

# Overlapping Resonances in Open Quantum Systems

by

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# Abstract

In this thesis, we study the dynamics of open quantum systems in the *overlapping resonances regime*. An open quantum system consists of a small, finite-dimensional system  $S$  and a reservoir  $R$ , which interacts with the system  $S$  via some operator. The reservoir is, for example, modeled by a spatially infinitely extended free Bose gas in thermal equilibrium at positive temperature. One of the main tasks in the theory of open quantum system is to examine quantum phenomena observed in the small system, such as return to equilibrium and decoherence, brought about by the interaction with the environment. To deal with such a problem, one tries to study the dynamics of the total system  $S + R$ , which is generated by the total Hamiltonian. By tracing out the degrees of freedom of the reservoir, we obtain the dynamics of the small system.

Over the last decade and a half, a perturbation theory based on quantum resonance methods has been developed in the *isolated resonances regime*. The perturbation theory developed so far permits a mathematically rigorous treatment of the dynamics for fixed, small system-reservoir coupling parameter  $\lambda$ . However, in complex quantum systems, e.g. when the dimension of small system is large, the problem belongs to the overlapping resonances regime. The theory mentioned above is not applicable in this regime. In this thesis, we adapt the perturbation theory for the treatment of such a regime. We first obtain a representation formula of the reduced dynamics involving

resonances of an associated operator. By analyzing the behaviour of resonances, we derive the reduced density matrix of the small system.

Furthermore, we consider the spin-boson model and analyze the transition from the isolated resonances regime to the overlapping one by using the Feshbach map. We also find a critical value, related to the parameters, which separates the two regimes, marking a sharp transition in the behaviour of the resonance energies.

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# Chapter 1

## Introduction

The theory of open quantum systems is an important and interesting topic in the field of mathematical physics. It has been applied to many different subjects, such as quantum optics [16, 22], quantum decoherence [30], quantum information theory [32], dissipative two state systems [21], and so on. In simple words, an *open quantum system* consists of a small system  $S$  and a reservoir (an environment)  $R$ , which interacts with the system  $S$ . A physical system is *closed* if it is isolated from others. The time evolution of a closed system is given by the Schrödinger equation  $i\partial_t\psi_t = H\psi_t$ . If the initial state  $\psi_0$  is known, then the configuration at any further time  $t$  can be derived,  $\psi_t = e^{-itH}\psi_0$ . In real physical world, any system  $S$  that is not closed can be viewed as part of a larger, but closed system. If we denote the larger closed system as  $S+R$ , then  $S$  is called an *open system* and  $R$  a *reservoir* (or environment). The joint system  $S+R$  is closed and from the knowledge of it, we can obtain all the information on its subsystem  $S$  by tracing out the degrees of freedom of  $R$ . Although the total system  $S+R$  is closed, it may be very large and it is not easy to solve.

An important phenomenon observed in open systems is decoherence, which is a quantum phenomenon brought about by the coupling of a small system under con-

sideration to the environment. Decoherence can be viewed as the loss of information from the small system into the reservoir or environment. Mathematically, decoherence means that the off-diagonal density matrix elements of the open system, in the energy basis, decay to zero as time goes to infinity. Decoherence is an elementary and important phenomenon in quantum mechanics. Also decoherence is a big challenge for the realization of quantum computers because the quantum computer will change to a classical one after decoherence [30, 32]. How to preserve coherence and manage the decoherence is an important and challenging problem. Open quantum systems theory may also be applied to quantum measurement theory. In quantum mechanics, although the Copenhagen interpretation has been commonly used, there are still many different interpretations to explain how quantum mechanics informs our understanding of the world. Quantum measurement is viewed in different ways among various interpretations of quantum mechanics. Because the observed (measured) system is open, the measurement is related to the phenomenon of decoherence [7].

The theory of open quantum systems is a broad topic. There are many different approaches, including but not limited to theoretical physics approach, Hamiltonian approach and Markovian approach, to study the theory. The theoretical physics approach to the description of open systems dynamics has a long tradition. Based primarily on master equation techniques, it relies on approximations that are not controlled mathematically, but are very popular and successful nevertheless [7, 15, 19, 32, 33, 35]. A rigorous approach is the van Hove, or weak coupling limit [1, 9, 10]. It describes the dynamics of the small system for times up to the order of  $\lambda^{-2}$ , where  $\lambda$  is the strength of the system-environment coupling. Given a fixed  $\lambda$ , the time-asymptotics,  $t \rightarrow \infty$ , cannot be resolved with the weak coupling method. More recently, over the last decade and a half, a perturbation theory based on quantum resonance methods has been developed [4, 5, 12, 14, 17, 18, 23, 27, 28, 29];



see also [13] for a different approach. Spectral deformation, positive commutator and renormalization group are three important tools of quantum resonance methods to describe the dynamics of open quantum systems.

In the classical paper [17], Jakšić and Pillet developed an energy-translation deformation technique to study the dynamical behavior of the open system at positive temperature. They got isolated eigenvalues of the unperturbed *Liouville operator*, and then applied standard analytic perturbation theory to the eigenvalues. Based on the spectrum analysis, they derived the return to equilibrium property under certain conditions. They also introduced a unitary equivalent representation in the so-called *Jakšić-Pillet gluing* Hilbert space, which is more convenient for the spectral deformation technique. Bach, Fröhlich and Sigal [4] developed the dilation deformation to analyze the spectrum of the system Liouville operator. They proved the return to equilibrium for the spin-boson system uniformly in positive temperature of the reservoir. Employing the spectral deformation theory developed in [17], Merkli, Sigal and Berman [29] present a rigorous analysis of the dynamics for general  $N$ -level systems coupled to reservoirs. By applying the general theory to energy conserving interactions, they derived the explicit solutions of the reduced density matrix elements. All these works base the mathematical description of the reservoir on the so-called Araki-Woods representation of the infinitely extended free bose gas, which was first found in [2].

It should be noted that all the resonances methods developed so far are based on a perturbation theory in the system-reservoir coupling parameter  $\lambda$ . The latter is assumed to be small relative to the spacing  $\sigma > 0$  between the energy levels of the small system:  $|\lambda| \ll \sigma$ . However, in complex systems, involving many particles (many energy levels), it is not the case that  $|\lambda| \ll \sigma$ . For *complex open systems*,

e.g. where the small system itself is composed of many individual parts (particles), the energy level spacing may become very small. For instance, the Hamiltonian of a system of  $N$  spins has  $2^N$  eigenvalues and the total energy of the spins is of the order of  $N$ . The generic energy spacing is thus of the order of  $\sigma \sim N/2^N$ , which is exponentially small in  $N$ . For such systems, the condition  $|\lambda| \ll \sigma$  is not reasonable. The resonance approach developed in the *isolated resonances regime*  $|\lambda| \ll \sigma$  has to be modified for the regime  $\sigma \ll |\lambda|$ . In this thesis, a perturbation theory for the treatment of this regime is developed and is used to investigate the dynamics of an open quantum system in the *overlapping resonances regime*  $\sigma \ll |\lambda|$ . Furthermore, we describe the transition between two regimes for a spin-boson model. We find a critical value of the ratio  $\sigma/\lambda^2$  which separates the two regimes, marking a sharp transition in the behaviour of the resonance energies. In my thesis, the small system  $S$  is a finite-dimensional system and the reservoir  $R$  is a spatially infinitely extended free Bose gas in thermal equilibrium at positive temperature.

## 1.1 Heuristic explanation of main results

**Resolvent representation of the dynamics.** Let the operator  $A$  be a system-reservoir observable,  $\rho_S \otimes \rho_R$  an initial system-reservoir state. Here,  $\rho_S$  is any density matrix of  $S$  and  $\rho_R$  is the thermal equilibrium state of the reservoir at temperature  $1/\beta > 0$ . The system-reservoir dynamics is given by  $e^{-itH} \rho_S \otimes \rho_R e^{itH}$ . Here  $H$  is the system-reservoir Hamiltonian after taking the infinite volume limit. The average value of the observable  $A$  at time  $t$  is given by  $\langle A \rangle_t = \text{Tr}(e^{-itH} \rho_S \otimes \rho_R e^{itH} A)$ . In my thesis, I show the resolvent representation

$$\langle A \rangle_t = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Psi_0, B(K(\sigma, \lambda) - z)^{-1} A \Omega \rangle dz. \quad (1.1)$$

Here  $\langle \cdot, \cdot \rangle$  is the inner product in a suitable Hilbert space.  $K$  is an unbounded, non-normal operator.  $B$  is an operator depending on the initial state  $\Psi_0$  and  $\Omega$  is the vector representing the uncoupled equilibrium state. See Theorem 4.1.1 for the precise result. In particular, for mathematical rigour, we need to do some "spectrum deformation" in the theorem.

The proof of (1.1) uses the complex deformation theory, following [27, 29]. But this result is new, as in other papers, the formula is proved only for system observables  $A$  (in the bosonic case). Also, compared with the result in [27], our method reduced the regularity requirements on the interaction form factor, see remarks after Theorem 4.1.1.

**Spectral analysis of  $K(\sigma, \lambda)$ .**  $K(\sigma, \lambda)$  has spectrum in the upper complex half-plane. In the isolated resonances regime, the imaginary parts of all eigenvalues of  $K(\sigma, \lambda)$  are proportional to  $\lambda^2$ , except for a simple one at the origin, see Figure 1.1. The spectrum of  $K(\sigma, 0)$  is known explicitly, and then applying analytic perturbation

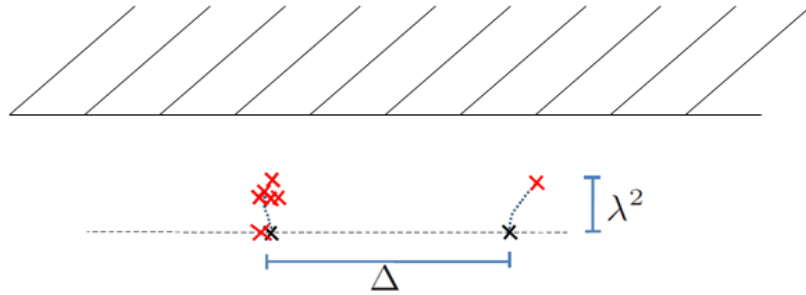


Figure 1.1: Eigenvalues of  $K(\sigma, \lambda)$ : isolated resonances regime

theory in  $\lambda$ , one finds the spectrum of  $K(\sigma, \lambda)$ . The spectrum of  $K(\sigma, 0)$  is easy to be obtained because in this unperturbed case, the systems  $S$  and  $R$  are uncoupled. As showed in Figure 1.1, the spectrum of  $K(\sigma, 0)$  consists of eigenvalues, indexed as dark

crosses, and continuous spectrum, located in the shadow area above a fixed horizontal line. Zero is an eigenvalue of  $K(\sigma, 0)$  with multiplicity  $N$ . As the perturbation switched on ( $\lambda \neq 0$ ), the continuous spectrum is still located in the shadow area but it moves a bit. The eigenvalues moved a little, from dark crosses to light crosses in Figure 1.1. At the same time,  $N - 1$  eigenvalues bifurcate out of the origin, and the multiplicity of zero disappears.

In the overlapping resonances regime, we start from  $K(0, \lambda)$  then apply perturbation theory for small  $\sigma$  (as  $\lambda \neq 0$ ). However, this case is more delicate since  $S + R$  are already coupled for  $\sigma = 0$ . Figure 1.2 shows the spectra of  $K(\sigma, 0)$  and  $K(\sigma, \lambda)$  in the

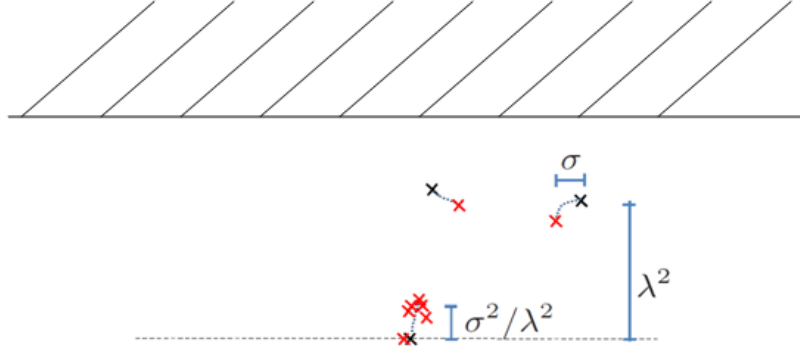


Figure 1.2: Eigenvalues of  $K(\sigma, \lambda)$ : overlapping resonances regime

overlapping resonances regime. As in the isolated resonances regime, all the continuous spectrum is located in the shadow area above a fixed horizontal line. However, for the motion of eigenvalues under perturbation, it is quite different from the isolated resonances case. As showed in Figure 1.2, the eigenvalues of  $K(\sigma, 0)$  are denoted as dark crosses, which are derived in Theorem 4.2.1. As the perturbation is switched on ( $\sigma \neq 0$ ), all eigenvalues are separated into two groups. The imaginary part of one group is proportional to  $\lambda^2$ , while the imaginary of the second group (close to zero) is proportional to  $\sigma^2/\lambda^2 \ll \lambda^2$  (and eigenvalue 0), shown as light crosses in Figure 1.2.

Please see Theorem 4.3.1 for the rigorous statement.

Consequently, the dynamics in equation (1.1) can be represented as

$$\langle A \rangle_t = \sum_{\epsilon} e^{it\epsilon} \langle \Psi_0, BQ_{\epsilon}A\Omega \rangle + O(e^{-\gamma t}), \quad (1.2)$$

where  $\epsilon$  are the eigenvalues below the continuous spectrum,  $Q_{\epsilon}$  is the spectral (Riesz) projection of  $K$  associated to  $\epsilon$ , and  $\gamma > \text{Im}\epsilon$  is obtained from (1.1) by moving the integral path from  $\mathbb{R} - i$  to  $\mathbb{R} + \gamma i$ , the sum comes from residues of poles at eigenvalues.

**Emergence of two time-scales in overlapping resonances regime.** In view of (1.2), we get two time-scales for the evolution:

- On the time scale  $t \propto \frac{1}{\lambda^2}$ , the  $e^{it\epsilon}$  with  $\text{Im}\epsilon \propto \lambda^2$  decay, but  $e^{it\epsilon}$  with  $\text{Im}\epsilon \propto \frac{\sigma^2}{\lambda^2}$  persist (see also Figure 1.2). So on this quick time-scale,  $\langle A \rangle_t$  relaxes towards

$$\langle A \rangle_t \approx \sum_{\epsilon: \text{Im}\epsilon \propto \frac{\sigma^2}{\lambda^2}, \epsilon=0} e^{it\epsilon} \langle \Psi_0, BQ_{\epsilon}A\Omega \rangle. \quad (1.3)$$

- On the longer time-scale  $t \propto \frac{\lambda^2}{\sigma^2}$ , all directions  $e^{it\epsilon}$  decay except for the one with  $\epsilon = 0$  :

$$\lim_{t \rightarrow \infty} \langle A \rangle_t = \langle \Psi_0, BQ_0A\Omega \rangle. \quad (1.4)$$

One may check that the r.h.s. is the equilibrium state of the joint system, see Section 5.1.

Upshot: in the isolated resonances regime, the dynamics converges to the equilibrium state at time-scale  $\propto \frac{1}{\lambda^2}$ . In the overlapping resonances case, there are two time-scales: on the first one,  $t \propto \frac{1}{\lambda^2}$ , the dynamics converges to a manifold of "quasi-stationary states" (which would be stationary if  $\sigma = 0$ ), determined by (1.3) (resonance data close to zero). Then, on the longer time-scale,  $t \propto \frac{\lambda^2}{\sigma^2}$ , the quasi-stationary

manifold decays into the single equilibrium state.

**Dynamics of the system.** In Theorem 2.2.1, we give the elements  $[\rho_t]_{ab}$  of density matrix at time  $t$  in the orthonormal eigenbasis of the interaction operator  $G$  (see (2.12)). The *off-diagonal* density matrix elements evolve as

$$[\rho_t]_{a,b} = e^{it\varepsilon_{a,b}(\sigma,\lambda)}[\rho_0]_{a,b} + O_\lambda(\sigma) + O(\lambda), \quad a \neq b$$

where

$$\varepsilon_{a,b}(\sigma, \lambda) = \lambda^2 \delta_{a,b} + \sigma r_{a,b} + \frac{\sigma^2}{\lambda^2} z_{a,b} + O\left(\frac{\sigma^2}{\lambda}\right) + O_\lambda(\sigma^3).$$

Here,  $\delta_{ab}$  are complex numbers with  $\text{Im}\delta_{ab} \geq 0$ , and  $r_{a,b} \in \mathbb{R}$ . Consequently, if  $\text{Im}\delta_{ab} > 0$  (which holds generically for non-degenerate systems), off-diagonal density matrix elements disappear at a rate  $\propto \frac{1}{\lambda^2}$ , very quickly. The *diagonal* density matrix elements satisfy (see (2.22))

$$[\rho_t]_{a,a} = \frac{1}{N} + \sum_{b=2}^N D_{a,b}(t)[\rho_0]_{b,b} + O_\lambda(\sigma) + O(\lambda).$$

The decay speed of  $D_{a,b}$ , proportional to  $\sigma^2/\lambda^2$ , is very slow relative to the decay speed,  $\lambda^2$ , of off-diagonal elements. So, in the overlapping resonances case, thermalization (convergence of the diagonal of the density matrix) is much slower than decoherence (decay of off-diagonals) in the orthonormal eigenbasis of interaction operator  $G$ . In the isolated resonances regime, both diagonal and off-diagonal elements decay at the same speed  $\lambda^2$ .

**Transition between two regimes for a spin-boson model.** We consider a spin-boson model and find the transition between two regimes. The small system is

a spin with Hamiltonian and interaction operator given by

$$H_S = S^z \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad G = S^x \equiv \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. The parameters  $\sigma, \lambda$  are now considered to be small but independent of each other. We analyze the decoherence properties of the spin *in the energy basis* (the orthonormal eigenbasis of  $H_S$ ). Let  $\phi_{\pm}^z$  be the normalized energy eigenvectors, satisfying  $H_S \phi_{\pm}^z = \pm \frac{1}{2} \phi_{\pm}^z$ , and denote the spin density matrix elements in this basis by  $[\rho_t]_{+,-}^z := \langle \phi_+^z, \rho_t \phi_-^z \rangle$  (and similarly for other matrix elements). We show in Section 6.2 that

$$\begin{aligned} [\rho_t]_{+,+}^z &\doteq \frac{1}{2} + \frac{1}{2} e^{itw_2} ([\rho_0]_{+,+}^z - [\rho_0]_{-,-}^z), \\ [\rho_t]_{+,-}^z &\doteq \frac{r}{r^2+1} \left( (1+r) e^{itw_3} + (1/r - 1) e^{itw_4} \right) [\rho_0]_{+,-}^z, \end{aligned}$$

where  $\doteq$  means that terms of order  $O(\lambda^2)$  are disregarded (see (6.18)). It is assumed here that  $[\rho_0]_{+,-}^z \in \mathbb{R}$  (see (6.19) for the general expression) and we have set

$$r = \frac{-4i\gamma - \sqrt{\pi^2 \xi(0)^2 - 16\gamma^2}}{\pi \xi(0)} \quad \text{with} \quad \gamma = \frac{\sigma}{\lambda^2}.$$

Here, the square root is the principal branch with branch cut on the negative real axis and  $\xi(0) > 0$  is a constant proportional to the reservoir spectral density at zero (see (2.16)). The system has four resonance energies

$$w_1 = 0, \quad w_2 = i \frac{\lambda^2}{2} \pi \xi(0), \quad w_{3,4} = i \frac{\lambda^2}{4} \pi \xi(0) \pm i \sqrt{\frac{\lambda^4}{16} \pi^2 \xi(0)^2 - \sigma^2}.$$

These expressions interpolate the values of the previously known, isolated regime (lowest order in  $\lambda$  for  $\sigma$  fixed) and the overlapping resonances values derived here ( $\sigma$

small,  $\lambda$  fixed; see also the remark after Theorem 4.3.1).

The diagonal converges to  $\frac{1}{2}$  at the rate  $\text{Im}w_2 \propto \lambda^2$ , independently of  $\sigma$ . The decoherence rate (decay of the off-diagonal in the energy basis) is obtained as follows.

- *Overlapping resonances regime:*  $\gamma \ll 1$  and  $r \approx -1$ . Thus,  $[\rho_t]_{+,-}^z \approx e^{itw_4}[\rho_0]_{+,-}^z$ , which has decay rate  $\text{Im}w_4 \approx \frac{2}{\pi\xi(0)}\frac{\sigma^2}{\lambda^2}$ .
- *Isolated resonances regime:*  $1/\gamma \ll 1$  and  $r \approx -i\infty$ . Thus,  $[\rho_t]_{+,-}^z \approx e^{itw_3}[\rho_0]_{+,-}^z$ , which has decay rate  $\text{Im}w_3 \approx \frac{\pi\xi(0)}{4}\lambda^2$ .

In the isolated resonances regime, the decoherence rate is given by the system-reservoir coupling constant  $\lambda$  alone, while in the overlapping case, it depends also on the level splitting parameter  $\sigma$ . For a fixed  $\lambda$ , the decoherence rate increases quadratically in  $\sigma$  (for small  $\sigma$ ). The further its energy levels lie apart, the quicker the spin decoheres.

We define the critical value  $\gamma_*$  for which the square root in  $w_{3,4}$  vanishes,

$$\gamma_* := \frac{1}{4}\pi\xi(0).$$

This critical value separates two regimes with different qualitative behaviour of the resonances  $w_3$  and  $w_4$ . As  $\gamma$  increases from zero to  $\gamma_*$ , the resonance  $w_3$  moves down the imaginary axis, decreasing from the initial value  $\frac{1}{2}i\pi\xi(0)\lambda^2$  to  $\frac{1}{4}i\pi\xi(0)\lambda^2$ , while  $w_4$  moves up the imaginary axis, from the origin to  $\frac{1}{4}i\pi\xi(0)\lambda^2$ . The two resonances meet for  $\gamma = \gamma_*$ . As  $\gamma > \gamma_*$  increases further, the resonances  $w_3$  and  $w_4$  move horizontally away from the imaginary axis, their imaginary parts stay constant, equal to  $\frac{1}{4}\pi\xi(0)\lambda^2$ . This motivates the sharp definition of the overlapping resonances regime, in the spin-boson model, to be given by  $\gamma < \gamma_*$  and of the isolated resonances regime to be given by  $\gamma > \gamma_*$ .

It is interesting to note that in nuclear physics, there is a (to our knowledge not rigorously defined) notion of overlapping resonances, used in the description of pro-



cesses involving unstable nuclei by non-hermitian Hamiltonians [8, 36]. It is observed that in the overlapping regime, the resonance widths (imaginary parts of resonance energies) segregate into two clusters, one located close to the origin (slow channels), the other at a much larger value (fast channels). The same occurs in our system: in the overlapping regime, we have one resonance at zero and another one,  $w_4$ , close to it. The other two,  $w_2$  and  $w_3$ , are much larger, both close to  $\frac{1}{2}i\pi\xi(0)\lambda^2$ . As the system transitions into the isolated resonances regime, the two clusters mix.

## 1.2 Structure of the thesis

We end this chapter by listing the structure of the thesis briefly. With a simple introduction of background, we present and analyze main results in Chapter 2. It contains three subsections, introduction of background, reduced dynamics of the system, and invariant initial system states. Here we should note that the return to equilibrium is a known result, and we just present a different proof.

Operator algebras provide a convenient framework for open quantum systems. Chapter 3 is devoted to a basic introduction to it used in this thesis.

Chapters 4 to 6 correspond to three subsections of Chapter 1, respectively. They are the main body of the thesis by providing all the proofs of main results with details.

Appendix A contains the proofs of some results used in chapter 4. Appendix B is a brief overview of Tomita-Takesaki modular theory used in this thesis.

# Chapter 2

## Main Results

In this chapter, we present and discuss the main results of the thesis, which are reduced dynamics of the system, invariant initial system states and return to equilibrium. The mathematical proofs are provided in the following chapters. We start this chapter with some background [4, 5, 17, 18, 29].

### 2.1 Background

The  $W^*$ -dynamical system we consider consists of a Hilbert space

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R, \tag{2.1}$$

of a von Neumann algebra of observables

$$\mathfrak{M} = \mathfrak{M}_S \otimes \mathfrak{M}_R, \tag{2.2}$$

and of a Heisenberg dynamics of  $\mathfrak{M}$ ,

$$A \mapsto e^{itL(\sigma,\lambda)} A e^{-itL(\sigma,\lambda)}, \quad A \in \mathfrak{M}. \quad (2.3)$$

The small system is an  $N$ -level system having a Hamiltonian  $H_S$ . In the GNS (Gelfand-Naimark-Segal) representation, the Hilbert space is  $\mathcal{H}_S = \mathbb{C}^N \otimes \mathbb{C}^N$  and the algebra of observables is given by  $\mathfrak{M}_S = \mathcal{B}(\mathbb{C}^N) \otimes \mathbb{1}_{\mathbb{C}^N}$  (bounded linear operators). The dynamics is implemented as

$$A_S \mapsto e^{itL_S} (A_S \otimes \mathbb{1}_{\mathbb{C}^N}) e^{-itL_S}, \quad A_S \in \mathcal{B}(\mathbb{C}^N), \quad (2.4)$$

where

$$L_S = H_S \otimes \mathbb{1}_{\mathbb{C}^N} - \mathbb{1}_{\mathbb{C}^N} \otimes \mathcal{C}H_S\mathcal{C} \quad (2.5)$$

is the self-adjoint system Liouville operator. Here,  $\mathcal{C}$  is the operator taking the complex conjugate of components of vectors represented in the orthonormal eigenbasis  $\{\varphi_a\}_{a=1}^N$  of a self-adjoint matrix  $G$  in the interaction operator, see (2.12),

$$G\varphi_a = g_a\varphi_a, \quad a = 1, \dots, N. \quad (2.6)$$

The procedure of doubling of the Hilbert space is well known in the physics literature, also called the ‘Liouville Representation’, see e.g. [31, Chapter 3].

The reservoir state is the thermodynamic (infinite volume) limit of a free Bose gas in equilibrium at inverse temperature  $\beta$ . Its Hilbert space representation was first constructed in [2] and a unitarily equivalent representation, suitable for the use of spectral translation techniques, has been given in [17]. In the latter, the GNS Hilbert space is  $\mathcal{H}_R = \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\vartheta)) = \bigoplus_{n \geq 0} L^2_{\text{symm}}((\mathbb{R} \times S^2)^n, (du \times d\vartheta)^n)$ , the symmetric Fock space over the one-particle function space  $L^2(\mathbb{R} \times S^2, du \times d\vartheta)$ . The

thermal field operator is given by

$$\Phi(f_\beta) = \frac{1}{\sqrt{2}}(a^*(f_\beta) + a(f_\beta)), \quad (2.7)$$

where  $a^*(f_\beta) = \int_{\mathbb{R} \times S^2} f_\beta(u, \vartheta) a^*(u, \vartheta) du d\vartheta$  is the creation operator acting on the Fock space  $\mathcal{H}_R$  and  $a(f_\beta)$  is its adjoint, smoothed out with  $f_\beta \in L^2(\mathbb{R} \times S^2, du \times d\vartheta)$  defined by

$$f_\beta(u, \vartheta) := \sqrt{\frac{u}{1 - e^{-\beta u}}} |u|^{1/2} \begin{cases} f(u, \vartheta), & \text{if } u \geq 0, \\ \bar{f}(-u, \vartheta), & \text{if } u < 0. \end{cases} \quad (2.8)$$

Here,  $f \in L^2(\mathbb{R}^3, d^3k)$  is represented in polar coordinates. The thermal Weyl CCR algebra  $\mathfrak{M}_R \subset \mathcal{B}(\mathcal{H}_R)$  is the von Neumann algebra generated by the unitary Weyl operators  $W(f_\beta) := e^{i\Phi(f_\beta)}$ . The dynamics on  $\mathfrak{M}_R$  is given by the Bogoliubov transformation  $t \mapsto W(e^{itu} f_\beta) = e^{itL_R} W(f_\beta) e^{-itL_R}$ . It is implemented by the self-adjoint reservoir Liouvillian

$$L_R = d\Gamma(u) := \int_{\mathbb{R} \times S^2} u a^*(u, \vartheta) a(u, \vartheta) du d\vartheta, \quad (2.9)$$

the second quantization of the operator of multiplication by  $u \in \mathbb{R}$ . The vacuum vector  $\Omega_R \in \mathcal{H}_R$  represents the  $\beta$ -KMS state w.r.t. the dynamics generated by (2.9).

The Liouville operator  $L(\sigma, \lambda)$  determining the full dynamics, (2.3), has the form

$$L(\sigma, \lambda) = L_0(\sigma) + \lambda V, \quad (2.10)$$

with a free part

$$L_0(\sigma) = \sigma L_S + L_R \quad (2.11)$$

(see (2.5), (2.9)) and where the system-reservoir interaction is

$$\lambda V = \lambda G \otimes \mathbb{1}_{\mathbb{C}^N} \otimes \Phi(g_\beta). \quad (2.12)$$

Here,  $\sigma$  and  $\lambda$  are two real parameters,  $G$  is a self-adjoint matrix on  $\mathbb{C}^N$  and  $g_\beta \in L^2(\mathbb{R} \times S^2)$  is obtained from a *form factor*  $g \in L^2(\mathbb{R}^3)$  using the relation (2.8). It is well known that  $L(\sigma, \lambda)$  is self-adjoint for all  $\lambda, \sigma \in \mathbb{R}$  (this can be proven by the Glimm-Jaffe-Nelson commutator theorem, see e.g. [23, Theorem A.2]). The infrared behaviour of the form factor is characterized by the unique  $p \geq -\frac{1}{2}$  satisfying

$$0 < \lim_{|k| \rightarrow 0} \frac{|g(k)|}{|k|^p} = C < \infty. \quad (2.13)$$

The power  $p$  depends on the physical model considered, e.g. for quantum optical systems,  $p = 1/2$ , and for the quantized electromagnetic field,  $p = -1/2$ . We assume the following regularity of the form factor.

**Assumption A1.** (Analyticity) There is a  $\theta_0 > 0$  such that  $\theta \mapsto g_\beta(u + \theta, \vartheta)$  has an analytic extension to the domain  $\{\theta \in \mathbb{C} : |\theta| < \theta_0\}$ , as a map from  $\mathbb{C}$  to  $L^2(\mathbb{R} \times S^2, du \times d\vartheta)$ .

**Assumption A2.** (Ultra-violet decay) There is an  $\epsilon > 0$  such that  $e^{a|k|}g(k) \in L^2(\mathbb{R}^3, d^3k)$  for an  $a > (1/2 + \epsilon)\beta$ , where  $\beta$  is the inverse temperature.

Over the last decade and a half, to study the dynamics generated by the Liouvilian (2.10), a perturbation theory based on quantum resonance methods has been developed [4, 12, 14, 17, 18, 23, 27, 28, 29]; see also [13] for a different approach. It is implemented in various forms, using spectral deformation, positive commutator and renormalization group techniques and permits a mathematically rigorous treatment of the dynamics, for *fixed, small* coupling  $\lambda$  and for *all* times  $t \geq 0$ . This approach

has been useful in the analysis of the dynamics of decoherence and entanglement in open systems [24, 25] and in applications to quantum chemistry [26].

The resonance approach developed so far [17, 18, 29] is based on a perturbation theory in the system-reservoir coupling parameter  $\lambda$ . The latter is assumed to be small relative to the spacing  $\sigma > 0$  between the energy levels of the small system:  $|\lambda| \ll \sigma$ . We call this the *isolated resonances regime*. However, for *complex open systems*, e.g. where the small system itself is composed of many individual parts (particles), the energy level spacing may become very small. For instance, the Hamiltonian of a system of  $N$  spins has  $2^N$  eigenvalues and the total energy of the spins is of the order of  $N$ . The generic energy spacing is thus of the order of  $\sigma \sim N/2^N$ , which is exponentially small in  $N$ . For such systems, the condition  $|\lambda| \ll \sigma$  is not reasonable. In the present work, we develop the resonance method in the *overlapping resonances regime*  $\sigma \ll |\lambda|$ . We study here the simplest case, in which all the system energies lie close together relative to  $|\lambda|$ .

## 2.2 Reduced dynamics of the small system

In this section, we present and analyze the expression of reduced density of the small system, which is the most important result in this thesis. The density matrix  $\rho_t$  of the small system, acting on  $\mathbb{C}^N$ , is defined by the relation

$$\mathrm{Tr}_{\mathbb{C}^N}(\rho_t A) = \langle \chi_0 \otimes \Omega_{\mathrm{R}}, e^{itL(\sigma, \lambda)}(A \otimes \mathbb{1}_{\mathbb{C}^N} \otimes \mathbb{1}_{\mathrm{R}})e^{-itL(\sigma, \lambda)}\chi_0 \otimes \Omega_{\mathrm{R}} \rangle,$$

valid for all  $A \in \mathcal{B}(\mathbb{C}^N)$ . The inner product is that of the GNS space  $\mathcal{H}_S \otimes \mathcal{H}_R$ . The initial state of the total system is a product of a system state  $\rho_0$ , represented by the vector  $\chi_0 \in \mathcal{H}_S$  in the GNS space, and the equilibrium state of the reservoir,

represented by  $\Omega_R \in \mathcal{H}_R$ . To express the dynamics of the density matrix elements

$$[\rho_t]_{a,b} \equiv \langle \varphi_a, \rho_t \varphi_b \rangle, \quad a, b = 1, \dots, N, \quad (2.14)$$

we introduce the following quantities. For  $a, b = 1, \dots, N$ , set

$$\delta_{a,b} = -\frac{1}{2}(g_a^2 - g_b^2) \langle g, |k|^{-1} g \rangle + i\frac{\pi}{2}(g_a - g_b)^2 \begin{cases} 0 & \text{if } p > -1/2 \\ \xi(0) > 0 & \text{if } p = -1/2 \end{cases}, \quad (2.15)$$

with

$$\xi(0) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}^3} \coth\left(\frac{\beta|k|}{2}\right) |g(k)|^2 \frac{\epsilon}{|k|^2 + \epsilon^2} d^3k. \quad (2.16)$$

Recall (2.6) that  $\{g_a\}_{a=1}^N$  is the set of spectrum of  $G$ . The number  $p$  is the infra-red parameter given in (2.13). We prove in Theorem 4.2.1 below that the  $\lambda^2 \delta_{a,b}$  are the *resonance energies* for  $\sigma = 0$ .

**Assumption A3.** (Non-degeneracy) The spectrum  $\{g_a\}_{a=1}^N$  of  $G$  is such that all non-zero  $\delta_{a,b}$  are distinct.

This assumption simplifies the presentation of our results. However, our analysis is readily generalized to the case of degenerate resonances (see the proof of Theorem 4.3.1). Indeed, we do this for the spin-boson model, in which the two non-zero resonances are given by  $\delta_{1,2} = \delta_{2,1} = i\frac{\pi}{2}\xi(0)$ .

For  $a, b = 1, \dots, N$ ,  $a \neq b$ , define the complex numbers

$$\eta_{a,b}(\sigma, \lambda) = \lambda^2 \delta_{a,b} + \sigma \left( [H_S]_{a,a} - [H_S]_{b,b} \right) - \frac{\sigma^2}{\lambda^2} \left( \sum_{c=1, \dots, N; c \neq a} \frac{|[H_S]_{a,c}|^2}{\delta_{c,b} - \delta_{a,b}} + \sum_{c=1, \dots, N; c \neq b} \frac{|[H_S]_{b,c}|^2}{\delta_{a,c} - \delta_{a,b}} \right), \quad (2.17)$$

where  $[H_S]_{b,c} = \langle \varphi_b, H_S \varphi_c \rangle$  are the matrix elements of the system Hamiltonian. Also,

for  $a = 1, \dots, N$ , set

$$\eta_{a,a}(\sigma, \lambda) = 2i \frac{\sigma^2}{\lambda^2} \xi_a, \quad (2.18)$$

where  $\xi_a \geq 0$  are the eigenvalues of the real symmetric  $N \times N$  matrix  $T$  with matrix elements

$$[T]_{a,b} = \begin{cases} -\frac{\text{Im}\delta_{a,b}}{|\delta_{a,b}|^2} |[HS]_{a,b}|^2, & \text{if } a \neq b \\ \sum_{c=1, \dots, N; c \neq a} \frac{\text{Im}\delta_{a,c}}{|\delta_{a,c}|^2} |[HS]_{a,c}|^2, & \text{if } a = b. \end{cases} \quad (2.19)$$

The vector  $\frac{1}{\sqrt{N}}(1, \dots, 1)$  is in the null space of  $T$ . We enumerate the eigenvalues of  $T$  s.t.  $\xi_1 = 0$ .

We show in Theorem 4.3.1 that the resonances are, for  $\sigma \ll |\lambda|$ , given by

$$\varepsilon_{a,b}(\sigma, \lambda) \equiv \eta_{a,b}(\sigma, \lambda) + O\left(\frac{\sigma^2}{|\lambda|}\right) + O_\lambda(\sigma^3). \quad (2.20)$$

Here,  $O\left(\frac{\sigma^2}{|\lambda|}\right)$  is a function satisfying the following equality

$$\lim_{\frac{\sigma^2}{|\lambda|} \rightarrow 0} \frac{O\left(\frac{\sigma^2}{|\lambda|}\right)}{\frac{\sigma^2}{|\lambda|}} = C_1$$

with  $C_1 \in \mathbb{R}$ , and for each fixed  $\lambda > 0$ ,  $O_\lambda(\sigma^3)$  is a function with the property that

$$\lim_{\sigma^3 \rightarrow 0} \frac{O_\lambda(\sigma^3)}{\sigma^3} = C_2$$

and  $C_2 \in \mathbb{R}$ .

**Theorem 2.2.1** (Reduced dynamics). *There is a  $\lambda_0 > 0$  such that for fixed  $\lambda$  satisfying  $0 < |\lambda| < \lambda_0$ , the following holds. There is a  $\sigma_0 > 0$  (depending on  $\lambda$ ) s.t. if  $0 \leq \sigma < \sigma_0$ , then we have, uniformly in  $t \geq 0$ :*



– For  $a, b = 1, \dots, N$ ,  $a \neq b$ ,

$$[\rho_t]_{a,b} = e^{it\varepsilon_{b,a}(\sigma,\lambda)}[\rho_0]_{a,b} + O_\lambda(\sigma) + O(\lambda). \quad (2.21)$$

– For  $a = 1, \dots, N$ ,

$$[\rho_t]_{a,a} = \frac{1}{N} + \sum_{b=2}^N D_{a,b}(t)[\rho_0]_{b,b} + O_\lambda(\sigma) + O(\lambda). \quad (2.22)$$

Let  $\{\varphi_a^T\}_{a=1}^N$  be an orthonormal basis of eigenvectors of  $T$ , with  $T\varphi_a^T = \xi_a\varphi_a^T$  and denote by  $[\varphi_a^T]_c$ ,  $c = 1, \dots, N$ , the components of  $\varphi_a^T$  (in the canonical basis). Then

$$D_{a,b}(t) = \sum_{c=2}^N e^{it\varepsilon_{c,c}(\sigma,\lambda)} \overline{[\varphi_c^T]_b} [\varphi_c^T]_a.$$

### Discussion:

1. The *off-diagonal* density matrix elements evolve as

$$[\rho_t]_{a,b} = e^{it\varepsilon_{a,b}(\sigma,\lambda)}[\rho_0]_{a,b} + O_\lambda(\sigma) + O(\lambda), \quad a \neq b.$$

The remainder terms are uniform in  $t \geq 0$ . The complex *resonance energies*  $\varepsilon_{a,b}$  satisfy  $\text{Im } \varepsilon_{a,b} \geq 0$ . They are of the form (see (2.20))

$$\varepsilon_{a,b}(\sigma, \lambda) = \lambda^2 \delta_{a,b} + \sigma r_{a,b} + \frac{\sigma^2}{\lambda^2} z_{a,b} + O\left(\frac{\sigma^2}{\lambda}\right) + O_\lambda(\sigma^3).$$

having the following interpretation:

- $\lambda^2 \delta_{a,b}$  is a resonance energy for  $\sigma = 0$ . The imaginary part of  $\delta_{a,b}$  is strictly positive, proportional to the eigenvalue difference  $(g_a - g_b)^2$ . All off-diagonal density matrix elements tend (modulo an error term) to zero as  $t \rightarrow \infty$ . The

system exhibits *decoherence in the eigenbasis of  $G$* , regardless of whether the system energy is degenerate or not. The contribution to the decoherence rate of this term is proportional to  $\lambda^2$ .

- The term linear in  $\sigma$  is real, with  $r_{a,b} = [H_S]_{a,a} - [H_S]_{b,b}$  and where  $[H_S]_{a,b} = \langle \varphi_a, H_S \varphi_b \rangle$ . The decay rates of matrix elements do not depend on the first order in the energy splitting parameter  $\sigma$ .
- The second order term in  $\sigma$  has generally non-vanishing real and imaginary parts. The complex  $z_{a,b}$  are determined by the ratio of matrix elements  $[H_S]_{c,d}$  and differences of  $\delta_{c,d}$  (see (2.17)). The factor  $1/\lambda^2$  is due to the presence of the reduced resolvent in second order perturbation theory in  $\sigma$  (here, the ‘non-degenerate energies’ are  $\lambda^2 \delta_{a,b}$ ). The sign of  $\text{Im } z_{a,b}$  can be positive or negative, depending on the model.

2. The *diagonal* density matrix elements satisfy (see (2.22))

$$[\rho_t]_{a,a} = \frac{1}{N} + \sum_{b=2}^N D_{a,b}(t) [\rho_0]_{b,b} + O_\lambda(\sigma) + O(\lambda),$$

where

$$D_{a,b}(t) = \sum_{c=2}^N e^{it\varepsilon_{c,c}(\sigma,\lambda)} x_{a,b,c} \quad \text{with} \quad \varepsilon_{c,c}(\sigma,\lambda) = 2i\frac{\sigma^2}{\lambda^2}\xi_c + O\left(\frac{\sigma^2}{\lambda}\right) + O_\lambda(\sigma^3).$$

The  $\xi_c$ ,  $c = 2, \dots, N$ , are non-negative. The resonance energies bifurcating out of the origin, as  $\sigma \neq 0$ , are given by  $i\xi_c$ . The complex numbers  $x_{a,b,c}$  are overlaps of resonance vectors. Under a generic condition (for instance if all matrix elements  $[H_S]_{a,b}$  are non-zero) we have  $\xi_c > 0$  for  $c = 2, \dots, N$ , and so  $D_{a,b}(t)$  decays exponentially quickly in time.

3. Contrary to the off-diagonals, the diagonal of the density matrix evolves as a

group: the value of a given diagonal entry depends on the initial condition of all of them. While the convergence rate of off-diagonals is proportional to  $\lambda^2$ , that of the diagonal is proportional to  $\sigma^2/\lambda^2$ . Hence the convergence of the diagonal, the part of the density matrix in the manifold  $\mathcal{M}_{0,\lambda}$  (see (2.23)), is driven by the level splitting, while that of the off-diagonals is driven by the system-reservoir interaction.

## 2.3 Invariant initial system states

In general, the dynamics of the system depends on the initial states. For factorized initial conditions  $\rho_0 \otimes \omega_{R,\beta}$ , denote the reduced evolution of the system by

$$T_{\sigma,\lambda}(t)\rho_0 = \rho_t,$$

and the manifold of initial system states which are invariant under the evolution, by

$$\mathcal{M}_{\sigma,\lambda} = \{\rho_0 : T_{\sigma,\lambda}(t)\rho_0 = \rho_0 \quad \forall t \geq 0\}. \quad (2.23)$$

For  $\sigma = 0$  one can find the dynamics of the reduced density matrix exactly [29, 30, 33] (see (4.17)). The manifold  $\mathcal{M}_{0,\lambda}$  is the set of all system density matrices which are *diagonal in the eigenbasis of the interaction operator  $G$* . Moreover, there is a constant  $C$  such that, for all initial system states  $\rho_0$  and all times  $t \geq 0$ ,

$$\text{dist}(\mathcal{M}_{0,\lambda}, T_{0,\lambda}(t)\rho_0) \leq C e^{-\lambda^2 \gamma_G \Gamma(t)} \text{dist}(\mathcal{M}_{0,\lambda}, \rho_0).$$

The distance  $\text{dist}(\mathcal{M}_{0,\lambda}, \rho) = \inf\{\|\tau - \rho\|_1 : \tau \in \mathcal{M}_{0,\lambda}\}$  is measured in trace norm,  $\|x\|_1 = \text{Tr}\sqrt{xx^*}$  for linear operators  $x$  on  $\mathbb{C}^N$ . Here,  $\Gamma(t) \geq 0$  is the *decoherence*

function defined by (4.17),

$$\Gamma(t) = \int_{\mathbb{R}^3} |g(k)|^2 \coth\left(\frac{\beta|k|}{2}\right) \frac{\sin^2\left(\frac{|k|t}{2}\right)}{|k|^2} d^3k, \quad (2.24)$$

and  $\gamma_G = \min\{(g_a - g_b)^2 : a \neq b\} > 0$ , where  $\{g_a\}_{a=1}^N$  is the spectrum of  $G$ . We assume that the eigenvalues of  $G$  are simple (this eases the analysis, but is not necessary for it). The manifold  $\mathcal{M}_{0,\lambda}$  is *orbitally stable*, meaning that a state initially close to  $\mathcal{M}_{0,\lambda}$  remains so for all times. If  $\Gamma(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , then the system undergoes full decoherence in the eigenbasis of  $G$ . In this case the manifold  $\mathcal{M}_{0,\lambda}$  is dynamically attractive, or *asymptotically stable*. One shows that for suitable infra-red behaviour of the interaction form factor  $g(k)$ , the decoherence function satisfies  $\lim_{t \rightarrow \infty} \Gamma(t)/t = \Gamma_\infty$ , with  $\Gamma_\infty > 0$ . The manifold  $\mathcal{M}_{0,\lambda}$  is then approached exponentially quickly, at the rate  $\lambda^2 \gamma_G \Gamma_\infty$ . We give further detail in Section 5.1, where we also identify the stationary states of the total system (i.e., the system plus the reservoir).

As the degeneracy is lifted, for small  $\sigma > 0$ , the manifold of invariant initial system states becomes empty,  $\mathcal{M}_{\sigma,\lambda} = \emptyset$ . All initial states approach a single asymptotic state, which is the reduction to the small system of the joint system-reservoir equilibrium state (see section 5.1). In the regime  $\sigma \ll |\lambda| \ll 1$ , the approach of the asymptotic state, and hence the dissolution of the manifold  $\mathcal{M}_{0,\lambda}$ , takes place at a rate proportional to  $\sigma^2/\lambda^2$ . To first approximation, the final state is the trace state,

$$\lim_{t \rightarrow \infty} \rho_t = \frac{1}{N} \mathbb{1}_{\mathbb{C}^N} + O_\lambda(\sigma) + O(\lambda),$$

where  $O_\lambda(\sigma)$  is a term  $f(\lambda, \sigma)$  satisfying  $\limsup_{\sigma \rightarrow 0} \sigma^{-1} \|f(\lambda, \sigma)\| = C_\lambda < \infty$ .

# Chapter 3

## Mathematical setup

### 3.1 Operator algebras in open quantum systems

Operator algebras provide a convenient framework for quantum statistical mechanics. In this section, we present a basic introduction to operator algebras used in this thesis. For detailed information on operator algebras in quantum statistics mechanics, we refer to the classical books [6].

#### 3.1.1 Observable, states and dynamics

Consider a quantum system determined by a Hilbert space  $\mathcal{H}$  and a Hamiltonian  $H$  on  $\mathcal{H}$ . The observables of the system are some bounded operators on  $\mathcal{H}$  and form a  $C^*$ -algebra  $\mathfrak{A}$ . Recall that a linear functional  $w$  on  $C^*$ -algebra  $\mathfrak{A}$  is a *state* if  $w$  is a positive linear functional of norm one, i.e.

$$A \in \mathfrak{A}^*, \|w\| = 1 \text{ and } w(A^*A) \geq 0, \forall A \in \mathfrak{A}.$$

If the system at time zero is in the vector state  $\psi \in \mathcal{H}$ , then its state at time  $t$  is given by  $\psi(t) = e^{-itH}\psi$  (Schrödinger evolution). The expected value of observable  $A$  at time  $t$  is

$$\langle A \rangle_t := \langle \psi(t), A\psi(t) \rangle = \langle \psi, e^{itH} A e^{-itH} \psi \rangle. \quad (3.1)$$

In quantum statistical systems, states are defined by

$$w(A) = \text{Tr}(\rho A), \forall A \in \mathfrak{A}, \quad (3.2)$$

where  $\rho$  is a so-called *density matrix*, a positive, self-adjoint trace-class operator on  $\mathcal{H}$  with  $\text{Tr}\rho = 1$ . Hence we can write  $\rho$  as

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|, \quad (3.3)$$

where  $\{\psi_n\}$  is a collection of eigenvectors of  $\rho$  with corresponding set  $\{p_n\}$  of eigenvalues. Equation (3.2) can be rewritten as

$$w(A) = \sum_n p_n \langle \psi_n, A\psi_n \rangle. \quad (3.4)$$

If the system at time zero is in the mixed state (3.4), then it evolves according to

$$\langle A \rangle_t = \sum_n p_n \langle \psi_n, e^{itH} A e^{-itH} \psi_n \rangle. \quad (3.5)$$

The physical explanation of  $\langle A \rangle_0$  is the system's initial state located at vector state  $\psi_n$  with a probability  $p_n$ . We define the time evolution of the density matrix  $\rho$  as

$$\rho_t = e^{-itH} \rho e^{itH}.$$

It follows from the cyclicity of the trace and (3.5) that

$$\langle A \rangle_t = \text{Tr}(\rho_0 e^{itH} A e^{-itH}) = \text{Tr}(\rho_t A). \quad (3.6)$$

We define the evolution of an observable  $A$  (Heisenberg evolution) by

$$\alpha_t(A) = e^{itH} A e^{-itH},$$

then  $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$  defines a group of  $*$ -automorphisms on the  $C^*$ -algebra  $\mathfrak{A}$  such that

$$\langle A \rangle_t = \text{Tr}(\rho_t A) = \text{Tr}(\rho_0 \alpha_t(A)).$$

The pair  $(\mathfrak{A}, \alpha_t)$  is called a  $C^*$ -*dynamical system* if the group  $\alpha_t$  is strongly continuous, i.e.,

$$\mathbb{R} \ni t \mapsto \alpha_t(A)$$

is continuous as a map from  $\mathbb{R}$  to  $\mathfrak{A}$  in the  $C^*$ -norm topology for all  $A \in \mathfrak{A}$ .

The triple  $(\mathfrak{A}, w, \alpha_t)$  is called a *quantum statistical system* if  $\mathfrak{A}$  is a  $C^*$ -algebra with unit,  $\alpha = \{\alpha_t\}$  an automorphism group, and  $w$  an  $\alpha$ -invariant state on  $\mathfrak{A}$ , i.e.,  $w \circ \alpha_t = w$ , so that for all  $A \in \mathfrak{A}$ ,

$$\mathbb{R} \ni t \mapsto w(A^* \alpha_t(A))$$

is continuous.

### 3.1.2 Equilibrium states

The equilibrium state plays an important role in quantum statistical mechanics. We start from finite-dimensional case. Let  $\mathcal{H}$  be a finite-dimensional Hilbert space and  $H = H^* \in \mathcal{B}(\mathcal{H})$  be the Hamiltonian. Consider the quantum system described by the

$C^*$ -dynamical system  $(\mathcal{B}(\mathcal{H}), \alpha)$ , where

$$\alpha_t(A) = e^{itH} A e^{-itH}, \quad \forall A \in \mathcal{B}(\mathcal{H}).$$

For any  $\beta \in \mathbb{R}$ , it follows from the classical statistical mechanics that this system has a unique thermal equilibrium state (Gibbs state)  $w_\beta$  at inverse temperature  $\beta$  given by the Gibbs-Boltzmann prescription

$$w_\beta(A) = \frac{\text{Tr}(e^{-\beta H} A)}{\text{Tr}(e^{-\beta H})}. \quad (3.7)$$

Note that the Gibbs state is, in general, only defined in finite-dimensional system because  $\text{Tr}(e^{-\beta H})$  makes no sense when the Hamiltonian  $H$  is not trace class. It is reasonable for us to derive an equilibrium condition from (3.7) to characterize the infinite-dimensional equilibrium state. Using the cyclicity of trace, we have

$$\begin{aligned} w_\beta(\alpha_t(A)B) &= \frac{1}{\text{Tr}(e^{-\beta H})} \text{Tr}(e^{-\beta H} e^{itH} A e^{-itH} B) \\ &= \frac{1}{\text{Tr}(e^{-\beta H})} \text{Tr}(e^{-\beta H} B e^{i(t+i\beta)H} A e^{-i(t+i\beta)H}) \\ &= w_\beta(B\alpha_{t+i\beta}(A)). \end{aligned} \quad (3.8)$$

It follows from (3.8) that  $w_\beta(\alpha_t(A)B) = w_\beta(B\alpha_{t+i\beta}(A))$ . This relation doesn't involve the trace of  $e^{-\beta H}$  and is then suitable as a generalized notion of an equilibrium (KMS) state. Let  $(\mathfrak{A}, \alpha_t)$  be a  $C^*$ -dynamical system, then a state  $w_\beta$  on  $\mathfrak{A}$  is called an  $(\alpha, \beta)$ -KMS state [6], if for each pair  $A, B \in \mathfrak{A}$ , there is an analytic function of  $z$ ,  $F_{A,B}(z)$ , within the domain  $\{z \in \mathbb{C} : 0 < \text{Im}z < \beta\}$ , such that  $F_{A,B}(z)$  is continuous on the closure of the domain and satisfies the KMS condition,

$$F_{A,B}(t) = w_\beta(B\alpha_t(A)), \quad F_{A,B}(t+i\beta) = w_\beta(B\alpha_{t+i\beta}(A)) = w_\beta(\alpha_t(A)B), \quad (3.9)$$



for all  $t \in \mathbb{R}$ . Here the parameter  $\beta = \frac{1}{T}$  is the inverse temperature. It turns out that the KMS state is an ideal choice for the equilibrium state in infinite-dimensional system. One can show that the  $(\alpha, \beta)$ -KMS state is time invariant (stationary) w.r.t the evolution  $\alpha_t$ , see Proposition 5.3.3 in [6]. Note that stationarity is not a sufficient condition for KMS states. As the definition of the KMS state is a generalization of Gibbs state, it is easy to show that the Gibbs state is an example of KMS state. For the Gibbs state defined by (3.7), and any observables  $A, B \in \mathcal{B}(\mathcal{H})$ , we define the function  $F_{A,B}(z) : \mathbb{C} \rightarrow \mathbb{C}$  as

$$F_{A,B}(z) = w_\beta(B\alpha_z(A)) = \frac{1}{\text{Tr}(e^{-\beta H})} \text{Tr}(e^{-\beta H} B e^{izH} A e^{-izH}). \quad (3.10)$$

It is easy to see that  $F_{A,B}(z)$  is analytic on the whole plane and  $F_{A,B}(t) = w_\beta(B\alpha_t(A))$ . The second equality in (3.9) can be verified as (3.8). So the Gibbs state (3.7) is an example of KMS state. Another important example of KMS state is the equilibrium state in an algebra generated by Weyl operators constructed by Araki and Woods [2]. We give a brief introduction in Section 3.2.2, see also [3, 6] for detailed presentation.

### 3.1.3 GNS representation

To study the dynamical behavior of a *quantum statistical system*  $(\mathfrak{A}, w, \alpha_t)$ , it is useful to represent the state  $w$  by a single vector in an appropriate Hilbert space. This is the so-called Gelfand-Naimark-Segal representation theory. Recall that  $(\mathcal{H}_w, \pi_w, \Omega_w)$  is a cyclic representation of  $(\mathfrak{A}, w)$  means that  $\pi_w : \mathfrak{A} \mapsto \mathcal{B}(\mathcal{H})$  is a  $*$ -morphism,  $\Omega_w \in \mathcal{H}_w$  a normalized vector, and  $\pi_w(\mathfrak{A})$  dense in the Hilbert space  $\mathcal{H}_w$ . We have the following GNS (Gelfand-Naimark-Segal) theorem [6].

**Theorem 3.1.1.** *Let  $w$  be any state on an abstract  $C^*$ -algebra  $\mathfrak{A}$ , then there exists*

a cyclic representation  $(\mathcal{H}_w, \pi_w, \Omega_w)$  of  $(\mathfrak{A}, w)$  such that

$$w(A) = \langle \Omega_w, \pi_w(A)\Omega_w \rangle$$

for all  $A \in \mathfrak{A}$  and, consequently,  $\|\Omega_w\| = \|w\| = 1$ . Moreover, the representation is unique up to unitary equivalence. The triple  $(\mathcal{H}_w, \pi_w, \Omega_w)$  is called the GNS-representation of  $(\mathfrak{A}, w)$ .

Uniqueness of the GNS-representation means that if  $(\mathcal{H}'_w, \pi'_w, \Omega'_w)$  and  $(\mathcal{H}_w, \pi_w, \Omega_w)$  are two cyclic representations of  $(\mathfrak{A}, w)$ , then there is a unique unitary map  $U : \mathcal{H}_w \mapsto \mathcal{H}'_w$  such that

$$\pi'_w(A) = U\pi_w(A)U^{-1} \text{ and } \Omega'_w = U\Omega_w.$$

If the state  $w$  is invariant under the evolution  $\alpha_t$ , then by the theorem there is a uniquely determined group of unitary operators  $U(t) : \mathcal{H} \mapsto \mathcal{H}$  s.t. for all  $A \in \mathfrak{A}$

$$\pi_w(\alpha_t(A)) = U(t)^{-1}\pi_w(A)U(t) \text{ and } U(t)\Omega_w = \Omega_w.$$

If, in addition,  $t \mapsto U(t)$  is strongly continuous, by Stone's theorem, there exists a self-adjoint operator  $L$  such that  $U(t) = e^{-itL}$ . We call  $L$  the *Liouvillian*. Since  $U(t)\Omega_w = \Omega_w$ , we know that  $L\Omega_w = 0$ , that means 0 is an eigenvalue of  $L$  with an associated eigenvector  $\Omega_w$ .

## 3.2 The non-interacting particle-field system

### 3.2.1 The particle system

In this thesis, we consider an  $N$ -level quantum system coupled to a bosonic heat reservoir at positive temperature. The  $N$ -level system is described on a finite-dimensional

Hilbert space  $\mathcal{H}_S = \mathbb{C}^N$ . The dynamics on  $\mathcal{H}_S$  is generated by the self-adjoint Hamiltonian  $H_S$  acting on  $\mathcal{H}_S$ . Now we represent any state on  $\mathcal{H}_S$  by a single vector in the doubled Hilbert space  $\mathcal{H}_S \otimes \mathcal{H}_S$  by GNS representation theory as the following.

Take an arbitrary density matrix  $\rho_S$  on  $\mathcal{H}_S$  and diagonalize it as

$$\rho_S = \sum_{j=1}^N p_j |\psi_j\rangle\langle\psi_j|, \quad (3.11)$$

where  $p_j$  is an eigenvalue of  $\rho_S$  with an associated normalized eigenvector  $\psi_j$ . The state corresponding to  $\rho_S$  is given by

$$A \mapsto \text{Tr}(\rho_S A), \quad \forall A \in \mathcal{B}(\mathcal{H}_S). \quad (3.12)$$

The evolution of the density matrix  $\rho_S$  is

$$\mathbb{R} \ni t \mapsto e^{-itH_S} \rho_S e^{itH_S}. \quad (3.13)$$

Now we define the Hilbert space  $L^2(\mathcal{H}_S)$  with a inner product  $\langle \cdot \rangle_2$  by

$$\begin{aligned} L^2(\mathcal{H}_S) &:= \{A \in \mathcal{B}(\mathcal{H}_S) : \text{Tr}(A^*A) < \infty\}, \\ \langle A, B \rangle_2 &:= \text{Tr}(A^*B), \quad \forall A, B \in \mathcal{H}_S. \end{aligned} \quad (3.14)$$

It is easy to show that  $L^2(\mathcal{H}_S) = \mathcal{B}(\mathcal{H}_S)$  since  $\mathcal{H}_S = \mathbb{C}^N$  is finite-dimensional. Note that the density matrix  $\rho_S$  is a positive self-adjoint trace class operator, one has  $\sqrt{\rho_S} \in L^2(\mathcal{H}_S)$ . This and the cyclicity of the trace imply that

$$\text{Tr}(\rho_S A) = \langle \sqrt{\rho_S}, A \sqrt{\rho_S} \rangle_2, \quad (3.15)$$

which means the state determined by  $\rho_S$  is represented by the vector  $\sqrt{\rho_S} \in L^2(\mathcal{H}_S)$ .

Now we construct an isometric isomorphism between  $L^2(\mathcal{H}_S)$  and  $\mathcal{H}_S \otimes \mathcal{H}_S$  so that we may represent  $\rho_S$  by a vector in  $\mathcal{H}_S \otimes \mathcal{H}_S$ .

Let  $M_1 = \text{span}\{|\chi_1\rangle\langle\chi_2|, \chi_1, \chi_2 \in \mathcal{H}_S\}$  and  $M_2 = \text{span}\{\chi_1 \otimes \chi_2, \chi_1, \chi_2 \in \mathcal{H}_S\}$ , and define the linear operator  $T : M_1 \mapsto M_2$  by

$$T : |\chi_1\rangle\langle\chi_2| \mapsto \chi_1 \otimes \mathcal{C}\chi_2,$$

where  $\mathcal{C}$  is a map on  $\mathcal{H}_S$  chosen to be antilinear (conjugate linear) and antiunitary, i.e.,

$$\mathcal{C}(\alpha\chi_1) = \bar{\alpha}(\chi_1) \quad \text{and} \quad \langle \mathcal{C}\chi_1, \mathcal{C}\chi_2 \rangle = \overline{\langle \chi_1, \chi_2 \rangle}$$

for all  $\chi_1, \chi_2 \in \mathcal{H}_S$  and  $\alpha \in \mathbb{C}$ . It is easy to see that  $T$  is an isometric isomorphism between  $L^2(\mathcal{H}_S)$  and  $\mathcal{H}_S \otimes \mathcal{H}_S$  due to the facts that  $M_1 = L^2(\mathcal{H}_S)$  and  $M_2 = \mathcal{H}_S \otimes \mathcal{H}_S$ . Therefore, we have

$$\langle A, A \rangle_2 = \|TA\|^2 = \langle TA, TA \rangle, \forall A \in \mathcal{H}_S.$$

This and the polarization identity imply that

$$\langle A, B \rangle_2 = \langle TA, TB \rangle, \forall A, B \in \mathcal{H}_S.$$

In particular, (3.15) becomes

$$\begin{aligned} \text{Tr}(\rho_S A) &= \langle \sqrt{\rho_S}, A \sqrt{\rho_S} \rangle_2 \\ &= \langle \Psi, (A \otimes \mathbb{I}) \Psi \rangle \end{aligned} \tag{3.16}$$

due to the fact that  $TA\sqrt{\rho_S} = (A \otimes \mathbb{I})\Psi$ , where

$$\Psi = T\sqrt{\rho_S} = \sum_{j=1}^N \sqrt{p_j} \psi_j \otimes \mathcal{C}\psi_j. \quad (3.17)$$

Hence the vector  $\Psi \in \mathcal{H}_S \otimes \mathcal{H}_S$  represents the state in the represented space. It follows from (3.11), (3.13) and (3.17), that the evolution of  $\Psi$  is given by

$$t \mapsto \sum \sqrt{p_j} e^{itH_S} \psi_j \otimes \mathcal{C}e^{-itH_S} \psi_j = e^{-it\mathcal{L}_S} \Psi, \quad (3.18)$$

where  $\mathcal{L}_S = H_S \otimes \mathbb{I} - \mathbb{I} \otimes \mathcal{C}H_S\mathcal{C}$  is called the *Liouville operator*. In the Heisenberg picture, the dynamics evolves as

$$A \mapsto e^{it\mathcal{L}_S} (A \otimes \mathbb{I}) e^{-it\mathcal{L}_S}. \quad (3.19)$$

### 3.2.2 The reservoir system

The reservoir system is given by a spatially infinitely extended free massless bosonic field. Since the one particle space of the field is given by  $L^2(\mathbb{R}^3, d^3k)$ , the reservoir system is the bosonic Fock space over  $L^2(\mathbb{R}^3, d^3k)$ :

$$\mathcal{H}_R := \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n},$$

where  $L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^0} := \mathbb{C}$  is the complex number space. The Hilbert space  $L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n}$  identified as a subspace of the bosonic Fock space  $\mathcal{F}(L^2(\mathbb{R}^3, d^3k))$  is called the *n-sector* (or the *n-th chaos* in quantum probability). The zero-sector is

also called the *vacuum sector*. The Hilbert space  $L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n}$  is defined by

$$L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n} = \{f(k_1, \dots, k_n) \in L^2(\mathbb{R}^3, d^3k)^{\otimes^n} :$$

$$f(k_1, \dots, k_i, k_j, \dots, k_n) = f(k_1, \dots, k_j, k_i, \dots, k_n), \forall 1 \leq i, j \leq n\},$$

with scalar product

$$\langle f, g \rangle = \int \overline{f(k_1, \dots, k_n)} g(k_1, \dots, k_n) dk_1 \cdots dk_n.$$

Here we should note that the tensor product of two functions can be written as one function with two variables, i.e.  $f(k_1, k_2) = f_1(k) \otimes f_2(k)$ . Each element  $\psi \in \mathcal{H}_R$  can be viewed as a sequence  $\{\psi_n\}_{n=0}^\infty$  satisfying

$$\psi_n \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n} \text{ and } \sum_{n=0}^{\infty} \|\psi_n\|^2 < \infty.$$

Now we define the creation and annihilation operators as the following

$$\begin{aligned} a^*(f)\psi_n &= \sqrt{n+1} \mathcal{S}f \otimes \psi_n \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^{n+1}} \\ a(f)\psi_n &= \sqrt{n} \langle f, \psi_n(\cdot, k_1, k_2, \dots, k_{n-1}) \rangle \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^{n-1}} \end{aligned} \quad (3.20)$$

for any  $f \in L^2(\mathbb{R}^3, d^3k)$  and  $\psi_n \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n}$ ,  $n \geq 1$ . Here,  $\mathcal{S}$  is the symmetrization operator defined by

$$\mathcal{S}f(k_1, \dots, k_n) = \frac{1}{n!} \sum_{\pi \in S_n} f(k_{\pi(1)}, \dots, k_{\pi(n)}) \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n},$$

where  $S_n$  is the group of all permutations  $\pi$  of  $n$  objects. When  $n = 0$ , i.e.,  $L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^0} = \mathbb{C}$ , we define

$$\begin{aligned} a^*(f)\Omega &= f \in L^2(\mathbb{R}^3, d^3k) \\ a(f)\Omega &= 0, \end{aligned} \tag{3.21}$$

where  $\Omega \in \mathbb{C}$ ,  $\|\Omega\| = 1$  is the *vacuum*. It is easy to show that

$$\begin{aligned} \|a^*(f)\psi_n\| &\leq \sqrt{n+1}\|f\|\|\psi_n\| \\ \|a(f)\psi_n\| &\leq \sqrt{n}\|f\|\|\psi_n\| \end{aligned} \tag{3.22}$$

for  $f \in L^2(\mathbb{R}^3, d^3k)$  and  $\psi_n \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n}$ . Now we define the number operator  $N$  on  $\mathcal{H}_R$  as

$$N\psi = \{n\psi_n\}_{n=0}^\infty \tag{3.23}$$

for each  $\psi = \{\psi_n\}_{n=0}^\infty \in \mathcal{H}_R$ . The domain of  $N$  is given by

$$D(N) = \{\{\psi_n\}_{n=0}^\infty \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{symm}^n} : \sum_{n=0}^\infty \|n\psi_n\|^2 < \infty\}. \tag{3.24}$$

It follows from (3.22) that  $a^*$  and  $a$  are relatively bounded w.r.t.  $N^{1/2}$  as

$$\begin{aligned} \|a^*(f)(N+1)^{-1/2}\| &\leq \|f\|, \\ \|a(f)(N+1)^{-1/2}\| &\leq \|f\|. \end{aligned} \tag{3.25}$$

The creation and annihilation operators satisfy the following canonical commutation relations (CCR),

$$\begin{aligned} [a^*(f), a^*(g)] &= [a(f), a(g)] = 0, \\ [a(f), a^*(g)] &= \langle f, g \rangle, \end{aligned} \tag{3.26}$$

where  $[a, b] = ab - ba$ . Let  $a^*(k)$  and  $a(k)$  be operator-valued distributions such that

$$\begin{aligned} a^*(f) &= \int f(k)a^*(k)d^3k \\ a(f) &= \int \overline{f(k)}a(k)d^3k. \end{aligned} \tag{3.27}$$

Then the CCR can be translated to

$$\begin{aligned} [a^*(k), a^*(k')] &= [a(k), a(k')] = 0 \\ [a(k), a^*(k')] &= \delta(k - k'), \end{aligned} \tag{3.28}$$

where  $\delta$  is the Dirac-delta function. The Hamiltonian  $H_R$  acting on  $\mathcal{H}_R$  is defined by

$$H_R\psi = \{H_R\psi_n\}_{n=0}^\infty \tag{3.29}$$

for all  $\psi = \{\psi_n\}_{n=0}^\infty \in \mathcal{H}_R$ , where  $\psi_n \in L^2(\mathbb{R}^3, d^3k)^{\otimes_{sym}^n}$  and

$$[H_R\psi_n](k_1, \dots, k_n) = (|k_1| + \dots + |k_n|)\psi_n(k_1, \dots, k_n). \tag{3.30}$$

Zero is the only eigenvalue of  $H_R$  and its kernel is  $\text{span}\{\Omega\} = \mathbb{C}$ .

We know that unbounded operators are difficult to deal with in mathematics. Note that  $a^*(f)$  and  $a(f)$  are unbounded operators, we replace them by a set of bounded operators, *Weyl operators*, which have a closed relationship with  $a^*(f)$  and  $a(f)$ . The *Weyl operators* are defined by

$$W(f) := e^{i\Phi(f)},$$



for  $f \in L^2(\mathbb{R}^3, d^3k)$ , where

$$\begin{aligned}\Phi(f) &:= \frac{1}{\sqrt{2}}(a^*(f) + a(f)) \\ \Pi(f) &:= \Phi(if) = \frac{i}{\sqrt{2}}(a^*(f) - a(f)).\end{aligned}\tag{3.31}$$

Since  $\Phi(f)$  is self-adjoint,  $W(f)$  is a unitary operator. We define the  $C^*$ -algebra of reservoir system observables by

$$\mathfrak{A}_R = \overline{\text{span}\{W(f) : f \in D_R\}},\tag{3.32}$$

where

$$D_R := \{f \in L^2(\mathbb{R}^3, d^3k) \mid \frac{1}{\sqrt{|k|}}f(k) \in L^2(\mathbb{R}^3, d^3k)\}.$$

The choice of  $D_R$  is determined by the fact that the generating functional of the equilibrium state  $w_R$  should be finite, i.e.

$$w_R(W(f)) = \exp\left\{-\frac{1}{4} \int_{\mathbb{R}^3} \frac{e^{\beta|k|} + 1}{e^{\beta|k|} - 1} |f(k)| d^3k\right\} < \infty,\tag{3.33}$$

(see for instance [17, 27]). Similar to the CCR of creation and annihilation operators, we have the following Baker-Campbell-Hausdorff formula

$$W(f)W(g) = e^{-\frac{i}{2}Im\langle f, g \rangle} W(f+g) = e^{-iIm\langle f, g \rangle} W(g)W(f).\tag{3.34}$$

The equilibrium state  $w_R$  of reservoir system is obtained by performing a thermodynamic limit of finite-volume equilibrium states with fixed temperature. The GNS-representation of  $(\mathfrak{A}_R, w_R)$  is more complicated than that of finite system  $(\mathfrak{H}_S, w_S)$ . It was first constructed explicitly by Araki and Woods, known as *Araki-Woods (AW)*

representation. The AW representation is given by the triple  $(\mathcal{H}_{AW}, \pi_{AW}, \Omega_{AW})$ , where

$$\begin{aligned}
\mathcal{H}_{AW} &= \mathcal{H}_R \otimes \mathcal{H}_R, \\
\pi_{AW}(W(f)) &= W(\sqrt{1 + \mu_\beta}f) \otimes W(\sqrt{\mu_\beta}\bar{f}), \\
\Omega_{AW} &= \Omega \otimes \Omega, \\
w_R(A) &= \langle \Omega_{AW}, \pi_{AW}(A)\Omega_{AW} \rangle, \quad \forall A \in \mathfrak{A}_R,
\end{aligned} \tag{3.35}$$

where  $\mu_\beta$  is given by the Planck's law

$$\mu_\beta(k) = \frac{1}{e^{\beta|k|} - 1}. \tag{3.36}$$

Araki and Woods show that  $\Omega_{AW}$  is a cyclic vector for  $\pi_{AW}$ , so the AW representation is a GNS-representation of  $w_R$ . Since

$$\alpha(A) = e^{itH_R} A e^{-itH_R}, \quad \forall A \in \mathfrak{A}_R, \tag{3.37}$$

It is easy to show that there exists a self-adjoint operator  $\mathcal{L}_R$ , the *Liouvilian*, such that

$$\pi_R(\alpha(A)) = e^{it\mathcal{L}_R} A e^{-it\mathcal{L}_R}, \quad \forall A \in \mathfrak{A}_R, \tag{3.38}$$

where

$$\mathcal{L}_R = H_R \otimes \mathbb{I} - \mathbb{I} \otimes H_R. \tag{3.39}$$

In terms of the creation and annihilation operators,  $\pi_{AW}$  is given by

$$\begin{aligned}
a_\beta(k) &:= \pi_{AW}(a(k)) = \sqrt{1 + \mu_\beta}a(k) \otimes \mathbb{I} + \sqrt{\mu_\beta}\mathbb{I} \otimes a^*(k), \\
a_\beta^*(k) &:= \pi_{AW}(a^*(k)) = \pi_{AW}(a(k))^* = \sqrt{1 + \mu_\beta}a^*(k) \otimes \mathbb{I} + \sqrt{\mu_\beta}\mathbb{I} \otimes a(k).
\end{aligned} \tag{3.40}$$

The equilibrium state  $w_R$  satisfies

$$w(a_\beta^*(k)a_\beta(k')) = \delta(k - k')\mu_\beta(k), \quad (3.41)$$

where  $\delta$  is the Dirac-delta function. The smoothed out thermal field operator is given by

$$\Phi_\beta(g) = \frac{1}{\sqrt{2}}(a_\beta^*(g) + a_\beta(g)). \quad (3.42)$$

### 3.3 The interacting particle-field system

Let  $\mathcal{H}_S \otimes \mathcal{H}_R$  be the Hilbert space of the system coupled to a reservoir, and let

$$H(\sigma, \lambda) = \sigma H_S + H_R + \lambda v \quad (3.43)$$

be its Hamiltonian.  $(\mathcal{H}_S, H_S)$  and  $(\mathcal{H}_R, H_R)$  have been discussed in sections 2.1 and 2.2, respectively. The system reservoir interaction has the form

$$v = G \otimes \Phi(g), \quad (3.44)$$

where  $G$  is a self-adjoint operator on  $\mathcal{H}_S = \mathbb{C}^N$  and  $\Phi(g)$  is the field operator, smoothed out with a form factor  $g \in L^2(\mathbb{R}^3, d^3k)$ :

$$\Phi(g) = \frac{1}{\sqrt{2}}(a^*(g) + a(g)).$$

The algebra of coupled system observables is given by  $\mathfrak{A} = \mathfrak{B}(\mathbb{C}^N) \otimes \mathfrak{A}_R$ , where  $\mathfrak{A}_R$  is defined by (3.32). The dynamics on  $\mathfrak{A}$  is generated by

$$\alpha(A) = e^{itH} A e^{-itH}, \forall A \in \mathfrak{A}. \quad (3.45)$$

It follows from the results in previous sections, we can construct a GNS-representation  $(\mathcal{H}, \pi, \Omega)$  of  $(\mathcal{H}_S \otimes \mathcal{H}_R, \alpha, w_0)$  as following:

$$\begin{aligned}\mathcal{H} &= \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathfrak{H}_R \otimes \mathfrak{H}_R, \\ \pi &= \pi_S \otimes \pi_R, \\ \Omega &= \Omega_S \otimes \Omega_R.\end{aligned}\tag{3.46}$$

The dynamics of  $\pi(\alpha(A))$  is generated by *the Liouvilian*  $\mathcal{L}$  as

$$\pi(\alpha(A)) = e^{it\mathcal{L}} A e^{-it\mathcal{L}}, \forall A \in \mathfrak{A},\tag{3.47}$$

where

$$\begin{aligned}\mathcal{L} &= \sigma \mathcal{L}_S + \mathcal{L}_R + \lambda \mathcal{V}, \\ \mathcal{V} &= G \otimes \mathbb{I} \otimes \varphi_\beta(g).\end{aligned}\tag{3.48}$$

### 3.4 Jakšić-Pillet gluing representation

In this section, we construct a unitary equivalent representation which is more convenient for us to use the spectral translation techniques. This procedure was first introduced by Jakšić and Pillet [17] and therefore known as *Jakšić-Pillet gluing*. Now we construct an isometric isomorphism  $\mathcal{U}$  to show that

$$\mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \cong \mathcal{F}(L^2(\mathbb{R} \times S^2, dud\Sigma)),\tag{3.49}$$

where  $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$  is the unit sphere in  $\mathbb{R}^3$ . We call  $\mathcal{H}_R := \mathcal{F}(L^2(\mathbb{R} \times S^2, dud\Sigma))$  the *Jakšić-Pillet glued space*. First, we construct a unitary map

$$\begin{aligned} \mathcal{V}_1 : \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) &\mapsto \mathcal{F}(L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k)), \\ \mathcal{V}_1(a^*(f_1) \cdots, a^*(f_m) \otimes a^*(g_1) \cdots, a^*(g_n))\Omega & \\ = a^*(f_1 \oplus 0) \cdots, a^*(f_m \oplus 0)a^*(0 \oplus g_1) \cdots, a^*(0 \oplus g_n)\Omega, & \quad (3.50) \\ \mathcal{V}_1(\Omega \oplus \Omega) &= \Omega, \end{aligned}$$

where  $\Omega$ s are vacua in corresponding spaces. Recall that the inner product in  $L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k)$  is defined by

$$\langle f_1 \oplus g_1, f_2 \oplus g_2 \rangle = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle.$$

Next, we construct an isometric isomorphism

$$\begin{aligned} \mathcal{V}_2 : L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k) &\mapsto L^2(\mathbb{R} \times S^2, dud\Sigma) \\ \mathcal{V}_2(f \oplus g)(u, \Sigma) &= \begin{cases} uf(u\Sigma), & u \geq 0, \\ ug(-u\Sigma), & u < 0, \end{cases} \end{aligned} \quad (3.51)$$

where  $f \oplus g, f$  and  $g$  are represented in polar coordinates. Then the second quantization  $\Gamma(\mathcal{V}_2)$  of  $\mathcal{V}_2$  is an isometric isomorphism between  $\mathcal{F}(L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k))$  and  $\mathcal{F}(L^2(\mathbb{R} \times S^2, dud\Sigma))$ . Therefore, the composite map  $\mathcal{U} = \mathcal{V}_2 \circ \mathcal{V}_1$  is an isometric isomorphism.

Next, we consider the unitary transformed Liouvillian acting on the glued Hilbert space  $\mathcal{F}(L^2(\mathbb{R} \times S^2, dud\Sigma))$  as the following:

$$L = \mathcal{U}\mathcal{L}\mathcal{U}^{-1}, \quad (3.52)$$

where the  $\mathcal{L}$  is defined by (3.48). Note that  $\mathcal{U}$  acts trivially on the particle space  $\mathcal{H}_S \otimes \mathcal{H}_S$ , we only need to calculate  $\mathcal{U}\mathcal{L}_R\mathcal{U}^{-1}$  and  $\mathcal{U}\mathcal{V}\mathcal{U}^{-1}$ . First, note that

$$\begin{aligned} \mathcal{V}_1 e^{itd\Gamma(w)} \otimes e^{-itd\Gamma(w)} \mathcal{V}_1^{-1} &= \mathcal{V}_1 \Gamma(e^{itw}) \otimes \Gamma(e^{-itw}) \mathcal{V}_1^{-1} \\ &= \Gamma(e^{itw} \oplus e^{-itw}) \end{aligned} \quad (3.53)$$

and

$$\mathcal{V}_2 \Gamma(e^{itw} \oplus e^{-itw}) \mathcal{V}_2^{-1} = \Gamma(e^{itu}) = e^{itd\Gamma(u)}, \quad (3.54)$$

where  $u$  is the first variable of functions in  $L^2(\mathbb{R} \times S^2)$ , we have

$$\begin{aligned} e^{it\mathcal{U}\mathcal{L}_R\mathcal{U}^{-1}} &= \mathcal{U} e^{it\mathcal{L}_R} \mathcal{U}^{-1} \\ &= \mathcal{V}_2 \circ \mathcal{V}_1 e^{itd\Gamma(w)} \otimes e^{-itd\Gamma(w)} \mathcal{V}_1^{-1} \circ \mathcal{V}_2^{-1} \\ &= e^{itd\Gamma(u)}, \end{aligned} \quad (3.55)$$

and so

$$L_R = \mathcal{U}\mathcal{L}_R\mathcal{U}^{-1} = d\Gamma(u). \quad (3.56)$$

Next, we calculate  $V = \mathcal{U}\mathcal{V}\mathcal{U}^{-1}$ . It follows from the constructions of  $\mathcal{V}_1$  and  $\mathcal{V}_2$  that

$$\mathcal{U}a^*(f) \otimes \mathbb{I}\mathcal{U}^{-1} = a^*(uf(u\Sigma)\mathbf{1}_{\geq 0}), \quad (3.57)$$

where  $\mathbf{1}_{\geq 0} = \mathbf{1}_{\geq 0}(u)$  is the indicator function of  $u \in [0, \infty)$ . Hence, we have

$$V = \mathcal{U}\mathcal{V}\mathcal{U}^{-1} = G \otimes \mathbb{I} \otimes \Phi(g_\beta), \quad (3.58)$$

where  $\mathcal{V}$  is defined by (3.48) and

$$\Phi(g_\beta) = \frac{(a^*(g_\beta) + a(g_\beta))}{\sqrt{2}}.$$

Here we construct the function  $g_\beta(u, \sigma) \in L^2(\mathbb{R} \times S^2)$  as the following:

$$g_\beta(u) = \begin{cases} \sqrt{1 + \mu(u)}ug(u\Sigma), & u \geq 0; \\ \sqrt{\mu(-u)}u\bar{g}(-u\Sigma), & u < 0, \end{cases} \quad (3.59)$$

where  $g(u\Sigma) \in L^2(\mathbb{R}^3)$  is represented in polar coordinates. Therefore, we have

$$L = \mathcal{U}\mathcal{L}\mathcal{U}^{-1} = \sigma L_S + L_R + \lambda V, \quad (3.60)$$

where  $V$  is given by (3.58) and  $L_R = d\Gamma(u)$ .

# Chapter 4

## Resonances and Dynamics

In this chapter, we prove three main results. In theorem 4.1.1, we show the resolvent representation of dynamics, which is key bridge connecting the dynamics and the spectrum. In Theorem 4.3.1, we obtain the spectrum of the associated operator  $K_\theta(\sigma, \lambda)$ . Based on Theorems 4.1.1 and 4.3.1, we present the proof of Theorem 2.2.1, which describes the representation of density matrix elements under the energy basis, at the end of this chapter.

### 4.1 Resolvent representation of the dynamics

For  $\theta \in \mathbb{R}$ , let  $U_\theta$  be the unitary translation on  $\mathcal{H}_R = \mathcal{F}(L^2(\mathbb{R} \times S^2))$  defined by sector-wise action  $U_\theta \Omega_R = \Omega_R$  and

$$U_\theta \psi_n(u_1, \Sigma_1, \dots, u_n, \Sigma_n) = \psi_n(u_1 + \theta, \Sigma_1, \dots, u_n + \theta, \Sigma_n), \quad (4.1)$$

for all  $\{\psi_n\}_{n=0}^\infty \in \mathcal{H}_R$ . A vector  $\psi \in \mathcal{H}_R$  is called  $U_\theta$ -analytic if the map  $\theta \mapsto U_\theta \psi$  is  $\mathcal{H}_R$ -valued analytic in  $\{\theta \in \mathbb{C} : |\theta| < \theta_0\}$ . It is easy to show that all vectors of the



form  $\psi \otimes \Omega_R$  are  $U_\theta$ -analytic for arbitrary  $\psi \in \mathcal{H}_S$ . We introduce the *reference state*

$$\Omega = \Omega_S \otimes \Omega_R, \quad (4.2)$$

where  $\Omega_R$  is the vacuum in  $\mathcal{H}_R$  and  $\Omega_S$  is the trace state

$$\Omega_S = \frac{1}{\sqrt{N}} \sum_{a=1}^N \varphi_a \otimes \varphi_a. \quad (4.3)$$

Recall that  $\{\varphi_a\}_{a=1}^N$  is the collection of normalized eigenbasis of  $G$  (see (2.6)). It follows from GNS-representation theory,  $\Omega$  is a cyclic and separating vector for  $\mathfrak{M}$ . We denote the associated modular operator and modular conjugation by  $\Delta$  and  $J$ , respectively. We give a brief overview of Tomita-Takesaki modular theory in Appendix B.1. The explicit expressions for the modular data  $\Delta$  and  $J$  are given by

$$\Delta = \Delta_S \otimes \Delta_R \text{ and } J = J_S \otimes J_R, \quad (4.4)$$

$$\Delta_S = e^{-\beta L_S} = \mathbb{I},$$

$$\Delta_R = e^{-\beta L_R}, \quad (4.5)$$

$$J_S \phi_l \otimes \phi_r = \mathcal{C} \phi_r \otimes \mathcal{C} \phi_l,$$

$$J_R \psi_n(u_1, \Sigma_1, \dots, u_n, \Sigma_n) = \overline{\psi_n(-u_1, \Sigma_1, \dots, -u_n, \Sigma_n)}.$$

The first equation in (4.5) is due to the fact that the trace state is KMS with inverse temperature  $\beta = 0$ . Here, the action of operator  $\mathcal{C}$  is to take the complex conjugate of vector components in the basis  $\{\varphi_a\}_{a=1}^N$ . On the dense set  $\mathfrak{M}\Omega$  we define the group  $\mathcal{U}(t)$  by

$$\mathcal{U}(t)A\Omega = e^{itL(\sigma, \lambda)} A e^{-itL(\sigma, \lambda)} \Omega, A \in \mathfrak{M}, t \in \mathbb{R}, \quad (4.6)$$

where  $L(\sigma, \lambda)$  is the Liouvillian given by (3.60):

$$L(\sigma, \lambda) = \sigma L_p + L_R + \lambda V.$$

We also introduce the linear space

$$\mathcal{D}_0 = \text{Dom}(L_R) \cap \text{Dom}(N^{1/2}) \cap \mathfrak{M}\Omega \subset \mathcal{H}, \quad (4.7)$$

where  $N = d\Gamma(\mathbb{I})$  is the number operator.

**Proposition 4.1.1.** (a)  $\mathcal{U}(t)$  is strongly differentiable on  $\mathcal{D}_0$  and its generator is given by

$$i \frac{d}{dt} \Big|_{t=0} \mathcal{U}(t) = K(\sigma, \lambda) := L_0(\sigma) + \lambda V - \lambda J \Delta^{1/2} V J \Delta^{1/2}, \quad (4.8)$$

where  $L_0(\sigma)$  is given by (2.10).

(b)  $\theta \mapsto U_\theta K(\sigma, \lambda) U_\theta^*$  has an analytic continuation from  $\theta \in \mathbb{R}$  to  $\{\theta \in \mathbb{C} : |\theta| < \theta_0\}$ , in the strong sense on  $\mathcal{D}_0$ . This continuation is given by

$$K_\theta(\sigma, \lambda) = L_{0,\theta}(\sigma) + \lambda I_\theta, \quad (4.9)$$

where

$$\begin{aligned} L_{0,\theta}(\sigma) &= L_0(\sigma) + \theta N, \\ I_\theta &= V_\theta - V'_\theta, \\ V_\theta &= \frac{1}{\sqrt{2}} G \otimes I \otimes (a^*(g_\beta(\cdot + \theta)) + a(g_\beta(\cdot + \bar{\theta}))), \\ V'_\theta &= \frac{1}{\sqrt{2}} I \otimes G \otimes (a^*(e^{\frac{\beta}{2}(\cdot + \theta)} \bar{g}_\beta(-\cdot - \bar{\theta})) + a(e^{-\frac{\beta}{2}(\cdot + \bar{\theta})} \bar{g}_\beta(-\cdot - \theta))). \end{aligned} \quad (4.10)$$

Here notation  $\bar{g}_\beta(z)$  is the complex conjugation of  $g_\beta(z)$ , i.e.  $\bar{g}_\beta(z) = \overline{g_\beta(z)}$ .

*Proof.* To simplify the notation, we do not write the dependence of the operators on  $(\sigma, \lambda)$  in this proof.

(a) Let  $A\Omega \in \mathcal{D}_0$ , then

$$\begin{aligned}
iKA\Omega &= \frac{d}{dt}\Big|_{t=0}U(t)A\Omega \\
&= \lim_{\delta \rightarrow 0} \frac{e^{iL\delta}Ae^{-iL\delta} - A}{\delta}\Omega \\
&= \lim_{\delta \rightarrow 0} \left[ \frac{e^{iL\delta}Ae^{-iL\delta} - e^{iL\delta}A}{\delta}\Omega + \frac{e^{iL\delta}A - A}{\delta}\Omega \right] \\
&= -iAL\Omega + iLA\Omega \\
&= -iA(L_0 + \lambda V)\Omega + i(L_0 + \lambda V)A\Omega.
\end{aligned}$$

So

$$\begin{aligned}
(K - L_0 - \lambda V)A\Omega &= -A(L_0 + \lambda V)\Omega \\
&= -\lambda AV\Omega \\
&= -\lambda J\Delta^{1/2}J\Delta^{1/2}AV\Omega \\
&= -\lambda J\Delta^{1/2}V^*A^*\Omega \\
&= -\lambda J\Delta^{1/2}VJ\Delta^{1/2}A\Omega.
\end{aligned}$$

This proves part (a).

(b) For real  $\theta$ , we have the following equation

$$\begin{aligned}
U_\theta KU_\theta^* &= L_0 + \theta N + \frac{\lambda}{\sqrt{2}}G \otimes \mathbb{1} \otimes (a^*(g_\beta(\cdot + \theta)) + a(g_\beta(\cdot + \theta))) \\
&\quad - \frac{\lambda}{\sqrt{2}}\mathbb{1} \otimes G \otimes (a^*(e^{\frac{\beta}{2}(\cdot + \theta)}\bar{g}_\beta(-\cdot - \theta)) + a(e^{-\frac{\beta}{2}(\cdot + \theta)}\bar{g}_\beta(-\cdot - \theta))).
\end{aligned} \tag{4.11}$$

By assumption (A) that  $\theta \rightarrow g_\beta(u + \theta)$  has analytic extension to a complex neighbourhood  $\{|z| < \theta_0\}$ ,  $U_\theta KU_\theta^*$  has an analytic continuation from  $\theta \in \mathbb{R}$  to the strip  $\theta \in S_{\theta_0}$ .

It follows from (4.11) that (4.9) and (4.10) hold. Note that in the argument of the annihilation operators, the analytic extension has the complex conjugate  $\bar{\theta}$ , since the annihilation operators are anti-linear in their argument.

□

**Theorem 4.1.1.** *Let  $\theta$  with  $0 < \text{Im}\theta < \theta_0$  be fixed. There is a  $\lambda_0 > 0$  such that for all  $|\lambda| < \lambda_0$  and all  $\sigma \in \mathbb{R}$ , we have the following. Let  $\phi \in \mathcal{H}$  and  $A \in \mathfrak{M}$  be such that  $\phi$  and  $A\Omega$  are  $U_\theta$ -analytic vectors, and such that  $\phi_{\bar{\theta}} \in \text{Dom}(|L_{\mathbb{R}}|^{\frac{1}{4}+\eta})$ , for some  $0 < \eta < \frac{1}{4}$ . Then we have for all  $t \geq 0$*

$$\langle \phi, e^{itL(\sigma,\lambda)} A e^{-itL(\sigma,\lambda)} \Omega \rangle = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \phi_{\bar{\theta}}, (K_\theta(\sigma, \lambda) - z)^{-1} (A\Omega)_\theta \rangle dz. \quad (4.12)$$

We give a proof of this result in Appendix A.1.

**Remarks.** 1. Vectors representing product states of an arbitrary small system state and the equilibrium reservoir states are of the form  $\phi = B\Omega$ , where  $B \in \mathcal{M}_S$  (and, recall,  $\Omega$  is given in (4.2)). The proof of (4.12) for such  $\phi$  and  $A \in \mathcal{M}_S$  is much easier than that of the full result. This is the situation of [29].

2. In [27] a spectral dilation deformation is performed simultaneously with the translation (see also [4, 28]). In this doubly-deformed situation, the analogue of Theorem 4.1.1 is proven in Section 8 of [27]. The dilation deforms the spectrum of  $K$  in a ‘sectorial way’, leading to useful decay estimates of the (deformed) resolvent  $(K - z)^{-n}$ , as  $|\text{Re}z| \rightarrow \infty$ . However, in the present work, we only use spectral translation and such decay estimates do not hold (as the distance between the spectrum of  $K_\theta$  and the real axis does not grow now, as  $|\text{Re}z| \rightarrow \infty$ ). We therefore need a new proof of this result.

## 4.2 Resonances of $K(\sigma = 0, \lambda)$

The operator  $K_\theta(0, \lambda)$  is defined in Proposition 4.1.1, with  $L_0 = L_R$ . Recall that  $\varphi_a$ ,  $a = 1, \dots, N$ , is the orthonormal eigenbasis of  $G$ , (2.6). The operator  $K_\theta(0, \lambda)$  is reduced by the decomposition

$$\mathcal{H} = \bigoplus_{a,b=1}^N \text{Ran} \left( |\varphi_a\rangle\langle\varphi_a| \otimes |\varphi_b\rangle\langle\varphi_b| \right) \otimes \mathcal{H}_R.$$

Namely,

$$K_\theta(0, \lambda) = \bigoplus_{a,b=1}^N K_{a,b}, \quad (4.13)$$

where  $K_{a,b}$  acts on  $\mathcal{H}_R$  as

$$K_{a,b} = L_R + \theta N + \lambda(g_a \Phi_\theta - g_b \tilde{\Phi}_\theta), \quad (4.14)$$

with

$$\begin{aligned} \Phi_\theta &= \frac{1}{\sqrt{2}} \left( a^*(g_\beta(\cdot + \theta)) + a(g_\beta(\cdot + \bar{\theta})) \right), \\ \tilde{\Phi}_\theta &= \frac{1}{\sqrt{2}} \left( a^*(e^{\frac{\beta}{2}(\cdot + \theta)} \bar{g}_\beta(-\cdot - \bar{\theta})) + a(e^{-\frac{\beta}{2}(\cdot + \bar{\theta})} \bar{g}_\beta(-\cdot - \theta)) \right). \end{aligned} \quad (4.15)$$

To abbreviate the notation we do not display  $\theta$  and  $\lambda$  in  $K_{a,b}$ .

**Theorem 4.2.1** (Spectrum of  $K_{a,b}$ ). *Let  $\theta$  with  $0 < \text{Im}\theta < \theta_0$  be fixed. There is a  $\lambda_0 > 0$  such that if  $0 \leq |\lambda| < \lambda_0$ , then for all  $a, b = 1, \dots, N$ , the operator  $K_{a,b}$  has a simple eigenvalue  $\lambda^2 \delta_{a,b}$ , where  $\delta_{a,b}$  is given in 2.15. All other spectrum of  $K_{a,b}$  lies in  $\{z \in \mathbb{C} : \text{Im}z > \frac{3}{4}\text{Im}\theta\}$ .*

**Remarks.** 1. It follows from Theorem 4.2.1 and the decomposition (4.13) that the spectrum of  $K_\theta(0, \lambda)$  in the strip  $\{z \in \mathbb{C} : \text{Im}z < \frac{3}{4}\text{Im}\theta\}$  consists precisely of the eigenvalues  $\{\lambda^2 \delta_{a,b}\}_{a,b=1}^N$  (there are no higher order terms in  $\lambda$ ). A simple expression

for the eigenvectors associated to the non-zero eigenvalues is not available, only a perturbation series is. However, it is readily seen that the eigenvalue zero has the eigenvectors  $\varphi_a \otimes \varphi_a \otimes \Omega_{\mathbb{R}}$ ,  $a = 1, \dots, N$ . Indeed, if  $a = b$ , then it follows directly from (4.14) that

$$K_{a,a}\Omega_{\mathbb{R}} = \lambda g_a U_\theta (\Phi - J\Delta^{1/2}\Phi J\Delta^{1/2})\Omega_{\mathbb{R}} = 0, \quad (4.16)$$

as  $J\Delta^{1/2}\Phi J\Delta^{1/2}\Omega_{\mathbb{R}} = \Phi\Omega_{\mathbb{R}}$ .

2. If the form factor  $g$  satisfies  $\|g_\beta/u\|_2^2 < \infty$ , then the operator  $K_{a,b}$ , (4.14), is unitarily equivalent to the operator  $L_{\mathbb{R}} + \text{const}$ . The condition on the form factor implies the infra-red behaviour  $g(k) \sim |k|^p$  for small  $k$ , with  $p > -1/2$ . Then  $K_{a,b}$  has a simple real eigenvalue, as also predicted by (2.15), saying that  $\text{Im}\delta_{a,b} = 0$ . In the infra-red singular case,  $p = -1/2$ , the unitary transformation ceases to exist and the eigenvalue becomes complex.

The following result, taken from [29], Proposition 4.2, is needed in the proof of theorem 4.2.1.

**Proposition 4.2.1** ([29]). *Let  $\psi_0 \in \mathcal{H}_S$ . Then*

$$\left| \int_{\mathbb{R} + i\frac{3}{4}\text{Im}\theta} e^{itz} \langle \psi_0 \otimes \Omega_{\mathbb{R}}, (K_\theta(\sigma, \lambda) - z)^{-1} \psi_0 \otimes \Omega_{\mathbb{R}} \rangle dz \right| \leq C\lambda^2 e^{-\frac{3}{4}t\text{Im}\theta},$$

*uniformly in  $\sigma$  varying in compact sets. The same bound holds if  $K_\theta(\sigma, \lambda)$  is replaced by  $K_{a,b}$ .*

**Proof of Theorem 4.2.1.** The spectrum of  $K_{a,b}$  for  $\lambda = 0$  consists of a single simple eigenvalue at zero, with eigenvector  $\Omega_{\mathbb{R}}$ , and of horizontal lines of continuous spectrum  $\{x + \text{Im}\theta n : x \in \mathbb{R}, n = 1, 2, \dots\}$ . The operators  $\Phi_\theta$  and  $\tilde{\Phi}_\theta$  are infinitesimally small w.r.t.  $N$  (relatively bounded with arbitrarily small relative bound). Analytic

perturbation theory implies that there exists a  $\lambda_0 > 0$  such that if  $0 \leq |\lambda| < \lambda_0$ , then the only spectrum of  $K_{a,b}$  in  $\{z \in \mathbb{C} : \text{Im}z < \text{Im}\theta/2\}$  is a single, simple eigenvalue. We show that this eigenvalue is  $\lambda^2\delta_{a,b}$ , with  $\delta_{a,b}$  given in (2.15).

The dynamics of the *reduced density matrix* of the small system has been calculated explicitly in Proposition 7.4 of [29]. Let  $\psi_0 = B\Omega_S \otimes \Omega_R$  be an initial state, where  $B \in \mathfrak{M}'_S$  (the commutant) is arbitrary (see also (4.2)). The reduced system density matrix at time  $t$ , in the basis  $\{\varphi_a\}$ , is given by  $[\rho_t]_{a,b} = \langle \psi_0, e^{itL}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_S)e^{-itL}\psi_0 \rangle$ . It is shown in the above reference that

$$[\rho_t]_{a,b} = [\rho_0]_{a,b} e^{i\lambda^2\alpha_{a,b}(t)}, \quad (4.17)$$

with  $\alpha_{a,b}(t) = (g_a^2 - g_b^2)S(t) + i(g_a - g_b)^2\Gamma(t)$ , where

$$\Gamma(t) = \int_{\mathbb{R}^3} |g(k)|^2 \coth\left(\frac{\beta|k|}{2}\right) \frac{\sin^2\left(\frac{|k|t}{2}\right)}{|k|^2} d^3k, \quad S(t) = \frac{1}{2} \int_{\mathbb{R}^3} |g(k)|^2 \frac{|k|t - \sin|k|t}{|k|^2} d^3k. \quad (4.18)$$

For large times,  $\alpha_{a,b}(t)$  becomes linear,

$$\lim_{t \rightarrow \infty} \frac{\alpha_{a,b}(t)}{t} = \delta_{a,b}, \quad (4.19)$$

with  $\delta_{a,b}$  given in (2.15). We express the reduced density matrix alternatively, using Theorem 4.1.1, as

$$[\rho_t]_{a,b} = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle B^* B \Omega_S \otimes \Omega_R, (K_\theta - z)^{-1} (|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_S) \Omega_S \otimes \Omega_R \rangle dz. \quad (4.20)$$

We use that  $e^{itL}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_S)e^{-itL}B = B e^{itL}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_S)e^{-itL}$ , which holds since  $B \otimes \mathbb{1}_R$  belongs to the commutant  $\mathfrak{M}'$ . It follows from the definition (4.3) that

$(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_S)\Omega_S = \frac{1}{\sqrt{N}}\varphi_b \otimes \varphi_a$ . Therefore, we obtain from (4.20) that

$$\begin{aligned} [\rho_t]_{a,b} &= \frac{1}{\sqrt{N}} \langle B^* B \Omega_S, \varphi_b \otimes \varphi_a \rangle \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Omega_R, (K_{b,a} - z)^{-1} \Omega_R \rangle dz \\ &= [\rho_0]_{a,b} \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Omega_R, (K_{b,a} - z)^{-1} \Omega_R \rangle dz. \end{aligned} \quad (4.21)$$

Comparing (4.21) and (4.17) yields the identity

$$e^{i\lambda^2 \alpha_{a,b}(t)} = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Omega_R, (K_{b,a} - z)^{-1} \Omega_R \rangle dz. \quad (4.22)$$

Denote the unique eigenvalue of  $K_{a,b}$  in  $\{z \in \mathbb{C} : \text{Im}z < \text{Im}\theta/2\}$  by  $\zeta_{a,b}(\lambda)$  and let  $\mathcal{C}_{a,b}$  be a small circle around  $\zeta_{a,b}(\lambda)$  not including any other point of the spectrum of  $K_{a,b}$ . By deforming the contour of integration, we have

$$\frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Omega_R, (K_{a,b} - z)^{-1} \Omega_R \rangle dz = \frac{-1}{2\pi i} \oint_{\mathcal{C}_{a,b}} e^{itz} \langle \Omega_R, (K_{a,b} - z)^{-1} \Omega_R \rangle dz + R_\lambda(t), \quad (4.23)$$

with a remainder term. It follows from Proposition 4.2.1 the remainder term is small in  $\lambda$  and decaying to zero exponentially quickly as  $t \rightarrow \infty$ , more specifically,  $|R_\lambda(t)| \leq C\lambda^2 e^{-\frac{3\text{Im}\theta}{4}t}$  for some constant  $C$ . Since  $\zeta_{a,b}(\lambda)$  is a simple pole of the resolvent  $(K_{a,b} - z)^{-1}$  we can replace  $e^{itz}$  by  $e^{it\zeta_{a,b}(\lambda)}$  in (4.23) and we obtain

$$\frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \Omega_R, (K_{a,b} - z)^{-1} \Omega_R \rangle dz = e^{it\zeta_{a,b}(\lambda)} c_{a,b}(\lambda) + R_\lambda(t), \quad (4.24)$$

where  $c_{a,b}(\lambda) = \frac{-1}{2\pi i} \oint_{\mathcal{C}_{a,b}} \langle \Omega_R, (K_{a,b} - z)^{-1} \Omega_R \rangle dz$ . Combining (4.24) and (4.22) gives

$$e^{i\lambda^2 \alpha_{a,b}(t) - it\zeta_{a,b}(\lambda)} = c_{a,b}(\lambda) + e^{-it\zeta_{a,b}(\lambda)} R_\lambda(t).$$

As  $\text{Im}\zeta_{a,b}(\lambda) < \frac{1}{2}\text{Im}\theta$ , we have  $\lim_{t \rightarrow \infty} e^{-it\zeta_{a,b}(\lambda)} R_\lambda(t) = 0$ . Thus the exponent on the



left hand side converges to a finite number, as  $t \rightarrow \infty$ , and so this exponent, divided by  $t$ , tends to zero as  $t \rightarrow \infty$ . (Note that  $c_{a,b}(\lambda)$  is not zero for small  $\lambda$ , by perturbation theory.) Then, due to (4.19), we have  $\zeta_{a,b}(\lambda) = \lambda^2 \delta_{a,b}$ . The proof of Theorem 4.2.1 is complete.  $\square$

### 4.3 Resonances of $K(\sigma, \lambda)$

We now examine the operator  $K_\theta(\sigma, \lambda)$ , defined in Proposition 4.1.1, (4.9), (4.10), with  $L_0$  given in (2.11). We consider  $K_\theta(\sigma, \lambda)$  as an unperturbed part,  $K_\theta(0, \lambda)$  plus a perturbation  $\sigma L_S$  (see (2.5)). Since the eigenvalues of  $K_\theta(0, \lambda)$  are isolated (Theorem 4.2.1), we can apply analytic perturbation theory to follow them as the perturbation is switched on ( $\sigma \neq 0$ ).

**Theorem 4.3.1. [Spectrum of  $K_\theta(\sigma, \lambda)$ ] Let  $\lambda$  be fixed, satisfying  $0 < |\lambda| < \lambda_0$ , where  $\lambda_0$  is given in Theorem 4.2.1. There is a  $\sigma_0 > 0$  (depending on  $\lambda$ ) s.t. if  $0 \leq \sigma < \sigma_0$ , then the spectrum of  $K_\theta(\sigma, \lambda)$  in the region  $\{z \in \mathbb{C} : \text{Im}z < \frac{1}{2}\text{Im}\theta\}$  consists of simple eigenvalues  $\varepsilon_{a,b}(\sigma, \lambda)$ . Those eigenvalues are analytic functions of  $\sigma$ , given by (2.19). Zero is an eigenvalue of  $T$ , (2.20). It is simple if  $[H_S]_{a,b} \neq 0$  for all  $a \neq b$ .**

We present the proof of this result in Appendix A.2.

*Remark.* The theorem assumes the non-degeneracy condition A3. An analysis in presence of degenerate non-zero resonances  $\lambda^2 \delta_{a,b}$  can be carried out along the same lines. We have done this for the spin-boson model. We have checked that the values for the resonances thus obtained coincide with those obtained in Section 6.2 (to order two in  $\sigma$ ).

## 4.4 Proof of Theorem 2.2.1

Starting from the representation of the dynamics (4.12) given in Theorem 4.1.1 we derive the following result. The proof goes in the same spirit as that of [29, Theorem 3.1].

**Theorem 4.4.1** (Reduced dynamics). *Let  $\chi_1$  be an arbitrary normalized vector in  $\mathcal{H}_S$  and let  $A \in \mathfrak{M}_S$  be a system observable. Then we have*

$$\begin{aligned} & \langle \chi_1 \otimes \Omega_{\mathbb{R}}, e^{itL(\sigma, \lambda)} A e^{-itL(\sigma, \lambda)} \Omega \rangle \\ &= \sum_{a,b=1}^N e^{it\varepsilon_{a,b}(\sigma, \lambda)} \langle \chi_1, Q_{a,b} A \Omega_S \rangle (1 + O_\lambda(\sigma) + O(\lambda)) + O(\lambda^2 e^{-\frac{3}{4}t\theta_0}), \end{aligned} \quad (4.25)$$

where the  $\varepsilon_{a,b}(\sigma, \lambda)$  are given in (2.20). Here,

$$Q_{a,b} = \begin{cases} |\varphi_{a,b}\rangle \langle \varphi_{a,b}| & \text{if } a \neq b \\ |\varphi_a^T\rangle \langle \varphi_a^T| & \text{if } a = b, \end{cases} \quad (4.26)$$

where  $\{\varphi_a^T\}_{a=1}^N$  is the orthonormal basis of eigenvectors of  $T$ , (2.19), so that  $T\varphi_a^T = \xi_a \varphi_a^T$ .

**Proof of Theorem 4.4.1.** Take the representation (4.12) for a fixed  $\theta$ . The integral over the path  $\mathbb{R} - i$  equals the integral over the path  $\mathbb{R} + \frac{3}{4}i\text{Im}\theta$  plus the sum of the integrals around circles  $\Gamma_{a,b}$ , each enclosing exactly one eigenvalue  $\varepsilon_{a,b}$  of  $K_\theta(\sigma, \lambda)$ . While the integral over  $\mathbb{R} + \frac{3}{4}i\text{Im}\theta$  is  $O(\lambda^2 e^{-\frac{3}{4}t\text{Im}\theta})$  (see Proposition 4.2.1), the integral around a given eigenvalue  $\varepsilon_{a,b}$  is

$$\frac{-1}{2\pi i} \oint_{\Gamma_{a,b}} e^{itz} \langle \chi_1 \otimes \Omega_{\mathbb{R}}, (K_\theta(\sigma, \lambda) - z)^{-1} A \Omega \rangle dz = e^{it\varepsilon_{a,b}} \langle \chi_1 \otimes \Omega_{\mathbb{R}}, \tilde{Q}_{a,b} A \Omega \rangle dz,$$

where  $\tilde{Q}_{a,b} = \frac{-1}{2\pi i} \oint_{\Gamma_{a,b}} (K_\theta(\sigma, \lambda) - z)^{-1} dz$  is the Riesz spectral projection. By pertur-

bation theory, we have, for  $a \neq b$ ,

$$\tilde{Q}_{a,b} = |\varphi_{a,b}\rangle\langle\varphi_{a,b}| \otimes |X_{a,b}\rangle\langle X_{a,b}^*| + O_\lambda(\sigma) = |\varphi_{a,b}\rangle\langle\varphi_{a,b}| \otimes |\Omega_R\rangle\langle\Omega_R| + O_\lambda(\sigma) + O(\lambda).$$

Similarly, we have  $\tilde{Q}_{a,a} = |\varphi_a^T\rangle\langle\varphi_a^T| \otimes |\Omega_R\rangle\langle\Omega_R| + O_\lambda(\sigma)$ . (Note that  $T$  is self-adjoint.)

This completes the proof of Theorem 4.4.1.  $\square$

We now prove Theorem 2.2.1. Let  $\rho_0$  be the initial density matrix of the small system. It is represented by a normalized vector  $\chi$  in the GNS space  $\mathcal{H}_S$ . By the cyclicity of  $\Omega_S$  there is a unique element  $B'$  in the commutant  $\mathfrak{M}'_{\mathbb{C}^N} = \mathbb{1}_{\mathbb{C}^N} \otimes \mathcal{B}(\mathcal{H}_S)$  such that  $\chi = B'\Omega_S$ . The evolution of the reduced density matrix elements  $[\rho_t]_{a,b} = \langle\varphi_a, \rho_t \varphi_b\rangle$  is given by

$$\begin{aligned} [\rho_t]_{a,b} &= \langle\chi \otimes \Omega_R, e^{itL(\sigma,\lambda)}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_{\mathbb{C}^N})e^{-itL(\sigma,\lambda)}\chi \otimes \Omega_R\rangle \\ &= \langle\chi \otimes \Omega_R, B'e^{itL(\sigma,\lambda)}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_{\mathbb{C}^N})e^{-itL(\sigma,\lambda)}\Omega\rangle. \end{aligned} \quad (4.27)$$

We can thus use Theorem 4.4.1. The main term on the right side of (4.25) is

$$\sum_{c,d=1}^N e^{it\varepsilon_{c,d}(\sigma,\lambda)} \langle\chi, B'Q_{c,d}(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_{\mathbb{C}^N})\Omega_S\rangle = \frac{1}{\sqrt{N}} \sum_{c,d=1}^N e^{it\varepsilon_{c,d}(\sigma,\lambda)} \langle\chi, B'Q_{c,d}\varphi_{b,a}\rangle, \quad (4.28)$$

by the definition (4.3) of  $\Omega_S$ . If  $a \neq b$  then, according to (4.26),  $Q_{c,d}\varphi_{b,a}$  vanishes, except when  $c = b$  and  $d = a$ , in which case it equals  $\varphi_{b,a}$ . Then we have  $\langle\chi, B'\varphi_{b,a}\rangle = \sqrt{N} \langle\chi, B'(|\varphi_b\rangle\langle\varphi_a| \otimes \mathbb{1}_{\mathbb{C}^N})\Omega_S\rangle = \sqrt{N}[\rho_0]_{a,b}$ . We conclude that for  $a \neq b$ , the main term of  $[\rho_t]_{a,b}$  is  $e^{it\varepsilon_{b,a}(\sigma,\lambda)}[\rho_0]_{a,b}$ . This shows (2.21). Relation (2.22) is proven in the same way.  $\square$

# Chapter 5

## Return to equilibrium and Invariant states

### 5.1 Return to equilibrium

Suppose that zero is a simple eigenvalue of  $K_\theta(\sigma, \lambda)$ . The corresponding eigenvector is  $\Omega = \Omega_S \otimes \Omega_R$ . Let  $\Omega_{SR} = e^{-\beta L(\sigma, \lambda)/2} \Omega_0 / \|e^{-\beta L(\sigma, \lambda)/2} \Omega_0\|$  be the KMS state for the interacting system (see [11]), where  $L(\sigma, \lambda)$  is given in (2.10),  $\Omega_0 = \Omega_{S, \beta} \otimes \Omega_R$  is the non-interacting KMS equilibrium state and  $\Omega_{S, \beta}$  represents the Gibbs state  $\propto e^{-\beta \sigma H_S}$ . Let  $\Psi_0$  be any normalized vector in  $\mathcal{H}$ , and let  $A \in \mathfrak{M}$  be any observable. If

$$\lim_{t \rightarrow \infty} \langle A \rangle_t \equiv \lim_{t \rightarrow \infty} \langle \Psi_0, e^{itL(\sigma, \lambda)} A e^{-itL(\sigma, \lambda)} \Psi_0 \rangle = \langle \Omega_{SR}, A \Omega_{SR} \rangle, \quad (5.1)$$

then we say that the system has the property of return to equilibrium. This property has been shown in [4, 17] using a representation of the dynamics involving the *standard Liouvillian*, and (in a weaker form) in [14, 23], using positive-commutator methods. We show here how to deduce (5.1) from the representation of the dynamics given in Theorem 4.1.1. The difference is that we use the non-selfadjoint Liouvillian  $K$

to represent the dynamics. This operator has been introduced in [18] for Fermion systems. It has been used in [27] for Boson systems, but using a spectral translation *and* dilation. In the present work, we use only spectral translation, which complicates some aspects of the analysis, but reduces the regularity requirements on the interaction form factor.

To show (5.1) using Theorem 4.1.1, we fix any  $\epsilon > 0$  and take a  $B \in \mathfrak{M}'$  (commutant) s.t.  $\|\Psi_0 - B\Omega\| < \epsilon$  (cyclicity of  $\Omega$ ). Then  $\langle A \rangle_t = \langle B^* B \Omega, e^{itL(\sigma, \lambda)} A e^{-itL(\sigma, \lambda)} \Omega \rangle + O(\epsilon)$ , uniformly in time. Choose  $\eta > 0$  so small that  $\Psi_\eta := e^{-\eta L_R^2} e^{-\eta D^2} e^{-4\eta \theta_0^2 N^2} B^* B \Omega$  satisfies  $\|B^* B \Omega - \Psi_\eta\| < \epsilon$ . Here,  $D = d\Gamma(-i\partial_u)$  is the generator of spectral deformation and  $N = d\Gamma(\mathbf{1})$  is the number operator. The regularization of  $\Psi_\eta$  has been chosen so that  $\Psi_\eta$  is  $U_\theta$ -analytic and  $U_{\bar{\theta}}\psi_\eta \equiv (\Psi_\eta)_{\bar{\theta}}$  is in the domain of  $e^{\eta|L_R|/2}$ . Therefore, we obtain from Theorem 4.1.1 that

$$\langle A \rangle_t = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle (\Psi_\eta)_{\bar{\theta}}, (K_\theta(\sigma, \lambda) - z)^{-1} (A\Omega)_\theta \rangle dz + O(\epsilon).$$

One now deforms the contour of integration into the upper half-plane (as in the proof of Theorem 4.4.1 – see also [4, 17, 27]) to pick up the contributions of the poles at the resonance energies of the resolvent. All contributions decay exponentially in  $t$ , except for the one associated with the eigenvalue zero. We obtain  $\lim_{t \rightarrow \infty} \langle A \rangle_t = \langle (\Psi_\eta)_{\bar{\theta}}, Q_0 (A\Omega)_\theta \rangle + O(\epsilon)$ , where  $Q_0 = |\Omega\rangle\langle \Omega_\theta^*|$  is the rank-one Riesz projection of  $K_\theta(\sigma, \lambda)$  associated to the eigenvalue zero. Here,  $\Omega_\theta^*$  is the vector in the kernel of the adjoint  $K_\theta^*(\sigma, \lambda)$  satisfying  $\langle \Omega, \Omega_\theta^* \rangle = 1$ . We have  $\langle (\Psi_\eta)_{\bar{\theta}}, \Omega \rangle = \langle \Psi_\eta, \Omega \rangle \rightarrow 1$  as  $\eta \rightarrow 0$ . Therefore, by first taking  $t \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ , we obtain  $\lim_{t \rightarrow \infty} \langle A \rangle_t = \langle \Omega_\theta^*, (A\Omega)_\theta \rangle$ . Since, on the other hand, the state defined by  $\Omega_{SR}$  is invariant, the asymptotic state has to be  $\Omega_{SR}$ . This implies the property of return to equilibrium (for observables  $A$  such that  $A\Omega$  is  $U_\theta$ -analytic).

## 5.2 Invariant states

**Invariant system-reservoir states.** There is a one-to-one correspondence between normalized vectors in the kernel of the *standard Liouvillian*  $L_{\text{standard}} = L_0(\sigma) + \lambda V - \lambda J V J$  and normal states on  $\mathfrak{M}$  which are invariant under the dynamics generated by  $L$ , (2.10) (see for instance [11]).

For  $\sigma = 0$ , the standard Liouvillian has a direct sum decomposition as in (4.13), with ‘blocks’  $L_{\text{standard},a,b} = L_R + \lambda\{g_a\Phi(g_\beta) - g_b J\Phi(g_\beta)J\}$ . One can perform the spectral analysis of this operator in the same way as we do for  $K(0, \lambda)$  to see that  $\text{Ker}L_{\text{standard}} = \text{span}\{\varphi_a \otimes \varphi_a \otimes \Omega_{R,a}\}_{a=1}^N$ , where

$$\Omega_{R,a} = e^{-\beta(L_R + \lambda g_a \Phi(g_\beta))/2} \Omega_R / \|e^{-\beta(L_R + \lambda g_a \Phi(g_\beta))/2} \Omega_R\|$$

is the ‘perturbed KMS state’ (i.e., the KMS state with respect to the dynamics generated by the Liouvillian  $L_R + \lambda g_a \Phi(g_\beta)$ ). In fact, the reservoir has the property of return to equilibrium for the dynamics of  $\mathfrak{M}_R$  generated by  $L_{\text{standard},a,a}$ . That the states  $\varphi_a \otimes \varphi_a \otimes \Omega_{R,a}$  are invariant can also be seen using Theorem 4.1.1 and an argument similar to the one given above to show return to equilibrium.

For  $\sigma > 0$  and under the condition that  $K_\theta(\sigma, \lambda)$  has one-dimensional kernel, the only invariant state is the coupled equilibrium  $\Omega_{SR}$  introduced in this section above.

**Invariant initial states of the small system for  $\sigma = 0$ .** The explicit expression (4.17) shows that  $\mathcal{M}_{0,\lambda}$ , the manifold of invariant initial system states introduced in the introduction (see (2.23)), is the set of density matrices which are diagonal in the eigenbasis  $\{\varphi_a\}_{a=1}^N$  of  $G$ . Let  $\rho_0$  be a given initial density matrix of the small system and set  $\tau = \sum_a [\rho_0]_{a,a} |\varphi_a\rangle\langle\varphi_a|$ . Then  $\text{dist}(\mathcal{M}_{0,\lambda}, \rho_0) = \|\tau - \rho_0\|_1$ . To see this, let  $\tau_n$  be a sequence in  $\mathcal{M}_{0,\lambda}$  such that  $\lim_{n \rightarrow \infty} \|\tau_n - \rho_0\|_1 = \text{dist}(\mathcal{M}_{0,\lambda}, \rho_0)$ . By the equivalence

of the trace norm and the norm  $\|\rho\|_{\max} = \max_{a,b} |\langle \varphi_a, \rho \varphi_b \rangle| \equiv \max_{a,b} |[\rho]_{a,b}|$ , we have

$$\|\tau_n - \rho_0\|_1 \geq c \|\tau_n - \rho_0\|_{\max} \geq c \max_a |[\tau_n]_{a,a} - [\rho_0]_{a,a}|,$$

for some constant  $c > 0$ . It follows that  $\lim_{n \rightarrow \infty} \max_a |[\tau_n]_{a,a} - [\rho_0]_{a,a}| = 0$  and therefore  $\lim_{n \rightarrow \infty} \|\tau_n - \tau\|_1 = 0$ . This shows that  $\text{dist}(\mathcal{M}_{0,\lambda}, \rho_0) = \|\tau - \rho_0\|_1$ . As the dynamics leaves the diagonal invariant, we also have  $\text{dist}(\mathcal{M}_{0,\lambda}, T_{0,\lambda}(t)\rho_0) = \|\tau - T_{0,\lambda}(t)\rho_0\|_1$ . Again by the equivalence of norms, there is a  $C > 0$  s.t.

$$\|\tau - T_{0,\lambda}(t)\rho_0\|_1 \leq C \max_{a,b:a \neq b} |[T_{0,\lambda}(t)\rho_0]_{a,b}| \leq C e^{-\lambda^2 \gamma_G \Gamma(t)} \max_{a,b:a \neq b} |[\rho_0]_{a,b}|,$$

where we use (4.17) in the last inequality. Finally,  $\max_{a,b:a \neq b} |[\rho_0]_{a,b}| \leq c \|\tau - \rho_0\|_1$ . The statement about orbital stability in the introduction follows. The asymptotic linearity of  $\Gamma(t)$  follows from (4.19). In three dimensions,  $\lim_{t \rightarrow \infty} \Gamma(t) = \infty$  if the infra-red behaviour of the coupling form factor is  $g(k) \sim |k|^{-1/2}$  as  $k \sim 0$ , see (2.15). See also [33].

**Absence of invariant initial system states for  $\sigma > 0$ .** Suppose that zero is a simple eigenvalue of  $K_\theta(\sigma, \lambda)$ . Then for  $\sigma > 0$ , the set of invariant initial system states  $\mathcal{M}_{\sigma,\lambda}$  is empty. Indeed, by the property of return to equilibrium,  $\lim_{t \rightarrow \infty} T_{\sigma,\lambda}(t)\rho_0 = \rho_*$  for all initial states  $\rho_0$ , where  $\rho_*$  is the reduction to the small system of the coupled system-reservoir KMS state  $\Omega_{SR}$  (see at the beginning of this section). Therefore,  $\rho_*$  is the only possible element in  $\mathcal{M}_{\sigma,\lambda}$ . However, that  $\rho_* \notin \mathcal{M}_{\sigma,\lambda}$  can be seen as follows. For any  $A \in \mathcal{B}(\mathbb{C}^N)$  we have

$$\left. \frac{d}{dt} \right|_{t=0} \text{Tr}_{\mathbb{C}^N}(T_{\sigma,\lambda}(t)\rho_* A) = \langle \Omega_* \otimes \Omega_R, i[L(\sigma, \lambda), A \otimes \mathbb{1}_S \otimes \mathbb{1}_R] \Omega_* \otimes \Omega_R \rangle,$$

where  $\Omega_*$  is the vector representative of  $\rho_*$ . The commutator in the last expression

equals  $\sigma[H_S, A] \otimes \mathbb{1}_S \otimes \mathbb{1}_R + \lambda[G, A] \otimes \mathbb{1}_S \otimes \Phi(g_\beta)$ . Therefore, the above derivative is zero if and only if  $\langle \Omega_*, ([H_S, A] \otimes \mathbb{1}_S) \Omega_* \rangle = \langle \Omega_{SR}, ([H_S, A] \otimes \mathbb{1}_S \otimes \mathbb{1}_R) \Omega_{SR} \rangle = 0$ . By expanding  $\Omega_{SR} \propto \Omega_0 - \frac{\lambda}{2} \int_0^\beta e^{-sL_0/2} V \Omega_0 + O(\lambda^2)$  (see at the beginning of this section), we obtain

$$\langle \Omega_{SR}, ([H_S, A] \otimes \mathbb{1}_S \otimes \mathbb{1}_R) \Omega_{SR} \rangle = \frac{\lambda^2 \sigma}{2} \sum_{k,l=1}^N (E_k - E_l) \langle GP_k A P_l G \rangle_{S,\beta} f_{k,l} + O(\lambda^4), \quad (5.2)$$

where  $P_k$  is the spectral projection associated to the eigenvalue  $E_k$  of  $H_S$ , the average  $\langle \cdot \rangle_{S,\beta}$  is taken in the state  $\Omega_{S,\beta}$  and where  $f_{k,l} = \int_{\mathbb{R} \times S^2} |g_\beta(u, \vartheta)|^2 \frac{(e^{\beta u/2} - 1)(e^{-\beta u/2} - 1)}{u^2} du d\vartheta + O(\sigma)$ . For small  $\sigma$ , we have  $f_{k,l} < 0$  for all  $k, l$ . By choosing an  $A$  s.t. the right side of (5.2) does not vanish we obtain  $\frac{d}{dt} \Big|_{t=0} \text{Tr}_{\mathbb{C}^N} (T_{\sigma,\lambda}(t) \rho_* A) \neq 0$ , so  $\rho_*$  is not invariant.



# Chapter 6

## Transition between two regimes

As we stated in previous chapters, our analysis located in overlapping resonances regime. In this chapter, we consider the spin-boson model and find the transition between two regimes. The small system is a spin with Hamiltonian and interaction operator given by

$$H_S = S^z \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad G = S^x \equiv \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. The parameters  $\sigma, \lambda$  are now considered to be small but independent of each other. We analyze the decoherence properties of the spin *in the energy basis*. Let  $\phi_{\pm}^z$  be the normalized energy eigenvectors, satisfying  $H_S \phi_{\pm}^z = \pm \frac{1}{2} \phi_{\pm}^z$ , and denote the spin density matrix elements in this basis by  $[\rho_t]_{+,-}^z := \langle \phi_+^z, \rho_t \phi_-^z \rangle$  (and similarly for other matrix elements). By applying the Feshbach map [5], we analyze the spectrum of operator  $K_{\theta}(\theta, \lambda)$  defined in Proposition 4.1.1 with  $H_S$  and  $G$  given above.

## 6.1 Using the Feshbach map

Now let's recall the theorem of Feshbach map, which is taken from [5].

**Theorem 6.1.1.** *Let  $P$  be a bounded projection on a separable Hilbert space  $\mathcal{H}$  and  $\bar{P} = \mathbb{1} - P$ . For any densely defined, closed operator  $H$  on  $\mathcal{H}$ , whose domain contains the range of  $P$ , we define  $\bar{H} = \bar{P}H\bar{P}$ . In addition, suppose  $z$  is in the resolvent of  $\bar{H}$ . Then*

$$z \in \sigma(H) \Leftrightarrow z \in \sigma(f_{P,z}(H)),$$

where  $\sigma(H)$  and  $\sigma(f_{P,z}(H))$  are the spectrum of  $H$  and  $f_{P,z}(H)$ , respectively, and the Feshbach map is given by

$$f_{P,z}(H) = P(H - H\bar{P}(\bar{H} - z)^{-1}\bar{P}H)P.$$

We know that zero is an eigenvalue of  $K_\theta(0, 0)$  of multiplicity  $N^2$ . By a simple Riesz projection argument, one shows that, for  $\sigma$  and  $\lambda$  small,  $K_\theta(\sigma, \lambda)$  has  $N^2$  eigenvalues in the vicinity of the origin. The size of the eigenvalues can be estimated as follows. Suppose that  $z \neq 0$  and  $\text{Im}z < \frac{1}{2}\text{Im}\theta$ , so that  $z$  is in the resolvent set of  $K_\theta(0, 0)$ . If the series

$$(K_\theta(0, 0) - z)^{-1} \sum_{n \geq 0} [(\sigma L_S + \lambda I_\theta)(K_\theta(0, 0) - z)^{-1}]^n \quad (6.1)$$

converges, then  $z$  belongs to the resolvent set of  $K_\theta(\sigma, \lambda)$  and (6.1) equals  $(K_\theta(\sigma, \lambda) - z)^{-1}$ . Therefore, if  $z$  is a (non-zero) eigenvalue of  $K_\theta(\sigma, \lambda)$ , then we must have

$$\|(\sigma L_S + \lambda I_\theta)(K_\theta(0, 0) - z)^{-1}\| \geq 1. \quad (6.2)$$

Using standard bounds on the interaction, we see that (6.2) implies that there are

constants  $C, c > 0$  s.t. if  $\sigma, |\lambda| < c$ , then

$$|z| < C(\sigma + |\lambda|). \quad (6.3)$$

Estimate (6.3) is a bound on the eigenvalues of  $K_\theta(\sigma, \lambda)$  in the vicinity of the origin. The eigenvalues can be tracked using the Feshbach map. Namely,  $z \in \mathbb{C}$ ,  $\text{Im}z < \frac{1}{2}\text{Im}\theta$  is an eigenvalue of  $K_\theta(\sigma, \lambda)$  if and only if it is an eigenvalue of the operator

$$\mathcal{F}_z = P_R (\sigma L_S - \lambda^2 I_\theta (K_\theta(\sigma, \lambda) - z)^{-1} I_\theta) P_R \quad (6.4)$$

which acts on the smaller space  $\text{Ran}P_R = \mathbb{C}^N \otimes \mathbb{C}^N$ . Recall that  $P_R = |\Omega_R\rangle\langle\Omega_R|$ . By expanding the resolvent around  $z = 0$ ,  $\sigma = 0$  and  $\lambda = 0$ , taking into account (6.3), we have

$$\begin{aligned} \mathcal{F}_z &= P_R (\sigma L_S - \lambda^2 I_\theta K_\theta(0, 0)^{-1} I_\theta) P_R + O\left(\lambda^2(|\sigma| + |\lambda|)\right), \\ &= \sigma P_R L_S P_R - \lambda^2 P_R I_\theta K_\theta(0, 0)^{-1} I_\theta P_R + O\left(\lambda^2(|\sigma| + |\lambda|)\right), \end{aligned} \quad (6.5)$$

provided  $z$  is an eigenvalue of  $K_\theta(\sigma, \lambda)$  and  $\sigma, |\lambda| < c$ .

The following result is derived by an elementary calculation.

**Proposition 6.1.1.** *The operator  $\mathcal{F}_z$ , viewed as acting on  $\text{Ran}P_R$ , has the form*

$$\mathcal{F}_z = \sigma L_S - \lambda^2 (\alpha G^2 \otimes \mathbb{1} - \alpha G \otimes \mathcal{C}G\mathcal{C} + \bar{\alpha} G \otimes \mathcal{C}G\mathcal{C} - \bar{\alpha} \mathbb{1} \otimes \mathcal{C}G^2\mathcal{C}) + O\left(\lambda^2(|\sigma| + |\lambda|)\right), \quad (6.6)$$

where  $\mathcal{C}$  is defined after (2.5) and  $\alpha = \frac{1}{2} \langle g, |k|^{-1} g \rangle - \frac{1}{2} \pi \xi(0)$ , with  $\xi(0)$  given in (2.16).

**Proof of Proposition 6.1.1** Let  $Y$  be the second term in (6.5),

$$\begin{aligned} Y &= \lambda^2 P_R I_\theta K_\theta(0, 0)^{-1} I_\theta P_R \\ &= \lambda^2 P_R I_\theta (L_R + \theta N)^{-1} I_\theta P_R, \end{aligned} \quad (6.7)$$

and

$$Y_\epsilon = \lambda^2 P_R I_\theta (L_R + \theta N + i\epsilon)^{-1} I_\theta P_R, \quad (6.8)$$

then  $Y_\epsilon \rightarrow Y$  as  $\epsilon \rightarrow 0^+$ . Note that the r.h.s. of (6.8) is analytic with respect to  $\theta$ , and it is constant for real  $\theta$ , we can set  $\theta = 0$  in (6.8), and so

$$Y_\epsilon = \lambda^2 P_R I (L_R + i\epsilon)^{-1} I P_R. \quad (6.9)$$

By the algebraic structure of interaction term  $I$ , we have

$$\begin{aligned} Y_\epsilon &= \lambda^2 P_R I (L_R + i\epsilon)^{-1} I P_R \\ &= \lambda^2 (\alpha_\epsilon G^2 \otimes \mathbb{1} \otimes P_R - \alpha_\epsilon G \otimes \mathcal{C}GC \otimes P_R + \bar{\alpha}_\epsilon G \otimes \mathcal{C}GC \otimes P_R - \bar{\alpha}_\epsilon \mathbb{1} \otimes \mathcal{C}G^2\mathcal{C} \otimes P_R), \end{aligned} \quad (6.10)$$

with

$$\begin{aligned} \alpha_\epsilon &= \langle \Omega, \Phi_\beta(g) (L_R + i\epsilon)^{-1} \Phi_\beta \Omega \rangle \\ &= \frac{1}{2} (\langle \Omega, a_\beta(g) (L_R + i\epsilon)^{-1} a_\beta^*(g) \Omega \rangle + \langle \Omega, a_\beta^*(g) (L_R + i\epsilon)^{-1} a_\beta(g) \Omega \rangle). \end{aligned} \quad (6.11)$$

By an elementary calculation and the facts  $L_R = H_R \otimes \mathbb{1} - \mathbb{1} \otimes H_R$ ,  $a(k)H_R = (H_R + |k|)a(k)$  and  $a(k)a^*(k') = a^*(k')a(k) + \delta(k - k')$ , we have  $\alpha_\epsilon \rightarrow \alpha = \frac{1}{2} \langle g, |k|^{-1}g \rangle - \frac{i}{2} \pi \xi(0)$ . This completes the proof of Proposition 6.1.1.  $\square$

Note that the quadratic term in  $\lambda$  is diagonal in the basis  $\varphi_{a,b}$ ,

$$\begin{aligned} & -\lambda^2 (\alpha G^2 \otimes \mathbb{1} - \alpha G \otimes \mathcal{C}GC + \bar{\alpha} G \otimes \mathcal{C}GC - \bar{\alpha} \mathbb{1} \otimes \mathcal{C}G^2\mathcal{C}) \\ &= -\frac{\lambda^2}{2} \sum_{a,b=1}^N (\langle g, |k|^{-1}g \rangle (g_a^2 - g_b^2) - i\pi \xi(0) (g_a - g_b)^2) |\varphi_{a,b}\rangle \langle \varphi_{a,b}|. \end{aligned} \quad (6.12)$$

We conclude from the isospectrality of the Feshbach map and (6.6), (6.12) that the

eigenvalues of  $K_\theta(0, \lambda)$  are given by  $-\frac{\lambda^2}{2} (\langle g, |k|^{-1}g \rangle (g_a^2 - g_b^2) - i\pi\xi(0)(g_a - g_b)^2)$ , modulo a remainder  $O(\lambda^3)$ . This is compatible with the result of Theorem 4.2.1. However, from that Theorem, we know in addition that the remainder actually vanishes.

## 6.2 The spin-boson system

The Feshbach operator (6.6) is represented in the energy basis  $\{\phi_{+,+}, \phi_{+,-}, \phi_{-,+}, \phi_{-,-}\}$ , where  $\phi_{+,-} = \phi_+ \otimes \phi_-$  (etc) and  $S^z\phi_\pm = \pm\frac{1}{2}\phi_\pm$ , by the matrix

$$\mathcal{F}_z = W + O(\lambda^2(\sigma + |\lambda|)), \quad (6.13)$$

$$W = \begin{pmatrix} i\frac{\lambda^2}{4}\pi\xi(0) & 0 & 0 & -i\frac{\lambda^2}{4}\pi\xi(0) \\ 0 & \sigma + i\frac{\lambda^2}{4}\pi\xi(0) & -i\frac{\lambda^2}{4}\pi\xi(0) & 0 \\ 0 & -i\frac{\lambda^2}{4}\pi\xi(0) & -\sigma + i\frac{\lambda^2}{4}\pi\xi(0) & 0 \\ -i\frac{\lambda^2}{4}\pi\xi(0) & 0 & 0 & i\frac{\lambda^2}{4}\pi\xi(0) \end{pmatrix}. \quad (6.14)$$

The four eigenvalues of  $W$  are

$$w_1 = 0, \quad w_2 = i\frac{\lambda^2}{2}\pi\xi(0), \quad w_{3,4} = i\frac{\lambda^2}{4}\pi\xi(0) \pm i\sqrt{\frac{\lambda^4}{16}\pi^2\xi(0)^2 - \sigma^2}, \quad (6.15)$$

where the square root is the principal branch with branch cut on the negative real axis. The corresponding eigenvectors of  $W$  are

$$\chi_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \chi_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \chi_3 = \frac{1}{1+r^2} \begin{bmatrix} 0 \\ 1 \\ r \\ 0 \end{bmatrix}, \quad \chi_4 = \frac{1}{1+r^2} \begin{bmatrix} 0 \\ -r \\ 1 \\ 0 \end{bmatrix}, \quad (6.16)$$

where  $r = \frac{-4i\gamma - \sqrt{\pi^2\xi(0)^2 - 16\gamma^2}}{\pi\xi(0)}$  with  $\gamma = \frac{\sigma}{\lambda^2}$ . The eigenvalues of the adjoint  $W^*$  are the complex conjugates  $\bar{w}_j$  and the corresponding eigenvectors are

$$\chi_1^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \chi_2^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \chi_3^* = \begin{bmatrix} 0 \\ 1 \\ \bar{r} \\ 0 \end{bmatrix}, \quad \chi_4^* = \begin{bmatrix} 0 \\ -\bar{r} \\ 1 \\ 0 \end{bmatrix}. \quad (6.17)$$

The eigenvectors are normalized as  $\langle \chi_i, \chi_i^* \rangle = 1$  and  $\langle \chi_i, \chi_j^* \rangle = 0$  if  $i \neq j$ . The reduced spin density matrix, represented in the energy basis  $\phi_{\pm}$ , is given by (proceed as for Theorem 2.2.1 or see [24, Theorem 2.1] and [29])

$$[\rho_t]_{m,n}^z \doteq \sum_{j=1}^4 e^{itw_j} \sum_{k,l=\pm} [\rho_0]_{l,k}^z \langle \phi_{k,l}, \chi_j \rangle \langle \chi_j^*, \phi_{n,m} \rangle. \quad (6.18)$$

Here, we take  $m, n, k, l$  to stand for either  $+$  or  $-$ , and  $\doteq$  means that we approximate the true resonances  $\varepsilon$  (the eigenvalues of  $\mathcal{F}_z$ ) by the  $w$  and we neglect additive  $O(\lambda^2)$  terms (uniform in  $t \geq 0$ ) on both sides. Using the explicit formulas (6.16), (6.17) for the eigenvectors  $\chi_j, \chi_j^*$ , we arrive at

$$\begin{aligned} [\rho_t]_{+,+}^z &\doteq \frac{1}{2} + \frac{1}{2} e^{itw_2} ([\rho_0]_{+,+}^z - [\rho_0]_{-,-}^z), \\ [\rho_t]_{+,-}^z &\doteq \frac{r}{r^2+1} e^{itw_3} (r[\rho_0]_{+,-}^z + [\rho_0]_{-,+}^z) + \frac{1}{r^2+1} e^{itw_4} ([\rho_0]_{+,-}^z - r[\rho_0]_{-,+}^z). \end{aligned} \quad (6.19)$$

It follows from (6.19) that the convergent speed of matrix elements depends on the imaginary part of eigenvalues (6.15). However, the eigenvalues involve parameter ratio  $\gamma = \frac{\sigma}{\lambda^2}$ . We define the critical value  $\gamma_*$  for which the square root in  $w_{3,4}$  vanishes,

$$\gamma_* := \frac{1}{4} \pi \xi(0).$$

This critical value separates two regimes with different qualitative behaviour of the resonances  $w_3$  and  $w_4$ . As  $\gamma$  increases from zero to  $\gamma_*$ , the resonance  $w_3$  moves down the imaginary axis, decreasing from the initial value  $\frac{1}{2}i\pi\xi(0)\lambda^2$  to  $\frac{1}{4}i\pi\xi(0)\lambda^2$ , while  $w_4$  moves up the imaginary axis, from the origin to  $\frac{1}{4}i\pi\xi(0)\lambda^2$ . The two resonances meet for  $\gamma = \gamma_*$ . As  $\gamma > \gamma_*$  increases further, the resonances  $w_3$  and  $w_4$  move horizontally away from the imaginary axis, their imaginary parts stay constant, equal to  $\frac{1}{4}i\pi\xi(0)\lambda^2$ . This motivates the sharp definition of the overlapping resonances regime, in the spin-boson model, to be given by  $\gamma < \gamma_*$  and of the isolated resonances regime to be given by  $\gamma > \gamma_*$ .

# Chapter 7

## Conclusion and future work

**Conclusion.** In this thesis, we consider an  $N$ -level quantum system coupled to a bosonic heat reservoir at positive temperature. We develop a new resonance method to analyze the dynamics in the overlapping resonances regime  $\sigma \ll \lambda^2$ . Here  $\sigma \ll \lambda^2$  means the system-reservoir coupling strength,  $\lambda^2$ , is much larger than the spacing  $\sigma$  of system energy levels. We derive the representation of the reduced density matrix of the  $N$ -level system in the overlapping resonances regime under the condition that  $|\lambda| < \lambda_0$ , where  $\lambda_0$  is a fixed number depending on the system-reservoir coupling. We find that for vanishing  $\sigma$  there is a manifold of states invariant under the coupled system-reservoir dynamics. The manifold dissolves as the energy level is split, for small  $\sigma > 0$ . The system converges then to a unique asymptotic (equilibrium) state, at a rate proportional to  $\sigma^2/\lambda^2$ . To compare the dynamics in two different regimes, we consider the spin-boson model using Feshbach map. We describe the transition from the isolated resonances regime,  $\sigma/\lambda^2 \gg 1$ , to the overlapping one,  $\sigma/\lambda^2 \ll 1$ . We find a critical value of the ratio  $\sigma/\lambda^2$  which separates the two regimes, marking a sharp transition in the behaviour of the resonance energies.

**Future work.** Since our analysis is based on the resonance method developed by



previous researchers, there is a restriction on the coupling parameter  $|\lambda| < \lambda_0$ . We are considering the problems without the restriction on  $\lambda$ . The difficult point is that even for vanishing  $\sigma$  we still have no methods to analyze the problem in general. However, for the spin-boson system, our idea is to use a suitable unitary transformation to transform the problem into one where  $\lambda$  (big or small) plays only a marginal role. Our hope is to be able to treat the unitary transformed system rigorously. Another direction of future work is the transition between two regimes for different models. In this thesis, we only consider the two-level system ( spin-boson model). Our next task is to study some more complicated models. We are considering a three-level system, which is used to characterize donor-acceptor problem [26]. There are many difficult and interesting problems in this field, I will keep working on them.

# Appendix A

## A.1 Proof of Theorem 4.1.1

Throughout the proof, we do not write the dependence of operators on  $(\sigma, \lambda)$  (i.e., we write  $L$  for  $L(\sigma, \lambda)$ , and so on).

Let  $s \in \mathbb{C}$ ,  $|s| < 1/2 + \epsilon$ , where  $\epsilon$  is the constant in Assumption A2. Using the expression  $\Delta = \mathbb{1}_{\mathcal{H}_S} \otimes e^{-\beta L_R}$  for the modular operator, we get

$$\begin{aligned} \Delta^{i\bar{s}} V \Delta^{-i\bar{s}} &= G \otimes \mathbb{1}_{\mathbb{C}^N} \otimes e^{-i\beta\bar{s}L_R} \Phi(g_\beta) e^{i\beta\bar{s}L_R} \\ &= G \otimes \mathbb{1}_{\mathbb{C}^N} \otimes \frac{1}{\sqrt{2}} (a^*(e^{-i\beta\bar{s}u} g_\beta) + a(e^{-i\beta s u} g_\beta)). \end{aligned} \quad (\text{A.1})$$

This operator is well-defined and strongly analytic in  $\bar{s}$  on  $\text{Dom}(N^{1/2})$ , due to assumption (A2). On  $\text{Dom}(L_0) \cap \text{Dom}(N^{1/2})$  we define the family of strongly analytic operators in  $s$ ,

$$K^{(s)} = L_0 + \lambda I^{(s)}, \quad (\text{A.2})$$

$$I^{(s)} = V - \lambda V'^{(s)}, \quad (\text{A.3})$$

$$V'^{(s)} = \Delta^{-is} J V J \Delta^{is} = J \Delta^{i\bar{s}} V \Delta^{-i\bar{s}} J. \quad (\text{A.4})$$

This family has been introduced in [27]. It interpolates between the self-adjoint  $K^{(0)}$

and the operator  $K^{(-i/2)} = K$  (see (4.8)).

**Proposition A.1.1.** *Let  $I^{(s)}(t) = e^{itL_0} I^{(s)} e^{-itL_0}$  and recall the definition (4.2) of the reference state  $\Omega$ . The Dyson series*

$$\sum_{n \geq 0} (i\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n I^{(s)}(t_n) I^{(s)}(t_{n-1}) \cdots I^{(s)}(t_1) \Omega \quad (\text{A.5})$$

converges for all  $\lambda \in \mathbb{R}$  and is analytic in  $s$  for  $|s| < 1/2 + \epsilon$ .

**Proof of Proposition A.1.1.** Let  $\psi_\nu \in \text{Ran } P(N \leq \nu)$  (spectral projection of  $N$  onto subspace with at most  $\nu$  particles). Since the interaction operator  $I^{(s)}$  changes the particle number by at most one, we have

$$\begin{aligned} & I^{(s)}(t_n) I^{(s)}(t_{n-1}) \cdots I^{(s)}(t_1) \psi_\nu \\ &= e^{it_n L_0} I^{(s)} P(N \leq \nu + n - 1) e^{-it_n L_0} \cdots e^{it_1 L_0} I^{(s)} P(N \leq \nu) e^{-it_1 L_0} \psi_\nu. \end{aligned}$$

The standard bounds  $\|a^*(f)(N+1)^{-1/2}\| \leq \|f\|$  and  $\|a(f)(N+1)^{-1/2}\| \leq \|f\|$  give  $\|I^{(s)}(N+1)^{-1/2}\| \leq 4M$ , where  $M := (\int |e^{(\frac{1}{2}+\epsilon)\beta|u}| g_\beta(u, \sigma)|^2 du d\sigma)^{\frac{1}{2}} < \infty$  due to assumption (A2). Hence

$$\|I^{(s)}(t_n) I^{(s)}(t_{n-1}) \cdots I^{(s)}(t_1) \psi_\nu\| \leq \sqrt{(\nu+1) \cdots (\nu+n)} (4M)^n \|\psi_\nu\|, \quad (\text{A.6})$$

uniformly in  $s$ . This and the analyticity of  $I^{(s)}(t_n) I^{(s)}(t_{n-1}) \cdots I^{(s)}(t_1) \psi_\nu$  imply that (A.5) is analytic in  $s$  for  $|s| < \frac{1}{2} + \epsilon$ . This proves Proposition A.1.1.  $\square$

We define an operator denoted  $e^{itK^{(s)}}$ , on the dense set  $\mathfrak{M}\Omega$ , by

$$e^{itK^{(s)}} \Omega := (\text{A.5}) \quad \text{and} \quad e^{itK^{(s)}} A \Omega := e^{itL} A e^{-itL} e^{itK^{(s)}} \Omega \quad (\text{A.7})$$

for  $A \in \mathfrak{M}$ .

**Proposition A.1.2.** *We have  $e^{itK^{(-i/2)}}A\Omega = e^{itL}Ae^{-itL}\Omega$ , for all  $A \in \mathfrak{M}$ .*

**Proof of Proposition A.1.2.** It suffices to show that  $e^{itK^{(-i/2)}}\Omega = \Omega$ . Note that  $(G \otimes \mathbb{1})\Omega_S = (\mathbb{1} \otimes \mathcal{C}G\mathcal{C})\Omega_S$  (see after (2.5) for the definition of  $\mathcal{C}$ ),  $J\Delta^{\frac{1}{2}}\Omega_R = \Omega_R$  and that  $\Phi(g_\beta)$  is selfadjoint. Thus,

$$\begin{aligned} I^{(-i/2)}\Omega &= [G \otimes \mathbb{1} \otimes \Phi(g_\beta) - \mathbb{1} \otimes G \otimes J\Delta^{\frac{1}{2}}\Phi(g_\beta)J\Delta^{\frac{1}{2}}]\Omega_S \otimes \Omega_R \\ &= (G \otimes \mathbb{1})\Omega_S \otimes [\Phi(g_\beta)\Omega_R - J\Delta^{\frac{1}{2}}\Phi(g_\beta)J\Delta^{\frac{1}{2}}\Omega_R] = 0. \end{aligned}$$

It now follows directly from (A.7) and (A.5) that  $e^{itK^{(-i/2)}}\Omega = \Omega$ .  $\square$

Let  $\psi = A\Omega$ . Since  $K^{(s)}$  is self-adjoint for  $s \in \mathbb{R}$ , we have

$$\left\langle \phi, e^{itK^{(s)}}\psi \right\rangle = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \left\langle \phi, (K^{(s)} - z)^{-1}\psi \right\rangle dz, \quad s \in \mathbb{R}. \quad (\text{A.8})$$

Next we perform the spectral deformation. By analyticity the scalar product in the integrand of (A.8) equals  $\left\langle \phi_{\bar{\theta}}, (K_{\theta}^{(s)} - z)^{-1}\psi_{\theta} \right\rangle$ , for all  $|\theta| < \theta_0$ . Here,  $K_{\theta}^{(s)} = L_{0,\theta} + \lambda I_{\theta}^{(s)}$  is the analytic extension of  $U_{\theta}K^{(s)}U_{\theta}^*$  to complex  $|\theta| < \theta_0$ . Thus we obtain

$$\left\langle \phi, e^{itK^{(s)}}\psi \right\rangle = \frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \left\langle \phi_{\bar{\theta}}, (K_{\theta}^{(s)} - z)^{-1}\psi_{\theta} \right\rangle dz, \quad s \in \mathbb{R}. \quad (\text{A.9})$$

From now on we take  $\theta$  to be a fixed  $i\theta$ , for some  $0 < \theta < \theta_0$ .

**Proposition A.1.3.** *Both sides in (A.9) have an analytic extension to  $s \in \mathbb{C}$ ,  $|s| < 1/2 + \epsilon$ . Since they are equal for real  $s$  we have (by the identity principle) that (A.9) stays valid for all  $|s| < 1/2 + \epsilon$ .*

Taking the value  $s = -i/2$  in (A.9), together with Proposition A.1.2, gives relation (4.12) and hence proves Theorem 4.1.1.

**Proof of Proposition A.1.3.** Analyticity of the l.h.s. of (A.9) is immediate

from Proposition A.1.1 and relations (A.7). To prove the analyticity of r.h.s. of (A.9), we first prove the convergence of the improper Riemann integral. The second resolvent equation gives

$$(K_\theta^{(s)} - z)^{-1} = (L_{0\theta} - z)^{-1} + (L_{0\theta} - z)^{-1} \lambda I_\theta^{(s)} (K_\theta^{(s)} - z)^{-1}. \quad (\text{A.10})$$

Accordingly, the right side of (A.9) consists of two terms. The first one, coming from the uncoupled resolvent, equals  $\langle \phi, e^{itL_0} \psi \rangle$ . Hence we only need to show the convergence of the integral

$$\frac{-1}{2\pi i} \int_{\mathbb{R}-i} e^{itz} \langle \phi_\theta, (L_{0\theta} - z)^{-1} \lambda I_\theta^{(s)} (K_\theta^{(s)} - z)^{-1} \psi_\theta \rangle dz. \quad (\text{A.11})$$

Consider

$$\begin{aligned} (K_\theta^{(s)} - z)^{-1} &= (L_{0\theta} + \lambda I_\theta^{(s)} - z)^{-1} \\ &= (L_{0\theta} - z)^{-\frac{1}{2}} [\mathbb{1} - (L_{0\theta} - z)^{-\frac{1}{2}} \lambda I_\theta^{(s)} (L_{0\theta} - z)^{-\frac{1}{2}}]^{-1} (L_{0\theta} - z)^{-\frac{1}{2}}. \end{aligned} \quad (\text{A.12})$$

Since  $I_\theta^{(s)}(N+1)^{-\frac{1}{2}}$  is bounded and  $(z = x - i)$

$$\|(N+1)^{\frac{1}{2}}(L_{0\theta} - z)^{-\frac{1}{2}}\| = \sup_{n \geq 0, l \in \mathbb{R}} \frac{\sqrt{n+1}}{\sqrt[4]{(l-x)^2 + (\theta n + 1)^2}} \leq \frac{2}{\sqrt{\theta}}, \quad (\text{A.13})$$

we have  $\|(L_{0\theta} - z)^{-\frac{1}{2}} \lambda I_\theta^{(s)} (L_{0\theta} - z)^{-\frac{1}{2}}\| < 1/2$ , for  $|\lambda|$  small enough. It follows from (A.12) that

$$(K_\theta^{(s)} - z)^{-1} = (L_{0\theta} - z)^{-\frac{1}{2}} B (L_{0\theta} - z)^{-\frac{1}{2}}, \quad (\text{A.14})$$

where  $B$  is a bounded operator satisfying  $\|B\| \leq \frac{1}{1-1/2} = 2$ . This and (A.13) imply that

$$\|\lambda I_\theta^{(s)} (K_\theta^{(s)} - z)^{-1} (L_{0\theta} - z)^{\frac{1}{2}}\| \leq C|\lambda|, \quad (\text{A.15})$$

for some constant  $C$ . We estimate the integrand in (A.11) as

$$\begin{aligned}
& \left| \left\langle \phi_{\bar{\theta}}, (L_{0\theta} - z)^{-1} \lambda I_{\theta}^{(s)} (K_{\theta}^{(s)} - z)^{-1} \psi_{\theta} \right\rangle \right| \\
& \leq C |\lambda| \| (L_{0\theta}^* - \bar{z})^{-1} \phi_{\bar{\theta}} \| \| (L_{0\theta} - z)^{-\frac{1}{2}} \psi_{\theta} \| \\
& \leq C |\lambda| \left\{ (1 + |x|)^{\frac{1}{2} + \eta} \| (L_{0\theta}^* - \bar{z})^{-1} \phi_{\bar{\theta}} \|^2 + (1 + |x|)^{-\frac{1}{2} - \eta} \| (L_{0\theta} - z)^{-\frac{1}{2}} \psi_{\theta} \|^2 \right\} \\
& = C |\lambda| \{ S_1(x) + S_2(x) \}. \tag{A.16}
\end{aligned}$$

The last line defines the two functions  $S_1$  and  $S_2$  of  $x = \text{Re}z$ . Here we use the inequality  $ab \leq \alpha a^2 + b^2/\alpha$ , for  $\alpha = (1 + |x|)^{1/2 + \eta}$ , where  $0 < \eta < 1/2$ . We have

$$\begin{aligned}
S_1(x) &= (1 + |x|)^{\frac{1}{2} + \eta} \left\langle \phi_{\bar{\theta}}, (L_{0\theta} - z)^{-1} (L_{0\theta}^* - \bar{z})^{-1} \phi_{\bar{\theta}} \right\rangle \\
&= \sum_{n=0}^{\infty} (1 + |x|)^{\frac{1}{2} + \eta} \left\langle \phi_{\bar{\theta}}, (L_{0\theta} - z)^{-1} (L_{0\theta}^* - \bar{z})^{-1} P(N = n) \phi_{\bar{\theta}} \right\rangle \tag{A.17} \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{(1 + |x|)^{\frac{1}{2} + \eta}}{(l - x)^2 + (\theta n + 1)^2} d\mu_n(l),
\end{aligned}$$

where  $d\mu_n$  is the spectral measure of  $L_R$  associated to the vector  $P(N = n)\phi_{\bar{\theta}}$  and  $P(N = n)$  is the spectral projection onto the  $n$  particle sector. By Fubini's theorem,

$$\int_{\mathbb{R}} S_1(x) dx = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{(1 + |x|)^{\frac{1}{2} + \eta}}{(l - x)^2 + (\theta n + 1)^2} dx \right] d\mu_n(l). \tag{A.18}$$

The integral over  $x$  is bounded above by

$$\int_{\mathbb{R}} \frac{(1 + |x + l|)^{\frac{1}{2} + \eta}}{x^2 + 1} dx \leq \int_{\mathbb{R}} \frac{(1 + |x|)^{\frac{1}{2} + \eta} + |l|^{\frac{1}{2} + \eta}}{x^2 + 1} dx \leq C_{\eta} + \pi |l|^{\frac{1}{2} + \eta}.$$

We use here that  $(a + b)^r \leq a^r + b^r$  for  $a, b \geq 0$ ,  $0 < r < 1$ . It follows from (A.18) and

this estimate that

$$\int_{\mathbb{R}} S_1(x) dx \leq \left\langle \phi_{\bar{\theta}}, (C_\eta + \pi |L_{\mathbb{R}}|^{\frac{1}{2} + \eta}) \phi_{\bar{\theta}} \right\rangle < \infty. \quad (\text{A.19})$$

We treat the second term in (A.16) in a similar fashion.

$$\begin{aligned} \int_{\mathbb{R}} S_2(x) dx &= \int_{\mathbb{R}} (1 + |x|)^{-\frac{1}{2} - \eta} \left\langle \psi_\theta, (L_{0\theta}^* - \bar{z})^{-\frac{1}{2}} (L_{0\theta} - z)^{-\frac{1}{2}} \psi_\theta \right\rangle dx \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{(1 + |x|)^{-1/2 - \eta}}{\sqrt{(l - x)^2 + (\theta n + 1)^2}} dx \right] d\nu_n(l), \end{aligned} \quad (\text{A.20})$$

where  $d\nu_n$  is the spectral measure of  $L_{\mathbb{R}}$  associated to the vector  $P(N = n)\psi_\theta$ . The integral over  $x$  is bounded above by

$$\int_{\mathbb{R}} \frac{(1 + |x|)^{-1/2 - \eta}}{\sqrt{(l - x)^2 + 1}} dx \leq \int_{\mathbb{R}} \left\{ (1 + |x|)^{-1 - 2\eta} + \frac{1}{(l - x)^2 + 1} \right\} dx \leq C_\eta + \pi,$$

uniformly in  $l \in \mathbb{R}$ . It follows from the last estimate and (A.20) that

$$\int_{\mathbb{R}} S_2(x) dx \leq (C_\eta + \pi) \|\psi_\theta\|^2 < \infty. \quad (\text{A.21})$$

The bounds (A.19) and (A.21) finish the proof that the integral on the right side of (A.9) converges.

In order to complete the proof of Proposition A.1.3 (and hence that of Theorem 4.1.1), we need to show that the integral on the right side of (A.9) is analytic in  $s$ , for  $|s| < \frac{1}{2} + \epsilon$ . To do so, let  $\nu > 0$  and set

$$F_\nu(s) = \frac{-1}{2\pi i} \int_{-\nu - i}^{\nu - i} e^{itz} \left\langle \phi_{\bar{\theta}}, (K_\theta^{(s)} - z)^{-1} \psi_\theta \right\rangle dz, \quad (\text{A.22})$$

which is analytic in  $s$ , for  $|s| < \frac{1}{2} + \epsilon$ . Denote by  $F(s)$  the right side of (A.9). We

have

$$|F_\nu(s) - F(s)| = \frac{1}{2\pi} \left| \left( \int_{-\infty-i}^{-\nu-i} + \int_{\nu-i}^{\infty-i} \right) e^{itz} \left\langle \phi_{\bar{\theta}}, (K_\theta^{(s)} - z)^{-1} \psi_\theta \right\rangle dz \right|. \quad (\text{A.23})$$

The above analysis shows that the integrals converge uniformly in  $s$  and hence (A.23) converges to zero uniformly in  $s$ . Therefore,  $F(s)$  is analytic. This completes the proof of Proposition A.1.3 and that of Theorem 4.1.1.  $\square$

## A.2 Proof of Theorem 4.3.1

**Proof of Theorem 4.3.1. (A) Non-zero eigenvalues.** The non-zero eigenvalues of  $K_\theta(0, \lambda)$  are simple, given by  $\varepsilon_{a,b}(0, \lambda) = \lambda^2 \delta_{a,b}$ , for  $a \neq b$ . We denote by  $\varphi_{a,b} \otimes X_{a,b}$  the eigenvector associated to  $\varepsilon_{a,b}(0, \lambda)$ , where  $\varphi_{a,b} = \varphi_a \otimes \varphi_b$  and  $X_{a,b}$  is a normalized vector in  $\mathcal{H}_R$ , depending on  $\lambda$  and  $\theta$ . The adjoint operator satisfies  $K_\theta(0, \lambda)^* \varphi_{a,b} \otimes X_{a,b}^* = \lambda^2 \bar{\delta}_{a,b} \varphi_{a,b} \otimes X_{a,b}^*$  for a vector  $X_{a,b}^*$  satisfying  $\langle X_{a,b}, X_{a,b}^* \rangle = 1$ . We denote the Riesz projection of  $K_\theta(0, \lambda)$  associated to  $\varepsilon_{a,b}(0, \lambda)$  by

$$P_{a,b} = |\varphi_{a,b} \otimes X_{a,b}\rangle \langle \varphi_{a,b} \otimes X_{a,b}^*|. \quad (\text{A.24})$$

By analytic perturbation theory,  $K_\theta(\sigma, \lambda)$  has a simple eigenvalue in the vicinity of  $\lambda^2 \delta_{a,b}$ , for small  $\sigma$ . It is given by

$$\varepsilon_{a,b}(\sigma, \lambda) = \lambda^2 \delta_{a,b} + \sigma \varepsilon_{a,b}^{(1)} + \sigma^2 \varepsilon_{a,b}^{(2)} + O_\lambda(\sigma^3), \quad (\text{A.25})$$

where (see [20, Sect. II.2.2] and also [34, Thm. XII.12])

$$\varepsilon_{a,b}^{(1)} = \text{Tr}(L_S P_{a,b}) = [H_S]_{a,a} - [H_S]_{b,b}. \quad (\text{A.26})$$



Here, we have set  $[H_S]_{a,b} = \langle \varphi_a, H_S \varphi_b \rangle$ . The second order correction is

$$\varepsilon_{a,b}^{(2)} = -\text{Tr}(L_S(K_\theta(0, \lambda) - \lambda^2 \delta_{a,b})^{-1} \bar{P}_{a,b} L_S P_{a,b}). \quad (\text{A.27})$$

We write  $\bar{P}$  for  $\mathbb{1} - P$  for general projections  $P$ . We set  $P_{a,b}^S = |\varphi_{a,b}\rangle\langle\varphi_{a,b}|$  and  $P_{a,b}^R = |X_{a,b}\rangle\langle X_{a,b}^*|$ . Then  $P_{a,b} = P_{a,b}^S \otimes P_{a,b}^R$  and  $\bar{P}_{a,b} = \bar{P}_{a,b}^S \otimes \mathbb{1}_R + P_{a,b}^S \otimes \bar{P}_{a,b}^R$ . It follows that  $\bar{P}_{a,b} L_S (\varphi_{a,b} \otimes X_{a,b}) = (\bar{P}_{a,b}^S L_S \varphi_{a,b}) \otimes X_{a,b}$ . Using this and  $\bar{P}_{a,b}^S = \sum_{(c,d) \neq (a,b)} P_{c,d}^S$  in expression (A.27) yields

$$\varepsilon_{a,b}^{(2)} = - \sum_{(c,d) \neq (a,b)} \langle \varphi_{a,b} \otimes X_{a,b}^*, L_S (K_\theta(0, \lambda) - \lambda^2 \delta_{a,b})^{-1} \varphi_{c,d} \otimes X_{a,b} \rangle \langle \varphi_{c,d}, L_S \varphi_{a,b} \rangle.$$

‘Replacing’  $\varphi_{c,d} \otimes X_{a,b}$  by the eigenvector  $\varphi_{c,d} \otimes X_{c,d}$ , we obtain

$$\varepsilon_{a,b}^{(2)} = - \sum_{(c,d) \neq (a,b)} \frac{1}{\lambda^2 (\delta_{c,d} - \delta_{a,b})} |\langle \varphi_{a,b}, L_S \varphi_{c,d} \rangle|^2 \langle X_{a,b}^*, X_{c,d} \rangle + \xi, \quad (\text{A.28})$$

where

$$\xi = \sum_{(c,d) \neq (a,b)} \langle \varphi_{a,b} \otimes X_{a,b}^*, L_S (K_\theta(0, \lambda) - \lambda^2 \delta_{a,b})^{-1} \varphi_{c,d} \otimes (X_{c,d} - X_{a,b}) \rangle \langle \varphi_{c,d}, L_S \varphi_{a,b} \rangle. \quad (\text{A.29})$$

By perturbation theory, we have  $X_{a,b} = \Omega_R + O(\lambda)$ . Therefore,  $X_{c,d} - X_{a,b} = O(\lambda)$  and  $\langle X_{a,b}^*, X_{c,d} \rangle = 1 + O(\lambda)$ . Together with the bound (A.33) of Corollary A.2.2 below, we obtain

$$|\xi| \leq \frac{C}{|\lambda|}. \quad (\text{A.30})$$

Finally,

$$\langle \varphi_{a,b}, L_S \varphi_{c,d} \rangle = \chi_{b=d} [H_S]_{a,c} - \chi_{a=c} [H_S]_{d,b}. \quad (\text{A.31})$$

Relation (2.20) for  $a \neq b$  follows from (A.28), (A.30) and (A.31) and a little algebra.

**Proposition A.2.1** (Bound on the resolvent). *There are constants  $C$  and  $\lambda_0$  (depending on  $\text{Im}\theta$  only) such that if  $0 < |\lambda| < \lambda_0$ , then we have the following. Fix any  $\alpha > 0$  and take complex  $z$  satisfying  $|z| < C\alpha$ ,  $\text{Im}z < \frac{1}{4}\text{Im}\theta$ , and  $\text{dist}(\mathcal{E}, z) \geq \alpha\lambda^2$ , where  $\mathcal{E} = \{\lambda^2\delta_{a,b} : a, b = 1, \dots, N\}$  is the set of eigenvalues of  $K_\theta(0, \lambda)$ . Then we have*

$$\|(K_\theta(0, \lambda) - z)^{-1}\| \leq C_1 \left( \frac{1}{\text{Im}\theta} + \frac{1}{\text{dist}(\mathcal{E}, z)} \right), \quad (\text{A.32})$$

where  $C_1$  is a constant depending only on  $\text{Im}\theta$ .

Knowing the bound on the resolvent we can obtain a bound on the *reduced resolvent*.

**Corollary A.2.2.** *For any  $a, b = 1, \dots, N$  we have*

$$\|(K_\theta(0, \lambda) - \lambda^2\delta_{a,b})^{-1}\bar{P}_{a,b}\| \leq C_2 \left( \frac{1}{\text{Im}\theta} + \frac{1}{\lambda^2} \right), \quad (\text{A.33})$$

for some constant  $C_2$  depending on  $\text{Im}\theta$ .

**Proof of Corollary A.2.2.** The reduced resolvent has the representation

$$(K_\theta(0, \lambda) - \lambda^2\delta_{a,b})^{-1}\bar{P}_{a,b} = \frac{-1}{2\pi i} \oint_{\Gamma_{a,b}(\lambda)} (z - \lambda^2\delta_{a,b})^{-1} (K_\theta(0, \lambda) - z)^{-1} \bar{P}_{a,b} dz,$$

where  $\Gamma_{a,b}(\lambda) = \{z = \lambda^2\delta_{a,b} + \lambda^2 r e^{i\phi} : \phi \in [0, 2\pi]\}$ , with an appropriate radius  $r$  (independent of  $\lambda$ ) such that  $\Gamma_{a,b}(\lambda)$  encircles only the eigenvalue  $\lambda^2\delta_{a,b}$  and such that  $\Gamma_{a,b}(\lambda)$  lies within the region of  $z$  for which the bound (A.32) holds, according to Proposition A.2.1. Then  $\text{dist}(\mathcal{E}, z)$  is a constant times  $\lambda^2$ . It follows that

$$\|(K_\theta(0, \lambda) - \lambda^2\delta_{a,b})^{-1}\bar{P}_{a,b}\| \leq C \left( \frac{1}{\text{Im}\theta} + \frac{1}{\lambda^2} \right) (1 + \|P_{a,b}\|),$$

for some constant  $C$ . The bound (A.33) follows from  $\|P_{a,b}\| = 1 + O(\lambda)$ .  $\square$

**Proof of Proposition A.2.1.** Let  $P_R = |\Omega_R\rangle\langle\Omega_R|$ ,  $\bar{P}_R = \mathbb{1} - P_R$ , and  $R(z) = (K_\theta(0, \lambda) - z)^{-1}$ .

*Step 1.* For any  $\psi \in \mathcal{H}$  we have

$$\begin{aligned} |\langle \psi, \bar{P}_R(K_\theta(0, \lambda) - z)\bar{P}_R\psi \rangle| &\geq \operatorname{Im} \langle \psi, \bar{P}_R(K_\theta(0, \lambda) - z)\bar{P}_R\psi \rangle \\ &= \langle \psi, \bar{P}_R(N^{1/2}\{\operatorname{Im}\theta + \lambda\operatorname{Im}N^{-1/2}I_\theta N^{-1/2}\}N^{1/2} - \operatorname{Im}z)\bar{P}_R\psi \rangle \\ &\geq (\operatorname{Im}\theta - C|\lambda| - \operatorname{Im}z)\|\bar{P}_R\psi\|^2 \\ &\geq \frac{1}{2}\operatorname{Im}\theta\|\bar{P}_R\psi\|^2. \end{aligned}$$

By the Cauchy-Schwartz inequality, it follows that  $\|\bar{P}_R(K_\theta(0, \lambda) - z)\bar{P}_R\psi\| \geq \frac{1}{2}\operatorname{Im}\theta\|\bar{P}_R\psi\|$  and therefore

$$\|\bar{P}_R R(z)\bar{P}_R\| \leq \frac{2}{\operatorname{Im}\theta}. \quad (\text{A.34})$$

*Step 2.* Consider the Feshbach map

$$\begin{aligned} \mathcal{F}_z &= P_R(-z - \lambda^2 I_\theta \bar{P}_R R(z) \bar{P}_R I_\theta) P_R \\ &= P_R(-z - \lambda^2 I_\theta \bar{P}_R R(0) \bar{P}_R I_\theta) P_R + O(\lambda^2 |z|). \end{aligned} \quad (\text{A.35})$$

Let

$$\mathcal{G}_z = -\lambda^2 P_R I_\theta \bar{P}_R R(z) \bar{P}_R I_\theta P_R. \quad (\text{A.36})$$

By the isospectrality property of the Feshbach map (see e.g. [5, Theorem IV.1]) we know that

$$\mathcal{G}_{\lambda^2 \delta_{a,b}} \varphi_{a,b} \otimes \Omega_R = \lambda^2 \delta_{a,b} \varphi_{a,b} \otimes \Omega_R,$$

for all  $a, b = 1, \dots, N$ . We also have  $\mathcal{G}_z - \mathcal{G}_\zeta = O(\lambda^2 |z - \zeta|)$ , as long as  $\operatorname{Im}z, \operatorname{Im}\zeta < \frac{1}{4}\operatorname{Im}\theta$ . It follows that  $\mathcal{G}_0 \varphi_{a,b} \otimes \Omega_R = \lambda^2 \delta_{a,b} \varphi_{a,b} \otimes \Omega_R + O(\lambda^4)$ , for all  $a, b = 1, \dots, N$ .

Therefore,  $\mathcal{G}_0 = \sum_{a,b=1}^N \lambda^2 \delta_{a,b} |\varphi_{a,b}\rangle \langle \varphi_{a,b}| \otimes P_{\mathbb{R}} + O(\lambda^4)$ , and so

$$\mathcal{G}_z = \sum_{a,b=1}^N \lambda^2 \delta_{a,b} |\varphi_{a,b}\rangle \langle \varphi_{a,b}| \otimes P_{\mathbb{R}} + O(\lambda^4 + \lambda^2 |z|). \quad (\text{A.37})$$

Using (A.37) and (A.36) in (A.35) shows that

$$\mathcal{F}_z = \sum_{a,b=1}^N (\lambda^2 \delta_{a,b} - z) |\varphi_{a,b}\rangle \langle \varphi_{a,b}| \otimes P_{\mathbb{R}} + O(\lambda^4 + \lambda^2 |z|). \quad (\text{A.38})$$

The sum on the right side is an invertible operator, the norm of the inverse being

$$\max_{a,b=1,\dots,N} |\lambda^2 \delta_{a,b} - z|^{-1} = [\text{dist}(\mathcal{E}, z)]^{-1}.$$

Therefore, there is a constant  $C$  s.t. if

$$\lambda^4 + \lambda^2 |z| < C \text{dist}(\mathcal{E}, z), \quad (\text{A.39})$$

then  $\mathcal{F}_z$  is invertible and

$$\|\mathcal{F}_z^{-1}\| \leq \frac{2}{\text{dist}(\mathcal{E}, z)}. \quad (\text{A.40})$$

Let  $\alpha > 0$  be fixed, and take  $z$  s.t.  $\text{dist}(\mathcal{E}, z) \geq \alpha \lambda^2$ . Then (A.39) is satisfied provided  $\lambda$  is small enough and  $|z| < C\alpha$ .

*Step 3.* The resolvent  $R(z)$  is related to  $\bar{P}_{\mathbb{R}} R(z) \bar{P}_{\mathbb{R}}$  and  $\mathcal{F}_z^{-1}$  by (see e.g. [5, Eqn. (IV.14)])

$$R(z) = (P_{\mathbb{R}} - \bar{P}_{\mathbb{R}} R(z) \bar{P}_{\mathbb{R}} K_{\theta}(0, \lambda) P_{\mathbb{R}}) \mathcal{F}_z^{-1} (P_{\mathbb{R}} - P_{\mathbb{R}} K_{\theta}(0, \lambda) \bar{P}_{\mathbb{R}} R(z) \bar{P}_{\mathbb{R}}) + \bar{P}_{\mathbb{R}} R(z) \bar{P}_{\mathbb{R}}.$$

We combine this equation with the bounds  $\|\bar{P}_{\mathbb{R}} K_{\theta}(0, \lambda) P_{\mathbb{R}}\|$ ,  $\|P_{\mathbb{R}} K_{\theta}(0, \lambda) \bar{P}_{\mathbb{R}}\| \leq C|\lambda|$  and (A.34), (A.40) to arrive at the estimate (A.32). This completes the proof of

Proposition A.2.1. □

**(B) Zero eigenvalue.** Let  $P(\sigma)$  be the group projection associated to the eigenvalues of  $K_\theta(\sigma, \lambda)$  bifurcating out of the origin as  $\sigma \neq 0$ . Here, we consider  $\lambda$  fixed and  $\sigma$  small. The null space of  $K_\theta(0, \lambda)$  is known exactly, see (4.16). Let  $X_{a,a}^* \in \mathcal{H}_R$  be the vector satisfying  $K_{a,a}^* X_{a,a}^* = 0$  and  $\langle \Omega_R, X_{a,a}^* \rangle = 1$ . We have  $X_{a,a}^* = \Omega_R + O(\lambda)$ . Then  $P(0) = \sum_{a=1}^N |\varphi_{a,a}\rangle \langle \varphi_{a,a}| \otimes |\Omega_R\rangle \langle X_{a,a}^*|$ . Note that  $P(0)L_S P(0) = 0$ . Analytic perturbation theory gives

$$\begin{aligned} K_\theta(\sigma, \lambda)P(\sigma) &= \sigma^2 T_2 + O_\lambda(\sigma^3) \\ T_2 &= -P(0)L_S K_\theta(0, \lambda)^{-1} L_S P(0). \end{aligned} \quad (\text{A.41})$$

We have  $L_S P(0) = \sum_{a=1}^N \sum_{c,d=1,\dots,N;c \neq d} |\varphi_{c,d}\rangle \langle \varphi_{a,a}| \otimes P_R \langle \varphi_{c,d}, L_S \varphi_{a,a} \rangle + O(\lambda)$ . Next,

$$\begin{aligned} K_\theta(0, \lambda)^{-1} \varphi_{c,d} \otimes \Omega_R &= K_\theta(0, \lambda)^{-1} \varphi_{c,d} \otimes (X_{c,d} + \Omega_R - X_{c,d}) \\ &= \frac{1}{\lambda^2 \delta_{c,d}} \varphi_{c,d} \otimes \Omega_R + O(\lambda^{-1}), \end{aligned} \quad (\text{A.42})$$

where we use Corollary A.2.2 in the last step. Starting from (A.41) and using (A.42), we arrive at

$$T_2 = \frac{2i}{\lambda^2} T + O(\lambda^{-1}), \quad (\text{A.43})$$

where the operator  $T$  has matrix elements  $[T]_{a,b} = \langle \varphi_{a,a} \otimes \Omega_R, T \varphi_{b,b} \otimes \Omega_R \rangle$  given by (2.19). In this derivation, we also use that  $\delta_{b,a} = -\overline{\delta_{a,b}}$ , see (2.15). Note that  $T$  is a real symmetric matrix,  $[T]_{a,b} < 0$  for  $a \neq b$ , and  $[T]_{a,a} = -\sum_{b \neq a} [T]_{a,b}$ . These properties imply that for  $x = (x_1, \dots, x_N) \in \mathbb{C}^N$ ,  $\langle x, Tx \rangle = \sum_{a,b=1}^N |[T]_{a,b}| |x_a - x_b|^2 \geq 0$ . Therefore, if  $[T]_{a,b} \neq 0$  for all  $a \neq b$ , then zero is a simple eigenvalue of  $T$ , with eigenvector proportional to  $(1, \dots, 1)$  and all other eigenvalues of  $T$  are strictly positive. This completes the proof of Theorem 4.3.1. □

# Appendix B

## B.1 Tomita-Takesaki theory

We present a brief overview of Tomita-Takesaki modular theory in this section. We refer to [3, 6] for a more detailed exposition.

Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$  containing a unit vector  $\Omega$  which is cyclic and separating for  $\mathcal{M}$  and  $\mathcal{M}'$ . Hence we may define operators  $S_0 : \mathcal{M}\Omega \rightarrow \mathcal{M}\Omega$  and  $F_0 : \mathcal{M}'\Omega \rightarrow \mathcal{M}'\Omega$  as the following:

$$\begin{aligned} S_0 A \Omega &= A^* \Omega, \quad \forall A \in \mathcal{M} \\ F_0 B \Omega &= B^* \Omega, \quad \forall B \in \mathcal{M}'. \end{aligned} \tag{B.1}$$

Since  $\Omega$  is cyclic and separating,  $S_0$  and  $F_0$  are well-defined on dense domains. Now we state the following well known result [3, 6].

**Proposition B.1.1.** *The operators  $S_0$  and  $F_0$  are closable and  $\overline{F_0} = S_0^*$ ,  $\overline{S_0} = F_0^*$ .*

To simplify the notations, let  $S = \overline{S_0}$  and  $F = \overline{F_0}$ . It follows from the polar decomposition theorem that there exists a unique positive self-adjoint operator  $\Delta$  and a unique anti-unitary operator  $J$  such that

$$S = J \Delta^{1/2} = \Delta^{-1/2} J. \tag{B.2}$$

Here,  $\Delta$  is the so-called *modular operator* and  $J$  the *modular conjugation* associated with  $(\mathcal{M}, \Omega)$ . The following theorem states many important equalities related to the polar decomposition which can be found in [6].

**Theorem B.1.2.**

$$\begin{aligned}
 \Delta &= FS, \Delta^{-1} = SF, J^2 = I, \\
 S &= J\Delta^{1/2} = \Delta^{-1/2}J, \\
 F &= J\Delta^{-1/2} = \Delta^{1/2}J, \\
 J\Delta^{it} &= \Delta^{it}J, \\
 J\Omega &= \Delta\Omega = \Omega.
 \end{aligned} \tag{B.3}$$

Note that the modular operator  $\Delta$  may be unbounded. The following remarkable Tomita-Takesaki theorem plays an important role in operator algebra and mathematical physics.

**Theorem B.1.3.**

$$\begin{aligned}
 J\mathcal{M}J &= \mathcal{M}', \\
 \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \quad \forall t \in \mathbb{R}.
 \end{aligned} \tag{B.4}$$

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