THE THEORY OF SEMI-SIMPLICIAL COMPLEXES

NICHOLAS JOHN RICKETTS

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BY

NICHOLAS RICKETTS

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This thesis has been examined and approved by

. . . .

Supervisor

. . . .

Internal Examiner

. . . .

External Examiner

. . . .

Date

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INTRODUCTION

The concepts of Semi-Simplicial Complexes first came from the works of S. Eilenberg and S. Maclane who used them as a tool in their work on Homology and Cohomology theories. A paper by Eilenberg and Zilber [2] then gave a presentation of the basic definitions and some methods for their use. An important construction - the Geometric Realization - was introduced by Milnor [15]. The Geometric Realization associates to each semi-simplicial complex in the category of sets a topological space which has the same homology as the homology of the semi-simplicial complex. The purpose of this paper is to unify the language and concepts related to semi-simplicial complexes on a categorical basis.

Chapter I presents the basic categorical language we shall be using and gives a definition of semi-simplicial complexes in terms of functors and natural transformations. This definition enables us to use a construction introduced by Godement [4] to present a relationship between adjoint functors and semi-simplical complexes.

Chapters II and III introduce the categories of CW-complexes and K-spaces ($\underline{Top}_K$) where the former is a subcategory of the latter. These categories are shown by Steenrod [18] to be "convenient", which is to say, they admit a large variety of constructions without having to make very many assumptions in the hypothesis. The basic concepts are introduced from a categorical standpoint. A CW-complex is defined to be a colimit of a convenient diagram in <u>Top</u>. Both CW-complexes and K-spaces have the final topology with respect to certain inclusions.

A functor K is defined from Top to Top_K, which is left adjoint to the inclusion functor. We use K to define products in Top_K. If A and B are objects of Top_K, their product in Top_K is defined to be $K(A \times_c B)$ - the functor applied to the cartesian product in Top. This definition of product will satisfy the usual commutative and associative laws of products in Top.

As mentioned before the Geometric Realization is an important tool in Homology and Homotopy Theory. Here we give a description of the realization which shows that it is a CW-Complex and thus a K-space. The topology on the product of the realizations of two semi-simplicial complexes defined by the functor K will coincide with that defined when we take it as a product of CW-Complexes. In particular, if X and Y are two semi-simplicial complexes and $|X|$ and $|Y|$ denote their geometric realizations then $K(|X| \times |Y|)$ is homeomorphic to $|X \times Y|$.

In the Appendix we will show that the Geometric Realization can be used to define new semi-simplicial complexes.

CHAPTER I

Semi-Simplicial Complexes (S.S. Complexes)

§1. Our basic category will be the category $\underline{\Delta}$;

(i) Objects: $\Delta_n = \{0,1, \ldots, n\}$ = the set of integers from 0 to n inclusive, for $n = 0,1,2, \ldots$.

(ii) Morphisms: $\underline{\Delta}(\Delta_p, \Delta_q)$ = the set of all monotonic functions $\alpha : \Delta_p \to \Delta_q$; in other words, $\alpha(i) \leq \alpha(j)$ for $0 \leq i \leq j \leq p$.

Let $\underline{\Delta}^{opp}$ be the opposite category to $\underline{\Delta}$.

(1.1.1) Definition: For any category $\underline{R}$, a semi-simplicial complex in $\underline{R}$ is an object of the functor category $SC(\underline{R}) = \underline{R}^{\Delta^{opp}}$; a morphism $f \in SC(\underline{R})(F,K)$ will be called a semi-simplicial map from F to K.

Certain relations will be evident in the category $\underline{\Delta}$ and will be carried over by F to the category $\underline{R}$.
First we define:

(1.1.2) $$\sigma_n^i : \Delta_n \to \Delta_{n+1} \quad (0 \leq i \leq n+1)$$

with $\sigma_n^i(\{0,1, \ldots, n\}) = \{0,1, \ldots, \hat{i}, \ldots, n+1\}$

i.e. $$\sigma_n^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

We also define

$$\delta^i_n : \Delta_{n+1} \to \Delta_n \quad (0 \le i \le n)$$

by $\delta^i_n(\{0,1, \ldots, n+1\}) = \{0,1, \ldots, i-1, i, i, i+1, \ldots, n\}$

$= \{0,1, \ldots, i-1, i, i+1, \ldots, n\}$

i.e. $\delta^i_n(j) = \begin{cases} j & \text{if } j \le i \\ j-1 & \text{if } j > i \end{cases}$

(1.1.3) Lemma: The following relations hold in the category Δ:

(a) $\sigma^j_{n+1}\sigma^i_n = \sigma^i_{n+1}\sigma^{j-1}_n \quad (i < j)$

(b) $\delta^j_n\delta^i_{n+1} = \delta^i_n\delta^{j+1}_{n+1} \quad (i \le j)$

(c) $\delta^j_{n+1}\sigma^i_{n+1} = \sigma^i_n\delta^{j-1}_n \quad (i < j)$

(d) $\delta^i_n\sigma^i_n = \delta^i_n\sigma^{i+1}_n = 1$

(e) $\delta^j_{n+1}\sigma^i_{n+1} = \sigma^{i-1}_n \cdot \delta^j_n \; (j+1 < i)$

Proof

(a) for $i < j$ $\sigma^j_{n+1}\sigma^i_n(\{0,1, \ldots, i, \ldots, n\}) =$

$= \sigma^j_{n+1}(\{0,1,\ldots,\hat{i},\ldots,j,\ldots,n+1\})$

$= \sigma^j_{n+1}(\{0,1,\ldots,i-1,i+1,\ldots,j,\ldots,n+1\})$

$= \{0,1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,n+2\}$

$\sigma^i_{n+1}\sigma^{j-1}_n(\{0,1, \ldots, i, \ldots, j, \ldots, n\}) =$

$= \sigma^i_{n+1}(\{0,\ldots,i,\ldots,j-2,j,\ldots,n+1\})$

$= \{0,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,n+2\}$

$\Rightarrow \sigma^j_{n+1}\sigma^i_n = \sigma^i_{n+1}\sigma^{j-1}_n \quad (i < j)$

(b) for $i \leq j$ $\delta^j_n \delta^i_{n+1}(\{0,1, \ldots, i, \ldots, j, \ldots, n+2\}) =$

$$= \delta^j_n(\{0,\ldots,i-1,i,i,i+1,\ldots,j,\ldots,n+1\})$$

$$= \{0,\ldots,i-1,i,i,i+1,\ldots,j-1,j,j,\ldots,n\}$$

$$= \{0,1,\ldots,n\}$$

$$\delta^i_n \delta^{j+1}_{n+1}(\{0,\ldots, i, \ldots, j, \ldots, n+2\}) =$$

$$= \delta^i_n(\{0,\ldots,i,\ldots,j,j+1,j+1,j+2,\ldots,n+1\})$$

$$= \{0,\ldots,j-1,i,i,i+1,\ldots,j-1,j,j,j+1,\ldots,n\}$$

$$= \{0,1,\ldots,n\}$$

$$\Rightarrow \quad \delta^j_n \delta^i_{n+1} = \delta^i_n \delta^{j+1}_{n+1} \quad (i \leq j)$$

(c) for $i < j$ $\delta^j_{n+1} \sigma^i_{n+1}(\{0, \ldots, i, \ldots, j, \ldots, n+1\}) =$

$$= \delta^j_{n+1}(\{0,\ldots,i-1,i+1,\ldots,j,\ldots,n+2\})$$

$$= \{0,\ldots,i-1,i+1,\ldots,j-1,j,j,j+1,\ldots,n+1\}$$

$$= \{0,\ldots,i-1,i+1,\ldots,j-1,j,j+1,\ldots,n+1\}$$

$$\sigma^i_n \cdot \delta^{j-1}_n(\{0,\ldots, i, \ldots, j, \ldots, n+1\}) =$$

$$= \sigma^i_n(\{0,\ldots,i,\ldots,j-2,j-1,j-1,j,\ldots,n\})$$

$$= \{0,\ldots,i-1,i+1,\ldots,j-1,j,j,j+1,\ldots,n+1\}$$

$$= \{0,\ldots,i-1,i+1,\ldots,j-1,j,j+1,\ldots,n+1\}$$

$$\Rightarrow \quad \delta^j_{n+1}\sigma^i_{n+1} = \sigma^i_n \delta^{j-1}_n \quad (i < j)$$

(d) $\delta^i_n\sigma^i_n(\{0,\ldots,i,\ldots,n\}) = \delta^i_n(\{0,\ldots,i-1,i+1,\ldots,n+1\})$

$= \{0,\ldots,i-1,i,i+1,\ldots,n\}$

$= \{0,1,\ldots,n\}$

$\delta^i_n\sigma^{i+1}_n(\{0,\ldots,i+1,\ldots,n\}) = \delta^i_n(\{0,\ldots,i,i+2,\ldots,n+1\})$

$= \{0,\ldots,i,i+1,\ldots,n\}$

$= \{0,1,\ldots,n\}$

$\Rightarrow \delta^i_n\sigma^i_n = \delta^i_n\sigma^{i+1}_n = 1.$

(e) $\delta^j_{n+1}\sigma^i_{n+1}(\{0,\ldots,j,\ldots,i,\ldots,n+1\}) =$

$= \delta^j_{n+1}(\{0,\ldots,j,\ldots,i-1,i+1,\ldots,n+2\})$

$= \{0,\ldots,j-1,j,j,j+1,\ldots,i-2,i,\ldots,n+1\}$

$= \{0,\ldots,j-1,j,j+1,\ldots,i-2,i,\ldots,n+1\}$

$\sigma^{i-1}_n\delta^j_n(\{0,\ldots,j,\ldots,i,\ldots,n+1\}) =$

$= \sigma^{i-1}_n(\{0,\ldots,j-1,j,j,j+1,\ldots,i-1,\ldots,n\})$

$= \{0,\ldots,j-1,j,j+1,\ldots,i-2,i,\ldots,n+1\}$

$\Rightarrow \delta^j_{n+1}\sigma^i_{n+1} = \sigma^{i-1}_n\delta^j_n \quad (j+1<i)$ //

(1.1.4) Proposition. Every monotonic function from $\Delta_n \to \Delta_q$ is composed of morphisms of the form δ^i_p, σ^i_p.

Proof: By induction on n

(i) $n = 0$. Let $f : \Delta_0 \to \Delta_q$ be given by $f(0) = s \leq q$.

Then by (1.1.2) $f = \sigma^{s+1}_{q-1}\cdot\sigma^{s+1}_{q-2}\cdot\ldots\cdot\sigma^{s+1}_{s}\cdot\sigma^{s}_{s-1}\cdot\ldots\cdot\sigma^1_1\cdot\sigma^0_0$

(ii) Assume proposition is true for all $k \leq n$.

Let $g : \Delta_{n+1} \to \Delta_q$ be a monotonic function defined by $g(\{0,1,\ldots,n,n+1\}) = \{s_0, s_1, \ldots, s_n, s_{n+1}\}$ where $0 \le s_0 \le s_1 \le s_2 \le \ldots \le s_n \le s_{n+1} \le q$;

define $g_1 : \Delta_n \to \Delta_{s_n}$ by setting $g_1(i) = s_i$ for $0 \le i \le n$.

Then by the induction argument g_1 is composed of morphisms of the type σ^i, δ^i.

Define f to be the function obtained by increasing by 1 the subscript of each σ^i and $\delta^i_.$ in g_1 and composing them in the same order. Then $f : \Delta_{n+1} \to \Delta_{s_n+1}$

and

$$f(\{0,1\ldots,n,n+1\}) = \{s_0, s_1, \ldots, s_n, f(n+1)\}$$

increasing the subscripts by 1 means that $f(n+1) \in \Delta_{s_n+1}$.

By definition of the σ^i and δ^j, $f(n+1) = s_n+1$ since $0 \le i \le n+1$ and $0 \le j \le n$.

If $s_n+1 < s_{n+1}$ then

$$g = \sigma_{q-1}^{s_{n+1}+1} \cdot \sigma_{q-2}^{s_{n+1}+1} \cdot \ldots \cdot \sigma_{s_{n+1}}^{s_{n+1}+1} \cdot \sigma_{s_{n+1}-1}^{s_{n+1}-1} \cdot \sigma_{s_{n+1}-1}^{s_{n+1}-2} \cdot \ldots \cdot \sigma_{s_n+2}^{s_n+2} \cdot \sigma_{s_n+1}^{s_n+1} \cdot f$$

If $s_n + 1 > s_{n+1}$ then $S_{n+1} = s_n$ and

$$g = \sigma_{q-1}^{s_n+1} \cdot \sigma_{q-2}^{s_n+1} \cdot \sigma_{q-3}^{s_n+1} \cdot \ldots \cdot \sigma_{s_n}^{s_n+1} \cdot \sigma_{s_n}^{s_n} \cdot f$$

This concludes the proof of the proposition. //

There are other equivalent definitions of an ss-complex, two of which we shall give here.

(1.1.5) Definition: An s.s.complex F in a category $\underline{R}$ is a sequence of objects $F_0, F_1, F_2, \ldots$ in $\underline{R}$, together with the following maps:

the i-th face operator $d_i^q = d_i : F_{q+1} \to F_q$, $i = 0,1,\ldots,q+1$; $q \geq 0$

the j-th degeneracy operator $s_i^q = s_i : F_q \to F_{q+1}$, $i = 0,1,\ldots,q$: $q \geq 0$

subject to the identities:

(1) $d_i d_j = d_{j-1} d_i \quad (i < j)$

(2) $s_i s_j = s_{j+1} s_i \quad (i \leq j)$

(3) $s_i s_j = s_{j-1} d_i \quad (i < j)$

(4) $d_i s_i = d_{i+1} s_i = 1$

(5) $d_i s_j = s_j d_{i-1} \quad (j + 1 < i)$

If F and K are s.s. complexes, an s.s. map $f : F \to K$ is a sequence of maps $f_q : F_q \to K_q$, commuting with the face and degeneracy operators.

(1.1.6) Definition: An s.s. complex F in the category $\underline{R}$ is a sequence of objects $F_0, F_1, F_2, \ldots$ in the category $\underline{R}$, together with maps $\alpha^* : F_q \to F_p$ associated with each monotonic $\alpha : \Delta_p \to \Delta_q$ such that if $\Delta_p \xrightarrow{\alpha_1} \Delta_q \xrightarrow{\alpha_2} \Delta_n$ then $(\alpha_2 \cdot \alpha_1)^* = \alpha_1^* \cdot \alpha_2^*$, and $(id)^* = id$.

An s.s. map $f : F \to K$ is then characterized by the condition $\alpha^* \cdot f = f \cdot \alpha^*$ for all monotonics α.

(1.1.7) Theorem [13] p.

The three definitions of an s.s. complex given in (1.1.1), (1.1.5) and (1.1.6) are equivalent.

<u>Proof</u>: (1.1.1) => (1.1.5). Given a functor $F : \underline{\Delta}^{opp} \to R$, we construct an s.s. complex consisting of the sequence of objects $F_0 = F(\Delta_0)$, $F_1 = F(\Delta_1)$, $F_2 = F(\Delta_2)$, ... and the maps

$d_i^q = F(\sigma_q^i) : F_{q+1} \to F_q$, $i = 0,1, \dots, q+1$

$s_i^q = F(\delta_q^i) : F_q \to F_{q+1}$, $i = 0,1,\dots,q$.

These are the face and degeneracy operators of (1.1.5) and because of the identities in (1.1.3) they satisfy the required conditions making the sequence an s.s. complex.

(1.1.5) => (1.1.6)

We are given an s.s. complex with face and degeneracy operators. We must now associate to each monotonic $\alpha : \Delta_p \to \Delta_q$ a map $\alpha^* : F_q \to F_p$. By (1.1.4) we can write $\alpha = \sigma^{j_1}\sigma^{j_2} \dots \sigma^{j_t}\delta^{i_1}\delta^{i_2} \dots \delta^{i_r}$ for some t and r.

Then put $\alpha^* = s_{i_r} \dots s_{i_2} s_{i_1} d_{j_t} \dots d_{j_2} d_{j_1} : F_q \to F_p$.

Obviously if $\Delta_p \xrightarrow{\alpha_1} \Delta_q \xrightarrow{\alpha_2} \Delta_r$ then $(\alpha_2 \cdot \alpha_1)^* = \alpha_1^* \cdot \alpha_2^*$.

Since $1 = \delta_n^i \cdot \sigma_n^i$ then $(1)^* = d_i^n \cdot s_i^n = 1$.

(1.1.6) => (1.1.1)

We are given a sequence $F_0, F_1, \dots$ and a map $\alpha^* : F_q \to F_p$ associated to each monotonic α. Define a functor $F : \underline{\Delta}^{opp} \to \underline{R}$ as follows:

$(\forall \Delta_q \in |\underline{\Delta}^{opp}|)\ F(\Delta_q) = F_q$.

$(\forall f \in \underline{\Delta}(\Delta_p, \Delta_q))\ F(f) = f^* : F_q \to F_p$.

We see that because of the conditions in (1.1.6)

$$F(f \cdot g) = F(g) \cdot F(f)$$

and $F(1) = 1$,

so that F is indeed a functor from $\underline{\Delta}^{opp}$ to $\underline{R}$. //

(1.1.8) Definition: An s.s. complex $L : \underline{\Delta}^{opp} \to \underline{R}$ is a sub-complex of an s.s. complex F iff for $q = 0,1,\ldots,$

$$L(\Delta_q) \subseteq F(\Delta_q)$$

and $L(\sigma_n^i) = F(\sigma_n^i) \mid L(\Delta_{n+1}) \qquad i = 0,1,\ldots,n+1$

$$L(\delta_n^i) = F(\delta_n^i) \mid L(\Delta_n) \qquad i = 0,1,\ldots,n.$$

§2 Functors and Natural Transformations

(1.2.1) Lemma: If $\theta : F \to G$ and $\phi : G \to H$ are natural transformations of the functors $F,G,$ and H from the category $\underline{R}$ into the category $\underline{R}'$, then $\phi \cdot \theta : F \to H$ is a natural transformation defined by $(\phi \cdot \theta)_X = \phi_X \cdot \theta_X : FX \to GX$ for each $X \in |\underline{R}|$.

Proof: $(\forall X,Y \in |\underline{R}|)$, $(\forall f \in \underline{R}(X,Y))$ consider the following diagram

$$\begin{array}{ccc} X & \qquad & FX \xrightarrow{\theta_X} GX \xrightarrow{\phi_X} HX \\ \downarrow f & & F(f)\downarrow \qquad \downarrow G(f) \qquad \downarrow H(f) \\ Y & & FY \xrightarrow{\theta_Y} GY \xrightarrow{\phi_Y} HY \end{array}$$

The smaller squares are commutative since θ and ϕ are natural. Therefore the larger square is also commutative. Thus the definition of $\phi \cdot \theta$ give a natural transformation from F to H. //

Lemma: Let $F,G : \underline{R} \to \underline{R}'$, $U : \underline{R}' \to \underline{R}''$, and $U' : \underline{R}'' \to \underline{R}$ be functors on the categories $\underline{R}, \underline{R}', \underline{R}''$. If $\theta : F \to G$ is a natural transformation defined by $\theta_X : FX \to GX$ then $(\forall X \in |\underline{R}|)$ and $(\forall Y \in |\underline{R}''|)$.

(1.2.2) (i) $U\theta : UF \to UG$ is a natural transformation defined by $(U\theta)_X = U(\theta_X) : UFX \to UGX$

and

(1.2.3) (ii) $\theta U' : FU' \to GU'$ is a natural transformation defined by $(\theta U')_Y = \theta_{U'Y} : FU'Y \to GU'Y$.

Proof: The proofs are similar to (1.2.1). The commutative diagram for (1.2.2) is the following: $(\forall X,Z \in |\underline{R}|,\ \forall f \in \underline{R}(X,Z))$ we have

$$\begin{array}{c} X \\ \downarrow f \\ Z \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{\theta_X} & GX \\ \downarrow F(f) & & \downarrow G(f) \\ FZ & \xrightarrow[\theta_Z]{} & GZ \end{array} \qquad \begin{array}{ccc} UFX & \xrightarrow{U(\theta_X)} & UGX \\ \downarrow UF(f) & & \downarrow UG(f) \\ UFZ & \xrightarrow[U(\theta_Z)]{} & UGZ \end{array}$$

The commutative diagram for (1.2.3) is the following: $(\forall X,Y \in |\underline{R}''|,\ \forall \alpha \in \underline{R}''(X,Y))$ we have

$$\begin{array}{c} X \\ \alpha \downarrow \\ Y \end{array} \qquad \begin{array}{c} U'X \\ U'(\alpha) \downarrow \\ U'Y \end{array} \qquad \begin{array}{ccc} FU'X & \xrightarrow{\theta_{U'X}} & GU'X \\ FU'(\alpha) \downarrow & & \downarrow GU'(\alpha) \\ FU'Y & \xrightarrow{\theta_{U'Y}} & GU'Y \end{array}$$

Lemma: Given functors $F,G,H : \underline{R}' \to \underline{R}''$, $V : \underline{R}'' \to \underline{M}'$, $V' : \underline{R}'' \to \underline{M}''$ $U : \underline{M}' \to \underline{M}''$, $U' : \underline{M}' \to \underline{R}'$ and natural transformations $\theta : F \to G$, $\theta' : G \to H$, then the following statements hold:

(1.2.4) (i) $(U \cdot V)\theta = U(V\theta)$

(1.2.5) (ii) $\theta(U'V) = (\theta U')V$

(1.2.6) (iii) $(V'\theta)U' = V'(\theta U')$

(1.2.7) (iv) $V'(\theta' \cdot \theta)U' = (V'\theta'U') \cdot (V'\theta U')$

where $\cdot$ is composition of functors.

Proof: (i) $(\forall X,Y \in |\underline{R}'|,\ \forall \alpha \in \underline{R}'(X,Y))$ we have the commutative diagrams

$$\begin{array}{c} X \\ \alpha\downarrow \\ Y \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{\theta_X} & GX \\ F(\alpha)\downarrow & & \downarrow G(\alpha) \\ FY & \xrightarrow{\theta_Y} & GY \end{array} \qquad \begin{array}{ccc} (U\cdot V)FX & \xrightarrow{(U\cdot V)\theta_X} & (U\cdot V)GX \\ (U\cdot V)F(\alpha)\downarrow & & \downarrow (U\cdot V)G(\alpha) \\ (U\cdot V)FY & \xrightarrow{(U\cdot V)\theta_Y} & (U\cdot V)GY \end{array} \qquad \begin{array}{ccc} U(VFX) & \xrightarrow{U(V\theta_X)} & U(VGX) \\ U(VF(\alpha))\downarrow & & \downarrow U(VG(\alpha)) \\ U(VFY) & \xrightarrow{U(V\theta)_Y} & U(VGY) \end{array}$$

By (1.2.2),
$$\begin{aligned}[(U \cdot V)\theta]_X = (U \cdot V)(\theta_X) &= U(V\theta_X) \\ &= U[(V\theta)_X] \\ &= [U(V\theta)]_X .\end{aligned}$$

(ii) $(\forall X,Y \in |\underline{R}''|,\ \forall \alpha \in \underline{R}''(X,Y))$ we have the commutative diagrams

$$\begin{array}{c} X \\ \alpha\downarrow \\ Y \end{array} \qquad \begin{array}{c} U'VX \\ U'V(\alpha)\downarrow \\ U'VY \end{array} \qquad \begin{array}{ccc} F(U'VX) & \xrightarrow{\theta_{U'VX}} & G(U'VX) \\ F(U'V(\alpha))\downarrow & & \downarrow G(U'V(\alpha)) \\ F(U'VY) & \xrightarrow{\theta_{U'VY}} & G(U'VY) \end{array} \qquad \begin{array}{ccc} FU'(VX) & \xrightarrow{\theta U'_{VX}} & GU'(VX) \\ \downarrow & & \downarrow \\ FU'(VY) & \xrightarrow{\theta U'_{VY}} & GU'(VY) \end{array}$$

By (1.2.3)
$$\begin{aligned}[\theta(U'V)]_X = \theta_{U'VX} &= \theta_{U'(VX)} \\ &= \theta U'_{VX} \\ &= (\theta U')_{VX} \\ &= [(\theta U')V]_X .\end{aligned}$$

(iii) $(\forall X,Y \in |\underline{M}'|,\ \forall \alpha \in \underline{M}'(X,Y))$ we have the commutative diagram

$$\begin{array}{ccccc} X & U'X & V'F(U'X) & \xrightarrow{V'\theta_{U'X}} & V'G(U'X) \\ \alpha\downarrow & \downarrow U'(\alpha) & \downarrow & & \downarrow \\ Y & U'Y & V'F(U'Y) & \xrightarrow{V'\theta_{U'Y}} & V'G(U'Y) \end{array}$$

By (1.2.2) and (1.2.3), $[(V'\theta)U']_X = (V'\theta)_{U'X} = V'(\theta_{U'X})$

$= V'[(\theta U')_X]$

$= [V'(\theta U')]_X$.

(iv) $(\forall X,Y \in |\underline{M'}|, \forall \alpha \in M'(X,Y))$

$$\begin{array}{ccccccc} X & U'X & V'FU'X & \xrightarrow{V'\theta U'_X} & V'GU'X & \xrightarrow{V'\theta' U'_X} & V'HU'X \\ \alpha\downarrow & \downarrow U'(\alpha) & \downarrow & & \downarrow & & \downarrow \\ Y & U'Y & V'FU'Y & \xrightarrow{V'\theta U'_Y} & V'GU'Y & \xrightarrow{V'\theta' U'_Y} & V'HU'Y \end{array}$$

By (1.2.1), (1.2.2) and (1.2.3), $[V'(\theta' \cdot \theta)U']_X = V'(\theta' \cdot \theta)_{U'X} =$

$= V'(\theta'_{U'X} \cdot \theta_{U'X})$

$= V'(\theta'_{U'X}) \cdot V'(\theta_{U'X})$

$= V'(\theta' U'_X) \cdot V'(\theta U'_X)$

$= (V'\theta' U')_X \cdot (V'\theta U')_X$

$= [(V'\theta' U') \cdot (V'\theta U')]_X$.

//

(1.2.8) Lemma: Given functors $F, G : \underline{R'} \to \underline{R''}$ and $U, V : \underline{R''} \to \underline{M}$, with natural transformations $\phi : F \to G$ and $\psi : U \to V$, then

$$(\psi G) \cdot (U\phi) = (V\phi) \cdot (\psi F) ,$$

Proof: $(\forall X \in |\underline{R'}|)$ since ψ is natural

$$\begin{array}{cccc} FX & UFX & \xrightarrow{\psi_{FX}} & VFX \\ \downarrow \phi_X & \downarrow U(\phi_X) & & \downarrow V(\phi_X) \\ GX & UGX & \xrightarrow{\psi_{GX}} & VGX \end{array}$$ commutes.

But $[(\psi G)\cdot(U\phi)]_X = (\psi G)_X \cdot (U\phi)_X$

$= \psi_{GX} \cdot U(\phi_X)$

$= V(\phi_X) \cdot \psi_{FX}$ (from diagram)

$= (V\phi)_X \cdot (\psi F)_X$

$= [(V\phi)\cdot(\psi F)]_X$. //

§3 Cotriples and S.S. Complexes

In this section we show that if we have a cotriple (called a comonad in Maclane [12]) on a category $\underline{R}$, then for each object in $\underline{R}$ we can obtain a s.s. complex in $\underline{R}$. This development has its roots in Codement [4] and Kleisi [9].

(1.3.1) Definition: A cotriple $\mathcal{W} = (C,k,p)$ on the category $\underline{R}$ consists of a functor $C : \underline{R} \to \underline{R}$ and two natural transformations

$$k : C \to 1_{\underline{R}} \quad \text{and} \quad p : C \to C^2$$

such that

(A) $(Ck)\cdot p = (kC)\cdot p = 1$

and

(B) $(Cp)\cdot p = (pC)\cdot p$.

Let $SC(\underline{R})$ be the category of s.s. complexes of $\underline{R}$. Define $F^* : \underline{R} \to SC(\underline{R})$ as follows:

$(\forall X \in |\underline{R}|)$ $F^*(X) = F : \underline{\Delta}^{opp} \to \underline{R}$ where

$$F(\Delta_n) = F^n = C^{n+1}(X)$$

and $F(\sigma_n^i) = d_i^n = C^i k C^{n-i+1} : F^{n+1} = C^{n+2}X \to C^{n+1}X = F^n$

$$F(\delta_n^i) = s_i^n = C^i p C^{n-i} : F^n = C^{n+1}X \to C^{n+2}X = F^{n+1}$$

$(\forall f \in \underline{R}(X,Y))$ $F^*(f) = \{C^{n+1}(f) \mid n = 0,1,\ldots\}$.

(1.3.2) Proposition: F^* is a functor.

Proof: It needs only to be shown that $F^*(X)$ is an s.s. complex for each X, since the other conditions are obvious. We will show that d_i^n and s_i^n satisfy the relations given in (1.1.5).

(a) To show $d_i^n \cdot d_j^{n+1} = d_{j-1}^n \cdot d_i^{n+1} \quad (i < j)$

i.e. $(C^i k C^{n-i+1}) \cdot (C^j k C^{n-j+2}) = (C^{j-1} k C^{n-j+2}) \cdot (C^i k C^{n-i+2})$

Letting $j = i + r + 1$ for $r \geq 0$, we have to show that

$(C^i k C^{n-i+1}) \cdot (C^{i+r+1} k C^{n-i-r+1}) = (C^{i+r} k C^{n-i-r+1}) \cdot (C^i k C^{n-i+2})$

$$k : C \to 1 \quad C^r k : C^{r+1} \to C^r$$

(1.2.8) $\Rightarrow k C^r \cdot C^{r+1} k = C^r k \cdot k C^{r+1}$

$\Rightarrow C^i (k C^r \cdot C^{r+1} k) C^{n-i-r+1} = C^i (C^r k \cdot k C^{r+1}) C^{n-i-r+1}$ by (1.2.7).

Hence, $C^i (k C^r) C^{n-i-r+1} \cdot C^i (C^{r+1} k) C^{n-i-r+1} =$

$= C^i (C^r k) C^{n-i-r+1} \cdot C^i (k C^{r+1}) C^{n-i-r+1}$

$\Rightarrow C^i k C^{n-i+1} \cdot C^{i+r+1} k C^{n-i-r+1} = C^{i+r} k C^{n-i-r+1} \cdot C^i k C^{n-i+2}$

$\Rightarrow d_i^n \cdot d_j^{n+1} = d_{j-1}^n \cdot d_i^{n+1}$ for $i < j$.

(b) To show $s_i^{n+1} \cdot s_j^n = s_{j+1}^{n+1} \cdot s_i^n \quad (i \leq j)$

i.e. $(C^i p C^{n-i+1}) \cdot (C^j p C^{n-j}) = (C^{j+1} p C^{n-j}) \cdot (C^i p C^{n-i})$

In the case $i = j$, since (B) $\Rightarrow (pC) \cdot p = (Cp) \cdot p$

then $C^i [(pC) \cdot p] C^{n-i} = C^i [(Cp) \cdot p] C^{n-i}$

(1.2.7) $\Rightarrow C^i (pC) C^{n-i} \cdot C^i p C^{n-i} = C^i (Cp) C^{n-i} \cdot C^i p C^{n-i}$

$i = j \Rightarrow C^i p C^{n-i+1} \cdot C^j p C^{n-j} = C^{j+1} p C^{n-j} \cdot C^i p C^{n-i}$

If $i < j$, we let $j = i + r$ for $r > 0$. We then have to show that $(C^i pC^{n-i+1}) \cdot (C^{i+r} pC^{n-i-r}) =$

$$= (C^{i+r+1} pC^{n-i-r}) \cdot (C^i pC^{n-i})$$

$$p : C \to C^2 \qquad pC^{r-1} : C^r \to C^{r+1}$$

$(1.2.8) \Rightarrow pC^{r+1} \cdot C^r p = C^{r+1} p \cdot pC^r$

$\Rightarrow C^i (pC^{r+1} \cdot C^r p) C^{n-i-r} = C^i (C^{r+1} p \cdot pC^r) C^{n-i-r}$

$\Rightarrow C^i (pC^{r+1}) C^{n-i-r} \cdot C^i (C^r p) C^{n-i-r} = C^i (C^{r+1} p) C^{n-i-r} \cdot C^i (pC^r) C^{n-i-r}$

and so, $C^i pC^{n-i+1} \cdot C^{i+r} pC^{n-i-r} = C^{i+r+1} pC^{n-i-r} \cdot C^i pC^{n-i}$

$$\Rightarrow s_i^{n+1} \cdot s_j^n = s_{j+1}^{n+1} \cdot s_i^n \quad \text{for } i \leq j.$$

(c) to show that $d_i^{n+1} \cdot s_j^{n+1} = s_{j-1}^n \cdot d_i^n \quad (i < j)$

i.e. $C^i kC^{n-i+2} \cdot C^j pC^{n-j+1} = C^{j-1} pC^{n-j+1} \cdot C^i kC^{n-i+1}$

Letting $j = i + r + 1$ for $r \geq 0$ we have to show that

$C^i kC^{n-i+2} \cdot C^{i+r+1} pC^{n-i-r} = C^{i+r} pC^{n-i-r} \cdot C^i kC^{n-i+1}$

$$p : C \to C^2 \qquad kC^r : C^{r+1} \to C^r$$

$(1.2.8) \Rightarrow kC^{r+2} \cdot C^{r+1} p = C^r p \cdot kC^{r+1}$

$\Rightarrow C^i (kC^{r+2} \cdot C^{r+1} p) C^{n-i-r} = C^i (C^r p \cdot kC^{r+1}) C^{n-i-r}$

$(1.2.7) \Rightarrow C^i (kC^{r+2}) C^{n-i-r} \cdot C^i (C^{r+1} p) C^{n-i-r} =$

$$= C^i (C^r p) C^{n-i-r} \cdot C^i (kC^{r+1}) C^{n-i-r}$$

$\Rightarrow C^i kC^{n-i+2} \cdot C^{i+r+1} pC^{n-i-r} = C^{i+r} pC^{n-i-r} \cdot C^i kC^{n-i+1}$

$\Rightarrow d_i^{n+1} \cdot s_j^{n+1} = s_{j-1}^n \cdot d_i^n$ for $i < j$.

(d) to show that $d_i^n \cdot s_i^n = d_{i+1}^n \cdot s_i^n = 1$

i.e., $(C^i kC^{n-i+1}) \cdot (C^i pC^{n-i}) = (C^{i+1} kC^{n-i}) \cdot (C^i pC^{n-i}) = 1$

$(A) \Rightarrow kC \cdot p = Ck \cdot p = 1$

$\Rightarrow C^i(kC \cdot p)C^{n-i} = C^i(Ck \cdot p)C^{n-i} = C^i(1)C^{n-i}$

$\Rightarrow C^i kC^{n-i+1} \cdot C^i pC^{n-i} = C^{i+1}kC^{n-i} \cdot C^i pC^{n-i} = 1$

$\Rightarrow d_i^n \cdot s_i^n = d_{i+1}^n \cdot s_i^n = 1.$

(e) to show that $d_i^{n+1} \cdot s_j^{n+1} = s_j^n \cdot d_{i-1}^n \quad (j + 1 < i)$

i.e. $(C^i kC^{n-i+2}) \cdot (C^j pC^{n-j+1}) = (C^j pC^{n-j}) \cdot (C^{i-1}kC^{n-i+2})$

Letting $i = j + r + 2$ for $r \geq 0$ we have to show that

$(C^{j+r+2}kC^{n-j-r}) \cdot (C^j pC^{n-j+1}) = (C^j pC^{n-j}) \cdot (C^{j+r+1}kC^{n-j-r})$

$C^r k : C^{r+1} \to C^r \qquad p : C \to C^2$

$(1.2.8) \Rightarrow C^{r+2}k \cdot pC^{r+1} = pC^r \cdot C^{r+1}k$

$\Rightarrow C^j(C^{r+2}k \cdot pC^{r+1})C^{n-j-r} = C^j(pC^r \cdot C^{r+1}k)C^{n-j-r}$

$\Rightarrow C^j(C^{r+2}k)C^{n-j-r} \cdot C^j(pC^{r+1})C^{n-j-r} =$

$= C^j(pC^r)C^{n-j-r} \cdot C^j(C^{r+1}k)C^{n-j-r}$

$\Rightarrow C^{j+r+2}kC^{n-j-r} \cdot C^j pC^{n-j+1} = C^j pC^{n-j} \cdot C^{j+r+1}kC^{n-j-r}$

$\Rightarrow d_i^{n+1} \cdot s_j^{n+1} = s_j^n \cdot d_{i-1}^n$ for $j + 1 < i$.

§4 Adjointness and cotriples

We will now show that every set of adjoint functors defines a cotriple and every cotriple is induced by a pair (not necessarily unique) of adjoint functors. These results are essentially a dualization of the work done on triples in [5]. Cotriples were first used by Maclane (1956) and Godement (1958). The relationship between cotriples and adjunctions was first given by Kleisli [10] in 1965. He refers to the cotriple as a "standard construction". Eilenberg-Moore (1965) did the same for triples.

(1.4.1) <u>Definition</u>: Let $F : \underline{R}' \to \underline{R}$ and $G : \underline{R} \to \underline{R}'$ be functors. F is left adjoint to G (written $F \dashv G$) iff there is a natural isomorphism

$$\eta : \underline{R}\,(FA,B) \to \underline{R}'\,(A,GB)$$

for all $A \in |\underline{R}'|$ and for all $B \in |\underline{R}|$.

<u>Remark</u>: Each set of adjoint functors determines two natural transformations (we omit the proof):

(i) the unit $\varepsilon : 1 \to GF$ defined by $\varepsilon_A = \eta(1_{FA})$

and

(ii) the counit $\delta : FG \to 1$ defined by $\delta_B = \eta^{-1}(1_{GB})$.

(1.4.2) <u>Lemma</u>: If $F \dashv G$ with ε and δ the unit and counit of the adjunction η, then

$$\text{(i)}\quad \delta F \cdot F\varepsilon = 1$$

and

$$\text{(ii)}\quad G\delta \cdot \varepsilon G = 1.$$

<u>Proof</u>: For any $g \in \underline{R}'\,(A,GB)$, because η is natural we have the following commutative diagram.

$$\begin{array}{ccc} \underline{R}\,(FGB,B) & \xrightarrow{\ \eta\ } & \underline{R}'\,(GB,GB) \\ \Big\downarrow{\scriptstyle \underline{R}(F(g),1)} & & \Big\downarrow{\scriptstyle \underline{R}'(g,G(1))} \\ \underline{R}(FA,B) & \xrightarrow[\ \eta\ ]{} & \underline{R}'\,(A,GB) \end{array}$$

Thus, in particular

$$\eta^{-1} \cdot R'(g, G(1))(1_{GB}) = \underline{R}(F(g),1) \cdot \eta^{-1}(1_{GB}).$$

The left hand side is $\eta^{-1}(G(1) \cdot 1_{GB} \cdot g) = \eta^{-1}(g)$.

The right hand side is $\underline{R}(F(g),1)(\delta_B) = 1 \cdot \delta_B \cdot F(g)$,

$$= \delta_B \cdot F(g).$$

Letting $g = \varepsilon_A$ we get $\eta^{-1}(\varepsilon_A) = \delta_{FA} \cdot F(\varepsilon_A)$

$$1_{FA} = (\delta F)_A \cdot (F\varepsilon)_A$$

$$= [\delta F \cdot F\varepsilon]_A$$

and (i) is proven.

For (ii) we consider the following commutative diagram for any $f \in \underline{R}(FA, B)$:

$$\begin{array}{ccc} \underline{R}(FA,FA) & \xrightarrow{\eta} & \underline{R}'(A,GFA) \\ {\scriptstyle \underline{R}(F(1),f)} \downarrow & & \downarrow {\scriptstyle \underline{R}'(1,G(f))} \\ \underline{R}(FA,B) & \xrightarrow[\eta]{} & \underline{R}'(A,GB). \end{array}$$

We have $\eta \cdot \underline{R}(F(1),f)(1_{FA}) = \underline{R}'(1,G(f)) \cdot \eta(1_{FA})$

On the left we get $\eta(f \cdot 1_{FA} \cdot F(1)) = \eta(f)$

and right

$$R'(1,G(f))(\varepsilon_A) = G(f) \cdot \varepsilon_A \cdot 1$$

$$= G(f) \cdot \varepsilon_A$$

Thus for $f = \delta_B : FGB \to B$, we have

$$\eta(\delta_B) = G(\delta_B) \cdot \varepsilon_{GB}$$

$$1_{GB} = G\delta_B \cdot \varepsilon G_B$$

$$= [G\delta \cdot \varepsilon G]_B$$

which is (ii). //

(1.4.3) Let $F : \underline{R}' \to \underline{R}$ and $G : \underline{R} \to \underline{R}'$ with F left adjoint to G. Let $\varepsilon : 1 \to GF$ and $\delta : FG \to 1$ be the unit and counit of the adjunction; so (i) $\delta F \cdot F\varepsilon = 1$ and (ii) $G\delta \cdot \varepsilon G = 1$.

We now define a cotriple (C,k,p).

Set $C = FG : \underline{R} \to \underline{R}$.

$k : FG \to 1$ is the counit, δ, of the adjunction.

$p : FG \to FGFG$ is given by $p = F(\varepsilon G)$.

(1.4.4) Proposition: (FG,k,p) is a cotriple on $\underline{R}$.

Proof: We must show that the identities (A) and (B) of (1.3.1) hold. For (A) we have to show that the following diagram commutes for all $X \in |\underline{R}|$:

$$FGX \xrightarrow{F(\varepsilon G)_X} FGFGX \underset{(\delta F)G_X}{\overset{F(G\delta)_X}{\rightrightarrows}} FGX, \qquad FGX \xrightarrow{1_{FGX}} FGX$$

We obtain this result by applying F on the left of (ii) in (1.4.2) and G on the right of (i) in (1.4.2). For (B) we have to show that the following diagrams coincide for all $X \in |\underline{R}|$:

$$FGX \xrightarrow{F\varepsilon G_X} FGFGX \xrightarrow{(FG)(F\varepsilon G)_X} FGFGFGX$$

and

$$FGX \xrightarrow{F\varepsilon G_X} FGFGX \xrightarrow{(F\varepsilon G)(FG)_X} FGFGFGX$$

From the naturality of ε, if $f : X \to X'$, then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & GFX \\ \downarrow f & & \downarrow GF(f) \\ X' & \xrightarrow[\varepsilon_{X'}]{} & GFX' \end{array}$$

If $f = \varepsilon_X : X \to GFX$, then this diagram becomes

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & GFX \\ \downarrow \varepsilon_X & & \downarrow GF(\varepsilon_X) \\ GFX & \xrightarrow[\varepsilon_{GFX}]{} & GFGFX \end{array}$$

But $\varepsilon_{GFX} = \varepsilon GF_X$. Thus $\varepsilon GF \cdot \varepsilon = GF\varepsilon \cdot \varepsilon$.
Applying F on the left and G on the right we get

$F(\varepsilon GF \cdot \varepsilon)G = F(GF\varepsilon \cdot \varepsilon)G$ and thus by (1.2.1)
$F(\varepsilon GF)G \cdot F\varepsilon G = F(GF\varepsilon)G \cdot F\varepsilon G$; that is to say,
$(F\varepsilon G)(FG) \cdot F(\varepsilon G) = (FG)(F\varepsilon G) \cdot F(\varepsilon G)$ which is the required identity. //

Given a cotriple $\mathbb{W} = (C,k,p)$ on the category $\underline{R}$, we will now construct a pair of adjoint functors which induce $\mathbb{W}$.

Let $\underline{R}^{\mathbb{W}}$ be the category whose objects are pairs (X,ϕ) where $X \in |\underline{R}|$ and $\phi \in \underline{R}(X,CX)$ such that $k_X \cdot \phi = 1_X$ and $p_X \cdot \phi = C\phi \cdot \phi$.

A morphism $f : (X,\phi) \to (Y,\psi)$ is a morphism $f \in \underline{R}(X,Y)$ such that

$$\begin{array}{ccc} CX & \xrightarrow{Cf} & CY \\ \uparrow \phi & & \uparrow \psi \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

Define $F : \underline{R}^{\text{ш}} \to \underline{R}$ as follows:

$(\forall (X,\phi) \in |\underline{R}^{\text{ш}}|)\ F[(X,\phi)] = X$

$(\forall f \in \underline{R}^{\text{ш}}((X,\phi), (Y,\psi)))\ \ F(f) = f : X \to Y$

We define next $G : R \to R^{\text{ш}}$ by $G(X) = (CX, p_X)$ on objects;

notice that $\forall X \in |\underline{R}|$ since

$$k_{CX} \cdot p_X = (kC)_X \cdot p_X = 1$$

and

$$p_{CX} \cdot p_X = (pC)_X \cdot p_X = (Cp)_X \cdot p_X \ ,$$

(CX, p_X) is indeed an object of $\underline{R}^{\text{ш}}$.

As for the morphisms of $\underline{R}$, for every $f \in \underline{R}(X,Y)$

$G(f) = C(f) : CX \to CY$; here, since p is a natural transformation from C to C^2,

$$\begin{array}{ccc} C^2X & \xrightarrow{C^2(f)} & C^2Y \\ {\scriptstyle p_X}\uparrow & & \uparrow{\scriptstyle p_Y} \\ CX & \xrightarrow[C(f)]{} & CY \end{array}$$

commutes.

(1.4.5) Lemma: Given functors $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{C}$ and natural transformations $\varepsilon : 1 \to GF$ and $\delta : FG \to 1$ such that

(i) $\delta F \cdot F\varepsilon = 1_F$

and

(ii) $G\delta \cdot \varepsilon G = 1_G$,

then for every $f \in \underline{D}(FA,B)$, $\eta(f) = G(f) \cdot \varepsilon_A : A \to GB$ defines an adjunction $\eta : F \to G$ such that ε and δ are the unit and counit of the adjunction; in other words $(\forall A \in |\underline{C}|, \forall B \in |\underline{D}|)\ \eta : \underline{D}(FA,B) \to \underline{C}(A,GB)$ is a natural equivalence.

Proof: First, the naturality of η. We must show that

$\forall \alpha \in \underline{C}(A_1,A)$, $\forall \beta \in \underline{D}(B,B_1)$ and $\forall f \in \underline{D}(FA,B)$

$$\begin{array}{ccc} \underline{D}(FA,B) & \xrightarrow{\ \eta\ } & \underline{C}(A,GB) \\ \downarrow{\scriptstyle \underline{D}(F\alpha,\beta)} & & \downarrow{\scriptstyle \underline{C}(\alpha,G\beta)} \\ \underline{D}(FA_1,B_1) & \xrightarrow{\ \eta\ } & \underline{C}(A_1,GB_1) \end{array}$$

commutes, where

$\underline{D}(F\alpha,\beta)(f) = \beta \cdot f \cdot F\alpha$

and $\underline{C}(\alpha,G\beta)(g) = G\beta \cdot g \cdot \alpha$ for $g \in \underline{C}(A,G\beta)$.

So we must prove that $\eta(\beta \cdot f \cdot F\alpha) = G\beta \cdot \eta(f) \cdot \alpha$.

But

$$\begin{aligned} \eta(\beta \cdot f \cdot F\alpha) &= G(\beta \cdot f \cdot F\alpha) \cdot \varepsilon_{A_1} && \text{by definition of } \eta \\ &= G\beta \cdot G(f) \cdot GF\alpha \cdot \varepsilon_{A_1} \\ &= G\beta \cdot G(f) \cdot \varepsilon_A \cdot \alpha && \text{since } \varepsilon \text{ is natural} \\ &= G\beta \cdot \eta(f) \cdot \alpha && \text{by definition of } \eta. \end{aligned}$$

We must now show that η is an isomorphism.

Define $\bar{\eta} : \underline{C}(A,GB) \to \underline{D}(FA,B)$ by

$\bar{\eta}(g) = \delta_B \cdot F(g) : FA \to B$ for all $g \in \underline{C}(A,GB)$.

Now $\forall f \in \underline{D}(FA,B)$,

$$\begin{aligned} \bar{\eta}\eta(f) &= \delta_B \cdot F[\eta(f)] \\ &= \delta_B \cdot F(Gf \cdot \varepsilon_A) \\ &= \delta_B \cdot FGf \cdot F\varepsilon_A \\ &= f \cdot \delta_{FA} \cdot F\varepsilon_A && \text{since } \delta \text{ is natural} \\ &= f \cdot (\delta F)_A \cdot (F\varepsilon)_A \\ &= f \cdot (\delta F \cdot F\varepsilon)_A \\ &= f \cdot 1_{FA} && \text{by (i) above} \\ &= f . \end{aligned}$$

Thus $\bar{\eta}\eta = 1$.

If $g : A \to GB$, then

$$
\begin{aligned}
\eta\bar{\eta} &= G(\bar{\eta}(g)) \cdot \varepsilon_A \\
&= G(\delta_B \cdot F(g)) \cdot \varepsilon_A \\
&= G\delta_B \cdot GFg \cdot \varepsilon_A \\
&= G\delta_B \cdot \varepsilon_{GB} \cdot g && \text{since } \varepsilon \text{ is natural} \\
&= G\delta_B \cdot \varepsilon\, G_B \cdot g \\
&= (G\delta \cdot \varepsilon G)_B \cdot g \\
&= 1_{GB} \cdot g && \text{by (ii) above} \\
&= g \,,
\end{aligned}
$$

so $\eta\bar{\eta} = 1$

η is therefore also a bijection. //

(1.4.6) Theorem: Given $F : \underline{R}^{\mathbb{W}} \to \underline{R}$ and $G : \underline{R} \to \underline{R}^{\mathbb{W}}$ as described above, then $F \dashv G$ and, F and G define the cotriple $\mathbb{W}$.

Proof: To show that $F \dashv G$ we first find the unit and counit of the adjunction.

For each object X of $\underline{R}$, $FGX = CX$; thus, define δ_X to be k_X. On the other hand, $GF(X,\phi) = (CX, p_X)$. Hence $\varepsilon_{(X,\phi)}$ must be a morphism of the object (X,ϕ) into the object (CX, p_X); this means that $\varepsilon_{(X,\phi)} : X \to CX$ is such that

$$
\begin{array}{ccc}
CX & \xrightarrow{C\varepsilon_{(X,\phi)}} & C^2X \\
\uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle p_X} \\
X & \xrightarrow[\varepsilon_{(X,\phi)}]{} & CX
\end{array}
$$

commutes. Recalling that for every $(X,\phi) \in |\underline{R}^{\mathbb{W}}|$ $p_X \cdot \phi = C\phi \cdot \phi$, we set $\varepsilon_{(X,\phi)} = \phi$.

To check that ε and δ are indeed the unit and counit of the adjunction we must show that

$$\delta F \cdot F\varepsilon = 1_F \quad \text{and} \quad G\delta \cdot \varepsilon G = 1_G.$$

For every $(X,\phi) \in |\underline{R}^{\mathbb{W}}|$,

$$\begin{aligned}[\delta F \cdot F\varepsilon]_{(X,\phi)} &= (\delta F)_{(X,\phi)} \cdot (F\varepsilon)_{(X,\phi)} \\ &= \delta_{F(X,\phi)} \cdot F(\varepsilon_{(X,\phi)}) \\ &= \delta_X \cdot F(\phi) \\ &= k_X \cdot \phi \\ &= 1 \quad \text{(see definition of } \underline{R}^{\mathbb{W}}).\end{aligned}$$

As for the second equatity, for every $X \in |\underline{R}|$

$$\begin{aligned}[G\delta \cdot \varepsilon G]_X &= (G\delta)_X \cdot (\varepsilon G)_X \\ &= G(\delta_X) \cdot \varepsilon_{GX} \\ &= G(k_X) \cdot \varepsilon_{(CX,p_X)} \\ &= Ck_X \cdot p_X \\ &= 1_{CX} \text{ by } (1.3.1), (A) \\ &= 1_{(CX,p_X)}.\end{aligned}$$

By (1.4.5) then, we have $F \dashv G$ and $\eta : F \dashv G$ is given by

$$\eta(f) = G(f) \cdot \varepsilon_{(X,\phi)} = C(f) \cdot \phi$$

for $f : F(X,\phi) \to Y$

$: X \to Y$

and $\phi : X \to CX$.

Using (1.4.4) we see that F and G define the cotriple $\mathbb{W}$, since $(\forall X \in |\underline{R}|)\ FG(X) = F(CX,p_X) = CX$ and $k_X = \delta_X$ by definition, and $[F(\varepsilon G)]_X = F(\varepsilon G)_X = F\varepsilon_{GX}$

$$= F\varepsilon_{(CX,p_X)}$$
$$= F(p_X)$$
$$= p_X \quad //$$

5 Examples

We have seen that given a set of adjoint functors we can define a cotriple which in turn defines an s.s. complex. We will now take some categories and adjoint functors which can be used to construct s.s. complexes.

Our first example will provide a rather trivial s.s. complex.

(1.5.1) Example 1: Take the category Ab of abelian groups and group homomorphisms and the category ½Ab of abelian semi-groups and semi-group homomorphisms.

Let the functor $G : Ab \to ½Ab$ be the forgetfull functor.

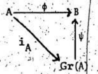
Define the functor $F : ½Ab \to Ab$ on objects by:

$(\forall A \in |½Ab|)\ F(A) = Gr(A) = A \times A/\sim$ where $\sim$ is an equivalence relation defined as follows:

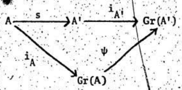
$(\forall a,b,c,d \in A)\ (a,b) \sim (c,d) \iff (\exists u \in A)$ such that $a + d + u = b + c + u$. Let $\overline{(a,b)}$ be the equivalence class defined by $\sim$, $Gr(A)$ together with the homomorphism $i_A : A \to Gr(A)$, with $i_A(a) = \overline{(a,o)}$, is called the Grothendiek Group of A. $\overline{(a,b)}$ has inverse $\overline{(b,a)}$.

As for the morphisms, we recall first the Universal Property

(1.5.2) of the Grothendiek Group. Given any abelian group B, semi-group A, and a semi-group homomorphism $\phi : A \to B$, there is a unique group homomorphism $\psi : Gr(A) \to B$ such that the following diagram commutes:



This implies that if $s : A \to A'$ is a semi-group homomorphism, there is a unique $\psi : Gr(A) \to Gr(A')$ making the following diagram commutative:

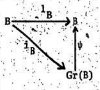


and thus,

(1.5.3) $(\forall A, A' \in |\tfrac{1}{2}\underline{Ab}|)\ (\forall s \in \tfrac{1}{2}\underline{Ab}\ (A, A'))\ F(s) = \psi$.

(1.5.4) Lemma: $(\forall B \in |\underline{Ab}|)$ $Gr(B) \cong B$ and the isomorphism is given by i_B.

Proof: By the universal property of (1.5.2) we have the following commutative diagram:



This implies that i_B = mono and ψ = epi.

$(\forall\, a, b \in B)$ (i) $\psi(\overline{a,o}) = \psi \cdot i_B(a) = 1_B(a) = a$

(ii) $\psi(\overline{a,b}) = \psi[(\overline{a,o}) + (\overline{o,b})]$

$= \psi[(\overline{a,o}) - (\overline{b,o})]$

$= \psi(\overline{a,o}) - \psi(\overline{b,o})$

$= a - b.$

(iii) Let $\psi(\overline{a,b}) = 0$

Then by (ii) $a - b = 0$. That is $a = b$.

But $(\overline{a,a}) = (\overline{o,o}) \Rightarrow (\overline{a,b}) = (\overline{o,o})$.

$\Rightarrow \psi =$ mono and epi

$\Rightarrow i_B =$ epi

$\Rightarrow i_B =$ isomorphism with inverse ψ. //

(1.5.5) Lemma; $F \dashv G$.

Proof: We have to show that for all $A \in |\tfrac{1}{2}\underline{Ab}|$ and for all $B \in |\underline{Ab}|$, there exists a natural isomorphism $\eta : \underline{Ab}\,(FA, B) \to \tfrac{1}{2}\underline{Ab}\,(A, GB)$.

For every $f \in \underline{Ab}\,(FA,B)$ we have

$$A \xrightarrow{i_A} FA = Gr(A) \xrightarrow{f} GB = B, \qquad A \xrightarrow{f \cdot i_A} B$$

Hence, define $\eta(f) = f \cdot i_A$.

To show that η is an isomorphism on each pair of objects $A \in |\tfrac{1}{2}\underline{Ab}|$, $B \in |\underline{Ab}|$

(i) η is mono.

Let $f \in Ab\,(FA,B)$ such that $\eta(f) = 0 =$ zero homomorphism.

Then $f \cdot i_A = 0$

$\Rightarrow f \cdot i_A(a) = 0 \quad \forall a \in A$

$\Rightarrow f(\overline{a,o}) = 0$

$(\forall a, b \in A)$ we have $f(\overline{a,b}) = f[(\overline{a,o}) + (\overline{o,b})]$

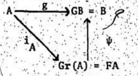
$= f(\overline{a,o}) - f(\overline{b,o})$

$= 0$

and so f is the zero homomorphism.

(ii) η is epi

Let $g \in \underline{\tfrac{1}{2}Ab}\,(A, GB)$; by the Universal Property we have the following commutative diagram:



But $\eta(\psi) = \psi \cdot i_A = g$, showing that η is an epimorphism.

(iii) η is natural.

By the proof of (1.4.3) we must show that

$(\forall A, A' \in |\underline{\tfrac{1}{2}Ab}|)\ (\forall B, B' \in |\underline{Ab}|)\ (\forall \beta \in \underline{Ab}\,(B, B'))\ (\forall \alpha \in \underline{\tfrac{1}{2}Ab}\,(A', A))$

and $\forall f \in \underline{Ab}\,(FA, B)$

$$\eta(\beta \cdot f \cdot F\alpha) = G\beta \cdot \eta(f) \cdot \alpha.$$

In fact $\eta(\beta \cdot f \cdot F\alpha) = \beta \cdot f \cdot F\alpha \cdot i_{A'}$

$= \beta \cdot (f \cdot i_A) \cdot \alpha$

$= \beta \cdot \eta(f) \cdot \alpha$

$= G\beta \cdot \eta(f) \cdot \alpha$. //

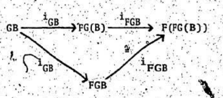
According to (1.4.3) the pair of adjoint functors $F = Gr$ and G defined a cotriple (C, k, p) on $\underline{Ab}$ with $C = FG$,

$k = \delta$ the counit of the adjunction, and $p = F(\varepsilon G)$. Notice that $(\forall B \in |\underline{Ab}|)$ by identifying $GB = B$ and by (1.5.4), $CB \cong B$; furthermore by (1.4.3),

$$p_B = F(\varepsilon G)_B = F(\varepsilon_{GB}) = F(\eta(1_{FGB}))$$
$$= F(1_{FGB} \cdot i_{GB}) = F(i_{GB})$$

(or writing $GB = B$, $p_B = F(i_B)$).

On the other hand, from the commutative diagram



the Universal Property and (1.5.3) we obtain $F(i_{GB}) = i_{FGB}$ and so $p_B = i_{FGB}$ is an isomorphism for every $B \in |\underline{Ab}|$.

For every $B \in |\underline{Ab}|$, $\delta_B = \eta^{-1}(1_{GB}) = \eta^{-1}(1_B)$ and thus $1_B = \eta(\delta_B)$; but (1.5.5) shows that $\eta(\delta_B) = \delta_B \cdot i_B$ and hence $\delta_B = i_B^{-1}$ is an isomorphism.

Using these facts and (1.3.2) we can now construct an s.s. complex, $H(B)$, for each $B \in |\underline{Ab}|$ with

$$H(B)^n = C^{n+1}(B) \cong B.$$
$$d_i^n = C^i k C^{n-i+1} = i^{n+1}_{Gr\,(B)} = \text{isomorphism}.$$
$$s_i^n = C^i p C^{n-i} = k_{Gr^{n+1}(B)} = \text{isomorphism}.$$

(1.5.6) Example 2: In this example we consider the category $\underline{Ab}$ and the category $\underline{M}_\Lambda$ (respectively ${}_\Lambda\underline{M}$) of right modules (respectively left modules) over a commutative ring Λ and Λ-homomorphisms.

Let B be a fixed object of ${}_\Lambda\underline{M}$. Define
$F = - \otimes_\Lambda B : \underline{M}_\Lambda \to \underline{Ab}$ as the functor which takes any $A \in |\underline{M}_\Lambda|$ into $A \otimes_\Lambda B$; if $f : A \to A'$ is a right Λ-module homomorphism, $F(f) : A \otimes_\Lambda B \to A' \otimes_\Lambda B$ takes $a \otimes b$ into $f(a) \otimes b$.

On the other hand, define $G : \underline{Ab} \to \underline{M}_\Lambda$ as the functor which takes any $H \in |\underline{Ab}|$ into $\mathrm{Hom}_Z(B,H)$ (with the right Λ-module structure given by $(\psi\lambda)(b) = \psi(\lambda b)$ for every $\lambda \in \Lambda$, $b \in B$ and the addition defined by ordinary addition of abelian group homomorphisms); if $h : G \to G'$, $G(h)$ is defined by composition with h, i.e. if $f : B \to G$ then $G(h)f = h \cdot f$.

(1.5.7) Lemma: $F = - \otimes_\Lambda B \dashv G = \mathrm{Hom}_Z(B,H)$.

Proof: For every $A \in |M_\Lambda|$ and every $H \in |\underline{Ab}|$ we show that there exists a natural equivalence
$\eta : \mathrm{Hom}_Z(A \otimes_\Lambda B,H) \to \mathrm{Hom}_\Lambda(A,\mathrm{Hom}_Z(B,H))$.
$(\forall a \in A)(\forall b \in B)(\forall f \in \mathrm{Hom}_Z(a \otimes_\Lambda B,H))$ define
$\eta(f)(a)(b) = f(a \otimes b)$.
We first show that $\eta(f)$ is a right Λ-homomorphism.

$$\begin{aligned}\forall\lambda \in \Lambda \quad \eta(f)\lambda(a)(b) &= f\lambda(a \otimes b)\\ &= f(\lambda(a \otimes b))\\ &= f(a\lambda \otimes b)\\ &= \eta(f)(a\lambda)(b)\end{aligned}$$

Thus $\eta(f)\lambda(a) = \eta(f)(a\lambda)$.
Let $f : A \otimes_\Lambda B \to H$ with $\eta(f) = 0 : A \to \mathrm{Hom}_Z(B,H)$
Then for all $a \in A$ and $b \in B$ $\eta(f)(a)(b) = 0$. This implies that $f(a \otimes b) = 0$ so that f is the zero morphism.
Thus η is mono.

Let $\phi \in \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_Z(B,H))$ be defined by $\phi(a)(b) = h$. Defining $f : A \otimes_\Lambda B \to H$ by $f(a \otimes b) = h$ we get $\eta(f) = \phi$ so that η is also an epimorphism.

To show that η is natural we must show $(\forall \alpha \in \mathrm{Hom}_\Lambda(A'A), \forall \beta \in \mathrm{Hom}_Z(H,H')$ and $\forall f \in \mathrm{Hom}_Z(A \otimes_\Lambda B,H))$ that

$$\eta(\beta \cdot f \cdot F(\alpha)) = G\beta \cdot \eta(f) \cdot \alpha.$$

But for all $a \in A$ and $b \in B$

$$\begin{aligned} \eta(\beta \cdot f \cdot F\alpha)(a)(b) &= \eta(\beta \cdot f \cdot \alpha \otimes 1_B)(a)(b) \\ &= (\beta \cdot f \cdot \alpha \otimes 1_B)(a \otimes b) \\ &= \beta \cdot f(\alpha(a) \otimes b) \\ &= G\beta(f(\alpha(a) \otimes b)) \\ &= G\beta \cdot \eta(f)(\alpha(a))(b) \\ & (G\beta \cdot \eta(f) \cdot \alpha)(a)(b). \quad // \end{aligned}$$

As in example 1 the adjoint functors $F = - \otimes_\Lambda B$ and $G = \mathrm{Hom}_Z(B,-)$ define a cotriple (C,k,p) on $\underline{M}_\Lambda$ with $C = FG = \mathrm{Hom}_Z(B,-) \otimes_\Lambda B$, $k = \delta$ and $p = F(\varepsilon G)$.
$(\forall H \in |\underline{AB}|)$ $k_H = \delta_H = \eta^{-1}(1_{GH}) = \eta^{-1}(1_{\mathrm{Hom}_Z(B,H)})$. Thus $\eta(k_H) = 1_{\mathrm{Hom}_Z(B,H)}$.

Thus $(\forall g \in \mathrm{Hom}_Z(B,H), \forall b \in B)$ $\eta(k_H)(g)(b) = 1_{\mathrm{Hom}_Z(B,H)}(g)(b)$

$$\begin{aligned} (1.5.7) \Rightarrow k_H(g \otimes b) &= 1_{\mathrm{Hom}_Z(B,H)}(g)(b) \\ &= g(b). \end{aligned}$$

Also by (1.4.3)

$$\begin{aligned} p_H = F(\varepsilon G)_H = F(\varepsilon_{GH}) &= \varepsilon_{GH} \otimes 1_B \\ &= \varepsilon_{\mathrm{Hom}_Z(B,H)} \otimes 1_B \\ &= \eta(1_{\mathrm{Hom}_Z(B,H)} \otimes_\Lambda B) \otimes 1_B. \end{aligned}$$

Using these facts and (1.3.2) we can now construct a semi-simplicial complex T_H for each $H \in |\underline{Ab}|$. Thus

$T_H : \underline{\Delta}^{opp} \to \underline{Ab}$ with $T_H(\Delta_n) = C^{n+1}(H) = \mathrm{Hom}_Z(B, C^n(H)) \otimes_\Lambda B$.

$d_i^n = C^i k C^{n-i+1} : C^{n+2}(H) \to C^{n+1}(H) \quad i = 0,1,\ldots,n+1$

$s_i^n = C^i p C^{n-i} : C^{n+1}(H) \to C^{n+2}(H) \quad i = 0,1,\ldots,n$

(1.5.8) Example 3: Consider the category BTop whose objects are toplogoical spaces with base point and morphisms are base preserving maps. The unit sphere s' can be viewed as the set of complex numbers $e^{i\theta}$, $0 \leq \theta < 2\pi$. The base point here is $* = e^{io} = 1$. In what follows we will leave out the base point where there is no confusion.

Define a functor Ω : BTop $\to$ BTop which acts on a space $(X, x_o) \in |\underline{BTop}|$ by $\Omega(X, x_o) = (\underline{BTop}(s', X), f_o)$ where $f_o(e^{i\theta}) = x_o$ for all θ. ΩX is given the compact-open topology. This topology is given by taking as a base for the open sets all finite intersections of sets $M_{K,U} = \{f \mid f(K) \subset U\}$ for K = compact, U = open. If $f : X \to Y$ then $\Omega(f)$ is given by composition, i.e. $\Omega(f)(g) = f \cdot g$.

Define a functor S : BTop $\to$ BTop as follows:
$(\forall (Y, y_o) \in |\mathrm{BTop}|)$ $SY = Y \wedge S' = Y \times S'/Y \vee S'$ where $Y \vee S' = Y \times * \cup y_o \times S'$. $Y \vee S'$ becomes the base point. The topology is that induced from Y and S'. If $g : (Y, y) \to (Z, z_o)$ then
$S(g)(|y, e^{i\theta}|) = g \wedge 1_s, \ (|y, e^{i\theta}|) = |g(y), e^{i\theta}|$ where $|y, e^{i\theta}|$ is the equivalence class of $(y, e^{i\theta})$.

(1.5.9) Lemma $S \dashv \Omega$.

Proof: We have to show that for all $X,Y \in |\underline{BTop}|$ there exists a natural equivalence $\eta : \underline{BTop}(SY,X) \to \underline{BTop}(Y,\Omega X)$. For every $f \in \underline{BTop}(SY,X)$ define $\eta(f) : Y \to \Omega X = \underline{BTop}(S^1,X)$ by $\eta(f)(y)(e^{i\theta}) = f(|y,e^{i\theta}|)$.

(1.5.10) Lemma: (Hilton-Wiley [6]).

Given functions $\bar{g} : A \to Y^X$, $g : X \times A \to Y$ related by $\bar{g}(a)(x) = f(x,a)$ then g = continuous $\Rightarrow \bar{g}$ is continuous. If $\bar{g}$ is continuous and X is Hausdorff and locally compact, then g is continuous.

Proof: Let g be continuous. To show that $\bar{g}$ is continuous we take an element $a \in \bar{g}^{-1}(M_{K,U})$ and show that there exists a neighbourhood of a contained in $\bar{g}^{-1}(M_{K,U})$. Since $a \in \bar{g}^{-1}(M_{K,U})$, $\bar{g}(a)(K) \subset U$. Thus for all $x \in K$, $\bar{g}(a)(x) = g(x,a) \in U$. Since g is continuous, $g^{-1}(U)$ is open. Thus there exist open sets $N(x)$ and $N_x(a)$ around x and a respectively such that $g(N(x) \times N_x(a)) \subseteq U$. Form the open covering $\{N(x)/x \in K\}$ of K, since K is compact we can select a finite subcovering $N(x_1), \ldots, N(x_k)$ and let $N(a) = \bigcap_{i=1}^{k} N_{x_i}(a)$. Then $g(K \times N(a)) \subseteq U$ so that $N(a) \subseteq \bar{g}^{-1}(M_{K,U})$ and $\bar{g}$ is thus continuous.

Conversely, suppose $\bar{g}$ is continuous and X is locally compact and Hausdorff. Let U be open in Y and $g(x,a) \in U$. Since $\bar{g}(a) \in Y^X$ is continuous, $\bar{g}(a)^{-1}(U)$ is open. But

$(x) \in \bar{g}(a)^{-1}(\overset{\circ}{U})$, so $\exists$ a compact neighbourhood $V(x)$ with $\bar{g}(a)(V) \subseteq U$, i.e. $g(V(x) \times a) \subseteq U$. Then $\bar{g}(a) \in M_{V(x),U}$ so that, by continuity of $\bar{g}$ there is a neighbourhood $U(a)$ of a such that $\bar{g}(U(a)) \subseteq M_{V(x),U}$. But then $g(V(x) \times U(a)) \subseteq U$ and g is continuous. //

Since $Y \times S' \xrightarrow{\pi} Y \times S'/Y \vee S' = \overset{\circ}{S}Y \xrightarrow{f} X$ and $Y \xrightarrow{\eta(f)} X^{S'}$ with $\eta(f)(y)(e^{i\theta}) = (f \cdot \pi)(y, e^{i\theta})$ and π = quotient map, S' = compact and Hausdorff, the lemma implies that $\eta(f)$ is continuous <=> $f \cdot \pi$ is continuous. But $f \cdot \pi$ is continuous iff f is continuous as SY has the quotient topology. Thus $\eta(f)$ is continuous <=> f is continuous.

Let $\eta(f_1) = \eta(f_2) : f_1, f_2 \in$ BTop (SY,X). Then $\eta(f_1)(y)(e^{i\theta}) = \eta(f_2)(y)(e^{i\theta})$ for every $y \in Y$ and $e^{i\theta} \in S'$. Thus by definition of η, $f_1(|y, e^{i\theta}|) = f_2(|y, e^{i\theta}|)$, so $f_1 = f_2$ and η is mono.

Let $g \in$ BTop $(Y, \Omega X)$; define $f : SY \to X$ by $f(|y, e^{i\theta}|) = g(y)(e^{i\theta})$. Lemma (1.5.10) implies that f is continuous. Also $\eta(f)(y)(e^{i\theta}) = f(|y, e^{i\theta}|) = g(y)(e^{i\theta})$. So $\eta(f) = g$ and η is an epimorphism.

To show that η is natural we take $\alpha \in$ BTop(Y',Y), $\beta \in$ BTop(X,X'); $f \in$ BTop(SY,X) and show that

$$\eta(\beta \cdot g \cdot S\alpha) = \Omega\beta \cdot \eta(g) \cdot \alpha$$

$$\begin{aligned} \text{But } \eta(\beta \cdot g \cdot S\alpha)(y)(e^{i\theta}) &= (\beta \cdot g \cdot S\alpha)(|y,e^{i\theta}|) \\ &= \beta \cdot g\,(|\alpha y,e^{i\theta}|) \\ &= \Omega\beta(g(|\alpha y,e^{i\theta}|)) \\ &= \Omega\beta \cdot \eta(g)(\alpha y)(e^{i\theta}) \\ &= (\Omega\beta \cdot \eta(g) \cdot \alpha)(y)(e^{i\theta}). \quad // \end{aligned}$$

The unit of the adjunction is given by $\varepsilon_Y = \eta(1_{SY})$ for every $Y \in |\underline{\mathrm{BTop}}|$. Then $\varepsilon_Y(y)(e^{i\theta}) = \eta(1_{SY})(y)(e^{i\theta}) = |y,e^{i\theta}|$.
Let $\delta : S\Omega \to 1$ be the counit of the adjunction.
Then for every $Y \in |\underline{\mathrm{BTop}}|$

$$\delta_Y = \eta^{-1}(1_{\Omega Y}) \text{ and so}$$

$$\eta(\delta_Y) = \eta\,\eta^{-1}(1_{\Omega Y}) = 1_{\Omega Y}.$$

So $\eta(\delta_Y)(g)(e^{i\theta}) = g(e^{i\theta})$ for $g \in \Omega Y$.
But by definition of η we have $\eta(\delta_Y)(g)(e^{i\theta}) = \delta_Y(|g,e^{i\theta}|)$.
Thus $\delta_Y(|g,e^{i\theta}|) = g(e^{i\theta})$.
The cotriple defined by Ω and S is (C,b,p) where $C = S\Omega$, $k = \delta$, and $p = S(\varepsilon\Omega)$. Thus

$$\begin{aligned} k_Y(|g,e^{i\theta}|) &= g(e^{i\theta}) \text{ and} \\ p_Y(|g,e^{i\theta}|) &= S(\varepsilon\Omega)_Y\,(|g,e^{i\theta}|) \\ &= (\varepsilon_{\Omega Y} \wedge 1_{S'})(|g,e^{i\theta}|) \\ &= (|\varepsilon_{\Omega Y}(g),e^{i\theta}|) \\ &= |\,|g,-|,\ e^{i\theta}|. \end{aligned}$$

The semi-simplicial functor T_Y induced by (C,k,p) for each Y is now defined as follows:

$$\begin{aligned} T_Y(\Delta_n) &= C^{n+1}(Y) = (S\Omega)^{n+1}(Y) \\ &= \underline{\mathrm{BTop}}\,(S', (S\Omega)^n(Y)) \wedge S' \end{aligned}$$

Some of the face and degeneracy maps are

$$s_o^o = C^o p C^o{}_Y : CY \to C^2Y$$

$$= p_Y$$

i.e. given $g : S' \to Y$, $e^{i\theta} \in S'$ Then

$$s_o^o(|g, e^{i\theta}|) = |\,|g,-|, e^{i\theta}|$$

$$d_o^o = C^o k C_Y : C^2Y \to CY$$

$$= k_{CY} = k_{S\Omega Y}$$

i.e. given $f : S' \to CY$, $e^{i\theta} \in S'$ Then

$$d_o^o(|f, e^{i\theta}|) = f(e^{i\theta})$$

$$d_1^o = C^1 k C_Y^o : C^2Y \to CY$$

$$= Ck_Y = S\Omega(k_Y) = \Omega(k_Y) \wedge 1_S$$

i.e. $d_1^o(|S, e^{i\theta}|) = \Omega(k_Y)(f) \wedge 1_{S'}(e^{i\theta})$

$$= |k_Y(f), e^{i\theta}|$$

CHAPTER II

CW-COMPLEXES

Most of the material in this section has been developed extensively in [16], which will be the major reference for this chapter.

§1. Colimits.

(2.1.1) Let $F : \underline{X} \to \underline{A}$ be a given diagram (a covariant functor from a small category $\underline{X}$) and let $I(\underline{A},F)$ be the category defined as follows:

the objects of $I(\underline{A},F)$ are the sets of morphisms $\{FX \xrightarrow{i(X)} A\}$, where X varies in $\underline{X}$ and $A \in |\underline{A}|$ is fixed for each set, such that $\forall f \in \underline{X}(X,X')$ $i(X')F(f) = i(X)$.

A morphism $\bar{u} : \{FX \xrightarrow{i(X)} A\} \to \{F(X) \xrightarrow{i'(X)} A'\}$ of $I(\underline{A},F)$ is given by a morphism $u \in \underline{A}(A,A')$ such that $(\forall X \in |\underline{X}|) ui(X) = i'(X)$.

(2.1.2) **Definition**: We define a colimit of F in $\underline{A}$ (denoted colim F) to be an initial object of $I(\underline{A},F)$. Provided they exist, colimits are unique up to isomorphism.

(2.1.3) Given a set $\{A_j\}_{j \in J}$ of objects of $\underline{A}$ we form the discrete category $\underline{X}$ with objects A_j, $j \in J$, and define the diagram $I : \underline{X} \to \underline{A}$ which takes A_j to A_j, for every $j \in J$. Then if I has a colimit, we say that the set $\{A_j\}_{j \in J}$ has a coproduct and write colim $I = \{A_j \to \coprod_{j \in J} A_j\}$. We denote the coproduct of the set $\{A_j\}_{j\in J}$ by $\coprod_{j\in J} A_j$.

(2.1.4) Lemma: Let $\underline{X}$ be a small category and let

$$\underline{X} \xrightarrow{F} \underline{A} \underset{T}{\overset{S}{\rightleftarrows}} \underline{B}$$

be given functors with $S \dashv T$. If colim F exists, then $S(\text{colim } F) \cong \text{colim}(SF)$.

Proof: If $\{F(X) \xrightarrow{i(X)} A\}$ is an initial object of $I(\underline{A},F)$ then we have to show that $\{SF(X) \xrightarrow{Si(X)} S(A)\}$ is an initial object of $I(\underline{B},SF)$.

Since $S \dashv T$, there exists a natural isomorphism $\theta : \underline{B}(SA,B) \to \underline{A}(A,TB)$ for each $A \in |\underline{A}|$ and $B \in |\underline{B}|$. Given an arbitrary object $\{SF(X) \xrightarrow{j(X)} B\}$ of $I(\underline{B},SF)$, since $SF(X) \in |\underline{B}|$ we can form the object $\{F(X) \xrightarrow{\theta j(X)} TB\}$ of $I(\underline{A},F)$. Since $\{F(X) \xrightarrow{i(X)} A\}$ is an initial object of $I(\underline{A},F)$, there exists a unique morphism

$$\alpha : A \to T(B)$$

such that $(\forall X \in |\underline{X}|)$

$$\begin{array}{ccc} & & A \\ & \overset{i(X)}{\nearrow} & \\ F(X) & & \downarrow \alpha \\ & \underset{\theta j(X)}{\searrow} & \\ & & T(B) \end{array}$$

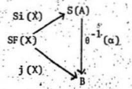
commutes. By naturality,

$$\begin{array}{ccc} \underline{B}(SF(X),B) & \xrightarrow[\cong]{\theta} & \underline{A}(F(X),T(B)) \\ \uparrow {\scriptstyle \_\circ Si(X)} & & \uparrow {\scriptstyle \_\circ i(x)} \\ \underline{B}(S(A),B) & \xrightarrow[\cong]{\theta} & \underline{A}(A,TB) \end{array}$$

commutes, and so

$$\theta j(X) = \alpha i(X) = \theta[\theta^{-1}\alpha Si(X)].$$

Thus for every $X \in |\underline{X}|$, since θ is an isomorphism



commutes and $\theta^{-1}(\alpha)$ is the only morphism making the diagram commutative. //

§2. <u>Colimits in |Top|</u>

Let $\{X_\alpha \mid \alpha \in J\}$ be a set of topological spaces. Let X be a set and, for each $\alpha \in J$, let

$$f_\alpha : X_\alpha \to X$$

be a given function. We define in X the following three topologies:

τ_1 : for every topological space Z and function $g : X \to Z$, g is continuous $\Leftrightarrow$ $(\forall \alpha \in J) g \cdot f_\alpha : X_\alpha \to Z$ is continuous;

τ_2 : the finest topology for which all f_α's are continuous;

τ_3 : $K \subset X$ is closed (open) $\Leftrightarrow$ $(\forall \alpha \in J) f_\alpha^{-1}(K)$ is closed (open) in X_α.

(2.2.1) <u>Theorem</u>: The topologies τ_1, τ_2, and τ_3 are equivalent.

<u>Proof:</u> If X has the topology τ_1 all f_α's are continuous, since the identity function of $X(\tau_1)$ unto itself is continuous.

So τ_2 is finer than τ_1. Conversely, if τ is any topology on X for which all functions f_α are continuous, form the commutative diagram

$$\begin{array}{ccc} X(\tau_1) & \xrightarrow{1} & X(\tau) \\ \uparrow f_\alpha & \nearrow f_\alpha & \\ X_\alpha & & \end{array}$$

If f_α = continuous, then $1 \cdot f_\alpha$ = continuous and 1 = continuous. Thus τ_1 is finer than τ. So τ_1 and τ_2 coincide.

By definition of continuity, τ_2 is finer than τ_3. Conversely, given $K \subset X(\tau_2)$ closed, $(\forall \alpha \in J) f_\alpha^{-1}(K)$ is closed in X_α. Then K is closed in $X(\tau_3)$ and so, τ_3 is finer than τ_2. //

Any of the equivalent topologies defined above is called the final topology of X with respect to the set $\{f_\alpha \mid \alpha \in J\}$.

(2.2.2) Theorem [6]: Given a diagram $F \in |\underline{Top}^{\underline{X}}|$, a space A_F defined by colim $F = \{F(X) \xrightarrow{i(X)} A_F\}$ has the final topology with respect to the set $\{i(X) \mid X \in |\underline{X}|\}$.

Proof: Let $g : A_F \to Z$, $Z \in |\underline{Top}|$, be a function such that $(\forall X \in |\underline{X}|) g \cdot i(X)$ is a map. We show that g is a map. $\{F(X) \xrightarrow{g \cdot i(X)} Z\}$ is an object of $I(\underline{Top}, F)$ and $g \cdot i(X) \in \{F(X) \xrightarrow{g \cdot i(X)} Z\}$ Since colim F is an initial object of $I(\underline{Top}, F)$, there exists a unique map $h : A_F \to Z$ such that

$$(\forall X \in |\underline{X}|) h \cdot i(X) = g \cdot i(X) .$$

As functions of sets g and h coincide; thus g is continuous. //

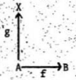
§3. Adjunction Spaces and CW-Complexes

Recall from general topology that

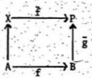
(2.3.1) Definition: A topological space is compact <=> each open cover has a finite subcover.

(2.3.2) Definition: A topological space X is a Hausdorff space <=> $(\forall x, y \in X) x \neq y \Rightarrow$ there exist disjoint, non-empty neighbourhoods of x and y.

A colimit of a diagram

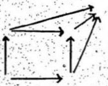


in |Top| is a commutative square called a pushout



(2.3.3)

satisfying the condition: for every $Q \in |\text{Top}|$ and for all $f' \in \text{Top}(X,Q)$, and all $g' \in \text{Top}(B,Q)$ such that $f'g = g'f$, there exists a unique map $u : P \to Q$ making the following diagram commutative:



(2.3.4) Theorem: Let

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & P \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ A & \xrightarrow[f]{} & B \end{array}$$

be a pushout in Top, with A a closed subspace of X and i the inclusion map. Then $\bar{i}$ is (1-1) closed and the restriction of $\bar{f}$ to $X \setminus A$ is (1-1) open.

Proof: Consider the coproduct $B \amalg X$ and let E be the equivalence relation defined by the relation

$xRb \iff x = b$ or $x = i(a)$ and $b = f(a)$ for some $a \in A$.

Let $F : B \amalg X \to B \amalg X/E$ (denoted by $B \amalg_f X$) be the quotient function and give to $B \amalg_f X$ the final topology with respect to F. Let $I(B) : B \to B \amalg X$ and $i(X) : X \to B \amalg X$ be the inclusions.

Claim:

$$\begin{array}{ccc} X & \xrightarrow{F \cdot i(X)} & B \amalg_f X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle F \cdot i(B)} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pushout.

For any $Q \in |\text{Top}|$ and for any $f' \in \text{Top}(X,Q)$ and $g' \in \text{Top}(B,Q)$ such that $f' \cdot i = g' \cdot f$ we construct $h : B \amalg_f X \to Q$ to be the function defined as follows;

$(\forall x \in X) h(\bar{x}) = f'(x)$ and $(\forall b \in B) h(\bar{b}) = g'(b)$ where $\bar{x}$ and $\bar{b}$ are the equivalence classes of x and b in $B \amalg_f X$. If $x \sim b$ then there exists an $a \in A$ such that $f(a) = b$ and

$i(a) = x$. Therefore $h(\bar{b}) = g'(b) = g' \cdot f(a) = f' \cdot i(a) = f'(x) = h(\bar{x})$. Thus h is well defined. $B \amalg_f X$ has the final topology with respect to F and $B \amalg X$ has the final topology with respect to $i(X)$ and $i(B)$ by (2.2.2). Therefore, since $h \cdot F \cdot i(X) = f'$ is continuous and $h \cdot F \cdot i(B) = g'$ is continuous, h itself is continuous. h is also obviously unique.

$B \amalg_f X$ is a pushout implies that $P \cong B \amalg_f X$ and by identifying $B \amalg_f X$ with P, $\bar{i}$ is (1-1) and $\bar{f} \mid X \setminus A$ is (1-1).

Let U be an open subset of $X \setminus A$ and set $V = \bar{f}(U)$; $\bar{i}^{-1}(V) = \phi$ and $\bar{f}^{-1}(V) = U$. Since P is a colimit it has the final topology with respect to $\bar{f}$ and $\bar{i}$. (2.2.1) => V is open in P.

If $W \subset B$ is closed and $\bar{i}(W) = Y$, then $\bar{i}^{-1}(Y) = W$ is closed in B and $\bar{f}^{-1}(Y) = f^{-1}(W)$ is closed in A and hence in X. Thus Y is closed in P. //

The space $B \amalg_f X$ is the <u>space obtained by the adjunction of X to B via the map f</u>. The map f is called an <u>attaching map</u> (or <u>adjunction map</u>) of X.

The first step in the construction of a CW-complex is to construct a diagram in <u>Top</u> of the following type:

$$K^0 \xrightarrow{\bar{i}_0} K^1 \xrightarrow{\bar{i}_1} K^2 \xrightarrow{\bar{i}_2} \cdots \longrightarrow K^{n-1} \xrightarrow{\bar{i}_{n-1}} K^n \xrightarrow{\bar{i}_n} \cdots \tag{2.3.5}$$

where the morphisms $\bar{i}_n$ are (1-1) closed maps and K^0 is a discrete space. Assume that K^{n-1} has been constructed. We construct K^n as follows:

Let Λ_n be a given set and to each $\lambda \in \Lambda_n$ we associate a sphere S_λ^{n-1} and a map $f_\lambda^{n-1} : S_\lambda^{n-1} \to K^{n-1}$. These maps now define a map $f^{n-1} : \coprod S^{n-1} \to K^{n-1}$.

But S_λ^{n-1} is a closed subspace of the cone

$$CS_\lambda^{n-1} = (S_\lambda^{n-1} \times I)/(S_\lambda^{n-1} \times 0) \text{ where } I = \text{unit interval } [0,1].$$

Then $\coprod_\lambda S_\lambda^{n-1}$ is a closed subspace of $\coprod_\lambda CS_\lambda^{n-1}$.

We define K^n to be the space obtained by the adjunction of $\coprod_\lambda CS_\lambda^{n-1}$ to K^{n-1} via f^{n-1}. So we have the following diagram

$$\begin{array}{ccc} \coprod_\lambda CS_\lambda^{n-1} & \xrightarrow{u=\bar{f}^{n-1}} & K^{n-1} \amalg_{f^{n-1}} (\coprod_\lambda CS^{n-1}) = K^n \\ \uparrow i & & \uparrow \bar{i}_{n-1} \\ \coprod_\lambda S_\lambda^{n-1} & \xrightarrow{f^{n-1}} & K^{n-1} \end{array}$$

where according to (2.3.2) $\bar{i}_{n-1}$ is (1-1) closed.

(2.3.6) Definition: A CW-complex with n-skeleton K^n, $n = 0,1,\ldots,$ is a space K-unique up to homeomorphism - defined by a colimit of a diagram of the type (2.3.5). If there is an integer $n_o \geq 0$ such that $(\forall n \geq n_o) K^n = K^{n_o}$, we say that K is of finite dimension n_o. If K is finite dimensional and all the sets Λ_n used in the construction of (2.3.3) are finite, then K is a finite CW-complex.

A couple of properties which can be deduced from this definition are the following:

(2.3.7) Theorem: Any CW-complex is a normal space. [16,I.3.6.].

(2.3.8) Theorem: Every point of a CW-complex is closed. [16,I.3.7].

As a consequence, any CW-complex is a Hausdorff space.

(2.3.9) Let K be a CW-complex given by a colimit of the diagram (2.3.5). Set $\bar{f}^{n-1}_\lambda = \bar{f}^{n-1} \mid CS^{n-1}_\lambda : CS^{n-1}_\lambda \to K^n$. Then $\bar{f}^{n-1}_\lambda(CS^{n-1}_\lambda \setminus S^{n-1}_\lambda) = \sigma^n_\lambda$ is an open subset of K^n by (2.3.4) and $\bar{f}^{n-1}_\lambda(CS^{n-1}_\lambda) = \bar{\sigma}^n_\lambda$ is closed in K^n and hence in K, as a compact subspace of a Hausdorff space. We call $\bar{\sigma}^n_\lambda$ a closed n-cell of K.

(2.3.10) Definition: Let $A = \bigcup_{n\geq 0} A^n$ and $X = \bigcup_{n\geq 0} X^n$ be CW-complexes in Top. A is a sub-CW-complex of X $\iff$ $(\forall n \geq 0)$ A^n is a closed subset of X^n and $X^n \cap A = A^n$. (Notice that the colimit of (2.3.6) coincides with the union with the weak topology).

(2.3.11) Remark 1: Let X^n be the n-skeleton of X. We regard X^n as a CW-complex by taking it as a colimit of the diagram

$$X^o \to X' \to \ldots\ldots \to X^n \overset{=}{\to} X^n \overset{=}{\to} \ldots\ldots ;$$

clearly X^n is a sub-CW-complex of X.

(2.3.12) Remark 2: Let X and Y be given CW-complexes. Then $X \amalg Y$ is a CW-complex because

$$\operatorname{colim}_X (\operatorname{colim}_Y F) \cong \operatorname{colim}_Y (\operatorname{colim}_X F). \quad [16].$$

Furthermore, X and Y are sub-CW-complexes of $X \amalg Y$.

4. Examples

Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ be the field of real, complex or quaternionic numbers, respectively. Since $\mathbb{H}$ is non-commutative, we wll consider only multiplications on the right.

Define an equivalence relation $\sim$ on $K^{n+1} \setminus \{0\}$ as follows:

$(\forall z, z' \in K^{n+1} \setminus \{0\}) z \sim z' \iff (\exists \lambda \in K \setminus \{0\}) z = \lambda z'$

thus if $z = (z_o, \ldots, z_n)$ and $z' = (z'_o, \ldots, z'_n)$ then

$(z_o, \ldots, z_n) \sim (z'_o, \ldots, z'_n) \iff (\exists \lambda \in K \setminus \{0\}) z_i = \lambda z'_i ;\ i = 0, \ldots, n.$

(2.4.1) Definition: The projective n-space, $P_n(K)$, is the quotient space $K^{n+1} \setminus \{0\}/\sim$ ([19] p.67). It is thus the space of all K-lines through 0 in K^{n+1}, since the equivalence relation sends all points on the same K-line to one point in $P_n(K)$.

Let $q : K^{n+1} \setminus \{0\} \to P_n(K)$ be the quotient map defined by $q(z_o, \ldots, z_n) = [z_o, \ldots, z_n] \in P_n(K)$ and give to $P_n(K)$ the induced quotient topology from $K^{n+1} \setminus \{0\}$.

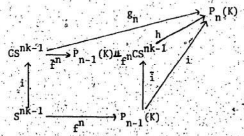
To show that $P_n(K)$ is a CW-complex we show that $P_n(K)$ can be obtained from $P_{n-1}(K)$, for each n, by adjunction of n-cells. Let $k = \dim_{\mathbb{R}} K$ be the dimension of K as a vector space over $\mathbb{R}$. Define a map $f^n : S^{nk-1} \to P_{n-1}(K)$ by $f^n(z_o, \ldots, z_{n-1}) = [z_o, \ldots, z_{n-1}]$ for $z_i \in K$. f^n is just the restriction of q and defines $P_{n-1}(K)$ as a quotient space of S^{nk-1} by identification of antipodal points. If $x', y' \in S^{nk-1}$ are such that $f^n(x') = f^n(y')$, then $x' = \lambda y'$ for some $\lambda \in K \setminus \{0\}$. But $|x'| = |y'| = 1 \Rightarrow |\lambda| = 1$ and thus $\lambda \in S^{k-1}$. Thus the inverse image by f^n of a point in $P_{n-1}(K)$ is homeomorphic to the sphere S^{k-1}.

The inclusions $K^o \subset K^1 \subset K^2 \subset \ldots \subset K^{n-1} \subset K^n \ldots$ induce inclusions $P_o(K) \hookrightarrow P_1(K) \hookrightarrow \ldots \hookrightarrow P_{n-1}(K) \hookrightarrow P_n(K) \hookrightarrow \ldots$. Define $g_n : CS^{nk-1} \to P_n(K)$ by

$$g_n(z_o, z_1, \ldots, z_{n-1}) = [z_o, z_1, \ldots, z_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2}]$$

If $(z_o, \ldots, z_{n-1}) \in S^{nk-1}$ then $g_n(z_o, \ldots, z_{n-1}) = [z_o, \ldots, z_{n-1}, 0] \in P_{n-1}(K)$.

Form the following pushout diagram as in (2.3.4):



There exists a unique map h making the diagram commute. We show that h is an open and bijective map. First we show that $P_n(K) \setminus P_{n-1}(K)$ is homeomorphic to $CS^{nk-1} \setminus S^{nk-1}$. For all $z_i \in K$ denote by $\bar{z}_i$ its conjugate. Define $\bar{g}_n : P_n(K) \setminus P_{n-1}(K) \to CS^{nk-1} \setminus S^{nk-1}$ by

$$\bar{g}_n[z_o, z_1, \ldots, z_n] = \left(\frac{z_o \bar{z}_n}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}}, \frac{z_1 \bar{z}_n}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}}, \ldots, \frac{z_{n-1} \bar{z}_n}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}} \right)$$

$$\text{Now } |\bar{g}_n[z_o, \ldots, z_n]| = \sqrt{\left| \frac{z_o \bar{z}_n}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}} \right|^2 + \cdots + \left| \frac{z_{n-1} \bar{z}_n}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}} \right|^2}$$

$$= \sqrt{\left(\frac{|z_o| \, |\bar{z}_n|}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}} \right)^2 + \cdots + \left(\frac{|z_{n-1}| \, |\bar{z}_n|}{|z_n| \sqrt{\sum_{i=0}^{n} |z_i|^2}} \right)^2}$$

$$g = \sqrt{\left(\frac{|z_o|}{\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right)^2 + \cdots + \left(\frac{|z_{n-1}|}{\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right)^2}$$

$$= \sqrt{\frac{\sum_{i=0}^{n-1}|z_i|^2}{\sum_{i=0}^{n}|z_i|^2}} = \sqrt{\frac{\sum_{i=0}^{n-1}|z_i|^2}{|z_n|^2 + \sum_{i=0}^{n-1}|z_i|^2}}$$

Thus since $z_n \neq 0$, $|\bar{g}_n[z_o,\ldots,z_n]| < 1$ and hence $\bar{g}_n[z_o,\ldots,z_n] \in CS^{nk-1} \setminus S^{nk-1}$.

The factor $\frac{\bar{z}_n}{|z_n|}$ serves to insure that $\bar{g}_n$ is one to one.

$$\bar{g}_n \cdot g_n(z_o,\ldots,z_{n-1}) = \bar{g}_n[z_o,\ldots,z_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}\,]$$

$$= \left(\frac{z_o\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}}{\left|\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}\right|\sqrt{\sum_{i=0}^{n}|z_i|^2}}, \ldots, \frac{z_{n-1}\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}}{\left|\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}\right|\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right)$$

(Here $z_n = \sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}$). Taking a general term we get

$$\frac{z_j\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}}{\sqrt{1 - \sum_{i=0}^{n-1}|z_i|^2}\sqrt{\sum_{i=0}^{n}|z_i|^2}} = \frac{z_j}{\sqrt{\sum_{i=0}^{n-1}|z_i|^2 + |z_n|^2}} =$$

$$= \frac{z_j}{\sqrt{\sum_{i=0}^{n-1}|z_i|^2 + \left(1 - \sum_{i=0}^{n-1}|z_i|^2\right)}} = z_j$$

Thus $\bar{g}_n \cdot g_n = 1$.

Now $g_n \cdot \bar{g}_n[z_o,\ldots,z_n] = g_n\left(\dfrac{z_o\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}},\ldots,\dfrac{z_{n-1}\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right)$

$$= \left(\frac{z_o\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}},\ldots,\frac{z_{n-1}\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}},\sqrt{1-\sum_{j=0}^{n-1}\left|\frac{z_j\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right|^2}\right)$$

But $\sqrt{1-\sum_{j=0}^{n-1}\left|\dfrac{z_j\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}}\right|^2} = \sqrt{1-\sum_{j=0}^{n-1}\dfrac{|z_j|^2\,|z_n|^2}{|z_n|^2\sum_{i=0}^{n}|z_i|^2}}$

$$= \sqrt{1-\frac{\sum_{j=0}^{n-1}|z_j|^2}{\sum_{i=0}^{n}|z_i|^2}} = \sqrt{\frac{\sum_{i=0}^{n}|z_i|^2-\sum_{j=0}^{n-1}|z_j|^2}{\sum_{i=0}^{n}|z_i|^2}}$$

$$= \sqrt{\frac{|z_n|^2}{\sum_{i=0}^{n}|z_i|^2}} = \frac{|z_n|}{|z_n|}\,\frac{|z_n|}{\sqrt{\sum_{i=0}^{n}|z_i|^2}}$$

$$= \frac{|z_n|^2}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}} = \frac{z_n\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}}$$ since

$z_n\bar{z}_n = |z_n|^2$.

Thus each term in the final expression has the same factor,

$\dfrac{\bar{z}_n}{|z_n|\sqrt{\sum_{i=0}^{n}|z_i|^2}}$, and therefore,

$g_n \cdot \bar{g}_n[z_o,\ldots,z_n] = [z_o,\ldots,z_n]$, that is to say

$g_n \cdot \bar{g}_n = 1.$

Thus we have indeed shown that $P_n(K) \setminus P_{n-1}(K)$ is isomorphic to $CS^{nk-1} \setminus S^{nk-1}$. This implies that h is a continuous bijection. But $P_n(K)$ and $P_{n-1}(K) \amalg_{f^n} CS^{nk-1}$ are compact spaces and $P_n(K)$ is Hausdorff, so h is an open map. We therefore have a CW-complex, $P_n(K)$, with one k-cell in each dimension n.

Notice that for $K = \mathbb{C}$ or $\mathbb{H}$, $P_n(K)$ is simply-connected since it has no 1-cell.

(2.4.2) Example 2: S^n is a CW-complex for $n = 0,1,\ldots$.

Starting from the zero-sphere S^o, we construct the following diagram:

$$S^o \xrightarrow{i_o} S^1 \xrightarrow{i} \cdots \xrightarrow{i_{n-1}} S^{n-1} \xrightarrow{i} S^n \longrightarrow \cdots$$

where S^n is constructed from S^{n-1} as a pushout of the following diagram:

$$\begin{array}{ccc} CS^{n-1} \amalg CS^{n-1} & & \\ \uparrow{\scriptstyle i} & & \\ S^{n-1} \amalg S^{n-1} & \xrightarrow{1 \amalg 1} & S^{n-1} \end{array}$$

By (2.3.4) we get

$$\begin{array}{ccc} CS^{n-1} \amalg CS^{n-1} & \longrightarrow & S^{n-1} \amalg_1 (CS^{n-1} \amalg CS^{n-1}) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ S^{n-1} \amalg S^{n-1} & \longrightarrow & S^{n-1} \end{array}$$

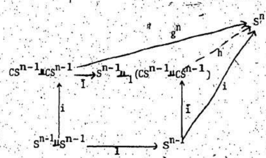
Define maps $g_1^n, g_2^n : CS^{n-1} \to S^n$ as follows:

$$g_1^n(x_o,x_1,\ldots,x_{n-1}) = (x_o,x_1,\ldots,x_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} x_i^2})$$

and

$$g_2^n(x_o,x_1,\dots,x_{n-1}) = (x_o,x_1,\dots,x_{n-1}, -\sqrt{1-\sum_{i=0}^{n-1} x_i^2})$$

This gives us a map $g^n = g_1^n \amalg g_2^n : CS^{n-1} \setminus CS^{n-1} \to S^n$ which makes the following diagram commutative.



There exists a unique map h making the triangles commutative. But $g^n \mid CS^{n-1} \amalg CS^{n-1} \setminus S^{n-1} \amalg S^{n-1}$ is a bijection. Thus h is bijective. But S^n is compact and Hausdorff. Thus h is a homeomorphism. Geometrically speaking, $S^{n-1} \amalg_1 (CS^{n-1} \amalg CS^{n-1})$ is just the attaching of the north and south hemispheres to the equator S^{n-1}. By (2.3.11) S^n is a CW-complex for each n.

CHAPTER III

K-SPACES

§1. Definitions and Examples

(3.1.1) Definition: A Hausdorff space X is a k-space or a compactly generated space iff it has the final topology with respect to the set of all inclusions, $\{i_K : K \to X \mid K = \text{compact subspace of } X\}$.

(3.1.2) Lemma: Let X be a k-space. $A \subset X$ is closed $\Leftrightarrow$ $A \cap K$ is closed for every compact $K \subset X$.

Proof: This statement is equivalent to saying that A is closed $\Leftrightarrow$ for all compact $K \subset X$, $i_K^{-1}(A)$ is closed in K. Thus it is one of the equivalent definitions of the final topology - see (2.2.1) - and is implied by the definition of a k-space. //

(3.1.3) Definition: A point x is a limit point of a subset A of a space X $\Leftrightarrow$ $(\forall U = \text{neighbourhoods of } x)(U \setminus \{x\} \cap A \neq \emptyset)$.

(3.1.4) Remark: A subset A of a topological space X is closed $\Leftrightarrow$ it contains the set of all its limit points.

We will henceforth denote by Top_K the category of k-spaces and the continuous functions between them. Recall that all k-spaces are Hausdorff.

(3.1.5) Lemma: Given any subset M of a Hausdorff space X, if for each limit point x of M, there exists a compact set K' in X such that x is a limit point of $M \cap K'$, then $X \in |\text{Top}_K|$.

Proof: Let $z \in |Top|$ and $g : X \to z$ be any map such that $g \cdot i_K$ is continuous for every compact $K \subset X$. We have to show that g is continuous. Thus, let $A \subset z$ be closed. Then $i_K^{-1} \cdot g^{-1}(A)$ is closed in K. Let x be a limit point of $g^{-1}(A)$. Then x is a limit point of $g^{-1}(A) \cap K' = i_{K'}^{-1}(g^{-1}(A))$ for some compact $K' \subset X$. Since $i_{K'}^{-1}(g^{-1}(A))$ is closed, $x \in i_{K'}^{-1}(g^{-1}(A)) = g^{-1}(A) \cap K'$. This implies that $x \in g^{-1}(A)$. That is, $g^{-1}(A)$ is closed, since it contains all its limit points. Thus g is continuous and $X \in |Top_K|$. //

The following examples and counterexamples were developed in [18] by Steenrod as a result of Kelly's work in [8].

(3.1.6) Lemma: The category $|Top_K|$ includes all locally compact spaces and all spaces satisfying the first axiom of countability.

Proof: (i) Let X be a locally compact space and M any subset of X. Let $x \in X$ be a limit point of M. By local compactness, x has a neighbourhood, N, whose closure, $\bar{N}$, is compact. Also $x \in \bar{N}$, so x is a limit point of $\bar{N}$ and of M, and thus of $\bar{N} \cap M$ also. X thus satisfies the conditions of (3.1.5) and is thus a k-space.

(ii) Again let X be first countable with x a limit point of $M \subset X$. There is then a sequence in $M \setminus \{x\}$ which converges to x. ([8], p. 73). Let K' be the set consisting of x and this sequence. Then any sequence in K' has a limit point and thus K' is countable compact [Kelly, Chapter 5, Problem E]. K' is countable then implies that it is compact. Thus x is a limit point of K' and of M. Again x is a limit point of

$M \cap K'$ and $X \in |\underline{Top_K}|$. //

(3.1.7) Examples: Let Ω' be the set of all ordinals less than or equal to the first uncountable ordinal Ω, and let X be $\Omega' \setminus \{\Omega\}$. Since X satisfies the first axiom of countability it is a k-space.

Let Y be the subspace of Ω' obtained by deleting all limit ordinals except Ω. Since Y is Hausdorff, the compact subsets must be closed. If $B \subset Y$ is infinite, it must contain a sequence converging to one of the deleted ordinals. Thus B does not contain all its limit points and is not compact. Since the only compact subsets of Y are the finite sets, the set $Y \setminus \{\Omega\}$ meets each compact set in a closed set. But $Y \setminus \{\Omega\}$ is not closed in Y because it has Ω as a limit point. Thus (3.1.2) is not satisfied and so $Y \notin |\underline{Top_K}|$.

The above shows that there are some open subsets of k-spaces which are not themselves k-spaces. However, we have the following:

(3.1.8) Lemma: If $X \in |\underline{Top_K}|$, then every closed subset of X is in $\underline{Top_K}$.

Proof: Let $A \subset X$ be closed.
Let $z \in |\underline{Top}|$ and $g : A \to z$ be such that $g \cdot i_K$ is continuous for all compact $K \subset A$.
If K' is any compact subset of X then $A \cap K'$ is compact in A.
Given $M \subset z$ closed, then

$$(g \cdot i_{A \cap K'})^{-1}(M) = g^{-1}(M) \cap (A \cap K') = g^{-1}(M) \cap K' \text{ is closed}$$

in A.

A = closed $\Rightarrow g^{-1}(M) \cap K'$ is closed in X

(3.1.2) $\Rightarrow g^{-1}(M)$ is closed in X

$\Rightarrow g^{-1}(M)$ is closed in A

$\Rightarrow g$ = continuous

$\Rightarrow A \in |\underline{Top}_K|$. //

2. The functor K

(3.2.1) Define a functor $K : \underline{Top} \to \underline{Top}_K$ as follows:

- $(\forall Y \in |\underline{Top}|)KY$ is the space with the same points as Y and with the compactly generated topology.
- $(\forall f \in \underline{Top}(Y,X))K(f) = f$ as a set theoretical function.

(3.2.2) Lemma: Kf is continuous.

Proof: For this it is enough to show that Y and KY have the same compact subspaces. In fact, a compact subspace of Y will be compact in KY by the definition of the compactly generated topology.

Since the closed sets of Y are closed in KY the identity function $\sigma_Y : KY \to Y$ is continuous. This implies that if $A \subset KY$ is compact, $\sigma_Y(A) = A$ is a compact subset of Y.

Let C be a compact set in KY. Then C is compact in Y as above. Since f is continuous $f(C)$ is compact in X and hence in KX. We have the following commutative diagram:

$$\begin{array}{ccc} KY & \xrightarrow{K(f)} & KX \\ \uparrow i_C & & \uparrow i_{f(C)} \\ C & \xrightarrow{f} & f(C) \end{array}$$

$Kf \cdot i_C$ is thus continuous for all compact C in KY. Therefore Kf is continuous.

(3.2.4) Remark: Let $i : \underline{Top}_K \to \underline{Top}$ be the inclusion functor and let $F : \underline{X} \to \underline{Top}_K$ be a given diagram. Take

$$\text{colim } iF = \{iF(X) \xrightarrow{\theta(X)} A\} \text{ in Top with}$$

A = Hausdorff.

Form the set $\{KiF(X) = F(X) \xrightarrow{K\theta(X)} K(A)\}$. Since colim iF is an initial object of $I(\underline{Top}, iF)$, there exists a unique map $u : A \to iK(A)$ such that

$$\begin{array}{ccc} iF(X) & \xrightarrow{\theta(X)} & A \\ & \searrow^{iK\theta(X)} & \downarrow u \\ & & iK(A) \end{array}$$

commutes. By uniqueness $1 : A \to iK(A)$ is continuous. Hence, $A \cong iK(A)$; that is $A \in \underline{Top}_K$.

(3.2.5) Remark: Spheres and cones over spheres satisfy the first axiom of countability and thus by (3.1.6) are compactly generated spaces. Diagrams of the type (2.3.5) are thus in $\underline{Top}_K$. CW-complexes are colimits of these diagrams and are Hausdorff. By (3.2.4) they are thus in $\underline{Top}_K$.

(3.2.6) Theorem: $i \dashv K$.

Proof: We have to show that there exists a natural isomorphism $\phi : \underline{Top}(i(A), B) \to \underline{Top}_K(A, KB)$ for all $A \in |\underline{Top}_K|$ and $B \in |\underline{Top}|$. For all $f \in \underline{Top}(iA, B)$ we have the following diagram;

$$\begin{array}{ccc} A = iA & \xrightarrow{\;f\;} & B \\ \uparrow \sigma_{iA} = 1 & & \uparrow \sigma_B \\ A = KiA & \xrightarrow[K(f)]{} & KB \end{array}$$

We define $\phi(f) = K(f)$.

Since f and $\phi(f)$ coincide as sets, ϕ is a monomorphism.

Let $g \in \underline{Top_K}(A, KB)$. Then

$$\begin{array}{ccc} iA & \xrightarrow{\;i(g)\;} & iKB = KB \\ & {\scriptstyle g'} \searrow & \downarrow \sigma_B \\ & & B \end{array}$$

$g' = \sigma_B \cdot i(g)$ is continuous and $\phi(g') = \phi(\sigma_B \cdot i(g)) = g$.

Thus ϕ is unto.

ϕ is obviously natural because of the definitions of K and i. //

Given a diagram G in $\underline{Top_K}$, iG is a diagram in $\underline{Top}$. Since $\underline{Top}$ is complete, iG has a limit in $\underline{Top}$. But K is right adjoint to i, so

$$K(\lim iG) = \lim(KiG) = \lim G.$$

Thus G has a limit in $\underline{Top_K}$, and therefore $\underline{Top_K}$ is complete.

§3. Products

Recall from general topology that the product of Hausdorff spaces is Hausdorff under the usual cartesian topology. As regards $\underline{Top_K}$ we then have

(3.3.1) Definition: If X and Y are in $\underline{Top_K}$, their product $X \times Y$ in $\underline{Top_K}$ is defined to be $K(X \times_c Y)$ where '$\times_c$' denotes the product in $\underline{Top}$ with the usual cartesian topology.

Notice that $X \times Y$ in $\underline{Top_K}$ satisfies the usual commutative and associative laws since these are satisfied in $\underline{Top}$.

The next two results are given in Steenrod [18].

(3.3.2) Theorem: If X is locally compact and $Y \in |\underline{Top_K}|$, then $X \times_c Y$ is in $\underline{Top_K}$; that is to say, $X \times Y = X \times_c Y$.

Proof: We will use (3.1.2). Suppose that $A \subset X \times_c Y$ meets each compact set in a closed set and let (x_o, y_o) be a point in $X \times_c Y \setminus A$. We will show that (x_o, y_o) is not a limit point of A and hence that A contains all it's limit points and is thus closed.

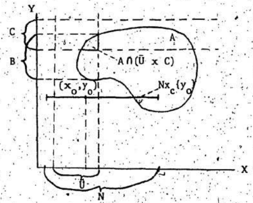
X is locally compact => x_o has a neighbourhood whose closure N is compact.

$N \times_c \{y_o\}$ = compact => $A \cap (N \times_c \{y_o\})$ = closed.

(x_o, y_o) is thus not a limit point of $A \cap (N \times_c \{y_o\})$ and there must exist a neighbourhood U of x_o, $\bar{U} \subset N$ such that $A \cap (\bar{U} \times_c \{y_o\}) = \phi$.

Let B be the projection of $A \cap (\bar{U} \times_c Y)$ on Y.

Let C be any compact set in Y.



$\bar{U}$ = compact => $\bar{U}\ x_c\ C$ = compact. Thus $A \cap (\bar{U}\ x_c\ C)$ is closed by assumption and thus compact. Since $B \cap C$ = the projection of $A \cap (\bar{U}\ x_c\ C)$ on Y, is a compact subspace of a Hausdorff space, it is closed. B thus intersects each compact subset of Y in a closed set. Since $Y \in |\underline{Top}_K|$, B is closed in Y. This implies that $Y \setminus B$ is open in Y. Since $y_o \in Y \setminus B$, $U\ x_c\ (Y \setminus B)$ is a neighbourhood of (x_o, y_o) not meeting A. Thus (x_o, y_o) is not a limit point of A. That is A is closed and $X\ x_c\ Y \in |\underline{Top}_K|$. //

(3.3.3) Lemma: If X, Y are Hausdorff spaces, then the two topologies $(KX) \times (KY)$ and $K(X\ x_c\ Y)$ on the product space, coincide.

Proof: $\sigma_X : KX \to X$, $\sigma_Y : KY \to Y$ are continuous => $g = \sigma_X \times \sigma_Y$ is continuous. Thus each compact subset of $(KX)\ x_c\ (KY)$ is compact in $X\ x_c Y$.

Let $A \subset X\ x_c\ Y$ be compact. Let $p_1 : X\ x_c Y \to X$ and $p_2 : X\ x_c Y \to Y$ be the projections. Then $B = p_1(A)$ and $C = p_2(A)$ are compact in X and Y, and hence in KX and KY. Thus $B\ x_c\ C$ is a compact subset of $KX\ x_c\ KY$.

$X, Y \in |\underline{Haus}| \Rightarrow X \times_c Y \in |\underline{Haus}| \Rightarrow A$ is closed in $X \times_c Y$. But $A \subset B \times_c C$ = compact, so A is compact in $(KX) \times_c (KY)$. Thus $(KX) \times_c (KY)$ and $(X \times_c Y)$ have the same compact sets, so their topologies, by Definition (3.1.1), coincide in $\underline{Top}_K$. //

CHAPTER IV

THE GEOMETRIC REALIZATION OF S.S. COMPLEXES

Let $\mathbb{R}^{n+1}$ be the (n + 1)-dimensional euclidean space. Given its orthogonal basis, $\{A_i = (0,\ldots,1,\ldots,0) \mid i = 0,1,\ldots,n\}$, let $\nabla_n = \{t = \sum_{i=0}^{n} t_iA_i \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$. We then define In $\nabla_n = \{t = \sum_{i=0}^{n} t_iA_i \mid t_i > 0, \sum_{i=0}^{n} t_i = 1\}$ and $\dot{\nabla}_n = \nabla_n - \mathrm{In}\ \nabla_n$.

If $\alpha \in \underline{\Delta}(\Delta_n, \Delta_q)$ define $|\alpha| : \nabla_n \to \nabla_q$ by

$|\alpha|(\sum_{i=0}^{n} t_iA_i) = \sum_{i=0}^{n} t_iA_{\alpha(i)}$. We then obtain

$$(\forall t = \sum_{i=0}^{n} t_iA_i \in \nabla_n)\ |\alpha\beta|(\sum_{i=0}^{n} t_iA_i) = \sum_{i=0}^{n} t_iA_{\alpha\beta(i)}$$

$$= |\alpha| \sum_{i=0}^{n} t_iA_{\beta(i)}$$

$$= |\alpha|\ |\beta|\ (\sum_{i=0}^{n} t_iA_i)$$

Thus $|\alpha\beta| = |\alpha|\ |\beta|$.

Also $|1|\ (\sum_{i=0}^{n} t_iA_i) = \sum_{i=0}^{n} t_iA_{1(i)} = \sum_{i=0}^{n} t_iA_i$

Thus $|1| = 1$.

The above implies that there is a covariant functor

$\mathcal{F} : \underline{\Delta} \to \underline{\mathrm{Set}}$ such that

$(\forall \Delta_n \in |\underline{\Delta}|)\mathcal{F}(\Delta_n) = \nabla_n$.

$(\forall \alpha \in \underline{\Delta}(\Delta_n, \Delta_q))\mathcal{F}(\alpha) = |\alpha| : \nabla_n \to \nabla_q$.

Given a semi-simplicial set $X \in |\underline{\mathrm{SSC}}|$ - $\underline{\mathrm{SSC}}$ = the category of semi-simplicial complexes in $\underline{\mathrm{Set}}$ - let $\tilde{X} = \coprod_n (X_n \times \nabla_n)$. Let $\sim$ be the equivalence relation R on $\tilde{X}$ induced by for $x \in X_q$, $t \in \nabla_n$ and $\alpha \in \underline{\Delta}(\Delta_n, \Delta_q)$

$$(\alpha^* x, t) R (x, |\alpha| t).$$

$(\alpha^* = X(\alpha))$

(4.1.1) Definition: The geometric realization of $X \in |\underline{SSC}|$, denoted by $|X|$, is given by $|X| = \bar{X}/\sim$.

This definition is due to Milnor [15].

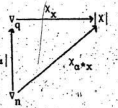
Let $\pi : \bar{X} \to |X|$ be the quotient map given by $\pi(x,t) = |x,t|$. Giving to each X_n the discrete topology and to each ∇_n the topology induced from $\mathbb{R}^{n+1}$, $|X|$ becomes a topological space with the quotient topology, i.e. the final topology with respect to π.

Given an s.s. map $f : X \to Y$ let $\bar{f} : \bar{X} \to \bar{Y}$ be the map defined by $\bar{f}(x,t) = (f(x),t)$. This induces a function $|f| : |X| \to |Y|$ on the quotients such that $|f|\pi = \pi\bar{f}$. Since $\pi\bar{f}$ is continuous and $|X|$ has the quotient topology, $|f|$ is continuous. We have shown.

(4.1.2) Lemma: $|\_| : \underline{SSC} \to \underline{Top}$ is a covariant functor.

Each simplex $x \in X_n$ defines a characteristic map $\chi_x : \nabla_n \to |X|$, where $\chi_x(t) = |x,t|$.

(4.1.3) Lemma: $(\forall \alpha \in \underline{\Delta}(\Delta_n, \Delta_q))(\forall x \in X_q)$ the following diagram commutes:



where $\alpha^* = X(\alpha)$. Moreover, $|X|$ has the finest topology for which all the χ_x are continuous.

Proof: $(\forall t \in \nabla_n)\chi_x \cdot |\alpha|(t) = \chi_x(|\alpha|t) = |x,|\alpha|t|$.

On the other hand $\chi_{\alpha * x}(t) = |\alpha * x, t|$.

But $(\alpha * x, t) \sim (x, |\alpha|t) \Rightarrow |\alpha * x, t| = |x, |\alpha| t|$

$$= \chi_x \cdot |\alpha|(t) = \chi_{\alpha * x}(t) .$$

Let τ_1 be the topology already defined on $|X|$ and τ_2 any other topology for which all the χ_x are continuous. Let $V \subset |X|$ be an open set under τ_2. Then $\chi_x^{-1}(V)$ is open in ∇_n for each $x \in X$. We have the following maps : the projection $p_n : X_n \times \nabla_n \to \nabla_n$, the inclusion $i_n : X_n \times \nabla_n \to \bar{X} = \coprod_n (X_n \times \nabla_n)$, and the quotient map $\pi : \bar{X} \to |X|$. We want to show that $\pi^{-1}(V)$ is open in $\bar{X}$. Since each X_n has the discrete topology it is enough to show that $p_n(i_n^{-1}(\pi^{-1}(V)))$ is open in ∇_n for each n.

Let $S_n = \{x \in X_n \mid (\exists t \in \nabla_n) \; |x,t| \in V\}$.

If $t \in p_n(i_n^{-1}(\pi^{-1}(V)))$, then $|x,t| \in V$ for some $x \in X_n$. Thus $x \in S_n$ and $\chi_x(t) \in V$. Therefore $t \in \bigcup_{x \in S_n} \chi_x^{-1}(V)$.

If $t \in \bigcup_{x \in S_n} \chi_x^{-1}(V)$, then $t \in \nabla_n$ and $|x,t| \in V$ for some $x \in X_n$. Thus $t \in p_n(i_n^{-1}(\pi^{-1}(V)))$.

Thus $p_n(i_n^{-1}(\pi^{-1}(V))) = \bigcup_{x \in S_n} \chi_x^{-1}(V)$.

$\chi_x^{-1}(V) =$ open for each $x \in S_n \Rightarrow \bigcup_{x \in S_n} \chi_x^{-1}(V) =$ open

$\Rightarrow p_n(i_n^{-1}(\pi^{-1}(V))) =$ open

$\Rightarrow V$ is open in τ_1.

This implies that τ_1 is finer than τ_2. //

(4.1.4) Lemma: $(\forall t \in \nabla_n)(\exists q \leq n)(\exists u \in \text{In } \nabla_q)(\exists \alpha : \Delta_q \to \Delta_n,$ α = injective and monotone) $t = |\alpha| u$.

Proof: Let $t = \sum_{i=0}^{n} t_i A_i$ and $B = \{i \mid t_i \neq 0\}$.
Let q = cardinality of B minus 1.

By labeling the non-zero t_i's from 0 to q we get a point $u = \sum_{i=0}^{q} t_i' A_i$ with $u \in \nabla_q$ and $\{t_i'\} = \{t_i / t_i \neq 0\}$.

Define $\alpha : \Delta_q \to \Delta_n$ as the combination of σ^i's which leave out in the image those $j \in \Delta_n$ for which $t_j = 0$.

Then $|\alpha| u = |\alpha| (\sum_{i=0}^{q} t_i' A_i) = \sum_{i=0}^{q} t_i' A_{\alpha(i)}$

$$= \sum_{i=0}^{n} t_i'' A_i \quad \text{where } t_i'' = \begin{cases} t_i' & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases}$$

$$= \sum_{i=0}^{n} t_i A_i = t \ . \quad //$$

(4.1.5) Definition: $x \in X_n$ is degenerate <=> ($\exists$ morhpism $\beta \in \underline{\Delta}(\Delta_n, \Delta_q)$, $q < n$, $\beta \neq id)(\exists y \in X_q)$ $x = \beta^* y$.

(4.1.6) Lemma: Every $x \in X_n$ can be written uniquely as $x = \beta^* y$ with β = surjective and y = non-degenerate, $y \in X_q$, $q < n$.

Proof: If x = non-degenerate then $x = 1^* x$ and we are finished. So assume x = degenerate. This implies that x has at least one factorization $x = \beta_1^* z$ for some $\beta_1 \in \underline{\Delta}(\Delta_n, \Delta_q)$, $z \in X_q$ and $q < n$. Let q' be the smallest such q and $\beta^* y$ the corresponding writing of x. Since $\beta \in \underline{\Delta}(\Delta_n, \Delta_{q'})$, β is a combination of δ^i's, each of which is surjective, and thus β is itself surjective.

<u>Claim:</u> y = non-degenerate and β^* = unique.

If y = degenerate, then by definition $y = \tau^* r$ for some $r \in X_p$ with $p < q'$. Then we have

$x = \beta^* y = \beta^*(\tau^* r) = (\tau\beta)^* r$ with

$r \in X_p$ and $p < q'$. This contradicts the assumption that q' is a minimum. Thus y = non-degenerate.

Suppose that we can write x in two ways; as $x = \tau^* s$ and $x = \beta^* y$. We know that both τ and β are surjective. This implies that there exist maps τ' and β' such that $\beta \cdot \beta' = 1$ and $\tau \cdot \tau' = 1$.

Since $x = \tau^* s = \beta^* y$ we have

$(\beta')^* \tau^* s = (\beta')^* \beta^* y = (\beta \cdot \beta')^* y = 1^* y = y$

and similarly

$$(\tau')^* \beta^* y = s$$

Since s and y are non-degenerate

$(\beta')^* \tau^* = (\tau')^* \beta^* = 1$ and thus $s = y$.

Suppose for some $k \in X_{q'}$, $\beta^*(k) \neq \tau^*(k)$.

Then $(\tau')^* \beta^*(k) \neq (\tau')^* \tau^*(k)$. Thus $k \neq (\tau')^* \beta^*(k)$, contradicting $(\tau')^* \beta^* = 1$. Therefore $\beta^* = \tau^*$. //

(4.1.7) <u>Definition:</u> Given $x \in X_n$, let $\mathrm{In}\, x = \{|x,t| \in |X| \mid t \in \mathrm{In}\, \nabla_n\}$.

(4.1.8) <u>Definition:</u> An element $(x,t) \in \tilde{X}$ is said to be <u>regular</u> <u>iff</u> x is non-degenerate and $t \in \mathrm{In}\, \nabla_n$ for some n.

(4.1.9) <u>Theorem:</u> If $X \in |SSC|$, $|X| = \amalg \mathrm{In}\, x$, where x runs over all non-degenerate simplexes of X.

Proof: To show the theorem, it is enough to construct a function $\phi : \bar{X} \to \bar{X}$ such that:

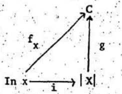
(1) $(\forall (x,t) \in \bar{X})$, $\phi(x,t)$ = regular

(2) if (x,t) = regular, $\phi(x,t) = (x,t)$

(3) $(\forall (x,t) \in \bar{X})\ \phi(x,t) \sim (x,t)$

(4) If $(x,t) \sim (y,s)$ then $\phi(x,t) = \phi(y,s)$.

Given such a ϕ, we show that $|X| = \amalg \text{In } x$ for x = non-degenerate by showing that for all $C \in |\underline{\text{Top}}|$ and for every family of maps $\{f_x : \text{In } x \to C\}$, there is a unique map $g : |X| \to C$ making commutative the diagram



where f_x is given.

Let $g(|x,t|) = f_y(|y,s|)$ where $(y,s) = \phi(x,t)$. $\phi(x,t)$ = regular, y = non-degenerate and s = interior.

If $|x,t| = |x',t'|$ then $(x,t) \sim (x',t')$ and $\phi(x,t) = \phi(x',t')$ by (4). Thus $g(|x,t|) = g(|x',t'|)$ and so g is well defined. Let $A \subset C$ be open.

By definition of g, $g^{-1}(A) = \bigcup_y f_y^{-1}(A)$, y = non-degenerate. Since each f_y is continuous, $\bigcup_y f_y^{-1}(A)$ is open. Thus $g^{-1}(A)$ = open and g = continuous.

$(\forall |x,t| \in \text{In } x)$ x is non-degenerate $\Rightarrow (x,t)$ = regular $\Rightarrow \phi(x,t) = (x,t) \Rightarrow g \cdot i(|x,t|) = g(|x,t|) = f_x(|x,t|)$

$\Rightarrow g \cdot i = f_x$ for each x = non-degenerate.

Let $h : |X| \to C$ with $h \cdot i = f_x$ for each x = non-degenerate.

$$(\forall |x,t| \in |X|)\ g(|x,t|) = f_y(|y,s|) \text{ where } (y,s) = \phi(x,t)$$
$$= h \cdot i(|y,s|)$$
$$= h(|y,s|)$$
$$= h(|x,t|) \text{ since } (x,t) \sim \phi(x,t) = (y,s).$$

Thus $g = h$ and g is unique. //

Construction of ϕ:

(4.1.10) Given $(x,t) \in \bar{X}$, by (4.1.4) $t = |\alpha|u$ for some $u \in$ interior. By (4.1.5) there exists a non-degenerate point y and a unique surjection β such that $\alpha^*x = \beta^*y$. Let $\phi(x,t) = (y, |\beta|u)$. We now show that ϕ satisfies the four conditions given above.

(1) Given $\phi(x,t) = (y, |\beta|u)$, y is non-degenerate, β is surjective $\Rightarrow |\beta|u$ is interior since u is interior. Thus $\phi(x,t)$ is regular.

(2) (x,t) = regular implies by definition that $\phi(x,t) = (x,t)$ since t is interior and x is non-degenerate.

(3) $$(\forall (x,t) \in \bar{X})\ (x,t) = (x, |\alpha|u) \sim (\alpha^*x, u) = (\beta^*y, u) \sim (y, |\beta|u)$$
$$\Rightarrow (x,t) \sim \phi(x,t).$$

(4) Given $(x,t) \sim (y,s)$ then $\phi(x,t) \sim \phi(y,s)$ by (3). Let $\phi(x,t) = (x', |\alpha|t')$ and $\phi(y,s) = (y', |\beta|s')$ with x', y' = non-degenerate and t', s' = interior. Then

$(x', |\alpha|t') \sim (y', |\beta|s') \Rightarrow x' = \gamma^* y'$ for some γ and $|\beta|s' = |\gamma|\ |\alpha|t'$. But x' = non-degenerate $\Rightarrow \gamma^* = 1$, $x' = y'$. Furthermore, $\beta^*\gamma^* = \beta^*$ implies, by uniqueness of β that $\gamma\beta = \beta$. Since β is an epimorphism, $\gamma = 1$. //

§2. Structure of $|\_|$.

(4.2.1) Given an s.s. complex X with $X_n = X(\Delta_n)$, let X^n be the sub-s.s. complex generated by X_n; that is to say,

$$X^n_p = X_p \text{ if } p \leq n$$
$$= \bigcup_{i=0}^{p-1} s_i^{p-1}(X_{p-1}), \text{ if } p > n,$$ where $\bigcup$ is the set union -

X^n is a sub-complex of X and $\emptyset \subset X^o \subset X' \subset \ldots \subset X^n \subset \ldots$ with $X = \operatorname{colim} X^n$.

(4.2.2) Define a covariant functor Σ : Top $\to$ SSC as follows:
Given $Y \in$ Top, $\Sigma(Y)_n$ = Top (∇_n, Y).
Given $f \in$ Top (Y, Z), $\Sigma(f)_n : \Sigma(Y)_n \to \Sigma(Z)_n$ is defined by $\Sigma(f)_n(g) = f \cdot g$ for every $g \in \Sigma(Y)_n$.

(4.2.3) Theorem: $|\_| \longrightarrow \Sigma$.

Proof: We must show that there exists a natural isomorphism θ : Top $(|X|, Y) \to$ SSC$(X, \Sigma(Y))$ for all $X \in |$SSC$|$ and $Y \in |$Top$|$

Given $f : |X| \to Y$ define a semi-simplical map $\theta(f) = f'$ as follows:

$(\forall x \in X_n)(\forall t \in \nabla_n)\ f'_n(x)(t) = f(|x,t|)$.

Given $\alpha : \Delta_q \to \Delta_n$, we have the following diagram:

$$\begin{array}{ccc} X_n & \xrightarrow{f'_n} & \Sigma(Y)_n \\ \Big\downarrow \alpha^* & & \Big\downarrow \Sigma(Y)(\alpha) = \alpha' \\ X_q & \xrightarrow[f'_q]{} & \Sigma(Y)_q \end{array}$$

where $\alpha'(\lambda) = \lambda \cdot |\alpha|$ for all $\lambda : \nabla_n \to Y$.

We have
$$\begin{aligned} \alpha'(f'_n(x))(t) &= f'_n(x)(|\alpha|(t)) \\ &= f(|x, |\alpha|t|) \\ &= f(|\alpha^* x, t|) \\ &= f'_q(\alpha^* x)(t) \end{aligned}$$
$$\Rightarrow \alpha' \cdot f'_n = f'_q \cdot \alpha^*.$$

Because the diagram commutes, f' is an s.s. map.

(i) If $f, g \in \underline{\text{Top}}\ (|X|, Y)$ are such that $\theta(f) = \theta(g)$ then for each n and $(\forall x \in X_n)(\forall t \in \nabla_n)$
$$f'_n(x)(t) = g'_n(x)(t)$$
$$\Rightarrow f(|x,t|) = g(|x,t|)$$
$$\Rightarrow f = g$$
$$\Rightarrow \theta \text{ is } 1\text{-}1.$$

(ii) Let $g : X \to \Sigma(Y)$ be a semi-simplicial map.

$(\forall x \in X_n)(t \in \nabla_n)$ define $f : |X| \to Y$ by $f(|x,t|) = g_n(x)(t)$.

Then $\theta(f) = g$ and thus θ is an epimorphism.

(iii) To show that θ is natural we must show for all $\alpha \in \underline{\text{SSC}}\ (X', X)$, $\beta \in \underline{\text{Top}}\ (Y, Y')$ and $g \in \underline{\text{Top}}\ (|X|, Y)$ that

$$\theta(\beta \cdot g \cdot |\alpha|) = \Sigma(\beta) \cdot \theta(g) \cdot \alpha.$$

But for each n and $\forall\, x \in X_n$, $\forall\, t \in \nabla_n$

$$\begin{aligned}\theta(\beta \cdot g \cdot |\alpha|)_n(x)(t) &= (\beta \cdot g \cdot |\alpha|)(|x,t|) \quad \text{by definition}\\ &= \beta \cdot g\,(|\alpha|(|x,t|))\\ &= \beta \cdot g\,(|\alpha x,t|)\\ &= (\Sigma(\beta)_n(g))(|\alpha x,t|)\\ &= (\Sigma(\beta)_n \cdot g)(|\alpha x,t|)\\ &= \Sigma(\beta)_n \cdot \theta(g)[\alpha(x))](t)\\ &= [\Sigma(\beta)_n \cdot \theta(g) \cdot \alpha](x)(t)\end{aligned}$$

$$\Rightarrow \theta(\beta \cdot g \cdot |\alpha|) = \Sigma(\beta) \cdot \theta(g) \cdot \alpha. \quad //$$

(4.2.4) Lemma: $|X| \cong \bigcup_n |X^n|$ with the weak topology.

Proof: By (4.2.1) $X = \operatorname{colimit} X^n$. Since $|-| \longrightarrow \Sigma$, Lemma (2.1.4) $\Rightarrow$ colimit $|X^n| = |\operatorname{colim} X^n| = |X|$. But the diagram $X^o \subset X^1 \subset X^2 \subset \ldots$ of Top has for colimit the union $\bigcup_{n \geq 0} X^n$ with the weak topology. //

(4.2.5) Theorem: The geometric realization $|X|$ of a semi-simplicial complex X is a CW-complex.

Proof: We give the following sequence:

$$|X|^o \longrightarrow |X|^1 \longrightarrow |X|^2 \longrightarrow \ldots$$

and show that $|X|^n$ is obtained from $|X|^{n-1}$ by means of a pushout diagram as in (2.3.2).

First we define $|X|^o$, the 0-skeleton of $|X|$, to be $|X^o|$. This set of 0-cells will actually be $\{\operatorname{In} x\}$ for $x \in X_o$

and x = non-degenerate, with the discrete topology. We also define $|X|^{n-1}$ to be $|X^{n-1}|$. We construct $|X|^n$ as follows:

For each non-degenerate $x \in X_n$, let $\phi^n_x = \chi_x|\dot{\nabla}_n$. Given a point $t \in \dot{\nabla}_n$, since t is not interior, there exists a map $\sigma^i : \Delta_{n-1} \to \Delta_n$ and a point $t' \in \nabla_{n-1}$ such that $t = |\sigma^i|t'$. Thus

$$\phi^n_x(t) = |x,t| = |x,|\sigma^i|t'| = |\sigma_i^* x, t'|.$$

Since $\sigma_i^* x \in X^{n-1}$, $\phi^n_x(t) \in |X^{n-1}|$.

Associating to each non-degenerate point $x \in X_n$ a copy $\dot{\nabla}^x_n$ of $\dot{\nabla}_n$ and forming $\coprod_x \dot{\nabla}^x_n$, the following diagram commutes for each x_i = non-degenerate:

$$\begin{array}{ccc}
\dot{\nabla}_n^{x^i} & & \\
\downarrow{\scriptstyle i} & \searrow{\scriptstyle \phi^n_x} & \\
\coprod_x \dot{\nabla}^x_n & \xrightarrow{\phi^n} & |X^{n-1}|
\end{array}$$

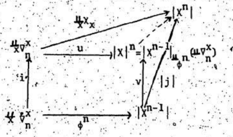
Take the inclusion $i_n : \coprod_x \dot{\nabla}^x_n \to \coprod_x \nabla^x_n$ and form the pushout diagram of i_n, ϕ^n in <u>Top</u>.

$$\begin{array}{ccc}
\coprod_x \nabla^x_n & \xrightarrow{u} & |X|^n = |X^{n-1}| \coprod_{\phi^x} (\coprod_x \nabla^x_n) \\
\uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle v} \\
\coprod_x \dot{\nabla}^x_n & \xrightarrow{\phi^n} & |X^{n-1}|
\end{array}$$

where u and v are as defined in (2.3.2). We obtain a space $|X|^n$. The colimit of $|X|^{n=0,1,\ldots}$ is then a CW-complex. Since

$|X| \cong \mathrm{Colim}\, |X^n|$ we must show that $|X|^n \cong |X^n|$ for all n.

If $j : X^{n-1} \to X^n$ is the inclusion we form the map $|j| : |X^{n-1}| \to |X^n|$ where $|j|(|x,t|) = |x,t|$. Also $\coprod_x \chi_x : \coprod_x \nabla^x_n \to |X^n|$. Then we have the following diagram:



By the definition of ϕ^n, this diagram commutes and thus there exists a unique $g : |X|^n \to |X^n|$ making the smaller triangles commutative. The topology on $|X|^n$ is the final topology with respect to u and v and thus g is continuous.

By (2.3.2) $u = F \cdot i_1$ and $v = F \cdot i_2$ where $i_1 : \coprod_x \nabla^x_n \to |X^{n-1}| \coprod (\coprod_x \nabla^x_n)$ and $i_2 : |X^{n-1}| \to |X^{n-1}| \coprod (\coprod_x \nabla^x_n)$ are the inclusions and

$$F : |X^{n-1}| \coprod (\coprod_x \nabla^x_n) \to |X^{n-1}| \coprod_{\phi^n} (\coprod_x \nabla^x_n)$$

is the quotient map. $|X^{n-1}| \coprod_{\phi^n} (\coprod_x \nabla^x_n) = |X^{n-1}| \coprod (\coprod_x \nabla^x_n)/E$ where E is the equivalence defined by the relation:

$(\forall |x,t| \in |X^{n-1}|, \ \forall s \in \coprod_x \nabla^x_n)$ $|x,t|\ R s \Leftrightarrow$

$|x,t| = \phi^n(a)$ and $s = i_n(a)$ for some $a \in \coprod_x \dot{\nabla}^x_n$.

<u>Claim:</u> g is bijective and open.

We first show that $|X^n| \setminus |X^{n-1}| \cong \coprod_x \nabla^x_n \setminus \coprod_x \dot{\nabla}^x_n$.

Let $|x,t| \in |X^n| \setminus |X^{n-1}|$ and $\phi(x,t) = (y, |\beta|u)$ as given in (4.1.9). Then $|x,t| = |y,|\beta|u|$ with $y \in X_r$, $r \geq n$. By definition of X^n, if $x \in X_q^n$ with $q \geq n$ then $x \in \bigcup_{i=0}^{q-1} s_i^{q-1}(X_{q-1}^n)$. Thus x is degenerate if $q > n$. Thus $y \in X_n^n$, $|\beta|u \in \text{In}\ \nabla_n^y$. Define a function $h : |X^n| \setminus |X^{n-1}| \to \coprod_x \nabla_n^x \setminus \coprod_x \dot{\nabla}_n^x$ by $h(|x,t|) = |\beta|u \in \text{In}\ \nabla_n^y$, where β, u and y are as defined above.

$(\forall |x,t| \in |X^n| \setminus |X^{n-1}|)\ \coprod_x \chi_x \cdot h(|x,t|) = \coprod_x \chi_x\ (|\beta|u)$

$= |y,|\beta|u|$ since $|\beta|u \in \text{In}\ \nabla_n^y$

$= |x,t|$

$\Rightarrow \coprod_x \chi_x \cdot h = 1.$

$(\forall t \in \text{In}\ \nabla_n^x$, x = non-degenerate)

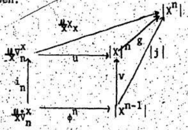
$h \cdot \coprod_x \chi_x(t) = h(|x,t|)$

$= t$ since t = interior and x = non-degenerate.

$= h \cdot \coprod_x \chi_x = 1.$

Since h is an isomorphism, g is also bijective. We now show that g is open.

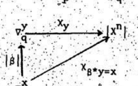
Recall that



is commutative. Let $V \subset |X|^n$ be open. By Lemma (4.1.3) $|X^n|$ has the finest topology for which all the χ_x are continuous.

Thus $g(V) \subset |X^n|$ is open $\Leftrightarrow$ $\chi_x^{-1}(g(V))$ is open for all $x \in X^n$.

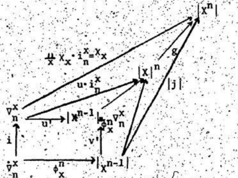
It is enough to show that $\chi_x^{-1}(g(V))$ is open for all non-degenerate $x \in X^n$, since if x is degenerate then there exists $y \in X^n$, non-degenerate and a unique surjection β, with $x = \beta^* y$. Thus if $x \in X_p^n$, $y \in X_q^n$, $q < p$



commutes. Then if $\chi_y^{-1}(g(V))$ is open,

$\chi_x^{-1}(g(V)) = |\beta|^{-1}[\chi_y^{-1}(g(V))]$ is also open.

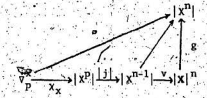
So we assume x = non-degenerate. Take $x \in X_n^n$ and form the following commutative diagram:



where the inner square is a pushout and $i_n^x : \nabla_n^x \to \amalg_x \nabla_n^x$ is the inclusion.

Now $\chi_x^{-1}(g(V)) = (u')^{-1}g^{-1}(V)$ = open in ∇_n^x. Thus for all x = non-degenerate in X_n^n, $\chi_x^{-1}(g(V))$ is open.

Take $x \in X_p^n$, non-degenerate and $p \neq n$. By definition of X^n, $p < n$. We have the following commutative diagram:



Thus $\chi_x^{-1}(g(V)) = \chi_x^{-1}|j|^{-1}v^{-1}(V)$ = open in ∇_p^x. //

(4.2.6) Example: We define a semi-simplicial complex $\Delta[n]$ as follows:

$\Delta[n] : \underline{\Delta}^{opp} \to \underline{Set}$ where

$\Delta[n](\Delta_p) = \underline{\Delta}(\Delta_p, \Delta_n)$

for $\sigma_p^i : \Delta_p \to \Delta_{p+1}$ and for $\alpha : \Delta_{p+1} \to \Delta_n$

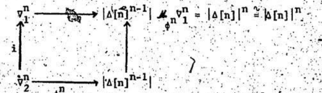
$\Delta[n](\sigma_p^i)(\alpha) = \alpha \cdot \sigma_p^i$

for $\delta_p^i : \Delta_{p+1} \to \Delta_p$ and for $\beta : \Delta_p \to \Delta_n$

$\Delta[n](\delta_p^i)(\beta) = \beta \cdot \delta_p^i$

(4.2.7) Lemma: $|\Delta[n]| = \nabla^n$

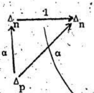
Proof: First we note that there is only one non-degenerate n-simplex and that is $1 : \Delta_n \to \Delta_n$. Thus we obtain $|\Delta[n]|^n$ from the following pushout diagram:



$\phi^n : \dot{\nabla}_1^n \to |\Delta[n]^{n-1}|$ is defined by $\phi^n(t) = |1,t|$.

Let $|\alpha,s| \in |\Delta[n]^{n-1}|$ with $\alpha \in \underline{\Delta}(\Delta_p,\Delta_n), s \in \nabla_p$, $p < n$.

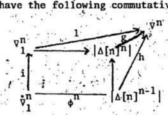
Since



commutes, we have $\alpha = \alpha^*(1)$. Thus $|\alpha,s| = |\alpha^*(1),s| = |1,|\alpha|s|$.

Define $h : |\Delta[n]^{n-1}| \to \nabla^n$ by $h(|\alpha,s|) = i \cdot |\alpha|s$. Since ϕ^n is an open map h is continuous. Furthermore, $h \cdot \phi^n = i$.

Thus we have the following commutative diagram:



This implies there exists a unique map $g : |\Delta[n]^n| \to \nabla^n$ making the small triangle commute. g is a homeomorphism since $1 : \nabla_1^n \to \nabla^n$ is a homeomorphism.

Since there are no non-degenerate q-simplices for $q > n$, $|\Delta[n]^q| \cong \nabla^n$. Thus $|\Delta[n]| = \nabla^n$. //

§3. Products

Given two semi-simplicial complexes X and Y, we know that their geometric realizations, $|X|$ and $|Y|$, are CW-complexes. However, the cartesian product of two CW-complexes may not be a CW-complex. We give $|X| \times |Y|$ a CW-structure by letting $|X| \times |Y| = \bigcup_{n\geq 0} (|X| \times |Y|)^n$

with the weak topology, i.e. $F \subset \bigcup_{n>0} (|X| \times |Y|)^n$ is closed $\Leftrightarrow$ $(\forall n \geq 0) F \cap (|X| \times |Y|)^n$ is closed in $(|X| \times |Y|)^n$.

(4.3.1) If $|X|^p$ and $|Y|^{n-p}$ $(p \leq n)$ are given by the pushouts

$$\begin{array}{ccc} \coprod_x \nabla_x^p & \xrightarrow{\bar{f}} & |X|^p \\ \uparrow i & & \uparrow \bar{i} \\ \coprod_x \dot{\nabla}_x^p & \xrightarrow[f]{} & |X^{p-1}| \end{array}$$

and

$$\begin{array}{ccc} \coprod_x \nabla_y^{n-p} & \xrightarrow{\bar{g}} & |Y|^{n-p} \\ \uparrow i & & \uparrow \bar{i} \\ \coprod_y \dot{\nabla}_y^{n-p} & \xrightarrow[g]{} & |Y^{n-p-1}| \end{array}$$

then $(|X| \times |Y|)^n = \bigcup_{p\leq n} |X|^p \times |Y|^{n-p}$ is given by the pushout

$$\begin{array}{ccc} \coprod_{p\leq n}\left(\coprod_{x,y} \nabla_{x,y}^n\right) \cong \coprod_{p\leq n}\left(\coprod_x \nabla_x^p \times \coprod_y \nabla_y^{n-p}\right) & \xrightarrow{\coprod_p(\bar{f} \times \bar{g})} & \bigcup_{p\leq n} |X|^p \times |Y|^{n-p} \\ \uparrow & & \uparrow \\ \coprod_{p\leq n}\left(\coprod_{x,y} \dot{\nabla}_{x,y}^n\right) \cong \coprod_{p\leq n}\left(\coprod_x \dot{\nabla}_x^p \times \coprod_y \dot{\nabla}_y^{n-p}\right) & \xrightarrow{\coprod_p(f\times\bar{g} \cup \bar{f}\times g)} & (|X| \times |Y|)^{n-1} \end{array}$$

as in [;6], p. 33.

(4.3.2) <u>Theorem</u>: $|X| \times |Y|$ is homeomorphic to $k(|X| \times_c |Y|)$, [16].

<u>Proof</u>: For each $n \geq 0$ the canonical projections

$$\Pi_1^n : (|X| \times |Y|)^n \longrightarrow |X|^n \longrightarrow |X|$$

$$\Pi_2^n : (|X| \times |Y|)^n \longrightarrow |Y|^n \longrightarrow |Y|,$$

define two maps

$$\pi_1 : |X| \times |Y| \to |X|$$
$$\pi_2 : |X| \times |Y| \to |Y|$$

Because of the Universal Property of the product, the identity function $1 : |X| \times |Y| \to k(|X| \times_c |Y|)$ is continuous. We now show that 1 is a closed map; we prove this by showing that every compact subspace of $k(|X| \times_c |Y|)$ is a compact subspace of $|X| \times |Y|$. Let C be a compact subspace of $k(|X| \times_c |Y|)$. Since $k(|X| \times_c |Y|)$ is Hausdorff, C is closed there and hence, C is a closed subset of $|X| \times |Y|$ by the continuity of 1. Take the compact spaces $C_1 = pr_1(C) \subset |X|$ and $C_2 = pr_2(C) \subset |Y|$. If $\bar{\sigma}_i$ and $\bar{\tau}_i$ are the closed n-cells of $|X|$ and $|Y|$, since a compact subspace of a CW-complex meets only a finite number of cells of the CW-complex, we may assume that $C_1 \subset \bigcup_{i=1}^{n} \bar{\sigma}_i$ and $C_2 \subset \bigcup_{j=1}^{n} \bar{\tau}_i$ and therefore, $C \subset C_1 \times C_2 \subset \bigcup_{i=j} \bar{\sigma}_i \times \bar{\tau}_i$. But $C_1 \times C_2$ is a compact subspace of $|X| \times |Y|$. C is closed implies that C is also a compact subspace of $|X| \times |Y|$. //

Given two semi-simplicial complexes X and Y, let p and p', defined by $p_n : X_n \times Y_n \to Y_n$ and $p'_n : X_n \times Y_n \to Y_n$, be the projection maps. Then $|p| : |X \times Y| \to |X|$ and $|p'| : |X \times Y| \to |Y|$. Define $\eta = |p| \times |p'| : |X \times Y| \to |X \times Y|$.

(4.3.3) Theorem : $\eta : |X \times Y| \to |X| \times_k |Y|$ is a homeomorphism.

Proof: First we show that η is an isomorphism.

Let $a = |x \times y, t| \in |X \times Y|$ $x \in X_n, y \in Y_n, t \in \text{In } \nabla_n$

with $(x \times y, t)$ = regular. We know that $x = \alpha^* x'$, $y = \beta^* y'$ for some unique α, β = surjective and $x' \in X_r, y' \in Y_s$; $r, s \leq n$. Thus

$$\begin{aligned} \eta(a) &= |x,t| \times |y,t| \\ &= |\alpha^*, x', t| \times |\beta^* y', t| \\ &= |x', |\alpha|t| \times |y', |\beta|t| \end{aligned}$$

Now let $|x,t| \in |X|$, $|y,s| \in |Y|$ with $x \in X_r$, $t \in \text{In } \nabla_r$; $y \in Y_m$, $s \in \text{In } \nabla_m$, with $t = (t_o, t_1, \ldots, t_r)$ and $s = (s_o, s_1, \ldots, s_m)$. Assuming $t_o \leq s_o$, define $w \in \nabla_{r+m}$ by

$$w = \left(t_o, t_1, \ldots, t_{p_o}, s_o - \sum_{i=0}^{p_o} t_i, s_1, s_2, \ldots, s_{p_1}, \sum_{i=0}^{p_o+1} t_i - \sum_{i=0}^{p_1} s_i, \right.$$
$$\left. t_{p_o+2}, \ldots, t_{p_2}, \sum_{i=0}^{p_1+1} s_i - \sum_{i=0}^{p_2} t_i, \ldots \right)$$

where $p_o < p_2 < p_4 < \ldots$; $p_1 < p_3 < p_5 < \ldots$;

and
$$\sum_{i=0}^{p_j+1} t_i - \sum_{i=0}^{p_{j+1}} s_i > 0, \quad \sum_{i=0}^{p_j+1} t_i - \sum_{i=0}^{p_{j+1}+1} s_i < 0 \quad \text{for each } j = \text{even},$$

and
$$\sum_{i=0}^{p_j+1} s_i - \sum_{i=0}^{p_{j+1}} t_i > 0, \quad \sum_{i=0}^{p_j+1} s_i - \sum_{i=0}^{p_{j+1}+1} t_i < 0 \quad \text{for each } j = \text{odd}.$$

Clearly, $\exists\, \alpha, \beta$ such that $t = |\alpha|w$ and $s = |\beta|w$.

Define $\bar{\eta} : |X| \times |Y| \to |X \times Y|$

by

$$\bar{\eta}(|x,t| \times |y,s|) = |(\alpha^* x \times \beta^* y, w)|.$$

Now $\quad \eta\bar{\eta}(|x,t| \times |y,s|) = \eta|\alpha * x \times \beta * y, \omega|$

$$= |\alpha * x, \omega| \times |\beta * y, \omega|$$

$$= |x, |\alpha|\omega| \times |y, |\beta|\omega|$$

$$= |x,t| \times |y,s|$$

$$\Rightarrow \eta\bar{\eta} = 1_{|X| \times |Y|}$$

Also $\quad \bar{\eta}\eta(|x \times y, t|) = \bar{\eta}(|x,t| \times |y,t|)$

$$= \bar{\eta}(|x', |\alpha|t| \times |y', |\beta|t|)$$

$$= |\alpha * x' \times \beta * y', t|$$

$$= |x \times y, t|$$

$$\Rightarrow \bar{\eta}\eta = 1_{|X \times Y|}$$

Next we show that $\bar{\eta}$ is continuous. It is enough to show that it is continuous on the n-cells. Thus we have to show that the function

$$h : \nabla^p_{x'} \times \nabla^{n-p}_{y'} \to \nabla^n_{x,y},$$

defined by $h(t \times s) = \omega$ as given in the definition of $\bar{\eta}$, is continuous.

Let $O \subset \nabla^n_{x,y}$ be open and $\omega \in O$ be a point such that $w = h(t \times s)$. Then $t = |\alpha|w$ and $s = |\beta|w$ and if $t = (t_o, t_1, \ldots, t_p)$, $s = (s_o, s_1, \ldots, s_{n-p})$ then

$$w = \Big(t_o, t_1, \ldots, t_{\ell_o}, s_o - \sum_{i=0}^{\ell_o} t_i, s_1, \ldots, s_{\ell_1}, \sum_{i=0}^{\ell_o+1} t_i - \sum_{i=0}^{\ell_1} s_i,$$

$$t_{\ell_o+2}, \ldots, t_{\ell_2}, \sum_{i=0}^{\ell_1+1} s_i - \sum_{i=0}^{\ell_2} t_i,$$

$$\ldots, s_{\ell_j+2}, \ldots, s_{\ell_{j+2}}, \sum_{i=0}^{\ell_{j+1}+1} t_i - \sum_{i=0}^{\ell_{j+2}} s_i, t_{\ell_{j+1}+2}, \ldots\Big).$$

We can represent w by $(w_o, w_1, \ldots, w_n)$

Let $\varepsilon > 0$ be a real number such that $\varepsilon < (\min\{w_i\})/n$.

We form an open neighbourhood around w with radius ε, denoted by $N(w)$. Let $u \in N(w)$ be denoted by $(u_o, u_1, \ldots, u_n)$.

Then $u_i = w_i + \varepsilon_i$ for some real number ε_i, for all i.

$$\sum_{i=0}^{n} u_i = 1 \Rightarrow \sum_{i=0}^{n} w_i + \sum_{i=0}^{n} \varepsilon_i = 1$$

$$\sum_{i=0}^{n} w_i = 1 \Rightarrow \sum_{i=0}^{n} \varepsilon_i = 0$$

Also, since $N(w)$ has center w and radius ε,

$$\sqrt{\sum_{i=0}^{n} (u_i - w_i)^2} < \varepsilon \Rightarrow \sum_{i=0}^{n} [(w_i + \varepsilon_i) - w_i]^2 < \varepsilon^2$$

$$\Rightarrow \sum_{i=0}^{n} \varepsilon_i^2 < \varepsilon^2.$$

Thus for all i, $|\varepsilon_i| < (\min\{w_i\})/n$.

We can now represent u as follows:

$$u = (t_o + \delta_o, \ldots, t_{\ell_o} + \delta_{\ell_o}, \delta_o - \sum_{i=0}^{\ell_o} t_i + \sigma_o, s_1 + \gamma_1, \ldots,$$
$$s_{\ell_1} + \gamma_{\ell_1}, \sum_{i=0}^{\ell_o+1} t_i - \sum_{i=0}^{\ell_1} s_i + \sigma_1,$$
$$t_{\ell_o+2} + \delta_{\ell_o+2}, \ldots, t_{\ell_2} + \delta_{\ell_2}, \sum_{i=0}^{\ell_1+1} s_i - \sum_{i=0}^{\ell_2} t_i + \sigma_2, \ldots)$$

For $j = 0,2,4,\ldots,$ define δ_{ℓ_j+1} recursively as

$$\delta_{\ell_j+1} = \sigma_{j+1} + \sum_{i=0}^{\ell_{j+1}} \gamma_i - \sum_{i=0}^{\ell_j} \delta_i ,$$

and for $j = -1,1,3, \ldots,$ define γ_{ℓ_j+1} recursively as

$$\gamma_{\ell_{j+1}} = \sigma_{j+1} + \sum_{i=0}^{\ell_{j+1}} \delta_i - \sum_{i=0}^{\ell_j} \gamma_i \quad . \quad \ell_{-1} = 0.$$

Now for $i = 0,1,\ldots,p$ define

$$t_i' = t_i + \delta_i$$

and for $i = 0,1,\ldots,n-p$ define

$$s_i' = s_i + \gamma_i .$$

Then $|\alpha|u = (t_0', t_1', \ldots, t_p') = t'$

and $|\beta|u = (s_0', s_1', \ldots, s_{n-p}') = s'$

Thus for all $u \in N(w)$, $h^{-1}(u) = |\alpha|u \times |\beta|u$

and hence $h^{-1}(N(w)) = (|\alpha| \times |\beta|)(N(w))$.

Since $|\alpha|$ and $|\beta|$ are open maps, $h^{-1}(N(w))$ is open.

Since $O = \bigcup_{j \in J} N_j(w_j)$, J = some indexing set, $h^{-1}(O)$ is the union of open sets and is therefore open.

Thus h is continuous. //

Appendix

In Chapter IV we showed that $|\_| \dashv \Sigma$ where $\Sigma Y_n = \underline{\text{Top}}(\nabla_n, Y)$. The adjunction $\theta : \underline{\text{Top}}(|X|, Y) \to \underline{\text{SSC}}\ (X, \Sigma Y)$ was given by

$$\theta(f)_n(x)(t) = f(|x,t|) \text{ for } f : |X| \to Y.$$

The cotriple defined by $|\_|$ and Σ is now (C,k,p) where $C = |\Sigma\_|$, $k = \delta$ the counit of the adjunction and $p_Y = |\varepsilon_{\Sigma Y}|$, where ε is the unit of the adjunction.

Thus, let $g \in \Sigma Y_n$ and $t \in \nabla_n$ for $Y \in |\underline{\text{Top}}|$

Since $k_Y = \eta^{-1}(1_{\Sigma Y})$ then $\eta(k_Y) = \eta\eta^{-1}(1_{\Sigma Y}) = 1_{\Sigma Y}$.

But $\eta(k_Y)(g)(t) = k_Y(|g,t|)$. Thus

$$k_Y(|g,t|) = 1_{\Sigma Y}(g)(t) = g(t)$$

Also, since $\varepsilon_X^n = \theta(1_{|X|})_n$, $\varepsilon_X^n(x)(t) = \theta(1_{|X|})_n(x)(t)$

$$= 1_{|X|}(|x,t|)$$

$$= |x,t|$$

$$= \chi_x(t)$$

and $\varepsilon_X^n(x) = \chi_x$. Thus

$$p_Y(|g,t|) = |\varepsilon_{\Sigma Y}|(|g,t|) = |\varepsilon_{\Sigma Y}(g),t|$$

$$= |\chi_g,t|$$

To every object $Y \in |\underline{\text{Top}}|$ there is now a semi-simplicial functor defined by $T_Y(\Delta_n) = C^{n+1}(Y)$ where $C = |\Sigma\_|$. Thus

$$T_Y(\Delta_n) = |\Sigma(C^nY)|$$

We will now give some of the face and degeneracy operators.

Let $g \in \Sigma Y_n$ and $t \in \nabla_n$.

$s_o^o = C^o p C_Y^o : C^1Y \to C^2Y$

$s_o^o(|g,t|) = p_Y(|g,t|) = |\chi_g,t|$.

Let $h \in \Sigma(|\Sigma Y|)_n = \underline{\mathrm{Top}}(\nabla_n, |\Sigma Y|)$ such that

$h(t) = |g_t, r_t|$ where $g_t : \nabla_m \to Y$, $r_t \in \dot{\nabla}_m$.

$d_o^o = C^o k C_Y : C^2Y \to CY$

$= k_{CY}$

$d_o^o(|h,t|) = h(t) = |g_t, r_t|$

$d_1^o = CkC_Y^o : C^2Y \to CY$

$= C(k_Y) = |\Sigma(k_Y)|$

$d_1^o(|h,t|) = |\Sigma(k_Y)|\,(|h,t|) = |\Sigma k_Y(h),t|$

$= |k_Y \cdot h,t|$

where $k_Y \cdot h(s) = k_Y(|g_s, r_s|) = g_s(r_s)$ for $s \in \nabla_n$.

$s_o^1 = C^o p C_Y : C^2Y \to C^3Y$

$= pC_Y = p_{CY} = p_{|\Sigma Y|}$

$s_o^1(|h,t|) = p_{|\Sigma Y|}(|h,t|) = |\chi_h,t|$

$s_1^1 = C^1 p C_Y^o : C^2Y \to C^3Y$

$= Cp_Y = |\Sigma(p_Y)|$

$s_1^1(|h,t|) = |\Sigma(p_Y)|\,(|h,t|)$

$= |\Sigma(p_Y)(h),t|$

$= |p_Y \cdot h,t|$

where $p_Y \cdot h(s) = p_Y(|g_s, r_s|) = |\chi_{g_s}, r_s|$ for $s \in \nabla_n$.

Let $g^3 : \nabla^n \to C^2Y$ with $g^3(t) = |g_t^2, r_t|$.

$g_t^2 : \nabla^m \to CY$ with $g_t^2(t') = |g_{t'}, r_{t'}|$

$g_{t'} : \nabla^q \to Y$

Now $d_o^1 = C^o k C_y^2 : C^3Y \to C^2Y$

$= k_{C^2Y}$

$$d_o^1(|g^3,t|) = k_{C^2Y}(|g^3,t|) = g^3(t) = |g_t^2, r_t|$$

$$d_1^1 = CkC_Y : C^3Y \to C^2Y$$

$$= Ck_{CY} = |\Sigma k_{|\Sigma Y|}|$$

$$d_1^1(|g^3,t|) = |\Sigma k_{|\Sigma Y|}|(|g^3,t|)$$

$$= |\Sigma k_{|\Sigma Y|}(g^3),t|$$

$$= |k_{|\Sigma Y|} \cdot g^3,t|$$

where $k_{|\Sigma Y|} \cdot g^3(t') = k_{|\Sigma Y|}(|g_{t'}^2, r_{t'}|)$

$$= g_{t'}^2(r_{t'})$$

$$d_2^1 = C^2kC_Y^o : C^3Y \to C^2Y$$

$$= C^2k_Y$$

$$d_2^1(|g^3,t|) = C^2k_Y(|g^3,t|)$$

$$= |\Sigma Ck_Y \cdot g^3,t|$$

$$= |Ck_Y \cdot g^3,t| = ||\Sigma k_U|(g^3),t|$$

where $|\Sigma k_Y|(g^3)(t') = |\Sigma k_Y|(|g_{t'}^2, r_{t'}|)$

$$= |\Sigma k_Y \cdot g_{t'}^2, r_{t'}|$$

$$= |k_Y \cdot g_{t'}^2, r_{t'}|$$

Similarly $d_j^n = C^i k C^{n+1-i} : C^{n+1}Y \to C^n Y$ for $i = 0, 1, \ldots, n+1$

and $s_j^n = C^i k C^{n-i} : C^n Y \to C^{n+1} Y$ for $i = 0, 1, \ldots, n$.

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