by a mapping from the spheroidal Earth domain V_E to an equivalent spherical domain (ESD). But, the coordinate transformations he used for regions near and away from boundaries seem to be inconsistent (Rochester 1993, unpublished notes). A detailed description of Smith's mapping is given in chapter 2.

Following Jeffreys & Vicente (1957), many researchers searched for evidence of the FCN in observations of Earth orientation. In contrast to the Chandler wobble, the amplitude of the nutation is too small to be detected directly by astronomical obeservations. Claimed discoveries of FCN involved estimations of the amplitude much higher than the possible upper limits (Rochester et al. 1974). However, its resonance effects show up in very long baseline interferometry (VLBI) data on the much larger forced nutations of the Earth's rotation axis, and in long-period superconducting gravimetry records of the solid Earth tides. Several studies of VLBI observations

(Herring et al. 1986, Gwinn et al. 1986, Jiang & Smylie 1995) showed that the FCN period is about 30 days shorter than its 460-day theoretical value (i.e. Smith 1977, Wahr 1981b). Other studies of stacked data from tidal superconducting gravimetry measurements (Neuberg et al. 1987, Richter & Zurn 1988. Merriam 1994) gave the same result. The consistency between the results of two different data sets suggests there exists a significant discrepancy between the observation and the conventional theory of the FCN period.

In search for the possible reasons for this discrepancy, Wahr & Sasao (1981) and Wahr & Bergen (1986) estimated the contributions from ocean tides and effects of mantle anelasticity respectively, and found them to be much smaller than this discrepancy. Gwinn et al. (1986) suggested that an extra flattening of the core mantle boundary relative to its hydrostatic equilibrium configuration could remove this discrepancy. Mathews et al. (1991), allowing for the rotational dynamics of the inner core, obtained an additional prograde nearly diurnal free wobble (also called free inner core nutation FICN) and a new free wobble of the inner core (ICW) with much longer period. Dehant et al. (1993) showed that the resonance from the FICN is almost negligible and cannot explain the FCN discrepancy. The explanation of a non-hydrostatic core-mantle boundary has become widely accepted.

However, before abandoning hydrostatic theory, we should be sure any shortcomings in its formulation (Smith 1974) have been removed. There appear to be three such problems.

1. The dynamical equations in interior regions were derived by coordinate trans-

formations inconsistent with those used for boundary conditions at an interface of discontinuity in material properties.

2. Material properties were treated improperly in formulating the boundary conditions for the normal stress components.

3. The displacement field chain was severely truncated i.e. to the three terms in equation (1.8) and the convergence of numerical results was not tested by extending the chain.

These problems may have unintentionally reduced the accuracy, and even prevented the appearance of a more complicated spectrum , of wobble/nutation resonances.

Problems 1 and 2 were brought about by incorrect coordinate transformations when rotation and oblateness of the Earth are considered. This thesis aims to remove them by a thorough reformulation of the hydrostatic Earth theory in a new coordinate system. Problem 3 should be dealt with by extending the chain (1.8) to $\mathbf{T}_5^m$ or $\mathbf{T}_7^m$ ($|m| = 1$).

Motivated by a suggestion by Jeffreys (1942), Rochester (unpublished notes 1993) proposed a non-orthogonal coordinate system (r_0, θ, ϕ), where r_0 is the mean radius of the equipotential surface through (r, θ, ϕ). He later realized that Kopal (1980) used this coordinate system to describe the free oscillation of a star (i.e. a fluid) and named it after Clairaut. We will continue to call it a Clairaut coordinate system. Rochester (unpublished notes) also proposed a new set of field variables dependent on the Clairaut coordinate r_0. These variables are modified from those suitable for a spherical Earth model (first defined by Alterman et al. (1959) and called AJP variables) so as to be continuous across ellipsoidal interfaces and include no derivatives of the material properties.

This thesis is divided into two parts, i.e. theoretical derivation and numerical application. The first part (chapters 2-4) covers the derivation of a set of governing ODEs and boundary conditions (BCs) for wobble/nutation modes of a rotating, spheroidal hydrostatic Earth model in the Clairaut coordinate r_0 and the new set of dependent variables. These ODEs and BCs should provide a precise consistent description for the free oscillation modes of a rotating oblate hydrostatic Earth model to first order in the ellipticity. These derivations form the major part of this thesis, with my original contributions being detailed in the Acknowledgements.

The second, and shorter, part of my thesis (chapter 5) describes the implementation of numerical calculations to search for wobble/nutation eigenperiods for the Preliminary Reference Earth Model (PREM), created by Dziewonski & Anderson (1981) to give a best overall fit to seismological data. The objective of these calculations is to check whether the results are closer to or farther away from the value inferred from observations than those based on Smith's (1974) theory.

Chapter 2

Clairaut Coordinate System

2.1 Smith's Mapping

A model of the Earth in hydrostatic equilibrium has its material properties constant on the equipotentials of its gravity field. The conventional theory of free oscillations (Alterman et al. 1959) disregards the rotation of the Earth and takes the surfaces of constant material properties as spherical. The surfaces of constant material properties in a rotating oblate hydrostatic Earth will be ellipsoidal, to first order in the ellipticity.

To compute the wobble/nutation modes of a rotating, oblate Earth model, the effects of Earth's ellipticity and rotation must be incorporated in the governing equations and boundary conditions. However the data for material properties of the Earth are tabulated in a spherical Earth model e.g. PREM. We need to properly adapt the latter Earth model to the former. In this chapter, we will analyse Smith's (1974) mapping and set up a new coordinate system , so as to establish a consistent transformation from spherical coordinates.

If a rotating, ellipsoidally-layered Earth model is in hydrostatic equilibrium and the point with location vector $\mathbf{r}(r, \theta, \phi)$ is on an equipotential of the equilibrium gravity field. then to the first order of ellipticity, the equation of this ellipsoidal equipotential is

$$r = r_0[1 - \frac{2}{3}f(r_0)P_2(\cos\theta)] \tag{2.1}$$

where r_0 is the mean radius of the equipotential, $f(r_0)$ is its ellipticity, P_2 is the second-degree Legendre polynomial and θ is the colatitude of vector $\mathbf{r}$. So, the mean radius r_0 can be expressed by the spherical polar coordinates of point $\mathbf{r}$:

$$r_0 = r[1 + \frac{2}{3}f(r)P_2(\cos\theta)] \tag{2.2}$$

Chandrasekhar & Roberts (1963) regarded the ellipsoidal Earth model as generated by modifying the spherical Earth model. They wrote the the material properties in an ellipsoidal Earth model, e.g. density $\rho_0(\mathbf{r})$ at a point $\mathbf{r}(r, \theta, \phi)$, as:

$$\rho_0(\mathbf{r}) = \rho_0(r) + \frac{2}{3}rf(r)\frac{d\rho_0}{dr}P_2(\cos\theta) \tag{2.3}$$

where $\rho_0(r)$ is the density at $\mathbf{r}$ in the corresponding spherical Earth model.

Based on the assumption that this ellipsoidal Earth is in hydrostatic equilibrium, which ensures that material properties in this Earth model are constant over equipotentials, this equation for density is:

$$\rho_0(\mathbf{r}) = \rho_0(r_0) \tag{2.4}$$

consistent with (2.3) and (2.2). There are similar relations for rigidity $\mu(\mathbf{r})$ and Lamé parameter $\lambda(\mathbf{r})$.

To deal with the elliptical configuration of the Earth, Smith (1974) used the following mapping from a point P (r, θ, ϕ) in the ellipsoidal Earth V_E to a point P'

(r_0, θ, ϕ) in the corresponding spherical volume called the "equivalent spherical domain" (ESD). Note we write Smith's mapping in the notation adapted in this thesis — his p, r correspond to our r, r_0 respectively.

1. If P is on an ellipsoidal internal or external boundary (i.e. an equipotential where any material property is not continuous) the location of P' is given by (2.2) i.e. a boundary with mean radius r_0 in V_E becomes a sphere of radius r_0 in the ESD.

2. If P is not at internal or external boundaries the location of P' is taken to be identical to that of P, i.e.

$$r_0 = r \tag{2.5}$$

When discussing material properties explicitly, Smith regarded such quantities, e.g. $\rho_0(\mathbf{r})$, at point $\mathbf{r}$ in V_E as formed in terms of those in the ESD by (2.3), i.e.

$$\rho_0(\mathbf{r}) \quad = \quad \rho_0(r) + \frac{2}{3} r f(r) \frac{d\rho_0}{dr} P_2(\cos\theta) \tag{2.6}$$

At a boundary, using (2.1)-(2.2), (2.6) reduces to

$$\rho_0(\mathbf{r}) = \rho_0(r_0) \tag{2.7}$$

in agreement with (2.4). However, at an interior point, the effect of using (2.5) is that

$$\rho_0(\mathbf{r}) = \rho_0(r_0) + \frac{2}{3} r_0 f(r_0) \frac{d\rho_0}{dr_0} P_2(\cos\theta) \tag{2.8}$$

In (2.7)-(2.8) $\rho_0(r_0)$ is prescribed as a function of radius r_0 in a spherical Earth model. Similar relations hold for other properties $\lambda(\mathbf{r})$ and $\mu(\mathbf{r})$.

Clearly, this mapping treated regions (i) in the interior between, and (ii) in the vicinity of, boundaries of the Earth in an inconsistent way, leading to an error of the first order in ellipticity in the vicinity of boundaries (Smith 1974). The existence of this error is shown by the discrepancy between (2.8) and (2.7).

This discrepancy will affect numerical calculations carried out using the foregoing mapping. The governing equations, based on the basic physical laws, are first written as a set of partial differential equations (PDEs). The orthogonality properties of spherical harmonics are applied to these PDEs to remove latitude and longitude dependences. Then a set of ODEs with radius r_0 as independent variable is formed, which will be numerically integrated between successive elliptical surfaces $r_0 =$ constant (2.2). As the integration approaches one side of an internal boundary, (2.8) has been used with density derivative appropriate to that side. The boundary conditions must be satisfied using (2.7). Then integration away from the other side of that boundary uses (2.8) with density derivative appropriate to that side. Smith describes this sequence of steps as requiring that material properties be continued smoothly across interfaces at which they are actually discontinuous. But this seems not to fully describe the consequences of using (2.8) rather than (2.7). Smith argues that the effects of this inconsistency should be small, but the contribution of this assumption is unknown before more precise computation is carried out. To avoid the above inconsistency, Rochester (1993 unpublished notes) proposed using (2.1)-(2.2) everywhere in V_E, not just at boundaries. Based on this idea, another coordinate system is introduced in the next section.

2.2 Clairaut Coordinates

Based on some work by Jeffreys (1942) on the description of tides, Rochester proposed to replace Smith's mapping of V_E into the ESD by simply making a coordinate transformation while remaining in V_E, i.e. replacing the orthogonal coordinates (r, θ, ϕ) of $\mathbf{r}$ by the non-orthogonal (r_0, θ, ϕ). Kopal (1980) used this set of coordinates to formulate the free oscillations of a star, but gave no numerical results. He named this coordinate system (r_0, θ, ϕ) after Clairaut, and we use that name in this thesis. The differences between Smith's mapping and Rochester's coordinate transformation are described in Fig. 2.1.

The transformation of coordinates (2.1)-(2.2) leaves the ellipsoidal Earth volume V_E unchanged, but with r_0, θ, ϕ (instead of r, θ, ϕ) as the coordinates of the point at the tip of vector $\mathbf{r}$. Comparing this transformation with the mapping used by Smith we see that points $\mathbf{r}$ in V_E which are (are not) on surfaces of discontinuity in material properties coincide (do not coincide) with points $\mathbf{r}$ in ESD. This is why we make a distinction between Smith's mapping and the transformation to Clairaut coordinates in Fig. 2.1, and why we prefer not to use the term "ESD" to describe the result of transforming to Clairaut coordinates.

Consistent with (2.4) and (2.7), the material properties at any point $\mathbf{r}(r, \theta, \phi)$ in V_E will be equated to the values on the equipotential surface with radius r_0 in ESD, which are only functions of r_0, e.g.,

$$\rho(\mathbf{r}) = \rho_0(r_0) \tag{2.9}$$

with similar relations for λ and μ.

Any field variable S in the ellipsoidal Earth V_E will be transformed by

$$S(r,\theta,\varphi) = S(r_0,\theta,\phi) - \frac{2}{3}r_0 f(r_0)\frac{\partial S}{\partial r_0}P_2(\cos\theta) \tag{2.10}$$

(to the first order of ellipticity).

To form the governing ODEs from their PDEs, the partial derivatives in spherical polar coordinates need to be related to those in Clairaut coordinates. To the first order of ellipticity,

$$\frac{\partial}{\partial r} = [1 + \frac{2}{3}(r_0 f)' P_2]\frac{\partial}{\partial r_0} \tag{2.11}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \frac{2}{3}r_0 f P_2^1 \frac{\partial}{\partial r_0} \tag{2.12}$$

where $()' = \frac{d}{dr_0}$.

In chapters 3 and 4, we will transform all material properties, field variables, governing equations and boundary conditions, so as to generate a description of wobble/nutation modes for a rotating, ellipsoidally-stratified Earth model in Clairaut coordinate system which is internally consistent to first order of ellipticity.

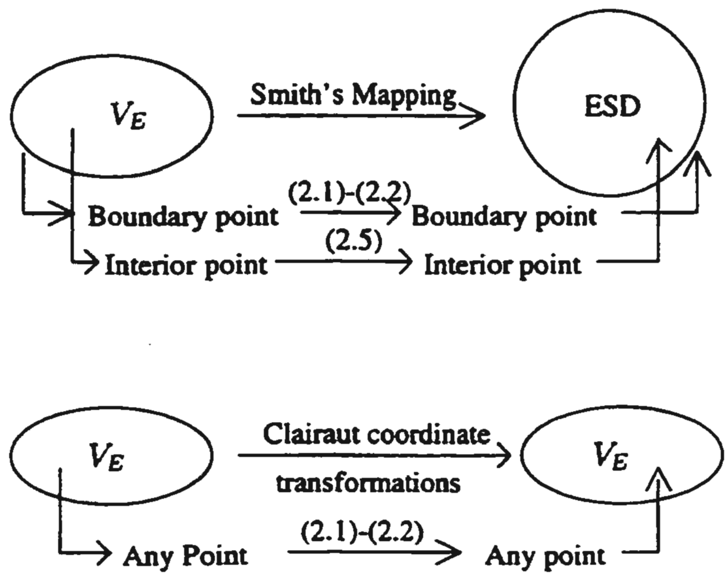

Figure 2.1: Smith's Mapping vs Clairaut Coordinate Transformation. (2.1) indicates application of equation (2.1).

Chapter 3

Boundary Conditions for an Ellipsoidally-Layered Earth with New Variables

3.1 Boundary Conditions in Clairaut Coordinates

The first application of Clairaut coordinates in this thesis is to write the boundary conditions for a rotating, ellipsoidally-layered Earth model. Since the equation of the equipotential through point $\mathbf{r}(r,\theta,\phi)$

$$r = r_0[1 - \frac{2}{3}f(r_0)P_2(\cos\theta)] \tag{3.1}$$

its outward unit normal $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} + \frac{2}{3}f(r)\frac{dP_2}{d\theta}\hat{\theta} \tag{3.2}$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are the unit vectors in r and θ directions.

For an Earth model in hydrostatic equilibrium, the boundary conditions at an internal or external equipotential surface of discontinuity in material properties require:

1. a. continuity in normal displacement component $\hat{\mathbf{n}} \cdot \mathbf{u}$ (at all internal boundaries); or,

b. continuity in displacement **u** (at internal boundaries between solid and solid);

2. continuity in Eulerian perturbations to gravitational potential V_1;

3. continuity in gravitational flux $\hat{\mathbf{n}} \cdot (\nabla V_1 - 4\pi G \rho_0 \mathbf{u})$;

4. continuity in components of normal stress $\hat{\mathbf{n}} \cdot \bar{\tau}$;

To express the boundary conditions in Clairaut coordinates we will expand the continuous quantities using spherical harmonics and transform the expansions to Clairaut coordinates by equation (2.10). Then we will simplify the resulting expressions by using the orthogonality of spherical harmonics to remove the θ- and ϕ- dependence and arrive at boundary conditions involving r_0 as the only independent variable.

3.1.1 Expansions of Continuous Quantities by Spherical Harmonics

In this section, each continuous quantity will be expanded by the spherical harmonics Y_n^m, where

$$Y_n^m \equiv P_n^m(\cos\theta)e^{im\phi}, \tag{3.3}$$

$$P_n^m(\cos\theta) = \frac{(-1)^m}{2^n n!}(1-x^2)^{m/2}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n, n \geq m \geq 0 \tag{3.4}$$

$$P_n^{-m}(\cos\theta) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta) \tag{3.5}$$

($x = \cos\theta$) and P_n^m is the associated Legendre function of degree n and azimuthal order m.

First of all, the expansion of displacement vector **u** is:

$$\mathbf{u} = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[u_n^m(r)\hat{\mathbf{r}} + rv_n^m\nabla - rt_n^m\hat{\mathbf{r}}\times\nabla]Y_n^m \tag{3.6}$$

where u_n^m, v_n^m and t_n^m, which are functions of r only, are respectively radial, transverse spheroidal and toroidal displacement components. Using (3.2)

$$\begin{aligned}\hat{\mathbf{n}}\cdot\mathbf{u} &= \hat{\mathbf{r}}\cdot\mathbf{u}+\frac{2}{3}f(r)P_2^1\hat{\theta}\cdot\mathbf{u}\\ &= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[u_n^mY_n^m+\frac{2}{3}fP_2^1(v_n^m\frac{\partial Y_n^m}{\partial\theta}+t_n^m\frac{imY_n^m}{\sin\theta})]\end{aligned}\tag{3.7}$$

is continuous across an ellipsoidal boundary.

Using (3.6), the continuity of displacement $\mathbf{u}$ at a solid/solid boundary reduces to the continuity of

$$\mathbf{u}-(\hat{\mathbf{n}}\cdot\mathbf{u})\hat{\mathbf{n}}=-\frac{2}{3}fP_2^1(\hat{\theta}\cdot\mathbf{u})\hat{\mathbf{r}}+(\hat{\theta}\cdot\mathbf{u}-\frac{2}{3}fP_2^1\hat{\mathbf{r}}\cdot\mathbf{u})\hat{\theta}+(\hat{\phi}\cdot\mathbf{u})\hat{\phi}\tag{3.8}$$

which is the tangential displacement vector ($\mathbf{u}_{tan}$). This continuity requires the separate continuities of its components in the $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$ directions. As we can see from (3.7), the continuity of $\hat{\mathbf{r}}$ component follows from the continuity of the $\hat{\theta}$ component, hence only the continuities of $\hat{\theta}$ and $\hat{\phi}$ components need to be considered. Furthermore, the continuities in the $\hat{\theta}$ and $\hat{\phi}$ components of the tangential displacement $\mathbf{u}_{tan}$ are equivalent to the continuities of its transverse spheroidal part $(\mathbf{u}_{tan})_{tr}$ and toroidal part $(\mathbf{u}_{tan})_{to}$, which are:

$$\begin{aligned}(\mathbf{u}_{tan})_{tr} &= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\frac{(-1)^m(2n+1)}{4\pi n(n+1)}\int_0^{\pi}\int_0^{2\pi}[(\hat{\theta}\cdot\mathbf{u}-\frac{2}{3}fP_2^1\hat{\mathbf{r}}\cdot\mathbf{u})\frac{\partial Y_n^{-m}}{\partial\theta}\\ &\quad -(\hat{\phi}\cdot\mathbf{u})\frac{imY_n^{-m}}{\sin\theta}]\sin\theta d\theta d\phi\end{aligned}\tag{3.9}$$

$$\begin{aligned}(\mathbf{u}_{tan})_{to} &= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\frac{(-1)^m(2n+1)}{4\pi n(n+1)}\int_0^{\pi}\int_0^{2\pi}[-(\hat{\theta}\cdot\mathbf{u}-\frac{2}{3}fP_2^1\hat{\mathbf{r}}\cdot\mathbf{u})\frac{imY_n^{-m}}{\sin\theta}\\ &\quad -(\hat{\phi}\cdot\mathbf{u})\frac{\partial Y_n^{-m}}{\partial\theta}]\sin\theta d\theta d\phi\end{aligned}\tag{3.10}$$

(Smylie 1965).

However, $(\mathbf{u}_{tan})_{tr}$ and $(\mathbf{u}_{tan})_{to}$ are continuous only at solid/solid boundaries.

The expansions of the perturbation gravitational potential V_1 and gravitational flux γ in spherical harmonics are:

$$
\begin{aligned}
V_1 &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \Phi_n^m(r) Y_n^m && (3.11)\\
\gamma &= \hat{\mathbf{n}} \cdot (\nabla V_1 - 4\pi G \rho_0 \mathbf{u}) \\
&= (\hat{\mathbf{r}} + \frac{2}{3} f(r) P_2^1 \hat{\theta}) \cdot (\nabla V_1 - 4\pi G \rho_0 \hat{\mathbf{n}} \cdot \mathbf{u}) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \{(\frac{d\Phi_n^m}{dr} - 4\pi G \rho_0 u_n^m) Y_n^m \\
&\quad + \frac{2}{3} f P_2^1 [(\frac{\Phi_n^m}{r} - 4\pi G \rho_0 v_n^m) \frac{\partial Y_n^m}{\partial \theta} - 4\pi G \rho_0 t_n^m \frac{im Y_n^m}{\sin\theta}]\} && (3.12)
\end{aligned}
$$

where $\Phi_n^m(r)$ is function of r only.

The normal stress vector

$$
\begin{aligned}
\hat{\mathbf{n}} \cdot \bar{\tau} &= (\hat{\mathbf{r}} + \frac{2}{3} f(r) P_2^1 \hat{\theta}) \cdot \bar{\tau} \\
&= (\tau_{rr} + \frac{2}{3} f(r) P_2^1 \tau_{r\theta}) \hat{\mathbf{r}} + (\tau_{r\theta} + \frac{2}{3} f(r) P_2^1 \tau_{\theta\theta}) \hat{\theta} \\
&\quad + (\tau_{r\phi} + \frac{2}{3} f(r) P_2^1 \tau_{\theta\phi}) \hat{\phi} && (3.13)
\end{aligned}
$$

where $\tau_{rr}, \tau_{r\theta}, \tau_{r\phi}, \tau_{\theta\theta}, \tau_{\theta\phi}$ are stress components. The continuity of $\hat{\mathbf{n}} \cdot \bar{\tau}$ requires that three components at $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$ directions separately be continuous across boundaries.

According to the Hooke's law relationship between stress and displacement, we can write the expansions of the stress components in spherical harmonics as:

$$
\begin{aligned}
\tau_{rr} &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} [\lambda(r_0) \Delta_n^m + 2\mu(r_0) \frac{du_n^m}{dr}] Y_n^m && (3.14)\\
\tau_{r\theta} &= \mu(r_0) \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} (\Psi_n^m \frac{\partial Y_n^m}{\partial \theta} + \xi_n^m \frac{im Y_n^m}{\sin\theta}) && (3.15)
\end{aligned}
$$

$$\tau_{r\phi} = \mu(r_0) \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} (\Psi_n^m \frac{imY_n^m}{\sin\theta} - \xi_n^m \frac{\partial Y_n^m}{\partial\theta}) \tag{3.16}$$

$$\begin{aligned} \tau_{\theta\theta} &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \{[\lambda(r_0)\Delta_n^m + 2\mu(r_0)\frac{u_n^m}{r}]Y_n^m \\ &+2\mu(r_0)[\frac{v_n^m}{r}\frac{\partial^2 Y_n^m}{\partial\theta^2} + im\frac{t_n^m}{r}\frac{\partial}{\partial\theta}(\frac{Y_n^m}{\sin\theta})]\} \end{aligned} \tag{3.17}$$

$$\begin{aligned} \tau_{\theta\phi} &= \mu(r_0) \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} [2im\frac{v_n^m}{r}\frac{\partial}{\partial\theta}(\frac{Y_n^m}{\sin\theta}) \\ &-\frac{t_n^m}{r}(\frac{\partial^2 Y_n^m}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial Y_n^m}{\partial\theta} + \frac{m^2}{\sin^2\theta}Y_n^m)] \end{aligned} \tag{3.18}$$

where

$$\Delta_n^m \equiv \frac{du_n^m}{dr} + \frac{2u_n^m - n(n+1)v_n^m}{r} \tag{3.19}$$

$$\Psi_n^m \equiv \frac{dv_n^m}{dr} - \frac{v_n^m - u_n^m}{r} \tag{3.20}$$

$$\xi_n^m \equiv \frac{dt_n^m}{dr} - \frac{t_n^m}{r} \tag{3.21}$$

are functions of r only.

So, the continuous radial component of normal stress $\hat{\mathbf{n}} \cdot \tilde{\tau}$, written as $(\hat{\mathbf{n}} \cdot \tilde{\tau})_{rad}$, can be expanded in spherical harmonics as:

$$\begin{aligned} (\hat{\mathbf{n}} \cdot \tilde{\tau})_{rad} &= (\tau_{rr} + \frac{2}{3}f(r)P_2^1\tau_{r\theta}) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \{[\lambda(r_0)\Delta_n^m + 2\mu(r_0)\frac{du_n^m}{dr}]Y_n^m \\ &+\frac{2}{3}fP_2^1\mu(r_0)[\Psi_n^m\frac{\partial Y_n^m}{\partial\theta} + \xi_n^m\frac{imY_n^m}{\sin\theta}]\} \end{aligned} \tag{3.22}$$

The continuities of $\hat{\theta}$ and $\hat{\phi}$ components of normal stress $\hat{\mathbf{n}} \cdot \tilde{\tau}$ are equivalent to the continuities of its transverse spheroidal part $(\hat{\mathbf{n}} \cdot \tilde{\tau})_{tr}$ and toroidal part $(\hat{\mathbf{n}} \cdot \tilde{\tau})_{to}$, which are (Smylie 1965)

$$(\hat{\mathbf{n}} \cdot \tilde{\tau})_{tr} = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{(-1)^m(2n+1)}{4\pi n(n+1)} \int_0^{\pi} \int_0^{2\pi} [(\tau_{r\theta} + \frac{2}{3}f(r)P_2^1\tau_{\theta\theta})\frac{\partial Y_n^{-m}}{\partial\theta}$$

$$-(\tau_{r\phi}+\frac{2}{3}f(r)P_2^1\tau_{\theta\phi})\frac{imY_n^{-m}}{\sin\theta}]\sin\theta d\theta d\phi \tag{3.23}$$

$$(\hat{\mathbf{n}}\cdot\vec{\tau})_{to} = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\frac{(-1)^m(2n+1)}{4\pi n(n+1)}\int_0^{\pi}\int_0^{2\pi}[-(\tau_{r\theta}+\frac{2}{3}f(r)P_2^1\tau_{\theta\theta})\frac{imY_n^{-m}}{\sin\theta}$$

$$-(\tau_{r\phi}+\frac{2}{3}f(r)P_2^1\tau_{\theta\phi})\frac{\partial Y_n^{-m}}{\partial\theta}]\sin\theta d\theta d\phi \tag{3.24}$$

3.1.2 Continuous Quantities in Clairaut Coordinates

After expanding the continuous quantities in spherical harmonics, we will transform all field variables in the above expansions in r into those in r_0 according to (2.10), i.e.

$$X(r) = X(r_0) - \frac{2}{3}r_0 f(r_0)P_2\frac{dX(r_0)}{dr_0} \tag{3.25}$$

where $X(r)$ can be any variable in r, e.g. u_n^m, $\frac{d\Phi_n^m}{dr}$ etc. Putting this transformation into (3.7), (3.9), (3.10), (3.11), (3.12), (3.22), (3.23), (3.24), we obtain expressions in Clairaut coordinates for eight quantities continuous across the ellipsoidal boundaries. Each expression has the form

$$X(r_0,\theta,\phi) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty} X_{mn} \tag{3.26}$$

where $X(r_0,\theta,\phi)$ represents respectively normal displacement, tangential displacement components , perturbation gravitational potential, gravitational flux and normal stress components. Their X_{mn} are respectively:

$$(\hat{n}\cdot\mathbf{u})_{mn} = u_n^m Y_n^m$$

$$+\frac{2}{3}f[-r_0\frac{du_n^m}{dr_0}P_2Y_n^m + P_2^1(v_n^m\frac{\partial Y_n^m}{\partial\theta}+t_n^m\frac{imY_n^m}{\sin\theta})] \tag{3.27}$$

$$\{(\mathbf{u}_{tan})_{tr}\}_{mn} = \frac{(-1)^m(2n+1)}{2n(n+1)}\sum_l\{v_l^m(I1)_{ln} + imt_l^m(I2)_{ln}$$

$$-\frac{2}{3}f[r_0\frac{dv_l^m}{dr_0}(I3)_{ln} + imr_0\frac{dt_l^m}{dr_0}(I4)_{ln} + u_l^m(I5)_{ln}]\} \tag{3.28}$$

$$
\begin{aligned}
\{(\mathbf{u}_{tan})_{to}\}_{mn} &= \frac{(-1)^m(2n+1)}{2n(n+1)}\sum_l \{t_l^m (I1)_{ln} - imv_l^m (I2)_{ln} \\
&\quad -\frac{2}{3}f[-imr_0\frac{dv_l^m}{dr_0}(I4)_{ln} + r_0\frac{dt_l^m}{dr_0}(I3)_{ln} - imu_l^m(I6)_{ln}]\} \qquad (3.29)\\
(V_1)_{mn} &= [\Phi_n^m - \frac{2}{3}fr_0P_2\frac{d\Phi_n^m}{dr_0}]Y_n^m \qquad (3.30)\\
(\gamma)_{mn} &= [\frac{d\Phi_n^m}{dr_0} - 4\pi G\rho_0 u_n^m]Y_n^m \\
&\quad +\frac{2}{3}f\{P_2r_0(-\frac{d^2\Phi_n^m}{dr_0^2} + 4\pi G\rho_0\frac{du_n^m}{dr_0})Y_n^m \\
&\quad +P_2^1[(\frac{\Phi_n^m}{r_0} - 4\pi G\rho_0 v_n^m)\frac{\partial Y_n^m}{\partial\theta} - 4\pi G\rho_0 t_n^m\frac{imY_n^m}{\sin\theta}]\} \qquad (3.31)\\
\{(\hat{\mathbf{n}}\cdot\bar{\tau})_{rad}\}_{mn} &= (\lambda\Delta_n^m + 2\mu\frac{du_n^m}{dr})Y_n^m + \frac{2}{3}f[P_2r_0(-\lambda\frac{d\Delta_n^m}{dr_0} - 2\mu\frac{d^2u_n^m}{dr_0^2})Y_n^m \\
&\quad +P_2^1\mu(\Psi_n^m\frac{\partial Y_n^m}{\partial\theta} + \xi_n^m\frac{imY_n^m}{\sin\theta})] \qquad (3.32)\\
\{(\hat{\mathbf{n}}\cdot\bar{\tau})_{tr}\}_{mn} &= \frac{(-1)^m(2n+1)}{2n(n+1)}\sum_l\{\mu[\Psi_l^m(I1)_{ln} + im\xi_l^m(I2)_{ln}] \\
&\quad -\frac{2}{3}f[-\lambda\Delta_l^m(I5)_{ln} + \mu < -2\frac{u_l^m}{r_0}(I5)_{ln} - 2\frac{v_l^m}{r_0}(I7)_{ln} \\
&\quad -im\frac{t_l^m}{r_0}(I8)_{ln} + r_0\frac{d\Psi_l^m}{dr_0}(I3)_{ln} + imr_0\frac{d\xi_l^m}{dr_0}(I4)_{ln} >]\} \qquad (3.33)\\
\{(\hat{\mathbf{n}}\cdot\bar{\tau})_{to}\}_{mn} &= \frac{(-1)^m(2n+1)}{2n(n+1)}\sum_l\{\mu[-im\Psi_l^m(I2)_{ln} + \xi_l^m(I1)_{ln}] \\
&\quad -\frac{2}{3}f[-3im\lambda\Delta_l^m(I10)_{ln} + \mu < -6im\frac{u_l^m}{r_0}(I10)_{ln} + 2im\frac{v_l^m}{r_0}(I11)_{ln} \\
&\quad -\frac{t_l^m}{r_0}(I12)_{ln} - imr_0\frac{d\Psi_l^m}{dr_0}(I4)_{ln} + r_0\frac{d\xi_l^m}{dr_0}(I3)_{ln} >]\} \qquad (3.34)
\end{aligned}
$$

where all variables X_l^m or X_n^m have been transformed into functions of r_0 and $(I1)-(I12)$ are about to be defined. To simplify the above expressions, we use the identities for associated Legendre functions,

$$
\begin{aligned}
P_2Y_n^m &= A_n^mY_{n-2}^m + B_n^mY_n^m + C_n^mY_{n+2}^m \qquad (3.35)\\
P_2^1\frac{\partial Y_n^m}{\partial\theta} &= 2(n+1)A_n^mY_{n-2}^m + 3B_n^mY_n^m - 2nC_n^mY_{n+2}^m \qquad (3.36)\\
\frac{P_2^1Y_n^m}{\sin\theta} &= -3(G_n^mY_{n-1}^m + H_n^mY_{n+1}^m) \qquad (3.37)
\end{aligned}
$$

where for brevity we define

$$A_n^m = \frac{3}{2}\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} \tag{3.38}$$

$$B_n^m = \frac{n(n+1)-3m^2}{(2n+3)(2n-1)} \tag{3.39}$$

$$C_n^m = \frac{3}{2}\frac{(n+2-m)(n+1-m)}{(2n+3)(2n+1)} \tag{3.40}$$

$$G_n^m = \frac{n+m}{2n+1} \tag{3.41}$$

$$H_n^m = \frac{n+1-m}{2n+1} \tag{3.42}$$

to obtain

$$\begin{aligned}(I1)_{ln} &= \int_0^\pi [\frac{dP_l^m}{d\theta}\frac{dP_n^{-m}}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m P_n^{-m}]\sin\theta d\theta \\ &= n(n+1)\frac{2(-1)^m}{2n+1}\delta_{ln} \end{aligned} \tag{3.43}$$

$$\begin{aligned}(I2)_{ln} &= \int_0^\pi \frac{1}{\sin\theta}[P_l^m\frac{dP_n^{-m}}{d\theta} + \frac{dP_l^m}{d\theta}P_n^{-m}]\sin\theta d\theta \\ &= 0 \end{aligned} \tag{3.44}$$

$$\begin{aligned}(I3)_{ln} &= \int_0^\pi P_2[\frac{dP_l^m}{d\theta}\frac{dP_n^{-m}}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m P_n^{-m}]\sin\theta d\theta \\ &= \frac{2(-1)^m}{2n+1}\{n(n+3)A_{n+2}^m\delta_{l,n+2} + [n(n+1)-3]B_n^m\delta_{l,n} \\ &\quad +(n-2)(n+1)C_{n-2}^m\delta_{l,n-2}\} \end{aligned} \tag{3.45}$$

$$\begin{aligned}(I4)_{ln} &= \int_0^\pi \frac{P_2}{\sin\theta}\frac{d}{d\theta}(P_l^m P_n^{-m})\sin\theta d\theta \\ &= \frac{2(-1)^m}{2n+1}3[G_{n+1}^m\delta_{l,n+1} + H_{n-1}^m\delta_{l,n-1}] \end{aligned} \tag{3.46}$$

$$\begin{aligned}(I5)_{ln} &= \int_0^\pi P_2^1 P_l^m \frac{dP_n^{-m}}{d\theta}\sin\theta d\theta \\ &= -\frac{2(-1)^m}{2n+1}[2nA_{n+2}^m\delta_{l,n+2} - 3B_n^m\delta_{l,n} - 2(n+1)C_{n-2}^m\delta_{l,n-2}] \end{aligned} \tag{3.47}$$

$$\begin{aligned}(I6)_{ln} &= \int_0^\pi \frac{P_2^1}{\sin\theta}P_l^m P_n^{-m}\sin\theta d\theta \\ &= -\frac{2(-1)^m}{2n+1}3[G_{n+1}^m\delta_{l,n+1} + H_{n-1}^m\delta_{l,n-1}] \end{aligned} \tag{3.48}$$

$$
\begin{aligned}
(I7)_{ln} &= \int_0^\pi P_2^1[\frac{d^2P_l^m}{d\theta^2}\frac{dP_n^{-m}}{d\theta} + \frac{m^2}{\sin\theta}\frac{d}{d\theta}(\frac{P_l^m}{\sin\theta})P_n^{-m}]\sin\theta d\theta \\
&= \frac{2(-1)^m}{2n+1}\{2n(n+3)^2A_{n+2}^m\delta_{l,n+2} + 3[n(n+1)-3]B_n^m\delta_{l,n} \\
&\quad -2(n+1)(n-2)^2C_{n-2}^m\delta_{l,n-2}\} \qquad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
(I8)_{ln} &= \int_0^\pi P_2^1[2\frac{d}{d\theta}(\frac{P_l^m}{\sin\theta})\frac{dP_n^{-m}}{d\theta} \\
&\quad +\frac{1}{\sin\theta}(\frac{d^2P_l^m}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dP_l^m}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m)P_n^{-m}]\sin\theta d\theta \\
&= -\frac{2(-1)^m}{2n+1}3[n(n+1)-6][G_{n+1}^m\delta_{l,n+1} + H_{n-1}^m\delta_{l,n-1}] \qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
(I9)_{ln} &= \int_0^\pi \cos\theta[\frac{dP_l^m}{d\theta}\frac{dP_n^{-m}}{d\theta} + \frac{m^2}{\sin^2\theta}P_m^lP_n^{-m}]\sin\theta d\theta \\
&= \frac{2(-1)^m}{2n+1}[n(n+2)G_{n+1}^m\delta_{l,n+1} + (n-1)(n+1)H_{n-1}^m\delta_{l,n-1}] \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
(I10)_{ln} &= \int_0^\pi \cos\theta P_l^mP_n^{-m}\sin\theta d\theta \\
&= \frac{2(-1)^m}{2n+1}[G_{n+1}^m\delta_{l,n+1} + H_{n-1}^m\delta_{l,n-1}] \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
(I11)_{ln} &= \int_0^\pi P_2^1[\frac{d^2P_l^m}{d\theta^2}\frac{P_n^{-m}}{\sin\theta} + \frac{d}{d\theta}(\frac{P_l^m}{\sin\theta})\frac{dP_n^{-m}}{d\theta}]\sin\theta d\theta \\
&= \frac{2(-1)^m}{2n+1}3[(n+4)G_{n+1}^m\delta_{l,n+1} - (n-3)H_{n-1}^m\delta_{l,n-1}] \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
(I12)_{ln} &= \int_0^\pi P_2^1[(\frac{d^2P_l^m}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dP_l^m}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m)\frac{dP_n^{-m}}{d\theta} \\
&\quad +2m^2\frac{P_n^{-m}}{\sin\theta}\frac{d}{d\theta}(\frac{P_l^m}{\sin\theta})]\sin\theta d\theta \\
&= \frac{2(-1)^m}{2n+1}[2n(n+3)(n+4)A_{n+2}^m\delta_{l,n+2} + 9(n-1)(n+2)B_n^m\delta_{l,n} \\
&\quad -2(n-3)(n-2)(n+1)C_{n-2}^m\delta_{l,n-2}] \qquad (3.54)
\end{aligned}
$$

For brevity, we also introduce the following operator symbols:

$$
\begin{aligned}
K_nX_n^m &= A_{n+2}^mX_{n+2}^m + B_n^mX_n^m + C_{n-2}^mX_{n-2}^m \qquad (3.55) \\
L_nX_n^m &= 2(n+3)A_{n+2}^mX_{n+2}^m + 3B_n^mX_n^m - 2(n-2)C_{n-2}^mX_{n-2}^m \qquad (3.56) \\
M_nX_n^m &= (n+2)(n+3)A_{n+2}^mX_{n+2}^m + n(n+1)B_n^mX_n^m
\end{aligned}
$$

$$+(n-1)(n-2)C^m_{n-2}X^m_{n-2} \quad (3.57)$$

$$Q_nX^m_n = -3[G^m_{n+1}X^m_{n+1} + H^m_{n-1}X^m_{n-1}] \quad (3.58)$$

$$N_nX^m_n = 2n(n+3)^2A^m_{n+2}X^m_{n+2} + 3[n(n+1)-3]B^m_nX^m_n$$
$$-2(n+1)(n-2)^2C^m_{n-2}X^m_{n-2} \quad (3.59)$$

$$R_nX^m_n = -3[(n+4)G^m_{n+1}X^m_{n+1} - (n-3)H^m_{n-1}X^m_{n-1} \quad (3.60)$$

$$O_nX^m_n = 2n(n+3)(n+4)A^m_{n+2}X^m_{n+2} + 9(n-1)(n+2)B^m_nX^m_n$$
$$-2(n-3)(n-2)(n+1)C^m_{n-2}X^m_{n-2} \quad (3.61)$$

where X^m_n is an arbitrary variable. By appealing to the linear independence of the spherical harmonics Y^m_n, the boundary conditions reduce to the continuity of the following quantities at an ellipsoidal boundary with mean radius r_0:

$$\tilde{u}^m_n = u^m_n + \frac{2}{3}f[-r_0K_n\frac{du^m_n}{dr_0} + L_nv^m_n + imQ_nt^m_n] \quad (3.62)$$

$$\tilde{v}^m_n = v^m_n + \frac{2}{3}\frac{f}{n(n+1)}[(L_n-6K_n)u^m_n - r_0(M_n-L_n)\frac{dv^m_n}{dr_0}$$
$$+imQ_nr_0\frac{dt^m_n}{dr_0}] \quad (3.63)$$

$$\tilde{t}^m_n = t^m_n + \frac{2}{3}\frac{f}{n(n+1)}[imQ_nu^m_n - imQ_nr_0\frac{dv^m_n}{dr_0} - (M_n-L_n)r_0\frac{dt^m_n}{dr_0}] \quad (3.64)$$

$$\tilde{\Phi}^m_n = \Phi^m_n - \frac{2}{3}fr_0K_n\frac{d\Phi^m_n}{dr_0} \quad (3.65)$$

$$\tilde{\Gamma}^m_n = \frac{d\Phi^m_n}{dr_0} - 4\pi G\rho_0u^m_n + \frac{2}{3}f[L_n(\frac{\Phi^m_n}{r_0} - 4\pi G\rho_0v^m_n)$$
$$-r_0K_n(\frac{d^2\Phi^m_n}{dr_0^2} - 4\pi G\rho_0\frac{du^m_n}{dr_0}) - 4\pi G\rho_0imQ_nt^m_n] \quad (3.66)$$

$$\tilde{R}^m_n = \lambda\Delta^m_n + 2\mu\frac{du^m_n}{dr_0} + \frac{2}{3}f[\mu(L_n\Psi^m_n - 2K_nr_0\frac{d^2u^m_n}{dr_0^2})$$
$$-\lambda K_nr_0\frac{d\Delta^m_n}{dr_0} + \mu imQ_n\xi^m_n] \quad (3.67)$$

$$\tilde{\Sigma}^m_n = \mu\Psi^m_n - \frac{2}{3}\frac{f}{n(n+1)}\{(L_n-6K_n)(\lambda\Delta^m_n + 2\mu\frac{u^m_n}{r_0})$$

$$
\begin{aligned}
&+\mu[(M_n - L_n)r_0\frac{d\Psi_n^m}{dr_0} - 2N_n\frac{v_n^m}{r_0} - (n(n+1)-6)imQ_n\frac{t_n^m}{r_0} \\
&-imQ_n r_0\frac{d\xi_n^m}{dr_0}]\} \qquad (3.68)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tau}_n^m &= \mu\xi_n^m - \frac{2}{3}\frac{f}{n(n+1)}\{imQ_n(\lambda\Delta_n^m + 2\mu\frac{u_n^m}{r_0}) \\
&+\mu[(M_n - L_n)r_0\frac{d\xi_n^m}{dr_0} - O_n\frac{t_n^m}{r_0} + r_0 imQ_n\frac{d\Psi_n^m}{dr_0} - 2imR_n\frac{v_n^m}{r_0}]\} \qquad (3.69)
\end{aligned}
$$

where all variables X_n^m are functions of r_0. Note that (3.63) and (3.64) are continuous only at a solid/solid boundary.

3.2 Boundary Conditions Expressed by a Set of New Variables

In order to simplify the boundary conditions, Rochester (unpublished notes 1995) proposed using $\tilde{u}_n^m(r_0)$, $\tilde{v}_n^m(r_0)$, $\tilde{t}_n^m(r_0)$, $\tilde{\Phi}_n^m(r_0)$, $\tilde{\Gamma}_n^m(r_0)$, $\tilde{R}_n^m(r_0)$, $\tilde{\Sigma}_n^m(r_0)$ and $\tilde{\tau}_n^m(r_0)$, whose definitions are in (3.62)-(3.69), as the dependent variables in Clairaut coordinates.

With this set of variables the boundary conditions are written in a much simpler form, i.e. to require that every new variable, i.e. $\tilde{u}_n^m$, $\tilde{v}_n^m$, $\tilde{t}_n^m$, $\tilde{\Phi}_n^m$, $\tilde{\Gamma}_n^m$, $\tilde{R}_n^m$, $\tilde{\Sigma}_n^m$, $\tilde{\tau}_n^m$, be continuous at an ellipsoidal boundary, e.g. at r_0,

$$\tilde{u}_n^m(r_0^-) = \tilde{u}_n^m(r_0^+) \qquad (3.70)$$

$$\tilde{v}_n^m(r_0^-) = \tilde{v}_n^m(r_0^+) \qquad (3.71)$$

$$\tilde{t}_n^m(r_0^-) = \tilde{t}_n^m(r_0^+) \qquad (3.72)$$

$$\tilde{\Phi}_n^m(r_0^-) = \tilde{\Phi}_n^m(r_0^+) \qquad (3.73)$$

$$\tilde{\Gamma}_n^m(r_0^-) = \tilde{\Gamma}_n^m(r_0^+) \qquad (3.74)$$

$$\tilde{R}_n^m(r_0^-) = \tilde{R}_n^m(r_0^+) \quad (3.75)$$

$$\tilde{\Sigma}_n^m(r_0^-) = \tilde{\Sigma}_n^m(r_0^+) \quad (3.76)$$

$$\tilde{\tau}_n^m(r_0^-) = \tilde{\tau}_n^m(r_0^+) \quad (3.77)$$

where of course (3.71) and (3.72) are invoked only at a solid/solid boundary.

In the next two sections we specify the continuity conditions at special boundaries. One is at the boundary between solid and liquid because rigidity μ is zero in liquid. Another is at the surface of the Earth where there is no stress at all.

3.2.1 Boundary Conditions at an Ellipsoidal Solid/Liquid Boundary

At a boundary between liquid and solid, only the normal displacement component $\hat{\mathbf{n}} \cdot \mathbf{u}$ is continuous and the components of tangential displacement $\tilde{v}_n^m$ and $\tilde{t}_n^m$ are not continuous. With $\mu = 0$, the transverse spheroidal and toroidal components of normal stress $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$ on the liquid side will be dependent on its radial part $\tilde{R}_n^m$. So, the boundary conditions will be,

$$\tilde{u}^m_{n\,solid} = \tilde{u}^m_{n\,liquid} \quad (3.78)$$

$$\tilde{\Phi}^m_{n\,solid} = \tilde{\Phi}^m_{n\,liquid} \quad (3.79)$$

$$\tilde{\Gamma}^m_{n\,solid} = \tilde{\Gamma}^m_{n\,solid} \quad (3.80)$$

$$\begin{aligned} \tilde{R}^m_{n\,solid} &= \tilde{R}^m_{n\,liquid} \\ &= \{\lambda\Delta_n^m - \frac{2}{3}f\lambda K_n r_0 \frac{d\Delta_n^m}{dr_0}\}_{liquid} \end{aligned} \quad (3.81)$$

$$\begin{aligned} \tilde{\Sigma}^m_{n\,solid} &= \tilde{\Sigma}^m_{n\,liquid} \\ &= -\frac{2}{3}\frac{f}{n(n+1)}(L_n - 6K_n)\{\tilde{R}_n^m\}_{liquid} \end{aligned} \quad (3.82)$$

$$
\begin{aligned}
\tilde{\tau}^m_{n\,solid} &= \tilde{\tau}^m_{n\,liquid} \\
&= -\frac{2}{3}\frac{f}{n(n+1)} imQ_n\{\tilde{R}^m_n\}_{liquid} \qquad (3.83)
\end{aligned}
$$

which have been reduced to six from eight for a solid/solid boundary.

3.2.2 Boundary Conditions at the Earth's Free Surface

At the surface of the Earth, the stress components vanish i.e. $\tilde{R}^m_n, \tilde{\Sigma}^m_n, \tilde{\tau}^m_n$ are all zero at $r_0 = R$, where R is the mean radius of the Earth.

$$
\tilde{R}^m_n = \tilde{\Sigma}^m_n = \tilde{\tau}^m_n = 0 \qquad (3.84)
$$

Since there is no restriction on the movement of the Earth's surface, the displacement components $\tilde{u}^m_n$, $\tilde{v}^m_n$ and $\tilde{t}^m_n$ are not continuous at $r_0 = R$.

The boundary conditions related to the gravitational potential require the continuity of $\tilde{\Phi}^m_n$ and $\tilde{\Gamma}^m_n$ across the Earth's surface. Beyond the surface of the Earth, the Poisson equation is:

$$
\nabla^2 V_1 = 0 \qquad (3.85)
$$

From the expansion of V_1 in spherical harmonics, we find that to satisfy (3.85) requires

$$
\Phi^m_n(r_0) \propto \frac{1}{r_0^{n+1}} \qquad (3.86)
$$

for $r_0 > R$. Using (3.65)-(3.66), and eliminating fields outside the Earth by (3.86), the continuity of $\tilde{\Phi}^m_n$ and $\tilde{\Gamma}^m_n$ together reduce to the following condition at the Earth's surface:

$$
(\tilde{\Gamma}^m_n + \frac{n+1}{R}\tilde{\Phi}^m_n)_{R-} = (\tilde{\Gamma}^m_n + \frac{n+1}{R}\tilde{\Phi}^m_n)_{R+} \qquad (3.87)
$$

$$= -\frac{2}{3}\frac{f}{R}[(n+3)A^m_{n+2}\tilde{\Phi}^m_{n+2} + (n-2)B^m_n\tilde{\Phi}^m_n$$
$$+(n-3)C^m_{n-2}\tilde{\Phi}^m_{n-2}]_{R^-} \quad (3.88)$$

where R^+ and R^- respectively represent just outside and inside the Earth's surface. This is the generalization of the BC at $r = R$ for a non-rotating spherical Earth model, namely

$$(\frac{d\Phi^m_n}{dr} - 4\pi G\rho_0 u^m_n) + \frac{n+1}{R}\Phi^m_n = 0 \quad (3.89)$$

3.3 Discrepancy Between Smith's Work and This Thesis, re Boundary Conditions

Since the mapping (2.1)-(2.2) used at boundaries by Smith (1974) is the same as the transformation (2.10) used everywhere in the Earth in this thesis, the boundary conditions (3.62)-(3.69) should be identical with Smith's. However, some differences are found between the boundary conditions involving the normal stress components (Rochester unpublished notes 1997).

When Smith (1974) transformed the normal stress [Smith's (2.22)] from the elliptical domain V_E into the ESD, he also made the transformations on the material properties. So, Smith's expressions of continuous normal stress components [Smith's (5.43), (5.44), (5.45)] include extra terms involving the derivatives of material properties of λ and μ. The extra terms are

$$\tfrac{2}{3}f \quad K_n r_0[\frac{d\lambda}{dr_0}\Delta^m_n + 2\frac{d\mu}{dr_0}\frac{du^m_n}{dr_0}]$$
$$\tfrac{2}{3}f \quad \frac{d\mu}{dr_0}r_0[(M_n - L_n)\Psi^m_n - imQ_n\xi^m_n]$$
$$\tfrac{2}{3}f \quad \frac{d\mu}{dr_0}r_0[(M_n - L_n)\xi^m_n + imQ_n\Psi^m_n] \quad (3.90)$$

in equations (3.64), (3.65) and (3.66) respectively. The reason for the extra terms is because, when transforming normal stress. Smith did not properly adapt the material properties listed in a spherical Earth model to an oblate Earth. In effect he transformed the elastic moduli by

$$\lambda(\mathbf{r}) = \lambda(r_0) - \frac{2}{3} f r_0 \frac{\partial \lambda}{\partial r_0} P_2 \tag{3.91}$$

$$\mu(\mathbf{r}) = \mu(r_0) - \frac{2}{3} f r_0 \frac{\partial \mu}{\partial r_0} P_2 \tag{3.92}$$

which are not correct because the material properties in the elliptical domain V_E as functions of r_0 should not be changed when being transformed into the ESD. The correct transformations are:

$$\lambda(\mathbf{r}) = \lambda(r_0) \tag{3.93}$$

$$\mu(\mathbf{r}) = \mu(r_0) \tag{3.94}$$

consistent with (2.7), i.e. the material properties at any point $\mathbf{r}$ in V_E are the same as the tabulated ones at radius r_0 in the ESD, where r_0 is the mean radius of the equipotential (3.1).

With incorrect mapping of the material properties, Smith (1974) arrived at incorrect boundary conditions for the normal stress components. The consequence of these errors for his numerical results (Smith 1977) are still unknown.

Chapter 4

Governing Equations with New Variables

In the previous chapter, we got the boundary conditions for a rotating, ellipsoidally-layered Earth in Clairaut coordinates. Based on the appearance of boundary conditions, a set of new field variables was introduced to automatically satisfy continuity conditions at surfaces where any material property is discontinuous. In this chapter, we will establish the governing equations for wobble/nutation modes for this Earth model in Clairaut coordinates and then reduce them to a set of coupled ODEs with the Clairaut coordinate r_0 as independent variable and the new field variables as dependent.

The free oscillations of the Earth are governed by four basic physical laws: conservation of mass, linear momentum, entropy and gravitational flux. On the basis of these laws, we find free oscillations to be governed by the momentum conservation equation (MCE) and the Poisson equation:

$$\nabla \cdot \bar{\tau} = -\rho_0 \omega^2 \mathbf{u} + 2i\rho_0 \omega \Omega \mathbf{k} \times \mathbf{u} - \rho_0 \nabla V_1 - \rho_0 \nabla(\mathbf{u} \cdot \mathbf{g}_0) + \rho_0 \mathbf{g}_0 \nabla \cdot \mathbf{u} \quad (4.1)$$

$$\nabla^2 V_1 = 4\pi G \nabla \cdot (\rho_0 \mathbf{u}) \quad (4.2)$$

where $\mathbf{k}$ is the unit vector of the Earth's figure axis, ω and Ω are respectively the angular frequency of free oscillation and the steady angular frequency of rotation of the reference frame about $\mathbf{k}$ (Wu & Rochester 1990).

In this chapter, to establish the governing equations from (4.1) and (4.2) we first write (4.1) in its radial, transverse spheroidal and toroidal components and expand the partial differential equations resulting from (4.1) and (4.2) in spherical harmonics Y_n^m. The expansions obtained from the above procedures will be in the elliptical domain V_E as functions of (r, θ, ϕ). The next step is to transform functions of r into those in r_0 by the mapping introduced in chapter 2. Using the recurrence relations and linear independence of Y_n^m, we finally obtain a set of ODEs in r_0.

4.1 Gravity in a Rotating, Ellipsoidally-Layered Hydrostatic Earth Model

Before entering into the next section to derive governing equations, we take this section to express the gravity for a rotating, ellipsoidal-layered Earth model.

Gravity $g_0(r)$ for a non-rotating Earth model is governed by the Poisson equation,

$$\nabla \cdot (\hat{\mathbf{r}} g_0(r)) = 4\pi G \rho_0(r) \tag{4.3}$$

where $g_0(r)$ is a function of r because of the purely radial distribution of density $\rho_0(r)$. Gravity for a rotating, oblate Earth model needs to be modified from the above situation because rotation and ellipticity will make gravity $\mathbf{g}_0(\mathbf{r})$ dependent on both r and θ.

To derive the expression of gravity properly, we need to make it satisfy both the

Poisson equation and the requirement that the gravity equipotentials be surfaces of constant density. First of all, we assume the gravity $\mathbf{g}_0(\mathbf{r})$, a downward vector, has this form.

$$-\mathbf{g}_0(\mathbf{r}) = [g_0(r) - \frac{2}{3}\Omega^2 r]\hat{\mathbf{r}} + \nabla\Psi \tag{4.4}$$

where $g_0(r)$ is the gravitational field for a spherical Earth, Ω is the angular frequency of rotation of the reference frame, and Ψ is to be determined. When we put this expression into the Poisson equation,

$$\nabla \cdot \mathbf{g}_0(\mathbf{r}) = -4\pi G\rho_0(\mathbf{r}) - 2\Omega^2 \tag{4.5}$$

we find that

$$\begin{aligned} \nabla^2\Psi &= 4\pi G[\rho_0(\mathbf{r}) - \rho_0(r)] \\ &= \frac{8\pi G}{3} r f \frac{d\rho_0}{dr} P_2 \end{aligned} \tag{4.6}$$

using (2.6). Obviously, Ψ is the same order as the ellipticity f.

To make sure that the gravitational equipotential is a surface of constant density we require

$$\nabla\rho_0(\mathbf{r}) \times \mathbf{g}_0(\mathbf{r}) = 0 \tag{4.7}$$

Substituting (4.4) into (4.7), we obtain

$$\frac{\partial\Psi}{\partial\theta} = \frac{2}{3} f r [g_0(r) - \frac{2}{3}\Omega^2 r] \frac{dP_2}{d\theta} \tag{4.8}$$

so

$$\Psi = \frac{2}{3} r f [g_0(r) - \frac{2}{3}\Omega^2 r] P_2 \tag{4.9}$$

Substitute (4.9) into (4.6). and use the relationship between spherical-Earth gravity $g_0(r)$ and average density $\bar{\rho}_0$ inside a sphere of radius r,

$$g_0 = \frac{4}{3}\pi G\bar{\rho}_0 r \tag{4.10}$$

$$\bar{\rho}_0 = \frac{3}{r^3}\int_0^r \rho_0(x)x^2dx \tag{4.11}$$

then we have this equation:

$$\frac{d^2f}{dr^2} + \frac{6}{r}\frac{df}{dr}[\frac{\rho_0-\sigma}{\bar{\rho}_0-\sigma}] - \frac{6f}{r^2}[\frac{\bar{\rho}_0-\rho_0}{\bar{\rho}_0-\sigma}] = 0 \tag{4.12}$$

where $\sigma = \frac{\Omega^2}{2\pi G}$. In fact this is the Clairaut equation, for if we replace $(\rho_0 - \sigma)$ and $(\bar{\rho}_0 - \sigma)$ by ρ_0 and $\bar{\rho}_0$ respectively, then (4.12) reduces to

$$\frac{d^2f}{dr^2} + \frac{6}{r}\frac{df}{dr}\frac{\rho_0}{\bar{\rho}_0} + \frac{6f}{r^2}[\frac{\rho_0}{\bar{\rho}_0} - 1] = 0 \tag{4.13}$$

This substitution has very small effects on the ellipticity f because $\frac{\sigma}{\bar{\rho}_0} \leq \frac{\sigma}{\rho_0} \leq \frac{1}{250}$.

Therefore, we have the expression for gravity in a rotating, ellipsoidally-layered Earth model as,

$$-\mathbf{g}_0(\mathbf{r}) = [g_0(r) - \frac{2}{3}\Omega^2 r]\hat{\mathbf{r}} + \frac{2}{3}\nabla[rf(g_0(r) - \frac{2}{3}\Omega^2 r)P_2] \tag{4.14}$$

$$= \hat{\mathbf{r}}\{(g_0(r) - \frac{2}{3}\Omega^2 r) + \frac{2}{3}\frac{d}{dr}[rf(g_0(r) - \frac{2}{3}\Omega^2 r)]P_2\} + \hat{\theta}[\frac{2}{3}rf(g_0(r) - \frac{2}{3}\Omega^2 r)P_2^1] \tag{4.15}$$

This derivation is a slight improvement on that given by Seyed-Mahmoud (1994).

4.2 Ordinary Differential Equations from the Momentum Conservation Equation, in Clairaut Coordinates

In this section, we will derive the ordinary differential equations from the MCE (4.1) in Clairaut coordinates by the steps we described above. The first step is to form the r, θ and ϕ components of MCE (4.1). To write the LHS of (4.1) we need Hooke's law for the relationship between the stress $\tilde{\tau}$ and strain $\tilde{\epsilon}$:

$$\tilde{\tau} = \lambda\Delta\tilde{\mathbf{I}} + 2\mu\tilde{\epsilon} \tag{4.16}$$

where $\tilde{\mathbf{I}}$ is the unit dyadic, $\Delta = \nabla \cdot \mathbf{u}$, and λ and μ are the Lamé parameters. So, the left hand side (LHS) of the MCE is:

$$\nabla \cdot \tilde{\tau} = \lambda\nabla\Delta + \nabla\lambda\Delta + 2\mu\nabla \cdot \tilde{\epsilon} + 2\nabla\mu \cdot \tilde{\epsilon} \tag{4.17}$$

where λ and μ are functions of r_0 only. Taking the inner product with unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$, we have r, θ and ϕ components of $\nabla \cdot \tilde{\tau}$,

$$\begin{aligned} \hat{\mathbf{r}} \cdot \nabla \cdot \tilde{\tau} &= \lambda\frac{\partial\Delta}{\partial r} + \frac{d\lambda}{dr_0}\nabla r_0 \cdot \hat{\mathbf{r}}\Delta \\ &\quad +2\mu(\nabla \cdot \tilde{\epsilon}) \cdot \hat{\mathbf{r}} + 2\frac{d\mu}{dr_0}\nabla r_0 \cdot \tilde{\epsilon} \cdot \hat{\mathbf{r}} \end{aligned} \tag{4.18}$$

$$\begin{aligned} \hat{\theta} \cdot \nabla \cdot \tilde{\tau} &= \frac{\lambda}{r}\frac{\partial\Delta}{\partial\theta} + \frac{2}{3}fP_2^1\frac{d\lambda}{dr_0}\Delta \\ &\quad +2\mu(\nabla \cdot \tilde{\epsilon}) \cdot \hat{\theta} + 2\frac{d\mu}{dr_0}\nabla r_0 \cdot \tilde{\epsilon} \cdot \hat{\theta} \end{aligned} \tag{4.19}$$

$$\hat{\phi} \cdot \nabla \cdot \tilde{\tau} = \frac{\lambda}{r\sin\theta}\frac{\partial\Delta}{\partial\phi} + 2\mu(\nabla \cdot \tilde{\epsilon}) \cdot \hat{\phi} + 2\frac{d\mu}{dr_0}\nabla r_0 \cdot \tilde{\epsilon} \cdot \hat{\phi} \tag{4.20}$$

Here

$$\nabla r_0 = \hat{\mathbf{r}}[1 + \frac{2}{3}\frac{d}{dr_0}(r_0 f)P_2] + \frac{2}{3}fP_2^1\hat{\theta} \tag{4.21}$$

$$
\begin{aligned}
(\nabla\cdot\tilde{\epsilon})\cdot\hat{\mathbf{r}} &= \nabla\cdot(\tilde{\epsilon}\cdot\hat{\mathbf{r}}) + \frac{\epsilon_{rr}-\Delta}{r} \\
&= \nabla\cdot(\hat{\mathbf{r}}\epsilon_{rr}+\hat{\theta}\epsilon_{r\theta}+\hat{\phi}\epsilon_{r\phi}) + \frac{\epsilon_{rr}-\Delta}{r} \qquad (4.22)\\
(\nabla\cdot\tilde{\epsilon})\cdot\hat{\theta} &= \frac{\partial\epsilon_{r\theta}}{\partial r} + \frac{3\epsilon_{r\theta}}{r} + \frac{1}{r}\frac{\partial\epsilon_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\epsilon_{\theta\phi}}{\partial\phi} + \frac{\cot\theta}{r}(\epsilon_{\theta\theta}-\epsilon_{\phi\phi}) \qquad (4.23)\\
(\nabla\cdot\tilde{\epsilon})\cdot\hat{\phi} &= \frac{\partial\epsilon_{r\phi}}{\partial r} + \frac{3\epsilon_{r\phi}}{r} + \frac{1}{r}\frac{\partial\epsilon_{\theta\phi}}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial\epsilon_{\phi\phi}}{\partial\phi} + \frac{2\cot\theta}{r}\epsilon_{\theta\phi} \qquad (4.24)
\end{aligned}
$$

where ϵ_{ij} $(i,j=r,\theta,\phi)$ are strain components in spherical polar coordinates.

The $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\phi}$ components of the RHS of the MCE can be written by inner products with these unit vectors. Its $\hat{\mathbf{r}}$ component is:

$$
\begin{aligned}
\rho_0\{-\omega^2(\hat{\mathbf{r}}\cdot\mathbf{u}) &- 2i\omega\Omega\sin\theta(\hat{\phi}\cdot\mathbf{u}) - \frac{\partial V_1}{\partial r} - [\bar{g_0}+\frac{2}{3}\frac{d}{dr}(rf\bar{g_0})P_2]\Delta \\
&+ \frac{\partial}{\partial r}[<\bar{g_0}+\frac{2}{3}\frac{d}{dr}(rf\bar{g_0})P_2>(\hat{\mathbf{r}}\cdot\mathbf{u}) + \frac{2}{3}f\bar{g_0}(\hat{\theta}\cdot\mathbf{u})P_2^1]\} \qquad (4.25)
\end{aligned}
$$

Its $\hat{\theta}$ component is:

$$
\begin{aligned}
\rho_0\{-\omega^2(\hat{\theta}\cdot\mathbf{u}) &- 2i\omega\Omega\cos\theta(\hat{\phi}\cdot\mathbf{u}) - \frac{1}{r}\frac{\partial V_1}{\partial\theta} - \frac{2}{3}f\bar{g_0}P_2^1\Delta \\
&+ \frac{1}{r}\frac{\partial}{\partial\theta}[<\bar{g_0}+\frac{2}{3}\frac{d}{dr}(rf\bar{g_0})P_2>(\hat{\mathbf{r}}\cdot\mathbf{u}) + \frac{2}{3}fP_2^1\bar{g_0}(\hat{\theta}\cdot\mathbf{u})]\} \qquad (4.26)
\end{aligned}
$$

Its $\hat{\phi}$ component is:

$$
\begin{aligned}
\rho_0\{-\omega^2(\hat{\phi}\cdot\mathbf{u}) &+ 2i\omega\Omega(\cos\theta\hat{\theta}\cdot\mathbf{u}+\sin\theta\hat{\mathbf{r}}\cdot\mathbf{u}) - \frac{1}{r\sin\theta}\frac{\partial V_1}{\partial\phi} \\
&+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[<\bar{g_0}+\frac{2}{3}\frac{d}{dr}(rf\bar{g_0})P_2>\hat{\mathbf{r}}\cdot\mathbf{u} + \frac{2}{3}f\bar{g_0}P_2^1\hat{\theta}\cdot\mathbf{u}]\} \qquad (4.27)
\end{aligned}
$$

where

$$\bar{g_0} = (g_0 - \frac{2}{3}\Omega^2 r_0) \qquad (4.28)$$

Therefore, the r, θ and ϕ components of the MCE, written as $(MCE)_{rad}$, $(MCE)_\theta$ and $(MCE)_\phi$, obtained from the above expressions are,

$$(MCE)_{rad}:\ (4.25)-(4.18)\ = 0 \qquad (4.29)$$

$$(MCE)_\theta: \quad (4.26) - (4.19) \quad = 0 \tag{4.30}$$

$$(MCE)_\phi: \quad (4.27) - (4.20) \quad = 0 \tag{4.31}$$

The next step is to expand all variables appearing in $(MCE)_{rad}$, $(MCE)_\theta$ and $(MCE)_\phi$ using spherical harmonics. The expansions of displacement and some strain components are,

$$\begin{aligned}
\mathbf{u} &= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[\hat{\mathbf{r}}u_n^m Y_n^m + \hat{\theta}(v_n^m \frac{\partial Y_n^m}{\partial\theta} + t_n^m \frac{1}{\sin\theta}\frac{\partial Y_n^m}{\partial\phi}) \\
&\quad + \hat{\phi}(v_n^m \frac{1}{\sin\theta}\frac{\partial Y_n^m}{\partial\phi} - t_n^m \frac{\partial Y_n^m}{\partial\theta})] && (4.32)\\
\epsilon_{rr} &= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\frac{du_n^m}{dr}Y_n^m && (4.33)\\
\epsilon_{r\theta} &= \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}(\Psi_n^m\frac{\partial Y_n^m}{\partial\theta} + \xi_n^m\frac{imY_n^m}{\sin\theta}) && (4.34)\\
\epsilon_{r\phi} &= \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}(\Psi_n^m\frac{imY_n^m}{\sin\theta} - \xi_n^m\frac{\partial Y_n^m}{\partial\theta}) && (4.35)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_n^m(r) &= \frac{dv_n^m}{dr} + \frac{v_n^m - u_n^m}{r} && (4.36)\\
\xi_n^m(r) &= \frac{dt_n^m}{dr} - \frac{t_n^m}{r} && (4.37)\\
\Delta_n^m(r) &= \frac{du_n^m}{dr} + \frac{2u_n^m - n(n+1)v_n^m}{r} && (4.38)
\end{aligned}$$

The expansions of other strain components can be formed from these equations,

$$\begin{aligned}
\epsilon_{\theta\theta} &= \frac{(\hat{\mathbf{r}}\cdot\mathbf{u})}{r} + \frac{1}{r}\frac{\partial(\hat{\theta}\cdot\mathbf{u})}{\partial\theta} && (4.39)\\
\epsilon_{\theta\phi} &= \frac{1}{r}(\frac{\partial}{\partial\theta} - \cot\theta)(\hat{\phi}\cdot\mathbf{u}) + \frac{1}{r\sin\theta}\frac{\partial(\hat{\theta}\cdot\mathbf{u})}{\partial\phi} && (4.40)\\
\epsilon_{\phi\phi} &= \frac{(\hat{\mathbf{r}}\cdot\mathbf{u})}{r} + \frac{(\hat{\theta}\cdot\mathbf{u})}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial(\hat{\phi}\cdot\mathbf{u})}{\partial\phi} && (4.41)
\end{aligned}$$

By substituting all above expansions into equation (4.29), (4.30) and (4.31), we will obtain the expansions of the MCE in spherical harmonics in the form

$$(MCE)_{rad}(r,\theta,\phi) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[(MCE)_{rad}]_n^m \tag{4.42}$$

$$(MCE)_{\theta}(r,\theta,\phi) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[(MCE)_{\theta}]_n^m \tag{4.43}$$

$$(MCE)_{\phi}(r,\theta,\phi) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}[(MCE)_{\phi}]_n^m \tag{4.44}$$

Instead of working with the $\{(MCE)_\theta\}_n^m$ and $\{(MCE)_\phi\}_n^m$, we will use the equivalent transverse spheroidal and toroidal components (written as $\{(MCE)_{tr}\}_n^m$ and $\{(MCE)_{to}\}_n^m$).

$$\begin{aligned}\{(MCE)_{tr}\}_n^m &= \frac{(-1)^m(2n+1)}{4\pi n(n+1)}\sum_l\int_0^{2\pi}\int_0^{\pi}[\{(MCE)_\theta\}_l^m\frac{\partial Y_n^{-m}}{\partial\theta} \\ &\quad -\{(MCE)_\phi\}_l^m\frac{im}{\sin\theta}Y_n^{-m}]\sin\theta d\theta d\phi \qquad (4.45)\\ \{(MCE)_{to}\}_n^m &= \frac{(-1)^m(2n+1)}{4\pi n(n+1)}\sum_l\int_0^{2\pi}\int_0^{\pi}[-\{(MCE)_\theta\}_l^m\frac{im}{\sin\theta}Y_n^{-m} \\ &\quad -\{(MCE)_\phi\}_l^m\frac{\partial Y_n^{-m}}{\partial\theta}]\sin\theta d\theta d\phi \qquad (4.46)\end{aligned}$$

The expansions $\{(MCE)_{rad}\}_n^m$, $\{(MCE)_{tr}\}_n^m$ and $\{(MCE)_{to}\}_n^m$, however, are in spherical polar coordinates. We will transform all involved variables into Clairaut coordinates by (2.10).

The resulting expressions from above derivations include lots of Legendre function P_n^m and trigonometric functions, which will be simplified with the following identities for P_n^m, (besides those in Chapter 2)

$$[\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}+n(n+1)-\frac{m^2}{\sin^2\theta}]P_n^m = 0 \tag{4.47}$$

$$\sum_l X_l^m \int_0^\pi \cos\theta[\frac{d}{d\theta}(\sin\theta\frac{dP_l^m}{d\theta}) - \frac{m^2}{\sin\theta}P_l^m]P_2P_n^{-m}d\theta = \frac{2(-1)^m}{2n+1}\frac{1}{3}Q_nM_nX_n^m \quad (4.48)$$

$$\sum_l X_l^m \int_0^\pi \sin^2\theta\frac{dP_l^m}{d\theta}P_n^{-m}d\theta = \frac{2(-1)^m}{2n+1}[-(n+2)G_{n+1}^mX_{n+1}^m + (n-1)H_{n-1}^mX_{n-1}^m] \quad (4.49)$$

$$\sum_l X_l^m \int_0^\pi \sin^2\theta\frac{dP_l^m}{d\theta}P_2P_n^{-m}d\theta = \frac{2(-1)^m}{2n+1}K_n[-(n+2)G_{n+1}^mX_{n+1}^m + (n-1)H_{n-1}^mX_{n-1}^m] \quad (4.50)$$

$$\sum_l X_l^m \int_0^\pi \sin\theta\cos\theta P_l^mP_2P_n^{-m}d\theta = \frac{1}{3}\frac{2(-1)^m}{2n+1}Q_nK_nX_n^m \quad (4.51)$$

$$\sum_l X_l^m \int_0^\pi \sin\theta\cos\theta[\frac{dP_l^m}{d\theta}\frac{dP_n^{-m}}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^mP_n^{-m}]d\theta = \frac{2(-1)^m}{2n+1}[n(n+2)G_{n+1}^mX_{n+1}^m + (n-1)(n+1)G_{n-1}^mX_{n-1}^m] \quad (4.52)$$

$$\sum_l X_l^m \int_0^\pi \sin^2\theta P_l^m\frac{dP_n^{-m}}{d\theta}d\theta = \frac{2(-1)^m}{2n+1}[nG_{n+1}^mX_{n+1}^m - (n+1)G_{n-1}^mX_{n-1}^m] \quad (4.53)$$

As we did in Chapter 3, we also introduce more operator symbols:

$$Q_n^{(1)}X_n^m = (n+2)G_{n+1}^mX_{n+1}^m - (n-1)H_{n-1}^mX_{n-1}^m \quad (4.54)$$

$$K_nQ_n^{(1)}X_n^m = (n+4)A_{n+2}^mG_{n+3}^mX_{n+3}^m + [(n+2)B_n^mG_{n+2}^m - (n+1)A_{n+2}^mH_{n+1}^m]X_{n+1}^m + [nC_{n-2}^mG_{n-1}^m - (n-1)B_n^mH_{n-1}^m]X_{n-1}^m - (n-3)C_{n-2}^mH_{n-3}^mX_{n-3}^m \quad (4.55)$$

$$Q_n^{(2)}X_n^m = n(n+2)G_{n+1}^mX_{n+1}^m + (n-1)(n+1)H_{n-1}^mX_{n-1}^m \quad (4.56)$$

$$Q_n^{(3)}X_n^m = -nG_{n+1}^mX_{n+1}^m + (n+1)H_{n-1}^mX_{n-1}^m \quad (4.57)$$

$$W_n = -\frac{1}{3}Q_nM_n - Q_n^{(1)} - 3K_nQ_n^{(1)} \quad (4.58)$$

$$V_n = \frac{4}{3}Q_nK_n - \frac{2}{3}Q_n + K_nQ_n^{(1)} \quad (4.59)$$

After applying the above identities, operator symbols and linear independence of the spherical harmonics Y_n^m, we obtain ODEs from the radial, transverse spheroidal and toroidal components of the MCE. The ODEs from the radial component of the MCE are:

$$
\begin{aligned}
&\frac{d}{dr_0}(\lambda\Delta_n^m + 2\mu\frac{du_n^m}{dr_0}) + \frac{2\mu}{r_0}(3\frac{du_n^m}{dr_0} - \Delta_n^m) - \frac{n(n+1)}{r_0}\mu\Psi_n^m + \rho_0(\frac{d\Phi_n^m}{dr_0} - 4\pi G\rho_0 u_n^m) \\
&+(\rho_0\omega^2 + \frac{4\rho_0 g_0}{r_0} - \frac{2}{3}\Omega^2\rho_0)u_n^m - [\rho_0(g_0 - \frac{2}{3}\Omega^2 r_0)\frac{n(n+1)}{r_0} + 2\omega\Omega\rho_0 m]v_n^m + 2\omega\Omega\rho_0 Q_n^{(1)} it_n^m \\
&-\tfrac{2}{3}f \quad \{r_0 K_n[\frac{d}{dr_0}(\lambda\frac{d\Delta_n^m}{dr_0} + 2\mu\frac{d^2u_n^m}{dr_0^2}) + \frac{2\mu}{r_0} < 3(\frac{d^2u_n^m}{dr_0^2} - \frac{1}{r_0}\frac{du_n^m}{dr_0}) - (\frac{d\Delta_n^m}{dr_0} - \frac{\Delta_n^m}{r_0}) >] \\
&\qquad -\frac{d\mu}{dr_0}(L_n\Psi_n^m + imQ_n\xi_n^m) - \mu M_n(\frac{d\Psi_n^m}{dr_0} - \frac{\Psi_n^m}{r_0}) + \rho_0 r_0 K_n\frac{d^2\Phi_n^m}{dr_0^2} \\
&\qquad +\rho_0\omega^2 K_n r_0\frac{du_n^m}{dr_0} - 2\omega\Omega\rho_0 r_0(mK_n\frac{dv_n^m}{dr_0} + K_n Q_n^{(1)}\frac{dit_n^m}{dr_0}) + \rho_0(g_0 - \frac{2}{3}\Omega^2 r_0)\cdot \\
&\qquad [K_n < r_0(\frac{d\Delta_n^m}{dr_0} - \frac{d^2u_n^m}{dr_0^2}) - \Delta_n^m + \frac{du_n^m}{dr_0} + \frac{2u_n^m}{r_0} > + L_n\frac{dv_n^m}{dr_0} + imQ_n\frac{dt_n^m}{dr_0}] \\
&\qquad -(4\pi G\rho_0 - \frac{2g_0}{r_0} - \frac{2}{3}\Omega^2)\rho_0[K_n r_0\frac{du_n^m}{dr_0} - L_n v_n^m - imQ_n t_n^m]\} \\
&+\tfrac{2}{3} \quad \{\frac{d(r_0 f)}{dr_0}K_n(\frac{d\lambda}{dr_0}\Delta_n^m + 2\frac{d\mu}{dr_0}\frac{du_n^m}{dr_0}) \\
&\qquad +r_0\frac{df}{dr_0}\rho_0(g_0 - \frac{2}{3}\Omega^2 r_0)[K_n(\Delta_n^m - \frac{du_n^m}{dr_0} + \frac{2u_n^m}{dr_0}) - L_n v_n^m - imQ_n t_n^m]\} \\
&= \quad 0 \qquad\qquad (4.60)
\end{aligned}
$$

The ODEs from the transverse spheroidal component of the MCE are:

$$
\begin{aligned}
&n(n+1)\{\frac{d}{dr_0}(\mu\Psi_n^m) + \frac{3\mu\Psi_n^m}{r_0} + \lambda\frac{\Delta_n^m}{r_0} + \frac{2\mu}{r_0^2}[u_n^m - < n(n+1) - 1 > v_n^m] \\
&+\rho_0[\omega^2 v_n^m + \frac{\Phi_n^m}{r_0} - (g_0 - \frac{2}{3}\Omega^2 r_0)\frac{u_n^m}{r_0} - \frac{2\omega\Omega}{n(n+1)} < m(u_n^m + v_n^m) + Q_n^{(2)} it_n^m >]\} \\
&+\tfrac{2}{3}f \quad \{(M_n - L_n)[\lambda(-\frac{d\Delta_n^m}{dr_0} + \frac{\Delta_n^m}{r_0}) - r_0\frac{d}{dr_0}(\mu\frac{d\Psi_n^m}{dr_0}) + \frac{d\mu}{dr_0}\Psi_n^m - 3\mu(\frac{d\Psi_n^m}{dr_0} - \frac{\Psi_n^m}{r_0})] \\
&\qquad +\frac{2\mu}{r_0} < (-\frac{du_n^m}{dr_0} + 2\frac{u_n^m}{r_0}) + (n(n+1) - 1)(\frac{dv_n^m}{dr_0} - 2\frac{v_n^m}{r}) >] \\
&\qquad +(6K_n - L_n)[\frac{d\lambda}{dr_0}\Delta_n^m + \frac{d\mu}{dr_0}\frac{2u_n^m}{r_0}] + 2\frac{d\mu}{dr_0}N_n\frac{v_n^m}{r_0} \\
&\qquad +imQ_n[r_0\frac{d}{dr_0}(\mu\frac{d\xi_n^m}{dr_0}) + \mu < 3\frac{d\xi_n^m}{dr_0} - 3\frac{\xi_n^m}{r_0} - \frac{(n(n+1) - 2)}{r_0}(\xi_n^m - \frac{t_n^m}{r_0}) > \\
&\qquad -\frac{d\mu}{dr_0}\xi_n^m] + [n(n+1) - 6]\frac{d\mu}{dr_0}imQ_n\frac{t_n^m}{r_0}\}
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{2}{3} f\rho_0 \quad & \{\omega^2 r_0[(M_n - L_n)\frac{dv_n^m}{dr_0} - imQ_n\frac{dt_n^m}{dr_0}] + (M_n - L_n)(\frac{d\Phi_n^m}{dr_0} - \frac{\Phi_n^m}{r_0}) \\
& -2\omega\Omega r_0[m < \frac{dv_n^m}{dr_0} + K_n(\frac{du_n^m}{dr_0} + 3\frac{dv_n^m}{dr_0}) > +W_n\frac{dit_n^m}{dr_0}] \\
& +(4\pi G\rho_0 - \frac{2g_0}{r_0} - \frac{2}{3}\Omega^2)(6K_n - L_n)u_n^m \\
& -(g_0 - \frac{2}{3}\Omega^2 r_0)[n(n+1)K_n\frac{du_n^m}{dr_0} + (6K_n - 2M_n + L_n)\frac{u_n^m}{r_0} \\
& - < 6M_n - L_n n(n+1) + n(n+1)L_n > \frac{v_n^m}{r_0} - n(n+1)imQ_n\frac{t_n^m}{r_0}]\} \\
+\tfrac{2}{3} \quad & \frac{df}{dr_0}\{r_0\frac{d\mu}{dr_0}[(M_n - L_n)\Psi_n^m - imQ_n\xi_n^m] - \rho_0(g_0 - \frac{2}{3}\Omega^2 r_0)n(n+1)K_n u_n^m\} \\
= \quad & 0 \qquad\qquad (4.61)
\end{aligned}
$$

The ODEs from the toroidal component of the MCE are:

$$
\begin{aligned}
& n(n+1)\{\frac{d}{dr_0}(\mu\xi_n^m) + \mu[3\frac{\xi_n^m}{r_0} - < n(n+1) - 2 > \frac{t_n^m}{r_0^2}] \\
& \quad +\rho_0\omega^2 t_n^m - \frac{2\omega\Omega\rho_0}{n(n+1)}[mt_n^m + i(Q_n^{(3)}u_n^m - Q_n^{(2)}v_n^m)]\} \\
+\tfrac{2}{3} f \quad & \{imQ_n[-\lambda(\frac{d\Delta_n^m}{dr_0} - \frac{\Delta_n^m}{r_0}) - (\frac{d\lambda}{dr_0}\Delta_n^m + \frac{d\mu}{dr_0}\frac{2u_n^m}{r_0}) - \frac{2\mu}{r_0} < (\frac{du_n^m}{dr_0} - 2\frac{u_n^m}{r_0}) \\
& +(n(n+1) - 1)(\frac{dv_n^m}{dr_0} - \frac{2v_n^m}{r_0}) > -\mu(r_0\frac{d^2\Psi_n^m}{dr_0^2} + 3\frac{d\Psi_n^m}{dr_0} - 3\frac{\Psi_n^m}{r_0})] \\
& +(M_n - L_n)[\frac{\mu}{r_0}(n(n+1) - 2)(\frac{dt_n^m}{dr_0} - \frac{2t_n^m}{r_0}) - (r_0\frac{d^2\xi_n^m}{dr_0^2} + \frac{3d\xi_n^m}{dr_0} - \frac{3\xi_n^m}{r_0})] \\
& +\frac{d\mu}{dr_0}[imQ_n(\Psi_n^m - r_0\frac{d\Psi_n^m}{dr_0}) + (M_n - L_n)(\xi_n^m - r_0\frac{d\xi_n^m}{dr_0}) + imR_n\frac{2v_n^m}{r_0} + O_n\frac{t_n^m}{r_0}]\} \\
-\tfrac{2}{3} f \quad & \rho_0\{\omega^2 r_0[imQ_n\frac{dv_n^m}{dr_0} + (M_n - L_n)\frac{dt_n^m}{dr_0}] + imQ_n[(\frac{d\Phi_n^m}{dr_0} - \frac{\Phi_n^m}{r_0}) \\
& +(\frac{g_0}{r_0} - \frac{2}{3}\Omega^2) < 3u_n^m - n(n+1)v_n^m > -(4\pi G\rho_0 - \frac{2g_0}{r_0} - \frac{2}{3}\Omega^2)u_n^m] \\
& +2\omega\Omega r_0[V_n i\frac{du_n^m}{dr_0} + W_n i\frac{dv_n^m}{dr_0} - (m + 3mK_n)\frac{dt_n^m}{dr_0}]\} \\
+\tfrac{2}{3} \quad & \frac{df}{dr_0}\frac{d\mu}{dr_0}r_0[(M_n - L_n)\xi_n^m + imQ_n\Psi_n^m] \\
= \quad & 0 \qquad\qquad (4.62)
\end{aligned}
$$

In (4.60)-(4.62), all field variables u_n^m, v_n^m, t_n^m, Φ_n^m, Δ_n^m, Ψ_n^m, ξ_n^m are functions of r_0

only.

Note that these ODEs are coupled in degree but not in azimuthal order m. Any normal mode, in an Earth model which is rotation-symmetric about $\mathbf{k}$, is characterized by a single m.

4.3 Ordinary Differential Equations from the Poisson Equation, in Clairaut Coordinates

In this section, we derive the ODEs from the Poisson equation (4.2) in Clairaut coordinates.

The Poisson equation (4.2),

$$\nabla^2 V_1 = 4\pi G(\rho_0 \nabla \cdot \mathbf{u} + \nabla \rho_0 \cdot \mathbf{u}) \tag{4.63}$$

In contrast to the sequence adopted for the MCE, we can transform the Poisson equation in V_E into Clairaut coordinates directly because of the simple scalar form of the Poisson equation.

$$\begin{aligned} \nabla^2 V_1(\mathbf{r}) &= \nabla^2 V_1(r_0) - \frac{2}{3} r_0 f P_2 \frac{\partial}{\partial r_0} \nabla^2 V_1 \\ &= 4\pi G\{\rho_0[\nabla \cdot \mathbf{u} - \frac{2}{3} r_0 f P_2 \frac{\partial}{\partial r_0}(\nabla \cdot \mathbf{u})] \\ &\quad + \frac{d\rho_0}{dr_0} \nabla r_0 \cdot (\mathbf{u} - \frac{2}{3} r_0 f P_2 \frac{\partial \mathbf{u}}{\partial r_0})\} \end{aligned} \tag{4.64}$$

where the expansion of $\nabla^2 V_1$ in spherical harmonics is:

$$\nabla^2 V_1(r_0) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} [\frac{d^2 \Phi_n^m}{dr_0^2} + \frac{2}{r_0} \frac{d\Phi_n^m}{dr_0} - \frac{n(n+1)}{r_0^2} \Phi_n^m] Y_n^m \tag{4.65}$$

where Φ_n^m is now a function of r_0 only.

As we stated earlier in this thesis, we will keep only terms of first order of ellipticity f. Because it appears multiplied by ellipticity the term $\frac{\partial}{\partial r_0}\nabla^2 V_1$ can be replaced by its zero-order approximation.

$$\frac{\partial}{\partial r_0}\nabla^2 V_1 \simeq 4\pi G \frac{\partial}{\partial r_0}(\rho_0 \nabla \cdot \mathbf{u} + \nabla \rho_0 \cdot \mathbf{u}) \tag{4.66}$$

$$\simeq 4\pi G \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{d}{dr_0}(\rho_0 \Delta_n^m + \frac{d\rho_0}{dr_0} u_n^m) Y_n^m \tag{4.67}$$

By combining equations (4.63)-(4.67) and using the recurrence relations for Legendre function listed in Chapter 3 and the linear independence of the spherical harmonics Y_n^m, we obtain the ODEs from the Poisson equation:

$$\begin{aligned}
&\frac{1}{4\pi G}[\frac{d^2\Phi_n^m}{dr_0^2} + \frac{2}{r_0}\frac{d\Phi_n^m}{dr_0} - \frac{n(n+1)}{r_0^2}\Phi_n^m] - (\rho_0 \Delta_n^m + \frac{d\rho_0}{dr_0} u_n^m) \\
&-\tfrac{2}{3}\ [\frac{d}{dr_0}(r_0 f \frac{d\rho_0}{dr_0}) K_n u_n^m + f\frac{d\rho_0}{dr_0}(r_0 K_n \Delta_n^m + L_n v_n^m + imQ_n t_n^m)] \\
&= 0
\end{aligned} \tag{4.68}$$

Again these ODEs are coupled in degree but decoupled in azimuthal order m.

4.4 Governing Equations in the New Variables

In this section, we substitute the new dependent variables introduced in Chapter 3 $\tilde{u}_n^m$, $\tilde{v}_n^m$, $\tilde{t}_n^m$, $\tilde{\Phi}_n^m$, $\tilde{\Gamma}_n^m$, $\tilde{R}_n^m$, $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$ into the ODEs (4.60), (4.61), (4.62) and (4.68).

The above ODEs involve variables u_n^m, v_n^m, t_n^m, Φ_n^m, $\Phi_n^{m\prime}$, Δ_n^m, Ψ_n^m and ξ_n^m , which will be replaced by the new variables, $\tilde{u}_n^m$, $\tilde{v}_n^m$, $\cdots$, $\tilde{\tau}_n^m$. Then from those dynamic equations, we will form ODEs for the first derivatives of $\tilde{\Gamma}_n^m$, $\tilde{R}_n^m$, $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$. Since $\tilde{\Gamma}_n^m$, $\tilde{R}_n^m$, $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$ depend on the first derivatives of $\tilde{\Phi}_n^m$, $\tilde{u}_n^m$, $\tilde{v}_n^m$ and $\tilde{t}_n^m$, other ODEs for the first derivatives of $\tilde{\Phi}_n^m$, $\tilde{u}_n^m$, $\tilde{v}_n^m$ and $\tilde{t}_n^m$ will be obtained from the definitions of new variables.

4.4.1 Ordinary Differential Equations from the Definitions of New Variables

The definitions of the new variables are listed in equation (3.62)-(3.69). Here, as an example we derive the first derivative of $\tilde{v}_n^m$ from (3.63):

$$\begin{aligned}\frac{d\tilde{v}_n^m}{dr_0} &= \frac{dv_n^m}{dr_0} + \frac{2}{3}\frac{f'}{n(n+1)}\{(L_n - 6K_n)u_n^m - r_0[(M_n - L_n)v_n^{m\prime} + imQ_nt_n^{m\prime}]\} \\ &+\frac{2}{3}\frac{f}{n(n+1)}\{(L_n - 6K_n)u_n^{m\prime} - [(M_n - L_n)v_n^{m\prime} + imQ_nt_n^{m\prime}] \\ &-r_0[(M_n - L_n)v_n^{m\prime\prime} + imQ_nt_n^{m\prime\prime}]\}\end{aligned} \tag{4.69}$$

At the same time, from equation (4.36), (3.68), (3.63) and (3.62), we have,

$$\begin{aligned}\frac{dv_n^m}{dr_0} &= \Psi_n^m + \frac{v_n^m - u_n^m}{r_0} \\ &= \frac{\tilde{\Sigma}_n^m}{\mu} + \frac{2}{3}\frac{f}{n(n+1)}\{(L_n - 6K_n)(\frac{\lambda}{\mu}\Delta_n^m + 2\frac{u_n^m}{r_0}) + (M_n - L_n)r_0[v_n^{m\prime\prime} \\ &\quad -\frac{v_n^{m\prime} - u_n^{m\prime}}{r_0} + \frac{v_n^m - u_n^m}{r_0^2}] - 2N_n\frac{v_n^m}{r_0} - im[n(n+1) - 6]Q_n\frac{t_n^m}{r_0} \\ &\quad -imQ_nr_0(t_n^{m\prime\prime} - \frac{t_n^{m\prime}}{r_0} + \frac{t_n^m}{r_0^2})\} \\ &+ \frac{\tilde{v}_n^m}{r_0} - \frac{2}{3}\frac{f}{n(n+1)}\{(L_n - 6K_n)\frac{u_n^m}{r_0} - [(M_n - L_n)v_n^{m\prime} + imQ_nt_n^{m\prime}]\} \\ &- \frac{\tilde{u}_n^m}{r_0} + \frac{2}{3}f[L_n\frac{v_n^m}{r_0} + imQ_n\frac{t_n^m}{r_0} - K_nu_n^{m\prime}]\end{aligned} \tag{4.70}$$

Within the terms with ellipticity f, the variables without tilde are the same as those with tilde, i.e. $u_n^m \sim \tilde{u}_n^m$ and so on, because we keep only terms of first order in ellipticity. Combining (4.69) and (4.70), we have the first derivative of $\tilde{v}_n^m$ expressed in new variables. After similar derivations are applied to $\frac{d\tilde{u}_n^m}{dr_0}$, $\frac{d\tilde{t}_n^m}{dr_0}$ and $\frac{d\tilde{\Phi}_n^m}{dr_0}$, we have the first order ODEs:

$$\frac{d\tilde{u}_n^m}{dr_0} = \frac{1}{\lambda+2\mu}\{-\frac{1}{r_0}[\lambda(2-\frac{4}{3}r_0f'K_n) - \frac{2}{3}f < 6\lambda K_n - (\lambda+2\mu)L_n >]\tilde{u}_n^m$$
$$+ [1-\frac{2}{3}\frac{d(r_0f)}{dr_0}K_n]\tilde{R}_n^m$$
$$+ \frac{1}{r_0}\{n(n+1)\lambda + \frac{2}{3}r_0f'[\lambda(L_n - M_n) + 2\mu L_n] + \frac{4}{3}f(\lambda+\mu)L_n\}\tilde{v}_n^m$$
$$+ \frac{2}{3}fL_n\tilde{\Sigma}_n^m$$
$$+ [\frac{2}{3}f'(\lambda+2\mu) + \frac{4}{3}\frac{f}{r_0}(\lambda+\mu)]mQ_n\tilde{it}_n^m$$
$$+ \frac{2}{3}fmQ_n\tilde{i\tau}_n^m\} \tag{4.71}$$

$$\frac{d\tilde{v}_n^m}{dr_0} = -\frac{1}{r_0}[1-\frac{2}{3}\frac{r_0f'}{n(n+1)}(M_n - 6K_n) - \frac{2}{3}\frac{f}{n(n+1)}(L_n - 6K_n)]\tilde{u}_n^m$$
$$+ \frac{1}{\mu}\frac{2}{3}\frac{f}{n(n+1)}(L_n - 6K_n)\tilde{R}_n^m$$
$$+ \frac{1}{r_0}\{1-\frac{2}{3}\frac{r_0f'}{n(n+1)}(M_n - L_n) + \frac{2}{3}f[L_n - \frac{2}{n(n+1)}N_n]\}\tilde{v}_n^m$$
$$+ \frac{1}{\mu}[1-\frac{2}{3}\frac{(r_0f)'}{(n+1)}(M_n - L_n)]\tilde{\Sigma}_n^m$$
$$+ \frac{1}{n(n+1)r_0}(\frac{2}{3}r_0f' + 4f)mQ_n\tilde{it}_n^m$$
$$+ \frac{2}{3}\frac{1}{n(n+1)\mu}(r_0f)'mQ_n\tilde{i\tau}_n^m \tag{4.72}$$

$$\frac{d\tilde{it}_n^m}{dr_0} = -\frac{1}{n(n+1)r_0}(\frac{4}{3}r_0f' + \frac{2}{3}f)mQ_n\tilde{u}_n^m$$
$$- \frac{2}{3}\frac{1}{\mu}\frac{f}{n(n+1)}mQ_n\tilde{R}_n^m$$
$$+ \frac{1}{r_0}\frac{1}{n(n+1)}(\frac{2}{3}r_0f'Q_n + \frac{4}{3}fR_n)m\tilde{v}_n^m$$
$$+ \frac{2}{3}\frac{1}{\mu}\frac{(r_0f)'}{n(n+1)}mQ_n\tilde{\Sigma}_n^m$$
$$+ \frac{1}{r_0}\{1-\frac{1}{n(n+1)}\frac{2}{3}[r_0f'(M_n - L_n) + fO_n]\}\tilde{it}_n^m$$
$$+ \frac{1}{\mu}[1-\frac{2}{3}\frac{1}{n(n+1)}(r_0f)'(M_n - L_n)]\tilde{i\tau}_n^m \tag{4.73}$$

$$
\begin{aligned}
\frac{d\tilde{\Phi}_n^m}{dr_0} &= 4\pi G\rho_0[1-\frac{2}{3}(r_0f)'\acute{K}_n]\tilde{u}_n^m \\
&- \frac{2}{3}\frac{f}{r_0}L_n\tilde{\Phi}_n^m \\
&+ [1-\frac{2}{3}(r_0f)'\acute{K}_n]\tilde{\Gamma}_n^m
\end{aligned}
\tag{4.74}
$$

where $f' = \frac{df}{dr_0}, (r_0f)' = \frac{d}{dr_0}(r_0f)$.

4.4.2 Other Ordinary Differential Equations in New Variables

When we substitute the definitions of new variables into the ODEs (4.60), (4.61), (4.62) and (4.68), we need to apply the following identities for the operator symbols to simply the algebra:

$$
Q_n^{(1)}\frac{1}{n(n+1)}(M_n - L_n) = -\frac{m^2}{n(n+1)}Q_n + \acute{K}_nQ_n^{(1)} \tag{4.75}
$$

$$
Q_n^{(1)}\frac{1}{n(n+1)}Q_n = -\frac{1}{n(n+1)}(M_n - L_n) + \acute{K}_n \tag{4.76}
$$

$$
Q_nQ_n^{(1)} = -L_n \tag{4.77}
$$

$$
\begin{aligned}
Q_n^{(3)}\acute{K}_n &= -(\frac{4}{3}Q_n\acute{K}_n - \frac{2}{3}Q_n + \acute{K}_nQ_n^{(1)} + Q_n^{(3)}) \\
&= -V_n
\end{aligned}
\tag{4.78}
$$

$$
(M_n - L_n)\frac{1}{n(n+1)}Q_n^{(3)} = -m^2Q_n\frac{1}{n(n+1)} - V_n \tag{4.79}
$$

$$
Q_n^{(2)}\frac{1}{n(n+1)}(L_n - 6\acute{K}_n) = m^2\frac{1}{n(n+1)}Q_n + 2V_n - Q_n^{(3)} \tag{4.80}
$$

$$
Q_n^{(3)}Q_n = L_n - 6\acute{K}_n \tag{4.81}
$$

$$
Q_n^{(3)}L_n = 2[Q_n^{(2)} - W_n + m^2Q_n] \tag{4.82}
$$

$$
Q_n\frac{1}{n(n+1)}Q_n^{(3)} = -(M_n - L_n)\frac{1}{n(n+1)} + \acute{K}_n \tag{4.83}
$$

$$
Q_n\frac{1}{n(n+1)}Q_n^{(2)} = (M_n - L_n)\frac{1}{n(n+1)} - 3\acute{K}_n - 1 \tag{4.84}
$$

$$n(n+1)Q_n^{(2)}\frac{1}{n(n+1)}Q_n = (M_n - L_n) - n(n+1)(3K_n + 1) \quad (4.85)$$

$$Q_n^{(2)}\frac{1}{n(n+1)}(M_n - L_n) = W_n + \frac{1}{n(n+1)}m^2 Q_n \quad (4.86)$$

$$(M_n - L_n)\frac{1}{n(n+1)}Q_n^{(2)} = W_n + m^2 Q_n \frac{1}{n(n+1)} \quad (4.87)$$

We obtain the first-order ODEs for $\tilde{R}_n^m$, $\tilde{\Sigma}_n^m$, $\tilde{\tau}_n^m$ and $\tilde{\Gamma}_n^m$,

$$\begin{aligned}
\frac{d\tilde{R}_n^m}{dr_0} = & \{-\rho_0\omega^2[1 - \frac{2}{3}(r_0 f)' K_n] - \rho_0\frac{g_0}{r_0}[4 - \frac{2}{3}f(8K_n - L_n)] \\
& +\frac{2}{3}\rho_0\Omega^2[1 + 2r_0 f' K_n - \frac{2}{3}f(5K_n - L_n)] \\
& +\frac{4\mu}{r_0^2}\frac{3\lambda + 2\mu}{\lambda + 2\mu}[1 - \frac{2}{3}r_0 f' K_n - \frac{1}{3}f(10K_n - L_n)]\}\tilde{u}_n^m \\
- & \frac{1}{r_0(\lambda + 2\mu)}[4\mu(1 - \frac{2}{3}r_0 f' K_n) - \frac{2}{3}f\lambda(L_n - 6K_n)]\tilde{R}_n^m \\
+ & \{\rho_0\omega^2\frac{2}{3}fL_n + \rho_0\frac{g_0}{r_0}[n(n+1) + \frac{2}{3}r_0 f' L_n + \frac{2}{3}f(2L_n + M_n)] \\
& -\frac{2}{3}\rho_0\Omega^2[n(n+1) + \frac{2}{3}r_0 f' L_n + \frac{2}{3}f(2L_n + M_n)] \\
& -\frac{2\mu}{r_0^2}[\frac{3\lambda + 2\mu}{\lambda + 2\mu} < n(n+1) - \frac{2}{3}r_0 f' M_n + \frac{2}{3}fL_n > \\
& +\frac{2}{3}f < N_n + \frac{\lambda}{\lambda + 2\mu}L_n n(n+1) - \frac{3\lambda - 2\mu}{\lambda + 2\mu}M_n >] \\
& +2\rho_0 m\omega\Omega[1 - \frac{2}{3}(r_0 f)' K_n]\}\tilde{v}_n^m \\
+ & \frac{1}{r_0}[n(n+1) + \frac{2}{3}r_0 f'(L_n - M_n) + \frac{4}{3}f\frac{3\lambda + 4\mu}{\lambda + 2\mu}L_n]\tilde{\Sigma}_n^m \\
+ & \frac{2}{3}f\frac{\rho_0}{r_0}L_n\tilde{\Phi}_n^m \\
- & \rho_0[1 - \frac{2}{3}(r_0 f)' K_n]\tilde{\Gamma}_n^m \\
+ & \{\rho_0\omega^2\frac{2}{3}fmQ_n + \rho_0\frac{g_0}{r_0}(\frac{2}{3}r_0 f' + \frac{4}{3}f)mQ_n - \frac{2}{3}\rho_0\Omega^2(\frac{2}{3}r_0 f' + \frac{4}{3}f)mQ_n \\
& +2\rho_0\omega\Omega[-Q_n^{(1)} + \frac{2}{3}(r_0 f)' K_n Q_n^{(1)}] \\
& -\frac{2}{3}f\frac{\mu}{r_0^2}\frac{n(n+1)\lambda + 2(n^2 + n - 4)\mu}{\lambda + 2\mu}mQ_n\}i\tilde{t}_n^m \\
+ & \frac{1}{r_0}[\frac{2}{3}r_0 f' + \frac{4}{3}f\frac{3\lambda + 4\mu}{\lambda + 2\mu}]mQ_n i\tilde{\tau}_n^m
\end{aligned} \quad (4.88)$$

$$
\begin{aligned}
\frac{d\tilde{\Sigma}_n^m}{dr_0} &= \{\frac{2}{3}\frac{f}{n(n+1)}\rho_0\omega^2(L_n - 6K_n) \\
&+\rho_0\frac{g_0}{r_0}[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(L_n - 6K_n) + \frac{2}{3}\frac{f}{n(n+1)}(M_n + 2L_n - 18K_n)] \\
&-\frac{2}{3}\rho_0\Omega^2[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(L_n - 6K_n) + \frac{2}{3}\frac{f}{n(n+1)}(M_n - L_n)] \\
&-\frac{2\mu}{r_0^2(\lambda+2\mu)}[(3\lambda+2\mu)(1 - \frac{2}{3}r_0 f' K_n) + \frac{2}{3}\frac{f}{n(n+1)} \\
&< (\lambda+2\mu)(N_n - L_n n(n+1)) - 6(17\lambda+14\mu)K_n + 2(13\lambda+12\mu)L_n \\
&-(3\lambda-2\mu)M_n >] + \frac{2\rho_0 m\omega\Omega}{n(n+1)}[1 - \frac{2}{3}r_0 f' K_n - \frac{2}{3}f(1+3K_n)]\}\tilde{u}_n^m \\
- & \frac{\lambda}{r_0(\lambda+2\mu)}[1 - \frac{2}{3}r_0 f' K_n + \frac{2f}{n(n+1)}(L_n - 6K_n)]\tilde{R}_n^m \\
+ & \{-\rho_0\omega^2[1 - \frac{2}{3}\frac{(r_0 f)'}{n(n+1)}(M_n - L_n)] \\
&+\frac{2}{3}\frac{f}{n(n+1)}\rho_0(\frac{g_0}{r_0} - \frac{2}{3}\rho_0\Omega^2)[6M_n - L_n n(n+1)] \\
&+\frac{2\mu}{r_0^2(\lambda+2\mu)}[< 2n(n+1) - 1 > \lambda + 2 < n(n+1) - 1 > \mu \\
&-\frac{2}{3}\frac{r_0 f'}{n(n+1)} < \lambda n(n+1)M_n + (\lambda+2\mu)((M_n - L_n)(n(n+1)-1) - N_n) > \\
&+\frac{2}{3}\frac{f}{n(n+1)} < 2(\lambda+\mu)n(n+1)M_n + 2\lambda(O_n + 2L_n n(n+1)) + (\lambda+6\mu)N_n \\
&-(31\lambda+2\mu)M_n + (7\lambda+2\mu)L_n >] \\
&+\frac{2\rho_0 m\omega\Omega}{n(n+1)}[1 - \frac{2}{3}r_0 f'(1+3K_n) - \frac{2}{3}f(1+3K_n+L_n)]\}\tilde{v}_n^m \\
- & \frac{1}{r_0}\{3[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(M_n - L_n)] + \frac{2}{3}f[\frac{3\lambda+4\mu}{\lambda+2\mu}L_n - 2M_n \\
&+\frac{2}{n(n+1)} < M_n n(n+1) - L_n n(n+1) - N_n >]\}\tilde{\Sigma}_n^m \\
- & \frac{\rho_0}{r_0}[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(M_n - L_n)]\tilde{\Phi}_n^m \\
+ & \frac{2}{3}\frac{f}{n(n+1)}\rho_0(L_n - 6K_n)\tilde{\Gamma}_n^m \\
+ & \{-\frac{2}{3}\frac{(r_0 f)'}{n(n+1)}\rho_0\omega^2 m Q_n + \frac{m\mu}{r_0^2 n(n+1)}\cdot
\end{aligned}
$$

$$
\begin{aligned}
& [\frac{2}{3}r_0 f' < (n(n+1)-6)Q_n + Q_n(n(n+1)-2) > +\frac{2}{3}f < n(n+1)\frac{5\lambda+6\mu}{\lambda+2\mu} \\
& -16 > Q_n] + \frac{2\rho_0\omega\Omega}{n(n+1)}[Q_n^{(2)} - \frac{2}{3}r_0 f' W_n - \frac{2}{3}f(W_n + m^2 Q_n)]\}i\tilde{t}_n^m \\
- \quad & \frac{m}{r_0 n(n+1)}\{2r_0 f' Q_n + \frac{2}{3}f[< 2n(n+1)\frac{\lambda+\mu}{\lambda+2\mu} + 4 > Q_n \\
& -Q_n < n(n+1)-2 >]\}i\tilde{\tau}_n^m \qquad (4.89)
\end{aligned}
$$

$$
\begin{aligned}
\frac{di\tilde{\tau}_n^m}{dr_0} = \quad & \{-\frac{2}{3}\frac{f}{n(n+1)}\rho_0\omega^2 m Q_n + \frac{\rho_0 g_0}{r_0 n(n+1)}(\frac{2}{3}r_0 f' - \frac{8}{3}f)mQ_n \\
& -\frac{2}{3}\frac{\rho_0\Omega^2}{n(n+1)}(\frac{2}{3}r_0 f' - \frac{2}{3}f)mQ_n \\
& +\frac{2m\mu}{r_0^2 n(n+1)}\frac{2}{3}f[R_n - Q_n n(n+1) + n(n+1)Q_n + \frac{8\lambda+4\mu}{\lambda+2\mu}Q_n] \\
& -\frac{2\rho_0\omega\Omega}{n(n+1)}[Q_n^{(3)} + \frac{2}{3}r_0 f' V_n + \frac{2}{3}f(3V_n - Q_n^{(3)})]\}\tilde{u}_n^m \\
+ \quad & \frac{2f}{n(n+1)r_0}\frac{\lambda}{\lambda+2\mu}mQ_n\tilde{R}_n^m \\
+ \quad & \{-\frac{2}{3}\frac{(r_o f)'}{n(n+1)}\rho_0\omega^2 m Q_n + \frac{2}{3}\frac{f}{n(n+1)}\rho_0(\frac{g_0}{r_0} - \frac{2}{3}\Omega^2)mQ_n n(n+1) \\
& -\frac{m\mu}{r_0^2 n(n+1)}[\frac{4}{3}r_0 f' < R_n - Q_n(n(n+1) + Q_n > \\
& +\frac{2}{3}f < 6R_n + (n(n+1)-2)Q_n + \frac{6\lambda}{\lambda+2\mu}Q_n n(n+1) >] \\
& +\frac{2\rho_0\omega\Omega}{n(n+1)}[Q_n^{(2)} - \frac{2}{3}r_0 f' W_n - \frac{2}{3}f(3W_n - 2Q_n^{(2)} - m^2 Q_n)]\}\tilde{v}_n^m \\
- \quad & \frac{m}{r_0 n(n+1)}\{2r_0 f' Q_n - \frac{2}{3}f[2(Q_n n(n+1) - R_n) - n(n+1)Q_n]\}\tilde{\Sigma}_n^m \\
- \quad & \frac{2}{3}\frac{f'}{n(n+1)}\rho_0 m Q_n\tilde{\Phi}_n^m \\
- \quad & \frac{2}{3}\frac{f}{n(n+1)}\rho_0 m Q_n\tilde{\Gamma}_n^m \\
+ \quad & \{-\rho_0\omega^2[1 - \frac{2}{3}\frac{(r_0 f)'}{n(n+1)}(M_n - L_n)] \\
& +\frac{\mu}{r_0^2}[n(n+1) - 2 + \frac{2}{3}\frac{r_0 f'}{n(n+1)} < O_n - (M_n - L_n)(n(n+1)-2) > \\
& +\frac{2}{3}\frac{f}{n(n+1)} < 3O_n + (n(n+1)-2)(M_n - L_n) >]
\end{aligned}
$$

$$+\frac{2\rho_0 m\omega\Omega}{n(n+1)}[1-\frac{2}{3}r_0 f'(1+3K_n)-\frac{2}{3}f(1+9K_n-L_n)]\}\tilde{it}_n^m$$

$$-\quad \frac{1}{r_0}\{3-\frac{2r_0 f'}{n(n+1)}(M_n-L_n)-\frac{2}{3}\frac{f}{n(n+1)}[O_n+n(n+1)(M_n-L_n)$$

$$-(M_n-L_n)n(n+1)]\}\tilde{i\tau}_n^m \tag{4.90}$$

$$\frac{d\tilde{\Gamma}_n^m}{dr_0} = -\frac{4\pi G\rho_0}{r_0}[n(n+1)+\frac{2}{3}r_0 f'(L_n-M_n)+\frac{4}{3}fL_n]\tilde{v}_n^m$$

$$+\quad \frac{1}{r_0^2}[n(n+1)-\frac{2}{3}r_0 f'(L_n-M_n)+\frac{2}{3}f(M_n+L_n)]\tilde{\Phi}_n^m$$

$$-\quad \frac{1}{r_0}\{2-\frac{4}{3}r_0 f'K_n-\frac{2}{3}f[L_n-M_n+n(n+1)K_n]\}\tilde{\Gamma}_n^m$$

$$-\quad \frac{4\pi G\rho_0}{r_0}[(\frac{2}{3}r_0 f'+\frac{4}{3}f)mQ_n]\tilde{it}_n^m \tag{4.91}$$

The ODEs (4.71)-(4.74) and (4.88)-(4.91) are valid in the mantle & solid inner core. The important feature of these new governing equations is that they contain no derivatives of material properties. This is a consequence of having used the new dependent variables.

4.4.3 Governing Equations in the Liquid Outer Core

In the liquid outer core, the governing equations will be reduced because $\mu = 0$ there. Setting $\mu = 0$, the ODEs for the first order derivatives of $\tilde{v}_n^m$ and $\tilde{t}_n^m$, i.e. equations (4.72) and (4.73), become

$$\tilde{\Sigma}_n^m = -\frac{2}{3}\frac{f}{n(n+1)}(L_n-6K_n)\tilde{R}_n^m \tag{4.92}$$

$$i\tilde{\tau}_n^m = \frac{2}{3}\frac{f}{n(n+1)}mQ_n\tilde{R}_n^m \tag{4.93}$$

which means two variables $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$ become dependent on the $\tilde{R}_n^m$. With this dependent property, the ODEs for $\tilde{\Sigma}_n^m$ and $\tilde{\tau}_n^m$ are reduced to algebraic equations for

$\tilde{v}_n^m$, $\tilde{t}_n^m$ involving variables $\tilde{u}_n^m$, $\tilde{R}_n^m$, $\tilde{\Phi}_n^m$, $\tilde{\Gamma}_n^m$:

$$
\begin{aligned}
0 = \; & \{\rho_0 \frac{g_0}{r_0}[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(L_n - 6\acute{K}_n) + \frac{2}{3}\frac{f}{n(n+1)}(M_n - 2L_n + 6\acute{K}_n)] \\
& -\frac{2}{3}\rho_0 \Omega^2 [1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(L_n - 6\acute{K}_n) + \frac{2}{3}\frac{f}{n(n+1)}(M_n - 2L_n + 6\acute{K}_n)] \\
& +\frac{2\rho_0 m\omega\Omega}{n(n+1)}[1 - \frac{2}{3} r_0 f' \acute{K}_n - \frac{2}{3} f(1 + 3\acute{K}_n)]\}\tilde{u}_n^m \\
& +[-\frac{1}{r_0} + \frac{2}{3}\frac{f'}{n(n+1)}(M_n - L_n]\tilde{R}_n^m \\
& +\{-\rho_0\omega^2[1 - \frac{2}{3}\frac{(r_0 f)'}{n(n+1)}(M_n - L_n)] \\
& +\frac{2\rho_0 m\omega\Omega}{n(n+1)}[1 - \frac{2}{3} r_0 f'(1 + 3\acute{K}_n) - \frac{2}{3} f(1 + 9\acute{K}_n)]\}\tilde{v}_n^m \\
& -\frac{\rho_0}{r_0}[1 - \frac{2}{3}\frac{r_0 f'}{n(n+1)}(M_n - L_n)]\tilde{\Phi}_n^m \\
& +\{\frac{2\rho_0 m\omega\Omega}{n(n+1)}[Q_n^{(2)} - \frac{2}{3} r_0 f' W_n - \frac{2}{3} f < W_n + m^2 Q_n + (L_n - 6\acute{K}_n) Q_n^{(1)} >] \\
& -\rho_0 \omega^2 \frac{2}{3}\frac{(r_0 f)'}{n(n+1)} m Q_n\} i\tilde{t}_n^m \qquad (4.94) \\
0 = \; & \{\frac{2}{3}\frac{r_0 f'}{n(n+1)}\rho_0(\frac{g_0}{r_0} - \frac{2}{3}\Omega^2) m Q_n \\
& -\frac{2\rho_0\omega\Omega}{n(n+1)}[Q_n^{(3)} + \frac{2}{3} r_0 f' V_n + \frac{2}{3} f(3V_n - Q_n^{(3)})]\}\tilde{u}_n^m \\
& -\frac{2}{3}\frac{f'}{n(n+1)} m Q_n \tilde{R}_n^m \\
& +\{-\frac{2}{3}\frac{r_0 f'}{n(n+1)}\rho_0\omega^2 m Q_n \\
& +\frac{2\rho_0\omega\Omega}{n(n+1)}[Q_n^{(2)} - \frac{2}{3} r_0 f' W_n + \frac{2}{3} f(2Q_n^{(3)} - 3W_n)]\}\tilde{v}_n^m \\
& -\frac{2}{3}\frac{f'}{n(n+1)}\rho_0 m Q_n \tilde{\Phi}_n^m \\
& +\{-\rho_0\omega^2[1 - \frac{2}{3}\frac{(r_0 f)'}{n(n+1)}(M_n - L_n)] + \frac{2\rho_0 m\omega\Omega}{n(n+1)} \cdot \\
& [1 - \frac{2}{3} r_0 f'(1 + 3K_n) + \frac{2}{3} f(L_n - 9K_n - 1 + Q_n Q_n^{(1)})]\}\tilde{t}_n^m \qquad (4.95)
\end{aligned}
$$

The existing ODEs in the liquid outer core are those for the first derivatives of $\tilde{u}_n^m$, $\tilde{R}_n^m$, $\tilde{\Phi}_n^m$, $\tilde{\Gamma}_n^m$:

$$
\begin{aligned}
\frac{d\tilde{u}_n^m}{dr_0} &= -\frac{1}{r_0}[(2-\frac{4}{3}r_0 f')K_n - \frac{2}{3}f(6K_n - L_n)]\tilde{u}_n^m \\
&+ \frac{1}{\lambda}[1-\frac{2}{3}\frac{d(r_0 f)}{dr_0}K_n]\tilde{R}_n^m \\
&+ \frac{1}{r_0}[n(n+1)+\frac{2}{3}r_0 f'(L_n - M_n)+\frac{4}{3}fL_n]\tilde{v}_n^m \\
&+ \frac{1}{\lambda}\frac{2}{3}fL_n\tilde{\Sigma}_n^m \\
&+ [\frac{2}{3}f' + \frac{4}{3}\frac{f}{r_0}]mQ_n\tilde{i}t_n^m \\
&+ \frac{1}{\lambda}\frac{2}{3}fmQ_n\tilde{i}\tau_n^m\} \qquad (4.96)\\
\frac{d\tilde{R}_n^m}{dr_0} &= \{-\rho_0\omega^2[1-\frac{2}{3}(r_0 f)'K_n] - \rho_0\frac{g_0}{r_0}[4-\frac{2}{3}f(8K_n - L_n)] \\
&\quad +\frac{2}{3}\rho_0\Omega^2[1-2r_0 f'K_n - \frac{2}{3}f(5K_n - L_n)]\}\tilde{u}_n^m \\
&+ \{\rho_0\omega^2\frac{2}{3}fL_n + \rho_0\frac{g_0}{r_0}[n(n+1)+\frac{2}{3}r_0 f'L_n + \frac{2}{3}f(2L_n + M_n)] \\
&\quad -\frac{2}{3}\rho_0\Omega^2[n(n+1)+\frac{2}{3}r_0 f'L_n + \frac{2}{3}f(2L_n + M_n)] \\
&\quad -2\rho_0 m\omega\Omega[1-\frac{2}{3}(r_0 f)'K_n]\}\tilde{v}_n^m \\
&+ \frac{2}{3}f\frac{\rho_0}{r_0}L_n\tilde{\Phi}_n^m \\
&- \rho_0[1-\frac{2}{3}(r_0 f)'K_n]\tilde{\Gamma}_n^m \\
&+ \rho_0\{\omega^2\frac{2}{3}fmQ_n + \frac{g_0}{r_0}(\frac{2}{3}r_0 f' + \frac{4}{3}f)mQ_n - \frac{2}{3}\Omega^2(\frac{2}{3}r_0 f' + \frac{4}{3}f)mQ_n \\
&\quad +2\omega\Omega[-Q_n^{(1)} + \frac{2}{3}(r_0 f)'K_n Q_n^{(1)}]\}i\tilde{t}_n^m \qquad (4.97)\\
\frac{d\tilde{\Phi}_n^m}{dr_0} &= 4\pi G\rho_0[1-\frac{2}{3}(r_0 f)'K_n]\tilde{u}_n^m \\
&- \frac{2}{3}\frac{f}{r_0}L_n\tilde{\Phi}_n^m \\
&+ [1-\frac{2}{3}(r_0 f)'K_n]\tilde{\Gamma}_n^m \qquad (4.98)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\tilde{\Gamma}_n^m}{dr_0} &= -\frac{4\pi G\rho_0}{r_0}[n(n+1) - \frac{2}{3}r_0 f'(M_n - L_n) + \frac{4}{3}fL_n]\tilde{v}_n^m \\
&+ \frac{1}{r_0^2}[n(n+1) - \frac{2}{3}r_0 f'(M_n - L_n) + \frac{2}{3}f(M_n + L_n)]\tilde{\Phi}_n^m \\
&- \frac{1}{r_0}\{2 - \frac{4}{3}r_0 f' K_n - \frac{2}{3}f[L_n - M_n + n(n+1)K_n]\}\tilde{\Gamma}_n^m \\
&- \frac{4\pi G\rho_0}{r_0}[(\frac{2}{3}r_0 f' + \frac{4}{3}f)mQ_n]i\tilde{t}_n^m
\end{aligned}
\tag{4.99}
$$

As we see, these ODEs involve variables $\tilde{u}_n^m$, $\tilde{R}_n^m$, $\tilde{v}_n^m$, $\tilde{t}_n^m$, $\tilde{\Phi}_n^m$ and $\tilde{\Gamma}_n^m$.

Therefore, the governing equations in the liquid outer core include a coupled set of ODEs (4.96)-(4.99) and algebraic equations (4.94)-(4.95).

Chapter 5

Computing the Earth's Free Wobble/Nutation Modes

Having obtained the governing equations (in Chapter 4) and boundary conditions (in Chapter 3) for the free wobble/nutation modes of a rotating, ellipsoidally-layered Earth model . we now establish the numerical approach to computing eigenperiods. For the Earth's free wobble/nutation, as we stated in Chapter 1, the displacement is an infinite chain of odd-degree toroidal and even-degree spheroidal fields:

$$\mathbf{T}_1^m + \mathbf{S}_2^m + \mathbf{T}_3^m + \cdots \tag{5.1}$$

with $|m| = 1$. If we set $m = +1$ then prograde modes will have frequency $\omega < 0$ and retrograde modes $\omega > 0$. When carrying out the numerical solutions, we need to truncate this chain appropriately to include only a finite number of members. In this thesis, we adopt Smith's (1977) truncation at degree $n = 3$ to write our governing equations and boundary conditions.

5.1 Governing Equations and Boundary Conditions with Truncation at $n = 3$

The displacement chain for the free wobble/nutation modes truncated at degree $n = 3$ is:

$$\mathbf{T}_1^m + \mathbf{S}_2^m + \mathbf{T}_3^m \tag{5.2}$$

We introduce a set of generalized AJP variables y_i, $(i = 1, 10)$ to represent the new variables $\tilde{u}_2^m$, $\tilde{R}_2^m$, $\tilde{v}_2^m$, $\tilde{\Sigma}_2^m$, $\tilde{\Phi}_2^m$, $\tilde{\Gamma}_2^m$, $\tilde{t}_1^m$, $\tilde{\tau}_1^m$, $\tilde{t}_3^m$ and $\tilde{\tau}_3^m$ respectively. We use y_1, y_2, y_3, y_4, y_5, y_6 to represent the degree 2 spheroidal field, y_7, y_8 to represent degree 1 toroidal field, y_9, y_{10} to represent degree 3 toroidal field.

To increase the precision of calculations, we will use the dimensionless variables $\tilde{y}_i, (i = 1, 10)$ in the governing equations and boundary conditions. In the solid inner core and liquid outer core, we take the inner core boundary as scaling reference,

$$\begin{aligned} y_i &= a\tilde{y}_i\{\delta_{i1} + \delta_{i3} + \delta_{i7} + \delta_{i9} + g_0(a)[\rho_0(a_+)(\delta_{i2} + \delta_{i4} + \delta_{i8} + \delta_{i10}) + \delta_{i5} + \frac{\delta_{i6}}{a}]\} \\ (i &= 1, 10) \end{aligned} \tag{5.3}$$

where δ_{ij} is the conventional delta symbol. In the mantle, we take the core-mantle boundary as scaling reference,

$$\begin{aligned} y_i &= b\tilde{y}_i\{\delta_{i1} + \delta_{i3} + \delta_{i7} + \delta_{i9} + g_0(b)[\rho_0(b_-)(\delta_{i2} + \delta_{i4} + \delta_{i8} + \delta_{i10}) + \delta_{i5} + \frac{\delta_{i6}}{b}]\} \\ (i &= 1, 10) \end{aligned} \tag{5.4}$$

Here a_+ and b_- denote locations in the outer core just above the inner-core boundary and just below the core-mantle boundary.

For convenience, we henceforth drop the tilde $\tilde{}$ and use y_i to replace the dimensionless variable $\tilde{y}_i$. At the same time, the radius r_0 and derivative of ellipticity f' need to be scaled:

$$\begin{aligned} x &= \frac{r_0}{R} \\ f'_x &= \frac{df}{dx} = R\frac{df}{dr_0} \end{aligned} \qquad (5.5)$$

Upon substituting these dimensionless AJP notations into the governing equations (4.61), (4.62), (4.63) and (4.68), we have the ODEs for $y_1, y_2, \cdots, y_{10}$:

$$\begin{aligned} \frac{dy_i}{dx} &= \sum_{j=1}^{10} A(i,j)y_j \\ (i &= 1, 10) \end{aligned} \qquad (5.6)$$

The coefficients $A(i,j)$ for the inner core of the Earth are,

$$\begin{aligned} A(1,1) &= -2(1-2RA)\frac{1}{x} + \frac{4}{3}f'(1-2RA)B_2^m + 2\frac{f}{x}(1-4RA)B_2^m \\ A(1,2) &= DIAD[1 - \frac{2}{3}(xf)'B_2^m] \\ A(1,3) &= 6(1-2RA)\frac{1}{x} - 2(1-4RA)f'B_2^m + 4\frac{f}{x}(1-RA)B_2^m \\ A(1,4) &= DIAD[2fB_2^m] \\ A(1,5) &= A(1,6) = 0 \\ A(1,7) &= [-2f' - 4\frac{f}{x}(1-RA)]mH_1^m \\ A(1,8) &= -DIAD[2fmH_1^m] \\ A(1,9) &= A(1,7)\frac{G_3^m}{H_1^m} \\ A(1,10) &= A(1,8)\frac{G_3^m}{H_1^m} \\ A(2,1) &= BA[-1 + \frac{2}{3}(xf)'B_2^m] + BB[\frac{2}{3} + (\frac{4}{3}f'x - \frac{8}{9}f)B_2^m] + BC[-\frac{4}{x} + \frac{10}{3}\frac{f}{x}B_2^m] \end{aligned} \qquad (5.7)$$

$$
\begin{aligned}
& +BD[4(3-4RA)][\frac{1}{x^2}-(\frac{7}{3}\frac{f}{x^2}+\frac{2}{3}\frac{f'}{x})B_2^m] \\
A(2,2) &= -4RA\frac{1}{x}+[-2\frac{f}{x}(1-2RA)+\frac{8}{3}f'RA]B_2^m \\
A(2,3) &= BA[2fB_2^m]+BB[-4-(\frac{16}{3}f+\frac{4}{3}f'x)B_2^m]+BC[\frac{6}{x}+(8\frac{f}{x}+2f')B_2^m] \\
&+ BD[12(3-4RA)(-\frac{1}{x^2}+\frac{2}{3}\frac{f'}{x}B_2^m)-24\frac{f}{x^2}B_2^m]+BE[1-\frac{2}{3}(xf)'B_2^m] \\
A(2,4) &= \frac{6}{x}+[4\frac{f}{x}(3-2RA)-2f']B_2^m \\
A(2,5) &= ROR[2\frac{f}{x}B_2^m] \\
A(2,6) &= ROA[-1+\frac{2}{3}(xf)'B_2^m] \\
A(2,7) &= [BA(-2f)+BB(\frac{8}{3}f+\frac{4}{3}f'x)+BC\cdot 6(-\frac{2}{3}\frac{f}{x}-\frac{1}{3}f') \\
& +BD(3-4RA)4\frac{f}{x^2}]mH_1^m+BED[H_1^m+\frac{2}{3}(xf)'(2C_0^mG_1^m-B_2^mH_1^m)] \\
A(2,8) &= -mH_1^m[2f'+4\frac{f}{x}(3-2RA)] \\
A(2,9) &= [BA(-2f)+BB(\frac{8}{3}f+\frac{4}{3}f'x)+BC\cdot 6(-\frac{2}{3}\frac{f}{x}-\frac{1}{3}f') \\
& +BD(3-4RA)4\frac{f}{x^2}]mG_3^m+BED[-4G_3^m+\frac{2}{3}(xf)'(4B_2^mG_3^m-3A_4^mH_3^m)] \\
A(2,10) &= A(2,8)\frac{G_3^m}{H_1^m} \qquad (5.8) \\
A(3,1) &= -\frac{1}{x}-\frac{1}{3}\frac{f}{x}B_2^m \\
A(3,2) &= -DIBD\frac{1}{3}fB_2^m \\
A(3,3) &= \frac{1}{x}-\frac{1}{3}f'B_2^m \\
A(3,4) &= DIBD[1-\frac{1}{3}(xf)'B_2^m] \\
A(3,5) &= A(3,6)=0 \\
A(3,7) &= -(\frac{1}{3}f'+2\frac{f}{x})mH_1^m \\
A(3,8) &= -DIBD\frac{1}{3}(xf)'mH_1^m
\end{aligned}
$$

$$
\begin{aligned}
A(3,9) &= A(3,7)\frac{G_3^m}{H_1^m} \\
A(3,10) &= A(3,8)\frac{G_3^m}{H_1^m} \qquad (5.9) \\
A(4,1) &= -BA(\frac{f}{3}B_2^m) - BB[\frac{2}{3} + \frac{2}{9}(xf)'B_2^m] + BC[\frac{1}{x} - (\frac{2}{3}\frac{f}{x} - \frac{1}{3}f')B_2^m] \\
&+ BD[(3-4RA)(-\frac{2}{x^2} + \frac{4}{3}\frac{f'}{x}B_2^m) + (17-28RA)\frac{2}{3}\frac{f}{x^2}B_2^m] \\
&+ BE[\frac{1}{6} - \frac{1}{9}(xf' + 3f)B_2^m - \frac{1}{9}f] \\
A(4,2) &= -(1-2RA)[\frac{1}{x} - (\frac{2}{3}f' + \frac{f}{x})B_2^m] \\
A(4,3) &= BA[-1 + \frac{1}{3}(xf)'B_2^m] + BB[-\frac{4}{3}fB_2^m] + BC[2\frac{f}{x}B_2^m] \\
&+ BD[(11-12RA)\frac{2}{x^2} - (7-12RA)\frac{4}{3}\frac{f'}{x}B_2^m + \frac{40}{3}\frac{f}{x^2}B_2^m] \\
&+ BE\{\frac{1}{6} - \frac{1}{9}[3(2f + xf')B_2^m + (xf)']\} \\
A(4,4) &= -\frac{3}{x} + (4\frac{f}{x}RA + f')B_2^m \\
A(4,5) &= ROR[-\frac{1}{x} + \frac{1}{3}f'B_2^m] \\
A(4,6) &= ROA[-\frac{1}{3}fB_2^m] \\
A(4,7) &= [BA\frac{1}{3}(xf)' - BD(7-12RA)\frac{2}{3}\frac{f}{x^2}]mH_1^m \\
&+ BED[\frac{H_1^m}{2} - \frac{1}{9}(xf)'(-4C_0^m G_1^m + 5B_2^m H_1^m + H_1^m) + \frac{1}{3}fm^2 H_1^m] \\
A(4,8) &= [\frac{4}{3}\frac{f}{x}(4-3RA) + f']mH_1^m \\
A(4,9) &= \{BA\frac{1}{3}(xf)' - BD[\frac{10}{3}\frac{f'}{x} + \frac{2}{3}\frac{f}{x^2}(7-12RA)]\}mG_3^m \\
&\quad +BED[\frac{4}{3}G_3^m - \frac{1}{9}(xf)'(21A_4^m H_3^m - 4G_3^m) + \frac{1}{3}fm^2 G_3^m] \\
A(4,10) &= [2\frac{f}{x}(1-2RA) + f']mG_3^m \qquad (5.10) \\
A(5,1) &= ETAIC \cdot ROA[1 - \frac{2}{3}(xf)'B_2^m] \\
A(5,2) &= A(5,3) = A(5,4) = 0
\end{aligned}
$$

$$
\begin{aligned}
A(5.5) &= -\frac{2f}{x}B_2^m \\
A(5.6) &= \frac{R}{a}[1-\frac{2}{3}(xf)'B_2^m] \\
A(5.7) &= A(5,8) = A(5.9) = A(5.10) = 0
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
A(6.1) &= A(6,2) = 0 \\
A(6.3) &= -ETAIC \cdot 2ROR[\frac{3}{x} + (\frac{2f}{x} - f')B_2^m] \\
A(6,4) &= 0 \\
A(6,5) &= \frac{a}{R}[\frac{6}{x^2} + 2(\frac{3f}{x^2} - \frac{f'}{x})B_2^m] \\
A(6.6) &= -\frac{2}{x} + (2\frac{f}{x} + \frac{4}{3}f')B_2^m \\
A(6,7) &= ETAIC \cdot 2ROR(2\frac{f}{x} + f')mH_1^m \\
A(6,9) &= A(6,7)\frac{G_3^m}{H_1^m} \\
A(6,8) &= A(6,10) = 0
\end{aligned} \tag{5.12}
$$

$$
\begin{aligned}
A(7,1) &= (\frac{f}{x} + 2f')mG_2^m \\
A(7,2) &= DIBD \cdot fmG_2^m \\
A(7,3) &= -(10\frac{f}{x} + f')mG_2^m \\
A(7,4) &= -DIBD(xf)'mG_2^m \\
A(7,5) &= A(7,6) = 0 \\
A(7,7) &= \frac{1}{x} + \frac{f'}{3}B_1^m \\
A(7,8) &= DIBD[1 + \frac{1}{3}(xf)'B_1^m] \\
A(7,9) &= -(\frac{40}{3}\frac{f}{x} + \frac{4}{3}f')A_3^m \\
A(7,10) &= -DIBD\frac{4}{3}(fx)'A_3^m
\end{aligned} \tag{5.13}
$$

$$
\begin{aligned}
A(8,1) &= [BA(f) + BB\frac{2}{3}(-f + f'x) + BC(-f' + 4\frac{f}{x}) - BD(3 - 4RA)\frac{6f}{x^2}]mG_2^m \\
&+ BED[\frac{G_2^m}{2} + \frac{1}{3}(3f + f'x)(B_1^m G_2^m + 6A_3^m H_2^m - 2G_2^m) - \frac{1}{3}fG_2^m] \\
A(8,2) &= -3\frac{f}{x}(1 - 2RA)mG_2^m \\
A(8,3) &= [BA(fx)' + BB(4f) - BC(6\frac{f}{x}) + BD(11 - 12RA)6\frac{f}{x^2}]mG_2^m \\
&+ BED[\frac{3}{2}G_2^m + \frac{1}{3}(xf' + 3f)(3B_1^m G_2^m - 12A_3^m H_2^m + 3G_2^m) + f(2 - m^2)G_2^m] \\
A(8,4) &= 3f'mG_2^m \\
A(8,5) &= ROR(f')mG_2^m \\
A(8,6) &= ROA \cdot fmG_2^m \\
A(8,7) &= -BA[1 + \frac{1}{3}(xf)'B_1^m] + BE[\frac{1}{2} - (2f + xf')B_1^m - \frac{1}{3}(xf)'] \\
A(8,8) &= -\frac{3}{x} - f'B_1^m \\
A(8,9) &= [BA\frac{4}{3}(fx)' + BD(40\frac{f}{x^2})]A_3^m - BE\frac{1}{3}[(f + 3xf')A_3^m] \\
A(8,10) &= 4A_3^m f' \qquad (5.14) \\
A(9,1) &= A(7,1)\frac{H_2^m}{6G_2^m} \\
A(9,2) &= A(7,2)\frac{H_2^m}{6G_2^m} \\
A(9,3) &= -\frac{1}{6}f'mH_2^m \\
A(9,4) &= A(7,4)\frac{H_2^m}{6G_2^m} \\
A(9,5) &= A(9,6) = 0 \\
A(9,7) &= -\frac{2}{9}f'C_1^m \\
A(9,8) &= -DIBD\frac{2}{9}(fx)'C_1^m \\
A(9,9) &= \frac{1}{x} - (5\frac{f}{x} + \frac{1}{2}f')B_3^m
\end{aligned}
$$

$$
\begin{aligned}
A(9,10) &= DIBD[1-\frac{1}{2}(fx)'B_3^m] && (5.15)\\
A(10,1) &= [BA(\frac{f}{6})+BB\frac{1}{9}(-f+xf')+BC(\frac{2}{3}\frac{f}{x}-\frac{1}{6}f')-BD(7-6RA)\frac{2}{3}\frac{f}{x^2}]mH_2^m\\
&\quad +BED[-\frac{1}{3}H_2^m+\frac{1}{18}(3f+xf')(C_1^mG_2^m+6B_3^mH_2^m-2H_2^m)+\frac{2}{9}fH_2^m]\\
A(10,2) &= -\frac{1}{2}\frac{f}{x}(1-2RA)mH_2^m\\
A(10,3) &= [BA\frac{1}{6}(fx)'+BB\frac{2}{3}f-BC\frac{f}{x}+BD(23-36RA)\frac{f}{3x^2}-BD\frac{5}{3}\frac{f'}{x}]mH_2^m\\
&\quad + BED[\frac{2}{3}H_2^m+\frac{1}{18}(3f+xf')(3C_1^mG_2^m-12B_3^mH_2^m-2H_2^m)+\frac{1}{18}f(16-3m^2)H_2^m]\\
A(10,4) &= \frac{1}{2}f'mH_2^m\\
A(10,5) &= ROR\frac{1}{6}f'mH_2^m\\
A(10,6) &= ROA\frac{1}{6}fmH_2^m\\
A(10,7) &= [BA\frac{2}{9}(xf)'+BD(\frac{20}{9}\frac{f}{x^2})]C_1^m-BE\frac{1}{18}[(11f+3xf')C_1^m]\\
A(10,8) &= (\frac{20}{9}\frac{f}{x}+\frac{2}{3}f')C_1^m\\
A(10,9) &= BA[-1+\frac{1}{2}(fx)'B_3^m]+BD(\frac{10}{x^2}+20\frac{f}{x^2}B_3^m)\\
&\quad + BE[\frac{1}{12}-\frac{1}{18}(fx)'-\frac{1}{6}(2f+xf')B_3^m]\\
A(10,10) &= -\frac{3}{x}+\frac{3}{2}[\frac{10}{3}\frac{f}{x}+f']B_3^m && (5.16)
\end{aligned}
$$

where

$$
\begin{aligned}
RA &= \frac{\mu}{\lambda+2\mu}, \qquad ETAIC = \frac{4\pi G a\rho_0(a_+)}{g_0(a)}\\
DIAD &= \frac{\rho(a_+)g_0(a)R}{\lambda+2\mu}, \qquad DIBD = \frac{\rho_0(a_+)g_0(a)R}{\mu}\\
ROR &= \frac{\rho_0}{\rho_0(a_+)}, \qquad ROA = \frac{\rho_0}{\rho_0(a_+)}\frac{R}{a}\\
BA &= \frac{\rho_0}{\rho_0(a_+)}\frac{\omega^2 R}{g_0(a)}, \qquad BB = \frac{\rho_0}{\rho_0(a_+)}\frac{\Omega^2 R}{g_0(a)}\\
BC &= \frac{\rho_0}{\rho_0(a_+)}\frac{g_0}{g_0(a)}, \qquad BD = \frac{\mu}{\rho(a_+)g_0(a)R}
\end{aligned}
$$

$$BE = \frac{\rho_0}{\rho_0(a_+)} \frac{R}{g_0(a)} 2\omega\Omega m, \qquad BED = \frac{BE}{m} \tag{5.17}$$

In the mantle, these ODEs will be modified by changing the scaling parameters from those in (5.3) to those in (5.4). The coefficients $A(i,j)$ will be modified by changing the parameters in (5.17) into:

$$\begin{aligned}
RA &= \frac{\mu}{\lambda + 2\mu}, \qquad ETAIC = \frac{4\pi G b \rho_0(b_-)}{g_0(b)} \\
DIAD &= \frac{\rho(b_-) g_0(b) R}{\lambda + 2\mu}, \qquad DIBD = \frac{\rho_0(b_-) g_0(b) R}{\mu} \\
ROR &= \frac{\rho_0}{\rho_0(b_-)}, \qquad ROA = \frac{\rho_0}{\rho_0(b_-)} \frac{R}{b} \\
BA &= \frac{\rho_0}{\rho_0(b_-)} \frac{\omega^2 R}{g_0(b)}, \qquad BB = \frac{\rho_0}{\rho_0(b_-)} \frac{\Omega^2 R}{g_0(b)} \\
BC &= \frac{\rho_0}{\rho_0(b_-)} \frac{g_0}{g_0(b)}, \qquad BD = \frac{\mu}{\rho(b_-) g_0(b) R} \\
BE &= \frac{\rho_0}{\rho_0(b_-)} \frac{R}{g_0(b)} 2\omega\Omega m, \qquad BED = \frac{BE}{m}
\end{aligned} \tag{5.18}$$

In the liquid outer core, the governing equations include four ODEs (for the first derivative of y_1, y_2, y_5, y_6) and three algebraic equations. They have the forms:

$$\begin{aligned}
\frac{dy_i}{dx} &= \sum_{j=1}^{9} A(i,j) y_j \\
(i &= 1,2,5,6); (j = 1,2,3,5,6,7,9)
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
0 &= \sum_{j=1}^{9} A'(i,j) y_j \\
(i &= 4,8,10); (j = 1,2,3,5,6,7,9)
\end{aligned} \tag{5.20}$$

In the above equations, the variables y_4, y_8 and y_{10} have been expressed by y_2 according to the following relations (see chapter 4)

$$y_4 = \frac{f}{3} B_2^m y_2$$

$$y_8 = -fmG_2^m y_2$$
$$y_{10} = -\frac{1}{6}fmH_2^m y_2 \quad (5.21)$$

The coefficients $A(i,j)$ and $A'(i,j)$ are:

$$A(1,1) = -\frac{2}{x} + (\frac{4}{3}f' + 2\frac{f}{x})B_2^m$$
$$A(1,2) = DOAD[1 - \frac{2}{3}(xf)'B_2^m]$$
$$A(1,3) = \frac{6}{x} + (-2f' + 4\frac{f}{x})B_2^m$$
$$A(1,5) = A(1,6) = 0$$
$$A(1,7) = [-2f' - 4\frac{f}{x}]mH_1^m$$
$$A(1,9) = A(1,7)\frac{G_3^m}{H_1^m} \quad (5.22)$$
$$A(2,1) = BA[-1 + \frac{2}{3}(xf)'B_2^m] + BB[\frac{2}{3} + (\frac{4}{3}f'x - \frac{8}{9}f)B_2^m] + BC[-\frac{4}{x} + \frac{10}{3}\frac{f}{x}B_2^m]$$
$$A(2,2) = 0$$
$$A(2,3) = BA[2fB_2^m] + BB[-4 - (\frac{16}{3}f + \frac{4}{3}f'x)B_2^m] + BC[\frac{6}{x} + (8\frac{f}{x} + 2f')B_2^m]$$
$$+BE[1 - \frac{2}{3}(xf)'B_2^m]$$
$$A(2,5) = ROR[2\frac{f}{x}B_2^m]$$
$$A(2,6) = ROA[-1 + \frac{2}{3}(xf)'B_2^m]$$
$$A(2,7) = [BA(-2f) + BB(\frac{8}{3}f + \frac{4}{3}f'x) + BC\cdot 6(-\frac{2}{3}\frac{f}{x} - \frac{1}{3}f')]mH_1^m$$
$$+BED[H_1^m + \frac{2}{3}(xf)'(2C_0^m G_1^m - B_2^m H_1^m)]$$
$$A(2,9) = [BA(-2f) + BB(\frac{8}{3}f + \frac{4}{3}f'x) + BC\cdot 6(-\frac{2}{3}\frac{f}{x} - \frac{1}{3}f')]mG_3^m$$
$$+BED[-4G_3^m + \frac{2}{3}(xf)'(4B_2^m G_3^m - 3A_4^m H_3^m)] \quad (5.23)$$
$$A(5,1) = ETAIC\cdot ROA[1 - \frac{2}{3}(xf)'B_2^m]$$

$$
\begin{aligned}
A(5,2) &= A(5,3) = 0 \\
A(5,5) &= -\frac{2f}{x}B_2^m \\
A(5,6) &= \frac{R}{a}[1 - \frac{2}{3}(xf)'B_2^m] \\
A(5,7) &= A(5,9) = 0
\end{aligned} \tag{5.24}
$$

$$
\begin{aligned}
A(6,1) &= A(6,2) = 0 \\
A(6,3) &= -ETAIC \cdot 2ROR[\frac{3}{x} + (\frac{2f}{x} - f')B_2^m] \\
A(6,5) &= \frac{a}{R}[\frac{6}{x^2} + 2(\frac{3f}{x^2} - \frac{f'}{x})B_2^m] \\
A(6,6) &= -\frac{2}{x} + (2\frac{f}{x} + \frac{4}{3}f')B_2^m \\
A(6,7) &= ETAIC \cdot 2ROR(2\frac{f}{x} + f')mH_1^m \\
A(6,9) &= A(6,7)\frac{G_3^m}{H_1^m}
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
A'(4,1) &= -BB[\frac{2}{3} + \frac{2}{9}(xf' + 2f)B_2^m] + BC[\frac{1}{x} + (\frac{1}{3}f' + \frac{2}{3}\frac{f}{x})B_2^m] \\
&\quad +BE[\frac{1}{6} - \frac{1}{9}(xf' + 3f)B_2^m - \frac{1}{9}f] \\
A'(4,2) &= -\frac{1}{x} + \frac{1}{3}f'B_2^m \\
A'(4,3) &= BA[-1 + \frac{1}{3}(xf)'B_2^m] + BE\{\frac{1}{6} - \frac{1}{9}[3(3f + xf')B_2^m + (xf)']\} \\
A'(4,5) &= ROR[-\frac{1}{x} + \frac{1}{3}f'B_2^m] \\
A'(4,6) &= 0 \\
A'(4,7) &= [BA\frac{1}{3}(xf)'mH_1^m + BED[\frac{H_1^m}{2} - \frac{1}{9}(xf)'(-4C_0^mG_1^m + 5B_2^mH_1^m + H_1^m) \\
&\quad +\frac{1}{3}f(m^2 - B_2^m)H_1^m] \\
A'(4,9) &= [BA\frac{1}{3}(xf)'mG_3^m + BED[\frac{4}{3}G_3^m - \frac{1}{9}(xf)'(21A_4^mH_3^m - 4G_3^m) \\
&\quad +\frac{1}{3}f(m^2 + 4B_2^m)G_3^m]
\end{aligned} \tag{5.26}
$$

$$
\begin{aligned}
A'(8,1) &= [BB\frac{2}{3}f'x + BC(-f')]mG_2^m + BED[\frac{G_2^m}{2} \\
&\quad +\frac{1}{3}(3f+f'x)(B_1^mG_2^m + 6A_3^mH_2^m - 2G_2^m) - \frac{1}{3}fG_2^m] \\
A'(8,2) &= f'mG_2^m \\
A'(8,3) &= BA(fx)'mG_2^m + BED[\frac{3}{2}G_2^m \\
&\quad +\frac{1}{3}(xf'+3f)(3B_1^mG_2^m - 12A_3^mH_2^m + 3G_2^m) + 2fG_2^m] \\
A'(8,5) &= ROR(f')mG_2^m \\
A'(8,6) &= 0 \\
A'(8,7) &= -BA[1+\frac{1}{3}(xf)'B_1^m] + BE[\frac{1}{2} - (2f+xf')B_1^m - \frac{1}{3}(xf)' + fG_2^mH_1^m] \\
A'(8,9) &= BA\frac{4}{3}(fx)'A_3^m - BE\frac{1}{3}[(f+3xf')A_3^m + 12fG_2^mG_3^m] \qquad (5.27) \\
A'(10,1) &= [BB\frac{1}{9}xf' - BC\frac{1}{6}f']mH_2^m + BED[-\frac{1}{3}H_2^m \\
&\quad +\frac{1}{18}(3f+xf')(C_1^mG_2^m + 6B_3^mH_2^m - 2H_2^m) + \frac{2}{9}fH_2^m] \\
A'(10,2) &= \frac{1}{6}f'mH_2^m \\
A'(10,3) &= BA\frac{1}{6}(fx)'mH_2^m + BED[\frac{2}{3}H_2^m \\
&\quad +\frac{1}{18}(3f+xf')(3C_1^mG_2^m - 12B_3^mH_2^m - 2H_2^m) + \frac{8}{9}fmH_2^m] \\
A'(10,5) &= ROR\frac{1}{6}f'mH_2^m \\
A'(10,6) &= 0 \\
A'(10,7) &= BA\frac{2}{9}(xf)'C_1^m - BE\frac{1}{18}[(11f+3xf')C_1^m - 3fH_2^mH_1^m] \\
A'(10,9) &= BA[-1+\frac{1}{2}(fx)'B_3^m] + BE[\frac{1}{12} - \frac{1}{18}(fx)' - \frac{1}{6}(2f+xf')B_3^m \\
&\quad -\frac{2}{3}fH_2^mG_3^m] \qquad (5.28)
\end{aligned}
$$

where

$$\begin{aligned}
DOAD &= \frac{\rho(a_+)g_0(a)R}{\lambda}, \qquad ETAIC = \frac{4\pi G a \rho_0(a_+)}{g_0(a)} \\
ROR &= \frac{\rho_0}{\rho_0(a_+)}, \qquad ROA = \frac{\rho_0}{\rho_0(a_+)}\frac{R}{a} \\
BA &= \frac{\rho_0}{\rho_0(a_+)}\frac{\omega^2 R}{g_0(a)}, \qquad BB = \frac{\rho_0}{\rho_0(a_+)}\frac{\Omega^2 R}{g_0(a)} \\
BC &= \frac{\rho_0}{\rho_0(a_+)}\frac{g_0}{g_0(a)}, \qquad BE = \frac{\rho_0}{\rho_0(a_+)}\frac{R}{g_0(a)}2\omega\Omega m \\
BED &= \frac{BE}{m}
\end{aligned} \tag{5.29}$$

The boundary conditions in Chapter 3 can also be written in dimensionless AJP notations as:

At the solid/solid interfaces, all variables $y_1, y_2, \cdots, y_{10}$ are continuous;

At the solid/fluid interfaces, y_1, y_2, y_5, y_6 are continuous, and

$$\begin{aligned}
(y_4)_{solid} &= (y_4)_{liquid} = \frac{f}{3}B_2^m(y_2)_{liquid} \\
(y_8)_{solid} &= (y_8)_{liquid} = -fmG_2^m(y_2)_{liquid} \\
(y_{10})_{solid} &= (y_{10})_{liquid} = -\frac{f}{6}mH_2^m(y_2)_{liquid}
\end{aligned} \tag{5.30}$$

At the Earth's surface,

$$\begin{aligned}
y_2, \ y_4, \ y_8, \ y_{10} &= 0 \\
y_5 + \frac{R}{3b}y_6 &= 0
\end{aligned} \tag{5.31}$$

5.2 Starting Solutions at the Geocentre

The above set of coupled ODEs need to be integrated from the Earth's centre to surface using the boundary conditions at the Earth's internal and external boundaries.

In this section, we will derive the starting solutions at the Earth's centre, a singular point of the ODE system, by generalizing the method of Crossley (1975).

First of all we need to know the number of regular starting solutions at the geocentre. i.e. which can be represented by power series expansions. Suppose there are N independent solutions at the geocentre. When integrating the ODEs across the Earth. N free constants will be reduced to $N-3$ at the inner-core boundary because of three BCs there. At the core-mantle coundary, they will be increased to N again because no BCs constrain y_3, y_7, y_9 there. At the Earth's surface, N free constants need to satisfy five BCs, then, a unique solution is $N = 5$. Therefore, in the vicinity of $r = 0$, we expand the solutions y_i, $(i = 1, 10)$ by power series as,

$$\begin{aligned} y_i &= \sum_{k=0}^{\infty} C_{i,k} x^{\nu+k} \\ (i &= 1, 10) \end{aligned} \tag{5.32}$$

To decide the coefficients $C_{i,k}$ and ν, we substitute this expansion into 10 ODEs in solid inner core (5.7-5.16) and group the terms with same power of variable x from the lowest order. Then we obtain their forms as,

$$y_1 = Ax + A'x^3 + \cdots \tag{5.33}$$

$$y_2 = B + B'x^2 + \cdots \tag{5.34}$$

$$y_3 = Cx + C'x^3 + \cdots \tag{5.35}$$

$$y_4 = D + D'x^2 + \cdots \tag{5.36}$$

$$y_5 = Ex^2 + E'x^4 + \cdots \tag{5.37}$$

$$y_6 = Fx + F'x^3 + \cdots \tag{5.38}$$

$$\begin{aligned} y_7 &= Gx + G'x^3 + \cdots && (5.39) \\ y_8 &= H + H'x^2 + \cdots && (5.40) \\ y_9 &= Ix^3 + \cdots && (5.41) \\ y_{10} &= Jx^2 + \cdots && (5.42) \end{aligned}$$

where there are only five independent constants , which are chosen as A, F, D', G, I. Each non-zero constant will be associated with a set of starting solutions. The solution with $A = 1$ is written as $y^{(1)}$,

$$\begin{aligned} A &= 1 \\ B &= 2\frac{\mu(0)}{\rho_0(a_+)g_0(a)R}(1 + \frac{2}{3}fB_2^m)A \\ C &= \frac{1}{2}(1 - \frac{4}{3}fB_2^m)A \\ D &= \frac{\mu(0)}{\rho_0(a_+)g_0(a)R}(1 + \frac{4}{3}fB_2^m)A \\ E &= 2\pi G\frac{\rho_0(0)R}{g_0(a)}(1 - \frac{5}{3}fB_2^m)A && (5.43) \end{aligned}$$

where $\mu(0)$ and $\rho_0(0)$ are elastic parameters at the Earth's centre. All other constants, $F, D', \cdots, G, I, J$ are zero.

When $F = 1$, we have a set of solutions written as $y^{(2)}$ with coefficients,

$$\begin{aligned} F &= 1 \\ E &= \frac{1}{2}\frac{R}{a}(1 - \frac{5}{3}fB_2^m)F && (5.44) \end{aligned}$$

All other constants, $A, D', \cdots, G, I, J$ are zero.

When $D' = 1$, we have a set of solutions written as $y^{(3)}$ with coefficients,

$$D' = 1$$

$$
\begin{aligned}
A' &= \frac{\rho_0(a_+)g_0(a)R}{\mu(0)}\frac{3(RA-2)}{8RA-9}[1+\frac{1}{3}fB_2^m\frac{8(RA)^2+10RA-3}{(8RA-9)(RA-2)}]D' \\
B' &= \frac{3(RA-2)}{8RA-9}[-1+\frac{1}{3}fB_2^m\frac{144(RA)^2-317RA+156}{(8RA-9)(RA-2)}]D' \\
C' &= \frac{\rho_0(a_+)g_0(a)R}{\mu(0)}\frac{5RA-3}{2(8RA-9)}[1-\frac{2}{3}fB_2^m\frac{44(RA)^2-57RA+36}{(8RA-9)(5RA-3)}]D' \\
E' &= -\frac{\rho_0(a_+)g_0(a)R}{\mu(0)}\frac{4\pi G\rho_0(0)R}{3g_0(a)}\frac{3}{2(8RA-9)}[1-fB_2^m\frac{16RA-11}{8RA-9}]D' \\
F' &= -\frac{\rho_0(a_+)g_0(a)a}{\mu(0)}\frac{4\pi G\rho_0(0)R}{3g_0(a)}\frac{3RA}{8RA-9}[1+\frac{2}{3}fB_2^m\frac{4RA-15}{8RA-9}]D' \qquad (5.45)
\end{aligned}
$$

where $RA = \frac{\lambda(0)+2\mu(0)}{\mu(0)}$. All other constants, $A, F, G, \cdots, I, J$, are zero.

When $G = 1$, we have a set of solutions written as $y^{(4)}$ with coefficients,

$$
\begin{aligned}
G &= 1 \\
H &= 0 \\
H' &= -\frac{1}{5}\{\frac{\omega^2 R}{g_0(a)}\frac{\rho_0(0)}{\rho_0(a_+)}(1+\frac{1}{3}fB_1^m) + \frac{2\omega\Omega m R}{g_0(a)}\frac{\rho_0(0)}{\rho_0(a_+)} \\
&\quad [-\frac{1}{2}+\frac{1}{3}f(1+6B_1^m)]\}G \qquad (5.46)
\end{aligned}
$$

All other constants, $A, F, D', \cdots, I, J$, are zero.

When $I = 1$, we have a set of solutions written as $y^{(5)}$ with coefficients,

$$
\begin{aligned}
I &= 1 \\
J &= \frac{2\mu(0)}{\rho_0(a_+)g_0(a)R}(1+3fB_3^m)I \qquad (5.47)
\end{aligned}
$$

All other constants, $A, F, D', \cdots, G$, are zero.

Therefore, the starting solution at the Earth's center is:

$$
y = Ay^{(1)} + Fy^{(2)} + D'y^{(3)} + Gy^{(4)} + Iy^{(5)} \qquad (5.48)
$$

We do not need to consider the terms with higher power of variable x than that appearing in (5.30)-(5.39) because their coefficients will be dependent on A, F, D', G, I.

5.3 Earth Model

In this thesis, we adopt the Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson 1981) with effects of dispersion included, and the upper mantle treated as isotropic. As explained in Chapter 1, we apply the material properties of this spherically-layered Earth model to represent those of a rotating, oblate Earth model as functions of mean radius r_0. We use the polynomial expressions of Dziewonski & Anderson (1981, Table I and footnote thereto) for the density ρ_0, the P-wave and S-wave velocities α and β:

$$\begin{aligned} \rho_0(x) &= c_1 + c_2 x + c_3 x^2 + c_4 x^3 \\ \alpha(x) &= a_1 + a_2 x + a_3 x^2 + c_4 x^3 \\ \beta(x) &= b_1 + b_2 x + b_3 x^2 + b_4 x^4 \end{aligned} \quad (5.49)$$

where coefficients $a_1, b_1, c_1, \cdots, a_n, b_n, c_n$ are listed in Table 5.1 and 5.2. Only the top layer of PREM, i.e. an oceanic layer, is modified to be a solid outer layer as Rochester & Peng (1993) did. They adjusted V_p and V_s velocities to preserve both the P-wave travel time to 15 km depth and the ratio $\frac{V_p}{V_s}$ for depths between 3 and 15 km, and modified the density to keep the same mass and moment of inertia as PREM. Due to the effects of dispersion, α and β need to be modified to α_0 and β_0,

$$\alpha = \alpha(x)\{1 - \frac{\ln T}{\pi}[\frac{(1-E)}{q_k} + \frac{E}{q_\mu}]\} \quad (5.50)$$

$$\beta = \beta(x)[1 - \frac{\ln T}{\pi q_\mu}] \quad (5.51)$$

where q_κ, q_μ are dispersive parameters for PREM (listed in Table 5.3) and $E = \frac{4}{3}(\frac{\beta(x)}{\alpha(x)})^2$. The Lamé parameters λ and μ can be formed from α and β:

$$\lambda = \rho_0(\alpha^2 - 2\beta^2) \tag{5.52}$$

$$\mu = \rho_0\beta^2 \tag{5.53}$$

Another quantity i.e. the gravitational field for a non-rotating Earth model $g_0(r)$. can be produced by the mass inside the equipotential level through this point, i.e. by knowing the density distribution.

However, the ellipticity for this oblate Earth was not given in Dziewonski & Anderson (1981). We calculate the ellipticity data based on the material properties in the next section.

5.3.1 Ellipticity of the Preliminary Reference Earth Model (PREM)

The Clairaut equation gives the ellipticity of the equipotential surfaces of a hydrostatically prestressed Earth as,

$$\frac{d^2f}{dr_0^2} + \frac{6}{r_0}\frac{\rho_0}{\bar{\rho}_0}\frac{df}{dr_0} + \frac{6}{r_0^2}[\frac{\rho_0}{\bar{\rho}_0} - 1]f = 0 \tag{5.54}$$

where f is the ellipticity of the equipotential with mean radius r_0 (Jeffreys 1970).

As discussed in Chapter 4, we modify this equation to (4.12),

$$\frac{d^2f}{dr_0^2} + \frac{6}{r_0}[\frac{\rho_0 - \sigma}{\bar{\rho}_0 - \sigma}]\frac{df}{dr_0} + \frac{6}{r_0^2}[\frac{\rho_0 - \sigma}{\bar{\rho}_0 - \sigma} - 1]f = 0 \tag{5.55}$$

where ρ_0 and $\bar{\rho}_0$ are replaced by $\rho_0 - \sigma$ and $\bar{\rho}_0 - \sigma$ respectively.

When integrating this second-order ODE from the Earth's centre to surface by Runge-Kutta integration method, the initial values of f and $\frac{df}{dr}$ need to be chosen to

Table 5.1: The coefficients for density (PREM) (gm/cm^3)

$r(km)$	c_1	c_2	c_3	c_4
0	13.0885	0	-8.8381	0
1221.5	12.5815	-1.2638	-3.6426	-5.5281
3480	7.9565	-6.4761	5.5283	-3.0807
5701	5.3197	-1.4836	0	0
5771	11.2494	-8.0298	0	0
5971	7.1089	-3.8045	0	0
6151	2.6910	0.6924	0	0
6346.6	2.900	0	0	0
6362.2	1.853	0	0	0

Table 5.2: The coefficients for P- and S- wave velocities (PREM) (km/s)

$r(km)$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	11.2622	0	-6.3640	0	3.6678	0	-4.4475	0
1221.5	11.0487	-4.0362	4.8023	-13.5732	0	0	0	0
3480	15.3891	-5.3181	5.5242	-2.5514	6.9254	1.4672	-2.0834	0.9783
3630	24.9520	-40.4673	51.4832	-26.6419	11.1671	-13.7818	17.4575	-9.2777
5600	29.2766	-23.6027	5.5242	-2.5514	22.3459	-17.2473	-2.0834	0.9783
5701	19.0957	-9.8672	0	0	9.9839	-4.9324	0	0
5771	39.7027	-32.6166	0	0	22.3512	-18.5856	0	0
5971	20.3926	-12.2569	0	0	8.9496	-4.4597	0	0
6151	4.1875	3.9382	0	0	2.1519	2.3481	0	0
6346.6	6.800	0	0	0	3.900	0	0	0
6362.2	2.728	0	0	0	1.505	0	0	0

Table 5.3: The dispersive constants (PREM)

$r(km)$	q_κ	q_μ
0	1327.7	84.6
1221.5	57823.0	∞
3480	57823.0	312.0
5701	57823.0	143.0
6151	57823.0	80.0
6291	57823.0	600

Table 5.4: Ellipticity and its second-order derivative for PREM

r(km)	f	f"
0.00	0.2408278E-02	0.5580976E-03
100.00	0.2408347E-02	0.5583393E-03
200.00	0.2408554E-02	0.5590651E-03
300.00	0.2408898E-02	0.5602765E-03
400.00	0.2409380E-02	0.5619769E-03
500.00	0.2410000E-02	0.5641707E-03
600.00	0.2410760E-02	0.5668633E-03
700.00	0.2411659E-02	0.5700617E-03
800.00	0.2412699E-02	0.5737743E-03
900.00	0.2413880E-02	0.5780106E-03
1000.00	0.2415203E-02	0.5827819E-03
1100.00	0.2416670E-02	0.5881007E-03
1200.00	0.2418282E-02	0.5939812E-03
1221.50	0.2418648E-02	0.5953204E-03
1221.50	0.2418648E-02	0.1905970E-01
1300.00	0.2421170E-02	0.9695567E-02
1400.00	0.2426472E-02	0.3575184E-02
1500.00	0.2432721E-02	0.7523793E-03
1600.00	0.2439187E-02	-0.4774150E-03
1700.00	0.2445551E-02	-0.9336708E-03
1800.00	0.2451693E-02	-0.1018499E-02
1900.00	0.2457587E-02	-0.9317544E-03
2000.00	0.2463253E-02	-0.7730724E-03
2100.00	0.2468729E-02	-0.5915888E-03
2200.00	0.2474059E-02	-0.4107302E-03
2300.00	0.2479287E-02	-0.2408042E-03
2400.00	0.2484457E-02	-0.8548192E-04
2500.00	0.2489604E-02	0.5483607E-04
2600.00	0.2494765E-02	0.1812705E-03
2700.00	0.2499970E-02	0.2955611E-03
2800.00	0.2505248E-02	0.3996126E-03
2900.00	0.2510625E-02	0.4952768E-03
3000.00	0.2516123E-02	0.5842594E-03
3100.00	0.2521765E-02	0.6680907E-03
3200.00	0.2527571E-02	0.7481288E-03
3300.00	0.2533562E-02	0.8255769E-03
3400.00	0.2539756E-02	0.9015067E-03
3480.00	0.2544870E-02	0.9618089E-03

r(km)	f	f"
3480.00	0.2544870E-02	0.2298054E-01
3530.00	0.2548799E-02	0.2092536E-01
3580.00	0.2554018E-02	0.1902547E-01
3630.00	0.2560409E-02	0.1726979E-01
3630.00	0.2560409E-02	0.1726979E-01
3700.00	0.2571122E-02	0.1503465E-01
3800.00	0.2589518E-02	0.1224648E-01
3900.00	0.2610940E-02	0.9875546E-02
4000.00	0.2634803E-02	0.7864417E-02
4100.00	0.2660609E-02	0.6163397E-02
4200.00	0.2687940E-02	0.4729445E-02
4300.00	0.2716440E-02	0.3525281E-02
4400.00	0.2745813E-02	0.2518624E-02
4500.00	0.2775809E-02	0.1681542E-02
4600.00	0.2806223E-02	0.9898841E-03
4700.00	0.2836884E-02	0.4227921E-03
4800.00	0.2867650E-02	-0.3772735E-04
4900.00	0.2898410E-02	-0.4071769E-03
5000.00	0.2929070E-02	-0.6988957E-03
5100.00	0.2959560E-02	-0.9243445E-03
5200.00	0.2989824E-02	-0.1093356E-02
5300.00	0.3019818E-02	-0.1214351E-02
5400.00	0.3049515E-02	-0.1294531E-02
5500.00	0.3078893E-02	-0.1340039E-02
5600.00	0.3107942E-02	-0.1356106E-02
5600.00	0.3107942E-02	-0.1356106E-02
5630.00	0.3116592E-02	-0.1355869E-02
5670.00	0.3128078E-02	-0.1352244E-02
5701.00	0.3136943E-02	-0.1346973E-02
5701.00	0.3136943E-02	0.8641343E-03
5725.00	0.3143799E-02	0.7582977E-03
5750.00	0.3150953E-02	0.6519260E-03
5771.00	0.3156970E-02	0.5655448E-03
5771.00	0.3156970E-02	0.5655462E-03
5840.00	0.3176787E-02	0.6887283E-03
5900.00	0.3194084E-02	0.7956095E-03

r(km)	f	f"
5971.00	0.3214644E-02	0.9220681E-03
5971.00	0.3214644E-02	0.1945158E-02
6030.00	0.3231855E-02	0.1782235E-02
6090.00	0.3249515E-02	0.1631454E-02
6151.00	0.3267618E-02	0.1492472E-02
6151.00	0.3267618E-02	0.1927945E-02
6190.00	0.3279271E-02	0.1677669E-02
6250.00	0.3297320E-02	0.1312538E-02
6291.00	0.3309721E-02	0.1076416E-02
6291.00	0.3309721E-02	0.1076416E-02
6310.00	0.3315484E-02	0.9705633E-03
6330.00	0.3321560E-02	0.8615311E-03
6346.60	0.3326609E-02	0.7728700E-03
6346.60	0.3326609E-02	0.3533490E-02
6350.00	0.3327644E-02	0.3510797E-02
6355.00	0.3329168E-02	0.3477551E-02
6362.20	0.3331367E-02	0.3429941E-02
6362.20	0.3331367E-02	0.9454379E-02
6365.00	0.3332224E-02	0.9430146E-02
6368.00	0.3333144E-02	0.9404226E-02
6371.00	0.3334066E-02	0.9378352E-02

ensure that the solution is regular at the geocentre, which is a regular singular point of Clairaut's equation. $\frac{df}{dr}$ at the geocentre is zero because f is a continuous function there. If f at the geocentre is assumed to be 1, the computed results need to be scaled by the ellipticity at the Earth's outer surface, which is the first integral of the Clairaut equation (Darwin 1899).

This computation produces a set of pointwise numerical data for ellipticity and its second-order derivative, listed in Table 5.4. We can interpolate this set of data by the method of cubic splines to give the ellipticity and its derivative at any position x. Discrepancies between the values of ellipticity from (5.55) and from (5.54) are no more than the order of σ, i.e. 1 in 300.

5.4 Programming Flow Scheme

I have designed a numerical approach to solve the governing ODEs (whose coefficients are derived in Section 5.1) with boundary conditions as set out at the end of Section 5.1. using starting conditions derived in Section 5.2. I built the program on a much simpler program created by Dr. Rochester in 1991 to find the eigenperiods of spheroidal free oscillations of a non-rotating spherical Earth model. That program involved integrating only 6 coupled ODEs with 6 dependent variables in the solid parts of the Earth, and 4 ODEs plus 1 algebraic equation (AE) in the outer core. The truncation scheme adopted in this thesis requires that differential system to be extended in two ways:

1. 10 coupled ODEs with 10 dependent variables are needed in the inner core and mantle, and 4 ODEs plus 3 AEs in the outer core;

2. The presence of ellipticity and rotation makes the coefficients of this 10×10 system far more complicated.

The linear combination of 5 fundamental solutions integrated across the mantle must satisfy the BCs at the Earth's surface. This requires a 5×5 determinant of appropriate coefficients to vanish. An oscillation period for which this happens is an eigenperiod.

The programming flow scheme is in Fig. 5.1.

5.5 Numerical Results and Discussion

Our plan was to compute the eigenperiods, for PREM, of the free core nutation (m=+1 for positive eigenfrequency) and Chandler wobble (m=-1 for positive eigen-

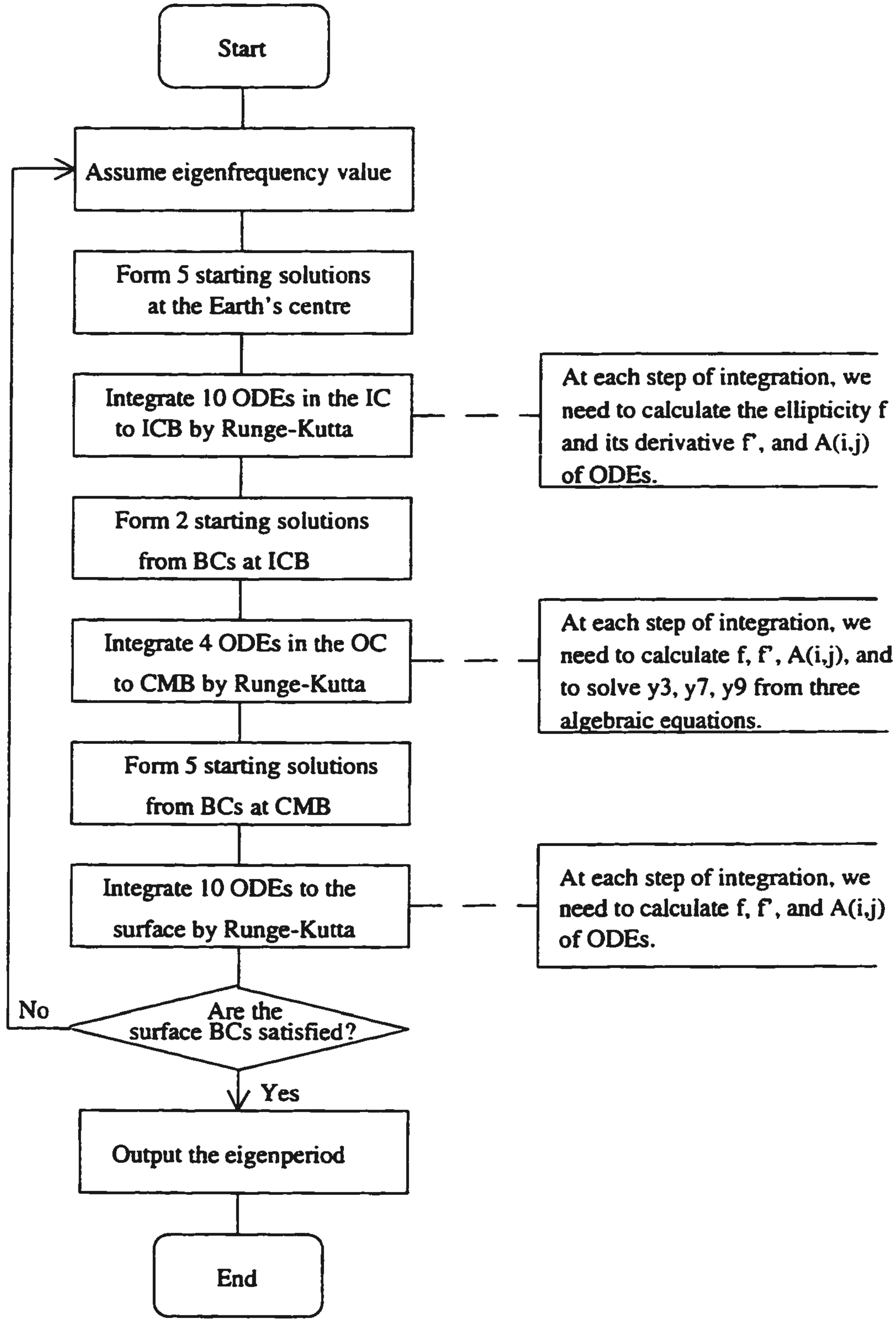

Figure 5.1: Programming Flow Chart

frequency, =+1 for negative eigenfrequency), as a test of this new description. As explained in Section 5.4, the eigenfrequency is identified by the vanishing of the 5×5 determinant formed from the BCs at the Earth's surface. The program to search for zero crossings of that determinant, based on the flow chart in Fig. 5.1, works but does not yet give satisfactory numerical results.

As observed from the terrestrial frame there is another retrograde mode, the so-called "tiltover mode" (TOM), corresponding to steady rotation of the Earth about an axis fixed in space but slightly displaced from the figure axis. Thus the TOM has eigenfrequency $\omega = -\Omega$ relative to the terrestrial frame. The TOM is of no physical interest, but its presence is a test of the computations. The search for zero values of the determinant from the surface BCs (Section 5.4), carried out by changing the (positive) input frequency ω for the case $m = +1$, should reveal both the TOM (period exactly one sidereal day = 23.93447 hr) and the FCN (period just over 3 minutes shorter than the sidereal day, i.e. close to 23.882 hr).

While two zero crossings are found near 1 sidereal day (Fig. 5.2[b]), they are not at the correct locations. Instead they are at 23.94401 hr, 23.93275 hr.

Fig. 5.2[a] is even more worrying. These is no evidence of the Chandler period in the vicinity of 400 sidereal days (viewed from the terrestrial frame).

Since great care was taken to check the mathematical derivations for this new description of wobble/nutation, it seems unlikely (though not impossible) that the cause of the numerical difficulty is an error or errors in the theory, and more likely that there remain so far undetected mistakes in programming. While I have found several such mistakes already, removing them has not solved the problem. The program would benefit from an independent check.

There is one other, perhaps more remote, possibility. The numerical convergence obtained by Smith (1977), for the heavily truncated representation of the displacement eigenfunction given by (5.2), might be due to the errors in his formulation. Perhaps the formulation in this thesis, which is more rigorous mathematically, requires that the coupling chain be extended to degree $n = 5$.

Further work on this subject should therefore:

1. make an independent check on the program for the truncation at degree $n = 3$, as in this thesis;

2. provide an independent check of the mathematical derivations;

3. continue the search for eigenperiods by extending the coupling chain to degree $n = 5$.

Project 3 would mean integrating 18 coupled ODEs in the solid parts of the Earth (6 each for the spheroidal displacement fields of degree $n = 2$ and 4, and 2 each for the toroidal displacement fields of degree $n = 1, 3$ and 5). This will not only test convergence but may also reveal extra normal modes which have been hidden by the deficiencies of conventional theory and heavy truncation. Only then will we have a firm conclusion as to the applicability of the theory based on a hydrostatic reference state.

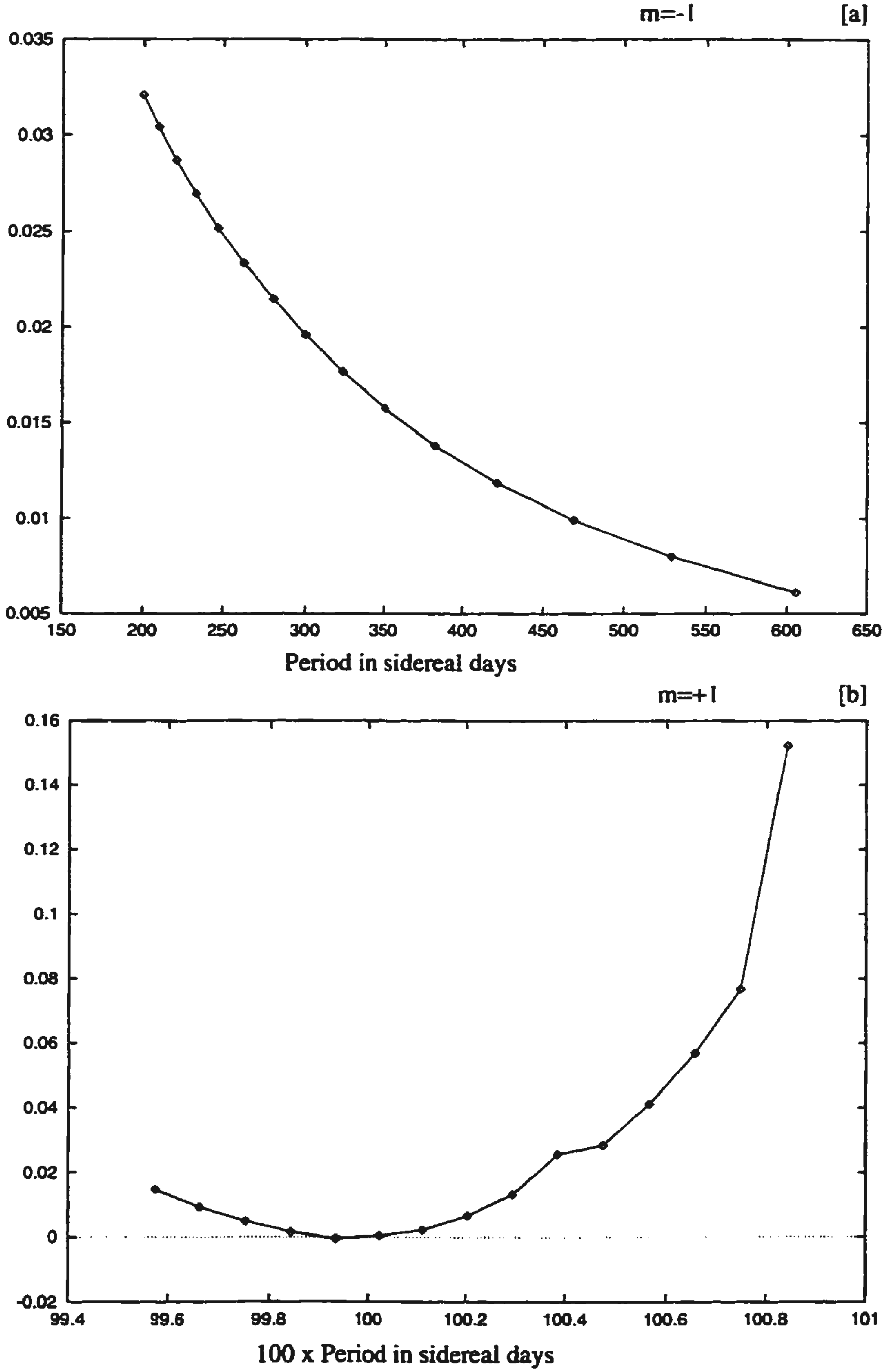

Figure 5.2: Computed determinant of the square matrix from the surface boundary conditions, as a function of period. Zero crossings denote eigenperiods.

5.6 Conclusions

The primary aim of the research program reported in this thesis was the derivation of the governing system of ODEs and BCs using Clairaut coordinates. This was successfully done (chapter 4). This description of the theory of free wobble/nutation in a hydrostatic Earth model is free of two shortcomings which mar the conventional formulation of the theory (Smith 1974), and ought to replace the latter.

A secondary objective of my research program was to obtain numerical values of eigenperiods for the FCN and Chandler wobble, in order to see the effects of this reformulation. En route to programming, I successfully obtained (Section 5.1) the coefficients of the scaled versions of the ODEs and AEs derived in Chapter 4, and also (Section 5.2) the conditions at the geocentre required to start numerical integration of the ODEs.

Finally, I built a program for computing eigenperiods. It does not yet give satisfactory numerical results. This may reflect (in decreasing order of likelihood):

- programming errors which I have not yet been able to detect;
- an error in the mathematical derivations which neither Dr. Rochester nor I have not been able to detect;
- too severe a truncation of the governing system of ODEs.

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